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PARTITION OF NUMBERS AND APPLICATION TO HIGHER DERIVATIVES OF COMPOSITE FUNCTIONS

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PARTITION OF NUMBERS AND APPLICATION TO HIGHER DERIVATIVES OF COMPOSITE FUNCTIONS*

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ABSTRACT

Formulae and algorithms are presented for computing the partitions and the number of partitions of positive integers. Results are given and applied through an equivalent form of the partition problem to the determination of higher derivatives of composite functions, in particular to the probability-generating functions in reactor noise analysis.

I. INTRODUCTION

In combinatory analysis one defines the unrestricted partitions, b(i), of a positive integral number, n, as the positive integer solutions to the equation

$$\sum_i b(i) = n \tag{1}$$

By collecting the terms of equal value on the left-hand side into subgroups, this equation can be written in an equivalent form,

$$\sum_{i=1}^{n} i\, c(i) = n\,. \tag{2}$$

Here each nonnegative integral coefficient, c(i), is the number of b(i)'s equal to i in Eq. 1. In the subsequent discussion we deal with the form given in the latter equation. We present computational algorithms for finding the solutions, c(i), and the number of different solutions of Eq. 2, and discuss some of their applications.

Nonnegative integer solutions c(i), also called c_i below, of Eq. 2 are needed in several problems. As an example consider the calculation of higher derivatives of a composite function

$$y = f_1(x_1), \tag{3}$$

where $x_1 = f(x)$.

It can be shown that the nth derivative of y with respect to x is expressed as[1,2]

$$D_x^n y = \sum \frac{n!}{c_1!\; c_2! \cdots c_n!} \left(D_{x_1}^s f_1\right) \left(\frac{D_x^1 f}{1!}\right)^{c_1} \times \left(\frac{D_x^2 f}{2!}\right)^{c_2} \cdots \left(\frac{D_x^n f}{n!}\right)^{c_n}$$

$$= \sum D_{x_1}^s f_1 \prod_{i=1}^{n} \frac{i}{c_i!} \left(\frac{D_x^i f}{i!}\right)^{c_i}\,, \tag{4}$$

where the summation is extended over all nonnegative integer solutions of Eq. 2 and where

$$s = \sum_{i=1}^{n} c_i. \tag{5}$$

In a similar way one may obtain the nth derivative of an r-fold composite function

$$y = f_r(x_r), \qquad (6)$$

where
$$x_r = f_{r-1}(x_{r-1}),$$
$$\vdots$$
$$x_2 = f_1(x_1),$$
$$x_1 = f(x),$$

by applying Eq. 4 repeatedly. This is done by substituting for $D^{\ell}_{x_k} f_k$, for $1 \le k \le r$,

$$D^{\ell}_{x_k} f_k = \sum D^{s}_{x_k} f_k \prod_{j=1}^{\ell} \frac{1}{c_j!}\left(\frac{D^{j}_{x_{k-1}} f_{k-1}}{j!}\right)^{c_j}, \qquad (7)$$

where the summation is done over all nonnegative integer solutions of

$$\sum_{j=1}^{\ell} jc_j = \ell \quad \text{and} \quad s = \sum_{j=1}^{\ell} c_j .$$

Another example of the solution of Eq. 2 is encountered in the statistics of rare events. Lüders[3] derives a generalized Poisson-type formula applicable to cases in which multiple events may occur simultaneously. If h_i is the given mean for the occurrence of an i-fold event, and the probabilities for the occurrence of single, double, and so on up to ℓ-fold events are independent, then the probability that exactly c_i i-fold events occur is given by Poisson's formula

$$P^{(i)}_{c_i} = \frac{e^{-h_i}\, h_i^{c_i}}{c_i!} . \qquad (8)$$

With the above assumptions the probability for the occurrence of n events is given by

$$P_n = \sum \prod_{i=1}^{n} P^{(i)}_{c_i} = e^{-\sum_{j=1}^{\ell} h_j} \sum \prod_{i=1}^{n} \frac{h_i^{c_i}}{c_i!} , \qquad (9)$$

where the summation is extended over all nonnegative integer solutions of

$$\sum_{i=1}^{n} i\, c_i = n.$$

Other aspects of the theory of partitions, as well as a comprehensive historical survey of the field with an extensive bibliography, has recently been given by Gupta.[4]

II. COMPUTING THE SOLUTIONS

The solutions of Eq. 2 are obtained by permuting the c_i's over an appropriate range of nonnegative integral values and comparing their sum with n. The number of permutations can be obviously limited to values $c_i \le [n/i]$, where the brackets indicate truncated integral quotients. For these ranges of c_i's the number of possible permutations is $\prod_{i=1}^{n} ([n/i] + 1)$. This number can be further reduced if the permutations for any $i < j$ are skipped whenever the sum $\sum_{i=j}^{n} i\, c_i$ exceeds n.

We have written a Fortran program DKOPF1 to perform the permutations described above and have run it on the CDC-6600 computers at Lawrence Radiation Laboratory–Berkeley. The permutations are done by starting with the smallest values of i. Table I lists the solutions and the sums

$$s = \sum_{i=1}^{n} c_i \quad \text{for } n \text{ from 1 to 10.}$$

With this algorithm it is necessary to compute the solutions to determine their number. For large values of n the computer time required becomes prohibitively long. We next present an algorithm in which the number of solutions of Eq. 2 is obtained without actually determining the solutions.

III. RECURRENCE FORMULA FOR THE NUMBER OF SOLUTIONS

By defining the number of solutions to Eq. 2, that is, the number of unrestricted partitions of n, as p(n), we can directly write

$$p(n) = 1 + \sum_{i=1}^{\left[\frac{n}{2}\right]} p(i) + \sum_{i=1}^{\left[\frac{n-1}{2}\right]} p(n,i) \,. \quad (10)$$

The three terms on the right-hand side correspond to different highest nonzero values, c_i, appearing in Eq. 2. The first term corresponds to the partition, which includes n, of which there is only one. The second term corresponds to the partitions that include as their highest integer a number between (n-1) and [(n+1)/2]. Only one such number, for instance (n-i), may appear in any one partition, and the remaining numbers, whose sum equals i, can be made up unrestrictedly in p(i) different ways. The third term on the right-hand side of Eq. 10 corresponds to the partitions that have as their highest number an integer between [(n-1)/2] and 1. Here we define p(n, i) as the number of restricted partitions of n that have i as their highest integer.

The quantity p(n, i) can be written as

$$p(n,i) = \sum_{k=1}^{i} p(n-i,k), \quad (11)$$

where the right-hand side corresponds to different restricted ways of making up the remainder (n-i) of a partition already containing one i. We can further split the sum above and write

$$\sum_{k=1}^{i} p(n-i,k) = \sum_{k=1}^{\left[\frac{n-i-1}{2}\right]} p(n-i,k) + \sum_{k=\left[\frac{n-i-1}{2}\right]+1}^{i} p(n-i,k). \quad (12)$$

The second term on the right-hand side corresponds to the restricted partitions of n with the highest integer greater than [(n-i-1)/2]. We next use the fact that the number of restricted partitions p(n-i, k) for k > [(n-i+1)/2] is equal to the number of unrestricted partitions of the quantity (n-i-k), that is, p(n-i-k). It then follows that

$$\sum_{k=\left[\frac{n-i-1}{2}\right]+1}^{i} p(n-i,k) = \sum_{k=1}^{i-\left[\frac{n-i-1}{2}\right]} p\left(n-i-k-\left[\frac{n-i-1}{2}\right]\right) = \sum_{k=1}^{i-\left[\frac{n-i-1}{2}\right]} p\left(\left[\frac{n-i}{2}\right] + 1 - k\right) \,, \quad (13)$$

where the integral arguments of the third form are equal to those of the middle form for any combinations of n and i.

Combining Eqs. 10 through 13, we obtain for the number of unrestricted partitions of n the recurrence formula

$$p(n) = 1 + \sum_{i=1}^{\left[\frac{n}{2}\right]} p(i) + \sum_{i=1}^{\left[\frac{n-1}{2}\right]} \left\{ \sum_{k=1}^{\left[\frac{n-i-1}{2}\right]} p(n-i,k) + \sum_{k=1}^{i-\left[\frac{n-i-1}{2}\right]} p\left(\left[\frac{n-i}{2}\right] + 1 - k\right) \right\}. \quad (14)$$

The number of unrestricted partitions according to Eq. 14 can be conveniently determined by setting up a table as shown in Table II. We start from the top of the table by filling first the columns for n, [n/2], and [(n-1)/2]. The columns following [n/2] give an appropriate number of unrestricted partitions, which have been computed earlier and are given in the column p(n). These terms correspond to the first sum on the right-hand side of Eq. 14. Similarily an appropriate number of restricted partitions corresponding to the latter part of Eq. 14 are written following the column $[(n-1)/2]$. Each entry in the columns of restricted partitions--for instance, $p(j,i)$-- is obtained by summing over i given restricted and unrestricted partitions in the (j-i)th row, as is indicated by Eq. 11. Thus for instance

$$\begin{aligned} p(7,2) &= p(5,1) + p(5,2) \\ &= 1 + 2 \\ &= 3 \end{aligned}$$

and

$$\begin{aligned} p(10,4) &= p(6,1) + p(6,2) + p(3) + p(2) \\ &= 1 + 3 + 3 + 2 \\ &= 9. \end{aligned}$$

In order to extend this computation to large values of n we have written a Fortran program DKOPF2 and run it on the CDC-6600 computers of Lawrence Radiation Laboratory–Berkeley. To save memory space the unrestricted and restricted partitions already computed are stored in opposite triangles of a square matrix of dimension equal to the maximum value of n.

The numbers of unrestricted partitions obtained for n up to 200 are given in Table III. With program DKOPF2 the computer time required with a CDC-6600 machine for computing the number of partitions for n up to 50 is 0.33 sec, and for n up to 200 is 4.45 sec. This compares with about 400 sec required for determining the solutions for n up to 50 with program DKOPF1.

An asymptotic formula for the number of unrestricted partitions has been given by Ramanujan.[5] This formula possesses an amazing accuracy and has been proven by Ramanujan to yield for large values of n a number nearest to the correct integer p(n). The results from the asymptotic formula have been checked by MacMahan and found to be in agreement for n up to 158, and in addition for n = 200 with the results obtained by using a recurrence formula to solve for the number of unrestricted partitions, p(n)'s, from the known expansion formula[7]

$$f(x) = 1 + \sum_{n=1}^{\infty} p(n)\, x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} \tag{15}$$

The values given by us in Table III for n up to 200 agree with these results.

IV. ILLUSTRATIVE EXAMPLE

As an example we consider the relationship between a probability-generating function y(x) and the probability densities $b_i(t)$,

$$y(x) = \sum_{i=0}^{\infty} b_i(t)\, x^i, \tag{16}$$

where t is a parameter. If y(x) is known, then $b_i(t)$ can be obtained as

$$b_i(t) = \frac{1}{i\,!}\left(\frac{d^i y(x)}{dx^i}\right)_{x=0}. \tag{17}$$

For instance, in the noise analysis of nuclear reactors, one encounters a specific probability-generating function given by Mogilner and Zolotukhin[8] of the form

$$\begin{aligned} y &= f_2(x_2) \\ &= c\, x_2^{-b/a}, \end{aligned} \tag{18}$$

$$\begin{aligned} x_2 &= f_1(x_1) \\ &= \frac{(1+x_1)^2\, e^{ax_1} - (1-x_1)^2\, e^{-ax_1}}{4\, x_1}, \end{aligned} \tag{19}$$

$$\begin{aligned} x_1 &= f(x) \\ &= [1 + d\,(1-x)]^{1/2}, \end{aligned} \tag{20}$$

where a, b, c, and d are experimentally measured parameters. In the interpretation of neutron noise measurements the probability densities up to approximately the 20th order are needed. In order to compute the nth probability density according to Eq. 17 one has to know the nth derivative with respect to x of y(x) given in Eq. 18 through 20. Using Eq. 4 once gives

$$D_x^n y = \sum D_{x_1}^s f_2 \prod_{i=1}^{n} \frac{i}{c_i!}\left(\frac{D_x^i f}{i!}\right)^{c_i}, \quad (21)$$

where the summation is extended over all nonnegative integer solutions of Eq. 2 and where $s = \sum_{i=1}^{n} c_i$. The terms of the form $D_x^\ell f$ for $1 \le \ell \le n$ in Eq. 21 can readily be calculated from Eq. 20. However, to find $D_{x_1}^s f_2$ Eq. 4 is used a second time,

$$D_{x_1}^s f_2 = \sum D_{x_2}^{s'} f_2 \prod_{i=1}^{s} \frac{i}{c_i'!}\left(\frac{D_{x_1}^i f_1}{i!}\right)^{c_i'}, \quad (22)$$

where the summation is extended over all nonnegative integer solutions of $\sum_{i=1}^{s} i\, c_i' = s$ and where $s' = \sum_{i=1}^{s} c_i'$. It is obvious that $D_{x_1}^s f_2$ in Eq. 22 has to be calculated first, because it is then used in Eq. 21. $D_{x_2}^{s'} f_2$ can easily be calculated from Eq. 18. The terms of the form $D_{x_1}^\ell f_1$ for $1 \le \ell \le s$ are calculated from Eq. 19.

This demonstrates in practice how Eq. 4 through 7 are used for the case in which the nth derivative of a twofold composite function is needed. A more detailed description, and numerical results of this example, are given elsewhere.[9]

<u>Appendix.</u> The Fortran Listings of programs DKOPF1 and DKOPF2.

```
      PROGRAM DKOPF1(INPUT,OUTPUT,TAPE2=INPUT,TAPE3=OUTPUT)

C THIS PROGRAM COMPUTES THE SOLUTIONS TO THE PARTITION PROBLEM BY
C FINDING THE NONNEGATIVE INTEGER SOLUTIONS L(I) OF THE EQUATIONS
C SUM OF I*L(I), I=1,2,....,K, EQUAL TO K, FOR K=M1,M1+1,....,M2.
C THE METHOD AND THE RESULTS ARE DISCUSSED BY ROUTTI AND SZELESS
C IN UCRL-REPORT, PARTITION OF NUMBERS AND APPLICATION TO HIGHER
C DERIVATIVES OF COMPOSITE FUNCTIONS, SEPTEMBER 1970.

C THE ONLY DATA CARD SPECIFIES M1 AND M2 IN FORMAT(2I10).
C THIS IS A SCHEMATIC LISTING FOR M2 UP TO 50,
C THE THREE REPETITIVE SECTIONS OF THE CODE
C ARE ONLY PARTIALLY LISTED AND SHOULD BE
C COMPLETED , AS INDICATED BELOW, BEFORE
C COMPILATION AND EXECUTION.

      DIMENSION L(51)
      K0=0
      WRITE(3,1000)
 1000 FORMAT(56H1NONNEGATIVE INTEGER SOLUTIONS L(I) OF THE EQUATIONS SUM
     1/57H OF I*L(I),I=1,2,....,K, EQUAL TO K, FOR K=M1,M1+1,....,M2.
     2/38H THE SUM OF L(I) IS ALSO PRINTED.          ///)
C READ INPUT CARD
      READ(2,1001)M1,M2
 1001 FORMAT(2I10)
C PERMUTE OVER DIFFERENT VALUES OF K
      DO 200 K=M1,M2
C ISC IS THE NUMBER OF SOLUTIONS FOR EACH K, IS IS THE SUM COMPUTED
      ISC=0
      IS=0
C DETERMINE MAXIMUM VALUES OF L(I)
      N50=K/50
      N49=K/49
      N48=K/48
.  .  .  .                                                      .  .  .  .
.  .  .  .      COMPLETE IN A SIMILAR FASHION DOWN TO N4        .  .  .  .
.  .  .  .                                                      .  .  .  .
      N3 =K/3
      N2 =K/2
      N1 =K/1
C JUMP INTO THE OUTMOST LOOP CONSIDERED
      GO TO(50,49,48,47,46,45,44,43,42,41,40,39,38,37,36,35,34,33,32,31,
     130,29,28,27,26,25,24,23,22,21,20,19,18,17,16,15,14,13,12,11,10,9,8
     2,7,6,5,4,3,2,1) K
C IN EACH LOOP COMPUTE THE INCREASED SUM AND CHECK IT AGAINST K
    1 DO 150 I50=0,N50
      IS=IS+I50*50
      KK=50
      L(50)=I50
      IF(IS.GE.K) I50=N50
      IF(IS.GE.K) GO TO 65
    2 DO 149 I49=0,N49
      IS=IS+I49*49
      KK=49
      L(49)=I49
      IF(IS.GE.K) I49=N49
      IF(IS.GE.K) GO TO 65
    3 DO 148 I48=0,N48
      IS=IS+I48*48
      KK=48
      L(48)=I48
      IF(IS.GE.K) I48=N48
      IF(IS.GE.K) GO TO 65
```

```
 .   .   .   .        COMPLETE IN A SIMILAR FASHION DOWN TO 47         .   .   .   .
 .   .   .   .                                                         .   .   .   .
 .   .   .   .                                                         .   .   .   .
   48 DO 103 I3 =0,N3
      IS=IS + I3*3
      KK=3
      L(3 )=I3
      IF(IS.GE.K) I3=N3
      IF(IS.GE.K) GO TO 65
   49 DO 102 I2 =0,N2
      IS=IS + I2*2
      KK=2
      L(2 )=I2
      IF(IS.GE.K) I2=N2
      IF(IS.GE.K) GO TO 65
   50 DO 101 I1 =0,N1
      IS=IS + I1*1
      KK=1
      L(1 )=I1
      IF(IS.GE.K) I1=N1
      IF(IS.GE.K) GO TO 65
      GO TO 69
   65 IF(IS.GT.K) GO TO 69
      ISC=ISC+1
      IF(ISC.NE.1) GO TO 66
      WRITE(3,1003)(I,I=1,K)
 1003 FORMAT(//17H           K      S   ,30I3/17X,20I3)
      WRITE(3,1004)
 1004 FORMAT(/)
C COMPUTE SUM OF L(I)
   66 IP=0
      DO 67 I=KK,K
   67 IP=IP+L(I)
C PRINT THE SOLUTIONS
      IF(KK.EQ.1)GO TO 68
      II=KK-1
      WRITE( 3,1005) ISC,K,IP,(K0,I=1,II),(L(I),I=KK,K)
      GO TO 69
   68 CONTINUE
      WRITE( 3,1005) ISC,K,IP,(L(I), I=KK,K)
 1005 FORMAT(4I5,30I3/20X,20I3)
   69 CONTINUE
C SUBTRACT THE LAST TERM FROM THE SUM
   70 GO TO(101,102,103,104,105,106,107,108,109,110,111,112,113,114,115,
     1         116,117,118,119,120,121,122,123,124,125,126,127,128,129,130,
     2         131,132,133,134,135,136,137,138,139,140,141,142,143,144,145,
     3         146,147,148,149,150) KK
  101 IS=IS-L(1)
  102 IS=IS-L(2)*2
  103 IS=IS-L(3)*3
 .   .   .   .
 .   .   .   .        COMPLETE IN A SIMILAR FASHION DOWN TO 147        .   .   .   .
 .   .   .   .                                                         .   .   .   .
  148 IS=IS-L(48)*48
  149 IS=IS-L(49)*49
  150 IS=IS-L(50)*50
  200 CONTINUE
      STOP
      END
```

```
      PROGRAM DKOPE2(INPUT,OUTPUT)
C THIS PROGRAM COMPUTES THE NUMBER OF UNRESTRICTED PARTITIONS, S(N), OF
C N, FOR N=NSTART,NSTART+1,...,NSTOP. A RECURRENCE FORMULA IS USED WHICH
C DOES NOT REQUIRE COMPUTING THE PARTITIONS. THE METHOD AND THE RESULTS
C ARE DISCUSSED BY ROUTTI AND SZELESS IN UCRL-REPORT, PARTITION OF
C NUMBERS AND APPLICATION TO HIGHER DERIVATIVES OF COMPOSITE FUNCTIONS,
C SEPTEMBER 1970.

C THE ONLY DATA CARD SPECIFIES NSTART AND NSTOP IN FORMAT(2I10).
C THE LOWER LEFT TRIANGLE OF B MATRIX STORES THE NUMBERS OF UNRESTRICTED
C PARTITIONS AND THE UPPER RIGHT TRIANGLE THE NUMBERS OF RESTRICTED
C PARTITIONS.

      COMMON B(200,200),S(200)
      NMAX=200
      READ 1002,NSTART,NSTOP
 1002 FORMAT(2I10)
      DO 10 I=1,200
      S(I)=0.
      DO 10 J=1,200
   10 B(I,J)=0.
      DO 500 N=NSTART,NSTOP
      N2=N/2
      IF(N2.LE.0) GO TO 110
      DO 100 J=1,N2
      JB=NMAX-J+1
  100 B(N,JB)=S(J)
  110 N12=(N-1)/2
      IF(N12.LE.0) GO TO 300
      DO 200 J=1,N12
      B(N,J)=0.
      K1=(N-J-1)/2
      IF(J.GT.K1) GO TO 150
      DO 120 K=1,J
  120 B(N,J)=B(N,J)+B(N-J,K)
      GO TO 200
  150 DO 160 K=1,K1
  160 B(N,J)=B(N,J)+B(N-J,K)
      K2=(N-J)/2+1
      JK1=J-K1
      DO 170 K=1,JK1
      KB=NMAX-K2+K+1
  170 B(N,J)=B(N,J)+B(N-J ,KB)
  200 CONTINUE
  300 S(N)=1.
      IF(N2.LE.0) GO TO 350
      DO 310 I=1,N2
      IB=NMAX-I+1
  310 S(N)=S(N)+B(N,IB)
  350 IF(N12.LE.0) GO TO 400
      DO 360 I=1,N12
  360 S(N)=S(N)+B(N,I)
 1000 FORMAT(I5,F25.0)
  500 CONTINUE
      STOP
      END
```

FOOTNOTES AND REFERENCES

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Table I. The solutions c_i and the sums of the solutions, s, for n = 1, 2, ..., 10.

	n	s	c_1	2	3	4	5	6	7	8
1	1	1	1							
1	2	2	2	0						
2	2	1	0	1						
1	3	3	3	0	0					
2	3	2	1	1	0					
3	3	1	0	0	1					
1	4	4	4	0	0	0				
2	4	3	2	1	0	0				
3	4	2	0	2	0	0				
4	4	2	1	0	1	0				
5	4	1	0	0	0	1				
1	5	5	5	0	0	0	0			
2	5	4	3	1	0	0	0			
3	5	3	1	2	0	0	0			
4	5	3	2	0	1	0	0			
5	5	2	0	1	1	0	0			
6	5	2	1	0	0	1	0			
7	5	1	0	0	0	0	1			
1	6	6	6	0	0	0	0	0		
2	6	5	4	1	0	0	0	0		
3	6	4	2	2	0	0	0	0		
4	6	3	0	3	0	0	0	0		
5	6	4	3	0	1	0	0	0		
6	6	3	1	1	1	0	0	0		
7	6	2	0	0	2	0	0	0		
8	6	3	2	0	0	1	0	0		
9	6	2	0	1	0	1	0	0		
10	6	2	1	0	0	0	1	0		
11	6	1	0	0	0	0	0	1		
1	7	7	7	0	0	0	0	0	0	
2	7	6	5	1	0	0	0	0	0	
3	7	5	3	2	0	0	0	0	0	
4	7	4	1	3	0	0	0	0	0	
5	7	5	4	0	1	0	0	0	0	
6	7	4	2	1	1	0	0	0	0	
7	7	3	0	2	1	0	0	0	0	
8	7	3	1	0	2	0	0	0	0	
9	7	4	3	0	0	1	0	0	0	
10	7	3	1	1	0	1	0	0	0	
11	7	2	0	0	1	1	0	0	0	
12	7	3	2	0	0	0	1	0	0	
13	7	2	0	1	0	0	1	0	0	
14	7	2	1	0	0	0	0	1	0	
15	7	1	0	0	0	0	0	0	1	
1	8	8	8	0	0	0	0	0	0	0
2	8	7	6	1	0	0	0	0	0	0
3	8	6	4	2	0	0	0	0	0	0
4	8	5	2	3	0	0	0	0	0	0
5	8	4	0	4	0	0	0	0	0	0
6	8	6	5	0	1	0	0	0	0	0
7	8	5	3	1	1	0	0	0	0	0
8	8	4	1	2	1	0	0	0	0	0
9	8	4	2	0	2	0	0	0	0	0
10	8	3	0	1	2	0	0	0	0	0
11	8	5	4	0	0	1	0	0	0	0
12	8	4	2	1	0	1	0	0	0	0
13	8	3	0	2	0	1	0	0	0	0
14	8	3	1	0	1	1	0	0	0	0
15	8	2	0	0	0	2	0	0	0	0
16	8	4	3	0	0	0	1	0	0	0
17	8	3	1	1	0	0	1	0	0	0
18	8	2	0	0	1	0	1	0	0	0
19	8	3	2	0	0	0	0	1	0	0
20	8	2	0	1	0	0	0	1	0	0
21	8	2	1	0	0	0	0	0	1	0
22	8	1	0	0	0	0	0	0	0	1

	n	s	c_1	2	3	4	5	6	7	8	9	10
1	9	9	9	0	0	0	0	0	0	0	0	
2	9	8	7	1	0	0	0	0	0	0	0	
3	9	7	5	2	0	0	0	0	0	0	0	
4	9	6	3	3	0	0	0	0	0	0	0	
5	9	5	1	4	0	0	0	0	0	0	0	
6	9	7	6	0	1	0	0	0	0	0	0	
7	9	6	4	1	1	0	0	0	0	0	0	
8	9	5	2	2	1	0	0	0	0	0	0	
9	9	4	0	3	1	0	0	0	0	0	0	
10	9	5	3	0	2	0	0	0	0	0	0	
11	9	4	1	1	2	0	0	0	0	0	0	
12	9	3	0	0	3	0	0	0	0	0	0	
13	9	6	5	0	0	1	0	0	0	0	0	
14	9	5	3	1	0	1	0	0	0	0	0	
15	9	4	1	2	0	1	0	0	0	0	0	
16	9	4	2	0	1	1	0	0	0	0	0	
17	9	3	0	1	1	1	0	0	0	0	0	
18	9	3	1	0	0	2	0	0	0	0	0	
19	9	5	4	0	0	0	1	0	0	0	0	
20	9	4	2	1	0	0	1	0	0	0	0	
21	9	3	0	2	0	0	1	0	0	0	0	
22	9	3	1	0	1	0	1	0	0	0	0	
23	9	2	0	0	0	1	1	0	0	0	0	
24	9	4	3	0	0	0	0	1	0	0	0	
25	9	3	1	1	0	0	0	1	0	0	0	
26	9	2	0	0	1	0	0	1	0	0	0	
27	9	3	2	0	0	0	0	0	1	0	0	
28	9	2	0	1	0	0	0	0	1	0	0	
29	9	2	1	0	0	0	0	0	0	1	0	
30	9	1	0	0	0	0	0	0	0	0	1	
1	10	10	10	0	0	0	0	0	0	0	0	0
2	10	9	8	1	0	0	0	0	0	0	0	0
3	10	8	6	2	0	0	0	0	0	0	0	0
4	10	7	4	3	0	0	0	0	0	0	0	0
5	10	6	2	4	0	0	0	0	0	0	0	0
6	10	5	0	5	0	0	0	0	0	0	0	0
7	10	8	7	0	1	0	0	0	0	0	0	0
8	10	7	5	1	1	0	0	0	0	0	0	0
9	10	6	3	2	1	0	0	0	0	0	0	0
10	10	5	1	3	1	0	0	0	0	0	0	0
11	10	6	4	0	2	0	0	0	0	0	0	0
12	10	5	2	1	2	0	0	0	0	0	0	0
13	10	4	0	2	2	0	0	0	0	0	0	0
14	10	4	1	0	3	0	0	0	0	0	0	0
15	10	7	6	0	0	1	0	0	0	0	0	0
16	10	6	4	1	0	1	0	0	0	0	0	0
17	10	5	2	2	0	1	0	0	0	0	0	0
18	10	4	0	3	0	1	0	0	0	0	0	0
19	10	5	3	0	1	1	0	0	0	0	0	0
20	10	4	1	1	1	1	0	0	0	0	0	0
21	10	3	0	0	2	1	0	0	0	0	0	0
22	10	4	2	0	0	2	0	0	0	0	0	0
23	10	3	0	1	0	2	0	0	0	0	0	0
24	10	6	5	0	0	0	1	0	0	0	0	0
25	10	5	3	1	0	0	1	0	0	0	0	0
26	10	4	1	2	0	0	1	0	0	0	0	0
27	10	4	2	0	1	0	1	0	0	0	0	0
28	10	3	0	1	1	0	1	0	0	0	0	0
29	10	3	1	0	0	1	1	0	0	0	0	0
30	10	2	0	0	0	0	2	0	0	0	0	0
31	10	5	4	0	0	0	0	1	0	0	0	0
32	10	4	2	1	0	0	0	1	0	0	0	0
33	10	3	0	2	0	0	0	1	0	0	0	0
34	10	3	1	0	1	0	0	1	0	0	0	0
35	10	2	0	0	0	1	0	1	0	0	0	0
36	10	4	3	0	0	0	0	0	1	0	0	0
37	10	3	1	1	0	0	0	0	1	0	0	0
38	10	2	0	0	1	0	0	0	1	0	0	0
39	10	3	2	0	0	0	0	0	0	1	0	0
40	10	2	0	1	0	0	0	0	0	1	0	0
41	10	2	1	0	0	0	0	0	0	0	1	0
42	10	1	0	0	0	0	0	0	0	0	0	1

Table II. Recurrence table for the number of unrestricted partitions p(n) of an integer n.

$$p(n) = 1 + \sum_{i=1}^{\left[\frac{n}{2}\right]} p(i) + \sum_{i=1}^{\left[\frac{n-1}{2}\right]} p(n,i)$$

n	p(n)	1	$\left[\frac{n}{2}\right]$	p(1)	p(2)	p(3)	p(4)	p(5)	p(6)	$\left[\frac{n-1}{2}\right]$	p(n, 1)	p(n, 2)	p(n, 3)	p(n, 4)	p(n, 5)
1	1	1	0							0					
2	2	1	1	1						0					
3	3	1	1	1						1	1				
4	5	1	2	1	2					1	1				
5	7	1	2	1	2					2	1	2			
6	11	1	3	1	2	3				2	1	3			
7	15	1	3	1	2	3				3	1	3	4		
8	22	1	4	1	2	3	5			3	1	4	5		
9	30	1	4	1	2	3	5			4	1	4	7	6	
10	42	1	5	1	2	3	5	7		4	1	5	8	9	
11	56	1	5	1	2	3	5	7		5	1	5	10	11	10
.	.		.	.	.	.	.	.	.	.	.	.	.	.	.
.	.		.	.	.	.	.	.	.	.	.	.	.	.	.

Table III. Number of unrestricted partitions for n from 1 to 200.

n	p(n)	n	p(n)	n	p(n)	n	p(n)
1	1	51	239943	101	214481126	151	45060624582
2	2	52	281589	102	241265379	152	49686288421
3	3	53	329931	103	271248950	153	54770336324
4	5	54	386155	104	304801365	154	60356673280
5	7	55	451276	105	342325709	155	66493182097
6	11	56	526823	106	384276336	156	73232243759
7	15	57	614154	107	431149389	157	80630964769
8	22	58	715220	108	483502844	158	88751778802
9	30	59	831820	109	541946240	159	97662728555
10	42	60	966467	110	607163746	160	107438159466
11	56	61	1121505	111	679903203	161	118159068427
12	77	62	1300156	112	761002156	162	129913904637
13	101	63	1505499	113	851376628	163	142798995930
14	135	64	1741630	114	952050665	164	156919475295
15	176	65	2012558	115	1064144451	165	172389800255
16	231	66	2323520	116	1188908248	166	189334822579
17	297	67	2679689	117	1327710076	167	207890420102
18	385	68	3087735	118	1482074143	168	228204732751
19	490	69	3554345	119	1653668665	169	250438925115
20	627	70	4087968	120	1844349560	170	274768617130
21	792	71	4697205	121	2056148051	171	301384802048
22	1002	72	5392783	122	2291320912	172	330495499613
23	1255	73	6185689	123	2552338241	173	362326859895
24	1575	74	7089500	124	2841940500	174	397125074750
25	1958	75	8118264	125	3163127352	175	435157697830
26	2436	76	9289091	126	3519222692	176	476715857290
27	3010	77	10619863	127	3913864295	177	522115831195
28	3718	78	12132164	128	4351078600	178	571701605655
29	4565	79	13848650	129	4835271870	179	625846753120
30	5604	80	15796476	130	5371315400	180	684957390936
31	6842	81	18004327	131	5964539504	181	749474411781
32	8349	82	20506255	132	6620830889	182	819876908323
33	10143	83	23338469	133	7346629512	183	896684817527
34	12310	84	26543660	134	8149040695	184	980462880430
35	14883	85	30167357	135	9035836076	185	1071823774337
36	17977	86	34262962	136	10015581680	186	1171432692373
37	21637	87	38887673	137	11097645016	187	1280011042268
38	26015	88	44108109	138	12292341831	188	1398341745571
39	31185	89	49995925	139	13610949895	189	1527273599625
40	37338	90	56634173	140	15065878135	190	1667727404093
41	44583	91	64112359	141	16670689208	191	1820701100652
42	53174	92	72533807	142	18440293320	192	1987276856363
43	63261	93	82010177	143	20390982757	193	2168627105469
44	75175	94	92669720	144	22540654445	194	2366022741845
45	89134	95	104651419	145	24908858009	195	2580840212973
46	105558	96	118114304	146	27517052599	196	2814570987591
47	124754	97	133230930	147	30388671978	197	3068829878530
48	147273	98	150198136	148	33549419497	198	3345365983698
49	173525	99	169229875	149	37027355200	199	3646072432125
50	204226	100	190569292	150	40853235313	200	3972999029388