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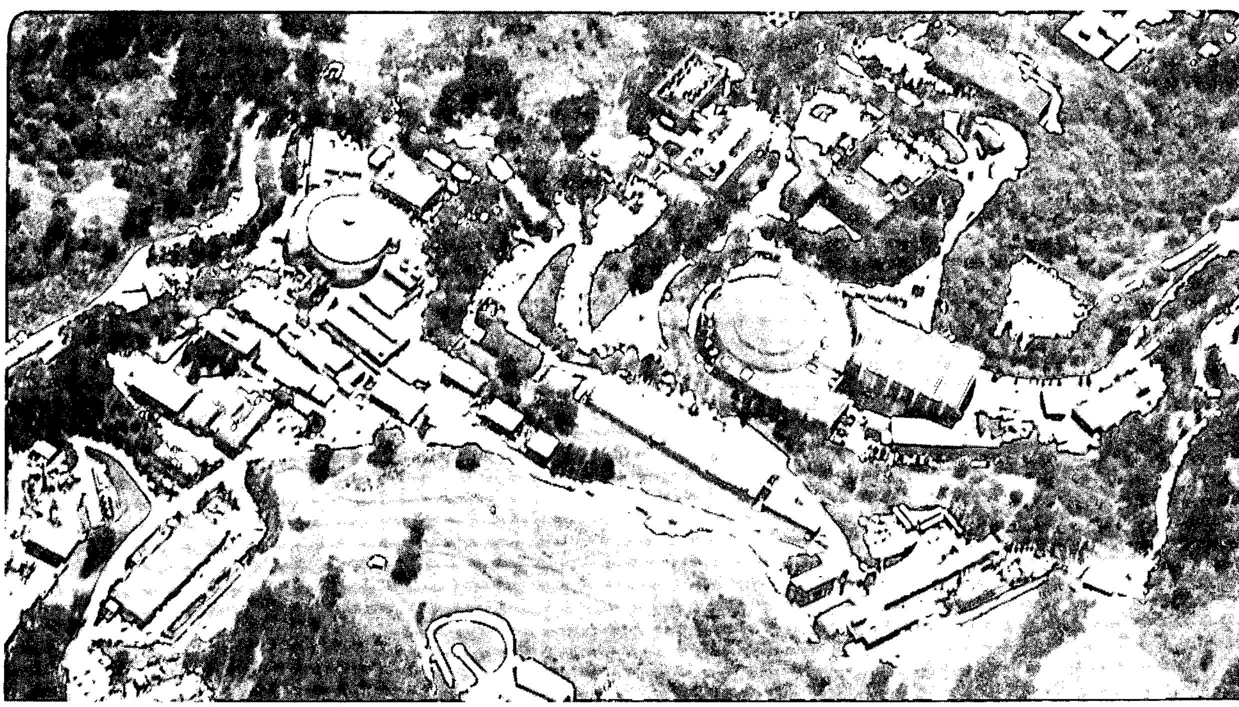

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ANALYZING GEOGRAPHIC CLUSTERED RESPONSE

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Analyzing Geographic Clustered Response

ABSTRACT

In the study of geographic disease clusters, an alternative to traditional methods based on rates is to analyze case locations on a transformed map in which population density is everywhere equal. Although the analyst's task is thereby simplified, the specification of the density equalizing map projection (DEMP) itself is not simple and continues to be the subject of considerable research. Here a new DEMP algorithm is described, which avoids some of the difficulties of earlier approaches. The new algorithm (a) avoids illegal overlapping of transformed polygons; (b) finds the unique solution that minimizes map distortion; (c) provides constant magnification over each map polygon; (d) defines a continuous transformation over the entire map domain; (e) defines an inverse transformation; (f) can accept optional constraints such as fixed boundaries; and (g) can use commercially supported minimization software. Work is continuing to improve computing efficiency and improve the algorithm.

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key words: cartogram; density equalizing map projection; disease clusters

I. Introduction

Studies of geographic disease clusters are frequently surrounded by controversy. Often such studies, being politically rather than scientifically motivated, are conducted with insufficient data and, predictably, yield inconclusive results. There is pressure to publish positive results and ignore negative results; as a result, causally unrelated results emerge which raise more questions than they answer. Rothman has gone so far as to state:[1]

> 1) with very few exceptions, there is little scientific or public health purpose to *investigate* individual disease clusters at all; 2) there is likewise very little reason to study overall patterns of disease clustering in space-time; and 3) as a consequent of points 1 and 2, no new statistical methodologies are needed to refine our study of disease clusters or clustering in general.

Rothman's statement, though perhaps true in a limited context, is so categorically stated that it demands discussion. To support the opposite view, we present Figures 1 and 2, which are examples of maps in which geographic variation provided important clues regarding disease etiology.

In Figure 1 is the well-known map of Dr. John Snow, in which he plotted cholera deaths in central London in 1854.[2] The observed clustering of deaths in the vicinity of the Broad Street pump (indicated by the circled X near the center of the map) led him to the correct conclusion that water from the pump was contaminated and responsible for the cholera epidemic.

In Figure 2 is a map of white male stomach cancer mortality rates, age-adjusted by county, 1950-1969.[2] The elevated rates observed in northern Minnesota and Michigan coincide geographically with high proportions of Northern European immigrants living in those areas. The observed correlation is compatible with high rates of stomach cancer prevailing in their countries of origin.[3]

The role of statistical analysis is to provide objective criteria for the evaluation of alternate hypotheses. A negative result can be as important as a positive one; either must be quantitatively stated if it is to contribute usefully to epidemiologic knowledge. For example, quantitative statistical techniques are required in order to avoid drawing possibly incorrect conclusions from Figures 1 and 2:

In Figure 1, cases rather than rates were plotted. Without knowledge of the population distribution, one cannot know whether the observed cluster of cholera cases demonstrates a real health risk, or is simply a reflection of higher population density in the vicinity of the Broad Street pump.

In Figure 2, without knowing the population of individual counties, one cannot know whether the high rate in a black-shaded county is statistically significant. Furthermore, large sparsely populated counties such as those in the Southwest are statistically less important than their size suggests.

I.A. Density equalizing map projections (DEMPs)

An early attempt to deal with such problems is illustrated in the maps of Figure 3, which were published in 1927.[4] In (a) the number of 1915-1924 smallpox deaths in each California county is represented by a number of dots; in (b) each county has been given an area proportional to its population, so that population density is everywhere equal. The author actually constructed his maps from lumps of plastocine; the weight of plastocine put into each county was proportional to the population; the density-equalized map in (b) was produced by rolling the counties flat with a rolling pin. The smallpox clusters observed in (a) are primarily an artifact of the large populations in one or two counties. In this example, the author did not discuss how the transformed map might be statistically analyzed.

Density-equalized maps, frequently called cartograms, have been widely used for graphic display, in various applications unrelated to epidemiology or public health. More often than not, cartograms have been drawn freehand by a graphic artist.[5,6] Others have been constructed with the aid of computers, but no specification has been given for linking the transformed map to a variable such as population.[7]

In recent years, a number of computerized algorithms for density equalizing map projections (DEMPs) have been described and used to display geographic data.[8,9,10,11,12] In each case, the algorithm iteratively adjusts polygon boundaries by *ad hoc* procedures until each polygon has the desired area. For example, maps of the United States before and after a DEMP, from Schulman *et al.*,[8] are shown in Figure 4. DEMPs in the references cited here, with the exception of Schulman *et al.*, have not been used for quantitative statistical analysis of spatial data.

I.B. Statistical analysis of DEMPs to study disease clusters

Previous work at Lawrence Berkeley Laboratory has focused on the application of DEMPs to the study of disease clustering.[8,13,14,15] A transformed map is particularly useful for epidemiologic analysis of disease because spatial patterns of disease cases are generally dominated by the distribution of populations at risk.

Clusters of disease cases can occur on an ordinary geopolitical map, even if every individual is equally likely to contract the disease. Examples are shown in Figure 1 and Figure 3(a). The apparent clusters can be the result of (a) non-uniform risk of disease, or (b) non-uniform distribution of the population at risk, or (c) both.

Typical procedures for eliminating the confounding effect of a non-uniform population distribution include calculation of rates (as shown in Figure 2), or Poisson regression analysis. Rates cannot be reliably calculated if the number of cases in each geographic subarea is very small, and combining subareas in order to create a region with a significant number of cases causes geographic information to be lost.

More importantly, combination of subareas can introduce bias, since the choice of subarea groups is arbitrary and can be affected by prior knowledge of the data. The notorious practice of "Texas sharpshooting" (for example defining a study area which would group together the high-rate counties of Figure 2) is guaranteed to produce a biased and meaningless result.

A DEMP in general provides an unbiased method of assessing geographic clustering both visually and statistically, free from the confounding distribution of the population at risk. Exact case locations can be used, so that no geographic detail is lost. The statistical analysis of a DEMP is simplified, since a disease whose risk is spatially uniform (the null hypothesis) produces a random distribution of cases over the transformed map. As a result, on a DEMP, the expected moments of the distribution of cases (calculated under the assumption of uniform risk) depend only on the external boundary of the transformed map and can be calculated analytically.[8,13,14]

I.C. Limitations of previous DEMP algorithms

In the present paper we discuss features of DEMP algorithms *per se,* without repeating previous work[8,13,14,15] concerning the statistical analysis of density equalized maps. Although the DEMP greatly simplifies the analyst's task, the specification of the DEMP itself is not simple and continues to be the subject of considerable research. With minor exceptions, the previously published DEMP algorithms[8,9,10,11,12] share the following drawbacks:

(a) Sparsely populated areas are necessarily transformed into thin snakelike areas, and repeated iterations are frequently unable to reduce their area to the desired value. Without special procedures, some transformed polygons overlap each other, corresponding to a multivalued mapping function and areas with negative population density.

(b) Even when correct polygon areas are achieved, there is an infinite number of possible solutions, and no objective criterion by which to judge their relative merit.

(c) Only the transformed polygon area is important for most display purposes. A stricter condition -- constant area magnification within each polygon -- is required if the transformed map is to be used for statistical analysis of point locations. This condition is not guaranteed by most of the previous algorithms.

(d) Only the transformation of boundary points is specified, and not the transformation of points *within* the polygons (for example, the locations of disease cases).

(e) An inverse transformation is not specified. This could be used, for example, to plot on the original map contours which include equal numbers of people.

(f) Imposition of additional constraints (for example, requiring a fixed boundary) is usually not possible.

(g) The previous algorithms incorporate *ad hoc* iteration procedures. They do not take advantage of important recently developed optimization techniques.

In this paper we introduce a new DEMP algorithm that eliminates all of the deficiencies (a)-(g).

Section II describes pre-DEMP map processing. Latitude and longitude are converted to Cartesian coordinates, unneeded geographic details are removed, and each polygon in the map is subdivided into triangles.

Section III describes the DEMP algorithm itself. A linear transformation is defined for every triangle in the map; then the parameters of the transformation are adjusted with standard nonlinear optimization techniques, so as yield the correct transformed area for each triangle while holding overall map distortion to a minimum.

Section IV describes post-DEMP map processing. The linear transformation within each triangle is used to determine the transformed location of arbitrary points in the map. The transformed locations of cases of disease are statistically analyzed by methods which have been described elsewhere.[8,13,14,15]

II. Pre-DEMP map processing

In this section we describe the methods used to prepare a geographic map, with coordinates given in latitude and longitude, for DEMP processing. Because the DEMP requires calculation of polygon areas, we first convert latitude and longitude to approximate distance units, *e.g.* kilometers. In order to reduce computation time we remove unnecessary detail by filtering out points which contribute little to map definition. Then we subdivide the map polygons into triangles so that the DEMP can be described as a continuous piecewise linear transformation with a finite number of parameters.

We assume that the map file to be processed is free of errors, and that unwanted features such as small islands and lakes have been removed. One can show that an error-free map of a simply connected region (*i.e.*, without islands or lakes) obeys the relationship (number of segments) - (number of points) = (number of polygons) - 1

II.A. Transformation of map coordinates

Coordinates in geographic base files (GBF) are customarily stored in degrees as (long,lat), where

$$\text{long} = |\text{ east longitude }| \quad \text{or} \quad -|\text{ west longitude }|$$

$$\text{lat} = |\text{ north latitude }| \quad \text{or} \quad -|\text{ south latitude }|$$

In order to facilitate area calculations, the geographic (long,lat) coordinates are first transformed into Cartesian coordinates (x,y), using any desired map projection. For example, for small areas far from the poles (noting that one degree of latitude is approximately 111.195 km) one might use the equirectangular projection

$$x = 111.195\, c_0\, (\text{long} - a_0) \cos(\, b_0\, \pi \,/\, 180\,)$$

$$y = 111.195\, c_0\, (\text{lat} - b_0)$$

where (a_0,b_0) are the (long,lat) coordinates of the approximate center of the map, and c_0 is a scale factor approximately equal to the number of map units in one kilometer. If (for example) $c_0 = 1000$, x and y are in meters, and areas are in square meters. We calculate maxkm, the maximum linear extent of the map in kilometers; then choose

$$c_0 = 1 \quad \text{if } 4000 \le \text{maxkm};$$

$$c_0 = 10 \quad \text{if } 400 \le \text{maxkm} < 4000;$$

$$c_0 = 100 \quad \text{if } 40 \le \text{maxkm} < 400;$$

$$c_0 = 1000 \quad \text{if } 4 \le \text{maxkm} < 40;$$

etc. With this choice, x and y can be stored as 16-bit integers, and exact area calculations can be performed with 32-bit integer arithmetic. (In order to use integer arithmetic and avoid division by 2, it is necessary to calculate and compare values equal to twice the area, rather than area.)

For larger areas such as the continental United States, a different projection such as a conic projection would be more suitable. The projection parameters (for example a_0, b_0 and c_0) and the specification of what projection was used should be stored along with the (x,y) map coordinates to permit later conversion back to (long,lat) coordinates.

II.B. Polygon area calculations

The DEMP algorithm relies on polygon area calculations, which are illustrated in Figure 5. Map files are stored in a Dual Independent Map Encoded (DIME) format, similar to that used by the U.S. Bureau of the Census for its 1990 Census TIGER (Topologically Integrated Geographic Encoding and Referencing) files.[16] Records corresponding to individual line segments, stored in arbitrary order, contain:

- code(s) of polygon on left side of segment
- code(s) of polygon on right side of segment
- x and y coordinates of start ("from") point of segment
- x and y coordinates of end ("to") point of segment

Each segment has an arbitrary "direction" indicated by its "from" and "to" point. The contribution to a given polygon's area is positive or negative if the polygon lies immediately to the left or right of the segment, respectively, and zero if the segment is not part of the polygon's boundary. The contribution of the segment to the polygon's area is

$$\pm \tfrac{1}{2} \left(x_{from}\, y_{to} - x_{to}\, y_{from} \right)$$

Polygon areas are calculated by summing over all segments in the file, in any order. "Polygon 00" in Figure 5 is the external area not included in polygons 01 and 02. By the convention just described, the areas of polygons 01, 02 and 00, respectively, are equal to

$$A_{01} = \tfrac{1}{2} (x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3) = +1$$

$$A_{02} = \tfrac{1}{2} (x_1y_3 - x_3y_1 + x_3y_4 - x_4y_3 + x_4y_1 - x_1y_4) = +1$$

$$A_{00} = \tfrac{1}{2} (x_2y_1 - x_1y_2 + x_1y_4 - x_4y_1 + x_4y_3 - x_3y_4 + x_3y_2 - x_2y_3) = -2$$

Ordinary polygons like 01 and 02 are traced counterclockwise and have positive area. Polygon "holes", like polygon 00, are traced clockwise and have negative area. Including the "holes", the net area of the entire map file is zero.

II.C. Point removal

Before proceeding with the DEMP, we remove unnecessary geographic detail from the map file in order to reduce computation time. The method illustrated in Figure 6 is simple, and retains the most important features of the map. The preservation of *nodes, i.e.* points that cannot be removed without altering the map topology, is an important feature of the point removal algorithm.

In (a), nodes A and F are connected by line segments through points B, C, D and E, which are candidates for removal. We wish to retain geographic features larger than a certain size, for example 10 km^2. The areas of all triangles involving three adjacent points (ABC, BCD, CDE, and DEF) are calculated, as shown. The smallest triangle is CDE (with an area of 3 km^2). We therefore remove point D; the segments CD and DE are replaced by the single segment CE.

In (b), the areas of the remaining affected triangles are recalculated and the process is repeated. The smallest triangle is CEF (with an area of 1 km^2) so point E is removed.

In (c), no triangles smaller than 10 km^2 remain, so points B and C are retained. Processing of the chain AF is complete.

The entire map file contains multiple chains similar to ABCDEF. In practice the smallest triangle in the entire map is always the next one considered for point removal, until the limit of 10 km^2 is reached. If removal of the considered point would cause two line segments anywhere in the map to intersect, the next smaller triangle in the map is considered instead. No polygon is reduced below three points.

In Figure 7 and Table I we illustrate point removal in a 1980 Census tract map of San Francisco. The map, with a total area of 120 km^2, has 152 polygons (excluding the external boundary polygon) and 415 chains connecting 264 nodes. All maps in Table I and Figure 7 obey the relationship

$$(\text{number of segments}) - (\text{number of points}) = (\text{number of polygons}) - 1$$

because each step in the point removal algorithm simultaneously removes one point and one segment, without altering the number of polygons.

The 0 km^2 map (not shown) differs from the original map in that collinear segments have been consolidated. The 0.01 km^2 map in (b) appears only slightly less detailed than the original map in (a) even though the number of points has been reduced from 1920 to 656. Even in the 1 km^2 map in (c), with only 268 points, the map still recognizably portrays San Francisco census tracts. The minimum map in (d), corresponding to an infinite point removal criterion, contains only the 264 nodes, which are necessary to preserve the original map topology; each of the 415 chains has been reduced to a single line segment.

Because excessive detail exacts a large computation penalty, we recommend a 1 km^2 point removal criterion for tract level DEMPs in dense urban areas like San Francisco. A point removal criterion between 100 km^2 and 1000 km^2 is appropriate for most county level maps.

II.D. Triangulation of polygons

We wish to represent the DEMP transformation by a finite number of parameters which are closely related to the available data; namely, the populations and polygon boundaries of the original map. Triangulation of each polygon allows a piecewise linear description of the DEMP transformation, while introducing no arbitrary new parameters.

It has been argued that, for purposes of interpolation,[17]

> the Delaunay triangulation is most appropriate because, besides being unique for a given set of data points, it simultaneously maximizes the number of triangles and produces triangles that are as equiangular as possible. This is important in the interpolation process since it ensures that triangle edge lengths and thus the distances between interpolation points are minimized.

A simple process is used which yields the uniquely defined Delaunay triangulation of every polygon. (In triangulating a quadrilateral all of whose points lie on a circle, there are two possible choices, both equally valid.) The algorithm is based on the properties of Voronoi (or Thiessen) polygons[17] and a theorem which is proved in Appendix A:

> Given *n* points in a plane, if the circle drawn through three of those points does not contain or touch any other of the *n* points, the triangle connecting those three points belongs to the Delaunay triangulation.

Application of the theorem is illustrated in Figure 8 for a sample polygon ABCDE. We want the triangulation of the plane to include all the pre-existing polygon boundaries, so we need only triangulate each polygon, one polygon at a time. Three adjacent points in the polygon boundary which form an interior angle less than 180 degrees, for example ABC, are arbitrarily selected. The (unique) circle is drawn which passes through all three of the points ABC. If the circle contains no other points of the ABCDE polygon boundary (which is the case), then triangle ABC is part of the Delaunay triangulation. Triangle ABC is separated out, and the same process is applied to the remaining polygon ACDE.

In the remaining polygon ACDE, the circle through triangle ADE contains no other points, so triangle ADE (and hence also ACD) is valid. On the other hand, the circle through triangle CDE contains points A and B, so triangle CDE is not valid (nor is ACE).

The triangulation process is repeated for every polygon in the entire map. Because the interior angle EAB is greater than 180 degrees, triangle EAB may be included in the triangulation of another polygon, but not of ABCDE.

In Figure 9 we illustrate the triangulated map of San Francisco, obtained by applying the triangulation algorithm to the 1 km^2 map shown in Figure 7(c). The triangulated map in Figure 9 contains 498 triangles, 268 points and 765 segments. Like all the maps in Figure 7 and Table I, it obeys the relationship

(number of segments) - (number of points) = (number of polygons) - 1

since each step in the triangulation algorithm simultaneously adds one segment and one polygon, without altering the number of points.

III. DEMP algorithm

The DEMP algorithm itself will now be described. The DEMP transformation is assumed to be continuous and piecewise linear; every triangle is mapped into another triangle, and adjacency of the triangles is preserved. The parameters of the DEMP are the transformed coordinates of the triangle vertices. The transformed area of each triangle is uniquely determined by requiring that (a) total area be unchanged (b) transformed areas be proportional to population, and (c) magnification be constant within each polygon. The DEMP parameters are optimized to determine the transformed map configuration which satisfies the area constraints and is least distorted relative to the original map.

III.A. Piecewise linear transformations

Once the Delaunay triangulation has been performed on the entire map file, our density equalizing map projection (DEMP) is performed. For simplicity, we require that the transformation within each Delaunay triangle be constant and linear; under such a transformation, each triangle maps into another triangle.

In the example of Figure 10, all points (x,y) within triangle 01 are mapped into points (u,v) by the linear transformation

$$u = a_{01}\, x + b_{01}\, y + e_{01}$$

$$v = c_{01}\, x + d_{01}\, y + f_{01}$$

In matrix notation,

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} a_{01} & b_{01} & e_{01} \\ c_{01} & d_{01} & f_{01} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

The area of triangle 01 is A_{01} before the transformation and B_{01} after the transformation.

Similarly, all points (x,y) within triangle 02 are mapped into points (u,v) by the linear transformation

$$u = a_{02}\,x + b_{02}\,y + e_{02}$$

$$v = c_{02}\,x + d_{02}\,y + f_{02}$$

In matrix notation,

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} a_{02} & b_{02} & e_{02} \\ c_{02} & d_{02} & f_{02} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

The area of triangle 02 is A_{02} before the transformation and B_{02} after the transformation.

The two linear transformations in Figure 10 can be defined by specifying the twelve coefficients a_{01} through f_{01} and a_{02} through f_{02}. Four constraints on the coefficients are required in order to preserve the adjacency of triangles 01 and 02 in the transformation. Necessary (but not sufficient) conditions are: $e_{01} = e_{02}$ and $f_{01} = f_{02}$. More economically, one can define the transformation by simply specifying the eight transformed triangle coordinates (u_1,v_1), (u_2,v_2), (u_3,v_3), (u_4,v_4). Expressions for $a \ldots f$ in terms of u and v (and the original coordinates x and y) are as follows. For example, for triangles 01 and 02:

$$\begin{pmatrix} a_{01} & b_{01} & e_{01} \\ c_{01} & d_{01} & f_{01} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} a_{02} & b_{02} & e_{02} \\ c_{02} & d_{02} & f_{02} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} u_1 & u_3 & u_4 \\ v_1 & v_3 & v_4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$$

By considering the transformation of an infinitesimally small map region, one can show that at any point the map magnification, *i.e.*, the ratio of transformed to original map area, is equal to the Jacobian

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = ad - bc$$

In order that spurious disease clusters not be produced by the DEMP, we require not only that transformed polygons have the correct area, but also that area magnification be constant within each polygon associated with a given population. This is automatically true within each *triangle,* since *a, b, c,* and *d* are constant for a linear transformation. In the next section we describe the additional constraints required to guarantee constant magnification within each entire *polygon.*

If $A_{01} \neq 0$, the DEMP transformation is defined. The Jacobian is equal to $B_{01}/A_{01} = a_{01}d_{01}-b_{01}c_{01}$, the area magnification factor of triangle 01.

If $B_{01} \neq 0$, the inverse transformation is defined, with an area magnification factor of $A_{01}/B_{01} = (a_{01}d_{01}-b_{01}c_{01})^{-1}$.

III.B. Area constraints

In Figure 11 we illustrate the area constraints that are applied to polygons in a DEMP. Two polygons, a triangle 01 and a quadrilateral 02, are defined by coordinates (x_1,y_1) through (x_5,y_5), which are transformed into coordinates (u_1,v_1) through (u_5,v_5). The Delaunay triangulation divides the quadrilateral 02 into triangles 02.1 and 02.2 (and the triangle 01 into a single triangle 01.1). Areas before and after the DEMP are denoted by A and B respectively. We require that:

- total map area be unchanged by the transformation; *i.e.*,

$$B_{tot} = A_{tot}$$

- after the transformation, transformed polygon areas be proportional to population; *i.e.*,

$$B_{01} / pop_{01} = B_{02} / pop_{02}$$

- areas of triangles within each polygon are in the same proportion before and after the transformation; *i.e.*,

$$B_{02.1} / A_{02.1} = B_{02.2} / A_{02.2}$$

Hence we require the target areas $B_{01.1}$, $B_{02.1}$ and $B_{02.2}$ to be equal to

$$B_{01.1} = pop_{01} \times [A_{tot} / pop_{tot}] \times [A_{01.1} / A_{01}]$$

$$B_{02.1} = pop_{02} \times [A_{tot} / pop_{tot}] \times [A_{02.1} / A_{02}]$$

$$B_{02.2} = pop_{02} \times [A_{tot} / pop_{tot}] \times [A_{02.2} / A_{02}]$$

where

$$pop_{tot} = pop_{01} + pop_{02}$$

$$A_{tot} = A_{01} + A_{02} = A_{01.1} + A_{02.1} + A_{02.2}$$

$$B_{tot} = B_{01} + B_{02} = B_{01.1} + B_{02.1} + B_{02.2}$$

$$A_{01.1} = \tfrac{1}{2} (x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$$

$$A_{02.1} = \tfrac{1}{2} (x_1y_3 - x_3y_1 + x_3y_5 - x_5y_3 + x_5y_1 - x_1y_5)$$

$$A_{02.2} = \tfrac{1}{2} (x_3y_4 - x_4y_3 + x_4y_5 - x_5y_4 + x_5y_3 - x_3y_5)$$

In the $(\boldsymbol{u},\boldsymbol{v})$ space, then, we require the transformed triangle areas to be equal to their target values, which sets up the following three quadratic constraints on (u_1,v_1) through (u_5,v_5):

$$H_{01.1} \equiv \tfrac{1}{2} (u_1v_2 - u_2v_1 + u_2v_3 - u_3v_2 + u_3v_1 - u_1v_3) - B_{01.1} = 0$$

$$H_{02.1} \equiv \tfrac{1}{2} (u_1v_3 - u_3v_1 + u_3v_5 - u_5v_3 + u_5v_1 - u_1v_5) - B_{02.1} = 0$$

$$H_{02.2} \equiv \tfrac{1}{2} (u_3v_4 - u_4v_3 + u_4v_5 - u_5v_4 + u_5v_3 - u_3v_5) - B_{02.2} = 0$$

The imposition of separate area constraints on triangles $B_{02.1}$ and $B_{02.2}$, and not just on the quadrilateral B_{02}, is a critical feature of the DEMP algorithm. This feature prevents the occurrence of negative-area triangles, which would produce double-valued regions of the mapping function and self-intersecting polygon boundaries.

III.C. Map distortion

The area constraints described in the previous section are insufficient to completely define the DEMP. In the example of Figure 11, the ten parameters (u_1,v_1) through (u_5,v_5) are restricted by the three constraints $H_{01.1} = 0$, $H_{02.1} = 0$, and $H_{02.2} = 0$. Three further constraints are imposed by requiring that the transformed map not be rotated or translated (in x or y) relative to the original map. Four degrees of freedom remain, which means that the number of possible solutions is infinite.

Tobler recommends that a unique optimum solution be defined by making the DEMP be as nearly conformal as possible.[18] Conformal transformations are those that locally preserve the shape (but not necessarily the size, location, or orientation) of each infinitesimal portion of the map. As illustrated in Figure 12, conformal *linear* transformations include all combinations of translations, rotations and magnifications, corresponding to the following transformation matrices:

$$\begin{pmatrix} 1 & 0 & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}; \quad \begin{pmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

but not reflections, compressions or shear transformations, corresponding to the following transformation matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \begin{pmatrix} A & 0 & 0 \\ 0 & A^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & B & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Tobler's suggested measure of non-conformality (which we write in more customary notation)[19] is the integral over the original map of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

This quantity is identically equal to zero for conformal transformations, which by definition obey the Cauchy-Riemann conditions[19]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

However Tobler's measure does not properly reflect the non-conformality of of reflections, compressions or shear transformations; *i.e.*, *linear* transformations of the form

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

where $a \neq d$ or $b \neq -c$, since his measure is equal to zero in this situation also.

Instead of using Tobler's measure, we define instead the non-conformal distortion of a triangle 01.1 as

$$G_{01.1} = (a_{01.1} - d_{01.1})^2 + (b_{01.1} + c_{01.1})^2$$

where $a_{01.1}$, $b_{01.1}$, $c_{01.1}$ and $d_{01.1}$ are functions of (x,y) and (u,v) defined as in section III.A. The resulting expression is quadratic in u_1, v_1, u_2, v_2, u_3 and v_3, the transformed coordinates of triangle 01.1:

$$G_{01.1} = \{ [u_1(y_2-y_3) + u_2(y_3-y_1) + u_3(y_1-y_2) - v_1(x_3-x_2) - v_2(x_1-x_3) - v_3(x_2-x_1)]^2$$

$$+ [u_1(x_3-x_2) + u_2(x_1-x_3) + u_3(x_2-x_1) + v_1(y_2-y_3) + v_2(y_3-y_1) + v_3(y_1-y_2)]^2 \} / \det^2$$

where

$$\det = 2 A_{01.1} = x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3$$

The distortion of the entire map is calculated as a sum of G over all triangles, weighted by the area of each triangle. We choose the weighting coefficients to be $A+B$, the sum of the original area A and the target area B. (Using A instead of $A+B$ yields unstable solutions for triangles which are initially small and which are magnified by large factors; using B alone yields unstable solutions for triangles which are magnified by small factors. In addition, the symmetric form $A+B$ has the aesthetic property that the DEMP transformation, and the inverse transformation to return to the original map, have equal distortion.)

The assumed transformation functions $u(x,y)$ and $v(x,y)$ are continuous and piecewise linear; however, they have discontinuous first and second derivatives at triangle boundaries. At this time it is not clear how to quantify that component of map distortion.

III.D. Summary of DEMP algorithm

In summary, the DEMP algorithm consists of finding the set of transformed coordinates u_1, v_1, u_2, v_2... which minimize overall map distortion subject to the triangle area constraints. That is, for the example of Figure 11, finding the values $u_1, v_1, \ldots u_5, v_5$ which minimize

$$G(u,v) \equiv g_0 \left[g_{01.1}\, G_{01.1} + g_{02.1}\, G_{02.1} + g_{02.2}\, G_{02.2} \right]$$

where

$$g_{01.1}=A_{01.1}+B_{01.1};\quad g_{02.1}=A_{02.1}+B_{02.1};\quad g_{02.2}=A_{02.2}+B_{02.2}$$

subject to the three quadratic constraints

$$H_{01.1} = 0;\quad H_{02.1} = 0;\quad H_{02.2} = 0$$

or, equivalently, to the single constraint

$$H(u,v) \equiv h_0 \left[h_{01.1}(H_{01.1})^2 + h_{02.1}(H_{02.1})^2 + h_{02.2}(H_{02.2})^2 \right] = 0$$

which is fourth order in $u_1, v_1, \ldots u_5, v_5$; g_0 and h_0 are arbitrary constants. The constants $h_{01.1}$, $h_{02.1}$ and $h_{02.2}$ must be positive and non-zero; good convergence is obtained for the choice

$$h_{01.1} = \frac{1}{B_{01.1} + B_{min}};\quad h_{02.1} = \frac{1}{B_{02.1} + B_{min}};\quad h_{02.2} = \frac{1}{B_{02.2} + B_{min}}$$

where $B_{min} = A_{tot}/pop_{tot}$; namely, the area on the transformed map equivalent to a population of one person. The B_{min} term is required to avoid infinitely large coefficents h, for triangles having population and target area B equal to zero.

During the DEMP process, three of the ten parameters $u_1, v_1, \ldots u_5, v_5$ should be fixed in order to prevent arbitrary x translation and/or y translation and/or rotation of the entire transformed map. As illustrated in Table II,

(number of parameters) = 2 * (number of points) - 3

In order to fix three parameters, a convenient choice for a map whose x range exceeds its y range is:

$$u_{west} = x_{west};\quad v_{west} = y_{west};\quad v_{east} = y_{east}$$

where (x_{west}, y_{west}) and (u_{west}, v_{west}) are respectively the original and transformed (x,y) coordinates of the westernmost point of the original map; y_{east} and v_{east} are respectively the original and transformed y coordinate of the easternmost point of the original map. A convenient choice (with similar definitions) for a map whose y range exceeds its x range is:

$$u_{south} = x_{south};\ v_{south} = y_{south};\ u_{north} = x_{north}$$

After the DEMP, the resulting map can then be translated and/or rotated as desired.

Optionally, additional constraints can be imposed during the DEMP optimization, provided a feasible solution of the area constraints exists. For example, one could require $u_i = x_i$ and $v_i = y_i$ for all points i on the external map boundary.

IV. Post-DEMP map processing

In analyzing the geographic distribution of disease, the primary purpose of the DEMP is to plot cases on a transformed map where population density has been equalized. In addition one may wish to transform the locations of various geographic features. For example, one would normally wish to plot on the transformed map the location of a supposed environmental hazard such as a microwave tower or a toxic waste dump. Contours of equal distance from a point source, although no longer circular, could still serve to identify areas of supposedly equal risk. A transformed latitude-longitude grid could help the analyst locate additional features on the transformed plotted map.

Case locations or geographic features are transformed as follows:

- With the transformation described in Section II.A, the (long,lat) location of a disease case or geographic feature k is first projected into (x_k, y_k), in the same coordinate system as the pre-DEMP polygon map.

- The DEMP solution, determined as in Section III.D, provides values of $(\boldsymbol{u}, \boldsymbol{v})$, the vertices of all triangles in the DEMP-transformed map. From these, as explained in Section III.A, one can calculate a_i, b_i, c_i, d_i for each triangle i, and a global e and f.

- A point-in-triangle routine, along with the pre-DEMP triangle coordinates $(\boldsymbol{x}, \boldsymbol{y})$, is used to determine in which polygon i each point k is located. A point k lies inside or on the boundary of a triangle with vertices (x_1,y_1), (x_2,y_2), (x_3,y_3) (labeled in counterclockwise order) if and only if all three of the following quantities are non-negative:

$$(x_2-x_1)(y_k-y_1) - (x_k-x_1)(y_2-y_1)$$

$$(x_3-x_2)(y_k-y_2) - (x_k-x_2)(y_3-y_2)$$

$$(x_1-x_3)(y_k-y_3) - (x_k-x_3)(y_1-y_3)$$

- Then the appropriate a_i, b_i, c_i, d_i (and e and f) are used to transform the point (x_k, y_k) into density-equalized coordinates (u_k, v_k).

In a density equalized map, analysis of disease distributions is relatively straightforward. As described in Section I.B, a number of quantitative techniques have been developed, which have certain advantages over traditional rate comparisons or relative risk calculations.[8,13,14,15]

V. Preliminary results

V.A. NAG optimization

In our preliminary tests of the DEMP algorithm we have used the NAG (Numerical Algorithms Group) Fortran library routine E04VDF, which minimizes the quadratic objective function $G(\boldsymbol{u},\boldsymbol{v})$ subject to the fourth order constraint condition $H(\boldsymbol{u},\boldsymbol{v}) = 0$.[20] (An attempt to separately apply the quadratic constraints $H_{01.1} = 0$, $H_{02.1} = 0$, $H_{02.2} = 0$ led to excessive memory usage when the size of the problem was increased.)

E04VDF uses a sequential quadratic programming method described by Gill *et al.*[21] Explicitly calculated first derivatives of the objective function $G(\boldsymbol{u},\boldsymbol{v})$ and constraint function $H(\boldsymbol{u},\boldsymbol{v})$, with respect to the variables u_1, v_1, u_2, v_2, *etc.* are required. We note that $G(\boldsymbol{u},\boldsymbol{v})$ and the individual $H_i(\boldsymbol{u},\boldsymbol{v})$ are sums of separate terms, each of which is a quadratic function of only six parameters; namely, the transformed u and v coordinates of the vertices of a single triangle.

In addition, E04VDF requires an initial estimate of the solution, for which we use the original map ($u_1 = x_1$, $v_1 = y_1$, etc.)

Without affecting the final solution, the objective function $G(\boldsymbol{u},\boldsymbol{v})$ and its derivatives can be multiplied by an arbitrary constant g_0. Similarly, the constraint function $H(\boldsymbol{u},\boldsymbol{v})$ and its derivatives can be multiplied by an arbitrary constant h_0. Larger values of g_0 and h_0 produce more precise solutions, with more iterations. The ratio h_0/g_0 determines the relative weights of the two functions. Large values of h_0/g_0 can produce solutions in which distortion may not be minimized; small values can produce solutions in which the final triangle areas are not exactly correct. Some trial and error is required to obtain satisfactory solutions without an excessive number of iterations.

V.B. Examples

To illustrate the results of the preliminary DEMP implementation, we have chosen the state of Vermont, which has 14 counties. A point removal criterion of 500 km^2 was applied as described in Section II.C; then the filtered map was triangulated as described in Section II.D. The filtered and triangulated Vermont map, which is shown in Figure 13(a), has 30 points and 43 triangles; *i.e.*, 57 parameters and 43 area constraints.

In Figure 13(b) the same map has been transformed by a DEMP as described in Section III.D; in this example the variable equalized by the DEMP is the 1980 total population. Note that in the transformed map (b) the areas of triangles within a given county are proportional to their areas in the original map (a).

Figure 14 illustrates the optimization path corresponding to the total 1980 population DEMP map of Figure 13(b), for three different values of h_0/g_0. Optimization proceeds from lower right to upper left. The numbers on the curves represent the number of major iterations taken by the NAG routine E04VDF.

For any DEMP, the ***original*** map by definition has distortion $G(\boldsymbol{u},\boldsymbol{v})$ equal to zero; in this example (1980 total population) the initial violation of the area criterion was such that $\log_{10} H(\boldsymbol{u},\boldsymbol{v}) = -1.62$. During minimization, the area violation $H(\boldsymbol{u},\boldsymbol{v})$ was forced smaller and smaller, so that $\log_{10} H(\boldsymbol{u},\boldsymbol{v})$ became more and more negative; at the same time, distortion $G(\boldsymbol{u},\boldsymbol{v})$ necessarily increased from zero to a value around 0.52.

Consider first the middle curve, the solid line labeled "optimum area constraint." After about 90 major iterations (2.8 minutes of central processor time on a Sun SPARCstation 1), $\log_{10} H(\boldsymbol{u},\boldsymbol{v})$ reached a value around -6, at which point further changes in the map were invisible. Repeated trials from different starting points (not shown) consistently converged to the same solution.

In the lower dashed curve, labeled "weak area constraint," major iterations are indicated in parentheses ( ). Here the area constraint was weakened relative to the distortion function by decreasing the value of h_0/g_0. Distortion was low throughout the optimization, but more iterations were required. The final solution was the same as for the middle curve.

In the upper dashed curve, labeled "strong area constraint," major iterations are indicated in square brackets [ ]. Here the area constraint was strengthened relative to the distortion function by increasing the value of h_0/g_0. The area constraint was satisfied more quickly but distortion was increased. At the final solution, distortion remained around 0.53 and could not be reduced by further iterations. The final map configuration from the upper curve (not shown) was visibly different from those from the two lower curves.

In general, the optimum value of h_0/g_0 (here, the middle curve) is that which yields the least distorted solution in the fewest iterations. The user must experiment to determine optimum values of g_0 and h_0, and to determine at what value of $\log_{10} H(\boldsymbol{u},\boldsymbol{v})$ a stable solution has been reached.

Occasionally two or more distinct local minima of the distortion function $G(\boldsymbol{u},\boldsymbol{v})$ were found, both of which satisfy the area constraint $H(\boldsymbol{u},\boldsymbol{v})=0$. Figure 15 illustrates the results of a DEMP (again Vermont with a 500 km^2 point removal criterion) which equalized the 1980 *Native American* population.

The optimization path is illustrated in Figure 15(a). In this example (1980 Native American population) the initial violation of the area criterion was such that $\log_{10} H(u,v) = -0.97$. Two distinct solutions were found, with distortion $G(u,v)=1.14$ and 1.01 in (b) and (c) respectively. The second solution, which is less distorted, is preferred.

The initial value of $\log_{10} H(u,v) = -0.97$ for the Native American population, compared with $\log_{10} H(u,v) = -1.62$ for the total population, indicates that for the Native American population the area constraint (in the original map) is more severely violated than for the total population. This is consistent with the result that distortion (in the final transformed map) for the Native American population ($G(u,v)=1.14$ or 1.01) is larger than for the total population ($G(u,v)=0.52$).

For the Native American population the less distorted solution in Figure 15(c) was consistently obtained for weaker area constraints (smaller values of h_0/g_0). However, in other cases (*e.g.*, Black population, not shown) the reverse situation occurred.

The numbers given in this section for h_0/g_0, $G(u,v)$, and $\log_{10} H(u,v)$ are for comparison purposes only; they have no meaning in an absolute sense. A necessary condition for a DEMP "solution" is that repeated trials from various starting points converge to that solution, and that no solutions with lower distortion be found. However, one cannot prove that better solutions do not exist. When performing repeated trials, note that $G(u,v)$ is always the distortion of (u,v) relative to the *true* starting point (x,y), not relative to the perturbed starting point.

V.C. Computing Requirements

Approximate computing requirements for the NAG routine E04VDF are summarized in Table II. In order to estimate computing time for larger problems, we created a triangulated Vermont map (not shown) with a point removal criterion of 50 km^2, which had more points and more triangles than the 500 km^2 triangulated map in Figure 13(a). We performed DEMPs on both Vermont maps, equalizing both on 1980 total population density and using the same g_0 and h_0 and termination criterion ($\log_{10} H(u,v) = -6$).

Other factors being equal, computation time increased from 2.8 to 30 CP minutes (on a Sun SPARC station 1) when the number of points increased from 30 to 66, and the number of triangles increased from 43 to 103. Consistent results (an increase from 1.3 to 14 CP minutes) were obtained on a VAX 6510 computer.

For the example studied, computation time increased approximately as the third power of map complexity. The San Francisco 1 km^2 triangulated map in Figure 9 has about four times as many points and five times as many triangles as the Vermont 50 km^2 map (not shown). Therefore, if other factors remain comparable, we estimate that a San

Francisco tract level DEMP may take 2 to 3 days on a SPARC station 1. It is not known how the number of required iterations will increase with map complexity.

Runs of more than a few hours are impractical and expensive, especially since multiple runs are required to check the validity of a solution. Problems of interest may have thousands or even tens of thousands of points. The NAG routine E04VDF is not intended for large sparse problems, of which our DEMP algorithm is an example. Work is continuing to improve computation efficiency, and to find minimization tools which are better suited to the particular characteristics of our DEMP algorithm.

VI. Conclusions

As explained in Section I.B, density equalizing map projections (DEMPs), correct for the confounding effect of varying population density, thereby providing a useful tool for analyzing the geographic distribution of disease. The statistical analysis of a density equalized map is straightforward, since under the null hypothesis of equal risk, the distribution of disease cases on such a map is expected to be uniform.

Although a DEMP simplifies the task of the analyst, the specification of the DEMP transformation itself is not simple and continues to be the subject of considerable research. The algorithm described here represents a radical new approach to DEMP calculations, and has important advantages over previous techniques. In particular (*cf.* Section I.C.):

(a) application of the area constraint separately to each triangle avoids overlapping of transformed polygons, corresponding to a multivalued mapping function and areas of negative population density;

(b) competing solutions can be quantitatively compared, permitting a comparative evaluation of various optimization techniques;

(c) constant magnification within each polygon ensures that the DEMP cannot enhance spurious disease clusters;

(d) the solution determines a continuous transformation over the entire map surface;

(e) unless zero-population areas are present, the inverse transformation is uniquely defined;

(f) optional boundary constraints can be easily imposed, provided a solution exists that is compatible with the area constraints;

(g) commercially supported optimization software can be used.

Work is continuing to improve computation efficiency. Provided that the difficulties of numeric optimization can be resolved, the new DEMP algorithm can be put to practical use in routinely analyzing specific geographic disease distributions.

Appendix A. Voronoi polygons and Delaunay triangles.

In this appendix we prove the theorem used in Section II.D:

> Given *n* points in a plane, if the circle drawn through three of those points does not contain or touch any other of the *n* points, the triangle connecting those three points belongs to the Delaunay triangulation.

For illustration, a set of Voronoi polygons is shown in Figure A-1. Given *n* points in the plane (for example A,B,C,D,E), the Voronoi polygon associated with one point *i* is defined as that region which is as close or closer to point *i* than to any other of the *n* points. For example, the quadrilateral indicated by dashed lines in Figure A-1 encloses the region which is closer to point A than to point B, C, D or E.

The Delaunay triangulation is defined by the set of pairwise connections between points *i* whose Voronoi polygons are contiguous. In Figure A-1, the Delaunay triangulation is the set of eight line segments AB, AC, AD, AE, BC, CD, DE and EB.

Consider the circle centered at O, drawn through the three points A, D and E. If the circle does not contain or touch any other of the *n* points in the plane, then points A, D, and E are equally close to point O, and no other of the *n* points (*e.g.* B and C) is as close. Therefore, O belongs to the Voronoi polygons associated with A, D and E, but to no other Voronoi polygons. Exactly three Voronoi polygons (A, D and E) coincide at point O, so the three polygons must be mutually contiguous. By definition, AD and DE and EA belong to the Delaunay triangulation, so ADE is a Delaunay triangle.

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Table I. Point removal and triangulation, San Francisco, 1980 census tracts

area criterion	number of polygons*	number of points	number of segments	figure
original map	152	1920	2071	7(a)
0 km^2	152	1547	1698	not shown
0.001 km^2	152	1268	1419	not shown
0.01 km^2	152	656	807	7(b)
0.1 km^2	152	321	472	not shown
1 km^2	152	268	419	7(c)
minimum map	152	264	415	7(d)
1 km^2, triangulated	498	268	765	9

* excluding external boundary polygon for "rest of world"

Table II. DEMP Computing requirements

	two tri-angles	three tri-angles	Ver-mont 500 km^2	Ver-mont 50 km^2	San Francisco 1 km^2
figure	5, 10	11	13	not shown	7(c), 9
points	4	5	30	66	268
polygons*	2	2	14	14	152
triangles	2	3	43	103	498
parameters	5	7	57	129	533
area constraints	2	3	43	103	498
degrees of freedom	3	4	14	26	35
iterations			90	128	200-300
CP time on SPARC 1			2.8 min	30 min	2-3 days
CP time on VAX 6510			1.3 min	14 min	1-2 days

* excluding external boundary polygon for "rest of world"

Figure 1. Cholera in London, 1854

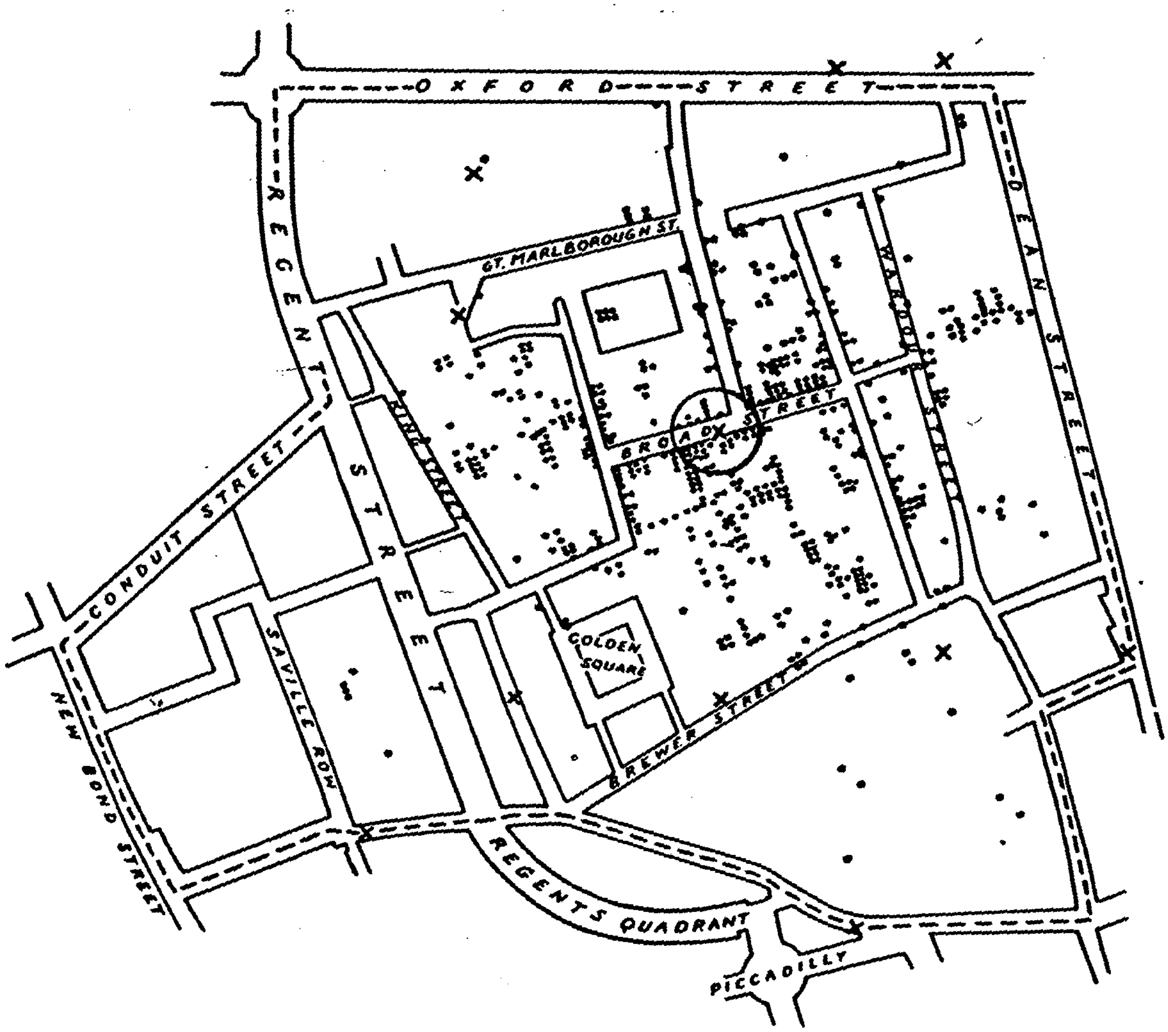


Figure 2. Stomach cancer, white males, age-adjusted rate by county, 1950-1969

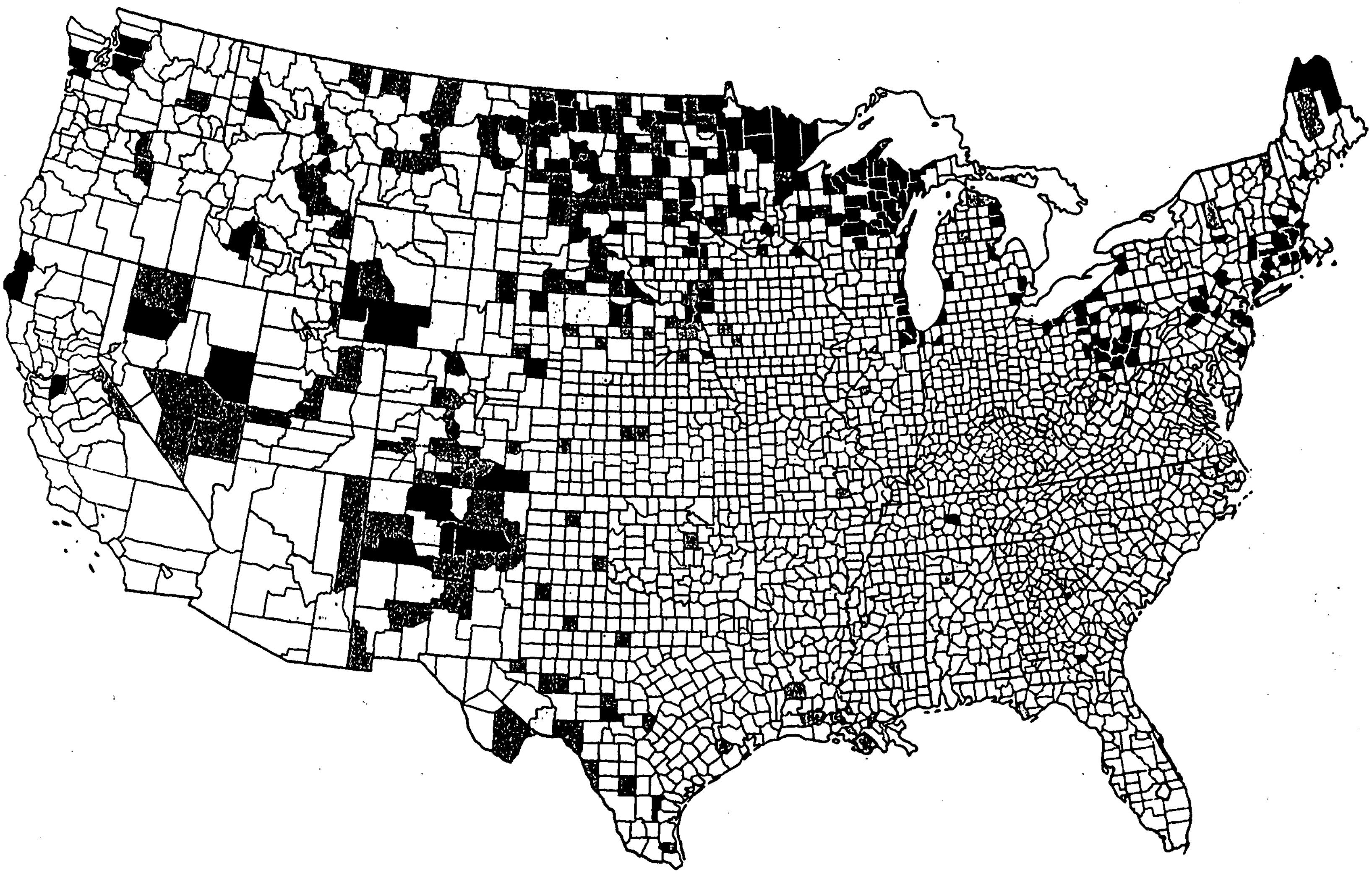


Figure 3. Ten years' deaths from smallpox, California, 1915-1924

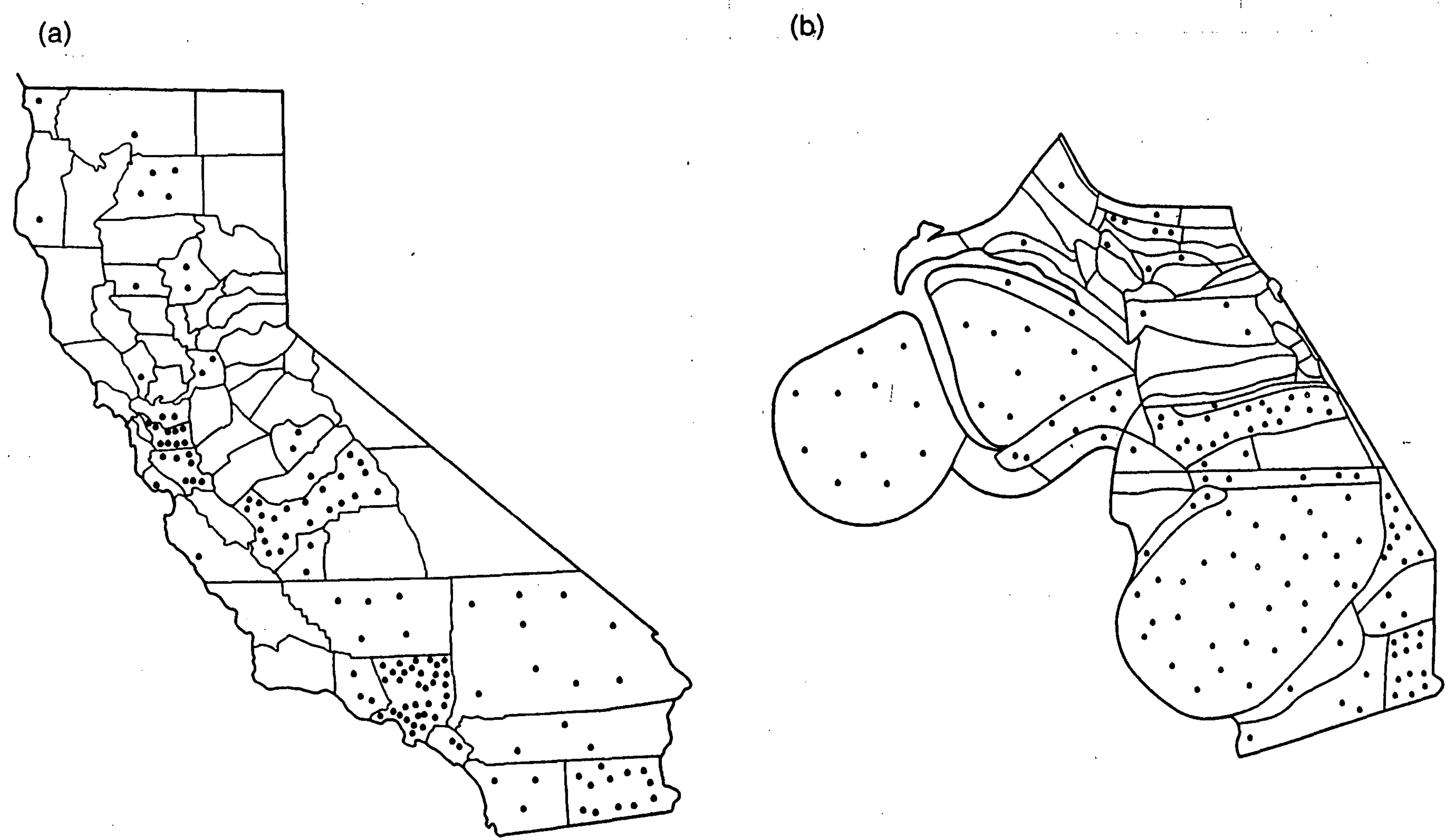


Figure 4. United States

(a) Geopolitical

(b) Transformed

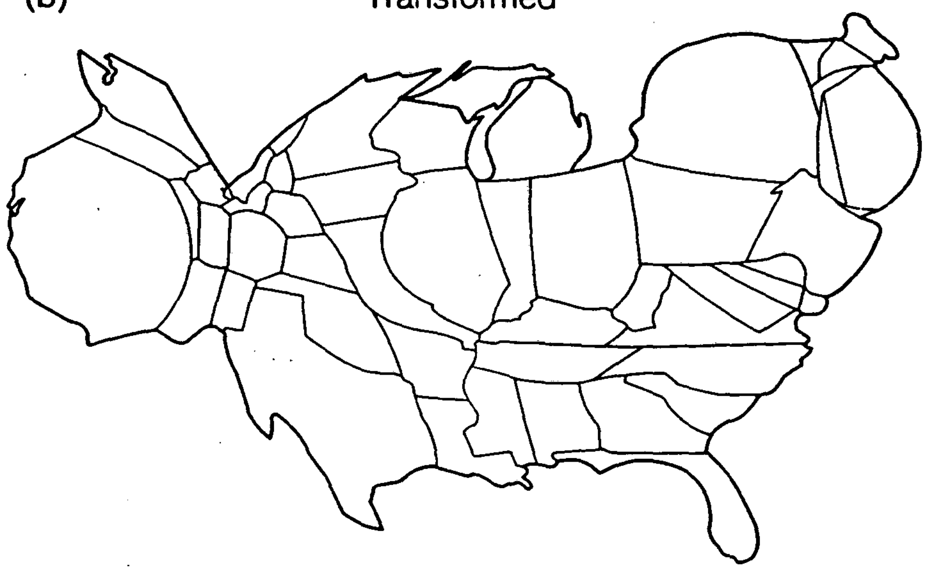
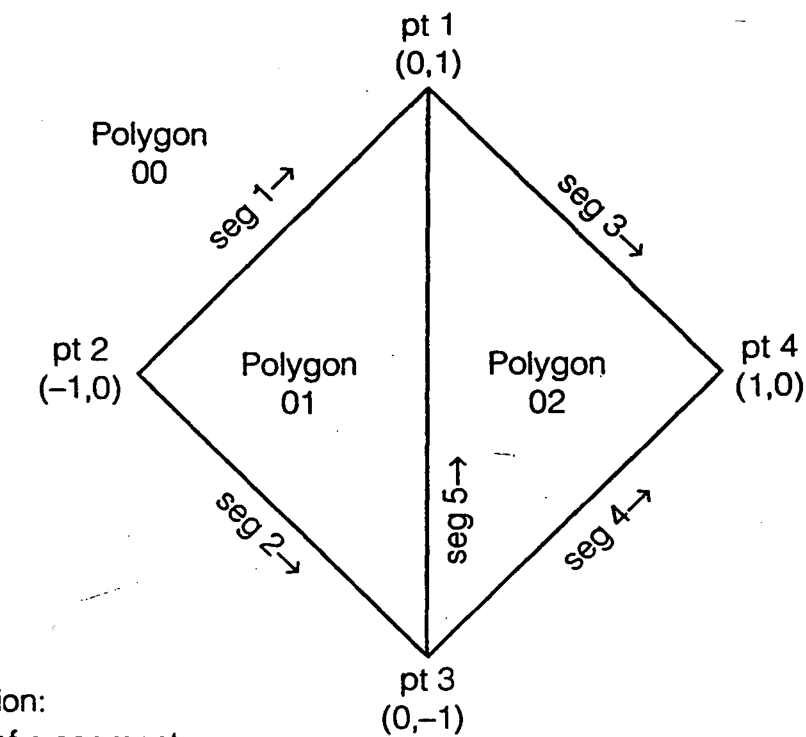


Figure 5. Polygon area calculations



Polygon area calculation:
If polygon is on (L,R) of a segment,
then (add, subtract) $1/2\,(x_{from}\,y_{to} - x_{to}\,y_{from})$.
Sum over all segments, in any order.

Segment	Polygon 01 on:	From point	To point	area contribution
1	R	2	1	$-1/2\,(x_2y_1 - x_1y_2)$
2	L	2	3	$+1/2\,(x_2y_3 - x_3y_2)$
3	no	1	4	0
4	no	3	4	0
5	L	3	1	$+1/2\,(x_3y_1 - x_1y_3)$

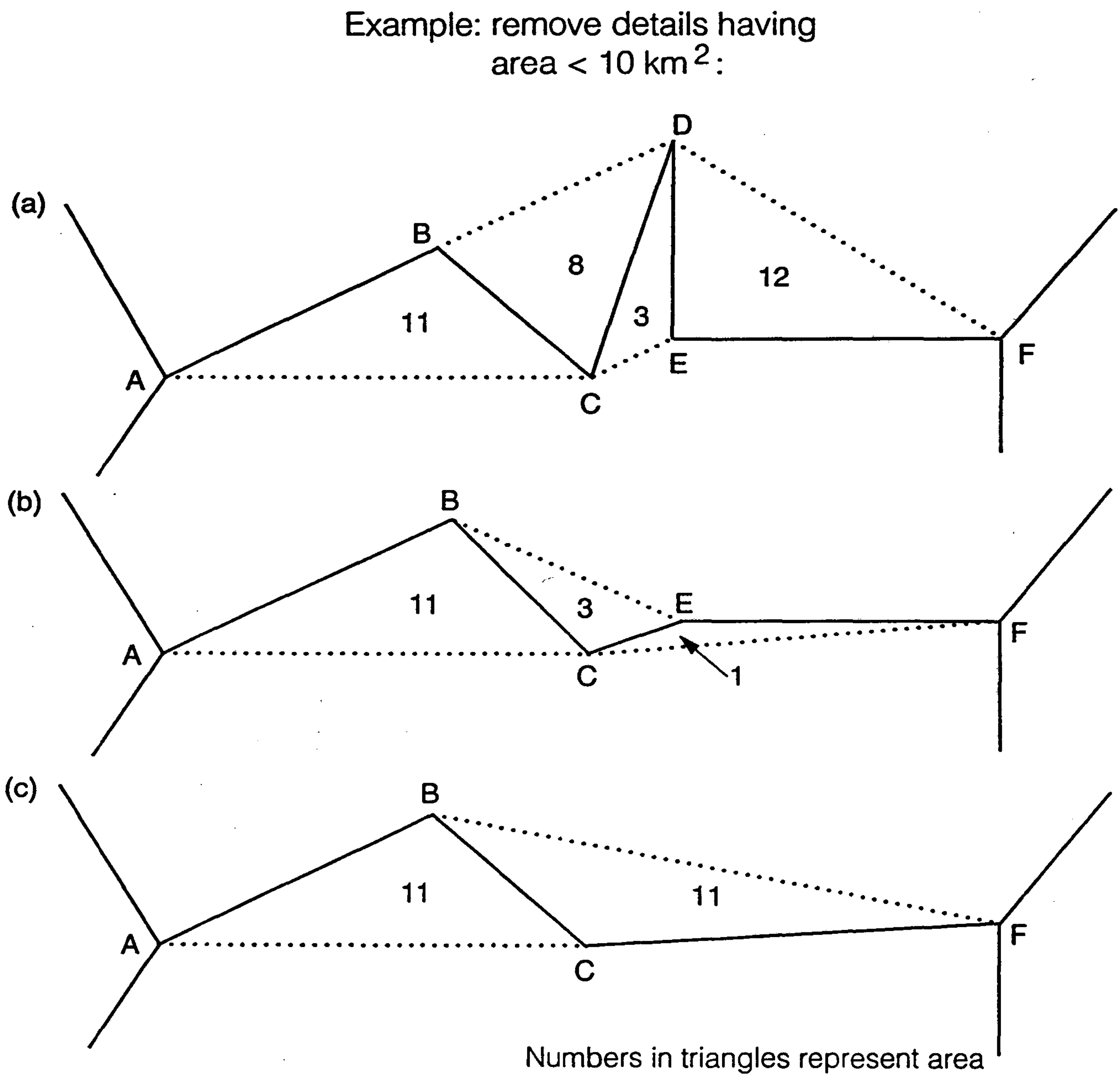
$$A_{01} = 1/2\,[\,x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\,] = +1$$

$$A_{02} = 1/2\,[\,x_1y_3 - x_3y_1 + x_3y_4 - x_4y_3 + x_4y_1 - x_1y_4\,] = +1$$

$$A_{00} = 1/2\,[\,x_2y_1 - x_1y_2 + x_1y_4 - x_4y_1 + x_4y_3 - x_3y_4 + x_3y_2 - x_2y_3\,] = -2$$

XBL916-7001

Figure 6. Point removal



XBL916-7005

Figure 7. San Francisco, 1980 census tracts

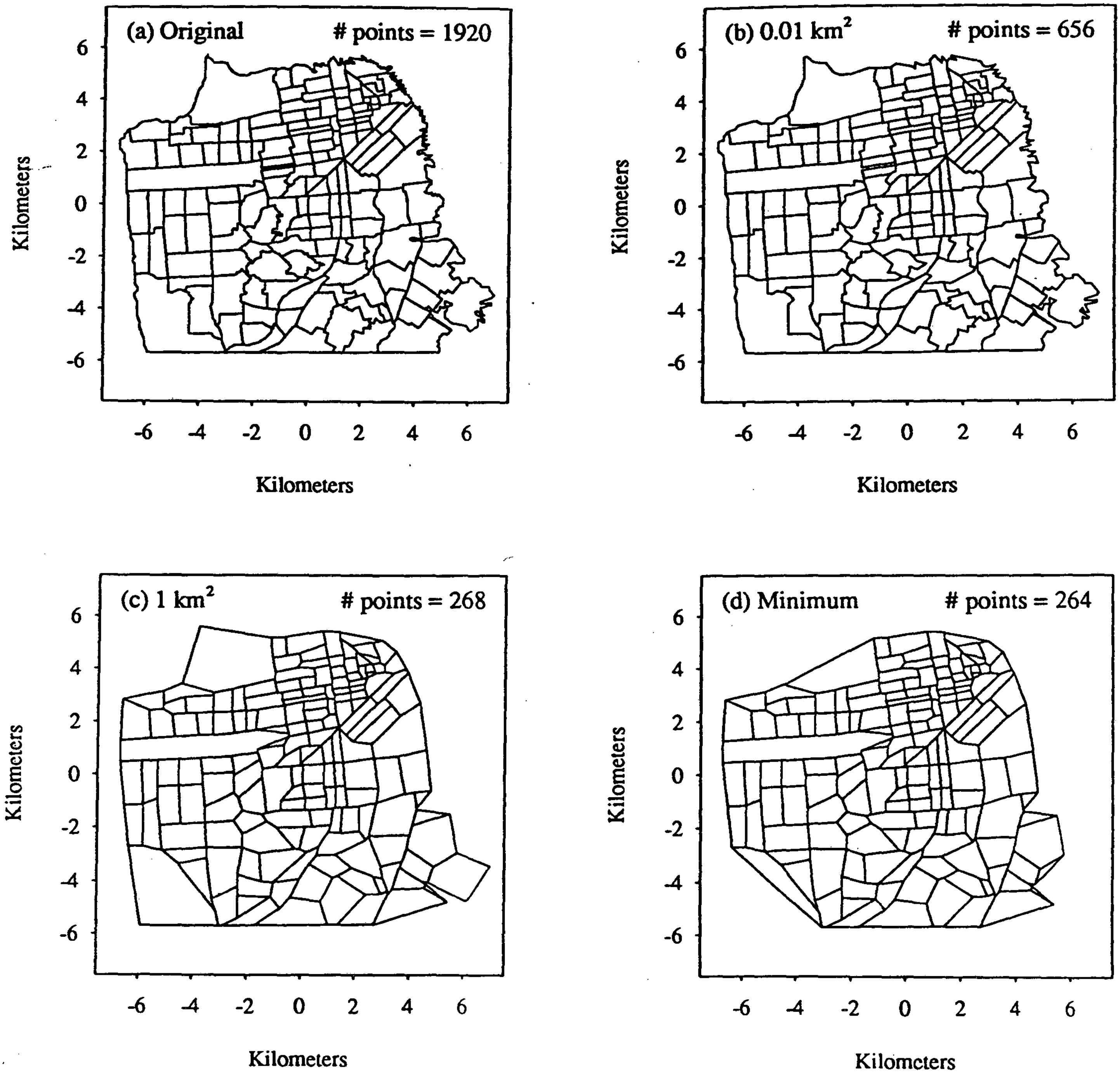
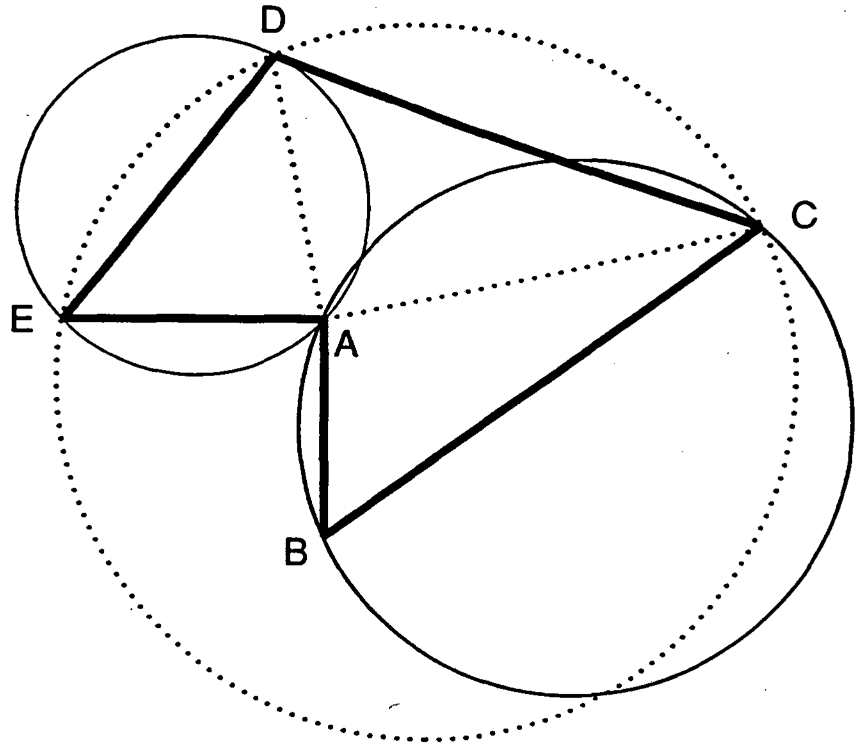


Figure 8. Triangulation of polygons



Circle ABC contains no other points,
so triangle ABC is valid.

Circle ADE contains no other points,
so triangle ADE (and hence also ACD)
is valid.

Circle CDE contains points A and B,
so triangle CDE is not valid (nor is ACE).

XBL916-7002

Figure 9. San Francisco, 1980 census tracts, 1 km^2, triangulated

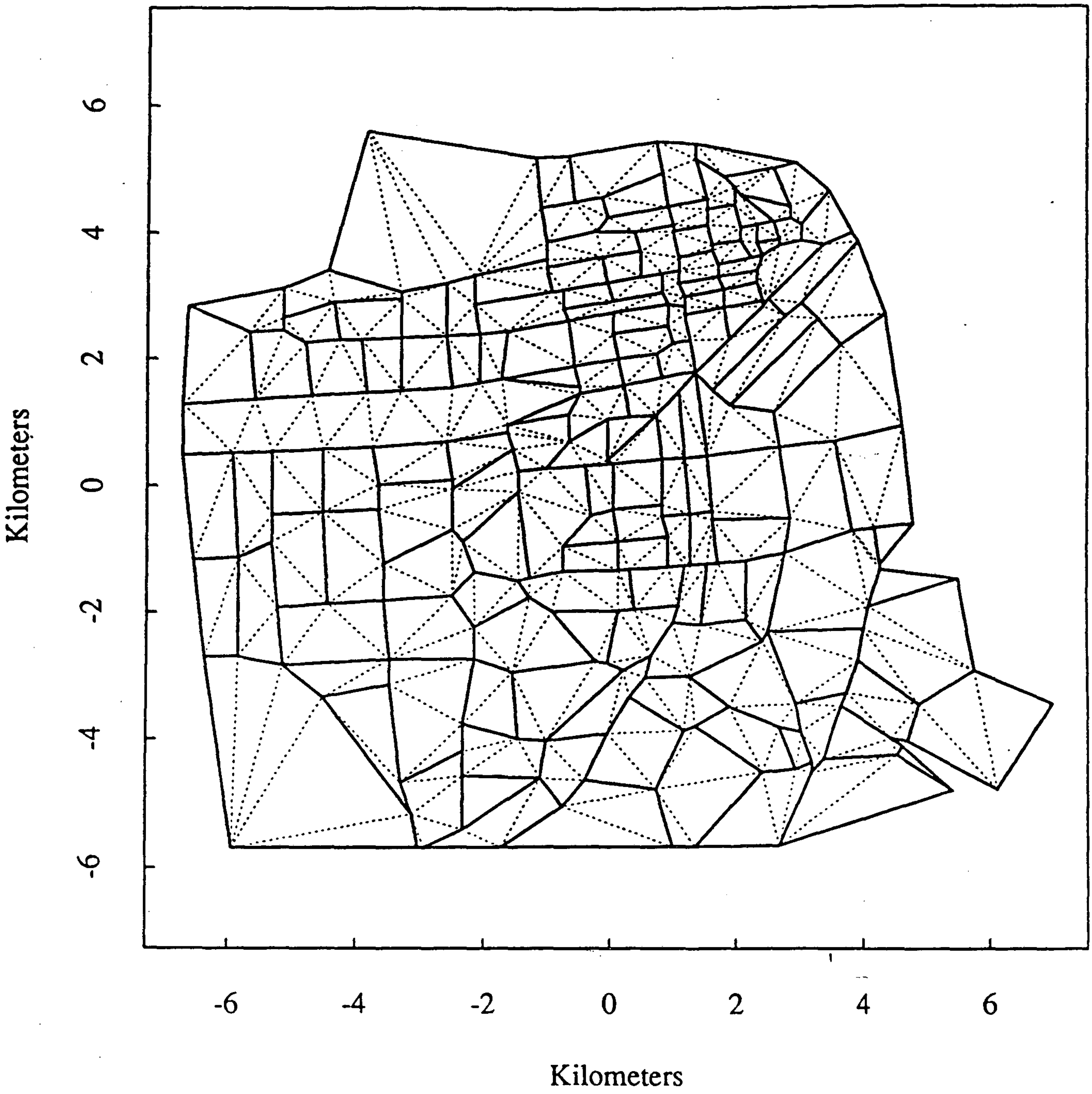
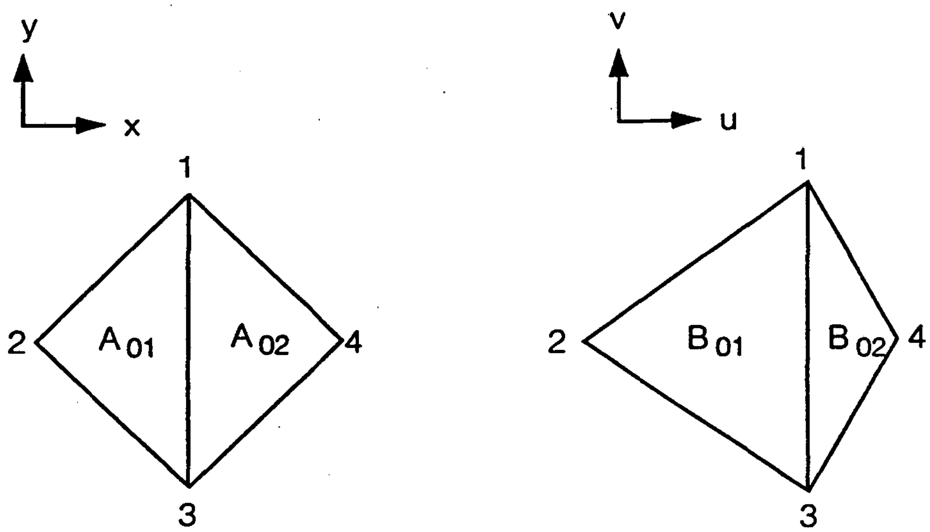


Figure 10. Piecewise linear transformations



a, b, c, d, e, f determine u, v:

$$\begin{pmatrix} a_{01} & b_{01} & e_{01} \\ c_{01} & d_{01} & f_{01} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}_{01} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}_{01} \quad \text{within triangle 01}$$

$$\begin{pmatrix} a_{02} & b_{02} & e_{02} \\ c_{02} & d_{02} & f_{02} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}_{02} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}_{02} \quad \text{within triangle 02}$$

Conversely, u and v determine a, b, c, d, e, f:

$$\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix}_{01} = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix}_{02} = \begin{pmatrix} u_1 & u_3 & u_4 \\ v_1 & v_3 & v_4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$$

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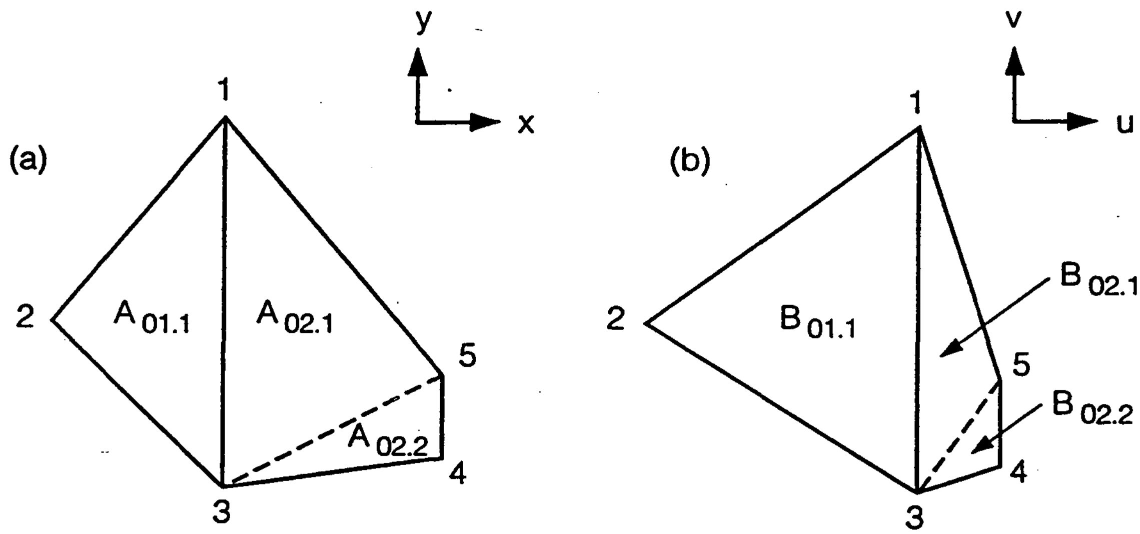
Figure 11. Area constraints



Given:

$(x_1, y_1) \ldots . (x_5, y_5), pop_{01}, pop_{02}$

$pop_{tot} = pop_{01} + pop_{02}$

$A_{tot} = A_{01} + A_{02} = A_{01.1} + A_{02.1} + A_{02.2}$

$B_{tot} = B_{01} + B_{02} = B_{01.1} + B_{02.1} + B_{02.2}$

Require:

$B_{tot} = A_{tot}$

$B_{01} / pop_{01} = B_{02} / pop_{02}$

$B_{02.1} / A_{02.1} = B_{02.2} / A_{02.2}$

Solution:

$$B_{01.1} = pop_{01} \cdot \frac{A_{tot}}{pop_{tot}} \cdot \frac{A_{01.1}}{A_{01}}$$

$$B_{02.1} = pop_{02} \cdot \frac{A_{tot}}{pop_{tot}} \cdot \frac{A_{02.1}}{A_{02}}$$

$$B_{02.2} = pop_{02} \cdot \frac{A_{tot}}{pop_{tot}} \cdot \frac{A_{02.2}}{A_{02}}$$

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Figure 12. Conformal and non-conformal linear transformations

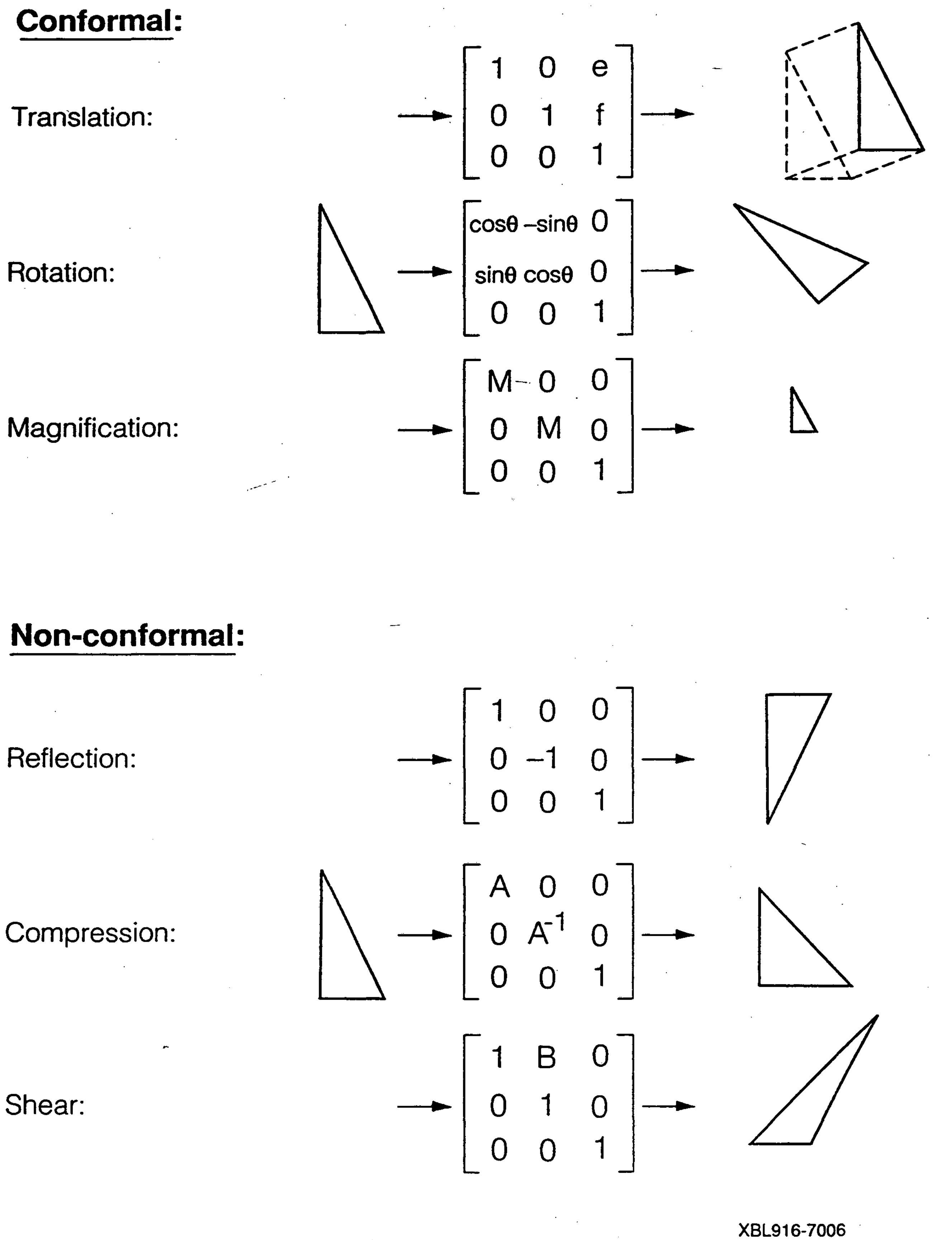

Figure 13. Vermont counties, 500 km^2

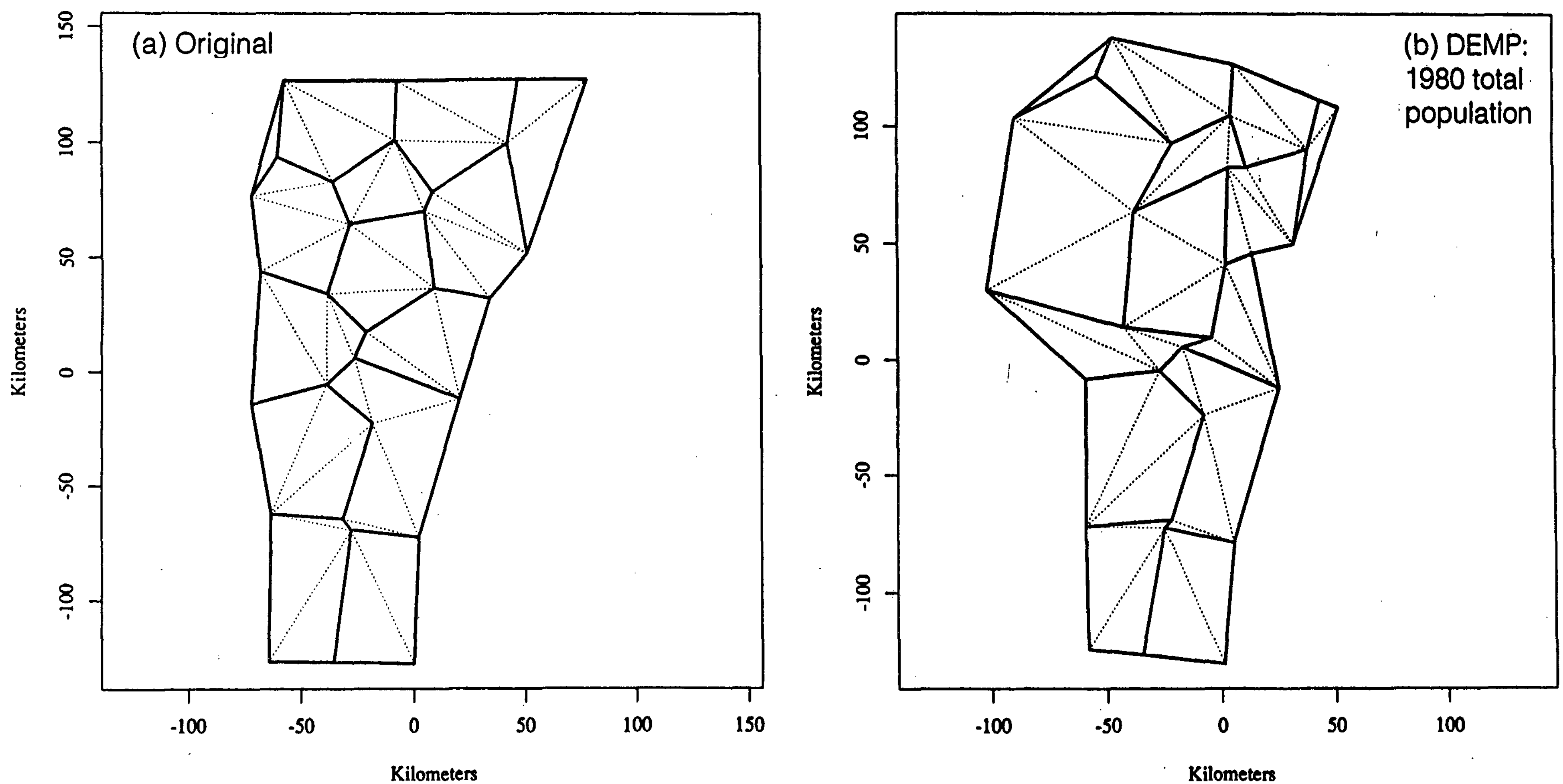

Figure 14. Vermont, 500 km^2, 1980 total population

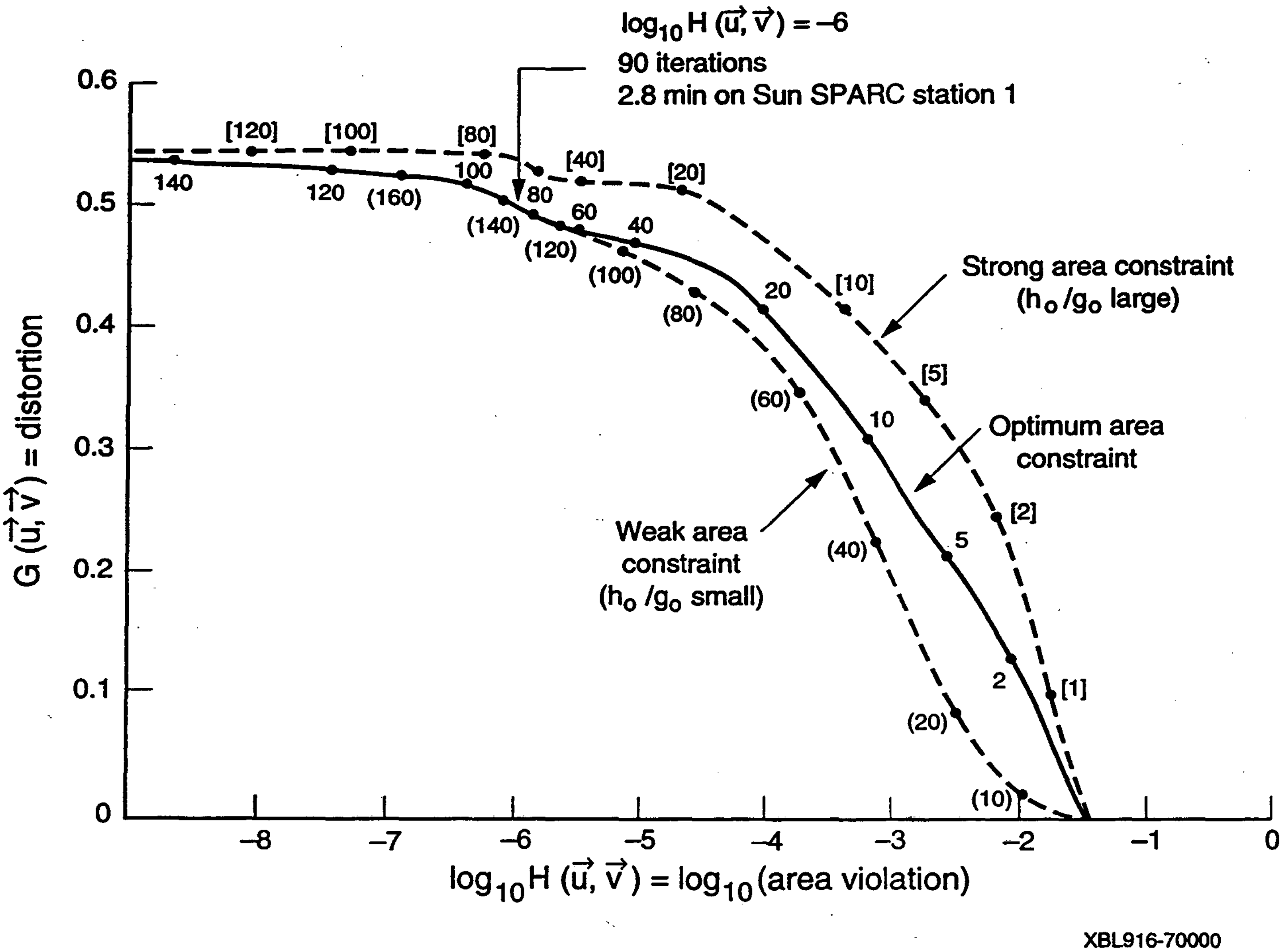

Figure 15. Vermont, 500 km², 1980 Native American population

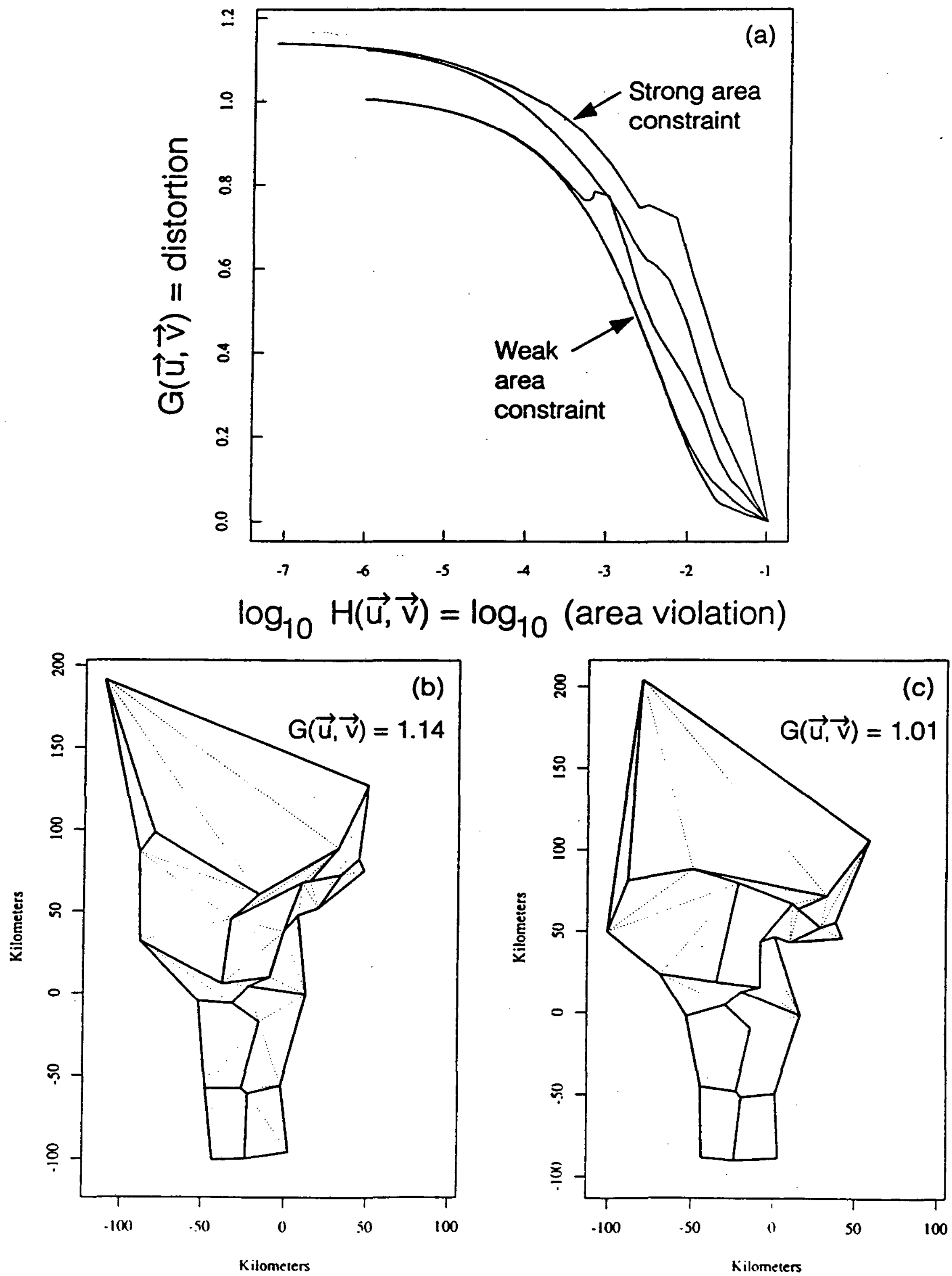

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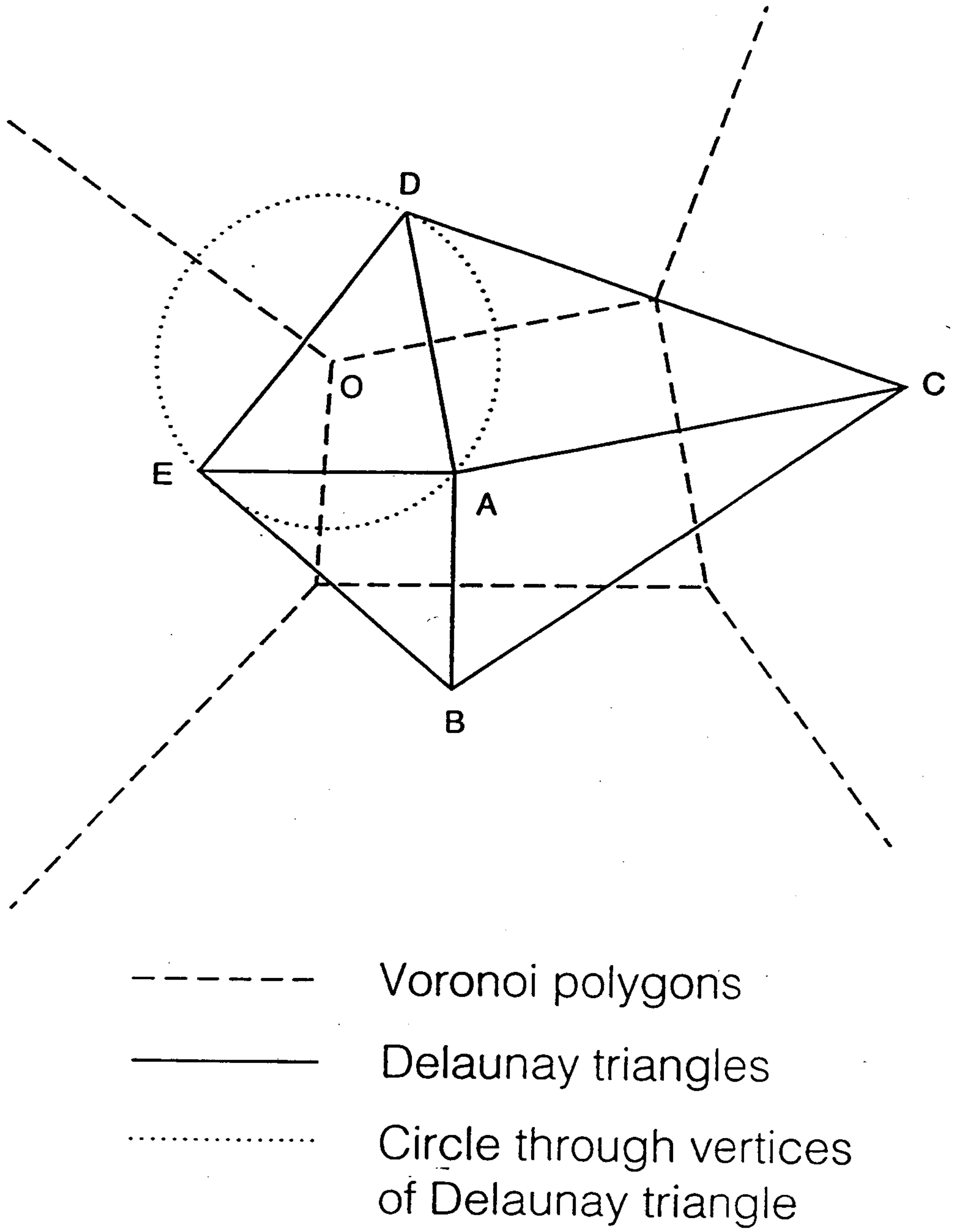
Figure A-1. Voronoi polygons and Delaunay triangles



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