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AN APPROXIMATE THEORY GOVERNING AXISYMMETRIC MOTIONS IN TRANSVERSELY ISOTROPIC RODS

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I. INTRODUCTION

In this study three approximate theories governing axisymmetric motions in transversely isotropic rods are developed. In choosing to develop these theories, we have taken the lead from isotropic rods for which three approximate theories have proved to be most useful.

The first is the classical theory which, while it accurately represents the motion for only a small range of wave lengths and frequencies, is very simple in form. It is used to advantage in much research such as building a theory of one-dimensional wave propagation in viscoelastic materials and for determining material functions (creep and relaxation functions) by means of experiments. The second is the classical theory with a term added representing radial inertia. This theory was suggested by Rayleigh[1] and developed by Love,[2] and is the simplest theory which reflects geometric dispersion in a rod. It is used much the same as classical theory but when somewhat more accuracy is desirable. When any substantial frequency range is required, such as in transient wave propagation, a more sophisticated approximate theory is necessary. Such a theory for isotropic elastic rods is due to Mindlin and McNiven[3] and is valid for frequencies that range from zero to a frequency just below the cut-off frequency of the second axial shear mode. Such detail requires that this approximate theory is much more complex than the first two.

The derivation of the classical theory is brief as there is nothing new in it. It is based on the assumption that plane cross sections remain plane. By neglecting radial inertia the surviving equation has the form of the classical wave equation. In this equation the phase velocity is given in terms of four elastic constants and the mass density.

The second theory which includes the term representing radial inertia is not developed exactly as Love[2] developed the comparable theory for isotropic rods. We choose to derive the governing equation by describing a potential energy functional and then requiring that the value of the functional be a minimum.

The bulk of the paper is devoted to the third theory. Before beginning, it was necessary to ascertain, at least for a limited number of transversely isotropic materials, how many modes such an approximate theory should contain. This judgment could only be made after examining the cut-off frequencies derived from the exact three-dimensional theory. This theory was developed in an earlier paper[4] and the frequency equation was explored numerically for two separate transversely isotropic materials; a fiber reinforced material and magnesium. Study of both materials indicated that a three-mode approximate theory was the most sensible when choosing a theory that extends beyond the fundamental mode. This conclusion is dictated by the fact that the second and third cut-off frequencies are close to one another, resulting in strong coupling between the two modes, whereas the cut-off frequency of the fourth mode is much higher indicating that the influence of this mode and all higher modes on the lowest three spectral lines would be much less pronounced.

The actual development of the theory needs little discussion as it follows in many respects the method used by Mindlin and McNiven in developing their theory for isotropic rods. We do, however, make one significant departure. After defining the generalized displacements and generalized strains we derive the equilibrium and constitutive equations and the kinematic relations using a variational theorem due to Hu and Washizu.[5] For completeness the theorem is stated in the body of the derivation of the theory.

To improve the approximate theory, we introduce adjustment factors into the theory and establish the values of the factors by matching, as did Mindlin and McNiven, the cut off frequencies and curvatures at cut-off of the second and third modes derived from both the approximate and exact theories. As it is the long wavelength motions that will be predominant when the rod is excited, it is logical to do the matching where the wavelength is infinite, but it is by no means the only way the matching could have been accomplished. Suggestions have been made, regarding the isotropic rod, that an alternate match could be made by equating the asymptotic phase velocity of the fundamental spectral line to the Rayleigh velocity or that points could be matched on the complex branch. In spite of these suggestions, the two theories were matched at cut-off for two reasons; first, the matching is easiest, and second, with such matching the two fundamental lines match extremely well, as do the complex branches at the frequency of the "end mode," at least for isotropic rods.

After the theory is developed, trial solutions are adopted which lead to a frequency equation. This equation is explored numerically for the two separate materials. Sufficient roots are established to show the three spectral lines on the frequency-wave propagation constant plane on which the propagation constant is real. These lines are shown in Figs. 1 and 2 on which are shown the same spectral lines from the exact theory. Inspection of the figures shows that the three spectral lines from the approximate theory match very closely those from the exact. It is interesting to note that for magnesium, the second branch displays the peculiar shape that it did for isotropic materials but that for fiber reinforced material the second branch has a positive curvature at cut-off and is a minimum there.

Finally, we present an Appendix A in which two things are established. First, the conditions under which the solution of the approximate theory will be unique, and second, the orthogonality conditions for mode shapes of free vibrations of finite rods are established.

II. DEVELOPMENT OF APPROXIMATE THEORIES

The rod is referred to a cylindrical coordinate system (r, θ, z) so that the origin lies on the axis of the rod and so that the z axis and the axis of the rod coincide. Further, we orient the material of the rod so that axes of isotropy are parallel to the axis of the rod. The rod is solid and of radius "a."

In the development which follows, when it is appropriate, we use indicial notation and all the rules that apply to its use. Because the field equations for transversely isotropic elasticity are common to all three theories, we begin by setting down these equations.

As the deformations are axisymmetric and torsionless, and because the axes of material isotropy are parallel to the axis of the rod, we may assume the displacement field in the form

$$\begin{aligned} u_r &= u_r(r,z,t) \\ u_z &= u_z(r,z,t) \\ u_\theta &\equiv 0 \ . \end{aligned} \tag{1}$$

It follows that the strain-displacement relations in cylindrical coordinates become

$$\begin{aligned} \epsilon_{rr} &= u_{r,r} & \epsilon_{r\theta} &\equiv 0 \\ \epsilon_{\theta\theta} &= \frac{u_r}{r} & \epsilon_{rz} &= \frac{1}{2}\left(u_{z,r} + u_{r,z}\right) \\ \epsilon_{zz} &= u_{z,z} & \epsilon_{\theta z} &\equiv 0 \ . \end{aligned} \tag{2}$$

The constitutive relation for transversely isotropic materials is given by

$$\tau_\alpha = c_{\alpha\beta}\epsilon_\beta \quad (\alpha, \beta = 1 - 6), \tag{3}$$

where

$$(\tau_\alpha) = (\tau_{rr}, \tau_{\theta\theta}, \tau_{zz}, \tau_{\theta z}, \tau_{zr}, \tau_{r\theta})$$

$$(\epsilon_\alpha) = (\epsilon_{rr}, \epsilon_{\theta\theta}, \epsilon_{zz}, 2\epsilon_{\theta z}, 2\epsilon_{zr}, 2\epsilon_{r\theta}) , \tag{4}$$

and

$$(c_{\alpha\beta}) = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11}-c_{12}) \end{bmatrix} . \tag{5}$$

For all elastic materials the coefficient matrix $c_{\alpha\beta}$ is symmetric, i.e.,

$$c_{\alpha\beta} = c_{\beta\alpha} , \tag{6}$$

and it is positive definite in the sense

$$c_{\alpha\beta}V_\alpha V_\beta \geq 0 \tag{7}$$

for all six-dimensional vectors V_α , where equality is satisfied only when $V_\alpha \equiv 0$.

We note that for a transversely isotropic material there are five independent elastic constants, namely, c_{11}, c_{12}, c_{13}, c_{33}, c_{44}, and that the necessary and sufficient conditions for positive definiteness of the $c_{\alpha\beta}$ are

$$c_{11} > |c_{12}|, \; (c_{11} + c_{12})\, c_{33} > 2c_{13}^{2}, \; c_{44} > 0. \tag{8}$$

From Eqs. (1-5) we see that $\tau_{r\theta} = \tau_{\theta z} \equiv 0$, so that the stress equations of motion, without body forces, become

$$\tau_{rr},_{r} + \tau_{rz},_{z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = \rho\, \ddot{u}_{r}$$

$$\tau_{rz},_{r} + \tau_{zz},_{z} + \frac{\tau_{rz}}{r} = \rho\, \ddot{u}_{z} \,, \tag{9}$$

where $(\dot{\ }) = \frac{\partial(\ )}{\partial t}$ and ρ is the mass density. The third equation is satisfied identically.

II-1. Classical Theory

The classical theory begins with the kinematical assumption that

$$u_{z} = u_{z}(z,t) \,, \tag{10}$$

i.e., during deformation plane sections remain plane. Furthermore, we neglect all the stresses except τ_{zz} . If we make use of the constitutive equations and strain-displacement relations, the above assumptions concerning kinematical variables and state of stress imply that

$$u_{r} = -\eta\, r\, u_{z},_{z} \,, \tag{11}$$

where

$$\eta = \frac{c_{13}}{c_{11} + c_{12}} \,.$$

If we neglect radial inertia, the first of stress equations of motion, Eq. (9), will be satisfied, whereas the second of Eq. (9) together with the constitutive equations and strain-displacement relations leads to the governing equation of the classical theory:

$$u_{z},_{zz} = \frac{1}{V_e^2} \ddot{u}_z , \tag{12}$$

where

$$V_e = \left\{ \frac{c_{33}(c_{11} + c_{12}) - 2c_{13}^2}{\rho(c_{11} + c_{12})} \right\}^{\frac{1}{2}} .$$

The frequency equation corresponding to Eq. (12) is

$$\omega = V_e \alpha , \tag{13}$$

where α is the wave propagation constant and ω the angular frequency.

In dimensionless form Eq. (13) is written

$$\Omega = \hat{V}_e \zeta , \tag{14}$$

where

$$\Omega = \frac{\omega}{\omega_1^s}$$

$$\zeta = \frac{a\alpha}{\delta} \tag{15}$$

$$\hat{V}_e = \frac{V_e}{G_{ns}} .$$

In Eqs.(15): $\omega_1^s = \frac{\delta\, G_{ns}}{a}$ is the first axial shear cut-off frequency,

$G_{ns} = \left(\frac{c_{44}}{\rho}\right)^{\frac{1}{2}}$ is the shear-wave velocity and δ is a constant defined as the first non-zero root of $J_1(\delta_m) = 0$, where J_1 is the Bessel function of the first kind.

As $\hat{V}_e$ is a constant, Eq. (14) represents a straight line on the $\Omega - \zeta$ plane emanating from the origin. As this line is tangent at the origin to the fundamental spectral line of the exact theory (see Ref. 4), it represents the correct relationship between Ω and ζ for a small neighborhood of the origin only (Figs. 1-2). It accommodates no dispersion.

II-2. <u>One-Dimensional Theory with Radial Inertia Correction</u>

In this section the classical theory is improved by the inclusion of a term representing radial inertia. The resulting equation will be equivalent to that suggested by Rayleigh for isotropic rods. The theory is based on the same assumptions concerning the kinematical variables and the stress field as the classical theory and the radial inertia term is introduced using a theorem of minimum potential energy. For completeness, the statement of the theorem is recorded here.

Given: A volume V completely bounded by a surface S. A body force per unit of mass f_i is specified in V, tractions $t_i = \bar{t}_i$ are specified over part of the boundary S_T, and displacements $u_i = \bar{u}_i$ are specified over the remaining boundary S_u. We introduce the functional

$$J_{PE}\{u_i\} = \int_V \{U(\epsilon_{ij}) - \rho f_i u_i\} dV - \int_{S_T} \bar{t}_i u_i dS . \quad (16)$$

In Eq. (16) the u_i are the state variables and U is the strain energy density. Implicit in Eq. (16) are the relations $\tau_{ij} = \partial U/\partial\epsilon_{ij}$, $2\epsilon_{ij} = u_{i,j} + u_{j,i}$ and the symmetry of the tensor τ_{ij}.

Theorem: Among all of the states satisfying the boundary condition $u_i = \bar{u}_i$ on S_u, the state which will satisfy the equilibrium equation $\partial/\partial x_j \, (\partial U/\partial \epsilon_{ij}) + \rho f_i = 0$ in V and the boundary condition $t_i = \bar{t}_i$ on S_T is the same state as the one which will make the J_{PE} stationary, i.e.,

$$\delta J_{PE} = 0 , \tag{17}$$

where δ denotes the first variation. We note that the positive definiteness of U implies that the stationary value here is the minimum value.

We now proceed by applying to this theorem the state of stress and kinematical variables adopted in the development of the classical theory. When we do so we obtain

$$U = \tfrac{1}{2} \tau_{zz} \epsilon_{zz} = \tfrac{1}{2} E_c u_{z,z}^2$$
$$f_i u_i = f_z u_z - \eta r f_r u_{z,z} \quad , \tag{18}$$

where

$$E_c = \frac{c_{33}(c_{11} + c_{12}) - 2c_{13}^2}{c_{11} + c_{12}} .$$

If we substitute Eqs. (18) into Eq. (16) and carry out integration with respect to the area of the cross section "A," we find

$$J_{PE} = \int_{z_1}^{z_2} \{- \tfrac{1}{2} E_c A \, u_{z,z}^2 + \rho F_z u_z - \rho \eta F_r u_{z,z}\} \, dz \tag{19}$$
$$+ \text{ boundary condition terms,}$$

where

$$F_z = \int_A f_z dA$$
$$F_r = \int_A r f_r \, dA . \tag{20}$$

When we take the first variation of Eq. (19), from the requirement that $\delta J_{PE} = 0$ we obtain

$$E_c A \, u_{z,zz} + \rho F_z + \rho \eta F_{r,z} = 0. \tag{21}$$

In order to find the equation of motion for the dynamic case we use D'Alambert's principle, i.e., we let

$$f_z = - \ddot{u}_z$$
$$f_r = - \ddot{u}_r = \eta \, r \ddot{u}_{z,z} . \tag{22}$$

Using Eqs. (20,22), Eq.(21) can be written as

$$V_e^2 u_{z,zz} = \ddot{u}_z - \frac{\eta^2 a^2}{2} \ddot{u}_{z,zz} . \tag{23}$$

In the governing equation, Eq. (23), the last term on the right hand side represents the radial inertia correction to the classical theory.

The corresponding frequency equation of Eq. (23) is

$$\omega^2 = \frac{V_e^2 \alpha^2}{1 + \frac{\eta^2 a^2}{2} \alpha^2} , \tag{24}$$

or in dimensionless form

$$\Omega^2 = \frac{\hat{v}_e^2 \zeta^2}{1 + \frac{\pi^2 \delta^2}{2} \zeta^2} \quad . \tag{25}$$

To get a feeling for the improvement over the classical theory, the spectral line representing solutions of Eq. (25) is shown in Figs. 1-2. It can be seen that it matches the line representing the exact theory over a somewhat more extensive length than did the classical theory. However, the range of frequencies for which the theory can be considered to be applicable is still quite limited. The principle advantage of this theory is that it introduces some dispersion, that is some nonlinearity in the relationship between frequency and wave length.

II-3. Three Mode Theory

From the above discussions, we have seen that the classical one-dimensional theory approximates the fundamental mode of the exact theory with a straight line whereas the one-dimensional theory with the radial inertia correction brings some improvement to it for small wave propagation constants and introduces some dispersion, but it does not represent the fundamental mode for large wave propagation constants and does not take into account higher modes and coupling between them. In this section we develop a three-mode theory which reproduces the first three modes of the exact theory remarkably well for a reasonable range of frequencies and wave propagation constants.

The choice of a three-mode theory is arrived at after a study of the exact three-dimensional theory developed in Reference 4. In that study we found that, for two sets of elastic constants assumed there, the cut-off frequencies of the second and third mode are close to one another meaning there will be a high degree of coupling between these two modes and that both of these spectral lines are low enough to have a pronounced influence on the fundamental spectral line. However, the cut-off frequency of the fourth mode ($\Omega_2^s = 1.83$) is high up on the spectrum indicating that its influence and the influence of the still higher modes, while present, will be much less pronounced.

In developing this theory, the method used by Mindlin and McNiven for isotropic rods will be closely followed except that, at the appropriate place in the development, a variational theorem will be used. The choice is to use a Variational theorem due to Hu and Washizu[5] which will be reviewed here.

Given: A volume V completely bounded by a surface S. A body force f_i is specified in V, tractions $t_i = \bar{t}_i$ are specified over part of the boundary S_T, and displacements $u_i = \bar{u}_i$ are specified over the remaining boundary S_u. We introduce the following functional

$$J_{HW}(\tau_{ij}, \varepsilon_{ij}, u_i, t_i) = \int_V \left\{\rho f_i u_i - U(\varepsilon_{ij}) + \tau_{ij}\varepsilon_{ij} - \tau_{ij}u_{i,j}\right\} dV + \int_{S_T} \bar{t}_i u_i dS + \int_{S_u} t_i(u_i - \bar{u}_i) dS \quad . \tag{26}$$

In Eq. (26) τ_{ij}, ε_{ij}, u_i and t_i are state variables. We define an admissible state as one in which τ_{ij} and ε_{ij} are symmetric.

Theorem: Among all admissible states, the state which will satisfy the field equations of linear elasticity in V and boundary conditions $t_i = \bar{t}_i$ on S_T, $u_i = \bar{u}_i$ on S_u is the same state as the one which will make J_{HW} stationary, i.e.,

$$\delta J_{HW} = 0 \,. \tag{27}$$

We begin developing the approximate theory by expressing the radial and axial components of the displacement vector in series of Jacobi polynomials in the radial coordinate:

$$u_r = \sum_{n=0}^{\infty} U_n(\bar{r}) u_n(z, t)$$

$$u_z = \sum_{n=0}^{\infty} W_n(\bar{r}) w_n(z, t) \tag{28}$$

$$u_\theta \equiv 0 \;,$$

where

$$\bar{r} = \frac{r}{a} \;,$$

and

$$U_0(\bar{r}) = \bar{r} \;, \quad U_1(\bar{r}) = \bar{r} - \frac{3}{2}\bar{r}^3 \;, \ldots,$$
$$U_n(\bar{r}) = \bar{r} + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{(n+2)_k}{(k+1)!} \bar{r}^{2k+1} \tag{29}$$

$$W_0(\bar{r}) = 1 \;, \quad W_1(\bar{r}) = 1 - 2\bar{r}^2 \;, \ldots,$$
$$W_n(\bar{r}) = 1 + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{(n+1)_k}{k!} \bar{r}^{2k} \;, \tag{30}$$

in which

$$\binom{n}{k} = \frac{n(n-1)\ \dots\ (n-k+1)}{k!}$$

$$(\beta)_k = \beta(\beta+1)(\beta+2)\ \dots\ (\beta+k-1)\ , \quad (\beta)_0 = 1\ .$$

The polynomials U_n and W_n have the properties

$$4(n+1)^3 \int_0^1 \bar{r}\, U_m U_n d\bar{r} = \delta_{mn}$$

$$2(2n+1) \int_0^1 \bar{r}\, W_m W_n d\bar{r} = \delta_{mn}\ ,$$

where δ_{mn} is the usual Kroneker delta.

Each term in the series, Eqs. (28) represents a mode of motion. We retain in our theory only the first three modes: namely, the longitudinal, first radial, and first axial shear. The displacements appropriate to this theory are derived from Eqs. $(28)_1$, and $(28)_2$. As there is only one mode having radial motions, we retain only the first term in Eq. $(28)_1$ and we retain the first two terms of Eq. $(28)_2$ representing the longitudinal and first axial shear modes. The displacements are

$$u_r = \bar{r}u(z,\ t)$$

$$u_z = w(z,\ t) + (1 - 2\bar{r}^2)\psi(z,\ t) \qquad (31)$$

$$u_\theta \equiv 0\ ,$$

where

$$u = u_0 \ , \quad w = w_0 \ , \quad \psi = w_1 \ .$$

From Eqs. (2) and (31), for strain-displacement relations we get

$$\begin{aligned}
\varepsilon_{rr} &= \frac{u}{a} \\
\varepsilon_{\theta\theta} &= \frac{u}{a} \\
\varepsilon_{zz} &= w,_z + (1 - 2\bar{r}^2)\psi,_z \\
\varepsilon_{rz} &= \frac{\bar{r}}{2}\left(u,_z - \frac{4}{a}\psi\right) \\
\varepsilon_{\theta z} &= \varepsilon_{\theta r} \equiv 0 \ .
\end{aligned} \tag{32}$$

If we introduce the following generalized strains

$$\begin{aligned}
\Gamma_r &= \frac{u}{a} \\
\Gamma_z &= w,_z \\
\Gamma_\psi &= \psi,_z \\
\Gamma_{rz} &= u,_z - \frac{4}{a}\psi \ ,
\end{aligned} \tag{33}$$

then, Eqs. (32) become

$$\begin{aligned}
\varepsilon_{rr} &= \Gamma_r \\
\varepsilon_{\theta\theta} &= \Gamma_r \\
\varepsilon_{zz} &= \Gamma_z + (1 - 2\bar{r}^2)\Gamma_\psi \\
\varepsilon_{rz} &= \frac{\bar{r}}{2}\Gamma_{rz} \ .
\end{aligned} \tag{34}$$

Now, we proceed to apply the Hu-Washizu variational theorem to the problem. Using Eqs. (31) , (34) we obtain

$$U(\varepsilon_{ij}) = \frac{1}{2} c_{\alpha\beta}\varepsilon_\alpha\varepsilon_\beta$$

$$= \frac{1}{2}\Big\{2(c_{11} + c_{12})\Gamma_r^2 + 4c_{13}\Gamma_r[\Gamma_z + (1 - 2\bar{r}^2)\Gamma_\psi]$$

$$+ c_{33}[\Gamma_z^2 + (1 - 2\bar{r}^2)\Gamma_z\Gamma_\psi + (1 - 2\bar{r}^2)^2\Gamma_\psi^2]$$

$$+ c_{44}\bar{r}^2\Gamma_{rz}^2\Big\} \qquad (35)$$

$$\tau_{ij}\varepsilon_{ij} = \tau_{rr}\Gamma_r + \tau_{\theta\theta}\Gamma_r + \tau_{zz}[\Gamma_z + (1 - 2\bar{r}^2)\Gamma_\psi] + \bar{r}\,\tau_{rz}\Gamma_{rz}$$

$$\tau_{ij}u_{i',j} = \tau_{rr}\frac{u}{a} + \tau_{\theta\theta}\frac{u}{a} + \tau_{zz}[w,_z + (1 - 2\bar{r}^2)\psi,_z] + \bar{r}\,\tau_{rz}\left(u,_z - \frac{4}{a}\psi\right)$$

$$f_iu_i = \bar{r}\,f_r\,u + [w + (1 - 2\bar{r}^2)\psi]f_z \quad .$$

We note that Eqs. (35) are written in cylindrical coordinates using physical components of the tensors involved. If we assume that on the lateral surface of the rod, only the tractions τ_{rr} and τ_{rz} are specified, then the functional, Eq. (26) takes the form

$$J_{HW} = 2\pi a^2\int_0^1\int_{z_1}^{z_2}\left\{\rho f_iu_i - U(\varepsilon_{ij}) + \tau_{ij}\varepsilon_{ij} - \tau_{ij}u_{i',j}\right\}\bar{r}\,dzd\bar{r}$$

$$+ 2\pi a\int_{z_1}^{z_2}\bar{t}_iu_idz + \text{end boundary condition terms.} \qquad (36)$$

At this stage of the development we introduce adjustment factors $K_i(i = 1 - 4)$ for the following reason. It may be observed that w, ψ

are the amplitudes of uniform and quadratic distributions of axial displacement, and u is that of a linear distribution of radial displacement along a radius of the rod. Omission of the higher-order terms limits the applicability, of what is left, to relatively long wave lengths. In this range, the uniform distribution of axial displacement, whose amplitude is w, is a good approximation to the distribution of u_z found in the lowest mode of the exact solution for waves in an infinite rod. In fact the latter distribution approaches uniformity as the wave length approaches infinity. The linear and quadratic distributions of the other two displacements, i.e., those with amplitudes of u and ψ, are only fair approximations of the Bessel-function distributions found in the second and third modes of the exact theory,[4] even at very long wave lengths. It is advisable, therefore, to introduce means for compensating for the omission of the polynomials of higher degree. To this end, we replace Γ_r by $K_1\Gamma_r$, Γ_{rz} by $K_2\Gamma_{rz}$, u by K_3u and ψ by $K_4\psi$ in Eqs. $(35)_1$ and $(35)_4$ where the K_i are constants whose values will be determined so that the three spectral lines of the approximate theory match more closely the lowest three branches of the exact theory. Then, substituting Eqs. (35) into the functional, Eq. (36), and carrying out integration with respect to $\bar{r}$, we find

$$J_{HW} = \int_{z_1}^{z_2} \Big\{ 2\pi a^2 [\rho \bar{F}(\underline{F}, \underline{\bar{u}}) - \bar{U}(\underline{\Gamma}) + T(\underline{P}, \underline{\Gamma}) - S(\underline{P}, \underline{A})] + 2\pi a B(\underline{\bar{u}}) \Big\} dz + \text{e.b.c.t.}, \qquad (37)$$

where

$$\bar{F}(\underline{F}, \bar{\underline{u}}) = K_3F_u u + F_w w + K_4F_\psi \psi$$

$$\bar{U}(\underline{\Gamma}) = \frac{1}{2}\left\{K_1^2(c_{11} + c_{12})\Gamma_r^2 + 2K_1c_{13}\Gamma_r\Gamma_z + \frac{c_{33}}{2}\Gamma_z^2 + \frac{c_{33}}{6}\Gamma_\psi^2 + \frac{c_{44}}{4}K_2^2\Gamma_{rz}{}^2\right\}$$

$$T(\underline{P}, \underline{\Gamma}) = P_r\Gamma_r + P_z\Gamma_z + P_{rz}\Gamma_{rz} + P_\psi\Gamma_\psi \tag{38}$$

$$S(\underline{P}, \underline{A}) = P_r\frac{u}{a} + P_z w,_z + P_\psi\psi,_z + P_{rz}\left(u,_z - \frac{4}{a}\psi\right)$$

$$B(\bar{\underline{u}}) = Z(w - \psi) + Ru \quad .$$

In Eqs. (38):

$$F_u = \int_0^1 \bar{r}^2 f_r d\bar{r}$$

$$F_w = \int_0^1 \bar{r} f_z d\bar{r} \tag{39}$$

$$F_\psi = \int_0^1 (1 - 2\bar{r}^2)\bar{r} f_z d\bar{r}$$

$$P_r = \int_0^1 (\tau_{rr} + \tau_{\theta\theta})\bar{r} d\bar{r}$$

$$P_z = \int_0^1 \tau_{zz}\bar{r} d\bar{r} \tag{40}$$

$$P_\psi = \int_0^1 \tau_{zz}(1 - 2\bar{r}^2)\bar{r} d\bar{r}$$

$$P_{rz} = \int_0^1 \tau_{rz}\bar{r}^2 d\bar{r}$$

$$Z = \tau_{rz}\big|_{r=a} \tag{41}$$

$$R = \tau_{rr}\big|_{r=a} \quad ,$$

and

$$\underline{\bar{u}} = (\bar{u}_1, \bar{u}_2, \bar{u}_3) = (u, w, \psi)$$

$$\underline{F} = (F_1, F_2, F_3) = (F_u, F_w, F_\psi)$$

$$\underline{P} = (P_1, P_2, P_3, P_4) = (P_r, P_z, P_\psi, P_{rz}) \tag{42}$$

$$\underline{\Gamma} = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4) = (\Gamma_r, \Gamma_z, \Gamma_\psi, \Gamma_{rz})$$

$$\underline{\hat{A}} = (A_1, A_2, A_3, A_4, A_5) = (u, w_{,z}, \psi_{,z}, u_{,z}, \psi) \ .$$

If we take the first variation of Eq. (37), and integrate by parts using Eq. $(38)_4$ and the last of Eqs. (42), we find

$$\begin{aligned} \delta J_{HW} = 2\pi a^2 \int_{z_1}^{z_2} \Bigg\{ & \left(\rho \frac{\partial \bar{F}}{\partial \bar{u}_m} - \frac{\partial \bar{S}}{\partial \bar{u}_m} + \frac{1}{a}\frac{\partial B}{\partial \bar{u}_m}\right)\delta \bar{u}_m \\ & + \left(\frac{\partial T}{\partial \Gamma_s} - \frac{\partial \bar{U}}{\partial \Gamma_s}\right)\delta\Gamma_s \\ & + \left(\frac{\partial T}{\partial P_s} - \frac{\partial S}{\partial P_s}\right)\delta P_s \Bigg\} dz \\ & + \text{e.b.c.t.} \ , \end{aligned} \tag{43}$$

where

$$\bar{S}(\underline{P}, \underline{\bar{u}}) = \left(\frac{P_r}{a} - P_{rz,z}\right)u - P_{z,z}w - \left(P_{\psi,z} + \frac{4}{a}P_{rz}\right)\psi \ . \tag{44}$$

From the requirement $\delta J_{HW} = 0$, we obtain

$$\rho \frac{\partial \bar{F}}{\partial \bar{u}_m} - \frac{\partial \bar{S}}{\partial \bar{u}_m} + \frac{1}{a}\frac{\partial B}{\partial \bar{u}_m} = 0 \qquad \text{(the equilibrium equations)}$$

$$\frac{\partial T}{\partial \Gamma_s} - \frac{\partial \bar{U}}{\partial \Gamma_s} = 0 \qquad \text{(the constitutive equations)} \qquad (45)$$

$$\frac{\partial T}{\partial P_s} - \frac{\partial S}{\partial P_s} = 0 \qquad \text{(the kinematic relations)}$$

$$(m = 1, 2, 3 \; ; \quad s = 1, 2, 3, 4) \; .$$

If we use Eqs. (42) and substitute Eqs.(38, 44) into Eqs. (45) we find

(a) the equilibrium equations:

$$\rho K_3 F_u - \frac{P_r}{a} + P_{rz},_z + \frac{R}{a} = 0$$

$$\rho F_w + P_z,_z + \frac{Z}{a} = 0 \qquad (46)$$

$$\rho K_4 F_w + P_\psi,_z + \frac{4}{a} P_{rz} - \frac{Z}{a} = 0 \; ,$$

(b) the constitutive equations:

$$P_r = K_1^2(c_{11} + c_{12})\Gamma_r + K_1 c_{13}\Gamma_z$$

$$P_z = K_1 c_{13}\Gamma_r + \frac{c_{33}}{2}\Gamma_z \qquad (47)$$

$$P_\psi = \frac{c_{33}}{6}\Gamma_\psi$$

$$P_{rz} = \frac{c_{44}}{4} K_2^2 \Gamma_{rz} \; ,$$

(c) the kinematic relations:

$$\Gamma_r = \frac{u}{a}$$

$$\Gamma_z = w,_z \qquad (48)$$

$$\Gamma_\psi = \psi,_z$$

$$\Gamma_{rz} = u,_z - \frac{4}{a}\psi .$$

In order to find the equations of motion, we use D'Alembert's principle, i.e., we let

$$f_r = -\ddot{u}_r = -K_3\bar{r}\ddot{u}$$

$$f_z = -\ddot{u}_z = -\ddot{w} - (1 - 2\bar{r}^2)K_4\ddot{\psi} . \qquad (49)$$

Then, from Eqs. (39) we have

$$F_u = -\frac{K_3}{4}\ddot{u}$$

$$F_w = -\frac{1}{2}\ddot{w} \qquad (50)$$

$$F_\psi = -\frac{K_4}{6}\ddot{\psi} .$$

Substituting Eqs. (50) into Eqs. (46) we obtain the stress equations of motion in the form

$$P_{rz},_z - \frac{P_r}{a} + \frac{R}{a} = \rho\frac{K_3^2}{4}\ddot{u}$$

$$P_z,_z + \frac{Z}{a} = \frac{\rho}{2}\ddot{w} \qquad (51)$$

$$P_\psi,_z + \frac{4}{a}P_{rz} - \frac{Z}{a} = \rho\frac{K_4^2}{6}\ddot{\psi} .$$

Using Eqs. (47, 48) we can write the equations of motion in terms of the generalized displacements, i.e.,

$$c_{44}K_2^2(u,_{\xi\xi} - 4\psi,_{\xi}) - 4K_1^2(c_{11} + c_{12})u - 4K_1c_{13}w,_{\xi} + 4aR = \rho a^2K_3^2\ddot{u}$$

$$c_{33}w,_{\xi\xi} + 2K_1c_{13}u,_{\xi} + 2aZ = \rho a^2\ddot{w} \tag{52}$$

$$c_{33}\psi,_{\xi\xi} + 6c_{44}K_2^2(u,_{\xi} - 4\psi) - 6aZ = \rho a^2K_4^2\ddot{\psi} \quad ,$$

where

$$\xi = \frac{z}{a} \quad .$$

Eqs. (52) constitute the second order approximate theory.

It should be noted here that when the material is reduced to an isotropic one by letting

$$c_{12} = c_{13} = \lambda \ ; \quad c_{11} = c_{33} = \lambda + 2\mu \ ; \quad c_{44} = \mu$$

Eqs. (52) reduce, as they should, to the corresponding equations obtained by Mindlin and McNiven for isotropic rods.

The Frequency Equation

In order to find the frequency equation corresponding to the three mode approximate theory, we substitute

$$u = H_1 \cos \alpha z e^{i\omega t}$$

$$w = H_2 \sin \alpha z e^{i\omega t} \tag{53}$$

$$\psi = H_3 \sin \alpha z e^{i\omega t}$$

$$R = Z = 0$$

into Eqs. (52) . Then, the condition for having a nontrivial solution gives

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & 0 \\ a_{13} & 0 & a_{33} \end{vmatrix} = 0 \quad , \tag{54}$$

where

$$a_{11} = \delta^2 K_2^2 \zeta^2 + 4K_1^2 \frac{2 - \gamma_k}{\gamma_m^2} - K_3^2 \delta^2 \Omega^2$$

$$a_{22} = 2\delta^2(\gamma_n^2 \zeta^2 - \Omega^2)$$

$$a_{33} = 6\left\{\delta^2 \gamma_n^2 \zeta^2 + 24K_2^2 - \delta^2 K_4^2 \Omega^2\right\} \tag{55}$$

$$a_{12} = 4K_1 \frac{\gamma_c}{\gamma_m^2} \delta\zeta$$

$$a_{13} = 12K_2^2 \delta\zeta \quad ,$$

and

$$\gamma_m = \left(\frac{c_{44}}{c_{11}}\right)^{\frac{1}{2}} \quad ; \quad \gamma_n = \left(\frac{c_{33}}{c_{44}}\right)^{\frac{1}{2}} \quad ;$$

$$\gamma_c = \frac{c_{13}}{c_{11}} \quad ; \quad \gamma_k = \frac{c_{11} - c_{12}}{c_{11}} \quad .$$

Eq. (54) governs the relationship between the normalized frequency Ω and the dimensionless propagation constant ζ for the three modes included in the approximate theory.

The Adjustment Factors K_i

The adjustment factors K_i are introduced into the theory as undetermined coefficients so that they may be used to make the three spectral lines of the approximate theory match more close the corresponding three from the exact. As there are four factors, four properties of one set can be made to match four properties of the other. Which four properties should be chosen for matching is a subject for speculation. There is no clear rationale to dictate the choice. In making the choice, we anticipate the immediate use that we intend to make of the approximate theory. It will be used to get the transient response in a rod to a time dependent input on its end. It appears that in such a response, the long waves will predominate, so that the approximate theory should match the exact as closely as possible at the long wave length end of the spectrum, that is where ζ is zero.

Accordingly we choose to match, as did Mindlin and McNiven, the cut-off frequencies of the second and third modes and the curvatures of these same two modes at the cut-off frequencies. The fundamental modes match near the origin with no adjustment, as do the slopes, at cut-off, of the second and third modes.

The matching is most easily achieved if we formulate from Eq. (54) frequency equations that are valid for the three branches in the neighborhood of cut-off. That is if we take $\zeta \ll 1$, Eq. (54) can be written

$$\Omega = \hat{v}_e \, \zeta$$

$$\Omega = \bar{\Omega}_1^s + \tfrac{1}{2} \bar{c}_s \, \zeta^2 \tag{56}$$

$$\Omega = \bar{\Omega}_1^r + \tfrac{1}{2} \bar{c}_r \, \zeta^2 ,$$

where

$$\bar{\Omega}_1^s = \sqrt{24}\ \frac{K_2}{\delta K_4}\ ;\quad \bar{\Omega}_1^r = \frac{2K_1(2-\gamma_k)^{\frac{1}{2}}}{K_3\delta\gamma_m}\ . \tag{57}$$

The same formulation for the lowest three branches for the exact theory was made in Reference (4). These equations are

$$\begin{aligned}\Omega &= \hat{V}_e\ \zeta \\ \Omega &= 1 + \tfrac{1}{2}\, C_s\ \zeta^2 \\ \Omega &= \Omega_1^r + \tfrac{1}{2}\, C_r\ \zeta^2\ ,\end{aligned} \tag{58}$$

where

$$\Omega_1^r = \chi/\delta\gamma_m\ , \tag{59}$$

and χ is the first root of

$$\frac{\chi\ J_0(\chi)}{J_1(\chi)} = \gamma_k\ , \tag{60}$$

and the expressions for the curvatures C_s and C_r are given in Reference 4.

To get the first two equations governing the K_i's we equate the cut of frequencies $(\zeta = 0)$ of the second and third mode from each of Eqs. (56) and (58). This leads to the equations

$$\begin{aligned}\frac{K_2^2}{K_4^2} &= \frac{\delta^2}{24} \\ \frac{K_1^2}{K_3^2} &= \frac{\chi^2}{4(2-\gamma_k)}\ .\end{aligned} \tag{61}$$

After this matching, the curvatures of the second and third branches at cut-off from the approximate theory become

$$\bar{c}_s = \frac{\delta^2 \gamma_n^2}{24} \left\{ \frac{1}{K_2^2} + \frac{K_2^2}{K_1^2} \frac{6}{\gamma_n^2 (2 - \gamma_k) \left(\frac{\delta^2}{\chi^2} - \frac{1}{\gamma_m^2} \right)} \right\}$$

$$\bar{c}_r = \frac{2\delta\chi}{\gamma_m (2 - \gamma_k)} \left\{ \frac{\gamma_c^2}{\chi^2} + \frac{K_2^2}{K_1^2} \frac{\chi^2}{8 \left(\frac{\chi^2}{\gamma_m^2} - \delta^2 \right)} \right\} . \tag{62}$$

The third and fourth equations governing the K_i's are obtained by equating the curvatures of the second and third branches at cut-off from the two theories. The curvatures will match if

$$\bar{c}_r = c_r \ , \quad \bar{c}_s = c_s \ . \tag{63}$$

Eqs. (61) and (63) fix the four coefficients K_i as functions of the $c_{\alpha\beta}$.

III. NUMERICAL RESULTS

We present the numerical results for two kinds of materials which were also used for numerical analysis of the exact frequency equation (Reference 4). The first is for a fiber reinforced rod and the second is for a rod made of magnesium. The choice was made because we are particularly interested in fiber reinforced materials or "filament resins" and because magnesium is a common material whose elastic constants differ considerably from the first.

A set of constants for a filament resin was not readily available. The values we adopted were extracted from a paper by L. R. Herrmann and K. S. Pister[6]. Some modifications were necessary as Herrmann and Pister studied a material in which the filaments are laid down in rows on layers which results in a material that is tetragonal. It appears that when circular rods are made the filaments will be laid out symmetrically about the axis of the rod which would result in a transversely isotropic material.

To get an accurate set of constants for a transversely isotropic filament-resin one should treat the material as an independent one and derive the constants as Herrmann and Pister did. However, the concern here is not to get an accurate description of such a material but to get a set of constants which would be reasonable for such a material. Accordingly, the constants suggested by Herrmann and Pister are used and they are modified rather crudely to represent a transversely isotropic filament resin. As the normalized form of the frequency equation will be studied here the relationship between normalized frequency and dimensionless propagation constant is influenced by only four constants. These are taken as

$$\frac{c_{11}}{c_{33}} = 0.472 \; ; \quad \frac{c_{12}}{c_{33}} = 0.165 \; ;$$
$$\frac{c_{13}}{c_{33}} = 0.176 \; ; \quad \frac{c_{44}}{c_{33}} = 0.104 \quad . \tag{64}$$

The elastic constants representing magnesium were readily available and in terms of the appropriate ratios are as follows:

$$\frac{c_{11}}{c_{33}} = 0.954 \; ; \quad \frac{c_{12}}{c_{33}} = 0.390 \; ;$$

$$\frac{c_{13}}{c_{33}} = 0.326 \; ; \quad \frac{c_{44}}{c_{33}} = 0.277 \quad . \tag{65}$$

Using a digital computer, the adjustment factors are evaluated according to the method which was described in the previous section. They are

(a) for fiber reinforced rod

$$K_1^2 = 0.88287 \; ; \quad K_2^2 = 0.89350 \; ;$$

$$K_3^2 = 1.10376 \; ; \quad K_4^2 = 1.46056 \quad , \tag{66}$$

(b) for magnesium rod

$$K_1^2 = 0.53161 \; ; \quad K_2^2 = 0.83557 \; ;$$

$$K_3^2 = 0.67051 \; ; \quad K_4^2 = 1.36587 \; . \tag{67}$$

For these values of adjustment factors the frequency equation corresponding to the three mode approximate theory, Eq. (54), is solved and its solution in the form of three spectral lines is shown in Figs. 1-2 (dotted lines).

Examination of Figs. 1-2 shows that all three lines of the approximate theory match quite well the same three lines from the exact. The fundamental spectral lines match exceptionally well. Having such a comparison it is possible to outline a region of the $\Omega - \zeta$ plane where the approximate theory could be considered to be valid.

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Captions for Figures

Fig. 1. Frequency spectra from the exact and approximate theories for a fiber reinforced rod.

Fig. 2. Frequency spectra from the exact and approximate theories for a magnesium rod.

APPENDIX A: UNIQUENESS AND ORTHOGONALITY

Uniqueness

In what follows the conditions under which the solution of the governing equations of the three mode approximate theory is unique will be established.

Consider two systems of generalized displacements, strains stresses which satisfy the generalized strain displacement relations, Eqs. (48), generalized stress-strain relations, Eqs. (47), and stress equations of motion, Eqs. (51). Define the difference system as the one which is the difference of the two systems. Let K*, U* be the kinetic and strain energy densities per unit of length, and κ, s be the total kinetic and strain energies of the difference system at time t. Then one has in a system where the energy is conserved,

$$\kappa + s = \kappa_0 + s_0 + \int_0^t d\tau \int_{z_1}^{z_2} (\dot{K}^* + \dot{U}^*)dz \quad , \qquad \text{(A-1)}$$

where κ_0 and s_0 are the initial values of κ and s at time t = 0.

Using Eqs. $(35)_1$, and introducing adjustment factors, one obtains for the strain energy density per unit of length,

$$U^* = \int_0^{2\pi} d\theta \int_0^a U(\epsilon_{ij}) r dr$$

$$= \pi a^2 \left\{ K_1^2 (c_{11} + c_{12}) \Gamma_r^2 + 2K_1 c_{13} \Gamma_r \Gamma_z + \frac{c_{33}}{2} \Gamma_z^2 + \frac{c_{33}}{6} \Gamma_\psi^2 + \frac{c_{44}}{4} K_2^2 \Gamma_{rz}^{\ 2} \right\} \quad . \qquad \text{(A-2)}$$

Differentiating Eq. (A-2) with respect to time and using generalized stress-strain and strain-displacement relations, Eqs. (47, 48), one gets

$$\dot{U}* = 2\pi a^2 \left\{ P_r \frac{\dot{u}}{a} + P_z \dot{w},_z + P_\psi \dot{\psi},_z + P_{rz} \left(\dot{u},_z - \frac{4}{a} \dot{\psi} \right) \right\} \quad . \tag{A-3}$$

Similarly, the kinetic energy density per unit of length is given by

$$\begin{aligned} K* &= \int_0^{2\pi} d\theta \int_0^a \frac{1}{2} \rho (\dot{u}_r^2 + \dot{u}_z^2) r dr \\ &= \pi a^2 \rho \left(\frac{1}{4} K_3^2 \dot{u}^2 + \frac{\dot{w}^2}{2} + \frac{K_4^2}{6} \dot{\psi}^2 \right) \quad . \end{aligned} \tag{A-4}$$

Differentiating Eq. (A-4) with respect to time one gets

$$\dot{K}* = 2\pi a^2 \rho \left(\frac{1}{4} K_3^2 \dot{u}\ddot{u} + \frac{1}{2} \dot{w}\ddot{w} + \frac{K_4^2}{6} \dot{\psi}\ddot{\psi} \right) \quad . \tag{A-5}$$

When $\dot{U}*$, $\dot{K}*$ given by Eqs. (A-3, 5) are substituted into Eq. (A-1) and the terms involving partial derivatives of z are integrated by parts, one has

$$\begin{aligned} \kappa + s &= \kappa_0 + s_0 \\ &+ 2\pi a^2 \int_0^t d\tau \int_{z_1}^{z_2} \left\{ \left(- P_{rz},_z + \frac{P_r}{a} + \rho \frac{K_3^2}{4} \ddot{u} \right) \dot{u} \right. \\ &\left. + \left(-P_z,_z + \frac{1}{2} \rho \ddot{w} \right) \dot{w} + \left(-P_\psi,_z - \frac{4}{a} P_{rz} + \rho \frac{K_4^2}{6} \ddot{\psi} \right) \dot{\psi} \right\} dz \\ &+ 2\pi a^2 \int_0^t (P_z \dot{w} + P_\psi \dot{\psi} + P_{rz} \dot{u}) \Big|_{z_1}^{z_2} d\tau \quad . \end{aligned} \tag{A-6}$$

Finally, when the stress equations of motion, Eqs. (51) , are used Eq. (A-6) can be put into the form

$$\kappa + s = \kappa_0 + s_0$$

$$+ 2\pi a^2 \int_0^t d\tau \int_{z_1}^{z_2} \left\{ \frac{R}{a}\,\dot{u} + \frac{Z}{a}\,(\dot{w} - \dot{\psi}) \right\} dz \qquad \text{(A-7)}$$

$$+ 2\pi a^2 \int_0^t (P_z \dot{w} + P_\psi \dot{\psi} + P_{rz} \dot{u}) \Big|_{z_1}^{z_2} d\tau \quad .$$

If

(i) throughout the rod the initial values of (u, w, ψ) and $(\dot{u}, \dot{w}, \dot{\psi})$,

(ii) throughout the rod one member of each of the producss Ru and $Z(w - \psi)$,

(iii) at each end of the rod one member of each of the products $P_z w$, $P_\psi \psi$, $P_{rz} u$

are specified, the right hand of Eq. (A-7) will vanish. Then from the positive definiteness of K* and U* and from the condition that (u, w, ψ) are specified initially it follows that the solution will be unique.

Orthonogality

In the following discussion the orthogonality conditions for mode shapes of free vibrations of finite rods, governed by the three mode approximate theory, are established. This knowledge can be used when one studies forced vibrations of a finite rod by means of a mode superposition technique.

When one puts R = Z = 0, the displacement equations of motion, Eq. (52) , can be put into the form

$$A_{ij}\bar{u}_{j,\xi\xi} + B_{ij}\bar{u}_{j,\xi} + C_{ij}\bar{u}_j - D_{ij}\ddot{\bar{u}}_j = 0 \quad . \qquad \text{(A-8)}$$

$$(i, j = 1, 2, 3)$$

In Eq. (A-8)

$$(\bar{u}_1, \bar{u}_2, \bar{u}_3) = (u, w, \psi)$$

$$(A_{ij}) = \begin{bmatrix} 3c_{44}K_2^2 & 0 & 0 \\ 0 & 6c_{33} & 0 \\ 0 & 0 & 2c_{33} \end{bmatrix}$$

$$(B_{ij}) = \begin{bmatrix} 0 & -12K_1c_{13} & -12c_{44}K_2^2 \\ 12K_1c_{13} & 0 & 0 \\ 12c_{44}K_2^2 & 0 & 0 \end{bmatrix} \qquad \text{(A-9)}$$

$$(C_{ij}) = \begin{bmatrix} -12K_1^2(c_{11}+c_{12}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -48c_{44}K_2^2 \end{bmatrix}$$

$$(D_{ij}) = \begin{bmatrix} 3\rho a^2K_3^2 & 0 & 0 \\ 0 & 6\rho a^2 & 0 \\ 0 & 0 & 2\rho a^2K_4^2 \end{bmatrix}$$

Note that A_{ij}, C_{ij}, and D_{ij} are symmetric whereas B_{ij} is anti-symmetric.

We assume that the solution of Eq. (A-8) has the form

$$(u, w, \psi) = (u^p, w^p, \psi^p)e^{i\omega_p t} \quad , \tag{A-10}$$

and substituting it into Eq. (A-8) one gets

$$A_{ij}\bar{u}^p_{j},_{\xi\xi} + B_{ij}\bar{u}^p_{j},_{\xi} + C_{ij}\bar{u}^p_{j} + \omega_p^2 D_{ij}\bar{u}^p_{j} = 0 \quad , \qquad \text{(no sum on p)} \tag{A-11}$$

or

$$L_{ij}\bar{u}^p_j + \omega_p^2 D_{ij}\bar{u}^p_j = 0 \qquad \text{(no sum on p)} \quad , \tag{A-12}$$

where

$$L_{ij}\bar{u}^p_j = A_{ij}\bar{u}^p_{j},_{\xi\xi} + B_{ij}\bar{u}^p_{j},_{\xi} + C_{ij}\bar{u}^p_j = 0 \quad . \tag{A-13}$$

Note that (u^p, w^p, ψ^p) are functions of ξ only.

Let V_i, Y_i be two vectors satisfying the same homogeneous end boundary conditions (displacement or stress or mixed-mixed), and define

$$\langle V_i, Y_i \rangle = \int_{\xi_1}^{\xi_2} V_i Y_i d\xi \quad ,$$

then, using Eq. (A-13) one obtains

$$\begin{aligned} \langle V_i, L_{ij}Y_j \rangle &= \int_{\xi_1}^{\xi_2} V_i(A_{ij}\, Y_j,_{\xi\xi} + B_{ij}Y_j,_{\xi} + C_{ij}Y_j)d\xi \\ &= V_iA_{ij}Y_j,_{\xi}\Big|_{\xi_1}^{\xi_2} - V_i,_{\xi}A_{ij}Y_j\Big|_{\xi_1}^{\xi_2} + V_iB_{ij}Y_j\Big|_{\xi_1}^{\xi_2} \\ &+ \int_{\xi_1}^{\xi_2} (A_{ij}V_i,_{\xi\xi} - B_{ij}V_i,_{\xi} + C_{ij}V_i)Y_j d\xi \quad . \end{aligned} \tag{A-14}$$

Since V_i, Y_i satisfy the same homogeneous end boundary conditions, the first three terms in Eq. $(A\text{-}14)_2$ vanish. Taking into account the symmetry of A_{ij} and C_{ij}, and antisymmetry of B_{ij}, Eq. (A-14) can be written as

$$\begin{aligned}\langle V_i, L_{ij}Y_j\rangle &= \int_{\xi_1}^{\xi_2} (A_{ji}V_{i,\xi\xi} + B_{ji}V_{i,\xi} + C_{ji}V_i)Y_j d\xi \\ &= \langle L_{ji}V_i, Y_j\rangle \quad .\end{aligned} \tag{A-15}$$

From Eq. (A-15) one concludes that the operator L_{ij} is self adjoint.

Let $\bar{u}_i^p$, $\bar{u}_i^m$ satisfy the same homogeneous boundary conditions and be solutions of Eq. (A-12) with ω_p^2 and $\omega_m^2(\omega_p \neq \omega_m)$ respectively, i.e., $\bar{u}_i^p$ satisfies Eq. (A-12), and $\bar{u}_i^m$ satisfies the equations

$$L_{ij}\bar{u}_j^m + \omega_m^2 D_{ij}\bar{u}_j^m = 0 \quad . \tag{A-16}$$

If Eq. (A-12) is multiplied by $\bar{u}_i^m$ and Eq. (A-16) by $\bar{u}_i^p$ and they are integrated with respect to ξ over the interval $[\xi_1, \xi_2]$, one has

$$\begin{aligned}\langle \bar{u}_i^m, L_{ij}\bar{u}_j^p\rangle + \omega_p^2\langle \bar{u}_i^m, D_{ij}\bar{u}_j^p\rangle &= 0 \\ \langle \bar{u}_i^p, L_{ij}\bar{u}_j^m\rangle + \omega_m^2\langle \bar{u}_i^p, D_{ij}\bar{u}_j^m\rangle &= 0 \quad .\end{aligned} \tag{A-17}$$

Noting that from Eq. (A-15) one has $\langle \bar{u}_i^m, L_{ij}\bar{u}_j^p\rangle = \langle \bar{u}_i^p, L_{ij}\bar{u}_j^m\rangle$ and D_{ij} is symmetric, subtraction of Eq. $(A\text{-}17)_1$ from Eq. $(A\text{-}17)_2$ gives

$$(\omega_m^2 - \omega_p^2)\langle \bar{u}_i^m, D_{ij}\bar{u}_j^p\rangle = 0 \quad . \tag{A-18}$$

Since $\omega_m \neq \omega_p$, one concludes that

$$\langle \bar{u}_i^m, D_{ij}\bar{u}_j^p \rangle = 0 \qquad \text{for} \quad m \neq p \quad . \tag{A-19}$$

Using Eqs. $(A\text{-}9)_1$ and $(A\text{-}9)_5$, the orthogonality condition, Eq. (A-19), can be written in expanded form as

$$\int_{\xi_1}^{\xi_2} (3K_3^2 u^m u^p + 6w^m w^p + 2K_4^2 \psi^m \psi^p) d\xi = 0 \tag{A-20}$$

for $m \neq p$.

Now we investigate the sign of the eigenvalues ω_p^2 in Eq. (A-12). For this Eq. (A-12) will be multiplied by $\bar{u}_i^p$ and integrated with respect to ξ over the interval $[\xi_1, \xi_2]$. Then one obtains

$$\omega_p^2 = - \frac{\langle \bar{u}_i^p, L_{ij}\bar{u}_j^p \rangle}{\langle \bar{u}_i^p, D_{ij}\bar{u}_j^p \rangle} \quad . \tag{A-21}$$

But D_{ij} is a positive definite matrix; hence

$$\langle \bar{u}_i^p, D_{ij}\bar{u}_j^p \rangle > 0 \text{ for } \bar{u}_i^p \neq 0 \quad . \tag{A-22}$$

The sign of ω_p^2 therefore depends on the sign of the numerator of Eq. (A-21).

The numerator of Eq. (A-21) can be written in expanded form as

$$\langle \bar{u}_i^p, L_{ij}\bar{u}_j^p \rangle = \int_{\xi_1}^{\xi_2} \bar{u}_i^p (A_{ij}\bar{u}_{j,\xi\xi}^p + B_{ij}\bar{u}_{j,\xi}^p + C_{ij}\bar{u}_j^p) d\xi \quad . \tag{A-23}$$

Now, let

$$B_{ij} = B_{ij}^{(1)} + B_{ij}^{(2)} \quad , \tag{A-24}$$

where

$$
(B_{ij}^{(1)}) = \begin{bmatrix} 0 & -12K_1c_{13} & 0 \\ 0 & 0 & 0 \\ 12K_2^2c_{44} & 0 & 0 \end{bmatrix}
$$

(A-25)

$$
(B_{ij}^{(2)}) = \begin{bmatrix} 0 & 0 & -12K_2^2c_{44} \\ 12K_1c_{13} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

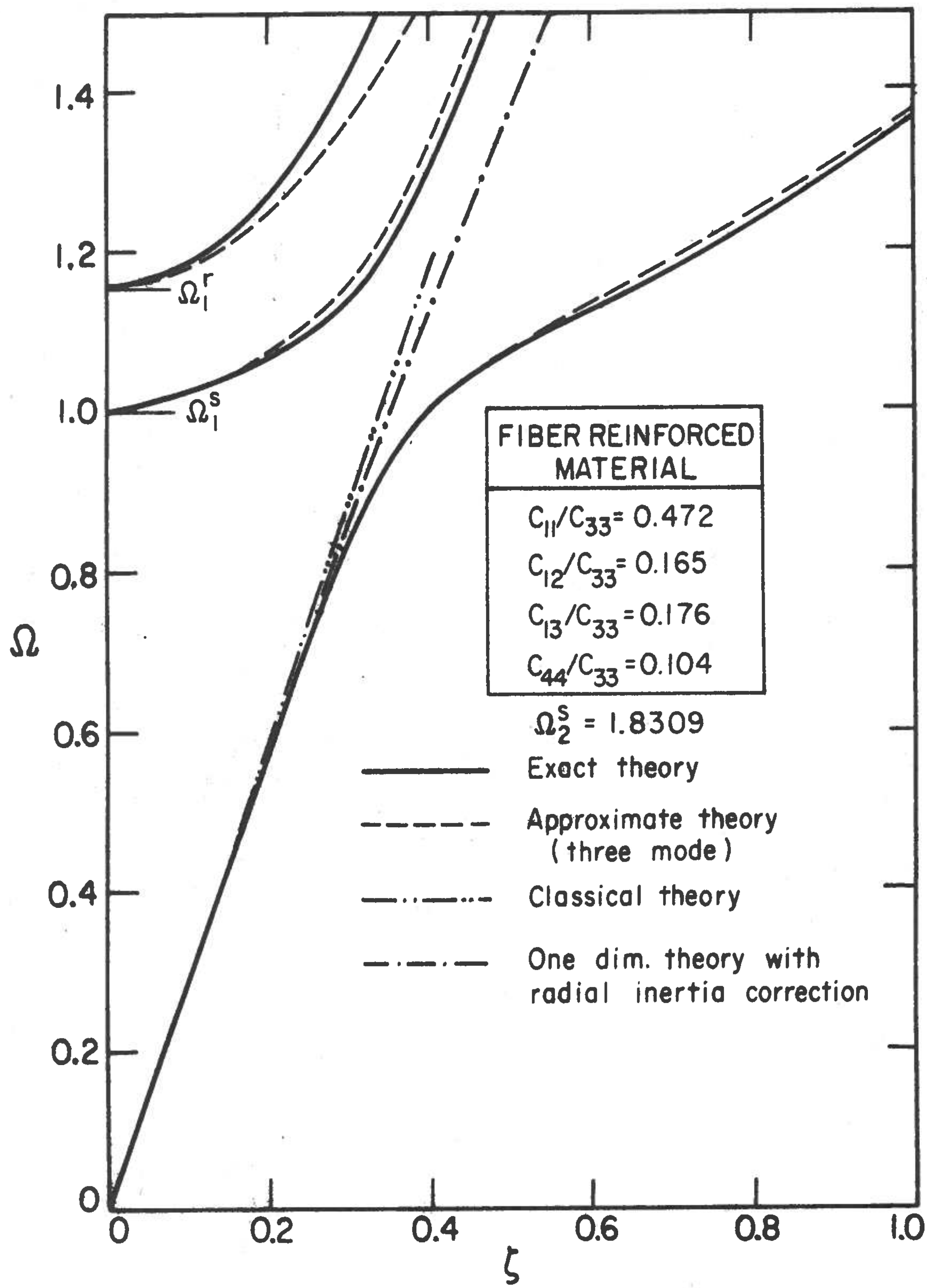
Then, integrating by parts Eq. (A-23) can be put into the form

$$
\langle \bar{u}_i^p, L_{ij}\bar{u}_j^p \rangle = \bar{u}_i^p A_{ij} \bar{u}_{j,\xi}^p \Big|_{\xi_1}^{\xi_2} + \bar{u}_i^p B_{ij}^{(2)} \bar{u}_j^p \Big|_{\xi_1}^{\xi_2}
$$

$$
+ \int_{\xi_1}^{\xi_2} (-\bar{u}_{i,\xi}^p A_{ij} \bar{u}_{j,\xi}^p + \bar{u}_i^p B_{ij}^{(1)} \bar{u}_{j,\xi}^p \qquad \text{(A-26)}
$$

$$
- \bar{u}_{i,\xi}^p B_{ij}^{(2)} \bar{u}_j^p + \bar{u}_i^p C_{ij} \bar{u}_j^p) d\xi \quad .
$$

When the integrand of the last term in Eq. (A-26) is expanded, it can be shown that it is proportional to the strain energy density per unit length and is opposite in sign. In Eq. (A-26), the first two terms will vanish because $\bar{u}_j^p$ satisfies homogeneous end boundary conditions (displacement or stress or mixed-mixed); and because of the positive definiteness of the strain energy, the integral will have a negative value for $\overline{u}_i^p \neq 0$. Thus the numerator of Eq. (A-21) will be negative. Then it follows that eigenvalues ω_p^2 will be positive for $\overline{u}_i^p \neq 0$.



1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
Ω
Ω_1^r
Ω_1^s
0
0.2
0.4
0.6
0.8
1.0
ζ
FIBER REINFORCED MATERIAL
C_{11}/C_{33}= 0.472
C_{12}/C_{33}= 0.165
C_{13}/C_{33}=0.176
C_{44}/C_{33}=0.104
Ω_2^s = 1.8309
Exact theory
Approximate theory (three mode)
Classical theory
One dim. theory with radial inertia correction

FIG. 1

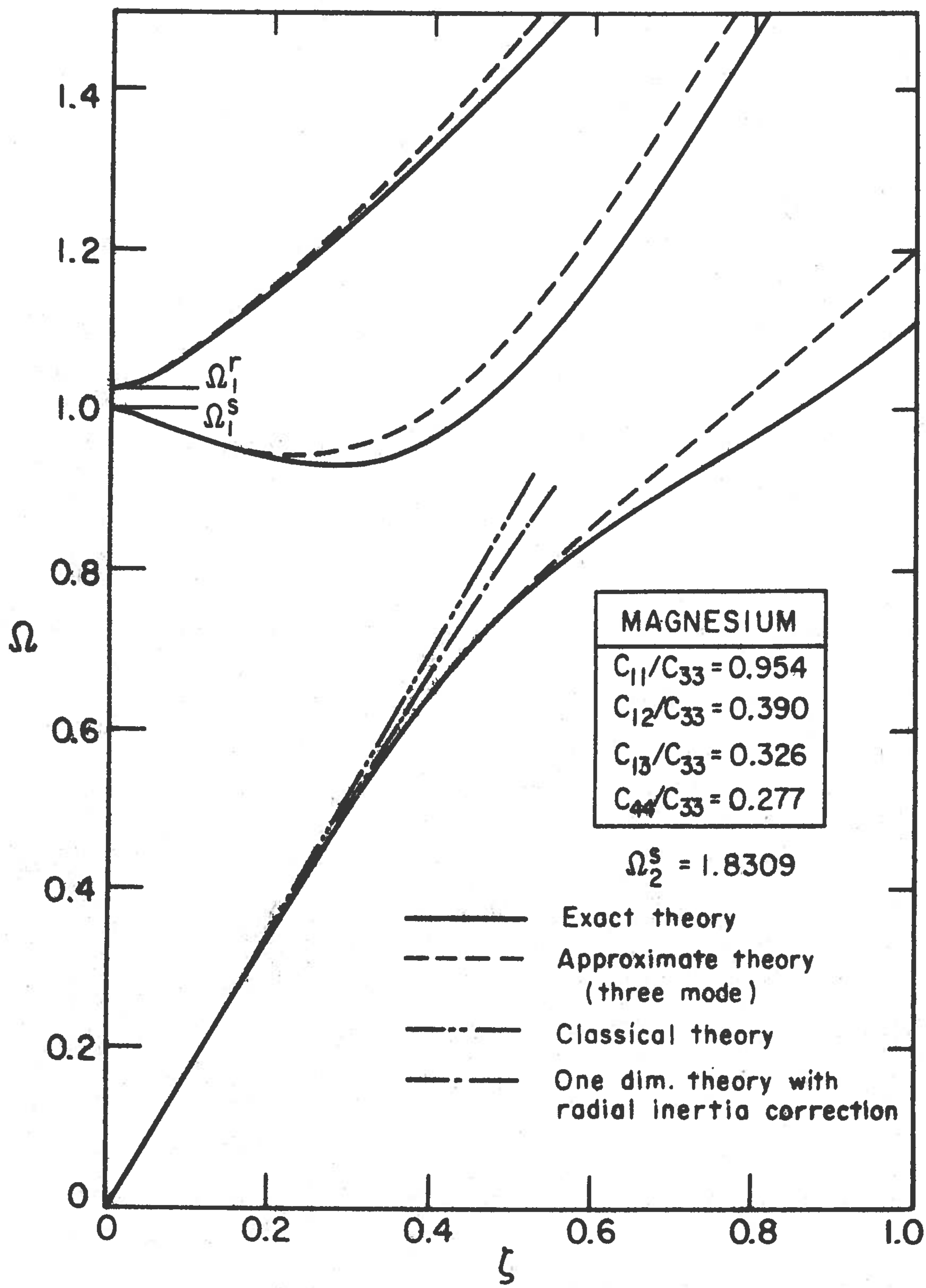

MAGNESIUM
C11/C33 = 0.954
C12/C33 = 0.390
C13/C33 = 0.326
C44/C33 = 0.277
Ω2s = 1.8309
Exact theory
Approximate theory (three mode)
Classical theory
One dim. theory with radial inertia correction
Ω1r
Ω1s
Ω
ζ
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4

FIG. 2