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# BASIC TIMING GAMES

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Abstract

We propose a new approach to modelling finite-move, closed-loop games in continuous time. Continuous time is modeled as "discrete-time, but with a grid that is arbitrarily fine." We construct a "calculus for continuous-time games" that enables us to solve a certain class of timing games with a minimum of computation. Our machinery yields sharp and intuitive results for a range of interesting economic problems. These problems include games that might be hard to solve in discrete time without a great deal of computation. The paper concludes with an example of a continuous-time game with a subgame perfect equilibrium that is far away from any approximate equilibrium of any nearby discrete-time game.

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This is the first of several papers in which we propose a new approach to modelling finite-move, closed-loop games in continuous time.[1] In the present paper, we focus on "basic timing games" (BTG)'s: in a BTG, each agent chooses a time to move, selecting from a finite set of alternative termination actions. Once one player has moved, the game is over. In the subsequent papers, we enrich the model by allowing finitely many moves in succession.[2]

A continuous-time formulation is desirable on several grounds. The most obvious is that usually, agents *in reality* can act virtually whenever they wish: we should therefore be concerned about the predictions of a model in which agents are confined to move once a day, say at 8.00 a.m., if in the actual situation being modelled, a great deal may be gained by moving at 7.59 a.m. The second argument in favor of continuous time is convenience. Economists usually model quantities and prices as continuous variables, because these are easier to work with than discrete ones. In particular, differential calculus techniques can be used to make marginal calculations. These reasons appear equally valid when time is the economic variable being analyzed.

There is a third argument in favor of continuous-time that is related to the second, but less well understood. It turns out that certain classes of games are much easier to solve in continuous time than in discrete time. The main goal of this paper is to make precise the sense in which this is true. Specifically, the paper develops a "calculus" for continuous-time games.[3] For a certain class of games, our methodology is surprisingly powerful. This class includes many kinds of preemption games that are of interest to economists. Solutions to these games can be obtained with a minimum of computation. By contrast, the corresponding discrete-time solutions might be extremely tedious to calculate. Moreover, in contrast to the technology for solving discrete-time games, our solution procedure invokes general formulae rather than computations that are specific to the problem being analyzed. In many cases,

[1] In a closed-loop game, agents can revise their decisions at every point in time; in an open-loop game, by contrast, decisions are made in advance and cannot be revoked.

[2] If agents can move at arbitrary points in time, then, without further restriction, they can also move arbitrarily frequently. As we discuss in [16] (henceforth SS1), this possibility raises severe technical problems. As a first step, we restrict attention to finite move games and thereby sidestep these problems completely.

[3] A calculus is a "particular method of calculation or reasoning." (Concise Oxford Dictionary).

these formulae are very simple to apply.

We conclude this introduction with some qualified caveats. First, while our "calculus" is straightforward to work with, its conceptual foundations are by no means simple. Nonetheless, the machinery can be used without a complete understanding of its underpinnings. Second, our model has little new to say about many kinds of timing problems--including, for example, games like the war of attrition. On the other hand, our machinery does yield sharp and intuitive results for a wide range of interesting economic problems.

The remainder of the paper is organized as follows. Section 0 argues for a closed- rather than an open-loop formulation of continuous-time games. Section I introduces our notion of a continuous-time game form. In section II, we illustrate the model with some classical examples and introduce the ideas underlying our "calculus." The formal model is presented in section III. Sections IV and V develop the "calculus" introduced above. Section VI contains an example, showing that in a certain sense, our model of continuous time is fundamentally different from the conventional discrete-time model. Proofs are gathered together in the Appendix.

## 0. Closed- vs. Open-loop Strategies in Continuous Time.

For the purposes of this heuristic section, we will focus on simplest kind of Basic Timing Game. The game is played on the unit interval. There are two agents. Each has two strategies, "continue" ($Ct$) and "terminate" ($Tm$). In the continuous-time version, agents can move at any point in time; in discrete-time, they can move at only a finite number of grid points. The game ends either when some player plays "$Tm$" for the first time or when time runs out. Agents choose closed-loop strategies, i.e., they can revise their decisions at any stage of the game. We will refer to a game in this class as a $2 \times 2$ BTG. When a game of this kind is played on a discrete-time grid, $R$--i.e., a finite subset of the unit interval--a behavior strategy is a function, $\xi_i^R$, that assigns a probability weight to each point in $R$. $\xi_i^R$ has the following interpretation: "for each $r \in R$, if the game has not already ended by $r$, I will play "$Tm$" at this time with probability $\xi_i^R(r)(Tm)$."

There is no obvious way to specify the continuous-time analog of the closed-loop game-form just described. The conceptual problems are discussed in SS1 (section III). Most studies of timing games finesse these difficulties by compromising in one of two ways.[4] (A notable exception is Fudenberg-Tirole [5].) Some consider closed-loop, discrete-time games, and study the limit of subgame perfect equilibria as the grid-length shrinks to zero (e.g. Dixit [2], Katz-Shapiro [11]). Others (e.g., Pitchik [12], Reinganum [13, 14], Hendriks-Wilson [9, 10]) analyze open-loop continuous-time games. In these games, strategies are declared in advance, and agents have no subsequent opportunity to revise their decisions. Formally, a open-loop timing game is simply a one-shot game with a continuum of strategies: it has no proper subgames. A pure strategy for such a game is a point in the unit interval, representing a time to move. A mixed strategy is a probability distribution over this interval. Traditionally, mixed strategies have been represented as cumulative distribution functions (c.d.f.'s).[5]

The open-loop formulation is sometimes, but not always, an adequate proxy for the more complicated closed-loop model. To illustrate the relationship between the two kinds of game, we consider three examples. The first makes the familiar point that the equilibria of open-loop games may involve incredible threats. The second and third illustrate a more subtle, and independent issue: there are mixed-strategy equilibria of closed-loop discrete-time games that have no analogs in the continuous-time, open-loop model. Indeed, the open-loop version of our third example has no equilibrium at all. In each example, we will use the following notation for payoffs. For $t \in [0, 1]$, let $L_i(t)$ denote the payoff to $i$, and $F_j(t)$ the payoff to $j$, if "$Tm$" is played for the first time by $i$ alone at $t$. If both move simultaneously for the first time at $t$, denote the payoff to $i$ by $S_i(t)$. In each example, if neither player moves, each earns a payoff of zero.

In our first example, there is a unique subgame perfect equilibrium outcome for the (closed-loop) BTG. The corresponding open-loop game has many additional Nash outcomes, all supported by incredible threats by player #2. Let $L_1(t) = (\frac{1}{4} - t)^2$, and $F_1(t) = S_1(t) = 0$. Let $F_2(t) = 2 - t$ and

[4] See Gilbert-Harris [7] for still another approach.

[5] A c.d.f. is a right continuous, non-decreasing function with range [0, 1]. A strategy $G_i$ is interpreted as follows: $G_i(t)$ is the probability that $i$ has chosen a termination time less than or equal to $t$.

$L_2(t) = S_2(t) = -1$. Clearly, the unique SGP equilibrium for the basic timing game is that player #1 terminates at time $t = 1$. Now consider the following strategy profile for the open-loop game: player #1 moves at $t_1 = 0$, and #2 moves at $0 < t_2 < \frac{1}{2}$. This is a Nash equilibrium, since against #1's strategy, #2 is indifferent between any two positive termination times. #2's proposed action is, however, not credible (unless he can precommit to $t_2$): if $t_2$ were ever reached, #2 would strictly prefer to continue at this time rather than terminate.

Our second example is the familiar "war of attrition"; the third is a stylized model of preemption, known as "grab the dollar."[6,7] In the war of attrition, each agent prefers that the other moves first. In "grab the dollar," each prefers to move first himself. In either game, the leader prefers to lead sooner rather than later. Payoffs for the symmetric war of attrition are: $L_i(t) = 1 - t$ and $F_i(t) = S_i(t) = -t$. If neither player ever moves, each receives -1. (Think of a declining industry: if two firms remain active, each makes losses; if one firm leaves, the other can make a profit.) Payoffs for the symmetric "grab the dollar" are: $L_i(t) = 1 - t$, $F_i(t) = 0$ and $S_i(t) = t - 1$. If neither player ever moves, each receives zero. (Think of entry into an industry that can profitably support only one firm.) We will play each of these games on a sequence of increasingly fine, discrete-time grids, $(R^n)$, where $R^n = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}\}$. Payoffs for the game played on $R^n$ are derived by restricting the above payoffs to $R^n$.

The symmetric SGP equilibrium for the closed-loop "war of attrition" on $R^n$ is: if the game has not yet ended by $r = \frac{k}{n}$, player $i$ terminates at $r$ with probability $\frac{1}{n-k+1}$. The resulting outcome is that the termination time of the game is distributed uniformly over $R^n$. As $n$ gets large, the probability that both agents move simultaneously shrinks to zero. There are, in addition, two pure strategy equilibria: player $i$ terminates with probability one at time $t = 0$. Each of these equilibria has a counterpart in the open-loop continuous-time game. The limit of the symmetric outcomes is implemented by the following open-loop strategies: player $i$'s termination time is distributed uniformly over $[0, 1]$. Thus, for games

[6] The discussion below was motivated by, and overlaps substantially with, section 4.A of Fudenberg-Tirole [5]. Our perspective on the issues, however, differs significantly from theirs. We believe these issues are sufficiently important and subtle to warrant the inevitable overlap.

[7] This game was first proposed by Richard Gilbert and is discussed in Fudenberg-Tirole [5].

like the war of attrition, the open-loop game is an acceptable proxy for the much more complex BTG.

Not surprisingly, this fortuitous relationship between a BTG and its open-loop analog does not hold in general. The "grab the dollar" game illustrates how it can break down. When "grab the dollar" is played on the grid $R^n$, the symmetric equilibrium is: if the game has not yet ended by $r = k/n$, player $i$ terminates at $r$ with probability ½. These strategies generate the following outcome: at the $k$'th grid point in $R^n$, player $i$ alone terminates with probability $(1/4)^k$; with the same probability, both players terminate simultaneously at this grid point. The limit of these outcomes has the game ending with probability one at time zero. Each of the three possible termination states is equally likely. This outcome is not a product distribution and, therefore, cannot be implemented as the noncooperative equilibrium of a game in which strategies are cumulative distribution functions (unless, of course, correlation is permitted).

The proposition below shows that the problem just described is pervasive. Consider a symmetric, $2 \times 2$ Basic Timing Game satisfying the following conditions: there exists $\bar{t}$ such that (i) the payoff to a player who terminates unilaterally at $\bar{t}$ strictly exceeds any payoff he could obtain if the game continued beyond $\bar{t}$; (ii) if a player terminates unilaterally at $\bar{t}$, his payoff strictly exceeds the payoff to the remaining player, who, in turn, strictly prefers continuing at this time to moving simultaneously with the first player. Both conditions are typically satisfied in, say, preemption games. In any discrete-time, symmetric equilibrium of such a game, players must be randomizing with probability bounded away from zero on an interval of time, regardless of the period length. The limit of the outcomes generated by these equilibria will not be a product measure, and so cannot be implemented as the outcome of a open-loop game.

**Prop'n I**: Consider a symmetric $2 \times 2$ BTG with continuous payoffs. Assume that for some $\bar{t} \in [0, 1]$, the following conditions are satisfied:

- $L_i(\bar{t}) > L_i(s) \vee F_i(s) \vee S_i(s)$, for all $s > \bar{t}$;[8]
- $L_i(\bar{t}) > F_i(\bar{t}) > S_i(\bar{t})$.

Let $(\xi_1^n, \xi_2^n)$ be a symmetric SGP equilibrium when this game is played on $(R^n)$. There exists $\bar{n} \in \mathbf{N}$, $\epsilon > 0$ and $\underline{t} < \bar{t}$ such that for all $n > \bar{n}$, $\xi_i^n(r)(Tm) > \epsilon$, for all $r \in R^n \cap (\underline{t}, \bar{t})$.

[8] For $x, y \in \mathbf{R}$, $x \vee y$ is the maximum of $x$ and $y$. Similarly, $x \wedge y$ is the minimum of $x$ and $y$.

The above discussion indicates the need for a continuous-time closed-loop framework for analyzing timing games. The only attempt to provide one has been Fudenberg-Tirole [5]. They propose a model of symmetric $2 \times 2$ BTG's. Each agent announces a family of "simple strategies." A simple strategy is a pair of real-valued functions. A "closed-loop" equilibrium is a family of Nash equilibria in simple strategies, one for every moment in time. The payoff function is designed so that the "sensible" SGP equilibria of discrete-time games have closed-loop counterparts in their model.

The model presented in this paper was motivated by, but differs substantially from Fudenberg-Tirole's. Ours is simpler to work with and can readily be generalized. Moreover, it is more closely related than theirs to the conventional discrete-time framework. In particular, strategies in our model are mapped to outcomes rather than directly to payoffs. The payoff function can then be defined by extending the discrete-time payoff functions in the natural way. The model is introduced in the following section.

## I. Introduction to the Model.

We take as our starting point the idea that continuous-time should be thought of as "discrete-time, but with a grid that is *arbitrarily* fine," that is, finer than any *given* grid. To formalize this idea, we specify a restricted class of *continuous-time-behavior-strategies*. These strategies are interpreted as "master plans" that instruct agents how to play the game on every conceivable discrete-time grid. Specifically, the restriction of a master plan to a finite grid is a well-defined discrete-time behavior strategy. Thus any profile of master plans generates a well-defined discrete-time outcome on every discrete-time grid. We will define an *outcome* for the continuous-time game to be the limit of the outcomes generated by restricting a profile of strategies to an increasingly fine sequence of grids. For this construction to be coherent, we must restrict agents to choose from a family of strategies with the following property: whenever an arbitrary profile of strategies is drawn from this family, there exists a unique "limit" outcome such that when the profile is restricted to an *arbitrary* sequence of grids, the resulting sequence of outcomes converges to the identified outcome. Once we have identified this family of strategies, we have a well-defined game-form. We can now assign payoffs to outcomes in the customary way and study the subgame perfect equilibria of the resulting game. When specified in this way, a continuous-time game is, in a literal sense, the limit of corresponding discrete-time games.

In the general version of our model (see [17], henceforth SS2), the discrete-time instructions encoded in master plans take into account both time and the fineness of the discrete-time grid. Specifically, a master plan assigns to each $t$ a probability distribution over actions, together with a vector of "derivatives." For example, in a $2 \times 2$ game, a master-plan $\zeta_i = (\zeta_i^{(0)}, \zeta_i^{(1)})$ has the following interpretation: "if I am playing on a discrete-time grid and reach $t$, and the distance to the next grid-point is $dt$, then I will terminate at this time with probability $\begin{cases} \zeta_i^{(0)}(t)(Tm) & \text{if } \zeta_i^{(0)}(t)(Tm) > 0 \\ \min\left(1, \zeta_i^{(1)}(t)(Tm)dt\right) & \text{otherwise} \end{cases}$.[9] For example, a symmetric equilibrium for the continuous time war of attrition is defined as follows: for $t \in [0, 1]$, $\zeta_i(t)(Tm) = (0, \frac{1}{1-t})$. To see that these strategies form an equilibrium, consider the sequence of grids $(R^n)$, defined above (p. 3). At $\frac{k}{n} \in R^n$, $i$ is instructed to terminate with probability $(1 - \frac{k}{n})^{-1}dt = \frac{1}{n-k}$. Thus, when $n$ is large, the restriction of $\zeta_i$ to $R^n$ virtually coincides with the discrete-time equilibrium strategy specified on p. 3! Obviously, therefore, the outcome generated by $(\zeta_1, \zeta_2)$ will be limit of the discrete-time symmetric equilibrium outcomes.

In this paper, we restrict attention to a much simpler class of master plans, which we call *grid-independent* (GI) strategies. A GI strategy depends only on time: the intensity with which an agent randomizes is required to be independent of the fineness of the grid on which he is playing. Moreover, we require that strategies be piecewise continuous functions of time. These two restrictions together exclude strategies such as $\zeta_i$ above. On the other hand, the following strategy is admissible: $\xi_i(t)(Tm) = \frac{1}{2}$, for all $t$. It is straightforward to verify that $\xi = (\xi_1, \xi_2)$, is an an equilibrium profile for for the continuous time version of "grab the dollar."

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9 $\zeta_i^{(1)}(t)(Tm)$ is interpreted as a derivative--actually a hazard rate--and may therefore be any nonnegative number. We maximize $\zeta_i^{(1)}(t)(Tm)dt$ with unity, to ensure that agents announce a *probability* of moving at every point in time. The problem does not arise with $\zeta_i^{(0)}(t)(Tm)$, which is assumed to be a probability from the start.

Our grid-independence restriction may be challenged on several grounds. It is clearly severe and ad hoc. Moreover, there are games for which no equilibrium in GI strategies exists. Finally, there are familiar solutions to familiar games--for example, the symmetric outcome in the war of attrition--that cannot be implemented by GI strategies. On the other hand, there are compelling pragmatic reasons for imposing the restriction. First and most obvious, it is a natural first step in a larger research program. The only consequence of excluding grid-dependent strategies is that the set of equilibria is kept artificially small: since we are not excluding any *pure* strategies, any equilibrium in the smaller strategy space will still be an equilibrium when the strategy space is expanded. Second, GI strategies are particularly easy to work with and yield sharp and intuitive results. The third reason is more delicate. When the strategy space is expanded to include grid-*dependent* strategies, the equilibrium set may explode. We illustrate this in the following section, by constructing a game that has a unique GI equilibrium, together with a continuum of equilibria in grid-dependent strategies. Moreover, these additional equilibria are often nonintuitive and artificial. It appears, therefore, that in certain contexts, the price of obtaining sharp and intuitive results is that the strategy space must be restricted in some way.

## II. Examples.

In this section we illustrate our model with two examples. Each has two players. The first is a classical game known as the noisy duel.[10] The second is a dynamic version of Cournot's quantity-setting game. In each case, our BTG is a reduced-form representation of a more primitive game, in which each agent can move once. Once one player has moved, the other faces a single-person decision problem. We reduce the primitive game to a BTG in the usual way, by replacing nodes in which one player has not yet moved with the payoffs that arise when the second plays his optimal response.

In the noisy duel, two protagonists start moving toward each other at time zero. If $i$ shoots at time $t \in [0, 1]$, he hits $j$ with probability $t$. If he misses, $j$ hears the bullet, waits until time $t = 1$,

[10] We are grateful to Carolyn Pitchik for introducing us to this example. A noisy duel is so named because each player can hear his opponent's bullet at the instant it is fired. In a silent duel, by contrast, the shots cannot be heard, so that at any given time, $i$ cannot determine whether $j$ has fired and missed or is has been holding his fire. Clearly, the two games have very different properties. For other discussions of this game, see Pitchik [12], Hendriks-Wilson [10] and Dixit [2]. Pitchik's paper contains a useful list of references.

shoots at point blank range and scores a hit with certainty. This game can be modeled in reduced form as a BTG. We first consider a symmetric version: each player assigns a value 1 to a hit and -1 to being hit. If both players hit each other simultaneously, each player earns -1. If neither player scores a hit, both earn zero. Using the notation of p. 3, we have $L_i(t) = t + (-1)(1 - t) = 2t - 1$, $F_i(t) = 1 - 2t$ and $S_i(t) = -t^2$.

The equilibria of this game have two obvious properties. First, neither player will shoot before $t = ½$. Second, at every $t > ½$, the probability that the game is already over must be positive. A less obvious fact is that the open-loop version of the game has no Nash equilibrium.[11]

When modeled as a continuous-time BTG, the game has three equilibrium outcomes. In each case, each agent's expected payoff is zero. For each $i$, there is an equilibrium in which $i$ alone terminates at $t = ½$ with probability one. The third equilibrium is symmetric: each player terminates alone with probability ½. As in grab-the-dollar, this last outcome is not a product distribution over terminal states. In all three equilibria, each player's expected payoff is zero. A strategy profile that implements the symmetric outcome is $\xi^0 = (\xi_1^0, \xi_2^0)$, defined as follows:

$$\xi_i^0(t)(Tm) = \begin{cases} 0 & \text{if } t \le ½ \\ \dfrac{L_i(t) - F_i(t)}{L_i(t) - S_i(t)} = \dfrac{2(2t-1)}{t^2 + 2t - 1} & \text{if } t > ½ \end{cases}.$$

To implement the outcome in which player $i$ terminates with probability one, define $\xi^i$ identically to $\xi^0$, except that $\xi_i^i(½)(Tm) = 1$.[12]

We now verify that $\xi^0$ indeed implements the symmetric outcome specified above. For each $\epsilon > 0$, if $\xi^0$ is restricted to a sufficiently fine grid, the probability that some agent will have moved before $½ + \epsilon$ will exceed $1 - \epsilon$. In the limit, therefore, the game must end with probability one at $t = ½$.

[11] This fact is noted by Pitchik [12]. To see that no equilibrium exists, suppose $(G_1, G_2)$ is an equilibrium in c.d.f.'s. From the preceding observations, $G_i(\cdot)$ must be zero on $[0, ½)$ and positive on $(½, 1]$. Moreover, obviously, $G_1(½)G_2(½) < 1$. Finally, by a standard argument, the $G_i$'s must be continuous on $(0, 1]$. Now assume that the support of $G_1$ is $[½, 1]$, while $G_2(½) < 1$. It follows that # 1's expected payoff is zero. But this cannot be an equilibrium because if $i$ were to move at $½ + \epsilon$, his payoff would be $2\epsilon(1 - G_2(½ + \epsilon)) - \epsilon^2$, which will be strictly positive for $\epsilon$ sufficiently close to zero.

[12] Because $\xi^i$ is discontinuous w.r.t. time, the interpretation of this strategy as a master plan is a subtle matter. If we simply restricted $\xi^i$ to an arbitrary grid, the outcome would depend critically on whether or not the grid contained the discontinuity point $t = ½$. Accordingly, we "adapt" the profile to each grid, ensuring that if the grid is sufficiently fine, each of the profile's discontinuities will be "captured" by the grid. Specifically, we define an operator called a "graph preserving restriction" (g.p.r.), which "shifts to the right" the discontinuities of a profile in the appropriate way. For example, the g.p.r. of $\xi^i$ to the grid $R$ has player $i$ terminating with probability one at the first grid point in $R$ greater than or equal to ½.

Also, if $\epsilon$ is sufficiently small, the probability that both agents move simultaneously will be arbitrarily small *relative to* the probability that only one agent moves. In the limit, therefore, the probability of a simultaneous move is zero.

In section V, we apply our theorems to prove rigorously that $\xi^0$ is an SGP equilibrium. The following observations outline the formal argument. We first verify that at every $t > ½$, player $i$ is indifferent between terminating at $t$ and continuing at and immediately after $t$. If $i$ terminates at $t$, his expected payoff is $\xi_j^0(t)(Tm)S_i(t) + (1 - \xi_j^0(t)(Tm))L_i(t) = F_i(t)$. Suppose now that he continues at and immediately after $t$ and consider the sequence of outcomes generated by restricting $\xi_j^0$ to an increasingly fine sequence of grids. Since $j$ is terminating with probability bounded away from zero, on each of an increasingly large number of grid-points, the limit outcome must have $j$ terminating with probability one exactly at $t$! Once again, therefore, $i$'s payoff is $F_i(t)$. To complete the argument, note that as $t \downarrow ½$, each player's expected payoff in the subgame beginning at $t$ converges to zero. Therefore, each player is indifferent between terminating and continuing at $t = ½$.

The example illustrates an important difference between continuous- and discrete-time games, which is central to the construction of our "calculus." The difference is that discrete-time games can only be solved by backward dynamic programming; to solve certain kinds of continuous-time games, on the other hand, only a minimal amount of backward programming is needed. To be concrete, consider a strategy profile for the noisy duel that has each agent terminating with positive, nonunitary probability at some $t > ½$. In discrete time, the next grid-point will be reached with positive probability. Therefore, to determine whether the profile is an equilibrium from $t$, one must know what payoffs agents will receive if they reach this next grid-point. For this reason, discrete-time games can be solved only by working backwards from the end of the game. In continuous time, by contrast, the outcomes generated by a profile from subgames after $t$ have no bearing on whether or not the profile solves the subgame at $t$.[13] This is true because, as we have seen, if one agent is terminating with positive probability at $t$, the other cannot unilaterally prevent the game from ending with probability one at $t$! In short, verifying

[13] This statement is precisely true iff each agent's strategy is continuous at $t$ and each is terminating with positive probability immediately after $t$,

that players are playing equilibrium strategies at $t$ is equivalent to verifying that their actions at $t$ form a Nash equilibrium for a kind of one-shot game--we will call it a surrogate game--derived from the parameters of the game at $t$.[14] Since each agent is terminating with positive probability in such an equilibrium, we will call it a plural termination Nash equilibrium (PTNE).

The preceding idea forms the cornerstone of our "calculus." To solve simple preemption games like the duel, we invoke the following formula:[15] let $\bar{t}$ denote the largest $t$ such that at most one agent wishes to preempt before $t$; before $\bar{t}$, have each agent play "continue;" at every $s > \bar{t}$, have them play PTNE strategies for the surrogate game at $s$; at $\bar{t}$, define the strategies in one of the three ways specified above (p. 9). An attractive feature of this formula is that it applies to any game in a class: it is not necessary to compute a specific solution for each specific problem.

We now consider an asymmetric version of the duel. Suppose that the payoffs are exactly as above, except that player #1 assigns a value +2 to a hit (provided he is not simultaneously hit himself). The reduced form payoffs are modified as follows: $L_1(t) = 3t - 1$ while all other payoffs are as above. This game has a unique equilibrium outcome: player $i$ terminates alone with probability 1 at $t = ½$. An equilibrium profile that implements this outcome is $(\hat{\xi}_1, \hat{\xi}_2)$, where $\hat{\xi}_1 = \xi_1^1$, defined above (p. 9), and

$$\hat{\xi}_2(t)(Tm) = \begin{cases} 0 & \text{if } t \leq ½ \\ \dfrac{L_1(t) - F_1(t)}{L_1(t) - S_1(t)} = \dfrac{5t - 2}{t^2 + 2t - 1} & \text{if } t > ½ \end{cases} .$$

Our second example is a reduced-form representation of the following dynamic quantity-setting game. As in Cournot's original problem [1], two firms produce mineral water at zero marginal cost. The market inverse demand function is given by $P(Q) = 1 - Q$, where $Q$ is the aggregate quantity produced. Each firm must specify a quantity at some time in the unit interval. (For technical reasons, we restrict agents to choose from a finite set of quantities, and consider the limit of the equilibria as this set

14 We add the qualifier "a kind of..." because the *outcome function* for this "one-shot game" differs from the conventional one: it simulates the outcome that would have resulted, had the strategies been played in continuous-time. Specifically, if players terminate at $t$ with probability $(\sigma_i, \sigma_j)$, the probability that $i$ terminates alone at $t$ will be $\frac{\sigma_i(1 - \sigma_j)}{1 - (1 - \sigma_i)(1 - \sigma_j)}$ rather than simply $\sigma_i(1 - \sigma_j)$.

15 Note that when applied to the preemption game studied in Fudenberg-Tirole [5], the answer given by our formula agrees with their solution.

grows large.) Once a quantity has been chosen, it cannot be changed. If a firm moves at $t$, a cost $c(t)$ is incurred. Tthis cost is independent of the quantity that the firm announces: we assume $c(0) = \infty$ and, for positive $t$, $c(t) = 1/t - 1$. At time $t = 1$, firms receive their one-shot Cournot payoffs minus their "movement costs."

If only one player moves at $t < 1$, the other will, obviously, wait until $t = 1$ and then respond optimally. As noted above, we can reduce this two-stage game to a BTG, by declaring that the game ends as soon as one player moves; if the other player continues at this time, assign to this ending the payoffs that result in the original game, when the second player responded optimally at $t = 1$.

Conceptually, this reduced-form game is a straightforward preemption game, very similar to the noisy duel. From a computational standpoint, however, it is an order of magnitude more complicated. Agents must now choose not only *when* to move but also *how* to move. In discrete time, the only way to find the symmetric solution to this game is, once again, to compute the appropriate difference equation, working backwards from the end of the game.[16] On the other hand, the problem is simple to solve in continuous-time, using our "calculus." With one difference, we follow *exactly* the formula laid out on p. 11 above. We identify $\bar{t}$ as above and invoke a theorem that guarantees that PTNE's exists for every $t > \bar{t}$. We now draw an appropriately-behaved selection from the PTNE correspondence to define the equilibrium strategies beyond $\bar{t}$. Finally, we define strategies at $\bar{t}$ in one of three ways, just as in the duel. The difference between the procedure in this complicated game and in the noisy duel is that we do not attempt to construct the strategies explicitly. This difference is unimportant, however: our strategies *will* be explicitly defined along the equilibrium path. Off this path, all that matters is that strategies satisfy certain properties. (This point will be amplified in section V below.) The theorems in the following sections guarantee that these properties are indeed satisfied by strategies drawn from the PTNE correspondence.

Modulo relabelling of agents, there is a unique equilibrium outcome for our quantity-setting

[16] There are trivial, asymmetric "chattering" solutions to this game, in which agents take turns to be the leader in alternate periods. We view such solutions as artifacts of the discrete-time formulation, that have no coherent counterparts in continuous time.

game.[17] This outcome corresponds to the Stackleberg solution for the one-shot quantity-setting game. Specifically, the reduced form ends at $t^{Stk} = {}^{16}/_{17}$. The leader produces one-half unit of output; in the continuation subgame, the remaining firm will produce a quarter unit. Note that the cost incurred by leading at $t^{Stk}$ (i.e., ${}^{1}/_{16}$), exactly equals the difference between Stackleberg leader's and the follower's payoffs. Thus, as one would expect, the rents to leading in this game are exactly dissipated when agents can compete with each other to determine who leads.

To conclude this section, we emphasize an important consequence of our restriction that strategies be grid-independent. If grid-dependent strategies (defined on p. 7) were introduced in, say, the noisy duel, the equilibrium set for this game would be much larger. For every $\tau \in ({}^{1}/_{3}, {}^{1}/_{2})$, the profile $\zeta$ below would be an SGP equilibrium, implementing the outcome: player #1 terminates alone with probability 1 at $\tau$. $\zeta$ is defined as follows: $\zeta_1(t, dt)(Tm) = \begin{cases} (0, 0) & \text{if } t < \tau \\ (1, 0) & \text{if } t \geq \tau \end{cases}$, while $\zeta_2(t, dt)(Tm) = \begin{cases} (0, 0) & \text{if } t < \tau \\ (0, \alpha(t)) & \text{if } t \geq \tau \end{cases}$, where $\alpha(t) \geq {}^{3}/_{5t-2}$. To see that $\zeta$ is an equilibrium, observe first that for any $t \in (\tau, {}^{1}/_{2})$, the outcome generated by $\zeta$ from $t$ is that player #1 terminates with probability one at $t$ (since #2's probability of termination shrinks with the size of the grid, while #1's does not). Therefore, #2 is indifferent between playing this grid-dependent strategy and not terminating at all. On the other hand, #2's strategy does deter #1 from waiting until $s > \tau$ to terminate, since $\alpha(\cdot)$ is chosen so that #1's gain to waiting until $s$ is offset by the risk that #2 will move between $\tau$ and $s$.

## III. The Formal Model.

A basic timing game is completely described by a quadruple, $\Gamma = (T, I, A, u)$. The set $T \subset \mathbf{R}_+$ represents the interval of time on which the game is played. We assume $0 \in T$. In applications, $T$ will usually be either $[0, 1]$ or $[0, \infty)$. Let $t^{end}$ denote the supremum of $T$. Let $I$ denote the set of agents, with generic element $i$. The finite set $A_i$ is called the <u>action set for player $i$</u>. $A_i$ contains a dis-

[17] More precisely, there is a unique limit to the outcomes of the games in which agents choose from finite sets of quantities. To avoid confusion, we emphasize that we model *time* as continuous but *quantities* as discrete and then take limits as the set of *quantities* converges to the continuum.

tinguished element, denoted $a_i^0$, that represents the action "continue," or "don't move." The remaining elements of $A_i$ will be called termination options for $i$.

Let $A = \prod_{i \in I} A_i$. An element $a = (a_i)_{i \in I} \in A$ will be called an action profile. A profile $a$ will frequently be written as $(a_i, a_{-i})$, or $(a_i, a_j, a_{-i,j})$, where $a_{-i} = (a_j)_{j \neq i}$ and $a_{-i,j} = (a_k)_{k \neq i,j}$. The profile $a^0 = (a_i^0)_{i \in I}$ will be called the continuation profile. All other profiles will be called termination profiles. The game ends as soon as some termination profile is chosen. A pair $(a, t) \in ((A - \{a^0\}) \times T) \cup \{(a^0, t^{end})\}$ will be called a terminal state of $\Gamma$. Denote the set of terminal states of $\Gamma$ by $TS(\Gamma)$.

Let $\Sigma_i$ denote the set of probability distributions over $A_i$, i.e., $\Sigma_i = \{\sigma_i = (\sigma(a_i))_{a_i \in A_i} \in [0, 1]^{\#A_i}: \sum_{a_i \in A_i} \sigma_i(a_i) = 1\}$. A point in $\Sigma_i$ will be called a random action for $i$.[18] Denote the support of $\sigma_i$ by $\text{supp}(\sigma_i)$, i.e., $\text{supp}(\sigma_i) = \{a_i \in A_i: \sigma_i(a_i) > 0\}$. For $a_i \in A_i$, let $\delta_{a_i}$ denote the degenerate distribution with support $\{a_i\}$. Let $\Sigma = \prod_i \Sigma_i$ denote the set of product distributions over $A$. A point in $\Sigma$ will be called a random action profile. For $\sigma \in \Sigma$ and $a \in A$, we will write $\sigma(a)$ to denote $\prod_i \sigma_i(a_i)$. Let $\delta_a$ denote the degenerate distribution with support $\{a\}$. In our subsequent analysis, we will be particularly concerned with strategy profiles in which at least two agents terminate with positive probability. We will say that $\sigma \in \Sigma$ is a plural termination profile if for all $i$ there exists $j \neq i$ such that $\sigma_j(a_j^0) < 1$.

The last component of $\Gamma$ is a "valuation function," $u: TS(\Gamma) \to \mathbf{R}^{\#I}$, that assigns a vector of utilities to each terminal state of the game. We normalize utilities so that all agents receive zero if no agent ever terminates:

$$u(a^0, t^{end}) = 0. \tag{u0}$$

We now impose a regularity condition on payoffs, to ensure that equilibrium strategies will be adequately well-behaved. Loosely, the assumption is that $u$ can be constructed by splicing together a finite

[18] The term "mixed strategy" is reserved for probability distributions over pure-strategies in the BTG game.

number of *real analytic* sections. More precisely, we assume:

$$\text{for all } i\,,\ \text{for all } a\,,\ u_i(a\,,\cdot) \text{ is piecewise extendible real analytic on } T.^{19} \qquad \text{(u1)}$$

Our next condition is

$$\text{for all } i\,,\ \text{for all } a \neq a^0,\ \text{for all } t < t^{end},\ u_i(a\,,\cdot) \text{ is right continuous at } t. \qquad \text{(u2)}$$

In the sequel to this paper, we will show that payoffs will automatically satisfy right continuity in the applications we consider, so that (u2) is innocuous. For the remainder of the game, we will assume implicitly that conditions (u0)-(u2) are satisfied.

Let $A_i^{lead}(t)$ denote the set of best termination options for $i$ at $t$, if all agents other than $i$ continue at this time. That is, $A_i^{lead}(t) = \underset{a_i \neq a_i^0}{\operatorname{argmax}}\, u_i((a_i\,, a_{-i}^0), t)$. It follows from (u1) and (u2) that $A_i^{lead}$ will be a right- and piecewise-continuous correspondence.[20] We will say that $i$ leads at $t$ if $i$ chooses (possibly randomly) from $A_i^{lead}(t)$, while other agents continue. Let $L_i(t)$ denote $i$'s payoff if he leads at $t$: $L_i(t) = \max_{a_i \neq a_i^0} u_i((a_i\,, a_{-i}^0), t)$. Let $F_i(t)$ denote the highest payoff that $i$ can attain, if some agent other than $i$ leads at $t$: $F_i(t) = \max_{j \neq i} \max_{a_j \in A_j(t)} u_i((a_j\,, a_{-j}^0), t)$.

*Behavior Strategies.*

A behavior strategy for agent $i$ is a function, $\xi_i$, assigning a random action to each $t \in T$. The scalar $\xi_i(t)(a_i)$ denotes the weight that $i$ assigns to the action $a_i$ at $t$. It is convenient to define $\xi_i$ at $t^{end}$, even if $t^{end}$ is not a member of $T$. In this case, we adopt the convention that agents assign probability one to "continue":

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19 Fix an open subset $O$ of $T$. A function $\theta$ defined on $O$ is real analytic if at every $t \in O$, its Taylor series converges. See Rudin [15], Ch. 8, for details and properties of real analytic functions. Now fix a subset $R$ of $T$ and a function $\phi$ defined on $R$. $\phi$ is called extendible real analytic if there exists an open set $O$ containing the closure of $R$ and a real analytic function $\theta$ defined on $O$ whose restriction to $R$ is $\phi$. (The function $\phi(t) = \sqrt{t}$ is real analytic on $(0, \infty)$ but is not extendible real analytic.) A function $\phi$: $T \rightarrow \mathbb{R}$ is called piecewise extendible real analytic on $T$ if for all $t < t^{end}$, there exists a finite partition of $[0, t)$ such for each member $R$ of the partition, $\phi_{|R}$ is an extendible real analytic function.

20 A *function* $\theta$ will be called piecewise continuous on $T$ if for all $t \in T$, $\{s \in [0, t): \theta(s) \neq \lim_{s' \to s} \theta(s')\}$ is finite. A *correspondence* mapping $T$ to a finite set $X$ will be called piecewise- (right-) continuous if, viewed as a function from from $T$ to the set of subsets of $X$, it is piecewise- (right-) continuous.

$$\text{if } t^{end} \notin T, \text{ then } \xi_i(t^{end})(a_i^0) = 1. \tag{X0}$$

To guarantee that our outcome function is well-defined--i.e., the discrete-time outcomes have well-defined limits--we need some regularity conditions on the way strategies vary with time. First, we require:

$$\text{for all } i, \text{ for all } a, \ \xi_i(\cdot)(a_i) \text{ is piecewise extendible continuous on } T.^{21} \tag{X1}$$

If an agent terminates with probability zero at $t$, and positive probability just beyond $t$, then the condition we need at $t$ is more stringent. We require that for all $t < t^{end}$:[22]

$$\xi_i(t)(a_i^0) = 1 \text{ and } \xi_i(\cdot)(a_i^0) < 1 \text{ on an interval after } t, \text{ then} \tag{X2}$$

there exists $r \in \mathbf{R}_+$ such that for all $a_i \neq a_i^0$, $\lim_{\delta \downarrow 0} \frac{\xi_i(t+\delta)(a_i)}{\delta^r}$ exists and, for at least one $a_i$, is positive.

This condition is not restrictive for the class of games that we will study. To see this, observe that (X2) will be certainly be satisfied if the restriction of $\xi$ to an interval after $t$ is extendible real analytic (fn. 19): simply choose $r$ to be the first nonzero derivative of its extension. Moreover, condition (u1) ensures that equilibrium strategy profiles can indeed be so extended.[23]

Our final restriction is imposed purely for pragmatic reasons. The assumption is very ad hoc. Moreover, it is not required in order to define the game. (We do not impose it in SS2.) It does, however, dramatically simplify the specification of the outcome function. Say that agent $i$ is <u>properly randomizing at $t$</u> if he continues at $t$ with some positive, non-unitary probability, i.e., if $\xi_i(t)(a_i^0) \in (0, 1)$. We will assume

$$\text{for all } t < t^{end}, \text{ if } i \text{ is properly randomizing at } t, \text{ then } \xi_i(\cdot) \text{ is right continuous at } t. \tag{X3}$$

Summarizing, a <u>behavior strategy for $i$</u> is a function $\xi_i : cl(T) \to \Sigma_i$ satisfying (X0)-(X3). Let $\Xi_i$ denote the set of behavior strategies for $i$ and let $\Xi = \prod_i \Xi_i$. A list of behavior strategies,

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[21] The definition of piecewise extendible continuity is exactly analogous to that of piecewise extentable real analyticity (see fn. 19 above).

[22] This condition is discussed in detail in SS2.

[23] See the proof of Lemma II.1 in the Appendix for details.

$\xi = (\xi_i)_{i \in I} \in \Xi$ will be called a behavior strategy profile. We will write $\xi(t)(a)$ to denote the product of the $\xi_i(t)(a_i)$'s, i.e., $\xi(t)(a) = \prod_i \xi_i(t)(a_i)$.

As noted earlier, we will be particularly concerned with strategy profiles in which at least two agents are randomizing at every point in some open interval. Accordingly, we say that a profile $\xi$ has plural termination just beyond $t$ if there exists $\delta > 0$ such that for all $s \in (t, t+\delta)$, $\xi(s)$ is a plural termination profile (p. 14).

*The Outcome Function.*

Our outcome function assigns to each strategy profile and point in time a probability distribution over terminal states of the game. As explained above (pp. 6-7, fn. 12), the outcome generated by $\xi$ from $t$ is the limit of the sequence of outcomes generated by restricting $\xi$ to an arbitrary, increasingly fine sequence of discrete-time grids, starting from $t$. For the class of strategies defined above, this limit distribution has an exceedingly simple structure. Assumptions (X1)-(X3) together guarantee that the game must end with probability one, as soon as some agent assigns positive probability to some termination action. Precisely, for $\xi \in \Xi$ and $t < t^{end}$,

$$\text{if for some } i,\ \xi_i(\cdot)(a_i^0) < 1, \text{ either at or immediately after } t \quad \text{then the game will end with probability 1 at } t.\text{[24]} \tag{3.1}$$

A consequence of (3.1) is that we can without loss of generality define an outcome to be a pair $(\mu, t)$, where $t \in cl(T)$ and $\mu$ is a probability distribution over $A$, i.e., an element of $\Delta^{\#A-1} = \{\mu = (\mu(a))_{a \in A} \in [0,1]^{\#A-1}: \sum_{a \in A} \mu(a) = 1\}$. Note that $\Delta^{\#A-1}$ contains measures that are not product distributions.

[24] To see that (3.1) is true, first assume that for some $i$, $\xi_i(\cdot)(a_i^0) < 1$ on an interval after $t$. For $\delta$ sufficiently small, there exists $\epsilon > 0$ such that $\xi_i(\cdot)(a_i^0) < 1 - \epsilon$ on the interval $(t+\delta, t+2\delta)$. Pick $n$ such that $(1-\epsilon)^n < \delta$. If $\xi$ is played on a grid with at least $n$ grid-points between $t+\delta$ and $t+2\delta$, then with probability exceeding $1-\delta$, agent $i$ will move before $t+2\delta$. Since $\delta$ was chosen arbitrarily, this establishes that in the limit, player $i$ must move with probability 1 at $t$. Now suppose that $\xi_i(\cdot)(a_i^0) = 1$ on an interval immediately after $t$. By hypothesis, $\xi_i(t)(a_i^0) < 1$. By (X3), $\xi_i(t)(a_i^0) = 0$. (3.1) now follows trivially.

Let $\nu(\sigma)(a)$ denote the *conditional* probability that agents choose $a$, *given* that they do *not* choose $a^0$ and let $\nu(\sigma)(a^0) = 1$ if agents continue with probability 1. That is, for $a \neq a^0$,

$$\nu(\sigma)(a) = \begin{cases} \frac{\sigma(a)}{1-\sigma(a^0)} & \text{if } \sigma(a^0) < 1 \\ 0 & \text{if } \sigma(a^0) = 1. \end{cases}$$

Now fix a strategy profile $\xi \in \Xi$ and consider the induced function of time, $\nu(\xi(\cdot)): T \rightarrow \Delta^{\#A-1}$. Assumption (X1) on strategies guarantees that

$$\nu(\xi(\cdot)) \text{ is piecewise continuous w.r.t. } t.^{25} \tag{3.2}$$

For $t \in T$ and $\xi \in \Xi$, we define $\nu^{t+}(\xi)$ by: $\nu^{t+}(\xi) = \begin{cases} \lim_{\delta \downarrow 0} \nu(\xi(t+\delta))^{26} & \text{if } t < t^{end} \\ \delta_{a^0} & \text{if } t = t^{end} \end{cases}$. It follows from (3.2) that this limit is always well-defined.

We can now specify the outcome function using the concepts just defined. Fix $t \in T$ and consider the outcome generated by $\xi$ from $t$. If all agents continue from $t$ to the end of the game, the outcome is that the state $(a^0, t^{end})$ is realized with probability one. Suppose that agents continue until $\tau$, and then some agent terminates with positive probability, either at or immediately after $\tau$. If $\tau = t^{end}$, then state $(a^0, t^{end})$ is realized with probability $\xi(t^{end})(a)$. If $\tau < t^{end}$, the probability that state $(a, \tau)$ is realized will be a weighted average of (i) the conditional probability that $a$ is chosen at $\tau$ and (ii) the limit of the conditional probabilities that $a$ is chosen immediately after $\tau$. Formally, we define the outcome generated by $\xi$ from $t$ to be the pair $o(\xi, t) = (o^{\Sigma}(\xi, t), o^T(\xi, t))$, where $o^{\Sigma}(\xi, t)$ is the distribution over $A$ described above and $o^T(\xi, t)$ denotes the first time after $t$ that some agent moves with positive probability. That is, for $(\xi, t) \in \Xi \times T$,

$$o^T(\xi, t) = \begin{cases} t^{end} & \text{if } \xi(\cdot)(a^0) = 1 \text{ on } [t, t^{end}) \\ \inf\{s > t: \xi(s)(a^0) < 1\} & \text{otherwise} \end{cases}$$

and, setting $\tau = o^T(\xi, t)$,

[25] For a proof of (3.2), see SS2. The key issue in the proof is the existence of $\lim_{\delta \downarrow 0} \nu(\xi(t+\delta))$ when $\lim_{\delta \downarrow 0} \xi(t+\delta)(a^0) \uparrow 1$. To prove that this limit exists, we generalize L'Hospital's rule and invoke assumption (X2).

[26] The symbol $\downarrow$ denotes converging from strictly above, while $\uparrow$ denotes converging from strictly below.

$$o^{\Sigma}(\xi, t)(a) \;=\; (1 - \xi(\tau)(a^0))\nu(\xi(\tau))(a) \;+\; \xi(\tau)(a^0)\nu^{\tau+}(\xi)(a).$$

See SS2 for a proof that $o(\xi, t)$ is the limit, in the sense defined above (pp. 6-7, fn. 12), of the outcomes generated by restricting $\xi$ to discrete-time grids. It is useful to note the following special cases. Again letting $\tau = o^T(\xi, t)$, we have:

$$o^{\Sigma}(\xi, t) \;=\; \begin{cases} \xi(\tau) & \text{if } either\ \tau = t^{end}\ \ or\ \ \xi(\tau)(a^0) = 0 \\ \nu(\xi(\tau)) & \text{if } \tau < t^{end},\ \xi(\tau)(a^0) < 1 \ \text{ and } \ \xi(\tau) = \xi(\tau+) \\ \nu^{\tau+}(\xi) & \text{if } \tau < t^{end} \ \text{ and } \ \xi(\tau)(a^0) = 1 \\ \delta_{a^0} & \text{if } \tau = t^{end} \notin T \end{cases} \tag{3.3}$$

Clearly, $o^{\Sigma}(\xi, t)$ is trivial to compute, except if $\lim_{\delta \downarrow 0} \xi(\tau + \delta)(a^0) \uparrow 1$. In this case, we need to generalize L'Hospital's rule to compute $\nu^{\tau+}(\xi)$. If $\xi$ has a real analytic extension at $t$, then $\nu^{\tau+}(\xi)$ has a simple form. Let $\hat{\xi}$ denote this extension and let $m \in \mathbb{N}$ be the smallest integer such that for at least one $i$, the $m$'th partial derivative, $\hat{\xi}_i^{(m)}(\bar{t})(a_i^0) < 1$. In SS2, we show that for $i \in I$ and $a_i \neq a_i^0$, $\nu^{\tau+}(\xi)(a_i, a_{-i}^0) \;=\; \dfrac{\hat{\xi}_i^{(m)}(\bar{t})(a_i)}{\sum_{j \in I} \sum_{a_j \in A_j} \hat{\xi}_j^{(m)}(\bar{t})(a_j, a_{-j}^0)}$.

Let $O(\Gamma)$ denote the image of the outcome function, i.e.,

$$O(\Gamma) \;=\; \{(\mu, \tau) \in \Delta^{\#A-1} \times cl(T) \text{: there exists } (\xi, t) \in \Xi \times T \text{ such that } o(\xi, t) = (\mu, \tau)\}.$$

A pair $(\mu, t) \in O(\Gamma)$ will be called an <u>outcome</u>. We will say that $(\mu, \tau)$ is <u>implemented by $\xi$ from $t$</u> if $(\mu, \tau) = o(\xi, t)$. If $(\mu, \tau)$ is implemented by $\xi$ from $t = 0$, we say simply that $(\mu, \tau)$ is <u>implemented by $\xi$</u>.

*Payoffs and Equilibrium Notions.*

Given a valuation function, $u$, the <u>expected valuation function</u>, $Eu$, assigns an expected payoff vector to each outcome. That is, $Eu(\mu, t) = \begin{cases} \sum_{a \neq a^0} u(a, t)\mu(a) & \text{if } t < t^{end} \\ \sum_{a \in A} u(a, t)\mu(a) & \text{if } t = t^{end} \end{cases}$. (Note that if $(\mu, t) \in O(\Gamma)$, then $\mu(a^0) > 0$ only if $t = t^{end}$.) The <u>payoff function</u> $P : \Xi \times T \to \mathbb{R}^{\#I}$ assigns a payoff vector to each strategy profile and each point in time. $P_i(\xi, t)$ is player $i$'s payoff if agents play $\xi$ from the subgame beginning at $t$. Thus, setting $\tau = o^T(\xi, t)$,

$$P_i(\xi, t) = Eu_i(o^\Sigma(\xi, t), \tau) = (1 - \xi(\tau)(a^0))Eu_i(v(\xi(\tau)), \tau) + \xi(\tau)(a^0)Eu_i(v^{\tau+}(\xi), \tau). \qquad (3.4)$$

For $t \in T$, we will say that $\bar{\xi}$ is Nash from $t$ if for all $i$, and all $\xi_i$, $P_i(\bar{\xi}, t) \geq P_i((\xi_i, \bar{\xi}_{-i}), t)$. Say that $(\mu, \tau)$ is Nash implementable from $t$ there exists a Nash equilibrium from $t$ that implements $(\mu, \tau)$ from $t$. A profile $\xi$ is a subgame perfect equilibrium (SGP equilibrium) if $\xi$ is Nash from each $t$. Finally, $(\mu, t)$ is SGP-implementable there exists an SGP equilibrium that implements it.

*Existence of Subgame Perfect Equilibria.*[27]

In addition to (u0)-(u2), a number of conditions are needed to guarantee existence. The first condition is: if some agent $j$ moves at $t$, then agent $i \neq j$ does at least as well by continuing at $t$ as he does by moving simultaneously with $j$. Formally, we will assume that for all $j$ and all $a_j \neq a_j^0$,

$$\text{for all } i \in I, \text{ for all } a_i \neq a_i^0, \text{ for all } t \in T, u_i((a_i^0, a_j, a_{-i,j}^0), t) \geq u_i((a_i, a_j, a_{-i,j}^0), t). \qquad (A1)$$

In many applications, this restriction will be satisfied automatically. In particular, most two-person BTG's are reduced-form representations of games in which the player who follows at $t$ has an opportunity to move *after* $t$. In such a game, if $j$ moves at $t$ and $i$ moves immediately afterwards, the outcome will be the same as if both had moved simultaneously. Therefore A1 is certainly satisfied. In the sequel to this paper, we show that the condition is also satisfied in many multi-player BTG's derived from more primitive "generalized" timing games.

The other two conditions are relatively innocuous. They are:

$$\text{for all } i, \text{ for all } t < t^{end}, L_i(\cdot) \text{ is upper semi-continuous at } t;^{28} \qquad (A2)$$

$$\text{there exists } s < t^{end} \text{ such that } \begin{cases} \textit{either} & \text{for all } i, L_i(\cdot) > F_i(\cdot) \text{ on } (s, t^{end}] \\ \textit{or} & \text{there exists } i \text{ such that } L_i(\cdot) \text{ is nonincreasing on } (s, t^{end}]. \end{cases} \qquad (A3)$$

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[27] I am greatly indebted to Bill Zame for introducing me to the mathematics needed to prove existence.

[28] A function $\theta: \mathbf{R}_+ \to \mathbf{R}$ is upper semi-continuous at $t$ if $\theta(t) \geq \limsup_{s \to t} \theta(s)$.

In many applications, $L_i(\cdot)$ will in fact be continuous, so that assumption (A) will be satisfied. Assumption (A) is satisfied in most of the applications that interest us: "either" is usually satisfied in "pure" preemption games with a finite time horizon; "or" is satisfied by almost all infinite horizon timing games.

These three conditions are sufficient for existence in *either* a $2 \times 2$ game *or* a game that is symmetric, in the following sense. First, each agent must have the same set of available actions. Now, for $a \in A$, let $a^{i \leftrightarrow 1}$ denote the action profile which is identical to $a$, except that the actions of players $i$ and 1 are interchanged. That is, $a^{i \leftrightarrow 1}$ is defined by: $a_j^{i \leftrightarrow 1} = \begin{cases} a_i & \text{if } j = 1 \\ a_1 & \text{if } j = i \\ a_j & \text{otherwise} \end{cases}$. In a symmetric game, player $i$'s payoff from the terminal state $(a, t)$ is identical to player 1's payoff from $(a^{i \leftrightarrow 1}, t)$. Summarizing, we will say that a game $\Gamma = (T, A, I, u)$ is <u>symmetric</u> if for all $i > 1$, $A_i = A_1$ and if for all $i$, all $a \in A$ and all $t \in T$, $u_i(a, t) = u_1(a^{i \leftrightarrow 1}, t)$. We can now state our existence result:

**Th'm II**: A $2 \times 2$ game, or a symmetric game, satisfying (A1)-(A3) has an SGP equilibrium.

Section V contains an example of an asymmetric game with no equilibrium. On the other hand, symmetry is by no means a necessary condition. It remains an open question, therefore, how far symmetry can be relaxed without losing existence.

*Introduction to Sections IV and V.*

In the next two sections, we formalize our computational procedure for solving a class of continuous-time games. Formally, this "calculus" is a characterization of the set of equilibria for a game in terms of the payoffs to leading and certain "local" properties of the game. (Call a property <u>local at $t$</u> if one can determine whether a game has the property by considering only the parameters of the game in a neighborhood of $t$.) In simple games such as the duel, these properties are easy to check. In more complicated games--e.g., our Cournot example above--additional machinery will be invoked to check them.

There are two steps to our characterization. First, we identify possible "equilibrium endings" for a game. That is, we fix $t$ and classify the Nash implementable outcomes from $t$ in which the game ends immediately. Second, we single out those equilibrium endings that no agent would wish to preempt.

A BTG in the class we consider can end in one of two ways. We will refer to these as "type $k$ endings," where $k$ is 1 or 2. Type 1 endings are the more interesting and less familiar: all of the examples in section II had endings of this kind. We focus on these in section IV. Section V begins with a description of the other kind of ending; this is followed by our characterization theorem. The section concludes with a counter-example to existence.

## IV. Type I Endings.

We begin by defining two kinds of one-shot games: a "stage-game" and a "surrogate game." Each is obtained by freezing a BTG at an instant of time. The first concept is familiar from discrete-time game theory; the second is novel. We identify sufficient conditions for existence of a "plural termination Nash equilibrium" (PTNE) for a surrogate game. We then establish our key result: if $\xi$ has plural termination just beyond $t$, then to verify that $\xi$ is Nash from $t$, we need only check that (i) $\xi(t)$ solves the stage-game induced by $\xi$ at $t$; (ii) for every $s$ in an interval after $t$, $\xi(s)$ is a PTNE for the surrogate game at $s$. We then say that an outcome is a "type 1 outcome from $t$" if it is implementable from $t$ by a profile satisfying the above conditions.

Fix a basic timing game $\Gamma = (T, I, A, u)$. For each $t < t^{end}$ and strategy profile $\xi \in \Xi$, we denote by $StgG(t, \xi)$ the <u>stage game induced by $(\Gamma, \xi)$ at $t$</u>. This is the one-shot game defined as follows: if agents play a termination profile "$a$", they earn earn the payoff vector $u(a, t)$; if they play the continuation profile, $a^0$, they receive the "continuation payoff vector" $P(\xi, t+) = \lim_{\delta \downarrow 0} P(\xi, t + \delta)$. (This is the payoff vector that would be generated in $\Gamma$, if all agents continued at $t$ and played $\xi$ thereafter.) Thus, $StgG(t, \xi)$ is the one-shot game identified by the triple $(I, A, P^{t,\xi})$, where, for each random action profile $\sigma$, $P^{t,\xi}(\sigma) = \sum_{a \neq a^0} u(a, t)\sigma(a) + P(\xi, t+)\sigma(a^0)$. We will say that $\sigma$ <u>solves</u> <u>$StgG(t, \xi)$</u> if it is a Nash equilibrium for this game (i.e., if for all $i$ and all $\sigma_i \in \Sigma_i$, $P_i^{t,\xi}(\bar{\sigma}) \geq P_i^{t,\xi}(\sigma_i, \bar{\sigma}_{-i})$).

A *surrogate game* is exactly like a regular one-shot game, except that the payoff function is specified in a novel way. We first define the concept and then motivate it. Denote by $SurrG(t)$ the surrogate game induced by $\Gamma$ at $t$. If agents playing this "game" choose any profile $\sigma$ *except* the continuation profile $\delta_{a^0}$, the outcome that results is the distribution over $A$ that *would have resulted at* $t$--i.e., $\nu(\sigma)$--had agents playing $\Gamma$ chosen a *right-continuous* profile $\xi$ such that $\xi(t) = \sigma$. Specifically, the payoff assigned to this $\sigma$ in $SurrG(t)$ is $Eu(\nu(\sigma), t) = (1 - \sigma(a^0))^{-1} \sum_{a \neq a^0} \sigma(a) u(a, t)$. If agents play the continuation profile with probability one, they each receive a payoff of zero. Summarizing $SurrG(t)$ is identified by the triple $(I, A, Q^t)$, where $Q^t(\sigma) = \begin{cases} 0 & \text{if } \sigma = \delta_{a^0}. \\ Eu(\nu(\sigma), t) & \text{otherwise} \end{cases}$. Note that $\nu$, and hence, possibly, $Q^t(\cdot)$, is discontinuous at $\sigma = \delta_{a^0}$. This does not concern us, however, because we will be concerned only with equilibria of $SurrG(t)$ in which at least two agents terminate with positive probability. We will say that $\bar{\sigma}$ is a Nash equilibrium with plural termination (PTNE) for $SurrG(t)$ if $\sigma$ is a plural termination profile and a Nash equilibrium for this surrogate game (i.e., if for all $i$ and all $\sigma_i \in \Sigma_i$, $Q^t(\bar{\sigma}) \geq Q^t(\sigma_i, \bar{\sigma}_{-i})$).

The reason for our interest in the PTNE's of a surrogate game will be apparent from the discussion in section II. To recapitulate, if in the original BTG, $\xi$ has plural termination just beyond $t$, then no individual agent can prevent the game from ending exactly at $t$. If in addition, $\xi$ is right-continuous at $t$, then the strategic possibilities facing agents in the original game at $t$ will be *exactly* equivalent to their possibilities in the one-shot game $SurrG(t)$.[29] Consequently, there is an intimate relationship between the Nash equilibria for subgames of a BTG and the solutions to the corresponding surrogate games.

The following Proposition is a useful tool for analyzing surrogate games. Fix $\sigma_{-i}$ such that $\sigma_{-i}(a^0_{-i}) < 1$. A random action $\sigma_i$ will be a best response against $\sigma_{-i}$ in a surrogate game if and only if each action in the support of $\sigma_i$ does at least as well against $\sigma_{-i}$ as any other *nonrandom* response. This fact is not at all obvious, since if $\sigma_{-i}(a^0_{-i}) \in (0, 1)$, $Q^t(\cdot, \bar{\sigma}_{-i})$ depends nonlinearly on $\sigma_i$. Stated

[29] This follows immediately from the second line of expression (3.3) above.

precisely, we have:

**Prop'n IV:** Fix $\bar{\sigma}_{-i} \in \Sigma_{-i}$ such that $\sigma_{-i}(a_{-i}^{0}) < 1$. A necessary and sufficient condition for $\sigma_i$ to maximize $Q_i^t(\cdot, \bar{\sigma}_{-i})$ is that $\mathrm{supp}(\sigma_i) \subset \underset{a_i \in A_i}{\mathrm{argmax}}\, Q_i^t(\delta_{a_i}, \bar{\sigma}_{-i})$.

Surrogate games of a $2 \times 2$ game are very easy to solve. Let $\Gamma$ be a $2 \times 2$ BTG in which $I = \{1, 2\}$ and $A_i = \{Ct, Tm\}$. In this case, $L_i = u_i((Tm, Ct), \cdot)$, $F_i = u_i((Ct, Tm), \cdot)$ and $S_i = u_i((Tm, Tm), \cdot)$. (Whenever we write $u_i((\cdot, \cdot), \cdot)$, the *first* argument refers to $i$'s action.) By definition of $\nu$, if $\sigma_j > 0$, then

$$Q_i^t(\delta_{Ct}, \sigma_j) = Eu_i(\nu(\delta_{Ct}, \sigma_j), t) = F_i(t) \tag{4.1a}$$

$$Q_i^t(\delta_{Tm}, \sigma_j) = Eu_i(\nu(\delta_{Tm}, \sigma_j), t) = \sigma_j(Tm)S_i(t) + (1 - \sigma_j(Tm))L_i(t). \tag{4.1b}$$

The proposition below characterizes the PTNE's of $SurrG(t)$. Since the proof is basic and instructive, we include it in the text. Define the profile $\sigma^t$ by: $\sigma_i^t(Tm) = \begin{cases} \frac{L_j(t) - F_j(t)}{L_j(t) - S_j(t)} & \text{if } L_j(t) > F_j(t) \\ 1 & \text{if } S_j(t) = F_j(t) \end{cases}$. We have:

**Prop'n V:** Let $\Gamma$ be a $2 \times 2$ BTG satisfying (A1) and fix $t < t^{end}$. The following conditions are necessary and sufficient for existence of a PTNE for $SurrG(t)$: for each $i$, either $S_i(t) = F_i(t)$, or $L_i(t) > F_i(t)$. If these conditions are satisfied then a solution is $\sigma^t$, defined above. If $L_i(t) > F_i(t) > S_i(t)$, for each $i$, then this solution is *unique*. In any PTNE for $SurrG(t)$, player $i$'s expected payoff is $F_i(t)$.

*Proof of Proposition V:* We first prove that the conditions are necessary. Suppose that for some $i$, $S_i(t) \neq F_i(t)$. *and* $L_i(t) \leq F_i(t)$. By (A1), $S_i(t) < F_i(t)$. Fix $\sigma_j$ such that $\sigma_j(Tm) > 0$. It follows immediately from (4.1) that continuing is a better response for $i$ than terminating against $\sigma_j$. From Proposition IV, if $\sigma_i$ is a best response to $\sigma_j$, then $\sigma_i(Tm) = 0$. Therefore, there is no PTNE for this surrogate game.

To prove that the conditions are sufficient, we will show that $\sigma^t$ is an equilibrium: If $\sigma_j^t(Tm) = 1$, then from (4.1) and the definition of $\sigma^t$, $Q_i^t(\delta_{Tm}, \sigma_j^t) = S_i(t) = F_i(t) = Q_i^t(\delta_{Ct}, \sigma_j^t)$. If $\sigma_j^t(Tm) < 1$, then

$$Q_i^t(\delta_{Tm}, \sigma_j^t) = \sigma_j(Tm)S_i(t) + (1 - \sigma_j(Tm))L_i(t) \quad (4.2)$$
$$= \frac{L_i(t) - F_i(t)}{L_i(t) - S_i(t)}S_i(t) + \frac{F_i(t) - S_i(t)}{L_i(t) - S_i(t)}F_i(t) = F_i(t) = Q_i^t(\delta_{Ct}, \sigma_j^t).$$

In either case, it follows from Proposition IV that $\sigma_i^t$ is a best response to $\sigma_j^t$.

Finally, assume that for each $i$, $L_i(t) > F_i(t) > S_i(t)$. We will show that $\sigma^t$ is the unique PTNE for $SurrG(t)$. If player $i$ terminates with probability one, $j$ will strictly prefer to continue. Therefore, in any plural solution to this surrogate game, both agents must be properly randomizing. But from the calculations in (4.2), $i$ will be indifferent between continuing and terminating only if $j$ plays $\sigma_j^t$. The proof of the last sentence of the proposition is straightforward, and therefore left to the reader. □

In games with multiple termination options, the PTNE's of surrogate games will generally be hard to compute. However, as we noted in section II, we do not usually need to have explicit solutions in applications (see Section V below). We only need to know that under certain conditions, such equilibria exist and satisfy certain properties. For a *symmetric* game, there is an simple sufficient condition for existence: the payoff to leading at $t$ must exceed the payoff to following.

**Prop'n VI:** Assume that $\Gamma$ is a symmetric game. Fix $t \in T$, $t < t^{end}$. If $L_1(t) > F_1(t)$, then a PTNE for $SurrG(t)$ exists.

We now come to the main result of the section. Informally, Proposition VII below states that if a profile $\xi$ has plural termination just beyond $t$, then $\xi$ is a Nash equilibrium for the subgames beginning at and immediately after $t$ if and only if: (i) $\xi(t)$ solves the stage-game induced by $\xi$ at $t$ and (ii) for $s$ immediately after $t$, $\xi(s)$ solves the surrogate game at $s$.

**Prop'n VII:** Fix $\xi \in \Xi$ and $t < t^{end}$. If $\xi$ has plural termination just beyond $t$, then statements (i) and (ii) below are equivalent:

(i) there exists $\delta > 0$ such that for all $s \in [t, t + \delta)$, $\xi$ is a Nash equilibrium for the subgame beginning at $s$;

(ii) (a) $\xi(t)$ solves $StgG(t, \xi)$ *and* (b) there exists $\delta > 0$ such that for all $s \in (t, t + \delta)$, $\xi(s)$ solves $SurrG(s)$.[30]

It is useful to note that

if $\xi(\cdot)$ is right continuous at $t$, then (i) above is equivalent to: (4.3)

there exists $\delta > 0$ such that for all $s \in (t, t + \delta)$, $\xi(s)$ solves $SurrG(s)$.

The verification of (4.3) is tedious and deferred to the Appendix (following the proof of Proposition VII).

We now define an outcome $(\mu, t)$ to be a <u>type 1 outcome from $t$</u> if it can be implemented by a profile that has plural termination just beyond $t$ and satisfies condition (ii) of Proposition VII. A <u>type 1 outcome</u> is a type 1 outcome from $t$, for some $t \in T$.

To illustrate, we will characterize the type 1 outcomes for the symmetric noisy duel. When $t < ½$, both $S_1(t)$ and $L_1(t)$ are strictly less than $F_1(t)$ From Proposition V, therefore, the surrogate games before $t = ½$ have no PTNE's. Therefore, there can be no type 1 outcome before $t = ½$. On the other hand, we will show that for every $t \geq ½$, there are exactly three type 1 outcomes from $t$. Define the profile $\xi^*$ as follows: each player continues at every $s \leq ½$; at $s > ½$, $\xi^*(s) = \sigma^s$ (defined above, p. 24). Now fix $t \geq ½$. Clearly, $\xi^*$ has plural termination just beyond $t$. Moreover, from Proposition V, we know that for every $s > t$, $\xi^*(s)$ is a PTNE for $SurrG(s)$. Since $\xi^*$ is continuous at $t$, it follows from (4.3) that $\xi^*$ satisfies condition (ii) of Proposition VII. The outcome implemented by $\xi^*$ from $t$--i.e., $v^{t+}(\xi^*)$--is therefore a type 1 outcome. Now define $\xi^{t,i}$ identically to $\xi^*$ except that exactly at $t$, player $i$ terminates, and $j$ continues, with probability one. Obviously, the outcome implemented by $\xi^{t,i}$ from $t$ is that $i$ leads with probability one at $t$. To see that $\xi^{t,i}$ satisfies condition (ii) of Proposition VII, observe that from (4.2), if $i$ were to continue at $t$, his payoff in the stage-game at $t$ would be $P(\xi^{t,i}, t+) = F_i(t) \leq L_i(t)$. Moreover, if $j$ were to terminate at $t$, his payoff would be $S_i(t) < F_i(t)$. Therefore, $\xi^{t,i}(t)$ indeed solves the stage-game induced by $\xi^{t,i}$ at $t$. Using condition (X3), it is straightforward to verify that there are no other type 1 outcomes from $t$.

To find type 1 outcomes in a more complicated game, we proceed exactly as above, except that some additional theorems must be invoked. To illustrate, we return to the Cournot example discussed in section II. Recall that leading is dominated by following at each $t < {}^{16}/_{17}$, so that no PTNE exists before this time.[31] On the other hand, we will show that for every $t > {}^{16}/_{17}$, there are exactly three type 1

[31] To see this, observe that in any surrogate game, any quantity level strictly exceeding ½ is a strictly dominated strategy. In a PTNE, therefore, no agent will announce a quantity strictly greater than ½. Therefore, the payoff to continuing in any surrogate game must be at least ¼. At $t < {}^{16}/_{17}$, however, the *largest possible* payoff an agent can obtain by terminating at $t$ is strictly less than ¼. Therefore, there can be no PTNE at $t$.

outcomes from $t$. Choose a right-continuous profile $\xi^*$ satisfying: each player continues at every $t \leq {}^{16}/_{17}$; at $t > {}^{16}/_{17}$, $\xi^*(t)$ is a PTNE for $SurrG(t)$. (From Proposition VI, such an equilibrium exists for every $s > {}^{16}/_{17}$. Moreover, it can be shown (see Lemma II.1) that the PTNE correspondence is sufficiently nicely behaved that $\xi^*$ can be chosen to satisfy restrictions (X1), (X2) and right-continuity). Once again, it follows from (4.3) that for every $t \geq {}^{16}/_{17}$, $\xi^*$ satisfies condition (ii) of Proposition VII, so that the outcome $\nu(\xi^*(t))$ is a type 1 outcome. Now, once again, construct the profile $\xi^{t,i}$ that agrees with $\xi^*$ except that at $t$, $i$ plays his Stackleberg output and $j$ continues. Observe that player $i$'s payoffs in the PTNE's immediately after $t$ are strictly lower than his payoff from leading at $t$.[32] Therefore, we can repeat the argument above that $\xi^{t,i}(t)$ solves the stage-game induced by $\xi$ at $t$. Summarizing, the example just analyzed shows that our simple technique for characterizing the type 1 outcomes of the duel can readily be generalized to handle much more complex problems.

## V. Characterizing the SGP equilibria of a class of BTG's

In this section, we characterize the equilibrium set of a game satisfying the following conditions: (i) there exists *some* SGP equilibrium for the game; (ii) each agent's payoff to leading is strictly increasing with time; (iii) assumption A1 is satisfied.

As we have observed, a game in this class can end in one of two ways. The first of these was described in the preceding section. The other kind of ending is very simple. We will say that $(\mu, t)$ is a <u>type 2 outcome from $t$</u> if it can be implemented by a profile in which at least two players terminate with probability one at $t$.

Our theorem states that for a game in the class specified above, an outcome will be SGP implementable if and only if it is one of the two kinds of outcomes described above *and* no agent wishes to preempt it.

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32 This fact is intuitive, yet tedious to verify formally. The following is a (terse) verbal summary of the algebra: $i$'s payoff from a PTNE at $s \approx t$ is a strict convex combination of $i$'s conditional payoffs given that (a) $i$ moves and $j$ doesn't, (b) $j$ moves and $i$ doesn't and (c) they both move. Call these numbers, respectively $\gamma_l$, $\gamma_f$ and $\gamma_s$. From A1, $\gamma_f > \gamma_s$. Therefore, $\gamma_l > \gamma_f$ (otherwise, "continue" would be the unique best response for $i$). But by definition, $L_i(t) \approx L_i(s)$, which is at least as great as $\gamma_l$.

**Th'm VIII:** Let $\Gamma$ be a BTG satisfying assumption A1. Assume that an SGP equilibrium for $\Gamma$ exists and that for each $i$, $L_i$ is strictly increasing with time. An outcome $(\mu, t)$ will be SGP implementable if and only if (i) it is a type $k$ outcome, for some $k \in \{1, 2\}$, *and* (ii) for each $j$,

$$Eu_j(\mu, t) \geq L_j(\cdot) \text{ on } [0, t) \qquad (5.1)$$

To illustrate the theorem, we first characterize the equilibrium sets of the symmetric and asymmetric versions of the noisy duel. First observe that there can be no type 2 outcome, since the $S_i$'s are strictly smaller than the $F_i$'s. Moreover, from Proposition V, there exist type 1 outcomes from $t$ iff $t \geq ½$. However, if $(\mu, t)$ is a type 1 outcome from $t > ½$, then there exists some $j$ such that $Eu_j(\mu, t) = F_j(t)$. Therefore, in either version of the game, the no-preemption condition--i.e., (5.1)--will be satisfied only if $t = ½$. Finally, as we showed in the preceding section, each version of the game has three type 1 outcomes at $t = ½$. In the symmetric game, condition (5.1) is satisfied by all three of these, so that all three are SGP implementable. In the asymmetric game, $L_1(½)$ exceeds $F_1(½)$, so that (5.1) will be satisfied only when player #1 leads at ½. Therefore, the asymmetric duel has a unique SGP implementable outcome.

An exactly similar argument eliminates all but three endings in the Cournot example. The three outcomes are implemented by the profiles $\xi^{16/17,i}$ and $\xi^*$ defined on pp. 26-27 above. For each $i$, $\xi^{16/17,i}$ implements the outcome: player $i$ plays his Stackleberg quantity at $16/17$, and $j$ continues. The third equilibrium is analogous to the symmetric equilibrium for the duel: each player is the Stackleberg leader with probability ½ at $16/17$; the players move simultaneously with probability zero. To see that this is indeed the outcome implemented by $\xi^*$, first note that as $t \downarrow 16/17$, the probability that agent $i$ terminates at $t$ must converge to zero. This is true because immediately after $16/17$, an agent is virtually indifferent between being the Stackleberg leader and being the follower; by continuity, therefore, if one player were terminating with probability bounded away from zero in a neighborhood of $t$, the other would strictly prefer to continue at this time. Moreover, by a similar argument--see expression (7.2) in the proof of Proposition VI for details--there must be an interval immediately after $\bar{t}$ on which agents are assigning probability zero to any quantity *other* than $q^{Sik} = ½$. It now follows immediately that the limit outcome is as claimed above.

We can now explain our earlier remark that computing the equilibrium strategies explicitly is often unnecessary. The argument we have just given will apply to a large class of applications, provided that the payoff to leading is continuous in, and increases with, $t$. As in the preceding example, the game will end at the moment $\bar{t}$ that the payoff to leading crosses the payoff to following. Moreover, in the PTNE's immediately after $\bar{t}$, agent $i$ will be terminating with vanishingly small probability, and assigning positive probability only to actions that are optimal given that other players are continuing (i.e., only to elements of the set $A_i^{lead}(\cdot)$). The limit outcome at $\bar{t}$ can therefore be computed from these general principles, without needing further information about the specific solutions to the PTNE's.

*An Asymmetric Game with no SGP Equilibrium.*

Except in the $2 \times 2$ case, it is difficult to guarantee existence in an asymmetric game. The reason is that the following property plays a key role in the existence proof:

if at least two agents strictly prefer leading to following at $t$, then a PTNE exists for $SurrG(t)$. (5.2)

This property is satisfied both by symmetric and by $2 \times 2$ games. In asymmetric games, it is very difficult to guarantee. The two-person BTG below illustrates what can happen when the property fails. Let $T = [0, 1]$.

| A BTG with No Equilibrium: $u(a, t)$ | | | |
|---|---|---|---|
| | $Ct$ | $Tm^1$ | $Tm^2$ |
| $Ct$ | (0, 0) | (0, 1 + 3$t$) | (10, −1 + 3$t$) |
| $Tm^1$ | (−2 + 3$t$, 0) | (−2, −3) | (−2, −1) |

In this game, $L_{ROW} = 3t - 2$, $L_{COL} = 1 + 3t$ and $F_{ROW} = F_{COL} = 0$. For $s > 2/3$, therefore, both agents strictly prefer leading to following. We will show that no PTNE exists for $SurrG(s)$, so that property

(5.2) fails. To see this, note that if one player terminates with probability one at $s$, the other strictly prefers to continue. If a PTNE exists, therefore, both players must be properly randomizing at $s$. If *COL* is properly randomizing, but puts no mass on $Tm^1$, then *ROW* earns 10 if he continues, and so strictly prefers to do so. Therefore, to induce *ROW* to randomize, *COL* must assign positive probability both to $Ct$ and to $Tm^1$. However, if *ROW* terminates with probability $\alpha$, COL's best responses are $\begin{cases} Tm^1 & \text{if } \alpha \le 1/3 \\ Tm^2 & \text{if } \alpha \in [1/3, 3t/(1+3t)] \\ Ct & \text{if } \alpha \ge 3t/(1+3t) \end{cases}$. Since $s > 2/3$, there exists no $\alpha$ such that player *COL* is indifferent between $Ct$ and $Tm^1$. This establishes that the surrogate game at $s$ has no PTNE.

To verify that this game has no SGP equilibrium, we need only check the necessary conditions identified by Theorem VIII. No type 2 outcome exists, because if *COL* terminates with probability one at $t$, *ROW* prefers continuing to terminating. No PTNE exists before $t < 2/3$, since following strictly dominates leading for *ROW*. Finally, we have checked that no PTNE exists after $2/3$. Therefore, no type 1 outcome exists.

## VI. Continuous- and Discrete-time Games.

A natural conjecture is that the equilibria of a continuous-time BTG will be close to approximate equilibria of nearby discrete-time games.[33] In SS1 we show that under very weak conditions, this conjecture is true when agents are constrained to play pure strategies.[34] This section demonstrates that the relationship no longer holds when behavior strategies are introduced. We present an example of a continuous-time game with two SGP-implementable outcomes. The first involves randomization; the second is a pure-strategy equilibrium. When the game is restricted to an arbitrarily fine discrete-time grid, nothing close to the former equilibrium can be implemented even as an approximate equilibrium.

[33] See Fudenberg-Levine [3, 4] for a related discussion of the relationship between the equilibria of infinite-horizon games and the approximate equilibria of nearby finite-horizon games.

[34] See also Fudenberg-Levine [3] for a discussion and examples of the relationship between discrete- and continuous-time games.

This result is at first sight surprising, for the following reason. If a piecewise-continuous strategy profile for a continuous-time BTG is restricted to a sufficiently fine grid, the graphs of the original function and its restriction will be almost identical. One would expect, therefore, that the two profiles would have very similar properties in their respective games. This intuition turns out to be valid only if agents are playing pure strategies. The reason is that when agents are randomizing, the *outcomes* implemented by the continuous time profile from certain subgames may differ significantly from the corresponding discrete-time outcomes. To see this, suppose that a master plan $\xi$ has exactly one discontinuity at, say, $\bar{t}$ and that agents are randomizing immediately before $\bar{t}$. From expression (3.3), we know that for every $t < \bar{t}$ the continuous-time outcome implemented by $\xi$ from $t$ is *completely determined* by the values of $\xi$ *before* $\bar{t}$. On the other hand, for *every* discrete-time grid, $R$, there will be grid points before $\bar{t}$ from which the discrete-time outcomes are partially determined by values of $\xi$ *after* $\bar{t}$. Obviously, these outcomes may differ significantly. For example, consider a 2 × 2 game in which each agent terminates with probability ½ at every $t < \bar{t}$, and continues with probability one thereafter. Fix an arbitrary grid and let $r$ be the last grid-point before ½. The continuous-time outcome from $r$ is that the game ends with probability one at $r$. In discrete time, the game ends with probability ¾ at $r$; with the remaining probability, players continue to the end.

If agents play pure strategies, this discrepancy between the discrete- and continuous-time outcome functions does not arise. To see this, suppose that $\xi$ is a pure-strategy profile with exactly one discontinuity point, at $\bar{t}$. (Since the range of $\xi$ is finite, the image of $\xi$ must be two points.) Let $R$ be a sufficiently fine grid that the graphs of $\xi$ and $\xi_{|R}$ are similar. In this case, the continuous- and the discrete-time outcomes are *both* completely determined by one or other of the values of $\xi$. Specifically, there are now only two possibilities. If some agent is terminating before $\bar{t}$, then for every $r < \bar{t}$, both the discrete- and continuous-time games will end at $r$. Alternatively, if all agents continue until $\bar{t}$ and then some agent terminates, the continuous-time game will end at $\bar{t}$, and the discrete-time one soon after.

The game is constructed in the following way. We begin with a two-player game. Each player has two termination options ($Tm^1$ and $Tm^2$). Both the discrete- and continuous-time versions of this game have a "grab-the-dollar-type" equilibrium (see section II) and a second, pure-strategy equilibrium. To implement the former, both agents must randomize at every point in time. We then augment the game by adding a third and fourth player. The new players have one termination option each ($Tm$) and their payoffs are independent of the original players' actions. In any approximate equilibrium of the augmented game, each of the new players continues with probability close to one at the beginning of the game, and terminates with probability close to one beyond time ½. The continuous-time augmented game has the same set of equilibria. This is not the case in discrete-time: the "grab-the-dollar-type" equilibrium completely disappears. The reason is that at the last grid point before the new players switch from "continue" to "terminate," the continuous- and discrete-time outcomes differ significantly and it is no longer even approximately optimal for the first two agents to randomize at this time. Backward induction now destroys every approximation to the original equilibrium.

Our augmented game is in no way pathological (the valuation function is linear w.r.t. time). Moreover, the discrepancy between continuous- and discrete-time survives the introduction of an arbitrarily large discount factor. The example thus highlights a serious difficulty with the discrete-time framework. A basic intuition is that an event occurring far out in the future will have a minimal effect in the present, if agents are sufficiently impatient. Applying this intuition to our example, we would expect that the set of *approximate* equilibria would be affected only slightly by the addition of players who are inactive at the beginning of the game. As we have seen, our continuous-time model supports this intuition, while the conventional discrete-time model does not.

We now specify the example. In the original two-person game, payoffs are independent of the time at which the game ends. The valuation function, $\bar{v}$, is defined as follows:

| The 2-person game: $\left(\bar{v}_1((a_1, a_2), t), \bar{v}_2((a_1, a_2), t)\right)$ | | | |
|---|---|---|---|
| | $Ct$ | $Tm^1$ | $Tm^2$ |
| $Ct$ | (0, 0) | (0, 1) | (−9, 3) |
| $Tm^1$ | (1, 0) | (−3, −3) | (−9, −9) |
| $Tm^2$ | (3, −9) | (−9, −9) | (1, 1) |

The following outcome is an equilibrium for the continuous-time version of this game: the game ends with probability one at time zero; player $i$ terminates alone with probability $^3\!/\!_8$; with probability $^1\!/\!_4$, both players terminate simultaneously. Strategies that implement this equilibrium are: at every $t$, each player plays $Tm^1$ with probability ¼ and continues with the remaining probability. Expected payoffs are (0, 0). The restriction of these strategies to any grid form an equilibrium for the corresponding discrete-time game. In the second equilibrium, each player plays $Tm^2$ with probability one at every point in time.

We now augment this game by adding a third and a fourth player. The valuation function for the augmented game is $\bar{u}$, defined as follows:

$$u_1(a, t) \;=\; u_2(a, t) \;=\; \begin{cases} 1 & \text{if } a_3 = a_4 = Tm \\ \bar{v}_1((a_1, a_2), t) & \text{otherwise} \end{cases}$$

$$u_3(a, t) \;=\; u_4(a, t) \;=\; \begin{cases} 1 & \text{if } a_3 = a_4 = Ct \\ 2t & \text{if } a_3 = a_4 = Tm \\ -9 & \text{otherwise} \end{cases}$$

In continuous time, both of the outcomes described above are equilibria for the augmented game. On the other hand, we will show that in any approximate equilibrium for the discrete-time version of the game, agents #1 and #2 must each be playing $Tm^2$ with probability close to one. To see this, fix a discrete-time grid, $R$, and an approximate equilibrium for the game played on this grid. First observe that players #3 and #4 must be playing the same action with probability close to one at every grid point in $R$. Moreover, there must exist $t^* \approx \frac{1}{2}$ such that at every $s < t^*$, #3 and #4 are almost certain to continue, while at $t^*$, they are almost certain to terminate. Let $\lfloor t^* \rfloor^R$ denote the last grid-point in $R$ before $t^*$ and consider the situation facing players #1 and #2 at this time. If they both play continue at this node, then $t^*$ will be reached and they will each earn a payoff of approximately 1. At $\lfloor t^* \rfloor^R$, therefore, $Tm^1$ is strictly dominated by $Ct$ for each player. Moreover, provided that player $j$ assigns probability zero to $Tm^1$, $Tm^2$ is a strictly better response for $i$ than $Ct$. Therefore each $i$ must be playing playing $Tm^2$ with probability close to one at this node. Summarizing, in any approximate equilibrium, the expected payoff vector for the subgame beginning at $\lfloor t^* \rfloor^R$ must be approximately (1, 1).

Now observe that at the grid point before $\lfloor t^* \rfloor^R$, if both agents continue at this time, the game will almost certainly end at the next grid point, and they will earn a payoff vector approximately equal to (1, 1). Thus, the situation at the grid point before $\lfloor t^* \rfloor^R$ is virtually identical at the situation at $\lfloor t^* \rfloor^R$. Now proceed by induction to establish that in any approximate equilibrium for the augmented game, the original players will almost certainly play $Tm^2$ at time zero.

APPENDIX: PROOFS.

The propositions are proved first, then the two theorems.

*Proof of Proposition 1:* Given $s \in [0, 1]$, we will denote by $\lfloor r \rfloor^n$ the largest grid point in $R^n$ strictly before $r$; similarly, $\lceil r \rceil^n$ is the small grid point in $R^n$ strictly after $r$. For convenience, we will assume that $\bar{t} \in \bigcap_n R^n$ and that $\bar{t} < 1$ (if $\bar{t} = 1$, the proof is slightly easier). Fix $\bar{\delta} > 0$ sufficiently small that $L(\cdot)$ is strictly decreasing on $(\bar{t}, \bar{t} + 2\bar{\delta})$. Pick a subsequence, indexed by $n$, such that the (symmetric) outcomes generated by $(\xi^n)$ from the subgame beginning at $\bar{t}$ converge. Let $(\phi^n) = (\phi_1^n, \phi_1^n)$ denote the sequence of payoffs generated by $(\xi^n)$ and let $\bar{\phi} = (\bar{\phi}_1, \bar{\phi}_1)$ be the limit of the $(\phi^n)$'s. Necessarily, $\bar{\phi}_1 \leq ½(L_1(\bar{t}) + F_1(\bar{t})) < L_1(\bar{t})$. Fix $\underline{t} < \bar{t}$ such that on $(\underline{t}, \bar{t})$, (i) $L_1(\cdot) > \bar{\phi}_1$ and (ii) $L_1(\cdot) > F_1(\cdot) > S_1(\cdot)$. Pick $n^0$ sufficiently large that for all $n > n^0$ and all $r \in R^n \cap (\underline{t}, \bar{t})$, $L_1(r) > ½(L_1(\lceil r \rceil^n) + F_1(\lceil r \rceil^n))$. By the obvious argument, agents must be terminating with positive probability at every node in the interval $(\underline{t}, \bar{t})$, for every $n > n^0$. Since $St_1(\cdot) < F_1(\cdot)$ on this interval, they must be terminating with probability less than 1. Pick $n'$ sufficiently large that for $n > n'$, $\phi_1^n < L_1(\lfloor \bar{t} \rfloor^n)$. Since player 1 is indifferent between moving at $\bar{t}$ and at at $\lfloor \bar{t} \rfloor^n$, we have

$$(1 - \xi_1^n(\lfloor \bar{t} \rfloor^n))L_1(\lfloor \bar{t} \rfloor^n) + \xi_1^n(\lfloor \bar{t} \rfloor^n)S_1(\lfloor \bar{t} \rfloor^n) = (1 - \xi_1^n(\lfloor \bar{t} \rfloor^n))\phi_1^n + \xi_1^n(\lfloor \bar{t} \rfloor^n)F_1(\lfloor \bar{t} \rfloor^n).$$

Therefore, there exists $\epsilon' > 0$ such that for all $n > n'$, $\xi_1^n(\lfloor \bar{t} \rfloor^n) > \epsilon'$.

Now fix $r \in (\underline{t}, \bar{t})$ such that $\lceil r \rceil^n \leq \bar{t}$. Also, fix $n > n'$. Once again, agents must be terminating with positive, nonunitary probability, so that

$$(1 - \xi_1^n(r))L_1(r) + \xi_1^n(r)S_1(r) = (1 - \xi_1^n(r))\left[(1 - \xi_1^n(\lceil r \rceil^n))L_1(\lceil r \rceil^n) + \xi_1^n(\lceil r \rceil^n)S_1(\lceil r \rceil^n)\right] + \xi_1^n(r)F_1(r)$$

Rearranging terms,

$$\begin{aligned} &L_1(r) - \left[(1 - \xi_1^n(\lceil r \rceil^n))L_1(\lceil r \rceil^n) + \xi_1^n(\lceil r \rceil^n)S_1(\lceil r \rceil^n)\right] \\ &= \xi_1^n(r)\left\{L_1(r) - S_1(r) + F_1(r) - \left[(1 - \xi_1^n(\lceil r \rceil^n))L_1(\lceil r \rceil^n) + \xi_1^n(\lceil r \rceil^n)S_1(\lceil r \rceil^n)\right]\right\} \end{aligned}$$

Therefore,

$$(L_1(r) - L_1(\lceil r \rceil^n)) + \xi_1^n(\lceil r \rceil^n)(L_1(\lceil r \rceil^n) - S_1(\lceil r \rceil^n))$$

$$= \xi_1^n(r)\Big[(L_1(r) - L_1(\lceil r \rceil^n)) + (F_1(r) - S_1(r)) + \xi_1^n(\lceil r \rceil^n)(L_1(\lceil r \rceil^n) - S_1(\lceil r \rceil^n))\Big]$$

Define $\delta = (L_1(r) - L_1(\lceil r \rceil^n))$, $\beta = (F_1(r) - S_1(r))$ and $\gamma = (L_1(\lceil r \rceil^n) - S_1(\lceil r \rceil^n))$. We have $\xi_1^n(r) = \dfrac{\delta + \xi_1^n(\lceil r \rceil^n)\gamma}{\delta + \beta + \xi_1^n(\lceil r \rceil^n)\gamma}$ so that

$$\xi_1^n(r) - \xi_1^n(\lceil r \rceil^n) = \frac{\delta(1 - \xi_1^n(\lceil r \rceil^n)) + \xi_1^n(\lceil r \rceil^n)\Big[\gamma(1 - \xi_1^n(\lceil r \rceil^n)) - \beta\Big]}{\delta + \beta + \xi_1^n(\lceil r \rceil^n)\gamma}$$

By assumption, $\beta < \gamma$. Let $\epsilon'' = \dfrac{\gamma - \beta}{2\gamma}$. We can pick $\bar{n} > n'$ such that for $n > \bar{n}$, $\delta$ will be sufficiently small that whenever $\xi_1^n(r) < \epsilon''$, $\xi_1^n(r) - \xi_1^n(\lceil r \rceil^n) > 0$. Let $\bar{\epsilon} = \epsilon' \wedge \epsilon''$. Since $\xi_1^n(\lfloor \bar{t} \rfloor^n) > \bar{\epsilon}$, we have established that for $n > \bar{n}$, player #2 must terminate with probability at least $\bar{\epsilon}$ at every grid point in $R^n \cap (\underline{t}, \bar{t})$. □

*Proof of Proposition III:* We begin with some definitions. The expressions defined below will also be used in subsequent proofs. Given a random action $\sigma_i$ for $i$, let $\psi(\sigma_i)$ denote that conditional distribution over $A_i - \{a_i^0\}$ induced by $\sigma_i$, given that $i$ chooses *not* to play $a_i^0$. That is, for $a_i \neq a_i^0$, $\psi(\sigma_i)(a_i) = \dfrac{\sigma_i(a_i)}{1 - \sigma_i(a_i^0)}$. Denote by $\langle c_i, \psi(\sigma_i) \rangle$ the mixed strategy $\sigma_i'$ defined by:

$$\sigma_i'(a_i) = \begin{cases} c_i & \text{if } a_i = a_i^0 \\ (1 - c_i)\sigma_i(a_i) & \text{otherwise} \end{cases}.$$

Now, fix a BTG, $\Gamma = (T, I, A, u)$ and $\bar{t} < t^{end}$ and consider the surrogate game $SurrG(\bar{t})$. Fix $\sigma \in \Sigma$ such that $\sigma_i(a_i^0) < 1$ and $\sigma_{-i}(a_{-i}^0) < 1$. Define the following terms:

$$
\begin{aligned}
Ld_i(\sigma) &= \sum_{a_i \neq a_i^0} u_i((a_i, a_{-i}^0), \bar{t}) \frac{\sigma_i(a_i)}{1 - \sigma_i(a_i^0)} \\
Fl_i(\sigma) &= \sum_{a_{-i} \neq a_{-i}^0} u_i((a_i^0, a_{-i}), \bar{t}) \frac{\sigma_{-i}(a_{-i})}{1 - \sigma_{-i}(a_{-i}^0)} \\
St_i(\sigma) &= \sum_{a_i \neq a_i^0} \Big[ \sum_{a_{-i} \neq a_{-i}^0} u_i((a_i, a_{-i}), \bar{t}) \frac{\sigma_{-i}(a_{-i})}{1 - \sigma_{-i}(a_{-i}^0)} \Big] \frac{\sigma_i(a_i)}{1 - \sigma_i(a_i^0)} \\
Mv_i(\sigma) &= \sigma_{-i}(a_{-i}^0) Ld_i(\sigma) + (1 - \sigma_{-i}(a_{-i}^0)) St_i(\sigma)
\end{aligned}
$$

Now fix $\bar{\sigma}_{-i} \in \Sigma_{-i}$ such that $\bar{\sigma}_{-i}(a_{-i}^0) < 1$ and define: $Term_i(\bar{\sigma}_{-i}) = \max_{a_i \in A_i} Mv_i(\delta_{a_i}, \bar{\sigma}_{-i})$ and $Cont_i(\bar{\sigma}_{-i}) = Fl_i(\delta_{a_i^0}, \bar{\sigma}_{-i})$. $Term_i(\bar{\sigma}_{-i})$ is the highest payoff $i$ can get by terminating, given that other agents are playing $\bar{\sigma}_{-i}$. Note that in particular,

$$Term_i(\bar{\sigma}_{-i}) = \max_{a_i \neq a_i^0} Q_i^{\bar{t}}(\delta_{a_i}, \sigma_{-i}). \tag{7.3}$$

$Cont_i$ is $i$'s payoff if he continues. The proof of Proposition III follows from the following three Lemmas.

*Lemma III.1.* If $\sigma_i \in \operatorname{argmax} Q_i^{\bar{t}}(\cdot, \bar{\sigma}_{-i})$ and $\sigma_i(a_i^0) \in (0, 1)$, then for all $c_i \in [0, 1]$, $\langle c_i, \psi(\sigma_i)\rangle \in \operatorname{argmax} Q_i^{\bar{t}}(\cdot, \bar{\sigma}_{-i})$.

*Proof of Lemma III.1:* Suppose that $\bar{\sigma}_i \in \operatorname{argmax} Q_i^{\bar{t}}(\cdot, \bar{\sigma}_{-i})$ and $\bar{\sigma}_i(a_i^0) \in (0, 1)$. Let $\bar{\sigma} = (\bar{\sigma}_i, \bar{\sigma}_{-i})$. Rearranging terms, we have

$$
\begin{aligned}
Q_i^{\bar{t}}(\bar{\sigma}) &= \frac{1}{1 - \bar{\sigma}(a^0)} \Big( \bar{\sigma}_i(a_i^0)(1 - \bar{\sigma}_{-i}(a_{-i}^0)) Fl_i(\bar{\sigma}) + (1 - \bar{\sigma}_i(a_i^0)) Mv_i(\bar{\sigma}) \Big) \\
&= \lambda(\bar{\sigma}) Fl_i(\bar{\sigma}) + (1 - \lambda(\bar{\sigma})) Mv_i(\bar{\sigma})
\end{aligned}
$$

where $\lambda(\bar{\sigma}) = \dfrac{\bar{\sigma}_i(a_i^0)(1 - \bar{\sigma}_{-i}(a_{-i}^0))}{1 - \bar{\sigma}(a^0)}$. That is, $Q_i^{\bar{t}}(\bar{\sigma})$ is a convex combination of $Fl_i(\bar{\sigma})$ and $Mv_i(\bar{\sigma})$. Therefore, $\sigma_i \in \operatorname{argmax} Q_i^{\bar{t}}(\sigma_i)$ implies $Fl_i(\sigma) = Mv_i(\sigma)$ so that any convex combination of these two terms yields the same payoff.

*Lemma III.2.* If $\sigma_i \in \operatorname{argmax} Q_i^{\bar{t}}(\cdot, \bar{\sigma}_{-i})$, then

(i) $Q_i^{\bar{t}}(\sigma_i, \bar{\sigma}_{-i}) \geq Cont(\sigma_{-i})$, with equality holding if $\sigma_i(a_i^0) > 0$;

(ii) $Q_i^{\bar{t}}(\sigma_i, \bar{\sigma}_{-i}) \geq Term(\bar{\sigma}_{-i}) \geq Mv_i(\sigma_i, \bar{\sigma}_{-i})$; if $\sigma_i(a_i^0) < 1$, then $Q_i^{\bar{t}}(\sigma_i, \bar{\sigma}_{-i}) = Mv_i(\sigma_i, \bar{\sigma}_{-i})$.

(iii) for each $a_i$ such that $\sigma_i(a_i) > 0$, $\delta_{a_i} \in \operatorname{argmax}_{\sigma_i \in \Sigma_i} Q_i^{\bar{t}}(\cdot, \bar{\sigma}_{-i})$.

*Proof of Lemma III.2:* Fix $\bar{\sigma}_i \in \operatorname{argmax} Q_i^{\bar{t}}(\cdot, \bar{\sigma}_{-i})$. Let $\bar{\sigma} = (\bar{\sigma}_i, \bar{\sigma}_{-i})$. The inequalities in part (i) and (ii) are obvious. The equalities follow from the fact that $Q_i^{\bar{t}}(\bar{\sigma})$ is a convex combination of $Fl_i(\bar{\sigma})$ and $Mv_i(\bar{\sigma})$. For $a_i = a_i^0$, part (iii) follows from part (i), since $Q_i^{\bar{t}}(\bar{\sigma}_i, \bar{\sigma}_{-i}) = Fl_i(\bar{\sigma}_i, \bar{\sigma}_{-i})$. Now assume that $\bar{\sigma}_i(a_i) > 0$, for some $a_i \neq a_i^0$. We have

$$Q_i^{\bar{t}}(\bar{\sigma}) = Q_i^{\bar{t}}(\langle 0, \psi(\bar{\sigma}_i)\rangle, \bar{\sigma}_{-i}) = \sum_{a_i \neq a_i^0} Q_i^{\bar{t}}(\delta_{a_i}, \bar{\sigma}_{-i})\psi(\bar{\sigma}_i)(a_i).$$

The first equality follows from Lemma III.1 the second from the definitions of $Q_i^{\bar{t}}$ and $\langle 0, \psi(\bar{\sigma}_i)\rangle$. Part (iii) now follows immediately from the fact that $\bar{\sigma}_i \in \operatorname{argmax} Q_i^{\bar{t}}(\cdot, \bar{\sigma}_{-i})$ and $Q_i^{\bar{t}}(\bar{\sigma}_i, \bar{\sigma}_{-i})$ is a convex combination of the $Q_i^{\bar{t}}(\delta_{a_i}, \bar{\sigma}_{-i})$'s.

*Lemma III.3:* For all $\bar{\sigma}_{-i}$ such that $\bar{\sigma}_{-i}(a_{-i}^0) < 1$, the set $\operatorname{argmax}_{\sigma_i \in \Sigma_i} Q_i^{\bar{t}}(\sigma_i, \bar{\sigma}_{-i})$ is nonempty and convex.

*Proof of Lemma III.3:* If $\bar{\sigma}_{-i}(a_{-i}^0) < 1$, then $v(\cdot, \bar{\sigma}_{-i})$ is continuous w.r.t. $\sigma_i$. Therefore $Q_i^{\bar{t}}(\cdot, \bar{\sigma}_{-i})$ attains a maximum. Now assume that $\sigma_i' \neq \sigma_i''$, but that both maximize $Q_i^{\bar{t}}(\cdot, \bar{\sigma}_{-i})$. Assume w.l.o.g. that $\sigma_i'(a_i^0) \leq \sigma_i''(a_i^0)$ and that $\sigma_i'(a_i^0) < 1$. Fix $\alpha \in (0, 1)$ and let $\bar{\sigma}_i = \alpha\sigma_i' + (1 - \alpha)\sigma_i''$. There are several cases to consider.

(1) If $\sigma_i'(a_i^0) = \sigma_i''(a_i^0) = 0$, then

$$Q_i^{\bar{t}}(\sigma_i', \bar{\sigma}_{-i}) = Mv_i(\sigma_i', \bar{\sigma}_{-i}) = Term_i(\bar{\sigma}_{-i}) = Mv_i(\sigma_i'', \bar{\sigma}_{-i}) = Q_i^{\bar{t}}(\sigma_i'', \bar{\sigma}_{-i}).$$

But $Mv_i(\cdot, \bar{\sigma}_{-i})$ is linear, and hence convex, in $\sigma_i$, so that $\bar{\sigma}_i$ yields the same payoff.

(2) If $\sigma_i''(a_i^0) = 1$, then $\bar{\sigma} = \langle \alpha\sigma_i'(a_i^0), \psi(\sigma_i)\rangle$ and the Lemma follows from Lemma III.1.

(3) If $\sigma_i'(a_i^0) > 0$, while $\sigma_i''(a_i^0) < 1$, then by Lemma III.2(i) and (ii), $Cont_i(\bar{\sigma}_{-i}) = Term_i(\bar{\sigma}_{-i})$. Also, by Lemma III.1 and Lemma III.2(ii), $Q_i^{\bar{t}}(\langle 0, \psi(\sigma_i')\rangle, \bar{\sigma}_{-i}) = Q_i^{\bar{t}}(\langle 0, \psi(\sigma_i'')\rangle, \bar{\sigma}_{-i}) = Term_i(\bar{\sigma}_{-i})$. Therefore $Q_i^{\bar{t}}(\langle 0, \psi(\bar{\sigma}_i)\rangle, \bar{\sigma}_{-i}) = Term_i(\bar{\sigma}_{-i})$. But $Q_i^{\bar{t}}(\bar{\sigma}_i, \bar{\sigma}_{-i})$ is a convex combination of $Cont_i(\bar{\sigma}_{-i})$ and $Term_i(\bar{\sigma}_{-i})$, so that $\bar{\sigma} \in \operatorname{argmax} Q_i^{\bar{t}}(\cdot, \bar{\sigma}_{-i})$.

We can now prove Proposition III. Since Lemma III.2(iii) establishes that the condition is necessary, we need only check that it is sufficient. By Lemma III.2(iii), if $\hat{\sigma}_i \in \operatorname{argmax}_{\sigma_i \in \Sigma_i} Q_i^{\bar{t}}(\sigma_i, \bar{\sigma}_{-i})$ and $a_i \in \operatorname{supp}(\hat{\sigma}_i)$, then $\delta_{a_i} \in \operatorname{argmax}_{\sigma_i \in \Sigma_i} Q_i^{\bar{t}}(\sigma_i, \bar{\sigma}_{-i})$. Therefore, $\hat{a}_i \in \operatorname{argmax}_{a_i \in A_i} Q_i^{\bar{t}}(\delta_{a_i}, \bar{\sigma}_{-i})$ implies $\delta_{\hat{a}_i} \in \operatorname{argmax}_{\sigma_i \in \Sigma_i} Q_i^{\bar{t}}(\sigma_i, \bar{\sigma}_{-i})$. Fix $\bar{\sigma}_i$ such that $\operatorname{supp}(\bar{\sigma}_i) \subset \operatorname{argmax}_{a_i \in A_i} Q_i^{\bar{t}}(\delta_{a_i}, \bar{\sigma}_{-i})$. From the previous observation, $\bar{\sigma}_i$ is a convex combination of $\{\delta_{a_i} \in \operatorname{argmax}_{\sigma_i \in \Sigma_i} Q_i^{\bar{t}}(\sigma_i, \bar{\sigma}_{-i})\}$. By Lemma III.3, therefore, $\bar{\sigma}_i \in \operatorname{argmax}_{\sigma_i \in \Sigma_i} Q_i^{\bar{t}}(\sigma_i, \bar{\sigma}_{-i})$. □

*Proof of Proposition V:* Fix $\bar{t} < t^{end}$ and assume that $L_1(\bar{t}) > F_1(\bar{t})$. We will use extensively the notation and Lemmas from the proof of Proposition III. First note that there exists $\bar{\delta} > 0$ sufficiently small that whenever $1 > \sigma_{-1}(a_{-1}^0) > 1 - 2\bar{\delta}$,

$$\text{if } \sigma_1 \text{ is a best response to } \sigma_{-1} \text{ then } \operatorname{supp}(\sigma_1) \subset A_1^{lead}(\bar{t}) \cup \{a_i^0\}; \tag{7.4}$$

$$\text{if } \operatorname{supp}(\sigma_{-1}) \subset (A_1^{lead}(\bar{t}) \cup \{a_1^0\})^{\#I-1} \text{ then } Term_1(\sigma_{-1}) > Cont_1(\sigma_{-1}) \tag{7.5}$$

(7.4) states that if other agents terminate with sufficiently low probability, then it cannot be a best response for #1 to play some termination action that is *not* an element of $A_1^{lead}(\bar{t})$. (7.5) states that if other agents terminate with sufficiently low probability *and* assign zero probability to nonoptimal termination actions, then #1 will strictly prefer moving to not moving. (7.4) follows the definition of $A_i^{lead}$ and continuity. (7.5) holds because for $\delta$ sufficiently small $Term_1(\sigma_{-1}) \approx L_1(\bar{t}) > F_1(\bar{t}) = \max_{a_2 \in A_2^{lead}(t)} u_1((a_2, a_{-2}^0), \bar{t}) \geq Cont_1(\sigma_{-1})$.

Define $\Sigma_1^{\bar\delta}$ by $\Sigma_1^{\bar\delta} = \{\sigma_1 \in \Sigma_1 : \sigma_1(a_1^0) \le 1 - \bar\delta\}$. Define the correspondence $BR_1^{\bar\delta} : \Sigma_1^{\bar\delta} \twoheadrightarrow \Sigma_1^{\bar\delta}$ by: $BR_1^{\bar\delta}(\sigma_1) = \operatorname{argmax}_{\sigma_1 \in \Sigma_1^{\bar\delta}} Q_1^{\bar t}(\cdot, (\sigma_1, \dots, \sigma_1))$. By modifying Lemma III.3 in the obvious way, it can be shown that $BR_1$ is convex-valued. Also, $Q_1^{\bar t}$ is continuous on $\Sigma_1^{\bar\delta}$. Therefore, by standard arguments, $BR_1^{\bar\delta}$ is u.h.c. and there exists $\bar\sigma_1$ such that $\bar\sigma_1 \in BR_1^{\bar\delta}(\bar\sigma_1)$. Let $\bar\sigma = (\bar\sigma_1, \dots, \bar\sigma_1)$. We will show that $\bar\sigma_1 \in \operatorname{argmax}_{\sigma_1 \in \Sigma_1} Q_1^{\bar t}(\cdot, \bar\sigma_{-1})$. We first show that

$$\bar\sigma_1(a_1^0) \le 1 - 2\bar\delta \tag{7.7}$$

Consider $\hat\sigma_1$ such that $\hat\sigma_1(a_1^0) > 1 - 2\bar\delta$. Let $\hat\sigma_{-1} = (\hat\sigma_1, \dots, \hat\sigma_1)$. By (7.4), $\mathrm{supp}(\hat\sigma_1) \subset A_1^{lead}(\bar t) \cup \{a_i^0\}$; Therefore,

$$Q_1^{\bar t}(\hat\sigma_1, \hat\sigma_{-1}) \ \ge\ \max_{a_1 \ne a_1^0} Q_1^{\bar t}(\delta_{a_1}, \sigma_{-i}) \ =\ Term_1(\hat\sigma_{-1}) \ >\ Cont_1(\hat\sigma_{-1}) \ =\ Q_1^{\bar t}(\delta_{a_1^0}, \hat\sigma_{-1})$$

The first inequality holds because for all $a_i \ne a_i^0$, $\delta_{a_i} \in \Sigma_1^{\bar\delta}$; the first equality from the definition of $Term_i$ (i.e., (7.3)); the next inequality from (7.5). But in this case, #1 should terminate with probability one, since $Q_1^{\bar t}(\langle 0, \psi(\hat\sigma_1)\rangle, \hat\sigma_{-1}) > Q_1^{\bar t}(\hat\sigma_1, \hat\sigma_{-1})$. This contradicts the assumption that $\hat\sigma_1 \in BR_1(\hat\sigma_1)$ and establishes that (7.7) must hold. It follows immediately from (7.7) that $Term_1(\bar\sigma_{-1}) \ge Cont_1(\bar\sigma_{-1})$. (Otherwise, $Q_1^{\bar t}(\langle 1 - \bar\delta, \psi(\bar\sigma_1)\rangle, \hat\sigma_{-1}) > Q_1^{\bar t}(\bar\sigma_1, \hat\sigma_{-1})$). Therefore, $\bar\sigma_1$ is indeed a best reply to $\bar\sigma_{-1}$.

To complete the proof that $\bar\sigma$ is a Nash equilibrium for $SurrG(\bar t)$, we need to show that for $i \ne 1$, $\bar\sigma_i$ is a best reply to $\bar\sigma_{-i}$. It follows from Proposition III that for each $a_1 \in \mathrm{supp}(\bar\sigma_1)$ and $a_1 \in A_1$, $Q_i^{\bar t}(\delta_{a_1}, \bar\sigma_{-1}) \ge Q_i^{\bar t}(\delta_{a_1}, \bar\sigma_{-1})$. Also, using the definition of symmetry, we have that for $\alpha \in A_i$:

$$\begin{aligned} Q_i^{\bar t}(\delta_\alpha, \bar\sigma_{-i}) &= \sum_{\alpha' \in A_1} \Big[ \sum_{a_{-1,i} \in A_{-1,i}} v_i(\alpha, \alpha', a_{-1,i}) \bar\sigma_{-i,1}(a_{-1,i}) \Big] \bar\sigma_1(\alpha') \\ &= \sum_{\alpha' \in A_1} \Big[ \sum_{a_{-1,i} \in A_{-1,i}} v_1(\alpha, \alpha', a_{-1,i}) \bar\sigma_{-i,1}(a_{-1,i}) \Big] \bar\sigma_1(\alpha') \ =\ Q_1^{\bar t}(\delta_\alpha, \bar\sigma_{-1}) \end{aligned}$$

Therefore, for $a_i \in \mathrm{supp}(\bar\sigma_i)$ and $a_i \in A_i$, $Q_i^{\bar t}(\delta_{a_i}, \bar\sigma_{-1}) \ge Q_i^{\bar t}(\delta_{a_i}, \bar\sigma_{-i})$. This verifies that $\bar\sigma_i$ is a best reply to $\bar\sigma_{-i}$. □

Before proving Proposition VI, we prove the following Lemma, which establishes the relationship between the Nash equilibria of subgames of a BTG and the solutions to the corresponding stage-games. It will be used in the proof of several of the following propositions. Define a random action profile $\sigma$ to be a strongly plural termination profile (SPTP) if for each $i$, there exists $j \neq i$ such that $\sigma_j(a_j^0) = 0$.

**Lemma VI.1:** Fix $t \in T$ and a profile $\xi$ for $\Gamma$.

(i) If $\xi$ is a Nash equilibrium for $\Gamma$ from $t$, then $\xi(t)$ solves $StgG(t, \xi)$.

(ii) If $\xi(t)$ solves $StgG(t, \xi)$ and there exists $\delta > 0$ such that for every $s \in (t, t+\delta)$, $\xi$ is a Nash equilibrium for the subgame of $\Gamma$ beginning at $s$, then $\xi$ is a Nash equilibrium for the subgame of $\Gamma$ beginning at $t$.

(iii) If $\xi(t)$ solves $StgG(t, \xi)$ *and* is an SPTP, then $\xi$ is a Nash equilibrium for the subgame of $\Gamma$ beginning at $t$.

*Proof of Lemma VI.1:* We first establish a preliminary fact. Fix $\xi \in \Xi$ and $t \in T$. If $o^T(\xi, t) = t$ then

$$P(\xi, t+) = Eu(\nu^{t+}(\xi), t). \tag{7.8}$$

To see that (7.8) is true, observe that we can pick $\delta > 0$ sufficiently small that for all $s \in (\bar{t}, \bar{t} + \delta)$: (i) if $\bar{\xi}(\bar{t})(a^0) > 0$, then by (X3), $\bar{\xi}(s)(a^0) < 1$, so that $o^T(\bar{\xi}, \bar{t} + \delta) = \bar{t} + \delta$; (ii) by (X1), $\bar{\xi}(\cdot)$ is continuous at $\bar{t} + \delta$, so that by (3.3), $o^{\Sigma}(\bar{\xi}, t + \delta) = \nu(\bar{\xi}(\bar{t} + \delta))$; Therefore, $P_i(\bar{\xi}, \bar{t} + \delta) = Eu_i(\nu(\bar{\xi}(\bar{t} + \delta)), \bar{t} + \delta)$. Finally, since $u$ is right-continuous (u1), $P_i(\bar{\xi}, \bar{t}+) = Eu_i(\nu^{\bar{t}+}(\bar{\xi}), \bar{t})$. This concludes the verification of (7.8).

We now prove part (i). Assume that $\bar{\xi} \in \Xi$ is a Nash equilibrium for the subgame of $\Gamma$ beginning at $\bar{t}$. Fix $i$ and $a_i \in A_i$. We need to establish that

$$P^{\bar{t},\xi}(\delta_{a_i}, \xi_{-i}(\bar{t})) \leq P^{\bar{t},\xi}(\xi(\bar{t})) \tag{7.9}$$

Let $\bar{\tau}_{-i} = \inf\{s > \bar{t} : \xi_{-i}(s)(a_{-i}^0) < 1\}$. First assume that $\bar{\tau}_{-i} > \bar{t}$. In this case,

$$P_i(\bar{\xi}, \bar{t}) \geq L_i(\bar{t}) \vee P_i(\bar{\xi}, \bar{t}+) \tag{7.10}$$

To see this, observe that since no player other than $i$ moves at $\bar{t}$, $i$ can attain the payoff $L_i(t)$, by playing some action from $A_i^{lead}(\bar{t})$. Now suppose that $P_i(\bar{\xi}, \bar{t}+) > L_i(\bar{t})$. Since $L_i$ is right-continuous, it follows that $P_i(\bar{\xi}, \bar{t}+) > L_i(\bar{t}+)$. Since $\bar{\tau}_{-i} > \bar{t}$, it must be that $\bar{\xi}_i(\cdot)(a_i^0) = 1$ on an interval after $\bar{t}$. But then $i$ can attain $P_i(\bar{\xi}, \bar{t}+)$ by playing $a_i^0$ at $\bar{t}$. (7.10) now follows from the fact that $\bar{\xi}$ is a Nash equili-

We now show that (ii) implies (i). Assume that $\bar{\xi}(\bar{t})$ is a Nash equilibrium for $StgG(\bar{t}, \bar{\xi})$ and that for all $s \in (\bar{t}, \bar{t} + \bar{\delta})$, $\xi(s)$ is a Nash equilibrium for $SurrG(s)$. Pick $\bar{s} \in (\bar{t}, \bar{t} + \bar{\delta})$. Since $o^{\Sigma}(\bar{\xi}, \bar{s})v(\bar{\xi}(\bar{s}))$, we have, for all $\sigma_i \in \Sigma_i$,

$$P_i(\bar{\xi}, \bar{s}) = Eu_i(v(\bar{\xi}(\bar{s})), \bar{s}) = Q_i^{\bar{s}}(\bar{\xi}(\bar{s})) \geq Q_i^{\bar{s}}(\sigma_i, \bar{\xi}_{-i}(\bar{s})) = Eu_i(v(\sigma_i, \bar{\xi}_{-i}(\bar{s})), \bar{s}). \qquad (7.16)$$

Pick $\hat{\xi}_i \in \Xi_i$ and let $\hat{\xi} = (\hat{\xi}_i, \bar{\xi}_{-i})$. Let $\sigma_i^0 = \hat{\xi}_i(\bar{s})$ and $\sigma_i^+ = \hat{\xi}_i(\bar{s}+)$. We have

$$\begin{aligned} P_i(\hat{\xi}, \bar{s}) &= (1 - \hat{\xi}(\bar{t})(a^0))Eu_i(v(\hat{\xi}), \bar{t}) + \hat{\xi}(\bar{t})(a^0)Eu_i(v^{\bar{t}+}(\hat{\xi}), \bar{t}) \\ &= (1 - \hat{\xi}(\bar{t})(a^0))Eu_i(v(\sigma_i^0, \bar{\xi}_{-i}(\bar{s})), \bar{t}) + \hat{\xi}(\bar{t})(a^0)Eu_i(v^{\bar{t}+}(\sigma_i^+, \bar{\xi}_{-i}(\bar{s})), \bar{t}) \leq P_i(\bar{\xi}, \bar{s}). \end{aligned}$$

The first equality because $\bar{\xi}(\bar{s})$ is a PTP, so that $o^T(\hat{\xi}, \bar{s}) = \bar{s}$; the second equality is obtained by substituting $\sigma_i^0$ and $\sigma_i^+$ in the appropriate places; the inequality follows because, using (7.16), $P_i(\bar{\xi}, \bar{s})$ is a convex combination of two scalars, and each one is weakly greater than the corresponding term in the definition of $P_i(\hat{\xi}, \bar{s})$. This establishes that for all $s \in (\bar{t}, \bar{t} + \bar{\delta})$, $\bar{\xi}$ is a Nash equilibrium for $\Gamma$ from $s$. Also, since $\bar{\xi}(\bar{t})$ is a Nash equilibrium for $StgG(\bar{t}, \bar{\xi})$, it follows from Lemma VI.1 that $\bar{\xi}$ is a Nash equilibrium for $\Gamma$ from $\bar{t}$. □

*Proof of (4.3):* Fix $\bar{t}$ and $\bar{\xi}$. Assume that $\bar{\xi}(\bar{t}) = \bar{\xi}(\bar{t}+)$ and $\bar{\xi}$ satisfies condition (ii)(b) of Proposition VI. We have

$$\begin{aligned} P^{\bar{t},\bar{\xi}}(\bar{\xi}(\bar{t})) &= \sum_{a' \neq a^0} u(a')\bar{\xi}(\bar{t})(a') + \bar{\xi}(\bar{t})(a^0)(\lim_{\delta \downarrow 0} Eu(v(\bar{\xi}(\bar{t} + \delta)), \bar{t} + \delta)) \\ &= \sum_{a' \neq a^0} u(a')\bar{\xi}(\bar{t})(a') + \bar{\xi}(\bar{t})(a^0)Eu(v(\bar{\xi}(\bar{t})), \bar{t}) \\ &= \frac{1 - \bar{\xi}(\bar{t})(a^0)}{1 - \bar{\xi}(\bar{t})(a^0)} \sum_{a' \neq a^0} u(a')\bar{\xi}(\bar{t})(a') + \frac{\bar{\xi}(\bar{t})(a^0)}{1 - \bar{\xi}(\bar{t})(a^0)} \sum_{a' \neq a^0} u(a')\bar{\xi}(\bar{t})(a') \\ &= \frac{\sum_{a' \neq a^0} u(a')\bar{\xi}(\bar{t})(a')}{1 - \bar{\xi}(\bar{t})(a^0)} = Eu(v(\bar{\xi}(\bar{t})), \bar{t}) \end{aligned}$$

Fix $a_i \in A_i$. Let $\hat{\sigma} = (\delta_{a_i}, \bar{\xi}_{-i}(\bar{t}))$. Since the $\bar{\xi}(s)$'s solve the surrogate games immediately after $t$, and since all variables are continuous, we have $Eu(v(\hat{\sigma}), \bar{t}) \leq Eu(v(\bar{\xi}(\bar{t})), \bar{t})$. Therefore,

$$
\begin{aligned}
P^{\bar{t},\xi}(\delta_{a_i}, \bar{\xi}_{-i}(\bar{t})) &= (1-\hat{\sigma}(a^0))Eu(\nu(\hat{\sigma}),\bar{t}) + \hat{\sigma}(a^0))(\lim_{\delta\downarrow 0} Eu(\nu(\bar{\xi}(\bar{t}+\delta),\bar{t}+\delta)) \\
&\le (1-\hat{\sigma}(a^0))Eu(\nu(\bar{\xi}(\bar{t})),\bar{t}) + \hat{\sigma}(a^0))Eu(\nu(\bar{\xi}(\bar{t})),\bar{t}) \\
&= Eu(\nu(\bar{\xi}(\bar{t})),\bar{t}) = P^{\bar{t},\xi}(\bar{\xi}(\bar{t}))
\end{aligned}
$$

This establishes that $\bar{\xi}(\bar{t})$ solves $StgG(\bar{t}, \bar{\xi})$. □

*Proof of Theorem II:* Fix a game $\Gamma$ that is either a $2 \times 2$ game or a symmetric game. Define $T^* = \{t < t^{end}: \forall i, \exists j \neq i \text{ s.t. } L_j(t) > F_j(t)\}$. We first establish the following Lemma.

**Lemma II.1:** There exists $\xi^* \in \Xi$ such that for all $t \in T^*$,

(i) $\prod_{i \in I} \Big( L_i(t) - Q_i^t(\xi^*(t)) \Big) \geq 0$.

(ii) $\xi^*(t)$ is a PTNE for $SurrG(t)$;

*Proof of Lemma II.1:* We first show that condition (i) can be satisfied. First suppose $\Gamma$ is a $2 \times 2$ game. From Proposition IV, for all $t \in T^*$, $Q_i^t(\xi^*(t)) = Eu_i(\nu(\sigma'), t) = F_i(t)$. If $t \in T^*$, then for all $i$, $L_i(t) > F_i(t)$. Therefore, the condition is certainly satisfied. Next assume that $\Gamma$ is a symmetric game. Clearly, in this case, we can take $\xi^*$ to be symmetric, so that for all $i$ and $j$, $L_i(t) - Q_i^t(\xi^*(t)) = L_j(t) - Q_j^t(\xi^*(t))$. Once again, therefore, condition (i) is satisfied.

We now establish part (ii) of the Lemma. From Propositions IV and V, we know that for every $t \in T^*$, a PTNE for $SurrG(t)$ exists. To obtain $\xi \in \Xi$ satisfying condition (ii) above, we need to be able to extract an extendible piecewise continuous selection from the PTNE correspondence on $T^*$ that satisfies restriction (X2) on strategies. We will prove that we can do this for symmetric games. The argument for $2 \times 2$ games is similar but much simpler.

For convenience, we will assume that the time interval $T$ is bounded. The extension to unbounded time intervals is trivial. We first establish that a piecewise continuous selection can be taken from the symmetric PTNE correspondence on $T^*$. The proof uses a result from the theory of semi-algebraic sets (see Van den Dries [18] for a lucid summary of the relevant facts).[1] A set is semi-algebraic if it is defined by a finite number of polynomial inequalities. An immediate consequence of Hardt's [8] "generic triviality" result for semi-algebraic maps is: if the graph of a correspondence is semi-algebraic, then

[1] I am grateful to Bill Zame for introducing me to this paper and the requisite mathematics.

there exists a finite, semi-algebraic partition of the domain of the correspondence such that the restriction of the correspondence to each set in the partition is a continuous correspondence.

Let $V = (\mathbf{R}^{\#A_1})^{\#I}$ denote the set of symmetric payoff vectors for surrogate games. A point $v \in V$ is a list of payoff vectors, $(v(a))_{a \in A} = ((v_i(a))_{a \in A})_{i \in I}$, where $v(a^0) = 0$. Let $\Phi$ denote the graph of the PTNE correspondence, i.e., $\Phi = \{(v, \sigma) \in V \times \Sigma: \sigma$ is a symmetric PTNE for $v\}$. Thus, $(v, \sigma) \in \Phi$ if (i) $\sigma_i = \sigma_1$, for all $i > 1$; (ii) $\sigma_1(a_1^0) < 1$; (iii) for all $a_1 \neq a_1^0$, $\sigma \circ v_1 \geq (1 - \sigma(a^0))\sigma_{-1} \circ v_1(a_1, \cdot)$; (iv) $(1 - \sigma_{-1}(a_{-1}^0))\sigma \circ v_1 \geq (1 - \sigma(a^0))\sigma_{-1} \circ v_1(a_1^0, \cdot)$. Clearly, $\Phi$ is defined by a finite number of quadratic inequalities and is hence a semi-algebraic set. It follows from the piecewise continuity result cited above that there exists a finite partition, $(V^1, \ldots, V^n, \ldots, V^{\bar{n}})$, of $V$, such that for each $n$, $V^n$ is semialgebraic and the correspondence $\phi: V \to \Sigma$ defined by $\phi(v) = \{\sigma \in \Sigma: (v, \sigma) \in \Phi\}$ is continuous on $V^n$. Now consider the payoff function $u: T \to V$. Since $u$ is piecewise real analytic, there exists a finite partition, $(T^1, \ldots, T^k, \ldots, T^{\bar{k}})$, of $T$ such that for each $k$, there exists $n$ such that $u(\cdot, T^k) \subset V^n$. (To see this, observe that the boundary of $V^n$ is defined by a finite set of polynomial equalities and the image of an analytic function can cross a polynomial only finitely many times.) It follows that there exists a piecewise continuous selection from $\phi_{|u(\cdot, T^*)}$, i.e., a function $\xi^*$ defined on $T^*$ such that for every $t \in T^*$, $\xi^*(t)$ is a symmetric PTNE for $SurrG(t)$.

We now show that $\xi^*$ satisfies condition (X2). We will prove this when there are at least three players. The argument for the two player case is similar, but simpler. Also, we consider only the case in which $A_i^{lead}(\bar{t})$ is a singleton set. The generalization is straightforward but very messy. For every $t \in T^*$, $\xi^*(t)$ is a PTNE, so that by definition, $\xi_i^*(t) < 1$. It follows that $\lim_{\delta \downarrow 0} \xi_1^*(a_1^0, \bar{t} + \delta) \uparrow 1$ only if $\bar{t}$ is a boundary point of $T^*$, so that $\lim_{\delta \downarrow 0} L(\bar{t} + \delta) - F(\bar{t} + \delta) \downarrow 0$. To simplify notation, we will consider the case in which $\frac{d(L(\bar{t}) - F(\bar{t}))}{dt} > 0$. (Since both functions are locally real analytic at $\bar{t}$, there must be some $r$ such that $\frac{d^r(L(\bar{t}) - F(\bar{t}))}{dt^r}$ is positive. If $\frac{d(L(\bar{t}) - F(\bar{t}))}{dt} = 0$, then find the first $r$ such that $\frac{d^r(L(\bar{t}) - F(\bar{t}))}{dt^r} > 0$ and replace the $dt$'s in the argument below with $dt^r$'s.) By (7.4), we can pick $\bar{\delta}$ sufficiently small that for $\bar{t} < t < \bar{t} + \bar{\delta}$, $\xi_1^*(\cdot, t)$ assigns positive probability only to $a_1^0$ and $a_1^{lead}$. Let

$\epsilon(t)$ denote the probability assigned to $a_1^{lead}$. $\epsilon(t)$ has been chosen so that these payoffs to playing these two strategies are equalized. Ignoring terms that are "little oh" of $\epsilon(t)$ and $dt$, and setting $dt = t - \bar{t}$, the payoff to moving at $t$ is approximately

$$\begin{aligned}&(1 - c_1\epsilon(t))u_1((a_1^0, a_2^{lead}, a_{-1,2}^0), t) + c_1\epsilon(t)u_1((a_1^0, a_2^{lead}, a_3^{lead}, a_{-1,2}^0), t)\\ \approx\ &(1 - c_1\epsilon(t))(L(\bar{t}) + \gamma_1 dt) + c_1\epsilon(t)(L(\bar{t}) + \gamma_2 dt + k_2)\end{aligned}$$

where $c_1$, $\gamma_1$, $\gamma_2$ and $k_2$ are constants, possibly zero. On the other hand, the payoff to terminating at $t$ is approximately

$$\begin{aligned}&(1 - c_3\epsilon(t))u_1((a_1^{lead}, a_{-1}^0), t) + c_3\epsilon(t)u_1((a_1^{lead}, a_2^{lead}, a_{-1,2}^0), t)\\ \approx\ &(1 - c_3\epsilon(t))(L(\bar{t}) + \gamma_3 dt) + c_3\epsilon(t)(L(\bar{t}) + \gamma_4 dt + k_4)\end{aligned}$$

where by assumption, $\gamma_3 - \gamma_1 > 0$. Also, since $\epsilon(t)$ is arbitrarily close to zero, $k_2 - k_4$ must be strictly positive. Since player #1 must be indifferent between moving and continuing at $t$, we have

$$\begin{aligned}&(1 - c_1\epsilon(t))(L(\bar{t}) + \gamma_1 dt) + c_1\epsilon(t)(L(\bar{t}) + \gamma_2 dt + k_2)\\ =\ &(1 - c_3\epsilon(t))(L(\bar{t}) + \gamma_3 dt) + c_3\epsilon(t)(L(\bar{t}) + \gamma_4 dt + k_4)\end{aligned}$$

Rearranging terms and cancelling, we have:

$$\gamma_1 dt + \epsilon(t)c_1\Big((\gamma_2 - \gamma_1)dt + k_2\Big) = \gamma_3 dt + \epsilon(t)c_3\Big((\gamma_4 - \gamma_3)dt + k_4\Big)$$

so that $\epsilon(t) \approx \dfrac{(\gamma_3 - \gamma_1)dt}{\Big(c_1(\gamma_2 - \gamma_1) - c_3(\gamma_4 - \gamma_3)\Big)dt + (c_1k_2 - c_3k_4)}$. Therefore, $\dfrac{\epsilon(t)}{dt} \approx \dfrac{(\gamma_3 - \gamma_1)}{\Big(c_1(\gamma_2 - \gamma_1) - c_3(\gamma_4 - \gamma_3)\Big)dt + (c_1k_2 - c_3k_4)} \in (0, \infty)$. establishing that

$$\lim_{\delta\downarrow 0}\frac{\xi_1(\bar{t} + \delta, a_1^{lead})}{\delta^1} = \lim_{t\downarrow\bar{t}}\frac{\epsilon(t)}{\delta}$$ exists and is positive.

The final step is to verify that $\xi$ is piecewise *extendible* continuous. This fact is an immediate consequence of the following facts about "finitely subanaltyic sets," contained in Van Den Dries [18]. VdD(i): the restriction of any real analytic function $f: U \to \mathbf{R}$, where $U \subset \mathbf{R}$, to a compact interval contained in $U$ is a finitely subanalytic map (i.e., its graph is a finitely subanalytic set.) VdD(ii): if each of the component functions of $f: U \to \mathbf{R}^n$ is a finitely subanalytic map, then the function itself is finitely subanalytic. VdD(iii): the composition of a finitely subanalytic function and a semialgebraic corresondence is a finitely subanalytic correspondence (i.e., its graph is a finitely subanalytic set). VdD(iv): the projection of a finitely subanalytic set is a finitely subanalytic set. VdD(v): finitely analytic functions of one variable are piecewise monotone.

We now relate these facts to our problem. By assumption, each component function $u_i(a, \cdot)$ of our payoff function $u$ is piecewise extendible real analytic function of time. By VdD(i), therefore,$u_i(a, \cdot)$, is a piecewise finitely subanalytic function. By VdD(ii), $u$ itself is a piecewise finitely subanalytic function. Now the correspondence $\Psi$, defined by $\Psi(t) = \{\sigma \in \Sigma$: $\sigma$is a symmetric PTNE for $SurrG(t)\}$ is the composition of a piecewise finitely subanalytic function and a piecewise semi-algebraic correspondence, and so, by VdD(iii), has a piecewise finitely subanalytic graph. Therefore, we can choose $\xi^*$ to be a piecewise finitely subanalytic selection from this graph. By VdD(iv), each component function, $\xi_1^*(\cdot)(a_i)$, is piecewise finitely subanalytic and so, by VdD(v), is this function is piecewise monotone. Moreover, $\xi_1^*(\cdot)(a_1)$ is clearly bounded. It follows immediately that each of the finitely many continuous sections of this function has a continuous extension. □

We can now prove the theorem. We begin with some more definitions. For $t > 0$, define the "extended-left-derivative" $\frac{\partial-}{\partial t}L_i(t)$ by: $\frac{\partial-}{\partial t}L_i(t) = \lim_{\delta \downarrow 0} \frac{L_i(t) - L_i(t-\delta)}{\delta}$ . If $L_i(\cdot)$ is continuous at $t$, $\frac{\partial-}{\partial t}L_i(t)$ is just the usual left derivative. If $L_i(\cdot)$ jumps up at $t$ then $\frac{\partial-}{\partial t}L_i(t)$ is $+\infty$. Define the "right-closure" of a set $X \subset \mathbf{R}$, denoted $cl^+(X)$, to be the closure of the set minus it's left boundary points, i.e., $cl^+(X) = \{x \in \mathbf{R}: \exists\, \delta > 0 \text{ s.t. } (x-\delta, x) \in X\}$. For $t \in (0, t^{end})$ and $\delta \in (0, t)$, define $I_b^\delta(t) = \{i \in I: \forall s \in (t-\delta, t],\ \frac{\partial-}{\partial t}L_i(s) \le 0\}$ and let $I_b(t) = \bigcup_{\delta > 0} I_b^\delta$. For $t = 0$, define $I_b(t) = \varnothing$. For each $j$, choose a right-continuous selection $a_j^{lead}(\cdot)$ from $A_j^{lead}(\cdot)$ such that for each $i \ne j$, and each $t$

$$F_i(\cdot) = u_i(a_j^{lead}(\cdot), a_{-j}^0) \text{ except at finitely many points in } T \cap [0, t]. \qquad (7.17)$$

If $\Gamma$ has only two players, such a selection obviously exists. If $\Gamma$ is symmetric, then $i$'s payoff will be independent of the identity of the leader, so that once again, the required selection will exist.[2]

[2] It may happen that $A_i^{lead}(s)$ contains an element that is not an element of $\lim_{\delta \downarrow 0} A_i^{lead}(s)$. In this case, it may be that

We now proceed to the proof. Choose $\xi^*$ that satisfies the two conditions of Lemma II.1. Set $t^0 = t^{end}$. If $t^0 \in T$, let $\sigma^0$ be a Nash equilibrium for the one-shot game with payoff matrix $u(\cdot, t^0)$; now set $\xi(t^0) = \sigma^0$. Define $t^1$ by $t^1 = \inf\{s \in T\colon$ assumption (A3) is satisfied for $s\}$. If $(t^1, t^0] \not\subseteq T^*$, then by (A3), there exists $\iota^0 \in I$ such that $L_{\iota^0}(\cdot)$ is nonincreasing on this interval. Define $\gamma^0$ by $\gamma^0 = \begin{cases} 1 & \text{if } (t^1, t^0] \in T^* \\ 2 & \text{otherwise} \end{cases}$. Define $\bar{\xi}(\cdot)$ on $(t^1, t^0]$ by:

$$\bar{\xi}(t) = \begin{cases} \xi^*(t) & \text{if } \gamma^0 = 1 \\ \delta_{(a_{\iota^0}^{lead}(t),\, a_{-\iota^0}^0)} & \text{if } \gamma^0 = 2 \text{ and } L_{\iota^0}(t) \geq 0 \\ \delta_{a^0} & \text{otherwise} \end{cases}.$$

Now, fix $n \in \mathbb{N}$. Assume that $t^n$ has been defined and that $\bar{\xi}(\cdot)$ has been defined on $(t^n, t^{end}]$.

We now distinguish between three states of the system. This state is flagged by $\gamma^n$. We will proceed in different ways, depending on the value of this flag. Define $\gamma^n$ as follows.

$$\gamma^n = \begin{cases} 1 & \text{if } t^n \in cl^+(T^*) \\ 2 & \text{if } t^n \notin cl^+(T^*) \text{ and } \exists i \in I_b(t^n) \text{ s.t. } L_i(t^n-) \geq P_i(\bar{\xi}, t^n+) \\ 3 & \text{otherwise} \end{cases}.$$

Note that for $t^n = 0$, $I_b(t^n) = \varnothing$. Also, $0 \notin cl^+(T^*)$. Therefore, $t^n = 0$ implies $\gamma^n = c$. If there exists $i \in I$ such that $L_i(t^n-) \geq P_i(\bar{\xi}, t^n+)$, define $\iota^n$ by:

$$\iota^n = \begin{cases} \min\{i \in I_b(t^n)\colon L_i(t^n-) \geq P_i(\bar{\xi}, t^n+)\} & \text{if } \gamma^n = 2 \\ \min\left\{ i \in I\colon \begin{matrix} L_i(t^n-) \geq P_i(\bar{\xi}, t^n+) \text{ and} \\ L_i(t^n-) > F_i(t^n+) \end{matrix} \right\} & \text{if } \gamma^n = 3 \text{ and this set is nonempty} \\ \min\{i \in I\colon L_i(t^n-) \geq P_i(\bar{\xi}, t^n+)\}^{3} & \text{otherwise} \end{cases}$$

Note that by assumption (A2), $L_i(t^n-) \geq P_i(\bar{\xi}, t^n+)$ implies $L_i(t^n) \geq P_i(\bar{\xi}, t^n+)$, so that, in particular, if $\iota^n$ is defined, then

$$L_{\iota^n}(t^n) \geq P_{\iota^n}(\bar{\xi}, t^n+). \tag{7.18}$$

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$\lim_{\delta \downarrow 0} u_i(a_j^{lead}(s + \delta), a_{-j}^0) < F_i(s)$. However, $A_i^{lead}$ can have only finitely many discontinuity points in $T \cap [0, t]$.

[3] If $t^n = 0$, define $L_i(t^n-) = L(0)$.

Define $\bar{\xi}(t^n)$ by $\bar{\xi}(t^n) = \begin{cases} \delta_{(a_{\iota^n}^{\text{best}}(t),\, a_{-\iota^n}^0)} & \text{if } \iota^n \text{ has been defined} \\ \lim_{\delta \downarrow 0} \xi^*(t^n + \delta) & \text{otherwise} \end{cases}$. Now define $t^{n+1}$ as follows:

$$t^{n+1} = \begin{cases} \inf\{t \in [0, t^n) \colon \forall (t, t^n) \subset T^*\} & \text{if } \gamma^n = 1 \\ \inf\{t \in [0, t^n) \colon \forall s \in (t, t^n), \frac{\partial}{\partial t} L_{\iota^n}(s) \le 0\} & \text{if } \gamma^n = 2 \\ \inf\{t \in [0, t^n) \colon \forall i, \forall s \in (t, t^n), L_i(s) \le P_i(\bar{\xi}, t^n)\} & \text{if } \gamma^n = 3 \end{cases}.$$

$$\text{Define } \bar{\xi}(\cdot) \text{ on } (t^{n+1}, t^n) \text{ by: } \bar{\xi}(t) = \begin{cases} \xi^*(t) & \text{if } \gamma^n = 1 \\ \delta_{(a_{\iota^n}^{\text{best}}(t),\, a_{-\iota^n}^0)} & \text{if } \gamma^n = 2 \\ \delta_{a^0} & \text{if } \gamma^n = 3 \end{cases}.$$

We now establish that $\bar{\xi}$ is an SGP equilibrium. We first establish that for all $n \in \mathbf{N}$, $t^{n+1} < t^n$. If $\gamma^n = 1$, this follows directly from the definition of $\gamma^n$ and $t^{n+1}$. If $\gamma^n = 2$, then $\iota^n \in I_b(t^n)$. By definition of $I_b(t^n)$, there exists $\delta > 0$ such that on $(t^n - \delta, t^n)$, $\frac{\partial}{\partial t} L_{\iota^n}(\cdot) \le 0$. Therefore, $t^{n+1} < t^n - \delta$. Finally, suppose that $\gamma^n = 3$. We need to establish that for all $i$:

$$L_i(\cdot) \le P_i(\bar{\xi}, t^n) \text{ on an interval before } t^n. \tag{7.19}$$

Since $t^n \notin cl^+(T^*)$, there exists at most one $i$ such that $L_i(t^n) > F_i(t^n+)$. Also, by definition of $\gamma^n$, it must be the case that for each $i$

$$either\ i \notin I_b(t^n)\ or\ L_i(t^n) < P_i(\bar{\xi}, t^n+). \tag{7.20}$$

Choose $i \in I$. There are several cases to consider.

Case (a): $L_i(t^n) < P_i(\bar{\xi}, t^n+)$, for all $i$. In this case, $\bar{\xi}$ is continuous at $t^n$, so that $P(\bar{\xi}, t^n) = P(\bar{\xi}, t^n+)$. Inequality (7.19) now follows from the upper semi-continuity of $L_i$ (A2). Now suppose there exists $i$ such that $L_i(t^n) \ge P_i(\bar{\xi}, t^n+)$, so that $\iota^n$ has been defined. We first show that $\gamma^{n-1} \in \{a, b\}$. By definition of $\iota^n$ and $t^n$, we have $\lim_{\delta \downarrow 0} L_{\iota^n}(t^n - \delta) \ge P_{\iota^n}(\bar{\xi}, t^{n-1}) \ge \lim_{\delta \downarrow 0} L_{\iota^n}(t^n + \delta)$. Also, since $L_i$ is upper- and right-continuous, $\lim_{\delta \downarrow 0} L_{\iota^n}(t^n - \delta) \le L_{\iota^n}(t^n) = \lim_{\delta \downarrow 0} L_{\iota^n}(t^n + \delta)$. Therefore, $L_{\iota^n}(t^n) = P_{\iota^n}(\bar{\xi}, t^{n-1}) = P_{\iota^n}(\bar{\xi}, t^n+)$. Moreover, by definition of $t^n$, there exists some interval on which $L_{\iota^n}(\cdot) > L_{\iota^n}(t^n) = P_{\iota^n}(\bar{\xi}, t^{n-1})$. Therefore, $\iota^n \in I_b(t^n)$, so that $\gamma^{n-1} \in \{a, b\}$. There are three remaining possibilities:

Case (b): $i = \iota^n$. By definition of $\iota^n$, we have $L_i(t^n) \geq P_i(\bar{\xi}, t^n+)$. By construction, $i$ leads at $t^n$, so that $L_i(t^n) = P_i(\bar{\xi}, t^n)$. Moreover, by (7.20), $\frac{\partial^-}{\partial t} L_{\iota^n}(t^n) \geq 0$ on an interval before $t^n$. Therefore, (7.19) must be satisfied.

Case (c): $\iota^n = j \neq i$; $L_i(t^n) < P_i(\bar{\xi}, t^n+)$. By construction, $L_{\iota^n}(t^n) \geq P_{\iota^n}(\bar{\xi}, t^n+)$; Lemma II.1(i) therefore implies that $\gamma^{n-1} \neq a$. Therefore $\gamma^{n-1} = b$. Moreover, since $L_i$ is u.s.c., $P_i(\bar{\xi}, t^n+) > L_i(t^n+)$. Therefore, $i \neq \iota^{n-1}$ so that by (7.17), $P_i(\bar{\xi}, t^n+) = F_i(t^n+)$. But in this case:

$$\begin{aligned} L_i(t^n) \ < \ P_i(\bar{\xi}, t^n+) \ &= \ \lim_{\delta \downarrow 0} u_i((a^{lead}_{\iota^{n-1}}(t+\delta), a^0_{-\iota^{n-1}})), t+\delta) \\ &= \ \lim_{\delta \downarrow 0} u_i((a^{lead}_{\iota^n}(t+\delta), a^0_{-\iota^n})), t+\delta) \ = \ u_i((a^{lead}_{\iota^n}(t), a^0_{-\iota^n})), t) \ = \ F_i(t^n) \end{aligned} \tag{7.21}$$

The second equality follows because either there are only two players, in which case $\iota^{n-1} = \iota^n$, or because $\Gamma$ is symmetric; the third equality follows because $u$ is right-continuous (u1) and because, by construction, $a^{lead}_{\iota^n}(t) = a^{lead}_{\iota^n}(t+)$; the last equality follows from (7.17) and the fact that $i \neq \iota^n$.

Case (d): $\iota^n = j \neq i$; $L_i(t^n) \geq P_i(\bar{\xi}, t^n+)$. Since $t^n \notin cl^+(T^*)$, there exists at most one $i$ such that $L_i(\cdot) > F_i(t^n+)$ on an interval before $t^n$. Therefore, by definition of $\iota^n$, if $i \neq \iota^n$ and $L_i(t^n) \geq P_i(\bar{\xi}, t^n+)$, then $L_i(t^n) \leq F_i(t^n+)$. Since $i$ continues at $t$, it follows from (7.21) that $P_i(\bar{\xi}, t^n) = F_i(t^n+) \geq L_i(t^n)$. Moreover, by (7.20), we must have $\frac{\partial^-}{\partial t} L_{\iota^n}(\cdot) \leq 0$ on an interval before $t^n$. Condition (7.19) now follows immediately.

Since payoffs are pera, the above procedure must terminate in finitely many steps.

Finally, we need to verify that $\bar{\xi}$ is an SGP equilibrium. If $t^0 \in T$, then by construction, $\bar{\xi}$ is a Nash equilibrium for $\Gamma$ from $t^{end}$. Now fix $n \in \mathbf{N} \cup \{0\}$ and assume that for every $s \in [t^n, t^0]$, $\bar{\xi}$ is a Nash equilibrium for $\Gamma$ from $s$. We will show that for every $t \in [t^{n+1}, t^n)$, $\bar{\xi}$ is a Nash equilibrium for $\Gamma$ from $t$. There are several cases to consider.

Case (a): $t \in (t^{n+1}, t^n)$ and $\gamma^n = 1$. By definition of $\xi^*(\cdot)$, $\xi^*$ is right continuous at $t$ and condition the condition in (4.3) is satisfied. By Proposition VI, therefore, $\bar{\xi}$ is Nash from $t$.

Case (b): $t \in (t^{n+1}, t^n)$ and $\gamma^n = 2$. First consider agent $\iota^n$. By assumption, $\frac{\partial-}{\partial t} L_{\iota^n}(\cdot) \le 0$ on $[t, t^n]$. By construction, $\iota^n$ leads at $t^{n+1}$ and, by assumption, $\bar{\xi}$ is a Nash for $\Gamma$ from $t^{n+1}$. Therefore, for all $s \in (t, t^n)$, $L_{\iota^n}(t) \ge L_{\iota^n}(s) \ge L_{\iota^n}(t^n) = P_{\iota^n}(\bar{\xi}, t^n)$. Therefore, $\iota^n$'s best action at $t$ is to lead with some action from $A_i^{lead}(t)$. For $j \ne \iota^n$, it follows from (A1) that $j$ weakly prefers following to moving simultaneously.

Case (c): $t \in (t^{n+1}, t^n)$ and $\gamma^n = 3$. For all $i$, by definition of $\gamma^n$, $L_i(\cdot) \le P_i(\bar{\xi}, t^n)$, so that $i$ cannot gain by preempting.

Case (d): $t = t^{n+1}$. First suppose that $\gamma^n = 1$. If $\iota^{n+1}$ is defined, then by construction, $L_{\iota^{n+1}}(t^{n+1}) \ge P_{\iota^{n+1}}(\bar{\xi}, t^{n+1}+)$. Since all other agents but $\iota^{n+1}$ are continuing at $t^{n+1}$, it is a best response for $\iota^{n+1}$ to lead at this time. Moreover, by (A1), following is optimal for $j \ne \iota^{n+1}$. Therefore, $\bar{\xi}(t^{n+1})$ is a Nash equilibrium for the stage-game at $t^{n+1}$. Since the $\bar{\xi}(\cdot)$'s are Nash equilibria for the surrogate games immediately after $t^{n+1}$, it follows from Proposition VI that $\bar{\xi}(t^{n+1})$ is a Nash equilibrium for $\Gamma$ from $t^{n+1}$. If $\iota^{n+1}$ is not defined, then for each $i$, $P_i(\bar{\xi}, t^{n+1}+) > L_i(t^{n+1})$. Since the $\bar{\xi}(\cdot)$'s are Nash equilibria for the surrogate games immediately after $t^{n+1}$, it follows from (4.3) that $\bar{\xi}(t^{n+1})$ must be Nash equilibrium for $SurrG(t^{n+1})$. Therefore, by Proposition VI, $\bar{\xi}$ is Nash from $t$. Next suppose that $\gamma^n = 2$, so that some agent is leading immediately after $t^{n+1}$. Since $L_i$ is u.s.c., $\iota^{n+1}$ is defined in this case. By (7.18), $L_{\iota^n}(t^n) \ge P_{\iota^n}(\bar{\xi}, t^n+)$. By (A1), therefore, $\bar{\xi}(t^{n+1})$ is a Nash equilibrium for $StgG(t^{n+1}, \bar{\xi})$. Also, from Case (b) above, $\bar{\xi}$ is a Nash equilibrium for $\Gamma$ for $s > t^{n+1}$. By Lemma VI.1, therefore, $\bar{\xi}$ is a Nash equilibrium for $\Gamma$ from $t^{n+1}$. Finally, suppose that $\gamma^n = 3$. In this case, there exists $i$ such that $L_i(t^{n+1}-) \ge P_i(\bar{\xi}, t^{n+1}+) = P_i(\bar{\xi}, t^n)$. Since $L_i$ is u.s.c., $\iota^{n+1}$ is again defined. Once again, it follows from Lemma VI.1 that $\bar{\xi}$ is a Nash equilibrium for $\Gamma$ from $t^{n+1}$. □

*Proof of Theorem VII (necessity):* Fix an SGP outcome $(\bar{\mu}, \bar{t})$ and a equilibrium profile $\bar{\xi}$ that implements this outcome. Obviously, (5.1) must be satisfied. Assume that $(\bar{\mu}, \bar{t})$ is not a type 1 outcome. We will establish that it must be a type 2 outcome. By assumption, for every $s \in T$, $\bar{\xi}$ is Nash from $s$. By Lemma VI.1(i), therefore, $\bar{\xi}(\bar{t})$ must solve the stage-game induced by $\bar{\xi}$ at $\bar{t}$. Therefore, $\bar{\xi}$ cannot have plural termination just beyond $\bar{t}$; if it did, then by Proposition VI, $(\bar{\mu}, \bar{t})$ would be a type 1 outcome, contradicting our hypothesis. Moreover, since payoffs are strictly increasing, there can be no interval after $\bar{t}$ on which exactly one player is moving (if there were, this player could do better by waiting.) Therefore, there exists an interval after $\bar{t}$ on which no agent is moving. Since the game ends at $\bar{t}$, restriction (X3) implies that some agent is moving with probability one at $\bar{t}$. But a second player must also be moving at this time, otherwise the first would rather wait. Applying (X3) again, the second must be moving with probability one.

*Proof of sufficiency:* Assume that $(\bar{\mu}, \bar{t})$ satisfies condition (5.1). We first assume that it is a type 1 outcome. Let $\xi^*$ be an SGP profile for $\Gamma$. Let $\bar{\xi}$ be a profile that implements $(\bar{\mu}, \bar{t})$, has plural termination just beyond $\bar{t}$ and satisfies condition (ii) of Proposition VI for $\bar{\delta} > 0$. Define $\xi$ by:

$$\xi(s) = \begin{cases} \delta_{a^0} & \text{if } s < \bar{t} \\ \bar{\xi}(s) & \text{if } s = \bar{t} \\ \bar{\xi}(s) & \text{if } s = (\bar{t}, \bar{t} + \bar{\delta}) \\ \xi^*(s) & \text{otherwise} \end{cases}.$$

Clearly, $\xi$ implements $(\bar{\mu}, \bar{t})$. From (5.1), $\xi$ is Nash from $s$, for every $s < \bar{t}$. From Lemma VI.1 and Proposition VI, $\xi$ is Nash from $s$, for every $s$ such that $\xi(s) = \bar{\xi}(s)$. Since $\xi^*$ is an SGP equilibrium, $\xi$ is Nash from the remaining decision nodes. If $(\bar{\mu}, \bar{t})$ is a type 2 outcome, proceed exactly as above, except choose $\bar{\xi}$ to be a SPTP that implements $(\bar{\mu}, \bar{t})$ and set $\bar{\delta} = 0$. □

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