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Patterns and Stability in the Coefficients of the Colored Jones Polynomial /

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Patterns and Stability in the Coefficients of the Colored Jones Polynomial

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Katherine Walsh

Committee in charge:

Professor Justin Roberts, Chair
Professor Benjamin Grinstein
Professor Kenneth A. Intriligator
Professor David Meyer
Professor Hans Wenzl

2014

The dissertation of Katherine Walsh is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2014

DEDICATION

To my family and friends, who have been there for me through it all.

EPIGRAPH

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-Jay Bedsole

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ACKNOWLEDGEMENTS

I would like to thank Professor Justin Roberts for his support as my advisor. I have learned a great deal from him in the past six years.

I would not have enjoyed my time here at UCSD nearly as much if it weren't for my friends here. I am grateful for all the support you have all given me.

I would also like to thank the San Diego ARCS foundation for their financial support through a fellowship I received during part of my time in graduate school.

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PUBLICATIONS

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A. Hicks, H. Parks, and K. Walsh, "Learning through Outreach: Activities of the UCSD Association for Women in Mathematics Student Chapter," *Education in Action: Experiential Learning in Higher Education*. Conference Proceedings, 2012.

ABSTRACT OF THE DISSERTATION

Patterns and Stability in the Coefficients of the Colored Jones Polynomial

by

Katherine Walsh

Doctor of Philosophy in Mathematics

University of California, San Diego, 2014

Professor Justin Roberts, Chair

The colored Jones polynomial assigns to each knot a sequence of Laurent polynomials. This dissertation will focus on the patterns in the coefficients of these polynomials. We will discuss a new formula for calculating the colored Jones polynomial of certain pretzel knots and the stabilization and higher-order stabilization of the coefficients, specifically discussing what the second N coefficients of the N colored Jones polynomial of certain knots stabilize to. We will also look at patterns in the middle coefficients and explore a new way of looking at the colored Jones polynomials of amphichiral knots.

Chapter 1

Introduction

This dissertation focuses mainly on the patterns in the coefficients of the colored Jones polynomial. The colored Jones polynomial is a knot invariant that assigns to each knot a sequence of Laurent polynomials indexed by $N \geq 2$, the number of colors. For a knot K, denote the Nth term in this sequence $J_{N,K}(q)$, where N corresponds with the N dimensional representation, i.e. we use the convention that when $N = 2$, we get the Jones polynomial.

We usually think of the N colored Jones polynomial as either the Jones polynomial of a linear combination of i-cablings of the knot for $0 \leq i \leq N-1$ or as the evaluation in the Temperley-Lieb algebra of the knot diagram decorated with the $N-1^{st}$ Jones-Wenzl idempotent. In what follows, the colored Jones polynomial is normalized so that its value on the unknot is 1.

One of the main open questions in the subject area is how to relate the colored Jones polynomial to the geometry of the knot. One such relation is the following "hyperbolic volume conjecture."

Conjecture 1.0.1 ([Mur10], Kashaev-Murakami-Murakami)**.** *For any hyperbolic knot K,*

$$2\pi \lim_{N\to\infty} \frac{\log|J_{K,N}(e^{2\pi i/N})|}{N} = vol(S^3 \backslash K)$$

where $J_{K,N}(e^{2\pi i/N})$ is the normalized Colored Jones Polynomial of a knot K evaluated at an N^{th} root of unity and $vol(S^3\backslash K)$ is the volume of the unique complete hyperbolic Riemannian metric on the knot complement.

The hyperbolic volume conjecture has been proved for torus knots, the figure-eight knot, Whitehead doubles of torus knots, positive iterated torus knots, Borromean rings, (twisted) Whitehead links, Borromean double of the figure-eight knot, Whitehead chains, and fully augmented links (see [Mur10]). It is still open for other knots and links.

In [DL07], Dasbach and Lin related the first and last two coefficients of the original Jones polynomial to the the volume of the knot in the following way:

Theorem 1.0.2 (Dasbach, Lin)**.** *Volume-ish Theorem: For an alternating, prime, non-torus knot K let*

$$J_{K,2}(q) = a_n q^n + \cdots + a_m q^m$$

be the Jones polynomial of K. Then

$$2v_8(max(|a_{m-1}|, |a_{n+1}|) - 1) \leq Vol(S^3 - K) \leq 10v_3(|a_{n+1}| + |a_{m-1}| - 1).$$

Here, $v_3 \approx 1.0149416$ *is the volume of an ideal regular hyperbolic tetrahedron and* $v_8 \approx 3.66386$ *is the volume of an ideal regular hyperbolic octahedron.*

They also proved that the first two and last two coefficients of the Jones Polynomial were also the first and last two coefficients of the N colored Jones polynomial for all N and noticed that the first and last N coefficents of the N colored Jones polynomial seemed to be the same, up to sign, as the first N coefficients of the k colored Jones polynomial for all $k > N$. These types of theorems encourage us to look more deeply in to what the coefficients of the colored Jones polynomial can tell us about the knot.

1.1 Patterns in the Coefficients of the Colored Jones Polynomial

When studying the coefficients of the colored Jones polynomial, I first looked at patterns in the entire set of coefficients. To be able to visualize these patterns, I used a formula initially proved by Habiro and reproved by Masbaum in [Mas03] to calculate the colored Jones polynomial of the figure 8 knot and twist knots and then plotted the coefficients of these polynomials. The plot of the coefficients for the 95^{th} colored Jones polynomial of the figure 8 knot is below. (The plot has the degree of the term on the $x-$axis and the coefficient on the $y-$axis. Degrees were shifted by multiplying by q^M for some M so that all the degrees were positive.)

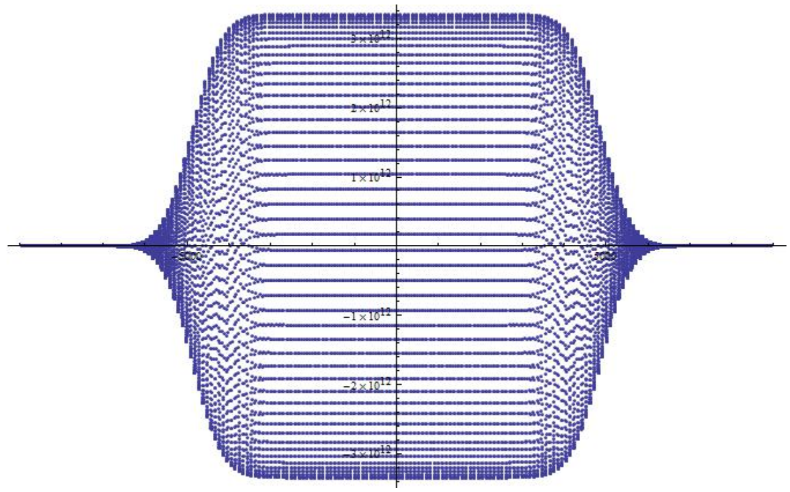


Figure 1.1: Coefficients of the 95^{th} Colored Jones Polynomial for the Figure Eight Knot

This led me to the following conjectures about the basic shape of the plot of the coefficients of the N^{th} colored Jones polynomial.

1. In the middle, the coefficients of $J_{K,N}$ are approximately periodic with period N.
2. There is a sine wave like oscillation with an increasing amplitude on the first and last quarter of the coefficients.
3. We can see that the oscillation persists throughout the entire polynomial. The amplitude starts small, grow steadily and then levels off in the middle and then goes back down in a similar manner.

I also looked at the growth rate of the maximum coefficients of each colored Jones polynomial of a knot. The maximum coefficients of the polynomials seemed to grow exponentially at a rate related to the hyperbolic volume of the knot.

Much of my research has been centered on trying to gain insight on where these patterns come from. This first led me to use the techniques from [Mas03] to find a formula for the colored Jones polynomial of pretzel knots of the form $(1, r-1, 2p-1)$ in order to have a larger class of knots for which I could easily calculate the colored Jones polynomial for large values of N.

Theorem 1.1.1. *A pretzel knot of the form $K_{p,r} = P(1,2p-1,r-1)$ has the colored Jones polynomial*

$$J_N(K_{p,r},a^2) = \sum_{n=0}^{N-1} c'_{n,p} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} \mu_n^* \sum_{k=0}^{n} \delta(2k;n,n)^r \frac{\langle 2k \rangle}{\langle n,n,2k \rangle} \frac{([k]!)^2}{[2k]!} \frac{\{2n+1\}!}{\{n\}!\{1\}}$$

$$= \sum_{n=0}^{N-1} (-1)^n \begin{bmatrix} N+n \\ N-n-1 \end{bmatrix} c'_{n,p} \frac{\{2n+1\}!\{n\}!}{\{1\}(a-a^{-1})^{2n}} \sum_{k=0}^{n} (-1)^{k(r+1)} \frac{[2k+1]}{[n+k+1]![n-k]!} \mu_{2k}^{r/2}.$$

Here

$$c'_{n,p} = \frac{1}{(a-a^{-1})^n} \sum_{k=0}^{n} (-1)^k \mu_{2k}^p [2k+1] \frac{[n]!}{[n+k+1]![n-k]!},$$

where $\mu_i = (-1)^i A^{i^2+2i}$ and,

$$\{n\} = a^n - a^{-n}, [n] = \frac{a^n - a^{-n}}{a-a^{-1}}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}.$$

Corollary 1.1.2. *When r is even this reduces to*

$$J_n(K_{p,r},a^2) = \sum_{n=0}^{N-1} (-1)^n \begin{bmatrix} N+n \\ N-n-1 \end{bmatrix} c'_{n,p} \frac{\{2n+1\}!}{\{1\}} c'_{n,r/2}.$$

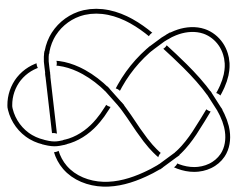
Corollary 1.1.3. *When r is odd this reduces to*

$$J_n(K_{p,r},a^2) = \sum_{n=0}^{N-1} (-1)^n \mu_n^{4p} c'_{n,p} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} \frac{\{2n+1\}!\{n\}!}{(a-a^{-1})^{2n}\{1\}} \sum_{k=0}^{n} \mu_{2k}{}^{\frac{r}{2}} \frac{[2k+1]}{[n+k+1]![n-k]!}$$

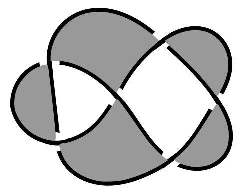
The formula for the case where r is even was independently proven by Garoufalidis and Koutschan in [GK12]. Using this formula, we are able to more quickly calculate the colored Jones polynomial for many knots with up to 9 crossings. This formula is discussed in Chapter 4.

I also was led to look at the current work studying the first and last coefficients of the colored Jones polynomial, known as the head and tail of the polynomial and look at the higher order stability of the coefficients. An overview of this research and how I have related it to my main questions is presented in Chapter 5.

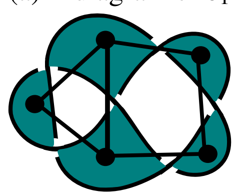
Given a sequence of Laurent polynomials, we say the *head* of this polynomial exists if the first N coefficients (of the highest order terms) of the N^{th} polynomial in the sequence are the same as the first N coefficients of the k^{th} polynomial for all $k \geq N$. The *tail* of the sequence of polynomials, if it exists, is the stabilized sequences of the coefficients of the lowest terms.



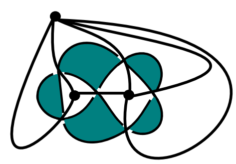
(a) A diagram of 6_1



(b) 6_1 with a checkerboard coloring



(c) The A-checkerboard graph



(d) The B-checkerboard graph

Figure 1.2: The Knot 6_1 and its associated graphs.

In [DL06, AD11, Arm13], Dasbach and Armond proved that the head and tail of the colored Jones exist for alternating and adequate knots and depend on the reduced checkerboard graphs of the knot diagrams.

In [GL11], Garoufalidis and Le independently proved that the head and tail of the colored Jones polynomial exist for alternating knots while proving (for alternating knots) a stronger version of this stability.

I hope to be able to use this stabilization to extend the work done on the first and last coefficients to the patterns in the middle that I originally observed. In particular, if I can find what the stabilized sequences are, I could extract which parts of these sequences contributed to the maximum coefficient, or to other coefficients I wanted to study.

For example, for the figure 8 knot, we know the that first coefficients stabilize to the pentagonal number sequence. By this, I mean that for the figure 8 knot,

$$\Phi_0 = \prod_{n=1}^{\infty}(1-q^n) = \sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{k}{2}(3k-1)}.$$

In the table below, I have listed out the first 16 coefficients of the N-colored Jones polynomial for the figure 8 knot for $N = 3,4$ and 5. We see that the first $N+1$ coefficients of the N-colored Jones polynomial are the same as the first $N+1$ coefficients of Φ_0.

Φ_0	1	-1	-1	0	0	1	0	1	0	0	0	0	-1	0	0	-1	$\cdots$
$N=3$	1	-1	-1	0	2	0	-2	0	3	0	-3	0	3	0	-3	0	$\cdots$
$N=4$	1	-1	-1	0	0	3	-1	-1	-1	-1	5	-1	-2	-2	-1	6	$\cdots$
$N=5$	1	-1	-1	0	0	1	2	0	-2	-1	-1	1	3	1	-2	-3	$\cdots$

Now, since we know all of Φ_0, we can subtract it from the shifted colored Jones polynomials. Now are coefficients are:

Φ_0	1	-1	-1	0	0	1	0	1	0	0	0	0	-1	0	0	-1	$\cdots$
$N=3$	0	0	0	0	2	-1	-2	-1	3	0	-3	0	4	0	-3	1	$\cdots$
$N=4$	0	0	0	0	0	2	-1	-2	-1	-1	5	-1	-3	-2	-1	7	$\cdots$
$N=5$	0	0	0	0	0	0	2	-1	-2	-1	-1	1	4	1	-2	-2	$\cdots$

Shifting these sequences back so that they start with a non-zero term, we can see that they again stabilize. The sequence they stabilize to is Φ_1.

Φ_1	2	-1	-2	-1	-1	1	$\cdots$						
$N=3$	2	-1	-2	-1	3	0	-3	0	4	0	-3	1	$\cdots$
$N=4$	2	-1	-2	-1	-1	5	-1	-3	-2	-1	7	$\cdots$	
$N=5$	2	-1	-2	-1	-1	1	4	1	-2	-2	$\cdots$		

I call the sequence Φ_1 the “neck of the tail" or the “tailneck" of the colored Jones polynomial of the figure 8 knot.

I calculated the tailneck of all three strand pretzel knots with negative twists in each region. For knots in this family, the B-checkerboard graph is a three cycle. These

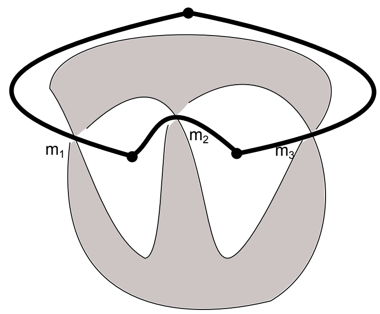


Figure 1.3: A trefoil knot with its checkerboard graph.

knots can be drawn like the trefoil in Figure 5.2, except we will have more crossings below the pictured crossings (and thus more parallel edges before we reduce the graph). The m_i represent the number of crossings in each section. As it is drawn, each $m_i = 1$. (If $m_1 = 2$ and the others are 1, we get the figure 8 knot.)

Theorem 1.1.4. *The tailneck of knots with reduce to the three cycle is:*

- $\prod_{n=1}^{\infty}(1-q^n)$*, i.e. the pentagonal numbers sequence, if all* $m_i = 1$ *(The only knot satisfying this is the trefoil).*
- $\prod_{n=1}^{\infty}(1-q^n) + \frac{\prod_{n=1}^{\infty}(1-q^n)}{1-q}$*, i.e. the pentagonal numbers plus the partial sum of the pentagonal numbers, if two* $m_i = 1$ *and one is 2 or more.*
- $\prod_{n=1}^{\infty}(1-q^n) + 2\frac{\prod_{n=1}^{\infty}(1-q^n)}{1-q}$*, i.e. the pentagonal numbers plus the 2 times the partial sum of the pentagonal numbers, if one* $m_i = 1$ *and two are 2 or more.*
- $\prod_{n=1}^{\infty}(1-q^n) + 3\frac{\prod_{n=1}^{\infty}(1-q^n)}{1-q}$*, i.e. the pentagonal numbers plus the 3 times the partial sum of the pentagonal numbers, if all* $m_i \geq 2$*.*

The proof of this theorem in is chapter 5.

1.2 Organization

This dissertation is organized as follows: In Chapter 2, we define knots and introduce the Jones polynomial. In Chapter 3 we introduce the the colored Jones polynomial and discuss its various definitions, some history and the hyperbolic volume conjecture.

In Chapter 4, we introduce pretzel knots and prove Theorem 1.1.1 and its corollaries. . In Chapter 5, we discuss the head and tail stability and prove Theorem 1.1.4. Finally, in Chapter 6, we discuss in more detail the patterns in the middle coefficients of the colored Jones polynomial.

There are two appendices. The first gives an overview of both kinds of Chebychev polynomials we use throughout this dissertation. The second discusses a way to use matrices to compute the Jones and Colored Jones polynomials.

There are a few different standard normalizations used in works related to the colored Jones polynomial, I have listed those I am using in section 3.3.

Chapter 2

Knots and Knot Invariants

2.1 Knots

Most everyone has a picture that comes to mind when you say *knot*. We begin with the mathematical definition of a knot so that our pictures are all basically the same.

Definition 2.1.1. *A* knot *is an embedding* $f : S^1 \to S^3$.

A knot is usually represented through projection into $\mathbb{R}^2$ such that at most two segments come together at any one point and, whenever two segments meet, we designate which arc is the over crossing and which is the under crossing.

If we have an embedding of multiple copies of S^1 into S^3 we call this a link instead of a knot. The image of each S^1, i.e. each connected segment, is called a component of the link. Thus a link with a single component is a knot. When we talk about a knot, we are referring to either the map itself or the image under this map. We call a specific projection of the knot into $\mathbb{R}^2$ the knot diagram.

Intuitively, we want to say that two knots are equivalent if, when made out of string, we can move the strings around to make one knot look like the other. The strands cannot pass through each other (it's string!) and you cannot cut the string and then retie it. Mathematically, we can define this as follows:

Definition 2.1.2 ([Lic97]). *Two knots are* equivalent *if there is an orientation preserving piecewise linear homeomorphism* $h : S^3 \to S^3$ *that maps one knot to the other.*

We will refer to the map as the ambient isotopy. This definition allows us to rule out pathological examples. Below are 5 knots diagrams. The first two are both diagrams of the unknot. The second two are both diagams of the trefoil knot 3_1 and the last is a diagram of the figure 8 knot 4_1.

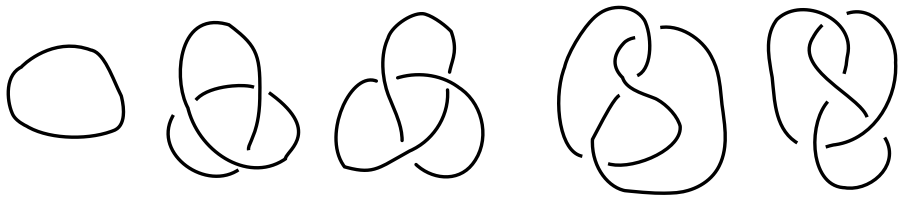


Figure 2.1: There are three different knot types in this figure. The first two knots are unknots, the third and fourth are trefoils and the fifth knot is a figure 8 knot.

2.2 Knot Invariants

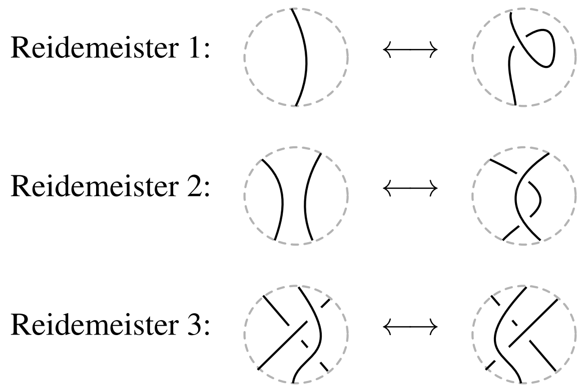
We can use knot invariants to help us tell whether or not two knot diagrams represent equivalent knots.

Definition 2.2.1. *A* knot or link invariant *is a property of a knot or link that does not change under ambient isotopy.*

If two knots have different values for any knot invariant, then it is impossible to transform one into the other, thus they are not equivalent. To prove a given property of a knot is a knot invariant, we need to show that it is invariant under any ambient isotopy. Luckily, we have a theorem from Reidemeister that tells us that any ambient isotopy can be achieved via a finite sequence of three basic moves and planar isotopy. Thus, to check a property is an knot invariant, we just need to check it is invariant under these three moves and planar isotopy.

Theorem 2.2.2 (Reidemeister 1928)**.** *Any two equivalent knots are related by planar isotopy and a sequence of the three Reidemeister moves.*

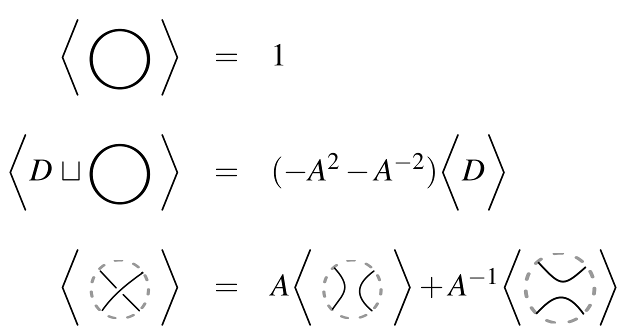
The three Reidemeister moves are pictured below. In each diagram, the knot diagrams are equivalent outside of the dotted circle and the change only occurs in the region pictured.



2.2.1 Kauffman Bracket

The first semi-invariant we will discuss is only invariant under planar isotopy and the second and third Reidemeister moves. We say a property invariant under these moves is an invariant of framed knots and links. This semi-invariant is called the Kauffman bracket is defined via a local skein relation, see [Kau87]

Definition 2.2.3. *The* Kauffman bracket *of a diagram of a knot or link D, denoted* $\langle D \rangle$*, is an invariant of framed knots which assigns to each link a Laurent polynomial in A. Ist is characterized by the skein relation below.*



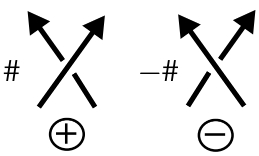
To find the Kauffman bracket polynomial of a link, we use the third relation to smooth out crossings and the second to remove disjoint unknots. Once all crossing and disjoint unknots are gone, we will only be left with a single unknot whose Kauffman bracket polynomial is 1.

2.2.2 The Jones Polynomial

We can adapt the Kauffman bracket to be a knot invariant by accounting for the change in polynomial due to the first Reidemeister move. The writhe of the diagram is only affected by the first move so by adding a factor depending on the writhe we can make the Kauffman bracket a knot invariant.

Definition 2.2.4. *The* writhe *of a diagram in the number of positive crossings in the diagram minus the number of negative crossings, i.e*

$w(D) = \#$



Once we adjust the Kauffman bracket using the writhe factor, we get a new polynomial which is an invariant. This polynomial, under a change of variable, if equivalent the the Jones polynomial and gives us one of the many ways of defining the Jones polynomial.

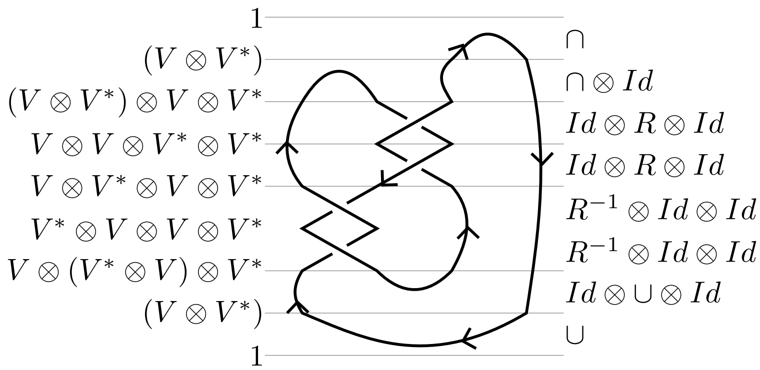
Definition 2.2.5. *The* Jones Polynomial *of a knot is a knot invariant of a knot K with diagram D defined by*

$$V(K) = \left((-A)^{3w(D)} \langle D \rangle \right)_{q^{1/2}=A^{-2}}$$

.

This definition extends to consistent definition of the Jones polynomial of oriented links. The orientation of the links is important since changing the orientation of a single component changes the Jones polynomial. There are many other ways of calculating the Jones polynomial of a knot, including another skein relation of oriented diagrams originally given by Jones, see [Jon85]. Below, we discuss a more formal way of defining the Jones polynomial which will be the definition we can easily extend to get the colored Jones polynomial.

We can view every knot as the closure of a braid. From the braid diagram, we can get a representation of the n^{th} braid group inside the n^{th} Temperly Lieb algebra.



This produces a map from 1 to itself and thus is just multiplication by an element in $\mathbb{Q}(q)$.

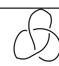
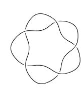
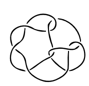
.

Figure 2.2: This is a Morsefication of a oriented diagram of the figure 8 knot. On each level, we associate a copy of V or V^* depending on whether the orientation is up or down. Between levels, we get cups, caps and R-matrices to move from one level to the next.

The trace of this representation gives the Jones Polynomial in the way it was originally defined by Vaughan Jones. Consider the Lie algebra $\mathfrak{sl}_2$ of traceless two-by-two complex matrices. It has universal enveloping algebra $U(\mathfrak{sl}_2)$, which has quantum deformation $U_q(\mathfrak{sl}_2)$. $U_q(\mathfrak{sl}_2)$ is an algebra over the ring $\mathbb{Q}(q)$ of rational functions in the variable q. Given a knot diagram, we can 'color' the diagram with a 2 dimensional representation of $U_q(\mathfrak{sl}_2)V$. Each strand oriented upward gets labeled with a V and those oriented downwards get labeled with a V^*, the dual representation. There are maps between these representations for cups, caps and the $R-matrix$ map for crossings. Since the overall maps if from the identity to itself, it is just a multiplication by an element in $\mathbb{Q}(q)$. This element is the Jones polynomial of the knot.

Table 2.1 shows the Jones Polynomials of various knots. When we take the mirror image of a knot, the Jones polynomial changes by interchanging q and q^{-1}. We see that the Jones polynomial of the trefoil is not symmetric in q and q^{-1} and thus the trefoil is not amphichiral, i.e. equivalent to its mirror image. The Jones polynomial of the figure 8 knot is symmetric in q and q^{-1}. This is a necessary but not sufficient condition for the figure 8 knot to be amphichiral. In fact, it is amphichiral. A diagram of figure 8 knot can be transformed into a diagram of its mirror image though a sequence of Reidemeister moves. Also notice that the mirror image of 5_1 and 10_{125} have the same Jones polynomials. Thus the Jones polynomial does not distinguish all knots. An important open question is whether or not there exists nontrivial knots with the same Jones polynomial as the unknot.

Table 2.1: Jones Polynomials of Different Knots

Knot	Knot Diagram	Jones Polynomial
Trefoil (3_1)		$q+q^3-q^4$
Mirror Image($\overline{3_1}$)		$q^{-1}+q^{-3}-q^{-4}$
Figure Eight (4_1)		$q^{-2}-q^{-1}+1-q+q^2$
Mirror Image ($\overline{4_1}$)		$q^2-q^1+1-q^{-1}+q^{-2}$
5_1		$q^2+q^4-q^5+q^6-q^7$
$\overline{5_1}$		$q^{-2}+q^{-4}-q^{-5}+q^{-6}-q^{-7}$
10_{132}		$q^{-2}+q^{-4}-q^{-5}+q^{-6}-q^{-7}$

Chapter 3

The Colored Jones Polynomial

3.1 Some historical highlights of the Jones and Colored Jones polynomial

In January 1985, Vaughan Jones published a paper, [Jon85], introducing the Jones Polynomial (although he did not call it that). While analyzing finite dimensional von Neumann algebras with specific properties, it was pointed out to him that these properties were similar to those of the n-string braid group, B_n. This leads to a representation, r_t of B_n. Jones noticed that the number

$$(-t+1)/\sqrt{(}t)^{n-1}tr(r_t(b))$$

for some b in B_n depends only on the isotopy class of the closed braid and thus when viewed as a function of t this gives us an invariant. He calls this the trace invariant, but everyone else began calling it the Jones polynomial. (He used the variable t in the same way we used the variable q or above.)

In this paper, Jones states some of the basic properties of the Jones polynomial, including that it is a Laurent polynomial in t when the link has an odd number of components and $\sqrt{t}$ times a Laurent polynomial when the link has an even number of components. He also shows how it changes over mirror image and connect sum and the relationship between the Jones polynomial of knots where a single crossing is replaced by the other crossing and its smoothing. This skein relation gives a recursive way to find the Jones polynomial of the any knot.

Based on this work of Jones, Freyd and Yetter, Hoste, Lickorish and Millett and Ocneanu (HOMFLY) developed a two variable polynomial invariant of knots and links which generalizes both the Jones polynomial and the Alexander polynomial. In fact, although working independently, these groups all submitted very similar papers to the Bulletin of the American Mathematical Society in a period of a few days. This led to a single paper[FYH$^+$85] on the idea being published with each as coauthors. This is unrelated to the colored Jones polynomial, but included here to give some historical perspective of the large amount of activity in this area at the time.

In 1986, in [Kau87], Kauffman showed a state sum model for calculating the Jones polynomial using a different skein relation, where each crossing can be smoothed in two different ways. In this paper, he defines the bracket polynomial, now called the Kauffman bracket which, under a simple scaling involving the writhe to make it invariant under the first Reidemeister move, is equivalent to the Jones polynomial. This gives us a state sum model for the Jones polynomial which leads to the proof of many more interesting properties of this polynomial. This is the definition we used to define the Jones polynomial is Chapter 1.

And now, this brings us to the introduction of the colored Jones polynomial. In 1988, in [Res88a] and [Res88b], (an unpublished pre-print in two parts which was only circulated by mail to interested parties but is now available on Reshetikhin's website), Reshetikhin shows that for any simple Lie algebra $\mathfrak{g}$, we can associate to the quantum deformation of the universal enveloping algebra of $\mathfrak{g}$ (i.e. $U_q(\mathfrak{g})$) a countable set of invariants of a link. When we take $\mathfrak{g} = \mathfrak{sl}_2$, we get the family of colored Jones polynomials of the link. He does this by constructing representations on the braid group which correspond to the R-matrices studied throughout the paper.

3.2 Definitions of the Colored Jones Polynomial

3.2.1 The Colored Jones Polynomial using R-matrices

Just as with the Jones polynomial defined above, we can get a representation of the n^{th} braid group inside the n^{th} Temperly Lieb algebra (defined in more detail below).

If we 'color' the diagram with an N-dimensional representation of $U_q(\mathfrak{sl}_2)V$, instead of a 2 dimensonal representation and do all the same steps we get the N colored Jones polynomial.

3.2.2 The Colored Jones Polynomial as a linear combination of Jones Polynomials of Cablings

We can think of the N-dimensional colored Jones polynomials as a linear combination of the original Jones polynomial on cablings of the knots, in the same way that the N-dimensional representation can be expressed as a linear combination of $V_2^{\otimes k}, k \leq N-1$.

We can express this linear combination recursively as:

$$g_1 = 1$$

$$g_2 = z$$

$$g_3 = zg_{i-1} - g_{i-2}.$$

For example, $g_3 = z^2 - 1$ so the 3-dimensional colored Jones polynomial is the Jones Polynomial of the two cabled figure 8 minus 1.

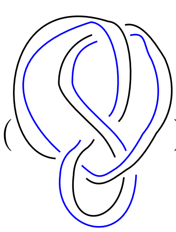
$J_{3,4_1} \quad = \quad V($  $) - 1$

Figure 3.1: Using the linear combination definiton of the colored Jones polynomials, we see the the 3 colored Jones polynomial of 4_1 is the Jones polynomial of the 2 cabling of 4_1 minus 1.

This linear combination gives us the Chebyshev polynomials of the 2nd kind. They are normalized differently in some contexts. See A.2 for more details.

3.2.3 Temperley-Lieb Algebra and the Colored Jones Polynomial

We can also give a definition of the colored Jones polynomial using the Temperley-Lieb algebra, TL_n and the Jones-Wenzl idempotent $f^{(n)} \in TL_n$.

Given an oriented disk D^2 with $2n$ marked points on it boundary, see Figure 3.2, we can draw arcs connecting the marked points. We can also can cycles inside of the diagram, see Figure 3.3.

Figure 3.2: To define the Temperley-Lieb algebra, we start with an oriented disk (D^2) with $2n$ marked points in its boundary.

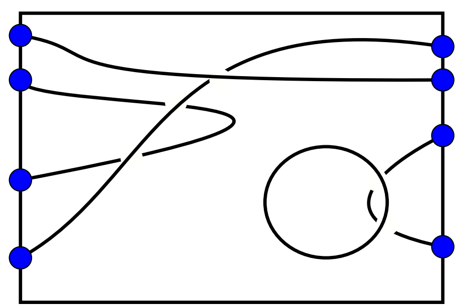


Figure 3.3: We draw arcs starting and ending at all of the marked points and cycles inside the surface. We have to mark at each crossing which strand is the over crossing and which is the under crossing.

We will say that two diagrams are the same is there is a homeomorphism of D^2 that is isotopic to the identity and keeps the boundary points fixed that maps ones to the other. These types of diagrams are the starting point for defining our linear skein. We also include rules reminiscent of the Kauffman bracket relations. We take the following definition:

Definition 3.2.1. *The* linear skein $S(D^2, 2n)$ *of* $(D^2, 2n)$ *is a vector space of formal linear sums over* $\mathbb{C}$ *of link diagrams in* $(D^2, 2n)$ *quotiented by the relations below.*

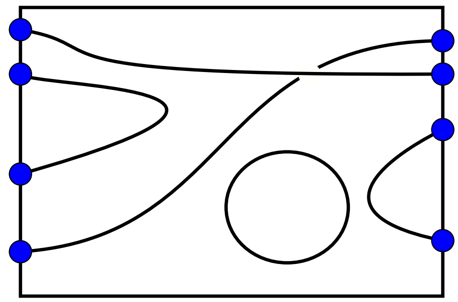
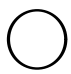
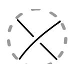
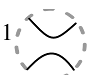


Figure 3.4: This diagram is the same in the Temperley-Lieb algebra as the one in Figure 3.3

$D \sqcup$  $= (-A^2 - A^{-2})D$

 $= A$  $+ A^{-1}$ 

The second relation allows us to undo all crossings and reduce our diagram to a series of crossing-less matches between the marked points and disjoint cycles inside the diagram. The first relation allows to to remove the disjoint cycles. This means that every diagram is equivalent to a linear combination of crossing-less matches between the marked points. The constants in this linear combination are Laurent polynomials in A, but we generally take A to be a fixed complex number.

The product of diagrams is by juxtaposition. An example is given in Figure 3.6.

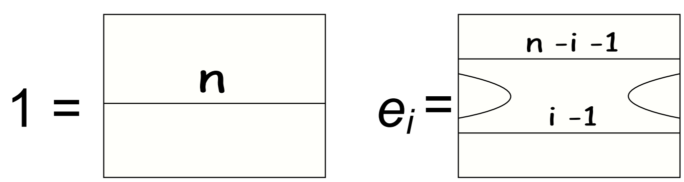


Figure 3.5: These are the generators of the nth Temperley-Lieb algebras. A strand with a number above it represents that many parallel strands.

This product extends to a well-defined bilinear map that turns $S(D^2, 2n)$ into an algebra. We call this algebra the n^{th} *Temperley-Lieb algebra* TL_n.

The algebra is generated by the elements $1, e_1, e_2, \ldots, e_{n-1}$ shown in Figure 3.5. In particular, the ith generator is has a backtrack between the ith and $i+1$st strand. Here, a strand with an n above it represents n parallel strands in the diagram.

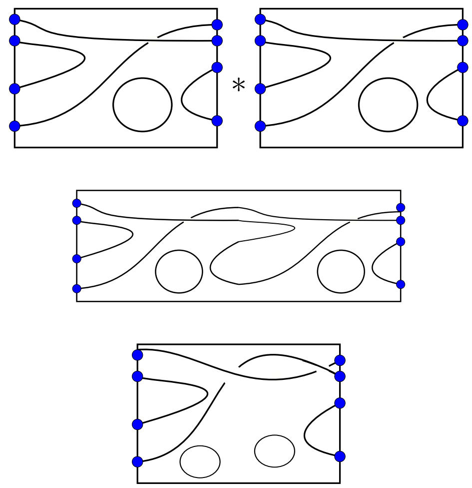


Figure 3.6: This figure illustrates what happens when we take the product of two elements in the Temperley-Lieb algebra

There is a special element in this algebra called the Jones-Wenzl idempotent. The Jones-Wenzl idempotent in the TL_n is denoted $f^{(n)}$. Diagrammatically, we usually draw it as a box labeled with n, as in Figure 3.7.

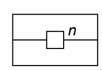
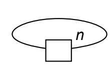
$f^{(n)} =$ 

Figure 3.7: This is how we will diagramatically represent the Jones-Wenzl idempotent inside the Temperley-Lieb algebra.

The Jones-Wenzl idempotent $f^{(n)} \in TL_n$ is the unique element such that:
(i) $f^{(n)}e_i = 0 = e_i f^{(n)}$ for $1 \leq i \leq n-1$ (kills backtracks)
(ii)$(f^{(n)} - 1)$ belongs to algebra generated by $\{e_1 \dots e_{n-1}\}$
(iii) $f^{(n)} f^{(n)} = f^{(n)}$
(iv)$\Delta_n = \frac{(-1)^n (A^{2(n+1)} - A^{-2(n+1)})}{(A^2 - A^{-2})}$ where

$\Delta_n =$ 

$\in S(\mathbb{R}^2)$

In [Wen87] Wenzl proved the recurrence relationship satisfied by the idempotents the we present in Figure 3.8. To give an idea of what these idempotents look like, we use the recursive definition to find $f^{(2)}$ and $f^{(n)}$. These are in Figures 3.9 and 3.10.

Using this Jones-Wenzl idempotent, we can define the colored Jones polynomial

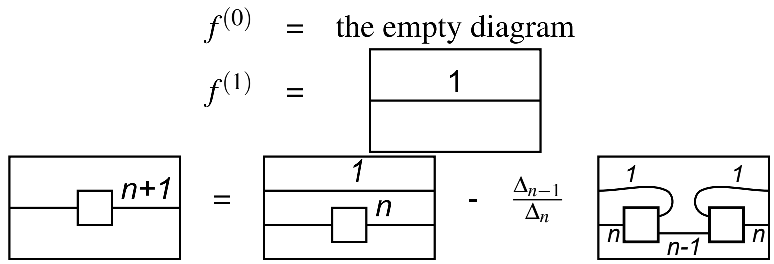


Figure 3.8: Here we show the three properties the Jones-Wenzl idempotents satisfy. This allows us to recursively find each one.

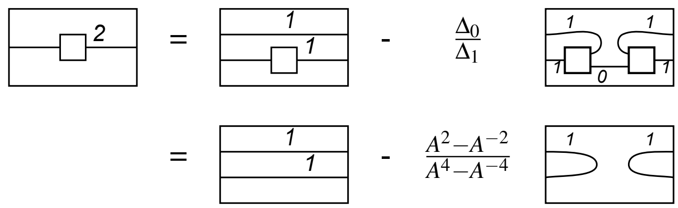


Figure 3.9: This figure shows the expansion of the second Jones-Wenzl idempotent in the Temperley-Lieb algebra.

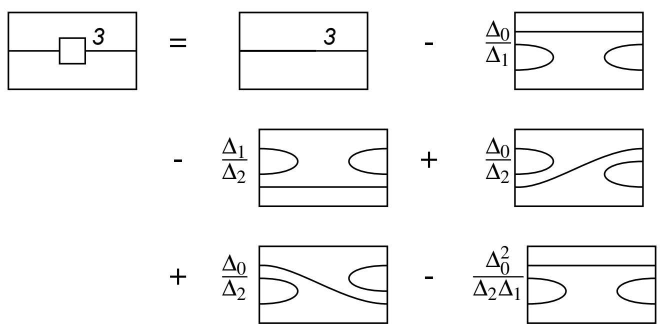


Figure 3.10: This figure shows the expansion of the third Jones-Wenzl idempotent in the Temperley-Lieb algebra.

of a knot in another way. In particular, the $n+1$ colored Jones polynomial of a knot K is the Jones polynomial of K decorated with the $f^{(n)}$, the Jones-Wenzl idempotent in TL_n.

3.2.4 Connection between these definitions

The definition of colored Jones polynomial in terms of Jones polynomials of cablings of the knot and the definition in terms of the Jones-Wenzl idempotents are two ways of looking at the same construction. To see this, note that when we expand the idempotent to its linear combination of diagrams, the diagrams we get are exactly those that we get from the recursive definition.

Recall that we can express this linear combination of cablings recursively as:

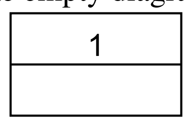
$$g_1 = 1$$

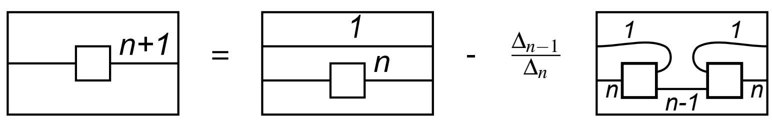
$$g_2 = z$$

$$g_i + 1 = zg_i - g_{i-1}$$

and that the Jones-Wenzl idempotent is defined by .

$f^{(0)}$ = the empty diagram

$f^{(1)}$ =




First, note that both recursions start with the same terms. Then in the recursive definition of the Jones-Wenzl idempotent, it is easy to see that the first term has an additional strand cabling the previous term, corresponding with the zg_{i-1} term in the cabling definition. Now, for the second part, since we are applying the projector to a knot, we can picture have the diagram inside an annulus. See Figure 3.11.

We can move the strands around to picture this in a different view. We cut along the dotted line to get a different perspective. See Figure 3.12. Now, using the rules of the projectors, we can see that for a knot decorated with a Jones Wenzl idempotent, we use one $f^{(n)}$ to cancel the other and then remove the loop by multiplying by $\frac{\Delta_n}{\Delta_{n-1}}$. We are left with -1 times a strand with the projector in $n-2$ strands, giving the $-g_{i-2}$ term in the cabling recursion. This, as described below, only works if we have the idempotent on a connected component.

Because of the extra properties, we know about the Jones-Wenzl idempotents (kills backtracks, absorb lower idempotents, etc.), it is sometimes convenient to use them in our calculations, even though they are the same as the cabling formula.

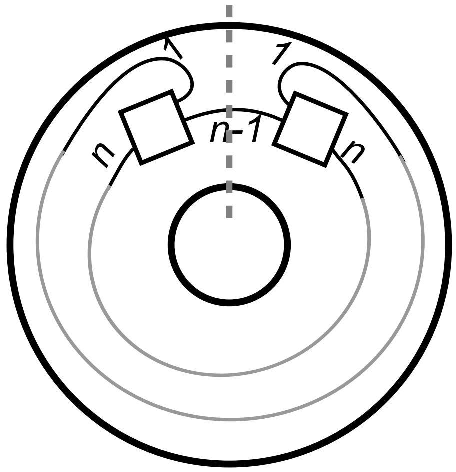


Figure 3.11: The second term in the recursive formula for the Jones Wenzl idempotent pictured inside an annulus

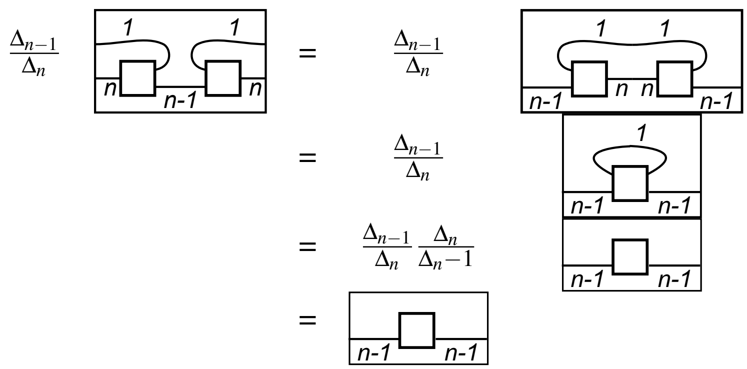


Figure 3.12: We get the above identity only if the Jones-Wenzl idempotent is on a knot. This helps us see why the definitions are equivalent.

3.3 Notes on Normalization

In an attempt to keep things clear, I will use the following conventions through this dissertation.

We will (almost exclusively) consider the normalized colored Jones polynomial $J'_{N,K}(q)$, which is normalized such that $J'_{N,\text{unknot}}(q) = 1$. We write $J_{N,K}(q)$ for the un-normalized colored Jones polynomial. The un-normalized colored Jones polynomial satisfies $J_{N,\text{unknot}}(q) = \Delta_{N-1} = (-1)^{N-1}[N]$.

$$J'_{N,K}(q) = \frac{J_{N,K}(q)}{\Delta_{N-1}}$$

We use the convention that $N = 2$ gives the standard Jones polynomial.

The variable $q = a^2$. This $q = 1/t$ in the standard Jones polynomial. The following definitions are standard and are used through this dissertation:

$$a = A^2$$

$$\{n\} = a^n - a^{-n}$$

$$[n] = \frac{a^n - a^{-n}}{a - a^{-1}}$$

$$\{n\}! = \{n\}\{n-1\}\dots\{1\}$$

$$[n]! = [n][n-1]\dots[1]$$

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}.$$

3.4 Hyperbolic Volume Conjecture

One of the major open questions relating to the colored Jones polynomial is the Hyperbolic Volume Conjecture.

Conjecture 3.4.1 ([Thu], Kashaev-Murakami-Murakami)**.** *For any hyperbolic knot K,*

$$2\pi * \lim_{n\to\infty} \frac{\log|J_{n,K}(e^{2\pi i/n})|}{n} = vol(S^3\backslash K)$$

where $J_{n,K}(e^{2\pi i/n})$ is the normalized Colored Jones Polynomial of a knot K evaluated at a n^{th} root of unity and $vol(S^3\backslash K)$ is the volume of the unique complete hyperbolic Riemannian metric on the knot complement.

This conjecture gives a connection between the quantum invariants of a knot K and its classical geometry. The conjecture is known to be true for torus knots (where both sides are 0) and for the figure eight knot. It has also been proven for certain other knots and classes of knots, see [vdV].

In [DL06], Dasbach and Lin prove the the first and last three coefficients for the colored Jones polynomial stabilize for alternating knots and give a formula for these coefficients.

From this, they up with a volume-ish theorem for the Jones Polynomial, [DL07].

Theorem 3.4.2 ([DL06], Volume-ish Theorem)**.** *For an alternating prime, non-torus knot K let*

$$V_K(t) = a_n t^n + \cdots + a_m t^m$$

be the Jones polynomial of K. Then

$$2v_0 max(|a_{m-1}|, |a_{n+1}| - 1) \leq Vol(S^3 - K) \leq 10v_0(|a_{n+1}| + |a_{m-1}| - 1).$$

Here, $v_0 \approx 1.0149416$ *is the volume of an ideal regular hyperbolic tetrahedron.*

This is done using bounds on the volume based on twist number and showing that the coefficients of the Jones polynomial can be expressed in terms of the twist number. In doing this work, Dasbach and Lin also noticed that the other coefficients of the colored Jones polynomial stabilize as n increases. We focus more of this in the Stability in the Coefficients of the Colored Jones Polynomial chapter.

Chapter 4

A Formula for the Colored Jones Polynomial of $(1,2p-1,r-1)$ Pretzel Knots

In this chapter, we give a formula for the colored Jones polynomial of $(1,2p-1,r-1)$ pretzel knots. In the case where r is even, Garoufalidis and Koutschan give a formula in [GK12]. The work presented here was done independently and works for all r. We begin with the definition of a pretzel knot.

4.1 What is a pretzel knot?

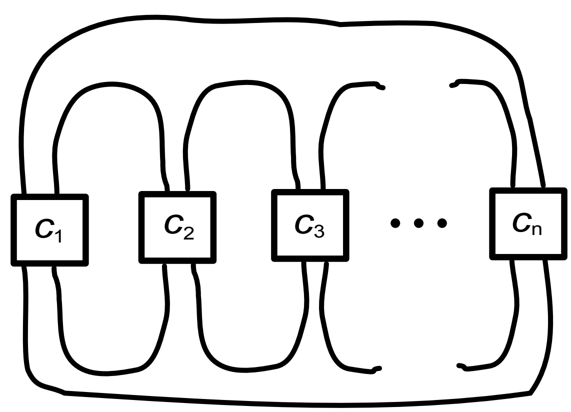


Figure 4.1: A $(c_1, c_2, \ldots, c_n)$ Pretzel Knot. A box with a c_i represents c_i half twists.

A pretzel knot or link is usually described by P$(c_1, c_2, \dots c_n)$ where each c_i is an integer corresponding the number of half twists within that section of the knot. These twisted parts are drawn vertically. Positive c_i correspond with positive half twists, while negative c_i correspond with negative half twists. See Figure 4.1. In order for this to form a knot (have only one component), we need either exactly one c_i to be even, or both n and each c_i to be odd. If n is even and each c_i is odd, we get a two component link. If at least one c_i is even, then the number of components is equal to the number of even c_i.

We consider pretzel knots of the form $P(1, 2p-1, r-1)$.

4.2 The Formula

Theorem 4.2.1. *A pretzel knot of the form $K_{p,r} = P(1, 2p-1, r-1)$ has the colored Jones polynomial*

$$J'_{N,K_{p,r}}(a^2) = \sum_{n=0}^{N-1} c'_{n,p} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} \mu_n^* \sum_{k=0}^{n} \delta(2k; n, n)^r \frac{\langle 2k \rangle}{\langle n, n, 2k \rangle} \frac{([k]!)^2}{[2k]!} \frac{\{2n+1\}!}{\{n\}!\{1\}}$$

Corollary 4.2.2. *When r is even this reduces to*

$$J_{N,K_{p,r}}(a^2) = \sum_{n=0}^{N-1} (-1)^n \begin{bmatrix} N+n \\ N-n-1 \end{bmatrix} c'_{n,p} \frac{\{2n+1\}!}{\{1\}} c'_{n,r/2}.$$

Corollary 4.2.3. *When r is odd this reduces to*

$$J_{N,K_{p,r}}(a^2) = \sum_{n=0}^{N-1} (-1)^n \mu_n^{4p} c'_{n,p} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} \frac{\{2n+1\}!\{n\}!}{(a-a^{-1})^{2n}\{1\}} \sum_{k=0}^{n} \mu_{2k}{}^{\frac{r}{2}} \frac{[2k+1]}{[n+k+1]![n-k]!}$$

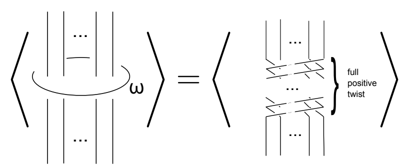
Corollary 4.2.4. *One reduced way to write the formula the works for all r is*

$$\sum_{n=0}^{N-1} (-1)^n \begin{bmatrix} N+n \\ N-n-1 \end{bmatrix} c'_{n,p} \frac{\{2n+1\}!\{n\}!}{\{1\}(a-a^{-1})^{2n}} \sum_{k=0}^{n} (-1)^{k(r+1)} \frac{[2k+1]}{[n+k+1]![n-k]!} \mu_{2k}^{r/2}.$$

4.2.1 The notation

We think of the N colored Jones Polynomial of a knot K as the Kauffman bracket of K cabled by e_{N-1}, where e_{N-1} is the Jones-Wenzl idempotent. We normalize so that the unknot has colored Jones Polynomial 1. For now, we turn our attention to the Kauffman bracket.

The Kauffman bracket gives an isomorphism from the skein module $K(M)$ of an oriented 3-manifold M to $\mathbb{Z}[A^{\pm}]$. It is normalized so that the bracket of the empty link is 1. The element $\omega \in K(M)$ as defined in [Mas03] has the property that the bracket of a link with ω linked around an even number of strands has the same Kauffman bracket as the same link with a positive full twist in place of the ω.



We can extend this to ω^p which has the property of inducing p full right hand twists. (If p is negative, we get left handed twists.) In [Mas03], the formula for ω is proved. We will simply restate it here and then define the necessary pieces of the formula.

$$\omega^p = \sum_{n=0}^{\infty} c'_{n,p} R'_n.$$

Here, R'_n is a basis for the skein module of the solid torus which is isomorphic to $\mathbb{Z}[A^{\pm}][z]$ Specifically, $R'_n = (n!)^{-1} \prod_{i=0}^{n-1} (z - \lambda_{2i})$ where $\lambda_i = -a^{i+1} - a^{-i-1}$. The coefficients are

$$c'_{n,p} = \frac{1}{(a-a^{-1})^n} \sum_{k=0}^{n} (-1)^k \mu_{2k}^p [2k+1] \frac{[n]!}{[n+k+1]![n-k]!},$$

where $\mu_i = (-1)^i A^{i^2+2i}$ and as usual

$$a = A^2, \{n\} = a^n - a^{-n}, [n] = \frac{a^n - a^{-n}}{a - a^{-1}}$$

Also, as can be expected

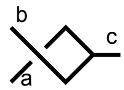
$$\{n\}! = \{n\}\{n-1\}\ldots\{1\}$$

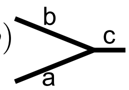
$$[n]! = [n][n-1]\ldots[1]$$

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}.$$

The other terms that appear in the formula come from the evaluation of standard pieces done in [Mas03]. There are three basic pieces. In each of the following the first equalities define the term while the last gives the equivalent term in [AD11] that we'll need in Chapter 5

The first is δ, the half twist coefficient.



$= \delta(c;a,b)$


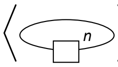
We know that

$$\delta(c;a,b) = (-1)^{\frac{a+b-c}{2}} A^{-a-b+c+\frac{-a^2-b^2+c^2}{2}} \quad \left(= \gamma(a,b,c)^{-1}\right)$$

and

$$\delta(c;a,b)^2 = \frac{\mu_c}{\mu_a \mu_b}.$$

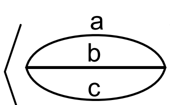
The term

$\langle n \rangle = \langle$  $\rangle = (-1)^n [n+1] \quad (= \Delta_n).$

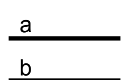
Assume (a,b,c) is an admissible triple, then let i,j,k be the internal colors, in particular

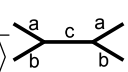
$$i = (b+c-a)/2 \qquad j = (a+c-b)/2 \qquad k = (a+b-c)/2.$$

The term $\langle a,b,c \rangle$ is the trihedron coefficient. In particular,

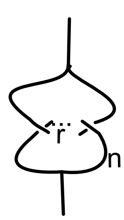
$\langle a,b,c \rangle = \Big\langle$  $\Big\rangle = (-1)^{i+j+k} \frac{[i+j+k+1]![i]![j]![k]!}{[i+j]![j+k]![i+k]!} \quad (= \theta(a,b,c)).$

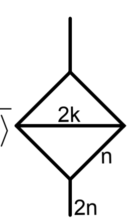
Using these we can get the fusion equation,



$= \sum_c \frac{\langle c \rangle}{\langle a,b,c \rangle}$


By combining the fusion and twist coefficients, we can also get



$= \sum_{k=0}^{n} \delta(2k;n,n)^{2p} \frac{\langle 2k \rangle}{\langle n,n,2k \rangle}$


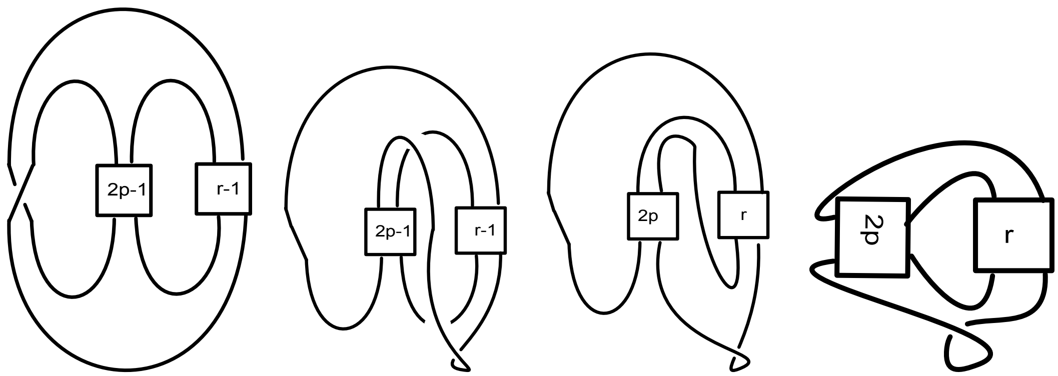


Figure 4.2: We can move the leftmost strand in the left-most image over the middle strand to view the$(1,2p-1,r-1)$ pretzel knot as a double twist knot

4.2.2 Proof

The first step is to redraw the $(1,2p-1,r-1)$ pretzel knot as a double twist knot. To do this, we pull the over-strand of the single crossing over the middle crossing section and then rotate this twist region a quarter turn in the clockwise direction. See Figure 4.2. Now we are ready to compute the colored Jones polynomial of this diagram. We will use the definition used in [Mas03], namely, $J_N(K_{p,r},a^2) = (-1)^{N-1}\langle K(e_{N-1})\rangle$. We will also use the fact that

$$\omega^p = \sum c'_{k,p} R'_k$$

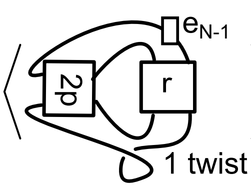
and

$$e_{N-1} = \sum_{n=0}^{N-1} (-1)^{N-1-n} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} R_n$$

Thus we have

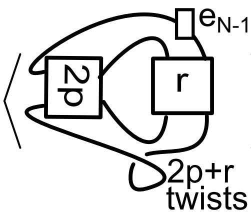
$$J_N(K_{p,r},a^2) = (-1)^{N-1}\langle K(e_{N-1})\rangle$$

$$= (-1)^{N-1} \left\langle \quad \right\rangle$$



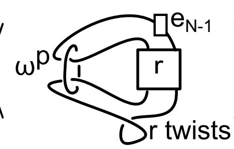
Since we are taking the Kauffman bracket, and this is not invariant under R1, we need to be careful to not remove kinks. We also want the knot to be zero-framed. The framing depends on whether r is even or odd. At this point, if r is even, the framing is $-2p-2+1$. When r is odd, the framing is $2p-r+1$. We now will add $2p+r-1$ twists. This gives us a 0-framing in the case where r is even. For the case where r is odd we should add $-2p+r-1$ twists. This will give us $-2p+r$ twists. The images below are for the r even case. We just need to relabel the number of twists to get the r odd case. Thus we have:

$$J_N(K_{p,r},a^2) = (-1)^{N-1}\left\langle \cdots \right\rangle$$



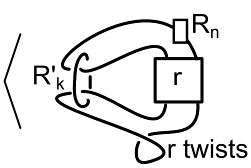
Now, we place ω^p around the part where there is $2p$ twists. This undoes the $2p$ full twists but also changes the framing (i.e. it undoes $2p$ of the extra twists.) In the undrawn case where r is odd we now have $4p+r$ twists. We get

$$J_N(K_{p,r},a^2) = (-1)^{N-1}\left\langle \cdots \right\rangle$$



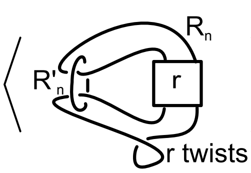
Now, we use the expansions of e_{N-1} and ω^p from above to get

$$J_N(K_{p,r},a^2) = (-1)^{N-1}\sum_k c'_{k,p}\sum_{n=0}^{N-1}(-1)^{N-1-n}\begin{bmatrix} N+n \\ N-1-n \end{bmatrix}\left\langle \cdots \right\rangle$$



Now, following [Mas03] Sec 5, when each component is a zero-framed unknot with a spanning disk pierced twice by the other component, the only terms that are nonzero are those where $k=n$. Thus we have

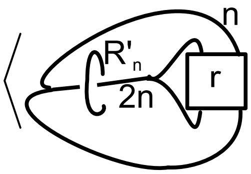
$$J_N(K_{p,r},a^2) = \sum_{n=0}^{N-1} c'_{n,p}(-1)^n\begin{bmatrix} N+n \\ N-1-n \end{bmatrix}\left\langle \cdots \right\rangle$$



Since $R_n - e_n$ has degree less than n we can replace R_n with e_n and then do fusion along the two strands inside the leftmost R_n.

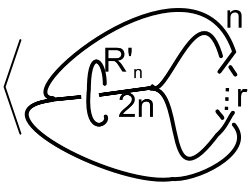
At this point, we will remove the extra twists. Each twist changes the colored Jones polynomial by μ_n. Thus, the change depends on if r is even. Define:

$$\mu_n^*(r) = \begin{cases} \mu_n^r & : r \text{ even} \\ \mu_n^{4p+r} & : r \text{ odd} \end{cases}$$

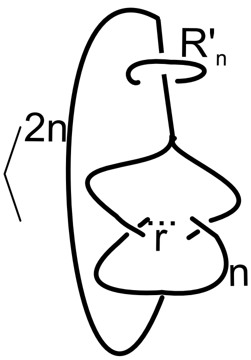
$$J_N(K_{p,r},a^2) = \sum_{n=0}^{N-1} c'_{n,p}(-1)^n \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} \mu_n^* \left\langle \vphantom{\sum} \right.$$


$$\left. \vphantom{\sum} \right\rangle$$

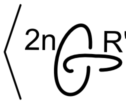
We will now draw out the r twist and rotate the diagram.

$$J_N(K_{p,r},a^2) = \sum_{n=0}^{N-1} c'_{n,p}(-1)^n \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} \mu_n^* \left\langle \vphantom{\sum} \right.$$


$$\left. \vphantom{\sum} \right\rangle$$

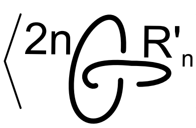
$$= \sum_{n=0}^{N-1} c'_{n,p}(-1)^n \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} \mu_n^* \left\langle \vphantom{\sum} \right.$$


$$\left. \vphantom{\sum} \right\rangle$$

$$J_N(K_{p,r},a^2) = \sum_{n=0}^{N-1} c'_{n,p}(-1)^n \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} \mu_n^* \sum_{k=0}^{n} \delta(2k;n,n)^r \frac{\langle 2k \rangle}{\langle n,n,2k \rangle} \frac{([k]!)^2}{[2k]!} \left\langle \vphantom{\sum} \right.$$


$$\left. \vphantom{\sum} \right\rangle$$

Finally,

$$\left\langle \vphantom{\sum} \right.$$


$$\left. \vphantom{\sum} \right\rangle = \frac{(-1)^n \{2n+1\}!}{\{n\}!\{1\}}$$

so we get

$$J_N(K_{p,r},a^2) = \sum_{n=0}^{N-1} c'_{n,p} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} \mu_n^* \sum_{k=0}^{n} \delta(2k;n,n)^r \frac{\langle 2k \rangle}{\langle n,n,2k \rangle} \frac{([k]!)^2}{[2k]!} \frac{\{2n+1\}!}{\{n\}!\{1\}}$$

4.2.3 Simplification

First we reorder the terms.

$$J_N(K_{p,r},a^2) = \sum_{n=0}^{N-1} c'_{n,p} \begin{bmatrix} N+n \\ N-1-n \end{bmatrix} \frac{\{2n+1\}!}{\{n\}!\{1\}} \sum_{k=0}^{n} \mu_n^* \delta(2k;n,n)^r \frac{\langle 2k \rangle}{\langle n,n,2k \rangle} \frac{([k]!)^2}{[2k]!}$$

Now we will simplify some of the pieces. Notice that:

$$\frac{\langle 2k\rangle}{\langle n,n,2k\rangle}\frac{([k]!)^2}{[2k]!} = \frac{(-1)^{2k}[2k+1][n]![n]![2k]![k]!^2}{(-1)^{n+k}[n+k+1]![k]![k]![n-k]![2k]!} \tag{4.1}$$

$$= \frac{(-1)^{n+k}[n]!^2[2k+1]}{[n+k+1]![n-k]!} \tag{4.2}$$

Now, we'll consider the cases when r is even and r is odd separately. First, consider the case where r is even. From [Mas03], we know $\delta(c;a,b)^2 = \frac{\mu_c}{\mu_a\mu_b}$. So

$$\delta(2k;n,n)^r = \left(\frac{\mu_{2k}}{\mu_n\mu_n}\right)^{r/2} = \frac{{\mu_{2k}}^{r/2}}{{\mu_n}^r} \tag{4.3}$$

When r is even $\mu_n^*(r) = \mu_n^r$. Thus our equation reduces to

$$J_N(K_{p,r},a^2) = \sum_{n=0}^{N-1} c'_{n,p}\begin{bmatrix} N+n \\ N-1-n\end{bmatrix}\frac{\{2n+1\}!}{\{n\}!\{1\}}\sum_{k=0}^{n}{\mu_{2k}}^{r/2}\frac{(-1)^{n+k}[n]!^2[2k+1]}{[n+k+1]![n-k]!}$$

$$= \sum_{n=0}^{N-1} c'_{n,p}\begin{bmatrix} N+n \\ N-1-n\end{bmatrix}\frac{\{2n+1\}!}{\{n\}!\{1\}}(-1)^n\frac{\{n\}!^2}{(a-a^{-1})^{2n}}\sum_{k=0}^{n}\frac{(-1)^k{\mu_{2k}}^{r/2}[2k+1]}{[n+k+1]![n-k]!}$$

Now, again from [Mas03], we have

$$c'_{n,\rho} = \frac{\{n\}!}{(a-a^{-1})^{2n}}\sum_{k=0}^{n}\frac{(-1)^k\mu_{2k}^{\rho}[2k+1]}{[n+k+1]![n-k]!} \tag{4.4}$$

Thus

$$J_N(K_{p,r},a^2) = \sum_{n=0}^{N-1} c'_{n,p}\begin{bmatrix} N+n \\ N-1-n\end{bmatrix}\frac{\{2n+1\}!}{\{1\}}(-1)^n c'_{n,r/2}$$

as claimed in Corollary 4.2.2.

Now when r is odd, we can again use (4.2) but we need to simplify $\delta(2k,n,n)$ differently. From [MV94]

$$\delta(2k;n,n) = (-1)^{n-k}a^{k^2+k}a^{-\left(\frac{n^2}{2}+n\right)}$$

so

$$\begin{aligned}
\delta(2k;n,n)^r &= \delta(2k;n,n)^{r-1}\delta(2k;n,n) \\
&= \frac{\mu_{2k}{}^{\frac{r-1}{2}}}{\mu_n{}^{r-1}}(-1)^{n-k}a^{k^2+k}a^{-\left(\frac{n^2}{2}+n\right)} \\
&= \frac{\mu_{2k}{}^{\frac{r-1}{2}}}{\mu_n{}^{r-1}}\frac{(-1)^k a^{k^2+k}}{(-1)^n a^{\left(\frac{n^2}{2}+n\right)}} \\
&= \frac{\mu_{2k}{}^{\frac{r-1}{2}}}{\mu_n{}^{r-1}}\frac{(-1)^k a^{(2k^2+2k)(1/2)}}{(-1)^n a^{\left(\frac{n^2}{2}+n\right)}} \\
&= \frac{\mu_{2k}{}^{\frac{r-1}{2}}}{\mu_n{}^{r-1}}\frac{(-1)^k \mu_{2k}^{1/2}}{\mu_n} \\
&= \frac{(-1)^k \mu_{2k}{}^{\frac{r}{2}}}{\mu_n{}^r}
\end{aligned}$$

Note this only differs from (4.3) by a factor of $(-1)^k$. This observation combined with the further simplification below will lead us to Corollary 4.2.4. Let's continue with the r odd case.

$$\begin{aligned}
J_N(K_{p,r},a^2) &= \sum_{n=0}^{N-1} c'_{n,p}\begin{bmatrix} N+n \\ N-1-n\end{bmatrix}\frac{\{2n+1\}!}{\{n\}!\{1\}}\sum_{k=0}^{n}\mu_n^*\delta(2k;n,n)^r\frac{\langle 2k\rangle}{\langle n,n,2k\rangle}\frac{([k]!)^2}{[2k]!} \\
&= \sum_{n=0}^{N-1} c'_{n,p}\begin{bmatrix} N+n \\ N-1-n\end{bmatrix}\frac{\{2n+1\}!}{\{n\}!\{1\}}\sum_{k=0}^{n}\mu_n^{4p+r}\frac{(-1)^k\mu_{2k}{}^{\frac{r}{2}}}{\mu_n{}^r}\frac{(-1)^{n+k}[n]!^2[2k+1]}{[n+k+1]![n-k]!} \\
&= \sum_{n=0}^{N-1} (-1)^n c'_{n,p}\begin{bmatrix} N+n \\ N-1-n\end{bmatrix}\frac{\{2n+1\}![n]!^2}{\{n\}!\{1\}}\sum_{k=0}^{n}\mu_n^{4p}\mu_{2k}{}^{\frac{r}{2}}\frac{[2k+1]}{[n+k+1]![n-k]!} \\
&= \sum_{n=0}^{N-1} (-1)^n \mu_n^{4p} c'_{n,p}\begin{bmatrix} N+n \\ N-1-n\end{bmatrix}\frac{\{2n+1\}!\{n\}!}{(a-a^{-1})^{2n}\{1\}}\sum_{k=0}^{n}\mu_{2k}{}^{\frac{r}{2}}\frac{[2k+1]}{[n+k+1]![n-k]!}
\end{aligned}$$

4.3 Which knots can we study?

Table 4.1 list out the knots up to 9 crossings that have diagrams of the form of a $(1,2p,r-1)$-pretzel knot.

These formulas were coded in to Mathematica to allow us to calculate out the colored Jones polynomials for these knots. The code is available at:

http://www.math.ucsd.edu/ k3walsh/research.php.

Table 4.1: Knots with up to 9 crossing that can be expressed as a $(1,2p,r-1)$-pretzel knot

Knot	Twists	Pretzel Notation	(p,r)
3_1	1	(1,3,0) or (1,1,1)	(2,1) or (1,2)
4_1		(1,1,2)	(1,3)
5_1		(1,5,0)	(3,1)
5_2	2	(1,3,1) or (1,1,3)	(2,2) or (1,4)
6_1		(1,1,4)	(1,5)
6_2		(1,3,2)	(2,3)
7_1		(1,7,0)	(4,1)
7_2	3	(1,1,5) or (1,5,1)	(1,6) or (3,2)
7_4		(1,3,3)	(2,4)
8_1		(1,1,6)	(1,7)
8_2		(1,5,2)	(3,3)
8_4		(1,3,4)	(2,5)
9_1		(1,9,0)	(5,1)
9_2	4	(1,1,7) or (1,7,1)	(1,8) or (4,2)
9_5		(1,3,5) or (1,5,3)	(2,6) or (3,4)

Chapter 5

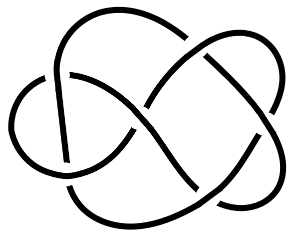
Stability in the Coefficients of the Colored Jones Polynomial

5.1 Stability in the Head and Tail

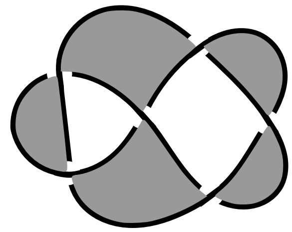
For a sequence of polynomials, we say the head of this sequence, if it exists, is the polynomial whose highest n terms agree with the highest n terms of the kth polynomial in the sequence, for all $k \geq n$. The tail of this sequence, if it exists, is the polynomial whose lowest n terms agree with the lowest n terms of the kth polynomial in the sequence, for all $k \geq n$.

Dasbach and Lin conjectured that the head and tail existed in [DL07]. Armond showed in [Arm13] that this conjecture was true for adequate links. In [GL11], Garoufalidis and Le independently proved this conjecture using a different technique.

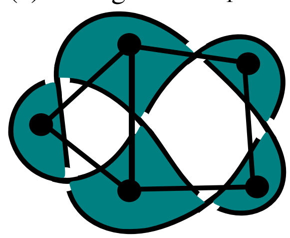
In [AD11], Armond and Dasbach show that the head and tail of the colored Jones polynomial of alternating links only depends on the reduced checkerboard graphs of the knot diagrams. Given an alternating diagram of a knot, we can assign a (gray/white) checkerboard coloring the faces in the diagram. We then place a vertex in each of the gray colored regions. We draw an edge between vertices for every crossing between the corresponding regions. Alternatively, we can start by placing a vertex in every white region to get the dual graph. If, when moving along an edge, the overcrossing starts on the right of the edge and ends on the left, this graph is the A-checkerboard graph. If the overcrossing goes from the left to the right, the graph is the B-checkerboard graph. See Figure 5.1. To get the reduced checkerboard, we can replace parallel edges in the graph, i.e. multiple edges between the same vertices, with a single edge.



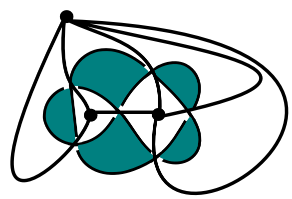
(a) A diagram of 6_1



(b) 6_1 with a checkerboard coloring



(c) The A-checkerboard graph



(d) The B-checkerboard graph

Figure 5.1: The Knot 6_1 and its associated graphs.

Theorem 5.1.1 ([AD11])**.** *Let K_1 and K_2 be the two alternating links with alternating diagrams D_1 and D_2 such that the reduced A-checkerboard (respectively B checkerboard) graphs of D_1 and D_2 coincide. Then the tails (respectively heads) of the colored Jones polynomial of K_1 and K_2 are identical.*

One only needs to prove this theorem for either the head or the tail since under taking a mirror image of a knot, the tail and head switch. The proof only focuses on the tail of the polynomial, i.e. the lowest powers of A in our polynomial. Since $q = A^{-4}$ proving that the tail of the polynomial with coefficients A stabilizes is equivalent to proving the head of the polynomial in coefficient q stabilizes. The main idea of the proof of this theorem is to show that the tail only depends on the highest term in the summand that gives the colored Jones polynomial, using a state sum formula based on fusion. Armond and Dasbach do this by proving three lemmas. Because these lemmas will be useful in the new theorems that follow, I will restate them here. But first, we will give an expression for the colored Jones polynomial of a knot K from its diagram D.

We identify all of the negative twist regions of the D. A negative twist region is a section of the knot with one or more consecutive negative twists. (Here, we think of negative twists without considering orientation so assume each strand is oriented in the same direction.) Let k be the number of twist regions.

Now, we will do a fusion along each of the k negative twist region and then use the twist coefficients to remove all crossings. See Chapter 4 for details on the fusion and twist coefficients. Here, to keep with the convention of the paper we are referring to we set $\langle n\rangle = \Delta_n$, $\langle a,b,c\rangle = \theta(a,b,c)$ and we let $\gamma(a,b,c)$ be our negative half twist coefficients so $\delta(c;a,b) = \gamma(a,b,c)^{-1}$.

Each of the k regions corresponding to the negative twists will be labeled with $2j_i$ for $1 \leq i \leq k$. The other edges will be labeled n. Thus we have a multi-sum of trivalent graphs, $\Gamma_{n,(j_1,\dots,j_k)}$ where each j_i runs from 1 to n. The evaluation of this graph gives us the $n+1$ colored Jones polynomial of the knot K. i.e.,

$$J_{n+1,K} = \sum_{j_1,\dots j_k=0}^{n} \prod_{i=1}^{k} \gamma(n,n,2j_i)^{m_i} \frac{\Delta_{2j_i}}{\theta(n,n,2j_i)} \Gamma_{n,(j_1,\dots,j_k)}$$

Armond and Dasbach show that when we decrease a single j_i from n to $n-1$, we increase the lowest degree by $n+1$ and we continue to increase it as we decrease j_1 further, so the only graphs that contribute to the first $n+1$ terms of the colored Jones polynomial, the tail, are the ones where all $j_i = n$.

We will use the notation $f(q) \stackrel{\cdot n}{=} g(q)$ if the first n terms of $f(q)$ agree with those of $g(q)$, i.e once we multiply by some power of q to get f^* and g^* so that $f^*(q)$ and $g^*(q)$ both have lowest degree equal to zero, then $f^*(q) - g^*(q)$ has lowest degree q^n.

Thus we have

$$J_{n+1,K} \stackrel{\cdot n+1}{=} \prod_{i=1}^{k} \gamma(n,n,2n)^{m_i} \frac{\Delta_{2n}}{\theta(n,n,2n)} \Gamma_{n,(n,\dots,n)}$$

For a rational function R, let $d(R)$ be the minimum degree of R considered as a power series when you expand $\mathbb{Z}(q) \hookrightarrow \mathbb{Z}[q^{-}1,q]]$. We choose to expand the power series allowing for infinite terms in the positive direction so that we get a minimum degree.

Lemma 5.1.2. *When we decrease j_i from n to $n-1$, when increase the minimum degree* *of* $\gamma(n,n,j_i)$ *by* $4n$*, i.e.,*

$$d(\gamma(n,n,2(n-1))) = d(\gamma(n,n,2n)) + 4n.$$

As we continue to decrease each j_i *the minimum degree continues to increase, i.e.,*

$$d(\gamma(n,n,2(j-1))) \geq d(\gamma(n,n,2j)).$$

Lemma 5.1.3. *Each time we decrease the* j_i*, we increase the minimum degree of* $\frac{\Delta_{2j}}{\theta(n,n,2j)}$ *by 2, i.e.,*

$$d\left(\frac{\Delta_{2(j-1)}}{\theta(n,n,2(j-1))}\right) = d\left(\frac{\Delta_{2j}}{\theta(n,n,2j)}\right) + 2.$$

Lemma 5.1.4. *When we decrease* j_i *from n to* $n-1$*, when increase the minimum degree of* $\Gamma_{n,(j_1,\ldots,j_{i-1},j_i,j_{i+1},\ldots,j_k)}$ *by at least 2, i,e.,*

$$d(\Gamma_{n,(n,\ldots,n-1,\ldots,n)}) \geq d(\Gamma_{n,(n,\ldots,n,\ldots,n)}) + 2.$$

We can only guarantee this change of two at the first step. As we continue to decrease j_i*, the best we get is:*

$$d(\Gamma_{n,(j_1,\ldots,j_{i-1},j_i-1,j_{i+1},\ldots,j_k)}) \geq d(\Gamma_{n,(j_1,\ldots,j_{i-1},j_i,j_{i+1},\ldots,j_k)}) \pm 2.$$

Thus when we decrease a single j_i from n to $n-1$, we increase the lowest degree in A by at least $(4n)m_i + 2 + 2 \geq 4n + 4$ which means we change the q degree by $n+1$ for the $(n+1)$ Colored Jones Polynomial. We also continue to increase the lowest degree in A as we continue to change the j_i so the only term that contributes to the lowest $4n+4$ terms of the polynomial in A are the ones where each $j_i = n$.

Let c be the number of crossings of the knot. Then $c = \sum_{i=1}^{k} m_i$. Now since $\gamma(n,n,2n)$ does not depend on j, we can pull each one out of the product. To get:

$$\begin{aligned} J_{n+1,K} &\overset{\cdot n+1}{=} \prod_{i=1}^{k} \gamma(n,n,2n)^{m_i} \frac{\Delta_{2n}}{\theta(n,n,2n)} \Gamma_{n,(n,\ldots,n)} \\ &\overset{\cdot n+1}{=} \gamma(n,n,2n)^{\sum_{i=1}^{k} m_i} \prod_{i=1}^{k} \frac{\Delta_{2n}}{\theta(n,n,2n)} \Gamma_{n,(n,\ldots,n)} \\ &\overset{\cdot n+1}{=} \gamma(n,n,2n)^{c} \prod_{i=1}^{k} \frac{\Delta_{2n}}{\theta(n,n,2n)} \Gamma_{n,(n,\ldots,n)} \\ &\overset{\cdot n+1}{=} \gamma(n,n,2n)^{c} \left(\frac{\Delta_{2n}}{\theta(n,n,2n)}\right)^{k} \Gamma_{n,(n,\ldots,n)} \end{aligned}$$

In fact, $\gamma(n,n,2j) = \pm A^{2n-2j+n^2-2j}$. Thus γ only contributes an overall shift to the polynomial and doesn't affect the sequence of leading coefficients. Thus:

$$J_{n+1,K} \stackrel{\cdot n+1}{=} \left(\frac{\Delta_{2n}}{\theta(n,n,2n)}\right)^k \Gamma_{n,(n,\ldots,n)}$$

This only depends on the overall graph structure, forgetting about the twists. This is the same information we forget when going to the reduced graph. This proves Theorem 5.1.1. It does not, however, tells us anything about the existence of the head and tail of alternating or adequate knots. It just tells us that if the head and tail exist, they only depend on the overall graph structure.

To show the head and tail exist, we must show that the evaluation of the n-trivalent graph is the same as the evaluation of the $n+1$-trivalent graph. Armond does this in [Arm13] by demonstrating a way to reduce the $n+1$-colored graph to the n-colored graph by peeling off one of the strands without changing the first n coefficients.

In [GL11], Garoufalidis and Le show a higher order stability described in the next section. We can use similar techniques to those used by Armond and Dasbach to find which knots will have the same higher order stabilizing sequences.

Corollary 5.1.5. *Let m be the minimum of the m_i. When we change a single j_i from n to $n-1$, we increase the lowest degree in A by at least $(4n)m+2+2 = 4nm+4$ and thus change the q degree by $n(m)+1$ for the $(n+1)$ Colored Jones Polynomial. In addition to the first $n+1$ terms only depending on the overall graph structure, the next $(m-1)n$ terms also depend only on the graph structure.*

Remark 5.1.6. *This only tells us they depend on a certain structure not that they stabilize. In [GL11], the stabilization is proved, but in chunks of $n+1$ not a single $n+1$ and then n.*

5.2 Higher Order Stability

In [GL11], Garoufalidis and Le define the property of k-stability for a sequence of polynomials as follows:

Definition 5.2.1. *Suppose $f_n(q), f(q) \in \mathbb{Z}((q))$, i.e. $f_n(q)$ and $f(q)$ are formal Laurent series, i.e*

$$f_n(q) = \sum_{m \geq M_n} a_{n,m} x^m \qquad a_{n,m} \in \mathbb{Z}$$

and

$$f(q) = \sum_{m \geq M} b_m x^m \qquad b_m \in \mathbb{Z}$$

We write that

$$\lim_{n \to \infty} f_n(q) = f(q)$$

if

- *there exists C such that $M_n \geq C$ for all n, and*
- *for each j, there exist N_j such that for all $n > N_j$.*

$$f_n(q) - f(q) \in q^j \mathbb{Z}[[q]]$$

In particular, $a_{n,j} = b_j$.

Definition 5.2.2. *A sequence $(f_n(q)) \in \mathbb{Z}((q))$ is* k-stable *is there exist $\Phi_j(q) \in \mathbb{Z}((q))$ for $j = 0, \ldots, k$ such that*

$$\lim_{n \to \infty} q^{-k(n+1)} \left(f_n(q) - \sum_{j=0}^{k} \Phi_j(q) q^{j(n+1)} \right) = 0.$$

We call $\Phi_k(q)$ the k-limit of $(f_n(q))$. We say that $(f_n(q))$ is stable *if it is k-stable for all k.*

For example, a sequence $(f_n(q))$ is *3-stable* if

$$\lim_{n \to \infty} q^{-3(n+1)} \left(f_n(q) - \left(\Phi_0(q) + q^{(n+1)} \Phi_1(q) + q^{2(n+1)} \Phi_2(q) + q^{3(n+1)} \Phi_3(q) \right) \right) = 0.$$

Consider the sequence of colored Jones polynomials of a knot K. We can shift these polynomials by multiplying by the appropriate power of q so that the minimum degree is 0. We call this sequence $(\hat{J}_{n,K}(q))$.

Theorem 5.2.3. *For every alternating link K, the sequence $(\hat{J}_{n,K}(q))$ is stable and its associated k-limit $\Phi_{k,K}(q)$ can be effectively computed from any reduced alternating diagram D of K.*

Remark 5.2.4. *Data from the figure 8 knot suggests that it would not be 4-stable and data from the knot 8_5 suggests it would not be 2-stable using this definition since it requires the stabilization in pieces of size $n+1$ instead of n for the n Colored Jones Polynomial.*

For example, for the figure 8 knot, we know the that first coefficients stabilize to the pentagonal number sequence. By this, I mean that for the figure 8 knot,

$$\Phi_0 = \prod_{n=1}^{\infty}(1-q^n) = \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{k}{2}(3k-1)}.$$

In the table below, I have listed out the first 16 coefficients of the N-colored Jones polynomial for the figure 8 knot for $N = 3,4$ and 5. We see that the first $N+1$ coefficients of the N-colored Jones polynomial are the same as the first $N+1$ coefficients of Φ_0.

Φ_0	1	-1	-1	0	0	1	0	1	0	0	0	0	-1	0	0	-1	$\cdots$
$N=3$	1	-1	-1	0	2	0	-2	0	3	0	-3	0	3	0	-3	0	$\cdots$
$N=4$	1	-1	-1	0	0	3	-1	-1	-1	-1	5	-1	-2	-2	-1	6	$\cdots$
$N=5$	1	-1	-1	0	0	1	2	0	-2	-1	-1	1	3	1	-2	-3	$\cdots$

Now, since we know all of Φ_0, we can subtract it from the shifted colored Jones polynomials. Now are coefficients are:

Φ_0	1	-1	-1	0	0	1	0	1	0	0	0	0	-1	0	0	-1	$\cdots$
$N=3$	0	0	0	0	2	-1	-2	-1	3	0	-3	0	4	0	-3	1	$\cdots$
$N=4$	0	0	0	0	0	2	-1	-2	-1	-1	5	-1	-3	-2	-1	7	$\cdots$
$N=5$	0	0	0	0	0	0	2	-1	-2	-1	-1	1	4	1	-2	-2	$\cdots$

Shifting these sequences back so that they start with a non-zero term, we can see that they again stabilize. The sequence they stabilize to is Φ_1.

Φ_1	2	-1	-2	-1	-1	1	$\cdots$						
$N=3$	2	-1	-2	-1	3	0	-3	0	4	0	-3	1	$\cdots$
$N=4$	2	-1	-2	-1	-1	5	-1	-3	-2	-1	7	$\cdots$	
$N=5$	2	-1	-2	-1	-1	1	4	1	-2	-2	$\cdots$		

I call the sequence Φ_1 the “neck of the tail" or the “tailneck" of the colored Jones polynomial of the figure 8 knot. In the next section, we will determine what this polynomial is for a certain class of knots.

5.3 Knots which reduce to a triangle graph

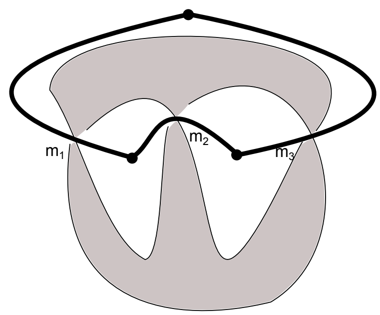
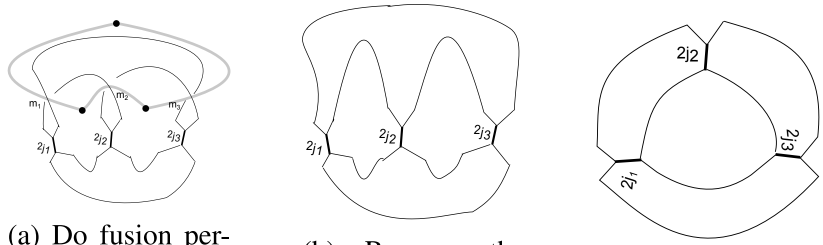


Figure 5.2: A trefoil knot with its checkerboard graph.

In this section, we will find the neck of the colored Jones polynomial of knots whose reduced checkerboard graph is a triangle graph. We define the neck to be the polynomial which when added in the right way to the head will have the property that its first $2N+1$ terms agree with the first $2N+1$ terms of the N+1 colored Jones polynomial. The knots we will focus on can be drawn like the trefoil in Figure 5.2, except we will have more crossings below the pictured crossings (and thus more parallel edges before we reduce the graph). The m_i represent the number of crossings in each section. As it is drawn, each $m_i = 1$. (If $m_1 = 2$ and the others are 1, we get the figure 8 knot.)

Following, [AD11], we can find the colored Jones polynomial by doing fusion perpendicular to the edges of the graph. We then can undo the twists above the fusion using the twist coefficients. See below for the overall picture and the next page for specifics.

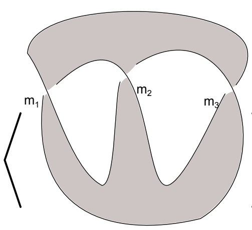


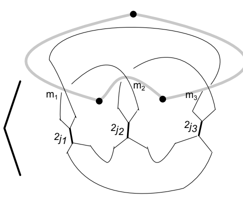
(a) Do fusion perpendicular to the edges of the graph

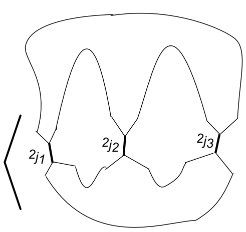
(b) Remove the twists using the twist coefficients

(c) Reorient the diagram

Figure 5.3: Steps to finding the colored Jones polynomial of the trefoil and similar knots

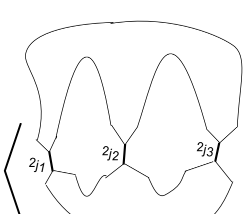
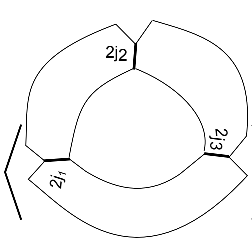
$$J_{N+1,K}(q) = \Big\langle$$  $$\Big\rangle$$

$$= \sum_{j_i=0}^{N}\prod_{i=1}^{3}\frac{\Delta_{2j_i}}{\theta(N,N,2j_i)}\Big\langle$$  $$\Big\rangle$$

$$= \sum_{j_i=0}^{N}\prod_{i=1}^{3}\gamma(N,N,2j_i)^{m_i}\frac{\Delta_{2j_i}}{\theta(N,N,2j_i)}\Big\langle$$  $$\Big\rangle$$

$$= \sum_{j_i=0}^{N}\prod_{i=1}^{3}\gamma(N,N,2j_i)^{m_i}\frac{\Delta_{2j_i}}{\theta(N,N,2j_i)}\Gamma_{N,(j_1,j_2,j_3)}$$

where

$$\Gamma_{N,(j_1,j_2,j_3)} = \Big\langle$$  $$\Big\rangle = \Big\langle$$  $$\Big\rangle$$

Now we want to compare this diagram to the diagram in Figure 5.4. Following [Lic97], we denote the evaluation of the graph in Figure 5.4 $\Gamma(x,y,z)$.

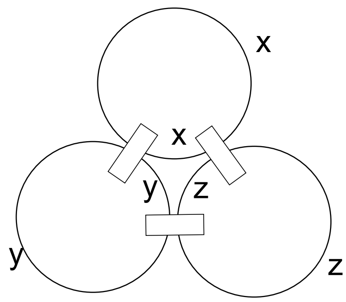


Figure 5.4: This diagram has x parallel copies of a one circle, y of another and z of a third. They are joined by the $x+y, y+z$, and $x+z$ idempotents. Its evaluation is $\Gamma(x,y,z)$

In what follows, we are interested in finding the first $2N+1$ coefficients (in the variable q) of the $N+1$ colored Jones polynomial. We will see that we only need to consider the cases where either all $j_i = N$ or where exactly one $j_i = N-1$. Thus, we will look at $\Gamma_{N,(j_1,j_2,j_3)}$ in these two cases.

In the case where each j_i is N, it is easy to see that

$$\Gamma_{N,(N,N,N)} = \Gamma(N,N,N).$$

In the case where one is $N-1$, we can expand the fusion into the idempotents, see Figure 5.5. The N idempotents can be absorbed in to the $2N$ idempotents. See Figure

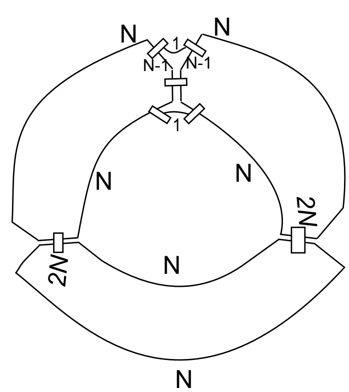


Figure 5.5: This diagram shows the expansion of the fusion with $j_i = N-1$ into the idempotent form.

5.6. We can then pull the outer and inner 1 strand down. Doing this we can see that

$$\Gamma_{N,(N-1,N,N)} = \Gamma(N+1,N-1,N-1)).$$

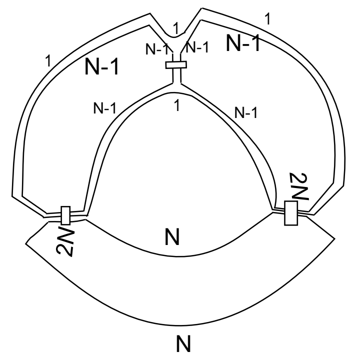


Figure 5.6: We can get rid of the N idempotents. Then after moving the 1 strands, we can see that $\Gamma_{N,(N-1,N,N)} = \Gamma(N+1,N-1,N-1))$.

5.3.1 Finding an expression for the neck

If two polynomials, $f(q)$ and $g(q)$ have the same coefficents for the n lowest order terms, we will write

$$f(q) \overset{\cdot n}{=} g_(q).$$

Lemma 5.3.1. *If* $f(q) = \frac{g(q)}{h(q)}$ *then* $f_n(q) \overset{\cdot n}{=} \frac{g_n(q)}{h_n(q)}$.

Proof of 5.3.1. For notational convenience in this proof, if $f(q)$ is a Laurent polynomial (or has a power series representation as a Laurent polynomial whose q^{-1} terms are bounded), let $f_n(q)$ represent a polynomial whose first n terms agree with $f(q)$.

If $f(q) = \frac{g(q)}{h(q)}$ then $f(q)h(q) = g(q)$. Since the first n terms of $f(q)$ and $g(q)$ determine the first n terms of $h(q)$, $f_n(q)h_n(q) = g_n(q)$ so $f_n(q) = \frac{g_n(q)}{h_n(q)}$. □

Lemma 5.3.2.

$$\{2N\}! \overset{\cdot 2N+1}{=} q^{-\left(\frac{3N^2+N}{4}\right)} \{N\}! \left(1 - \frac{q^{N+1}}{1-q}\right)$$

Now using this lemma, we can get an expression for the first $2N+1$ terms of $\{2N\}!^2$.

Corollary 5.3.3.

$$(\{2N\}!)^2 \overset{\cdot 2N+1}{=} q^{-\left(\frac{3N^2+N}{2}\right)} \{N\}!^2 \left(1 - \frac{2q^{N+1}}{1-q}\right)$$

Proof of 5.3.2. We'll expand the higher terms in the factorial. In the $\left(q^{-(N+i)/2} - q^{(N+i)/2}\right)$ terms changing from the $q^{-(N+i)/2}$ term to term $-q^{(N+i)/2}$ increases the degree by $N+i$. Thus, we can either not do this at all or only do this once.

$$
\begin{aligned}
\{2N\}! &= \{2N\}\{2N-1\}\cdots\{N+1\}\{N\}! \\
&= (q^{-(N+N)/2} - q^{(N+N)/2})\cdots(q^{-(N+1)/2} - q^{(N+1)/2})\{N\}! \\
&\overset{\cdot 2N+1}{=} q^{-\sum_{i=1}^{N}(N+i)/2}\left(\underbrace{\{N\}!}_{\text{no } q^+ \text{ terms}} \underbrace{-\sum_{i=1}^{N} q^{N+i}\{N\}!}_{\text{one } q^+ \text{ term}}\right) \\
&= q^{\frac{-(3N^2+N)}{4}}\left(\{N\}! - q^N\{N\}!\sum_{i=1}^{N} q^i\right) \\
&= q^{\frac{-(3N^2+N)}{4}}\left(\{N\}! - q^N\{N\}!\frac{q-q^{N+1}}{1-q}\right) \\
&= q^{\frac{-(3N^2+N)}{4}}\left(\{N\}! - q^{N+1}\{N\}!\frac{1}{1-q} + \underbrace{q^{2N+1}\{N\}!\frac{1}{1-q}}_{\text{does not affect first } 2N+1 \text{ terms}}\right) \\
&\overset{\cdot 2N+1}{=} q^{\frac{-(3N^2+N)}{4}}\left(\{N\}! - q^{N+1}\{N\}!\frac{1}{1-q}\right) \\
&= q^{-\left(\frac{3N^2+N}{4}\right)}\{N\}!\left(1 - \frac{q^{N+1}}{1-q}\right)
\end{aligned}
$$

□

Proof of 5.3.3.

$$
\begin{aligned}
(\{2N\}!)^2 &\overset{\cdot 2N+1}{=} \left(q^{-\left(\frac{3N^2+N}{4}\right)}\{N\}!\left(1-\frac{q^{N+1}}{1-q}\right)\right)^2 \\
&= q^{-\left(\frac{3N^2+N}{2}\right)}\{N\}!^2\left(1-\frac{q^{N+1}}{1-q}\right)\left(1-\frac{q^{N+1}}{1-q}\right) \\
&= q^{-\left(\frac{3N^2+N}{2}\right)}\{N\}!^2\left(1-\frac{2q^{N+1}}{1-q}+\frac{q^{2N+2}}{(1-q)^2}\right) \\
&= q^{-\left(\frac{3N^2+N}{2}\right)}\{N\}!^2\left(1-\frac{2q^{N+1}}{1-q}+\frac{q^{2N+2}}{1-(2q-q^2)}\right) \\
&= q^{-\left(\frac{3N^2+N}{2}\right)}\{N\}!^2\left(1-\frac{2q^{N+1}}{1-q}+\underbrace{q^{2N+2}(1+(2q-q^2)+\cdots)}_{\text{does not affect first } 2N+1 \text{ terms}}\right) \\
&\overset{\cdot 2N+1}{=} q^{-\left(\frac{3N^2+N}{2}\right)}\{N\}!^2\left(1-\frac{2q^{N+1}}{1-q}\right)
\end{aligned}
$$

□

5.3.2 Knots with $m_i > 2$

In the case where each m_i is greater than 2, the degree increases by more than $2N$ when we decrease j_i from N to $N-1$ thus we only need to deal with the case where each $j_i = N$. Thus we get

$$
\begin{aligned}
J_{N+1,K}(q) &= \sum_{j_i=0}^{N}\prod_{i=1}^{3}\gamma(N,N,2j_i)^{m_i}\frac{\Delta_{2j_i}}{\theta(N,N,2j_i)}\Gamma_{N,(j_1,j_2,j_3)} \\
&\overset{\cdot 2N+1}{=} \prod_{i=1}^{3}\gamma(N,N,2N)^{m_i}\frac{\Delta_{2N}}{\theta(N,N,2N)}\Gamma_{N,(N,N,N)} \\
&= \gamma(N,N,2N)^{m_1+m_2+m_3}\left(\frac{\Delta_{2N}}{\theta(N,N,2N)}\right)^3\left(\Gamma_{N,(N,N,N)}\right)
\end{aligned}
$$

Recall that $\gamma(a,b,c) = (-1)^{\frac{a+b-c}{2}}A^{a+b+c+\frac{a^2+b^2-c^2}{2}}$. This just has the effect of shifting polynomial but does not affect the sequence of coefficients. Also

$$
\Delta_n = \frac{(-1)^n(a^{n+1}-a^{-(n+1)})}{a-a^{-1}} = \frac{(-1)^n\{n+1\}}{\{1\}}.
$$

$$\Delta_n! = \Delta_n \Delta_{n-1} \cdots \Delta_1 = (-1)^{\frac{n(n+1)}{2}} \frac{\{n+1\}!}{\{1\}^{n+1}}$$

Following Lickorish [Lic97], we define $\Gamma(x,y,z)$ to be the diagram consisting of x parallel copies of a circle, y parallel copies of a circle and z parallel copies of a circle joined by the $f^{(x+y)}, f^{(y+z)}, and f^{(z+x)}$ idempotents. This is what our knot reduces to, i.e. $\Gamma_{N,(N,N,N)} = \Gamma(N,N,N)$.

Lemma 5.3.4. *[Lic97]*

$$\Gamma(x,y,z) = \frac{\Delta_{x+y+z}!\Delta_{x-1}!\Delta_{y-1}!\Delta_{z-1}!}{\Delta_{y+z-1}!\Delta_{z+x-1}!\Delta_{x+y-1}!}$$

Also

$$\theta(N,N,2N) = \Gamma(N,N,0) = \frac{\Delta_{2N}!\Delta_{N-1}!\Delta_{N-1}!}{\Delta_{N-1}!\Delta_{N-1}!\Delta_{2N-1}!} = \Delta_{2N}$$

So $\left(\frac{\Delta_{2N}}{\theta(N,N,2N)}\right) = 1$ And we have:

$$\begin{aligned}
J_{N+1,K}(q) &\overset{\cdot 2N+1}{=} \gamma(N,N,2N)^{m_1+m_2+m_3} \left(\frac{\Delta_{2N}}{\theta(N,N,2N)}\right)^3 \left(\Gamma_{N,(N,N,N)}\right) \\
&\overset{\cdot 2N+1}{=} \Gamma(N,N,N) \\
&= \frac{\Delta_{3N}!\Delta_{N-1}!\Delta_{N-1}!\Delta_{N-1}!}{\Delta_{2N-1}!\Delta_{2N-1}!\Delta_{2N-1}!} \\
&= \frac{\Delta_{3N}!\Delta_{N-1}!^3}{\Delta_{2N-1}!^3} \\
&= \frac{(-1)^N\{3N+1\}!\{N\}!^3}{\{2N\}!^3\{1\}}
\end{aligned}$$

Again since we only want the first $2N+1$ terms, we can reduce the $\{3N+1\}!$ term. We know we have to choose the $q^{-(2N+i)}$ term for each $1 \leq i \leq N+1$. This gives us a shift in degree that we can ignore. We have

$$\begin{aligned}
J_{N+1,K}(q) &\overset{\cdot 2N+1}{=} \frac{(-1)^N\{3N+1\}!\{N\}!^3}{\{2N\}!^3\{1\}} \\
&\overset{\cdot 2N+1}{=} \frac{(-1)^N\{2N\}!\{N\}!^3}{\{2N\}!^3\{1\}} \\
&= \frac{(-1)^N\{N\}!^3}{\{2N\}!^2\{1\}}
\end{aligned}$$

By Lemma 5.3.1 we can reduce the $\{2N\}!^2$ term to its lowest $2N+1$ terms, which we found in Corollary 5.3.3, again forgetting about the shifting. Thus we have:

$$\begin{aligned} J_{N+1,K}(q) &\overset{\cdot 2N+1}{=} \frac{(-1)^N \{N\}!^3}{\{2N\}!^2 \{1\}} \\ &\overset{\cdot 2N+1}{=} \frac{(-1)^N \{N\}!^3}{\{N\}!^2 \left(1 - \frac{2q^{N+1}}{1-q}\right) \{1\}} \\ &= \frac{(-1)^N \{N\}!}{\left(1 - \frac{2q^{N+1}}{1-q}\right) \{1\}} \end{aligned}$$

We ultimately want the tailneck of the normalized Colored Jones polynomial, so we will divide by $\Delta_N = \frac{(-1)^N \{N+1\}}{\{1\}}$ at this point.

$$\begin{aligned} J'_{N+1,K}(q) &\overset{\cdot 2N+1}{=} \frac{(-1)^N \{N\}!}{\left(1 - \frac{2q^{N+1}}{1-q}\right) \{1\}} \frac{\{1\}}{(-1)^N \{N+1\}} \\ &= \frac{\{N\}!}{\left(1 - \frac{2q^{N+1}}{1-q}\right) \{N+1\}} \\ &= \frac{\{N\}!}{q^{-(N+1)/2} \left(1 - \frac{2q^{N+1}}{1-q}\right) (1 - q^{N+1})} \\ &\overset{\cdot \infty}{=} \frac{\{N\}!}{1 - \frac{2q^{N+1}}{1-q} - q^{N+1} + \underbrace{\frac{2q^{2N+2}}{1-q}}_{\text{does not contribute}}} \end{aligned}$$

Now we do a power series expansion of the denominator. Since we only want the lowest $2N+1$ terms, we can ignore most of the terms in the expansion.

$$
\begin{aligned}
J'_{N+1,K}(q) &\overset{\cdot 2N+1}{=} \frac{\{N\}!}{1-\left(\frac{2q^{N+1}}{1-q}+q^{N+1}\right)} \\
&= \{N\}!\left(1+\left(\frac{2q^{N+1}}{1-q}+q^{N+1}\right)+\underbrace{\left(\frac{2q^{N+1}}{1-q}+q^{N+1}\right)^2+\cdots}_{\text{do not contribute to first } 2N+1 \text{ terms}}\right) \\
&\overset{\cdot 2N+1}{=} \{N\}!\left(1+\frac{2q^{N+1}}{1-q}+q^{N+1}\right)
\end{aligned}
$$

Note: The (minimum) degree of this term is $-\sum_{i=1}^{N}\frac{i}{2}=-\frac{N^2+N}{4}$ and its coefficient is 1.

Now we need to subtract off the stabilized tail. We still get the triangle graph as our reduced graph so any knot here will still have the same tail as this figure 8 knot. We again need the stabilized tail so we take the tail of the $2N+1$-dimensional colored Jones Polynomial of 4_1 which is $\{2N\}!$. By Lemma 5.3.2 we get

$$
\begin{aligned}
\text{stabilized tail} &\overset{\cdot 2N+1}{=} \{2N\}! \\
&\overset{\cdot 2N+1}{=} \{N\}!\left(1-\frac{q^{N+1}}{1-q}\right)
\end{aligned}
$$

Note: The (minimum) degree of this term is $-\sum_{i=1}^{N}\frac{i}{2}=-\frac{N^2+N}{4}$ and its coefficient is 1. Thus the minimum degree and sign of the first $2N+1$ coefficients we found above match so we are set to subtract.

$$
\begin{aligned}
J'_{N+1,K}(q)-\text{stabilized head} &\overset{\cdot 2N+1}{=} \{N\}!\left(1+\frac{2q^{N+1}}{1-q}+q^{N+1}\right)-\{N\}!\left(1-\frac{q^{N+1}}{1-q}\right) \\
&= \{N\}!\left(1+\frac{2q^{N+1}}{1-q}+q^{N+1}-\left(1-\frac{q^{N+1}}{1-q}\right)\right) \\
&= \{N\}!\left(q^{N+1}+\frac{3q^{N+1}}{1-q}\right) \\
&= q^{N+1}\left(\{N\}!+\frac{3\{N\}!}{1-q}\right)
\end{aligned}
$$

This tells us that the tailneck is the pentagonal numbers plus 3 times the partial sum of the pentagonal number.

5.3.3 When at least one of the m_i is 1

When we have an m_i which is 1, we need to consider the $j_i = N-1$ term as well as the $j_i = N$ term. We can only allow this for the i with $m_i = 1$ and only one can be $N-1$ at a time. Thus we need to determine what this term contributes to the first $2N+1$ terms and then add it once for each of the $m_i = 1$. For notation ease, we will refer to the j_i that we will consider the N and $N-1$ term of as j_1 and thus we label the edges so $m_1 = 1$

Because the degree increases by $N+1$ when j_1 decreases from N to $N-1$ we only need to consider the first N terms of the $j_1 = N-1$. Call the $j_1 = N-1, j_2 = j_3 = N$ summand $S_{N-1,N,N}$.

$$\begin{aligned}
S_{N-1,N,N} &= \prod_{i=1}^{3} \gamma(N,N,2j_i)\frac{\Delta_{2j_i}}{\theta(N,N,2j_i)}\Gamma(N+1,N-1,N-1) \\
&= \underbrace{\gamma(N,N,2N)^2\gamma(N,N,2N-2)}_{\text{shift, does not affect coefficients}}\underbrace{\left(\frac{\Delta_{2N}}{\theta(N,N,2N)}\right)^2}_{=1} \\
&\quad \cdot\frac{\Delta_{2N-2}}{\theta(N,N,2N-2)}\Gamma(N+1,N-1,N-1) \\
&\stackrel{\cdot\infty}{=} \frac{\Delta_{2N-2}}{\theta(N,N,2N-2)}\Gamma(N+1,N-1,N-1)
\end{aligned}$$

We know that

$$\begin{aligned}
\theta(N,N,2N-2) &= \frac{\Delta_{2N-1}!\Delta_{N-2}!\Delta_{N-2}!}{\Delta_{N-1}!\Delta_{N-1}!\Delta_{2N-3}!} \\
&= \frac{\Delta_{2N-1}\Delta_{2N-2}}{\Delta_{N-1}^2}
\end{aligned}$$

So

$$\begin{aligned}
\frac{\Delta_{2N-2}}{\theta(N,N,2N-2)} &= \frac{\Delta_{N-1}^2}{\Delta_{2N-1}} \\
&= \frac{-\{N\}^2}{\{1\}\{2N\}}
\end{aligned}$$

Also by Lemma 5.3.4 and simplifying the Δ_i in terms of $\{j\}$ we get:

$$\Gamma(N+1,N-1,N-1) = \frac{(-1)^{N-1}\{3N\}!\{N+1\}!\{N-1\}!^2}{\{1\}\{2N-2\}!\{2N\}!^2}$$

Thus we have:

$$\begin{aligned}
S_{N-1,N,N} &\stackrel{\cdot\infty}{=} \frac{\Delta_{2N-2}}{\theta(N,N,2N-2)}\Gamma(N+1,N-1,N-1) \\
&= \frac{-\{N\}^2}{\{1\}\{2N\}}\frac{(-1)^{N-1}\{3N\}!\{N+1\}!\{N-1\}!^2}{\{1\}\{2N-2\}!\{2N\}!^2} \\
&= \frac{(-1)^N\{N\}^2\{3N\}!\{N+1\}!\{N-1\}!^2}{\{1\}^2\{2N-2\}!\{2N\}!^2\{2N\}}
\end{aligned}$$

Let's normalize by dividing by $\frac{(-1)^N\{N+1\}}{\{1\}}$. Then we only want the first N terms so we can reduce $\{N+i\}$ to q^{-N-i}. We will forget about the overall shift that this reduction does.

$$\begin{aligned}
\overline{S_{N-1,N,N}} &\stackrel{\cdot\infty}{=} \frac{\{3N\}!\{N\}!^3}{\{1\}\{2N-2\}!\{2N\}!^2\{2N\}} \\
&\stackrel{\cdot N}{=} \frac{\{N\}!\{N\}!^3}{\{1\}\{N\}!\{N\}!^2(q^N)} \\
&\stackrel{\cdot\infty}{=} \frac{\{N\}!^4}{\{1\}\{N\}!^3} \\
&= \frac{\{N\}!}{\{1\}}
\end{aligned}$$

This gives us a copy of the pentagonal partial sums for each of the of the $m_i = 1$. Since the sign here is positive and for the other piece was negative these will cancel with the pentagonal partial sums we got from the other piece in conclusion:

Theorem 5.3.5. *The tailneck of knots whose reduces checkboard graph is the triangle graph is:*

- $\prod_{n=1}^{\infty}(1-q^n)$*, i.e. the pentagonal numbers sequence, if all* $m_i = 1$ *(The only knot satisfying this is the trefoil).*
- $\prod_{n=1}^{\infty}(1-q^n) + \frac{\prod_{n=1}^{\infty}(1-q^n)}{1-q}$*, i.e. the pentagonal numbers plus the partial sum of the pentagonal numbers, if two* $m_i = 1$ *and one is 2 or more.*
- $\prod_{n=1}^{\infty}(1-q^n) + 2\frac{\prod_{n=1}^{\infty}(1-q^n)}{1-q}$*, i.e. the pentagonal numbers plus the 2 times the partial sum of the pentagonal numbers, if one* $m_i = 1$ *and two are 2 or more.*

- $\prod_{n=1}^{\infty}(1-q^n)+3\frac{\prod_{n=1}^{\infty}(1-q^n)}{1-q}$, *i.e. the pentagonal numbers plus the 3 times the partial sum of the pentagonal numbers, if all* $m_i \geq 2$.

5.4 The Tail and Tailneck of the Figure 8 Knot

Remark 5.4.1. *This section does not prove anything new. It just shows that we get the same thing for the second stabilized sequence for the figure 8 knot if we analyze its colored Jones polynomial directly from its equation.*

We will use the formula for the normalized colored Jones polynomial of the Figure 8 knot to first determine the tail of this polynomial and then determine the next stabilized sequence, which for lack of a better term I will call the tailneck.

$$J'_{N,4_1}(q) = \sum_{n=0}^{N-1}\prod_{k=1}^{n}\{N-k\}\{N+k\}$$

5.4.1 The Tail of the Figure 8 Knot

We again define the q-degree of a polynomial to be the *minimum* degree of the polynomial with variable q. It is easy to check that the q-degree of each summand is Nn so decreasing the n by 1 increases the minimum degree by N thus only $n = N-1$ contributes to the tail.

$$\begin{aligned} J'_{N,4_1}(q) &= \sum_{n=0}^{N-1}\prod_{k=1}^{n}\{N-k\}\{N+k\} \\ &\stackrel{\cdot N}{=} \prod_{k=1}^{N-1}\{N-k\}\{N+k\} \\ &= \{1\}\ldots\{N-1\}\{\hat{N}\}\{N+1\}\ldots\{2N-1\} \\ &= \frac{1}{\{N\}}\{2N-1\}! \end{aligned}$$

We only want to lowest N terms (in q degree). Recall that $\{N+i\} = (q^{-(N+i)/2} - q^{(N+i)/2})$. Choosing the $-q^{(N+i)/2}$ term instead of the $q^{-(N+i)/2}$ term increases the degree by $N+i$. Thus for $1 \leq i \leq N-1$, we know we have to pick the $q^{-(N+i)/2}$ term to get the first N terms. So $\{2N-1\}! \stackrel{\cdot N}{=} \{N\}!$.

$$
\begin{aligned}
J'_{N,4_1}(q) &\overset{\cdot N}{=} \frac{1}{\{N\}}\{2N-1\}! \\
&\overset{\cdot N}{=} \frac{1}{\{N\}}\{N\}! \\
&= \{N-1\}! \\
&= \prod_{i=1}^{N-1} q^{-i/2} - q^{i/2} \\
&= q^{-\left(\frac{1}{2}+\frac{2}{2}+\cdots+\frac{N-1}{2}\right)} \prod_{i=1}^{N-1} 1-q^i \\
&\overset{\cdot N}{=} \prod_{i=1}^{N-1} 1-q^i
\end{aligned}
$$

Theorem 5.4.2 (Euler's Pentagonal Number Theorem)**.**

$$
\begin{aligned}
\prod_{n=1}^{\infty}(1-x^n) &= \sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k-1)/2} \\
&= 1-x-x^2+x^5+x^7-x^{12}-x^{15}+\cdots
\end{aligned}
$$

Corollary 5.4.3. *The tail of the colored Jones polynomial of the figure 8 knot is the pentagonal number sequence.*

5.4.2 The Tailneck of the Figure 8 Knot

We will now focus on the next N terms. We know from [GL11] that once we subtract off what the head will stabilize to, the next N coefficients stabilize. Again in the summation in the formula for the colored Jones polynomial of the figure 8 knot, the minimum degree increases by N each time n decreases by 1. Thus we only need to consider the first two summands. In addition, while we need to consider the first $2N$ terms of the $n = N-1$ summand, we only need to consider the first N terms in the

$n = N-2$ summand.

$$\begin{aligned} J'_{N,4_1}(q) &= \sum_{n=0}^{N-1}\prod_{k=1}^{n}\{N-k\}\{N+k\} \\ &\overset{\cdot 2N}{=} \prod_{k=1}^{N-1}\{N-k\}\{N+k\} + \prod_{k=1}^{N-2}\{N-k\}\{N+k\} \\ &= \underbrace{\frac{1}{\{N\}}\{2N-1\}!}_{\text{need first } 2N \text{ terms of this}} + \underbrace{\frac{1}{\{1\}\{N\}}\{2N-2\}!}_{\text{need first } N \text{ terms of this}} \end{aligned}$$

Let's consider the $\frac{1}{\{N\}}\{2N-1\}!$ piece first. We'll expand the higher terms in the factorial. In the $\left(q^{-(N+i)/2} - q^{(N+i)/2}\right)$ terms changing from the $q^{-(N+i)/2}$ term to term $-q^{(N+i)/2}$ increases the degree by $N+i$. Thus, we can either not do this at all or do this once.

$$\begin{aligned} \frac{1}{\{N\}}\{2N-1\}! &= \{2N-1\}\{2N-2\}\cdots\{N+1\}\{N-1\}! \\ &= (q^{\frac{-(N+(N-1))}{2}} - q^{\frac{(N+(N-1))}{2}})\cdots(q^{-(N+1)/2} - q^{(N+1)/2})\{N-1\}! \\ &\overset{\cdot 2N}{=} q^{-\sum_{i=1}^{N-1}(N+i)/2}\left(\underbrace{\{N-1\}!}_{\text{no } q^+ \text{ terms}} - \underbrace{\sum_{i=1}^{N-1} q^{N+i}\{N-1\}!}_{\text{one } q^+ \text{ term}}\right) \\ &= q^{\frac{-3N^2+3N}{4}}\left(\{N-1\}! - q^N\{N-1\}!\sum_{i=1}^{N-1} q^i\right) \\ &= q^{\frac{-3N^2+3N}{4}}\left(\{N-1\}! - q^N\{N-1\}!\frac{q-q^N}{1-q}\right) \\ &= q^{\frac{-3N^2+3N}{4}}\left(\{N-1\}! - q^{N+1}\frac{\{N-1\}!}{1-q} + \underbrace{q^{2N}\{N-1\}!\frac{1}{1-q}}_{\text{does not affect first } 2N \text{ terms}}\right) \\ &\overset{\cdot 2N}{=} q^{\frac{-3N^2+3N}{4}}\left(\{N-1\}! - q^{N+1}\{N-1\}!\frac{1}{1-q}\right) \end{aligned}$$

Now, let's consider the $\frac{1}{\{1\}\{N\}}\{2N-2\}!$ piece. Here we only need to consider the first N terms, so we need to pick the $q^{-(N+i)/2}$ term in each of the $N+i$ factors, $(1 \le i \le N-2)$.

$$\begin{aligned}
\frac{1}{\{1\}\{N\}}\{2N-2\}! &= \{2N-2\}\cdots\{N+1\}\frac{\{N-1\}!}{\{1\}} \\
&\overset{\cdot N}{=} q^{-\sum_{i=1}^{N-2}(N+i)/2}\frac{\{N-1\}!}{\{1\}} \\
&= q^{\frac{-3N^2+7N-2}{4}}\frac{\{N-1\}!}{q^{-1/2}-q^{1/2}} \\
&= q^{\frac{-3N^2+7N-2}{4}}\frac{1}{q^{-1/2}}\frac{\{N-1\}!}{1-q} \\
&= q^{\frac{-3N^2+7N}{4}}\frac{\{N-1\}!}{1-q}
\end{aligned}$$

Combining these we get:

$$\begin{aligned}
J'_{N,4_1}(q) &\overset{\cdot 2N}{=} \frac{1}{\{N\}}\{2N-1\}! + \frac{1}{\{1\}\{N\}}\{2N-2\}! \\
&\overset{\cdot 2N}{=} q^{\frac{-3N^2+3N}{4}}\left(\{N-1\}! - q^{N+1}\{N-1\}!\frac{1}{1-q}\right) + q^{\frac{-3N^2+7N}{4}}\frac{\{N-1\}!}{1-q} \\
&= q^{\frac{-3N^2+3N}{4}}\{N-1\}!\left(1-\frac{q^{N+1}}{1-q}+\frac{q^N}{1-q}\right)
\end{aligned}$$

We now need to subtract of the stabilized tail. The tail of the N-dimensional colored Jones Polynomial is:

$$\begin{aligned}
J'_{N,4_1}(q) &\overset{\cdot N}{=} \prod_{i=1}^{N-1} 1-q^i \\
&\overset{\cdot N}{=} \{N-1\}!
\end{aligned}$$

We need the stabilized tail so we will need to take the tail of the $2N$-dimensional colored Jones polynomial. This will agree with the first $2N$ terms of the stabilized sequence. Since we know the first $2N$ terms of $\frac{1}{\{N\}}\{2N-1\}!$, we can just multiply both sides by $\{N\}$.

$$
\begin{aligned}
(-1)^{2N-1}\{2N-1\}! &\stackrel{\cdot 2N}{=} q^{\frac{-3N^2+3N}{4}}\left(\{N-1\}!-q^{N+1}\{N-1\}!\frac{1}{1-q}\right)\{N\} \\
&= q^{\frac{-3N^2+3N}{4}}\{N-1\}!\left(1-\frac{q^{N+1}}{1-q}\right)\left(q^{-N/2}\right)\left(1-q^N\right) \\
&= q^{\frac{-3N^2+N}{4}}\{N-1\}!\left(1-\frac{q^{N+1}}{1-q}\right)\left(1-q^N\right) \\
&= q^{\frac{-3N^2+N}{4}}\{N-1\}!\left(1-\frac{q^{N+1}}{1-q}-q^N+\underbrace{\frac{q^{2N+1}}{1-q}}_{\text{can ignore for first } 2N}\right) \\
&\stackrel{\cdot 2N}{=} q^{\frac{-3N^2+N}{4}}\{N-1\}!\left(1-\frac{q^{N+1}}{1-q}-q^N\right)
\end{aligned}
$$

Our terms do not have the same minimum degree at this point so we need to shift them so they do in order for them to cancel correctly. To do this, we can simply drop the power of q in the front of each. Thus:

$$
\begin{aligned}
J'_{N,4_1}(q) &\stackrel{\cdot 2N}{=} \{N-1\}!\left(1-\frac{q^{N+1}}{1-q}+\frac{q^N}{1-q}\right) \\
\text{stabilized head} &\stackrel{\cdot 2N}{=} \{N-1\}!\left(1-\frac{q^{N+1}}{1-q}-q^N\right)
\end{aligned}
$$

$$
\begin{aligned}
J'_{N,4_1}(q)-\text{stabilized head} &\stackrel{\cdot 2N}{=} \{N-1\}!\left((1-\frac{q^{N+1}}{1-q}+\frac{q^N}{1-q})-\left(1-\frac{q^{N+1}}{1-q}-q^N\right)\right) \\
&= \{N-1\}!\left(q^N+\frac{q^N}{1-q}\right) \\
&\stackrel{\cdot 2N}{=} \underbrace{\{N+1\}!}_{\text{gives pentagonal number sequence}} \\
&\quad +\underbrace{\frac{\{N+1\}!}{1-q}}_{\text{gives partial sums of pentagonal number sequence}}
\end{aligned}
$$

Chapter 6

The Middle Coefficients of the Colored Jones Polynomial

6.1 Initial Observations for the Figure 8 Knot and Other Knots

We began by plotting the coefficients of the colored Jones polynomial of the figure 8 knot, 4_1. See Figure 6.1. This is a plot of the coefficients of $J'_{95,4_1}(q)$. We notice that the coefficients appear to be periodic in the middle third. The period is N, where N is the number of colors. This picture is representative of the pattern present in the coefficients independent of color.

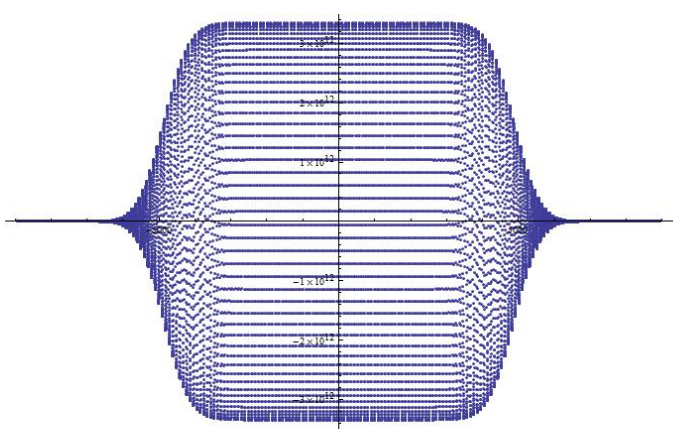
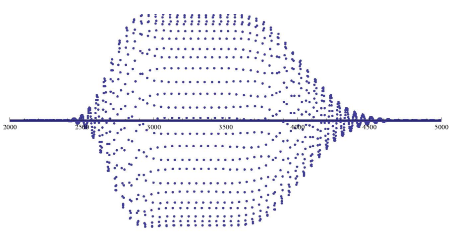


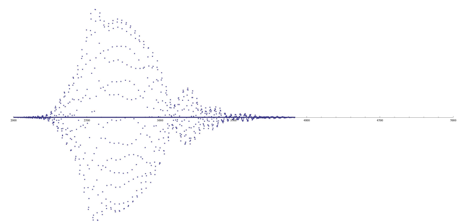
Figure 6.1: The coefficients of the 95 colored Jones polynomial of the figure 8 knot.

We also plotted the coefficients of the colored Jones polynomial of other knots. See Figure 6.2. In particular, we show the coefficients for the 25 colored Jones polynomial of the knot 5_2 (a 2 twist knot), 15 colored Jones polynomial of the knot 9_2 (a 4 twist knot), and the 20 colored Jones polynomial of the knot 9_5 (a (1,3,5) pretzel knot). For knots that we are able to calculate the colored Jones polynomial for large N, we begin to see a few patterns continuing to arise.

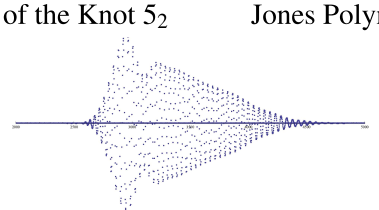
In particular, we notice that the overall magnitude of the coefficients seems to rise towards the middle of the polynomial and that there is a sine wave like oscillation in the coefficients of period N.



(a) Coefficients of the 25 Colored Jones Polynomial of the Knot 5_2



(b) Coefficients of the 15th Colored Jones Polynomial of the Knot 9_2



(c) Coefficients of the 20th Colored Jones Polynomial of the Knot 9_5

Figure 6.2: These are the coefficients of the the colored Jones polynomial of three different knots for relatively high N.

We also looked at the growth rate of the maximum coefficient of the N colored Jones polynomial as we increase N. We found that for the knots we checked, the maximum coefficient grows at an exponential rate. In particular, for the figure 8 knot, the maximum coefficients are plotted in Figure 6.3. We also observed that for the figure 8 knot, the maximum coefficient is the constant coefficient.

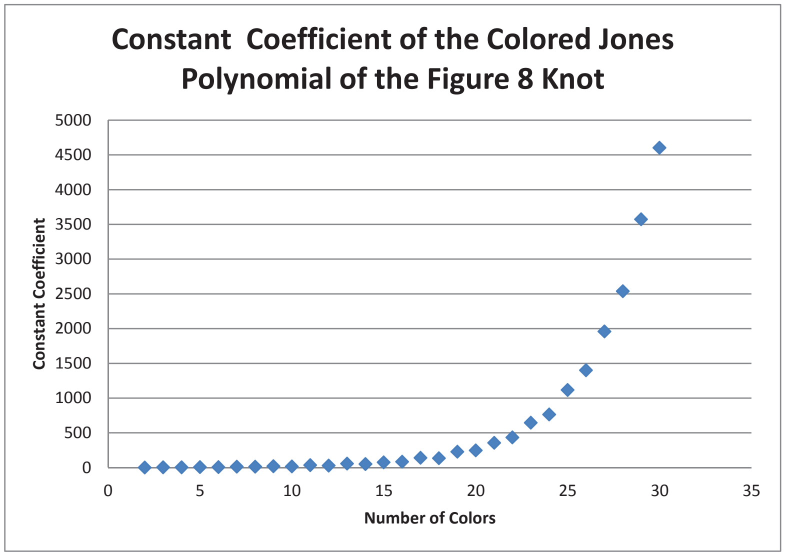


Figure 6.3: The Maximum Coefficient of the N Colored Jones Polynomial of the Figure 8 Knot as a function of N.

Now, since we notice that these coefficients are growing exponentially, presumably as e^{bN}, we'll take the log and divide by N to find b. We hope that this a relatively constant. We found that it is relatively constant and that it seems to be close to $\frac{\text{vol}(4_1)}{2\pi}$. Let $m(N)$ be the maximum coefficient of the N colored Jones polynomial of the figure 8 knot. The plot of $\frac{2\pi\log(m(N))}{N}$ is in Figure 6.4. We notice that the b value we are looking at is tending towards 2.02 which is the volume of the figure 8 knot.

6.1.1 The Idealized Polynomial

We wanted to know that if we had a idealized polynomial that satisfied these properties, what could we say about it evaluation at the Nth root of unity as $N \to \infty$, i.e. in the spirit of the Hyperbolic Volume Conjecture. We make the following conjectures about the colored Jones polynomials of a knot.

Assumption 6.1.1.

The maximum coefficient takes the form Ae^{bN} where N is the number of colors and A

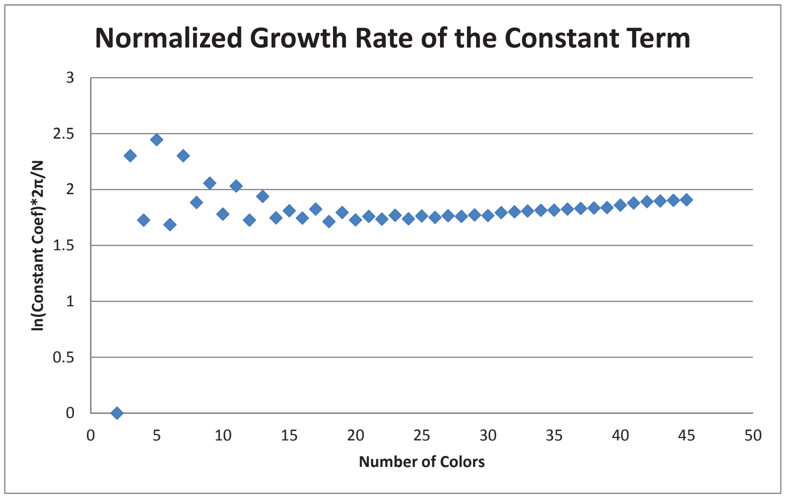


Figure 6.4: The plot of $\frac{2\pi\log(m(N))}{N}$ as a function of N.

and b depend on the knot.

Assumption 6.1.2.

The coefficients take the form of a normal distribution times a sine wave of period $2N$*.*

We also know the following fact:

Theorem 6.1.3 ([LÔ6]). *The growth rate in the breadth of the colored Jones polynomial is at most quadratic.*

Using our first two assumptions we can write out a formula for the coefficients of the colored Jones polynomial. Using this we write the Nth polynomial in the sequence as:

$$J'_{N,K}(q) = \sum_{n=0}^{M} q^{Nn} \sum_{k=0}^{N-1} a_n q^k \sin(\frac{2\pi}{N}k),$$

where M is the number of terms of the polynomial, and a_n is the normal distribution shaped function centered at $M/2$ scaled so that the max is Ae^{bN}. In particular:

$$a_n = \frac{Ae^{bN}}{p} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(n-M/2)^2}{2\sigma^2}},$$

where p is the max value of the distribution and σ^2 is the variance. We want to think of this as a colored Jones polynomial see what we get for the left hand side of the Hyperbolic Volume Conjecture. We first look at each term in the outer sum. We will let q be a N^{th} root of unity, i.e. $q = e^{2\pi i/N}$.

$$\begin{aligned}
\text{Summand} &= q^{Nn}\sum_{k=0}^{N-1} a_n q^k \sin(\frac{2\pi}{N}k) \\
&= a_n q^{Nn}\sum_{k=0}^{N-1} q^k \frac{1}{2i}(e^{\frac{2\pi}{N}ki)} - e^{-\frac{2\pi}{N}ki)}) \\
&= a_n q^{Nn}\sum_{k=0}^{N-1} q^k \frac{1}{2i}(q^k - q^{-k}) \\
&= \frac{a_n q^{Nn}}{2i}\sum_{k=0}^{N-1} q^{2k} - 1 \\
&= \frac{a_n q^{Nn}}{2i}(-N)\sum_{k=0}^{N-1} q^{2k} \\
&= \frac{-N a_n q^{Nn}}{2i}
\end{aligned}$$

The sum goes away since we are summing all the $N/2^{th}$ roots of unity (twice) and thus this is 1. Now plugging a_n and evaluating the sum we get:

$$\begin{aligned}
\sum_{n=0}^{M} \frac{-N a_n q^{Nn}}{2i} &= \sum_{n=0}^{M} \frac{-N\frac{Ae^{bN}}{p}\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(n-M/2)^2}{2\sigma^2}} q^{Nn}}{2i} \\
&= \frac{-NAe^{bN}}{p2i\sqrt{2\pi\sigma^2}}\sum_{n=0}^{M} e^{\frac{-(n-M/2)^2}{2\sigma^2}} \\
&= \frac{-NAe^{bN}}{p2i\sqrt{2\pi\sigma^2}} M \sum_{n=0}^{M} e^{\frac{-(n-M/2)^2}{2\sigma^2}} \frac{1}{M} \\
&= \frac{-NAe^{bN}}{p2i\sqrt{2\pi\sigma^2}} M
\end{aligned}$$

The last sum is a Riemann sum. As M gets large this approximates the area under a normal distribution curve and thus is approximately 1. Now we have

$$f_N(e^{2\pi i/N}) = \frac{-NAe^{bN}}{2pi\sqrt{2\pi\sigma^2}} M$$

so

$$|f_N(e^{2\pi i/N})| = \frac{NAe^{bN}}{2p\sqrt{2\pi\sigma^2}}M$$

and

$$\begin{aligned}\frac{\log|(f_N(e^{2\pi i/N}))|}{N} &= \frac{\log(\frac{NAe^{bN}}{2p\sqrt{2\pi\sigma^2}}M)}{N} \\ &= \frac{\log(NA)}{N} + \frac{bN}{N} - \frac{\log(2p\sqrt{2\pi\sigma^2})}{N} + \frac{\log(M)}{N}\end{aligned}$$

so taking the limit as N goes to infinity we get:

$$\begin{aligned}\lim_{N\to\infty}\frac{\log|(f_N(e^{2\pi i/N}))|}{N} &= \lim_{N\to\infty}\frac{\log(N)}{N} + \frac{bN}{N} + \frac{\log(2p\sqrt{2\pi\sigma^2})}{N} + \frac{\log(M)}{N} \\ &= b + \lim_{N\to\infty}\frac{\log(M)}{N}\end{aligned}$$

Since the growth rate for the breadth is at most quadratic, we have:

$$\begin{aligned}\lim_{N\to\infty}\frac{\log|(f_N(e^{2\pi i/N}))|}{N} &= b + \lim_{N\to\infty}\frac{\log(N(N-1)c/2)}{N} \\ &= b + \lim_{N\to\infty}\frac{\log(N)}{N} + \frac{\log(N-1)}{N} + \frac{\log(c/2)}{N} \\ &= b.\end{aligned}$$

Proposition 6.1.4. *If the colored Jones polynomial of a knot satisfies Assumption 6.1.1 and Assumption 6.1.2 then*

$$b = \lim_{N\to\infty}\frac{\log|(f_N(e^{2\pi i/N}))|}{N}.$$

If this is a knot for which the Hyperbolic Volume Conjecture holds,

$$b = \frac{vol(S^3\backslash K)}{2\pi}.$$

Now, if we don't make any initial assumptions about the growth rate of the maximum coefficients, we can see what we can conclude.

If we let $m_K(N)$ be the maximum coefficient of the Colored Jones Polynomial of a knot K, then

$$|J_k(N)(e^{2\pi i/N})| \leq \sum_{n=0}^{M}|m_K(N)| \leq Mm_K(N),$$

where M is the number of terms in the Colored Jones Coefficient. Using 6.1.3, we get

$$\begin{aligned}\lim_{N\to\infty}\frac{\log|J_k(N)(e^{2\pi i/N})|}{N} &\leq \lim_{N\to\infty}\frac{\log(a_2N^2+a_1N+a_0)m_K(N)}{N}\\ &= \lim_{N\to\infty}\frac{\log(a_2N^2+a_1N+a_0)}{N}+\frac{\log m_K(N)}{N}\\ &= \lim_{N\to\infty}\frac{\log m_K(N)}{N}.\end{aligned}$$

Now, if we include Assumption 6.1.1, so that $m(k)(N) = Ae^{BN}$, we get

$$\begin{aligned}\lim_{N\to\infty}\frac{\log|J_k(N)(e^{2\pi i/N})|}{N} &\leq \lim_{N\to\infty}\frac{\log m_K(N)}{N}\\ &= \lim_{N\to\infty}\frac{\log Ae^{bN}}{N}\\ &= \lim_{N\to\infty}\frac{\log A}{N}+\frac{bN}{N}\\ &= b\end{aligned}$$

So we can conclude the following:

Proposition 6.1.5. *The colored Jones polynomial of a knot K satisfies*

$$\lim_{N\to\infty}\frac{\log|J_k(N)(e^{2\pi i/N})|}{N} \leq \lim_{N\to\infty}\frac{\log m_K(N)}{N}.$$

If Assumption 6.1.1 holds, then the colored Jones polynomial satisfies

$$\lim_{N\to\infty}\frac{\log|J_k(N)(e^{2\pi i/N})|}{N} \leq b.$$

So for knots where the Hyperbolic Volume Conjecture holds we get to following

Proposition 6.1.6. *For knots for which the Hyperbolic Volume conjecture holds*

$$\frac{vol(S^3\backslash K)}{2\pi} \leq \lim_{N\to\infty}\frac{\log m_K(N)}{N}$$

.

Now, if we include Assumption 6.1.1, so that $m(k)(N) = Ae^{bN}$, we get

$$\frac{vol(S^3\backslash K)}{2\pi} \leq b.$$

6.2 Further Analysis on the Coefficients of the Figure 8 Knot

Since the coefficients appear to be periodic, we expect that when we multiply by $(1-q^N)$ or by $(q^{N/2}-q^{-N/2})$ the coefficients will become 0, or close to zero. We define the semi-(un)normalized colored Jones polynomial, $sJ_{N,K}(q)$ to be

$$sJ_{N,K}(q) = \{N\}J'_{N,K}(q).$$

Recall that the un-normalized colored Jones polynomial, $J_{N,K}(q)$ satisfies $J_{N,K}(q) = \frac{(-1)^{N-1}\{N\}}{\{1\}}J'_{N,K}(q)$ so we have that:

$$\pm J_{N,K}(q)\{1\} = sJ_{N,K}(q) = J'_{N,K}(q)\{N\}$$

Thus we expect that the coefficients of $sJ_{N,4_1}(q)$ will be zero or small in the middle. Below is a plot of these coefficients. First, we'll plot the whole sequence of coefficients, see Figure 6.5.

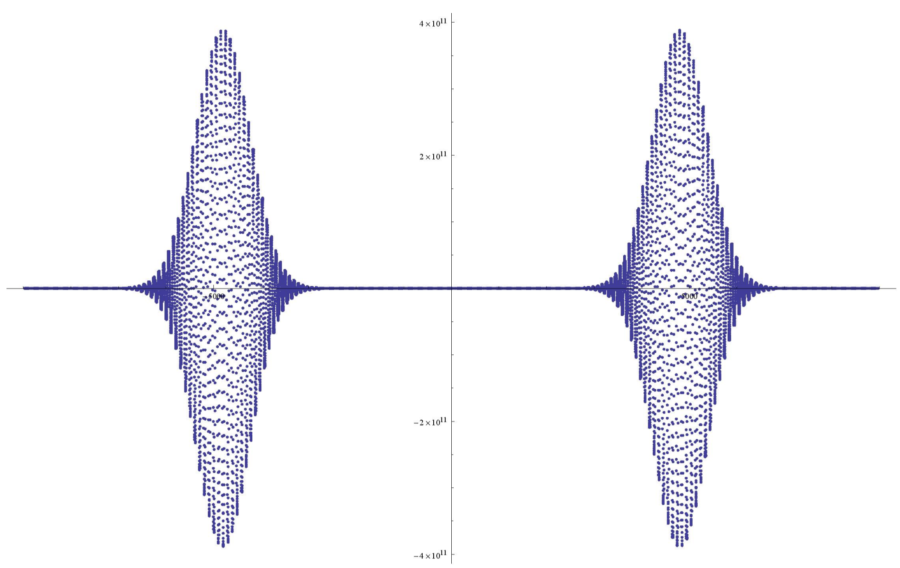
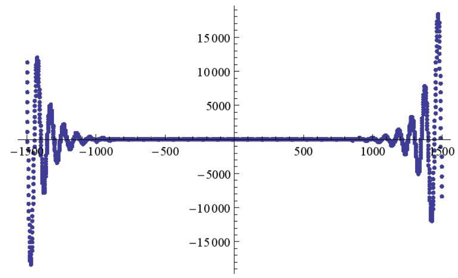
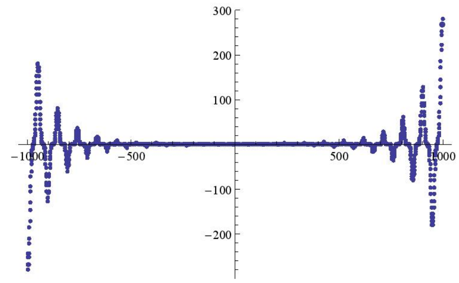


Figure 6.5: The coefficients of the 95 colored semi-(un)normalized Jones polynomial of the figure 8 knot.

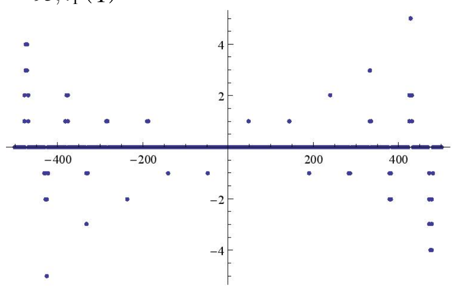
Now, we'll zoom in on the middle coefficients to see how close to zero they are. See Figure 6.6.



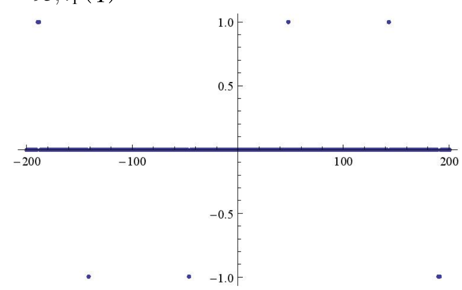
(a) The middle 3000 coefficients of the $sJ_{95,4_1}(q)$.



(b) The middle 2000 coefficients of the $sJ_{95,4_1}(q)$.



(c) The middle 1000 coefficients of the $sJ_{95,4_1}(q)$.



(d) The middle 400 coefficients of the $sJ_{95,4_1}(q)$.

Figure 6.6: We zoom in on the middle coefficients of the semi-(un)normalized colored Jones polynomial of the figure 8 knot.

In fact, from looking at different N, we observe the following conjecture is true for all n that we checked. (We checked many of the values of N less than 95. For $N > 95$, the calculation takes too long.)

Conjecture 6.2.1. *Let $c(q^i)$ be the coefficient of the q^i term of $sJ_{N,4_1}(q)$. When N is odd,*

$$c(q^i) = \begin{cases} \pm 1 & i = \pm N/2 \text{ or } \pm 3N/2 \\ 0 & |i| < 2N - 1/2 \text{ and } i \neq \pm N/2 \text{ or } \pm 3N/2 \end{cases}$$

When N is even,

$$c(q^i) = \begin{cases} \pm 1 & i = \pm N \text{ or } \pm 3N/2 \\ 0 & |i| < 2N \text{ and } i \neq \pm N \text{ or } \pm 3N/2 \end{cases}$$

We also want to know what patterns are evident in the coefficients of the unnormalized colored Jones polynomial of the figure 8 knot. We plot its coefficients below. See Figure 6.7.

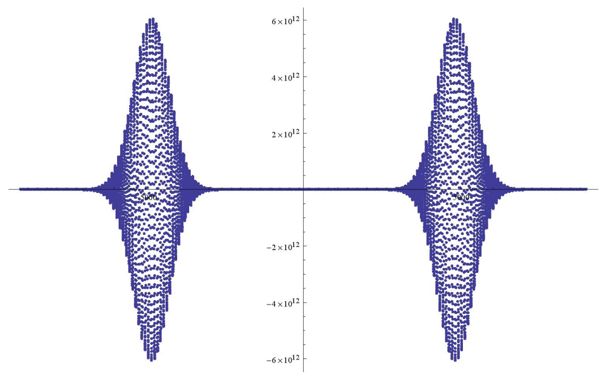


Figure 6.7: The coefficients of the 95 colored un-normalized Jones polynomial of the figure 8 knot.

Now we zoom in towards the middle coefficients. Below are the middle 1000 coefficients of $J_{95,4_1}(q)$. See Figure 6.8. Because of the difference in patterns depending on the parity of N, we also look at the middle 400 coefficients of $J_{50,4_1}(q)$. See Figure 6.9.

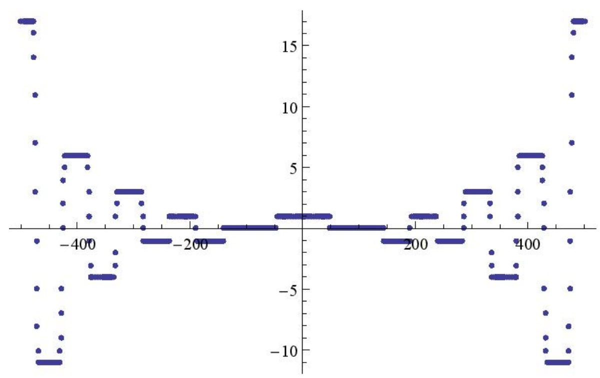


Figure 6.8: The middle 1000 coefficients of the 95 colored un-normalized Jones polynomial of the figure 8 knot.

Based on these and similar data for other values of N we make the following conjecture:

Conjecture 6.2.2. *Let $c(q^i)$ be the coefficient of the q^i term of $J_{N,4_1}(q)$. When N is odd,*

$$c(q^i) = \begin{cases} 1 & \frac{-(N-1)}{2} \leq i \leq \frac{N-1}{2} \\ 0 & \frac{(N+1)}{2} \leq |i| \leq \frac{3N-1}{2} \end{cases}$$

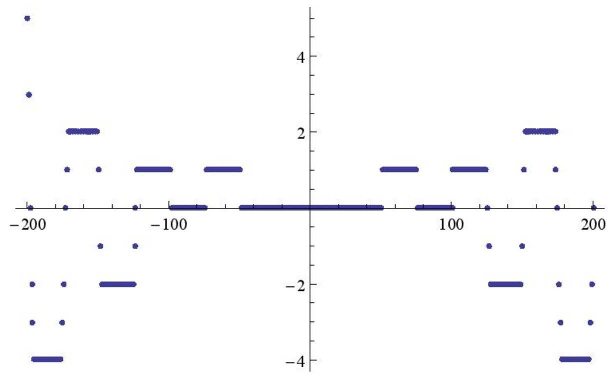


Figure 6.9: The middle 400 coefficients of the 50 colored un-normalized Jones polynomial of the figure 8 knot.

When N is even,

$$c(q^i) = \begin{cases} 0 & -N+1/2 \le i \le N-1/2 \\ 1 & N+1/2 \le |i| \le \dfrac{5N-3}{2} \end{cases}$$

Visually, we can see this as:

when N is odd:

i	$\frac{-3N+1}{2}$	$\cdots$	$\frac{-N-1}{2}$	$\frac{-N+1}{2}$	$\cdots$	$\frac{N-1}{2}$	$\frac{N+1}{2}$	$\cdots$	$\frac{3N-1}{2}$
$c(q^i)$	0	$0\cdots0$	0	1	$1\cdots1$	1	0	$0\cdots0$	0

when N is even:

i	$\frac{-5N+3}{2}$	$\cdots$	$-N-\frac{1}{2}$	$-N+\frac{1}{2}$	$\cdots$	$N-\frac{1}{2}$	$N+\frac{1}{2}$	$\cdots$	$\frac{5N-3}{2}$
$c(q^i)$	1	$1\cdots1$	1	0	$0\cdots0$	0	1	$1\cdots1$	1

6.3 What do the conjectures tell us?

First, we will show that Conjecture 6.2.2 implies Conjecture 6.2.1.

Assume N is odd. Then by Conjecture 6.2.2, the coefficients of $J_{N,4_1}(q)$ look like:

i	$\frac{-3N+1}{2}$	$\cdots$	$\frac{-N-1}{2}$	$\frac{-N+1}{2}$	$\cdots$	$\frac{N-1}{2}$	$\frac{N+1}{2}$	$\cdots$	$\frac{3N-1}{2}$
$c(q^i)$	0	$0\cdots0$	0	1	$1\cdots1$	1	0	$0\cdots0$	0

So the coefficients of $q^{1/2}J_{N,4_1}(q)$ look like:

i	$\frac{-3N+2}{2}$	$\cdots$	$\frac{-N}{2}$	$\frac{-N+2}{2}$	$\cdots$	$\frac{N}{2}$	$\frac{N+2}{2}$	$\cdots$	$\frac{3N-1}{2}$
$c(q^i)$	0	$0\cdots0$	0	1	$1\cdots1$	1	0	$0\cdots0$	0

i	$\frac{-3N}{2}$	$\cdots$	$\frac{-N-2}{2}$	$\frac{-N}{2}$	$\cdots$	$\frac{N-2}{2}$	$\frac{N}{2}$	$\cdots$	$\frac{3N-1}{2}$
$c(q^i)$	0	$0\cdots0$	0	1	$1\cdots1$	1	0	$0\cdots0$	0

And the coefficients of $q^{-1/2}J_{N,4_1}(q)$ look like:

Subtracting these, we get Conjecture 6.2.1. The proof for the N is even case is similar.

Now, let's see what these conjectures tell us about the coefficients of the normalized colored Jones polynomial, $J'_{N,4_1}(q)$ Recall that

$$\begin{aligned} J_{N,K}(q) &= (-1)^{N-1}\frac{\{N\}}{\{1\}}J'_{N,K}(q) \\ &= \pm(q^{\frac{-N+1}{2}} + q^{\frac{-N+2}{2}} + \cdots + q^{\frac{N-2}{2}} + q^{\frac{N-1}{2}})J'_{N,K}(q) \end{aligned}$$

By Conjecture 6.2.2, we know that for all N the coefficient of q^i is constant for $|i| \leq \frac{N-1}{2}$. If we look at what terms from $J'_{N,4_1}(q)$ will contribute to $J_{N,4_1}(q)$, we see that this is only true if for $|i| \leq \frac{N-1}{2}$ the coefficient of $q^{\frac{N-(i+1)}{2}}$ is equal to the coefficient of $q^{\frac{N+(i+1)}{2}}$ in $J'_{N,4_1}(q)$. This tells us that the coefficients aren't just symmetric about the constant term, but that for the middle N terms they are also symmetric about $q^{N/2}$.

6.4 A non-symmetric polynomial of amphichiral knots

Since the figure 8 knot is amphichiral (equivalent to its mirror image), we know that it's colored Jones polynomials are symmetric is q and q^{-1}, i.e. $J'_{N,4_1}(q) = J'_{N,4_1}(q^{-1})$. Thus we can rewrite $J'_{N,4_1}(q)$ as a polynomial in $x = q + q^{-1}$. To do this, we'll use the re-normalized Chebyshev polynomials of the first kind, $\tilde{T}_n$, see A.1. These are the unique set of polynomials satisfying

$$\tilde{T}_n(q+q^{-1}) = q^n + q^{-n}$$

.

Now, we will use these Chebyshev polynomials to find an expression for the non-symmetric polynomial of the Figure 8 knot, which we'll call $K'_{N,4_1}(x)$. We let $x = q + q^{-1}$. We want $K'_{N,4_1}(q+q^{-1}) = J'_{N,4_1}(q)$

We have that

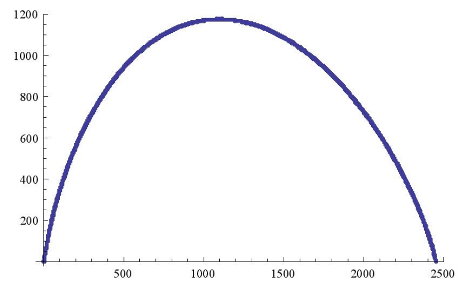
$$
\begin{aligned}
J'_{N,4_1}(q) &= \sum_{n=0}^{N-1}\prod_{j=0}^{n-1}\{N-j\}\{N+j\} \\
&= \sum_{n=0}^{N-1}\prod_{j=0}^{n-1} q^N - q^j - q^{-j} + q^{-N} \\
&= \sum_{n=0}^{N-1}\prod_{j=0}^{n-1} (q^N + q^N) - (q^j + q^{-j}) \\
&= \sum_{n=0}^{N-1}\prod_{j=0}^{n-1} T_N(q+q^{-1}) - T_j(q+q^{-1}) \\
K'_{N,4_1}(x) &= \sum_{n=0}^{N-1}\prod_{j=0}^{n-1} T_N(x) - T_j(x)
\end{aligned}
$$

We can define an analog of this for the un-normalized colored Jones polynomial. We will show the particular case when N is odd. Since, when N is odd, $J_{N,K}(q) = -[N]J'_{N,K}(q)$ we get that

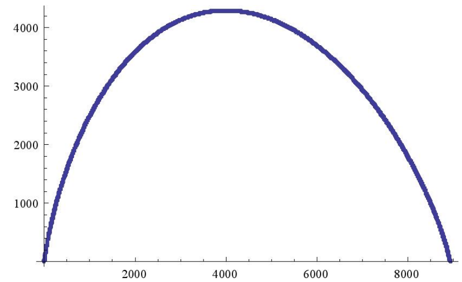
$$
\begin{aligned}
J_{N,4_1}(q) &= -[N]J'_{N,4_1}(q) \\
&= -\left(1 + \sum_{i=1}^{\frac{N-1}{2}} (q^{\frac{N-2i+1}{2}} + q^{-\frac{N-2i+1}{2}})\right) J'_{N,4_1}(q) \\
&= -\left(1 + \sum_{i=1}^{\frac{N-1}{2}} T_{\frac{N-1}{2}-i}(q+q^{-1})\right) J'_{N,4_1}(q) \\
K_{N,4_1}(x) &= -\left(1 + \sum_{i=1}^{\frac{N-1}{2}} T_{\frac{N-1}{2}-i}(x)\right) \sum_{n=0}^{N-1}\prod_{j=0}^{n-1} T_N(x) - T_j(x)
\end{aligned}
$$

Now, we want to look at the coefficients of this new polynomial and see if there are visible patterns in the coefficients and if we can relate these pattern back to the patterns visible in the colored Jones polynomial and also to the geometric properties of the knot.

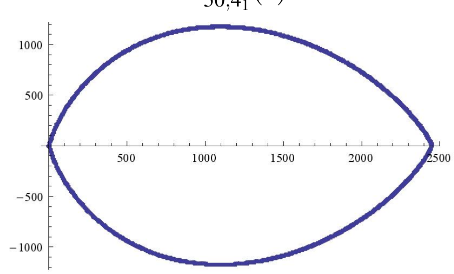
When we look at the coefficients of $K'_{N,4_1}(x)$, we notice that they grow very quickly. In order to visualize the patterns, for a polynomial of the form $\sum c_i x^i$ we will plot $\ln(|c_i|)$ and $\frac{c_i}{|c_i|}\ln(|c_i|)$, which we called the “signed log”. None off the coefficients are 0, so this is well defined. Below are plots of both visualizations for 50 and 95 colors.



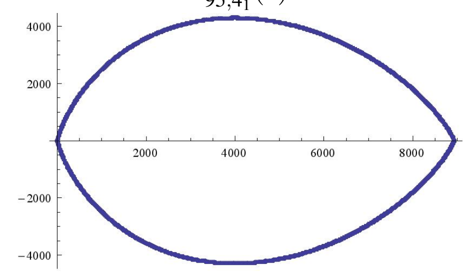
(a) The log of the absolute value of the coefficients of $K'_{50,4_1}(x)$



(b) The log of the absolute value of the coefficients of $K'_{95,4_1}(x)$



(c) The signed log of the coefficients of $K'_{50,4_1}(x)$



(d) The signed log of the coefficients of $K'_{95,4_1}(x)$.

Figure 6.10: These are plots of $\ln(|c_i|)$ and $\frac{c_i}{|c_i|}\ln(|c_i|)$ where the c_i are the coefficients of $K'_{N,4_1}(x)$ for $N = 50$ and $N = 95$

We notice that we get the same basic shape of the coefficients regardless of the number of colors. The breadth of the Nth polynomial is $N^2 - N$. We also want to look at the growth rate of the maximum coefficients of each of these polynomials. Below is a plot of the maximum coefficients of the first 50 polynomials for the figure 8 knot.

We again notice that the maximum coefficients seem to be growing exponentially. Once we take the log, the coefficients are still growing like a quadratic function. Once we take the square root of each coefficients, we notice that they seem to be growing almost exactly linearly. Let m_i be the maximum coefficient $K'_{i,4_1}(x)$. The plot of $f(n) = \sqrt{\log(m(n)}$ is below along with its line of best fit. The line of best fit has equation

$$g(n) = 0.694799n - 0.466332.$$

We have an R^2 value of .999999.

We want to know if we can relate the slope of this line to a geometric property of the knot, specifically to the hyperbolic volume of the knot. The hyperbolic volume of

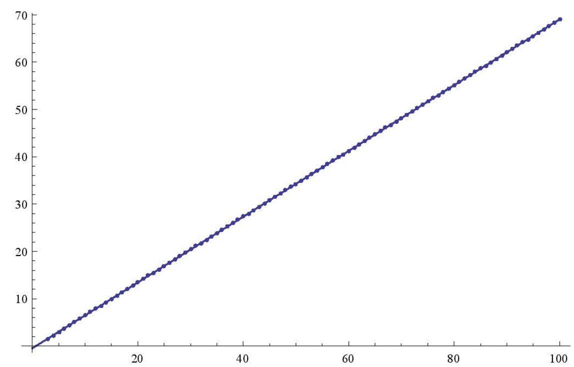


Figure 6.11: This is a plot of $f(n) = \sqrt{\log(m(n))}$ where m_i is the maximum coefficient $K'_{i,4_1}(x)$.

the figure 8 knot is 2.02988. The best relationship we can find is that

$$e^{0.694799^2}\sqrt{\pi/2} = 2.03292.$$

We now turn our focus to other amphichiral knots. The next one is 6_3. Since we don't have a nice formula for the colored Jones polynomial of 6_3, we could only find its colored Jones polynomial up to N=8. We found $K'_{N,6_3}$ for the same colors, we noticed that was had the same pattern arising in the coefficients. When scaled so that they can be plotted on the unit square by dividing by the maximum coefficient and relabeling the x-axis, we find the overall shape looks just like the one we had for 4_1. See Figure 6.12. We also found the non-symmetric version of other randomly generated polynomials (with positive and negative integer coefficients) and these also have the same basic shape when we use the Chebyshev polynomials to get the non-symmetic version. Thus, it seems the overall shape gives us no new information about the knot. But we can still, hopefully, get information from the growth rate of its maximum coefficient.

Let $m(n)$ be the maximum coefficient. We notice again that $g(n) = \sqrt{\log m(n)}$ grows linearly. See Figure 6.13. Here the line of best fit is

$$g(n) = 0.837346n + 0.370139$$

.

We also did the same analysis and reached the same conclusions with the knot 8_3. Table 6.1 show the comparison between the growth rates and the hyperbolic volumes. Our best guessed relationships are of the right type of form, but we do not have the same

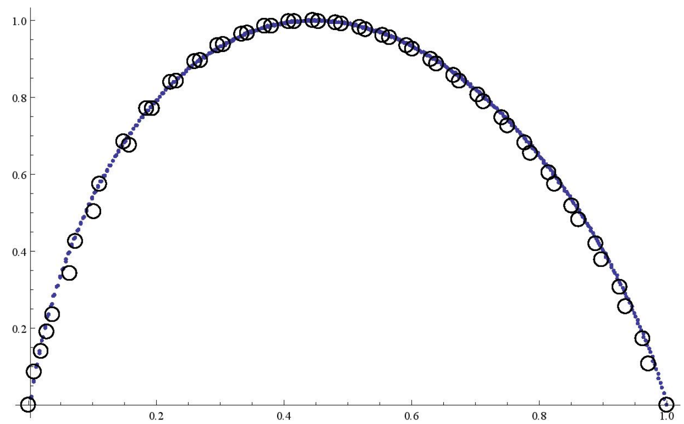


Figure 6.12: This is a plot of the log of the magnitude of the coefficients of the colored Jones polynomial of 4_1 and 6_3. The coefficients of 4_1 correspond to $N = 30$ and are plotted using small dots. The coefficients of 6_3 correspond to $N = 8$ and are plotted using larger open circles.

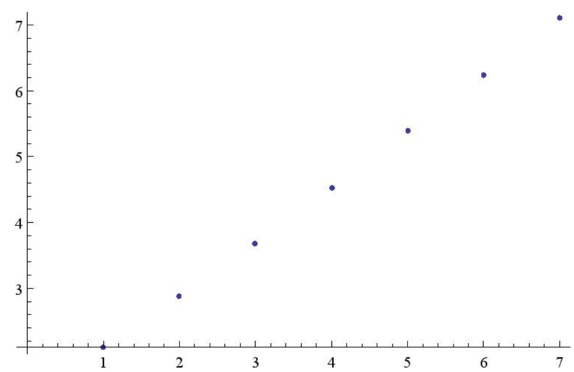


Figure 6.13: This is a plot of $f(n) = \sqrt{\log(m(n))}$ where m_i is the maximum coefficient $K'_{i,6_3}(x)$.

relationship for each knot. At this stage, we have no idea if these relationships give us any new information or are just coincidental. Since we could only get a few data points for the knots 6_3 and 8_3, it seems reasonable that those numbers are a bit farther off.

6.5 Extending this polynomial to non-amphichiral knots

Since we need $J'_{N,K}(q)$ to be symmetric in q and q^{-1} in order to define $K'_{N,K}$, we cannot define this in general for chiral knots. However, since we can get an amphichiral knot by taking any knot and connect summing it with its mirror image, we can use this

Table 6.1: Knots Hyperbolic Volumes compared to their maximum coefficient growth rate

Knot	Slope of Growth Rate	Hyperbolic Volume	Best Guess at Relationship
4_1	$m_{4_1} = 0.694799$	2.02988	$e^{(m_{4_1})^2}\sqrt{\pi/2} = 2.03292$
6_3	$m_{6_3} = 0.837346$	5.69302	$2e^{(m_{6_3})}\sqrt{\pi/2} = 5.79088$
8_3	$m_{8_3} = 0.994987$	5.23868	$\pi/2e^{(m_{8_3})}\sqrt{\pi/2} = 5.32473$

technique to study all knots. To find the colored Jones polynomial, use the fact that

$$J'_{N,K_1\#K_2} = J'_{N,K_1} J'_{N,K_2}.$$

The hyperbolic volume is also multiplicative under connect sum.

Chapter 7

Conclusion and Future Works

Throughout this dissertation, we've shown that there are many peculiar patterns visible in the coefficients of the colored Jones polynomial. We've seen that the highest order and lowest order terms stabilize as we increase the number of colors. We've also seen that are visible patterns in the middle coefficients that may be related to the hyperbolic volume of the knot.

Our hope is that we can use some of the tools developed in the dissertation to learn more about these middle patterns. The formula proved for the pretzel knots in Chapter 4 will hopefully be able to be used to help us gain insight about what happens for a larger class of knots. By studying what the higher order stability sequences stabilize to, we hope to be able to combine the higher order terms to give us information about the middle terms.

Using the new polynomial developed in the last chapter, we hope to gain further insight about the coefficients of the colored Jones polynomial. We would like to know whether the distinct pattern visible in the coefficients of the new polynomial for amphichiral knots tells us anything about the hyperbolic volume of the knot. Perhaps this pattern is just a consequence of the coefficients of the Chebyshev polynomials. Since we can get an amphichiral knot by taking any knot and taking its connect sum with its mirror image, this may help us gain information about not only amphichiral knots but about all knots.

We hope that the theorems proved in this dissertation will help push forward the work being done in the area and that the conjectures laid out in this dissertation will get people interested in these middle coefficients. There seems to be a lot of interesting phenomena related to the middle coefficients, they certainly warrant some future explorations.

Appendix A

Chebyshev polynomials

The Chebyshev polynomials come up in various places within this dissertation. The ones we use both in defining the colored Jones polynomial and in defining the new non-symmetric polynomial for the figure 8 knot, are slightly different then the standard Chebyshev polynomials. In this section, we'll describe the standard Chebyshev polynomials and how they relate to those that we are using.

A.1 Chebyshev polynomials of the 1st Kind

The standard Chebyshev polynomials of the 1st kind, $T_n(x)$ satisfy the following recurrence relation:

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

They are the unique set of polynomials that satisfy $T(\cos(\theta)) = \cos(n\theta)$, or $T(\frac{q+q^{-1}}{2}) = \frac{q^n - q^{-n}}{2}$.

When defining the non-symmetric polynomial for the figure 8 knot, we need a set of polynomials satisfying $\tilde{T}_n(q+q^{-1}) = q^n + q^{-n}$. Thus we can let $\tilde{T}_n(y) = 2T_n(y/2)$. Then we get the following recurrence relation:

$$\tilde{T}_0(x) = 2$$

$$\tilde{T}_1(x) = x$$

$$\tilde{T}_{n+1}(x) = x\tilde{T}_n(x) - \tilde{T}_{n-1}(x).$$

Now, $\tilde{T}_n(x)$ is the unique set of polynomials that satisfy $\tilde{T}_n(q+q^{-1}) = q^n + q^{-n}$ or $\tilde{T}_n(2\cos(\theta)) = 2\cos(n\theta)$.

A.2 Chebyshev polynomials of the 2nd Kind

The standard Chebyshev polynomials of the 2nd kind, $U_n(x)$ satisfy the following recurrance relation:

$$U_0(x) = 1$$

$$U_1(x) = 2x$$

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

They are the unique set of polynomials that satisfy

$$U_n(\cos(\theta)) = \frac{\sin((n+1)\theta)}{sin(\theta)} \quad \text{or} \quad U_n(\frac{q+q^{-1}}{2}) = \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}}$$

.

When defining the colored Jones polynomial, we need a set of polynomials satisfying $S_n(q+q^{-1}) = \frac{q^n+q^{-n}}{q-q^{-1}}$. Thus we can let $S_n(y) = U_n(y/2)$. Then we get the following recurrence relation:

$$S_0(x) = 1$$

$$S_1(x) = x$$

$$S_{n+1}(x) = xS_n(x) - S_{n-1}(x).$$

Now, $S_n(x)$ is the unique set of polynomials that satisfy $S_n(q+q^{-1}) = \frac{q^{n+1}-q^{-n-1}}{q-q^{-1}}$, or similarly, $S_n(2\cos(\theta)) = \frac{\sin((n+1)n\theta)}{\sin(\theta)}$.

Notice that the recursive step is the same for the first and second kind in both the standard and re-normalized Chebyshev polynomials, the only thing that changes is the initial conditions.

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