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Abstract.

The same topological argument that previously gave a sphaleron S (i.e. a static, but unstable, classical solution) suggests the existence of another sphaleron S*. There appear to be two alternatives: either S* is just a superposition of two S's infinitely far apart, or it is a truly new axisymmetric solution, probably with a single core. We propose an ansatz for S*. The resulting equations of motion can be solved asymptotically, but it is not clear if a general solution is possible.

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I. Introduction

The Weinberg-Salam model[1] is a relatively simple example of a field theory with non-Abelian gauge fields and the Higgs mechanism of spontaneous symmetry breaking. Still its non-linearity makes the theory highly non-trivial. A better understanding, or a complete solution even, of the Weinberg-Salam theory is all the more desirable, since the theory appears to give an excellent description[2] of the electroweak interactions, at least up to energies of order of 100 GeV. In this article we deal with the vacuum sector of the classical theory, knowledge about which will be a necessary ingredient for our eventual understanding of the quantum theory.

In a previous article[3] by N. Manton and the present author a close approximation was found to a static, but unstable, classical solution in the vacuum sector of the Weinberg-Salam theory without the fermionic fields of quarks and leptons. Being unstable we found it inappropriate to call it a soliton and we proposed for this solution, and others of the same kind, the word "sphaleron," which is derived from the Greek adjective for "unstable." In this article we look for a new sphaleron (S*), which is expected to be more complicated (axisymmetric) and heavier than the first sphaleron (S) discussed in Ref. 3, which was spherically symmetric and had a mass of order 10 TeV.

First let us recall how the sphaleron S was discovered, or rediscovered rather, since we learned afterwards that it was found many years ago by Dashen, Hasslacher and Neveu.[4] Inspired by work of Taubes[5] in the SU(2) Yang-Mills theory with <u>adjoint</u> Higgs, Manton[6] constructed in the space of classical static configurations of the Weinberg-Salam theory, where the Higgs fields are in the fundamental representation, a non-contractible loop passing through the unique vacuum solution at zero energy. This non-contractible loop, parametrized by $\mu \in [0,\pi]$, contains information of the behaviour of the fields at spatial infinity, which is covered by the standard spherical coordinates ϕ and θ. For a fixed value of μ this information is given by a SU(2) matrix $U(\phi,\theta;\mu)$. The configurations of the non-contractible loop are at infinity pure gauge as described by the matrix U and are interpolated inwards by use of two radial functions, one each for the gauge and Higgs fields. The $U(\phi,\theta;\mu)$ of Ref. 6 is a specific map from $S^1 \times S^2$ into $SU(2) \simeq S^3$, which can be deformed to the homotopically non-trivial map $S^3 \to S^3$ with winding number $n = 1$. The argument[5] of how this loop leads to a non-trivial solution of the field equations runs as follows. The energy of the configurations of the loop considered attains its maximum at $\mu = \pi/2$. Choosing the two radial functions appropriately it is possible to minimize that maximum energy. Morse theory arguments can be used to show that, because the loop in configuration space was non-contractible, the configuration obtained from this mini-max procedure is a saddle-point of the energy functional and thus a solution of the field equations. There is one possible loophole in this argument: the non-compactness and infinite dimensionality of the configuration manifold can make that the configuration approached by the mini-max procedure is rather trivial. An example of this was given in Ref. 7 for the Skyrme model. The non-contractible loop of configurations that was considered had the following structure: for $\mu = 0$ the vacuum, for $\mu = \pi/2$ a Skyrmion-antiSkyrmion pair at a separation d and with a relative isospin rotation, and for $\mu = \pi$ the vacuum again. For this loop the mini-max procedure never reached a new solution; rather the distance d kept on increasing indefinitely, always reducing the $\mu = \pi/2$ energy a little. Luckily, the Weinberg-Salam theory did not use a similar way out and the mini-max procedure did reach a new solution, i.e. the sphaleron S, for details of which the reader is referred to Ref. 3. Clearly the Weinberg-Salam theory is richer than the Skyrme model in that it possesses also gauge fields and it is precisely those extra degrees of freedom that allow for a solution. The situation is analogous to that of the 't Hooft-Polyakov magnetic monopole solution,[8] where the SU(2) gauge fields and the adjoint Higgs are subtly working together to give a solution.

Physically the rôle of the sphaleron S is the following. As argued by 't Hooft[9] there are instanton-like configurations in the Weinberg-Salam theory, starting at Euclidean time $t = -\infty$ from the vacuum and ending at $t = +\infty$ at the vacuum again, be it in a different gauge. The sphaleron S is just the maximum energy configuration at $t = 0$ or, in physical terms, it lies on the top of the energy barrier between the vacua. Our present knowledge[3] of S allows us to extend 't Hooft's approximate treatment, which was valid for small values of the Higgs quartic coupling constant λ, to all values of λ. The instanton has a winding number $n = 1$ and an approximate action $A(n = 1) \approx 8\pi/g^2$, where g is the SU(2) coupling constant. With the known[3] sphaleron energy $E = e\ 4\pi v/g$, where $e = e(\lambda/g^2)$ is a number of order 1 and v is the Higgs vacuum expectation value, we can set $A(1) = E_S \Delta t$, which gives the timelike extent of the instanton $\Delta t \approx e^{-1} M_W^{-1}$, where $M_W = \frac{1}{2}\,gv$ is the mass of the W boson. We could also start from a configuration[10] with $n > 1$ instantons on top of each other, which would have an action $A(n) \approx n\,A(1)$ + interaction terms. Assuming Δt not to change significantly, we are thus led to expect a tower of sphalerons with increasing energies. We expect them to be spherically symmetric, just as the original sphaleron S of Refs. 3 and 4.

These other sphalerons can perhaps be found explicitly by doing the mini-max procedure for non-contractible loops with winding number $n > 1$. Of course, other configurations of multiinstantons may lead to even more sphalerons.

These other possible sphalerons are not terribly exciting. Rather we will search for a really different sphaleron (S*). Its existence should be related to a non-contractible sphere in configuration space, which a priori is a possibility, since $\pi_4(SU(2)) = Z_2$. If S* exists, the cyclicity of this homotopy group perhaps implies that it is not directly related to the known instantons of the pure gauge theory. Anyway, first we have to establish the existence of such a solution S* and see if it is really different. In Section II we construct a non-contractible sphere in configuration space and obtain from it our tentative ansatz for S*. This ansatz has to be of a quite complicated form in order to stand a chance of being correct; specifically it is axisymmetric and involves three functions. In Section III we calculate the equations of motion for this ansatz. The general equations are certainly simplified by our ansatz, but it is not clear if a solution exists to them, since there appear two constraint equations on the polar dependence of the ansatz functions. Only at large distances are we able to construct an approximate solution (Section V). Hence there are two possibilities: either the ansatz allows miraculously for a solution of the field equations over all space or the ansatz is relevant only asymptotically and the fields in the inner region are different. Anyway, the ansatz is useful to calculate in Section IV for $\lambda/g^2 = 0$ an upper bound on the energy E_{S*} of 2.2 E_S. Finally, in Section V we compare the two ansätze for S* and S, and show that they are very similar. In fact, it looks as if S* may correspond to two sphalerons S infinitely far apart, which would be rather disappointing and an example of the loophole in the topological argument as discussed above. Still the other possiblity remains open that S* is a truly new axisymmetric solution with a single core and an energy certainly larger than E_S and perhaps close to $2E_S$.

II. Ansatz

Henceforth we consider the Weinberg-Salam theory with a restricted gauge group $G = SU(2)$ instead of the full $SU(2) \times U(1)$. In a previous paper[3] it was shown that inclusion of the U(1) field does not significantly modify the sphaleron S, whose existence is related to $\pi_3(G)$ being non-trivial and this homotopy group is insensitive to the U(1) factor. The reason that the energy, for example, changes by a small amount if the U(1) field is included is the smallness of the weak mixing angle $\Theta_W \equiv \tan g'/g$, where g' and g are the coupling constants of the U(1) and SU(2) gauge groups, respectively. So our use of a restricted gauge group SU(2) means that we set $g' = 0$ in the full theory. A more drastic amputation is that we shall omit the fermionic fields of the quarks and leptons. So we only have non-Abelian gauge fields given by an antihermitian potential $A_\mu(x)$, which takes values in the Lie algebra of SU(2), together with one doublet of Higgs scalars $\Phi(x)$, whose self interactions result in a vacuum expectation value $\langle 0|\Phi|0\rangle = v \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, with $v \approx 250$ GeV.

We will look at static configurations only and set $A_0 = 0$. The gauge we work in is given by the following two conditions: 1. Vanishing radial gauge fields $A_r = 0$, and 2. A fixed Higgs field at the north pole of the sphere at spatial infinity $\Phi(r = \infty, \theta = 0) = v \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which removes the global symmetry leftover by the first condition. Our starting point is the energy functional[6]

$$E = \frac{v}{g} \int \xi^2 d\xi \sin\theta \, d\theta \, d\phi$$

$$\left[\frac{1}{2} \left\{ (\partial_\xi \Phi)^\dagger \partial_\xi \Phi + \frac{1}{\xi^2} (D_\theta \Phi)^\dagger D_\theta \Phi + \frac{1}{\xi^2 \sin^2\theta} (D_\phi \Phi)^\dagger D_\phi \Phi \right\} \right.$$

$$+ \frac{1}{4} \frac{\lambda}{g^2} (\Phi^\dagger \Phi - 1)^2$$

$$\left. - \frac{1}{\xi^2} \mathrm{Tr} \left\{ \partial_\xi A_\theta \partial_\xi A_\theta + \frac{1}{\sin^2\theta} \partial_\xi A_\phi \partial_\xi A_\phi + \frac{1}{\xi^2 \sin^2\theta} F_{\theta\phi} F_{\theta\phi} \right\} \right] \quad , \qquad (1)$$

where the integral is made dimensionless by scaling the fields and using a dimensionless radial distance $\xi \equiv gvr$. Furthermore there are the following definitions of the covariant derivative and the field strength

$$D_i \Phi \equiv \partial_i \Phi + A_i \Phi$$

$$F_{ij} \equiv \partial_i A_j - \partial_j A_i + [A_i, A_j] \quad .$$

We turn now to the construction of the non-contractible sphere in configuration space, which, as mentioned in the Introduction, is described by a SU(2) matrix U. We take[11]

$$U(\hat{x};\mu,\nu) = (\sin\mu + i \cos\mu \, e^{i(\nu+\pi/2)\sigma_3} \, \hat{x} \cdot \vec{\sigma} \, e^{-i(\nu+\pi/2)\sigma_3})$$

$$\cdot (\sin\mu - i \cos\mu \, \hat{x} \cdot \vec{\sigma}) \quad , \qquad (2)$$

where $\hat{x} = (\hat{x},\hat{y},\hat{z})$ is a unit vector in 3-d space and $\mu,\nu \in [-\pi/2,\pi/2]$ are the parameters in configuration space. As the boundaries of the μ,ν square

are mapped in the same element (the identity), (2) describes in fact a non-trivial map $\tilde{S}^2 \times \bar{S}^2 \rightarrow SU(2) \simeq S^3$, where $\hat{x}$ lies on the spatial sphere $\tilde{S}^2$ and μ,ν are the coordinates of the parameter sphere $\bar{S}^2$. The map (2) has the property that the north pole $\hat{x} = (0,0,1)$ is mapped into the same element 1 for all possible values of μ and ν. This will guarantee that the second gauge condition is fulfilled by all configurations of the non-contractible sphere. These configurations are given by

$$
\begin{aligned}
A_\xi &= 0 \\
A_\theta &= -f(\xi,\theta)\partial_\theta U\, U^{-1} \\
A_\phi &= -g(\xi,\theta)\partial_\phi U\, U^{-1} \\
\Phi &= h(\xi,\theta)U\begin{pmatrix}0\\1\end{pmatrix} + (1 - h(\xi,\theta))\begin{pmatrix}0\\ \max(\sin^2\mu,\sin^2\nu)\end{pmatrix} \quad ,
\end{aligned}
\tag{3}
$$

where the functions f, g and h approach 1 for $\xi \rightarrow \infty$ and 0 for $\xi \rightarrow 0$ in order to assure finite energy (see below) and smoothness at the origin.

The pole μ or $\nu = \pm\pi/2$ of the parameter sphere $\bar{S}^2$ gives the vacuum[12] and on symmetry grounds we expect the maximum energy to be at the pole $\mu = \nu = 0$. Presently we will just assume this and verify it <u>a posteriori</u> (see (11) below) if we find a non-trivial solution S* based on the matrix $U(\hat{x},0,0)$. We have no interest in the map (2) <u>per se</u>, only as a guide to obtain an ansatz for S*. This ansatz then is (3) with the matrix U given by

$$
\begin{aligned}
U = U(\hat{x},0,0) &= \cos 2\theta - \sin 2\theta \cos\phi\, i\sigma_x \\
&\quad + \sin 2\theta \sin\phi\, i\sigma_y \quad ,
\end{aligned}
\tag{4}
$$

which is of a remarkably simple form and furthermore similar to the one of the original sphaleron S (see Section V). In the ansatz (3,4) we had to introduce different smoothing functions f and g for the A_θ and A_ϕ fields and allow the functions f, g and h to depend on θ also. These complications are required if we try to obtain valid field equations, which we will derive in Section III.

For this ansatz (3,4) the energy functional (1) becomes

$$E = \frac{2v}{g} \int d\xi \sin\theta \, d\theta \, d\phi \Big[4f'^2 + 4\cos^2\theta \, g'^2 +$$

$$4 \frac{\cos^2\theta}{\xi^2} \left\{ (\mathring{g} - 2 \,\mathrm{cotan}\, 2\theta \, (f - g))^2 + 4(f + g - 2fg)^2 \right\}$$

$$+ \frac{1}{4} \xi^2 h'^2 + \frac{1}{4} \mathring{h}^2 + (h(1 - f))^2 +$$

$$\cos^2\theta \, (h(1 - g))^2 + \frac{1}{8} \frac{\lambda}{g^2} \xi^2 (h^2 - 1)^2 \Big] \quad , \qquad (5)$$

where a prime or circle denotes differentiation with respect to ξ or θ, respectively. The energy density in (5) is axisymmetric and integration over ϕ gives a factor 4πv/g in front of the integrals that remain. As (5), appropriately, has non-negative terms in the integrand, the finite energy requirement must be applied to all terms separately. This leads to the following boundary conditions

$$\lim_{\xi \to 0} \left(\frac{f}{\xi}, \frac{g}{\xi}, h \right) = 0 \qquad (6a)$$

$$\lim_{\xi \to \infty} (f, g, h) = 1 \quad . \qquad (6b)$$

It is interesting that the $\frac{1}{4}\,\mathring{h}^2$ term in the integrand of (5) precludes having a non-trivial θ dependence in separated form $h(\xi,\theta) = \bar{h}(\xi)\tilde{h}(\theta)$, since for $\xi \to \infty$ the λ term would require that $h\tilde{h} \to 1$ and the $\xi^2 h'^2$ term that $h \to$ constant c, so that the $\mathring{h}^2$ term would give $\tilde{h} = c^{-1}$.

We now derive the variational equations for an extremum of (5). For later convenience we introduce a mnemonic notation for these equations; for example $[\delta f]$ stands for the expression that follows from variation of f. Furthermore we employ for the gauge field equations a matrix notation that saves space and clarifies their structure. The variational equations from (5) are then

$$[\delta h] \equiv (\xi^2 h')' + \mathring{\mathring{h}} + \cot\theta\,\mathring{h} - 4h(1-f)^2 - 4h\cos^2\theta(1-g)^2$$

$$- \frac{\lambda}{g^2}\,\xi^2(h^2 - 1)h = 0 \quad , \tag{7a}$$

$$\begin{pmatrix}[\delta f]\\ [\delta g]\end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & -\cot\theta\cos 2\theta & 0 \\ 0 & \cos^2\theta & -\cot\theta(\cos 2\theta - \sin^2\theta) & \cot\theta\cos 2\theta \end{pmatrix} \cdots$$

$$\cdots \begin{matrix} \cos^2 2\theta/\sin^2\theta & 4\cos^2\theta(1-2g) \\ -(\cos 2\theta + 4\cos^2\theta + \cos^2 2\theta/\sin^2\theta) & 4\cos^2\theta(1-2f) \end{matrix}\Bigg)\, C = 0 \quad , \tag{7b}$$

where we have used the column matrix C defined by

$$C \equiv \begin{pmatrix} -\xi^2 f'' - \frac{1}{4}\xi^2 h^2(1 - f) \\ -\xi^2 g'' - \overset{\circ\circ}{g} - \frac{1}{4}\xi^2 h^2(1 - g) \\ \overset{\circ}{g} \\ \overset{\circ}{j} \\ (f - g) \\ (f + g - 2fg) \end{pmatrix} .$$

Finding a solution of (7) is a necessary, but not automatically sufficient, condition for the ansatz to be a solution of the complete field equations. It may be helpful to have a simple picture in mind, where we visualize the configuration space as a two dimensional plane and the ansatz some linear section of it. The energy function then maps out a surface above this plane. Finding a minimum M in the energy function over the ansatz line, i.e. a solution of (7), does not preclude the possibliity that M is somewhere at the bottom of a valley with an overall gradient. We really need to find the points in the configuration plane, where variations in all directions give $\delta E = 0$. This amounts to solving the field equations, to which we turn in the following Section.

III. Field Equations

First we derive the general field equations in spherical coordinates and in the $A_\xi = 0$ gauge by variation of the energy functional (1):

$$[\delta\Phi] \equiv \partial_\xi(\xi^2\partial_\xi\Phi) + \frac{1}{\sin^2\theta} D_\phi D_\phi\Phi + \frac{1}{\sin\theta} D_\theta(\sin\theta\, D_\theta\Phi)$$

$$- \frac{\lambda}{g^2}\, \xi^2(\Phi^\dagger\Phi - 1)\Phi = 0 \qquad (8a)$$

$$[\delta A^a_\theta] \equiv \partial_\xi\partial_\xi A^a_\theta + \frac{1}{\xi^2\sin^2\theta}\,(D_\phi F_{\phi\theta})^a +$$

$$+ \frac{i}{4}\,[\Phi^\dagger\sigma^a D_\theta\Phi - \text{H.c.}] = 0 \qquad (8b)$$

$$[\delta A^a_\phi] \equiv \partial_\xi\partial_\xi A^a_\phi + \frac{1}{\xi^2}\,(D_\theta F_{\theta\phi} - \cot\text{an}\,\theta\, F_{\theta\phi})^a$$

$$+ \frac{i}{4}\,[\Phi^\dagger\sigma^a D_\phi\Phi - \text{H.c.}] = 0 \quad , \qquad (8c)$$

where

$$F \equiv F^a i\sigma^a/2 \qquad a = x,y,z \quad ,$$

$$D_i F_{ij} \equiv \partial_i F_{ij} + [A_i, F_{ij}] \quad .$$

It is a straightforward, but laborious, exercise to insert the ansatz (3,4) into (8). Let us first consider the equation (8a) for the Higgs field. After some manipulations this equation reduces to

$$[\overline{\delta\Phi}]\begin{pmatrix}\sin 2\theta\ e^{i\phi}\\ \\ -\cos 2\theta\end{pmatrix} + [\delta\tilde{\Phi}]\begin{pmatrix}\cos 2\theta\ e^{i\phi}\\ \\ \sin 2\theta\end{pmatrix} = 0 \quad ,$$

where the expressions in front of the doublets will be given shortly. A moment's thought reveals that the only solution of this equation is $[\overline{\delta\Phi}] = [\delta\tilde{\Phi}] = 0$. So we find for our ansatz (3,4) these two equations

$$[\overline{\delta\Phi}] \equiv (\xi^2 h')' + \mathring{\mathring{h}} + \text{cotan}\ \theta\ \mathring{h} - 4h(1 - f)^2 - \cos^2\theta\ 4h(1 - g)^2$$

$$- \frac{\lambda}{g^2}\ \xi^2(h^2 - 1)h = 0 \quad , \tag{9a}$$

$$[\delta\tilde{\Phi}] \equiv 2\mathring{h}(1 - f) + h\{-\mathring{f} + \sin 2\theta\ (1 - g) + \text{cotan}\ \theta\ (g - f)\} = 0 \quad , \tag{9b}$$

For our ansatz the gauge field equations (8bc) give, using the same matrix notation as in (7b),

$$\begin{pmatrix} [\delta A^x_\theta] \\ [\delta A^z_\phi] \\ [\delta A^x_\phi] \end{pmatrix} \equiv \left(\begin{matrix} \sin^2\theta & 0 & -\frac{1}{2}\sin 2\theta \cos 2\theta \\ 0 & 1 & \tan\theta - 4\mathrm{cotan}\, 2\theta\,(1-2f) - 2\mathrm{cotan}\, 2\theta \\ 0 & 1 & \tan\theta + 4\tan 2\theta\,(1-2f) - 2\mathrm{cotan}\, 2\theta \end{matrix} \right. \quad \dots$$

$$\dots \quad \begin{matrix} 0 & \cos^2 2\theta \\ 4g\,\mathrm{cotan}\, 2\theta & -\left\{4 + \frac{\cos 2\theta}{\sin^2\theta}\left(1 - \frac{\cos 2\theta}{\cos^2\theta}(1-2f)\right)\right\} \\ 2\mathrm{cotan}\, 2\theta + 2\tan 2\theta\,(1-2g) & -\left\{4 + \frac{\cos 2\theta}{\sin^2\theta} + 4(1-2f)\right\} \end{matrix} \quad \dots$$

$$\dots \quad \left. \begin{matrix} \sin^2 2\theta\,(1-2g) \\ 4(1-2f) + \frac{\cos 2\theta}{\cos^2\theta} \\ 4(1-2f) - \frac{4\sin^2\theta}{\cos 2\theta} \end{matrix} \right) C = 0 \tag{10}$$

These are the only equations that remain from the original six in (8bc), since our ansatz gives $[\delta A^y_\theta] = [\delta A^x_\theta]$ and $[\delta A^y_\phi] = [\delta A^x_\phi]$ and reduces $[\delta A^z_\theta]$ to a trivial identity.

Let us compare the field equations (9,10) with the three equations (7) from variations restricted to the ansatz section of configuration space. Heartingly the latter can be reproduced as follows

$$[\delta h] = [\overline{\delta \Phi}] \tag{11a}$$

$$\sin^2\theta\,[\delta f] = [\delta A^x_\theta] \tag{11b}$$

$$[\delta g] = \sin^2 2\theta \cos^2\theta\,[\delta A^z_\phi] + \cos^2 2\theta \cos^2\theta\,[\delta A^x_\phi] \quad . \tag{11c}$$

Analogously to (11c) a simple equation $[\delta\tilde{A}]$ results from the following difference

$$[\delta\tilde{A}] \equiv ([\delta A^{x}_{\phi}] - [\delta A^{z}_{\phi}])\sin^2 2\theta \cos 2\theta =$$

$$= \left(0 \quad 0 \quad 4\sin 2\theta\,(1 - 2f) \,\middle|\, 2\sin 2\theta(1 - 2g) \quad \ldots\right.$$

$$\left.\ldots \quad 4\cos^3 2\theta(1 - 2f) \quad -4\sin^2\theta\right) C = 0 \quad , \tag{12}$$

which relates the θ derivatives of f and g. For future use we simplify $[\delta f]$ by eliminating $\mathring{g}$ with (12) and $[\delta g]$ by eliminating $\mathring{f}$. The resulting equations

$$\begin{pmatrix} [\overline{\delta f}] \\ [\overline{\delta g}] \end{pmatrix} \equiv \left(\begin{matrix} (1 - 2f) & 0 \\ 0 & \cos^2\theta(1 - 2g) \end{matrix} \quad \cdots\right.$$

$$\cdots \quad \begin{matrix} 0 \\ -\text{cotan}\,\theta\{(\cos 2\theta - \sin^2\theta)(1 - 2g) + 2\cos 2\theta(1 - 2f)\} \end{matrix} \quad \cdots$$

$$\cdots \quad \begin{matrix} \frac{1}{2}\cos 2\theta\ \text{cotan}\,\theta(1 - 2g) \\ 0 \end{matrix} \quad \cdots$$

$$\cdots \quad \begin{matrix} \frac{\cos^2 2\theta}{\sin^2\theta}(1 + \frac{1}{2}\cos^2 2\theta)(1 - 2f) \\ -(\cos 2\theta + 4\cos^2\theta + \frac{\cos^2 2\theta}{\sin^2\theta})(1 - 2g) - \frac{\cos^4 2\theta}{\sin^2\theta}(1 - 2f) \end{matrix} \quad \cdots$$

$$\cdots \left.\begin{matrix} 4\cos^2\theta(1 - 2g)(1 - 2f) - \frac{1}{2}\cos 2\theta \\ 4\cos^2\theta(1 - 2g)(1 - 2f) + \cos 2\theta \end{matrix}\right) C = 0 \tag{13}$$

involve only derivatives of f in $[\overline{\delta f}]$ and of g in $[\overline{\delta g}]$.

We conclude that the ansatz (3,4) is the sphaleron S* provided a non-trivial solution to the three equations (7) exists, which simultaneously solves the two constraint equations (9b,12) on their θ dependence. Whether or not such a solution exists we do not know presently. Following Tyupkin et al.[13] we could introduce a minimizing sequence $\{(f_n,g_n,h_n)|n \in Z^+, E_{n+1} < E_n < \infty\}$. But to seek a <u>general</u> existence proof for non-trivial functions (f_o,g_o,h_o), so that $E(f_o,g_o,h_o) \leq \lim_{n\to\infty} E(f_n,g_n,h_n) = \inf E(f,g,h)$, does not seem worthwhile, since we need to know the <u>specific</u> form of (f_o,g_o,h_o) in order to check the constraint equations. Furthermore, if $f \neq g$ the term with $g(f - g)$ in (5) could be an obstacle for constructing the norm of the Hilbert space of the g functions, which would be needed in the proof.[13] It appears that the only way to prove existence of a non-trivial solution is by construction. In Section V we will philosophize on the possible form of S*, but we close this Section by showing that, asymptotically at least, there does exist a solution to the equations of motion of our ansatz.

For simplicity we take $\sigma^2 \equiv 2\lambda/g^2 < 1$. As $\xi \to \infty$ the $[\delta A_\phi]$ equations of (10) reduce to a single equation

$$-\xi^2 g'' \sim \frac{1}{4}\,\xi^2 h^2(1 - g) \qquad , \tag{14a}$$

the $[\delta A_\theta]$ equation gives the same one with g replaced by f, and the $[\overline{\delta\Phi}]$ equation (9a) simplifies to

$$(\xi^2 h')' \sim \frac{\lambda}{g^2}\,\xi^2(h^2 - 1)h \qquad . \tag{14b}$$

So asymptotically we have the solution

$$g \sim f \sim 1 - \tilde{\alpha}\, e^{-\xi/2} \tag{15a}$$

$$h \sim 1 - (\beta/\xi) e^{-\sigma\xi} \quad . \tag{15b}$$

Note that if we would have had $\sigma \geq 1$ the tail of h would still go as $e^{-\xi}$, because of the term $\sim 4h(1 + \cos^2\theta)(1 - f)^2$ in (9a), which would not be negligible.[14] Asymptotically we can take β to be a constant, so that $h \sim h(\xi)$. The θ dependence of $\tilde{\alpha}$ follows from the one equation $[\delta\tilde{\Phi}]$ (9b) that remains

$$\tilde{\alpha}(\theta) \sim \alpha\, e^{\frac{1}{2}\cos 2\theta} \quad , \tag{15c}$$

with α another constant. The values of α and β are to be determined from matching with the inside region. As far as $\tilde{\alpha}$ is concerned (15c) is only approximate, since, for example, it does not solve (12), and we conclude that $O(\xi^{-2})$ terms in β and $\tilde{\alpha}$ of (15ab), carrying θ dependence, are crucial for a complete solution.

IV. Energy

In this Section we obtain an upper bound on the energy of the new sphaleron S*. Based on the topological argument detailed in Sections I and II we expect such a solution S* to exist, but it may be that our ansatz (3,4) is only correct asymptotically and that the fields of S* in the inner region are excited away from the ansatz. Anyway, we can use the energy functional (5) to get an upper bound on E_{S*}, but solving the variational equations (7) is difficult, even numerically. Since the energy density in (5) only involves θ explicitly for the terms containing the function g, we may get a somewhat less sharp upper bound on the energy by restricting f, g and h to radial functions. This is so, provided the configuration with radial f, g and h is close enough to the true solution S*, or, to use the metaphoric language of the end of Section II, this radial configuration should belong to the same valley where S* is the lowest point, see Fig. 1. Henceforth we will assume this to be the case.

Setting f = g for simplicity, the energy functional (5) becomes for radial functions f and h

$$E = \frac{4\pi v}{g}\int d\xi \left[4\alpha f'^2 + \frac{8\beta}{\xi^2} f^2(1 - f)^2 + \frac{\gamma}{2}\xi^2 h'^2 + \delta h^2(1 - f)^2 + \frac{\varepsilon}{4}\ \frac{\lambda}{g^2}\xi^2(h^2 - 1)^2\right], \tag{16}$$

where $\alpha = \delta = 8/3$, $\beta = 16/3$ and $\gamma = \varepsilon = 1$. This energy is minimal for f and h that solve (for appropriate boundary conditions)

$$\alpha\xi^2 f'' = 2\beta\, f(1 - f)(1 - 2f) - \frac{1}{4}\,\delta\xi^2 h^2(1 - f) \tag{17a}$$

$$\gamma(\xi^2 h')' = 2\delta(1 - f)^2 h + \varepsilon\,\frac{\lambda}{g^2}\,\xi^2(h^2 - 1)h \quad . \tag{17b}$$

If all constants α ... ε would have equaled 1, the equations (16)(17) would have been precisely those[3] of the original sphaleron S. Yet this apparently minor modification leads to different behaviour of the functions near the origin,

$$f(\xi) = (\xi/\Xi)^n \quad , \quad h(\xi) = (\xi/\Omega)^m \quad , \quad \xi << 1 \quad , \tag{18}$$

where n and m solve the equations $n^2 - n - 2\alpha/\beta = 0$ and $m^2 + m - 2\delta/\gamma = 0$, so that in the present case $m = (\sqrt{201} - 3)/6 \approx 1.86$ and $n = (1 + \sqrt{17})/2 \approx 2.56$. The constants Ξ and Ω in (18) are determined by the boundary condition at $\xi = \infty$. An upper bound on (16) can be calculated analytically if we use for the functions f and h the form (18) for $\xi < \Xi$, where we set $\Xi = \Omega$ for simplicity, and $f = h = 1$ for $\xi \geq \Xi$. Inserting the numerical values for n and m the energy is then

$$E = \frac{4\pi v}{g} [19.17/\Xi + 0.4704\Xi + 0.03291 \frac{\lambda}{g^2} \Xi^3] \quad . \tag{19}$$

For the Higgs couplings[15] $\lambda/g^2 = 0$ and 1, this energy (for the rest of this section we use units of $4\pi v/g \approx 5.0$ TeV) has a minimum value ≈ 6.0 and 8.5 at $\Xi \approx 6.4$ and 3.4, respectively. To improve on these analytic bounds we have solved (17) numerically and found[16] $E(\lambda/g^2 = 0) = 3.31$ and $E(\lambda/g^2 = 1) \approx 5$, which, as said above, are upper bounds on $E_{S*}(\lambda/g^2)$. These bounds may be compared with the energies [3] of the original sphaleron $E_S(\lambda/g^2 = 0) = 1.52$ and $E_S(\lambda/g^2 = 1) = 2.07$. In the next Section we will continue the comparison further.

V. Discussion

Based on topological considerations[5] in configuration space, we expect the existence of a new sphaleron solution S*. The old sphaleron S was arrived at by the same topological argument. In that case the ansatz solving the equations of motion was found, and the sphaleron S turned out to be old indeed, since the ansatz was already written up[4] over 10 years ago. For the new sphaleron S* the situation is less clear for the moment. As discussed at the end of Section III, the problem is that for the ansatz proposed in this article we cannot be sure that the resulting equations of motion have a non-trivial solution. Even if there were no such solution, our ansatz would have some value asymptotically and the following picture for S* would hold. For $\xi > \Xi$ we would insert our approximate solution (15) in (5) to get $E^{>\Xi}_{S*}$, whereas for the inner region $E^{<\Xi}_{S*}$ would have a structure similar as (19). We expect Ξ to have such a value that $E^{<\Xi}_{S*}$, which contains a Ξ^{-1} piece, would balance with $E^{>\Xi}_{S*}$, whose dominant pieces go as positive powers of Ξ. So S* would be expected to have on its outside ($\xi > \Xi$) scalar fields and gauge fields close to our ansatz, while for the inside region the fields could differ appreciably from those of our ansatz.

Little more can be said generally, so henceforth we assume the ansatz to be correct for the inside region also. After all, working our ansatz through the field equations (8) some quite miraculous cancellations and simplifications occurred, thereby reducing the eight original equations (8) to precisely the ones (7) that followed from variations within the ansatz, and two additional equations (9b,12), that concern only the polar dependence of the functions involved. Furthermore, we will show in a moment that the present ansatz is similar to the one for the old sphaleron S, which gave a true solution.

The ansatz for S is the same as (3), but with f = g and h radial functions only and with U given by[17]

$$U^S \equiv u^S c \equiv (\cos\theta, \sin\theta, \sin\theta) \begin{pmatrix} 1 \\ -\cos\phi\, i\sigma_x \\ \sin\phi\, i\sigma_y \end{pmatrix} , \tag{20}$$

where we have introduced a 3-column c of 2 x 2 matrices. Remarkably the north pole ($\mu = \nu = 0$) of our map (2) gives U in (4) with u = (cos 2θ, sin 2θ, sin 2θ), which is just $u^S(2\theta)$. This apparently minor difference of U compared to U^S reduces the spherical symmetry of S to an axial symmetry; see the explicit $\cos^2\theta$ and cotan 2θ terms in (5). So we may think of S* as somehow doubling S, but in what sense precisely? There appear to be two alternatives: 1. S* is just a configuration with two sphalerons S infinitely far apart, or 2. S* is a truly new axisymmetric solution and may perhaps be viewed as an excitation of S or as the binding together of 2 S's.

Let us start with an heuristic picture of the first possibility. For simplicity we neglect the gauge fields in this picture (but see below), which may be reasonable for small values of λ, cf. (15ab). We look at the behaviour of the asymptotic Higgs field $\Phi \sim U \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv uc \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, where the column c of matrices was defined in (20). Consider the following configuration of two physically identical sphalerons at a large separation R: S^1 centered at (0,0,R/2) with U given by U^S of (20) and S^2 centered at (0,R,R/2) with U given by $-U^S$ and being flipped upside down. Note that this configuration complies with the gauge condition $\Phi = 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ at $(x,y,+\infty)$. Now we let an invisible hand push S^2 to the position (0,0,-R/2) and flip it upside down once more. In the vertical half plane through the z-axis and at fixed azimuth ϕ the resulting configuration of the Higgs field is then given schematically by the following values for u (abbreviating $r \equiv \sqrt{1/2}$).

$$
\begin{array}{llll}
(1,0,0) & & & \\
 & r(1,1,1) & & \\
S^1 & & (0,1,1) & \\
 & r(-1,1,1) & & \\
(-1,0,0) & & & \\
(-1,0,0) & & & \\
 & r(-1,-1,-1) & & \\
S^2 & & (0,-1,-1) & \\
 & r(1,-1,-1) & & \\
(1,0,0) & . & &
\end{array}
\tag{21}
$$

In the same half plane S* with u given by (4) has

$$
\begin{array}{llllll}
(1,0,0) & & & & & \\
 & r(1,1,1) & & & & \\
 & & (0,1,1) & & & \\
 & & & r(-1,1,1) & & \\
S^* & & & & (-1,0,0) & \\
 & & & r(-1,-1,-1) & & \\
 & & (0,-1,-1) & & & \\
 & r(1,-1,-1) & & & & \\
(1,0,0) & . & & & &
\end{array}
\tag{22}
$$

Pulling (-1,0,0) and its neighbours in (21) outwards, which does not cost energy since far outside the centers of S^1 and S^2 we have vacuum, we get the same sequence (22) as for S*. This argument shows that S* may be related to, or the same as, a configuration of two S's. But our neglect of the gauge fields, which certainly is doubtful for large values of λ, makes the above argument rather shaky. Only a complete solution for our ansatz, or a better one, can chose between the following options: 1. S* is just a configuration of two sphalerons S infinitely far apart, 2. S* has two energy cores and is the binding together of two S's, and 3. S* has a single energy core. The first possibliity could be ruled out if $E_{S*} < 2E_S$(=3.04 $4\pi v/g$ for λ/g^2 = 0). Disappointingly, the upper bound on E_{S*} of 3.31 $4\pi v/g$ for $\lambda/g^2 = 0$, which we derived in Section IV, does not decide the matter. Still, this bound is already so close to $2E_S$ to make the second or third possiblity quite real, that S* can smoothly distribute its energy density and make its total energy approximately $2E_S$ or less. Remark that it is possible, in principle, that option 3 has an energy somewhat above $2E_S$, but of course less than the bound of Section IV (hence the situation would be just as the full curve in Fig. 1, but with E_{S*} raised a little). Furthermore, we would like to mention that our ansatz really aims at a new solution (options 2 or 3), since it does <u>not</u> have vacuum fields at the origin, whereas (21) has. Should the first option be realized, and no new solution is found, that would be an example of the loophole in the topological argument as discussed in the Introduction.

To summarize, a topological argument in the space of classical configuration of the Weinberg-Salam theory leads us to expect a new saddle-point solution S*, although a loophole in the argument could make that S* is rather trivial. We have presented an ansatz (3,4) for a new sphaleron

S*, which has some promising features, but in the end its success is unclear, since the resulting equations of motion appear to be overdetermined. It seems that only a numerical study of these equations, which would be rather difficult, could establish whether or not a non-trivial solution exists. So, for the moment, we do not have a definite answer to the question contained in the title of this article.

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10. for the pure SU(2) gauge theory a true solution is known and has $A(n) = nA(1)$ precisely; see, for example, Section VII D1 of A. Actor, Rev. Mod. Phys. 51, 461 (1979).
11. We thank C. Taubes for suggesting this form to us; see Section VIII-5 of E.H. Spanier, Algebraic Topology (McGraw-Hill, 1966).
12. The (1-h) part of Φ in (3) is not unique. First, with the standard spherical coordinates $\bar{\phi},\bar{\theta}$ for $\bar{S}^2$, a more natural choice would be $(1-h)\begin{pmatrix} 0 \\ (1-\cos\bar{\theta})/2 \end{pmatrix}$, which also vanishes at the pole $\mu = \nu = 0$ or $\bar{\theta} = 0$. Secondly, since the ϕ,θ independent part of $U(\hat{x};\mu,\nu)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is simply $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, another possibility would have been $(1-h)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, c.f. Ref. 6. All of the

configurations would then have pure vacuum at the origin, but, as Section V will make clear, this does not seem promising for finding a new solution.

13. Yu.S. Tyupkin, V.A. Fateev, and A.S. Shvarts, Theor. Math. Phys. 26, 270 (1976). Their proof for the case of the magnetic monopole can be followed verbatim to show the existence of the sphaleron S, and incidentally also to prove the existence of a non-trivial solution to (17); see J. Burzlaff, Nucl. Phys. B233, 262 (1984).

14. The same holds for the sphaleron S, which corrects the statement below (11) of Ref. 3.

15. The reason for this choice is that we would like the energy (14) to be as close as possible to E_S, which has all constants $\alpha \ldots \varepsilon = 1$ (see Section V). To benefit from the fact that the kinetic and potential terms of the function h are multiplied by $\gamma = \varepsilon = 1$, we should have λ/g^2 significantly smaller than ∞, where h becomes a step function and both terms do not contribute.[3] For example, we see, without any calculation, that the energy (14) is $>(8/3)E_S$ for $\lambda/g^2 = \infty$.

16. We used a pocket calculator (HP15C) and time limitation prevented us from getting an accurate energy value. The $\lambda = 0$ value quoted in the text was calculated by J. Boguta and J. Kunz, for which we thank them. We have also considered other trial functions for f and h (for example $f = f(x)$, $h = h(x)$, with $x \equiv \xi/(1 + q \sin^2\theta)$), but could not find better energy bounds.

17. See (7abc) of Ref. 3, where in (7c) we have changed the sign of the σ_x term ($y \to -y$), so that (23) has the same signs as (4).

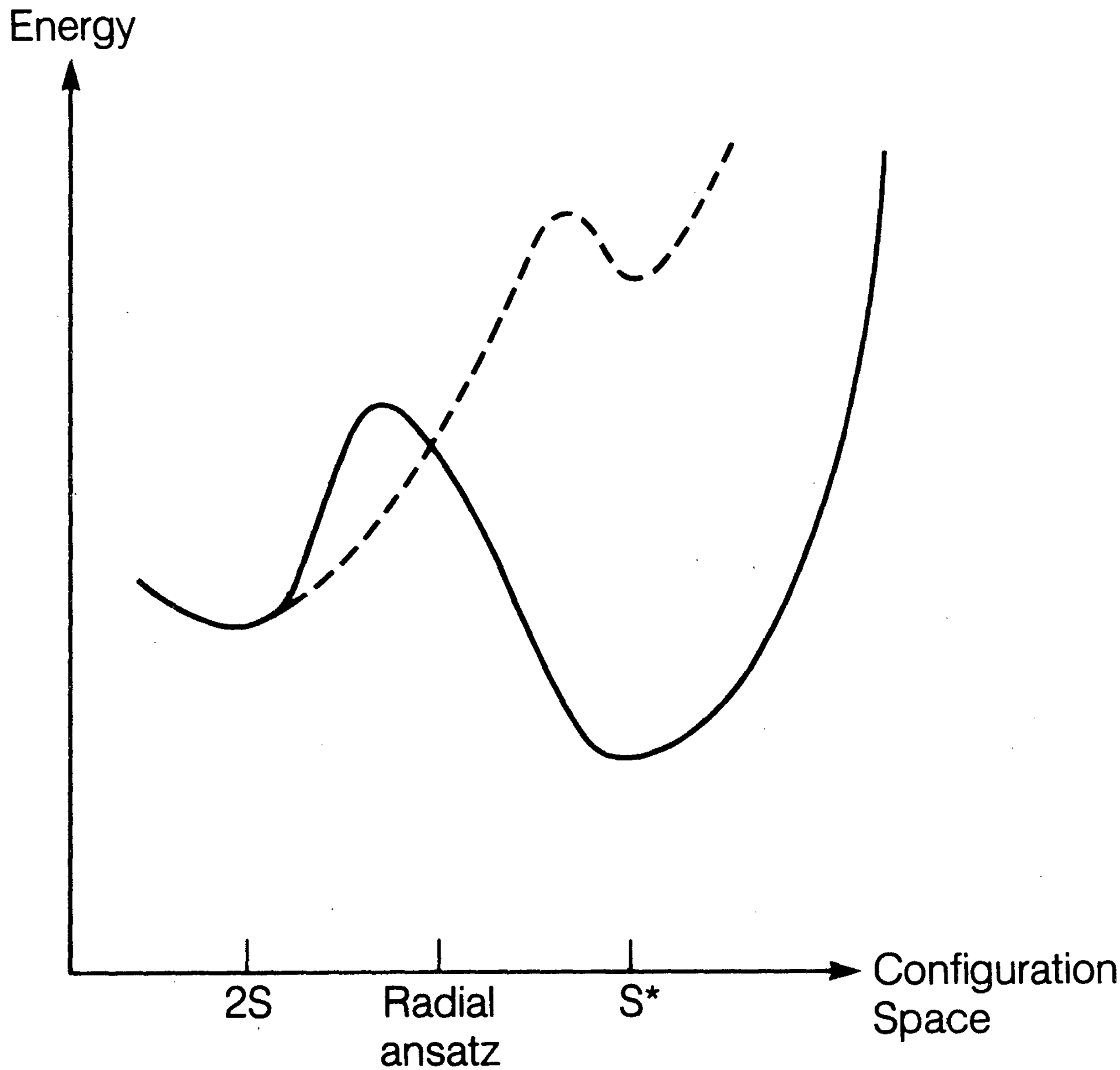

Fig. 1 Schematic view of a possible situation (dashed curve) which does not arise we hope. Rather we assume (full curve) that our ansatz with radial functions f,g,h is close enough to S*, so that $E_{S^*} \leq E$ (radial ansatz). Note that in drawing this figure we have assumed that there does exist a different solution, i.e. S*, in addition to the one of two sphalerons S infinitely far apart (2S).