Title
THEORY OF d -DECAY

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Publication Date
1959-10-01

UCRL 8931

UNIVERSITY OF CALIFORNIA

Ernest O. Lawrence Radiation Laboratory

BERKELEY, CALIFORNIA

UCRL-8931
Chemistry General
TID-4500 (15th Ed)

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

Contract No. W-7405-eng-48

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Printed for the U. S. Atomic Energy Commission

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*This work was performed under the auspices of the U. S. Atomic Energy Commission.

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Introduction

In an earlier paper[1] a theory of α-decay was developed whose aim was to take into account the influence of nuclear structure on α-decay. The theory was found to be successful in explaining the fine structure of the Po^{211} α-decay. (Relative intensities and coefficients in α-γ angular correlations were obtained in good agreement with measured values).

In this paper we shall briefly outline again the derivation of the formulas which are then applied to discuss the ground state transitions of even-even and odd-even nuclei in the region around Pb^{208} and the fine structure of some α-decays in this region.

The nuclear wave functions are approximated by shell model wave functions and good agreement is obtained with experimental data.[2] Finally we shall give the formulas that are necessary to treat α-decays of deformed nuclei on the basis of the unified model.

Derivation of an Expression for the Decay Constant

We consider a system of A nucleons (Z protons and N neutrons) and describe it by means of the time dependent Schroedinger equation,

$$H\,\Phi(1\ldots A,t) = i\hbar\,\dot{\Phi}(1\ldots A,t) \tag{1}$$

Assume for the Hamiltonian the following form,

$$H = \sum_i -\frac{\hbar^2}{2m}\Delta_i + V(1\ldots A) \tag{2}$$

V describes the interaction of all the particles. It is further assumed that the wave function $\Phi(1,\ldots A;\, t=0)$ is known and that this wave function is the wave function of the parent nucleus.

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1. H. J. Mang, Zs. f. Phys. 148, 5~~7~~82 (1957). For details also H. J. Mang, Sitzungsberichte der Heidelberger Akademie der Wissenschaften, in press (1958).

2. J. O. Rasmussen, Phys. Rev. 113, 1593 (1959), also UCRL report 8517.

Next we separate off the trivial center of mass motion of the system and decompose the remaining Hamiltonian in several terms.

$$H = H_\alpha(1234) + H_K(5\ldots A) - \frac{\hbar^2}{2M}\Delta_R + W(\alpha, K) \quad (3)$$

where 1, 2, are protons, 3, 4 are neutrons.

H_α describes the internal motion of a system consisting of two protons and two neutrons. H_k describes the internal motion of the remaining A-4 nucleons. R is the relative distance between the center of mass of particles 1; 2; 3; 4 and the center of mass of the remaining particles 5; 6; ... A; and $-\frac{h^2}{2M}\Delta_R$ is therefore the operator of the kinetic energy associated with the relative motion of the two groups nucleons. $W(\alpha K)$ accounts for the interaction between the two groups of particles. We should mention that the Hamiltonian H has not lost its symmetry properties.

For obvious reasons we shall refer from now on to these two groups of nucleons as α-particle and daughter nucleus. Consequently we call the solutions of the equations,

$$\begin{aligned} H_\alpha \chi_\alpha^\tau &= \epsilon_\alpha \chi_\alpha^\tau \\ H_K \Psi_K^\sigma &= E_K \Psi_K^\sigma \end{aligned} \quad (4)$$

the internal wave functions of the α-particle and the daughter nucleus. In these equations τ and σ are short hand notations for those sets of quantum numbers that are necessary to determine the solutions χ_α^τ and Ψ_K^σ completely.

Furthermore if χ_α^τ $\chi_\alpha^{\tau'}$ and Ψ_K^σ $\Psi_K^{\sigma'}$ are bound state solutions of (4) the following relation holds,

$$\langle \chi_\alpha^{\tau'} \Psi_K^{\sigma'} | W(\alpha K) | \chi_\alpha^\tau \Psi_K^\sigma \rangle = \delta_{\tau\tau'} \delta_{\sigma\sigma'} V(R); \; R > R_0 \quad (5)$$

The meaning of this relation is that always when the α-particle and the daughter nucleus are well separated in space the interaction between them is described by a simple potential. Furthermore V(R) approaches the Coulomb potential $\frac{2(Z-2)e^2}{R}$ rapidly. For most purposes therefore it will be sufficient to use $\frac{2(Z-2)e^2}{R}$ instead of V(R).

($R_o \approx 8.5 - 9.0 \cdot 10^{-13}$ cm for natural α emitters.)

These considerations indicate that it might be reasonable to write for the time-dependent wave function $\Phi(1\ldots A;t)$,

$$\Phi(1\ldots\ldots A;t) = a(t)\,\Phi_0(1\ldots\ldots A) + \sum_{\tau,\sigma,L,m} \int d\epsilon \; b^{(\epsilon,t)}_{\tau\sigma L m} \cdot \mathcal{A}\left\{ \chi^{\tau}_{\alpha}\, \psi^{\sigma}_{K}\, \varphi_L(R,\epsilon)\, Y^{m}_{L}\left(\frac{\mathbf{R}}{R}\right)\right\} \tag{6}$$

A is an antisymmetrization and normalization operator, while $\varphi_L(\epsilon R)$ is a solution of the equation,

$$\left\{ -\frac{\hbar^2}{2M}\frac{1}{R}\frac{d^2}{dR^2}R + \frac{\hbar^2}{2M}\frac{L(L+1)}{R^2} + \frac{2(Z-2)e^2}{R} - \epsilon \right\} \varphi_L(\epsilon R) = 0 \tag{7}$$

and is normalized as follows:

$$\int \varphi_L(\epsilon R)\, \varphi_L(\epsilon' R)\, R^2 dR = \delta(\epsilon-\epsilon') \tag{8}$$

To get unique solutions[3] for the coefficients a(t) and $b^{(\epsilon,t)}_{\tau\sigma L m}$ it is necessary to impose the condition,

$$\left\langle \Phi_0 \left| \sum_{\tau,\sigma,L,m} \int d\epsilon\, A\left\{ \chi^{\tau}_{\alpha}\psi^{\sigma}_{K}\varphi_L(\epsilon R) Y^{m}_{L}\right\} b^{(\epsilon,t)}_{\tau\sigma L m} \right. \right\rangle = 0 \tag{9}$$

Of course $\Phi(1\ldots A;t)$ may always be expanded in the above form (Eq. 6) as long as we sum over a complete set of function χ^{τ}_{α} and ψ^{σ}_{K}. But we know that for energetic reasons the terms in the sum over τ and σ corresponding to unbound states cannot contribute to the α-decay. Therefore we split the sum into one over bound states only and one in which τ or σ or both correspond to unbound states and neglect the latter sum. With this approximation and taking into account angular momentum conservation as well as the fact that there is only a single bound state of the α-particle we rewrite $\Phi(1\ldots\ldots A;t)$ in the following form,

3. H. Casimir, Physica 1, 193 (1934).

$$\Phi(1\ldots A;t) = a(t)\,\Phi_{0J}^{M}(1\ldots A) + \sum_{J\sigma L}\int d\epsilon\; b_{J\sigma L}(\epsilon,t)$$

$$A\left\{\chi_\alpha\,\varphi_L(R,\epsilon)\sum_m C(LJJ_j;m\,M-m)\,Y_L^m\left(\frac{\mathfrak{R}}{R}\right)\Psi_{KJ\sigma}^{M-m}\right\} \tag{10}$$

Introducing this expression for Φ into the Schroedinger equation (1) we get a system of coupled integrodifferential equations for a(t) and $b_{J\sigma L}(\epsilon,t)$ But this system of equations decouples and becomes rather simple if one more approximation is introduced. We neglect nondiagonal matrix elements of the type*

$$\left\langle \Phi_{JJ\sigma L\epsilon}^{M}\left| H - H_\alpha - H_K + \frac{\hbar^2}{2M}\Delta_{\mathfrak{R}} - \frac{2(Z-2)e^2}{R}\right| \Phi_{JJ'\sigma'L'\epsilon'}^{M}\right\rangle$$

which means we neglect the interaction of the α-particle with the daughter nucleus via the nuclear forces** and take only into account the Coulomb interaction; a reasonable approximation at the α-energies involved in natural α-decays.

The simplified equations read:

$$i\hbar\,\dot{a}(t) = a(t)E_0 + \sum_{J\sigma L}\int d\epsilon\; b_{J\sigma L}(\epsilon t)\left\langle \Phi_{0J}^{M}\right| H - E_0 \left|\Phi_{JJ\sigma L\epsilon}^{M}\right\rangle$$

$$i\hbar\left[\dot{b}_{J\sigma L}(\epsilon t) + \dot{a}(t)\left\langle \Phi_{JJ\sigma L\epsilon}\middle|\Phi_{0J}^{M}\right\rangle\right] = \tag{11}$$

$$(E_K + \epsilon_\alpha + \epsilon)\,b_{J\sigma L}(\epsilon,t) + a(t)\left\langle \Phi_{JJ\sigma L\epsilon}^{M}\right| H \left|\Phi_{0J}^{M}\right\rangle$$

where

$$E_0 = \left\langle \Phi_{0J}^{M}\right| H \left|\Phi_{0J}^{M}\right\rangle$$

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* $\Phi_{JJ\sigma L\epsilon}^{M} = A\left\{\chi_\alpha\,\varphi_L(R,\epsilon)\sum_m C(LJJ;m\,M-m)\,Y_L^m\,\Psi_{KJ\sigma}^{M-m}\right\}$

**If we use V(R) as defined in Eq. (5) instead of the Coulomb potential we take into account a part of the nuclear interaction.

Using standard techniques[4] to solve the equations approximtely we get,

$$a(t) = e^{-\frac{i}{\hbar}(E_0 + \Gamma - i\gamma)t}$$

$$b_{j\sigma L}(\epsilon,t) = \frac{e^{-\frac{i}{\hbar}(E_0+\Gamma-i\gamma)t} - e^{-\frac{i}{\hbar}(E_K+\epsilon_\alpha+\epsilon)t}}{E_0+\Gamma-E_K-\epsilon_\alpha-\epsilon-i\gamma}\langle\Phi_{0J}^M|H-E_0|\Phi_{JjsL\epsilon}^M\rangle$$

$$\gamma = \pi \cdot \sum_{j,L,\sigma}' \left|\langle\Phi_{Jj\sigma L\epsilon}^M|H-E_0|\Phi_{0J}^M\rangle\right|^2_{\epsilon = E_0-\epsilon_\alpha-E_K+\Gamma} \tag{12}$$

$$\Gamma = \sum_{jL\sigma}' \int d\epsilon \frac{1-e^{-\frac{i}{\hbar}(\epsilon_\alpha+E_K+\epsilon-E_0-\Gamma+i\gamma)t}}{E_0+\Gamma-E_K-\epsilon_\alpha-\epsilon-i\gamma}\left|\langle\Phi_{0J}^M|H-E_0|\Phi_{Jj\sigma L\epsilon}^M\rangle\right|^2$$

The conditions for the approximate solution to be a good one are,

$$\gamma \ll |E_0-E_K-\epsilon_\alpha| = \epsilon_0 \tag{13}$$
$$\Gamma \ll |E_0-E_K-\epsilon_\alpha|$$

and

$\left|\langle\Phi_{Jj\sigma L\epsilon}^M|H-E_0|\Phi_{0J}^M\rangle\right|^2$

has to vary slowly compared to $[(E_0+\Gamma-E_K-\epsilon_\alpha-\epsilon)^2+\gamma^2]^{-1}$

if ϵ varies between $\epsilon_0+\gamma$ and $\epsilon_0-\gamma$. For all natural α-emitters these conditions are well fulfilled.

The problem of determining the decay constant is now reduced to the calculation of the matrix elements $\langle\Phi_{0J}^M|H-E_0|\Phi_{Jj\sigma L\epsilon}^M\rangle$

But those matrix elements cannot be calculated unless Φ_{0J}^M is defined in a rigorous way. The somewhat vague statement "the wave function Φ_{0J}^M should describe the parent nucleus of the α-decay" cannot be considered as a definition of Φ_{0J}^M.

A natural way to define Φ_{0J}^M would be to obtain it from a consideration of the formation process of the α-emitter. But on a nuclear time scale all α-emitters are nearly stable. If it were not for the repulsive Coulomb-interaction between the α-particle and the daughter nucleus there would be no

4. W. Heitler, Quantum Theory of Radiation, 3rd edition.

α-decay at all. Therefore another definition of Φ_{0J}^{M} is suggested. It has to be a bound state solution of a Schroedinger equation.

$$H_0 \Phi_{0J}^{M} = E^{(0)} \Phi_{0J}^{M} \tag{14}$$

H_0 is defined as follows:

$H_0 = H$ if all nucleons are confined in a spherical* volume Ω_0 with radius r_0.

$H_0 = H - V_c(r) + V_c(r_0)$ if one or more nucleons are outside the above defined volume Ω_0. V_c is the Coulomb interaction between the nucleons inside and those outside Ω_0

r_0 is so defined that the interaction between one selected proton (neutron) and all the others that are in Ω_0 is repulsive (zero) if the selected proton (neutron) is outside Ω_0 but mostly attractive if the proton (neutron) is inside Ω_0. Of course there is a maximum value for r_0 because of the condition that the eigen value problem with H_0 (Eq. 14) should have bound state solutions. This definition assures that H_0 is equal to the exact Hamiltonian H in that part of the configuration space where the nuclear forces between the nucleons play an important role and therefore Φ_{0J}^{M} should be a very close approximation to an exact wave function as derived for instance from considering the formation process of an α-emitter. By means of these assumptions about Φ_{0J}^{M} the matrix element can be simplified considerably.

$$\begin{aligned} \langle \Phi_{0J}^{M} | H - E_0 | \Phi_{J j \sigma L \epsilon} \rangle &= \left[\binom{N}{2} \binom{Z}{2} \right]^{1/2} \cdot \Big\langle \Phi_{0J}^{M\,(1 \ldots A)} \Big| H - H_0 \Big| \\ &\Big| \chi_\alpha^{(1 \ldots 4)} \varphi_L(\epsilon R) \cdot \sum_m C(LjJ; m\, M-m)\, Y_L^m\, \Psi_{j\sigma}^{M-m\,(5 \ldots A)} \Big\rangle = \\ &\left[\binom{N}{2} \binom{Z}{2} \right]^{1/2} \cdot \int_{R_0}^{\infty} R^2 dR \int d\Omega_R \int d\xi_\alpha\, d\xi_K\, \Phi_{0J}^{*M} (H - H_0)\, \chi_\alpha\, \varphi_L(R,\epsilon) \\ &\sum_m C(LjJ; m\, M-m)\, Y_L^m\, \Psi_{j\sigma}^{M-m} \end{aligned} \tag{15}$$

---

*Of course one may assume Ω_0 to be nonspherical if one deals with a deformed nucleus.

The first equality holds because of H - E_o being a symmetric operator and $\bar{\Phi}_{c\gamma}^{m}$ an antisymmetric wave function and the fact that $E_o = E^{(o)}$ in a very good approximation. The second equality holds because of the definition of $\bar{\Phi}_{c\gamma}^{m}(1 \dots A)$ (Eq. 14).

R_o is somewhat smaller than r_o because of the finite size of the α-particle. The relation between R_o and r_o is schematically illustrated in Fig. 1.

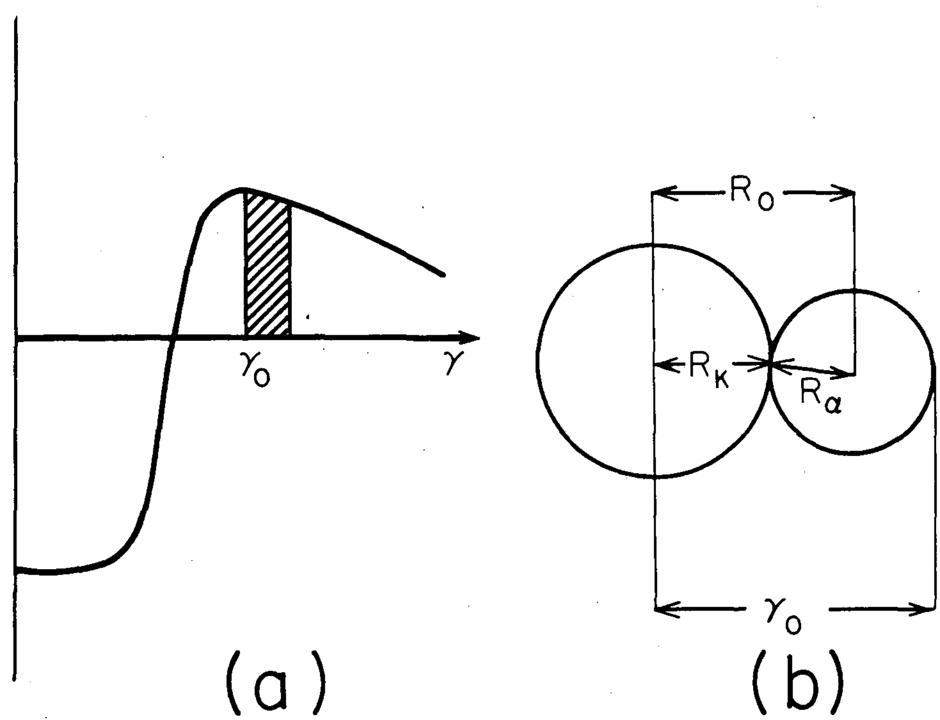

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Fig. 1.

Because of the freedom in choosing r_o and consequently R_o, r_o and hence R_o is chosen so that for $R \geq R_o$.

$$H \cdot \chi_\alpha \varphi_L(R,\epsilon) \sum_m C(LJJ;m\,M-m)\, Y_L^m\, \Psi_{J\sigma}^{M-m} = \tag{16}$$

$$(E_K + \epsilon_\alpha + \epsilon)\, \chi_\alpha\, \varphi_L(R\epsilon) \sum_m' C(LJJ;m\,M-m)\, Y_L^m\, \Psi_{J\sigma}^{M-m}$$

Using (15, (16) and integrating by parts on R the following expression is obtained for the matrix element in question:

$$\langle \bar{\Phi}_{0J}^M | H - E_0 | \bar{\Phi}_{JJ\sigma L\,\epsilon_0}^M \rangle = \left[\binom{N}{2}\binom{Z}{2} \right]^{1/2} \frac{\hbar^2}{2M}$$

$$\int d\xi_\alpha\, d\xi_K\, R_0^2\, d\Omega_R \left\{ \bar{\Phi}_{0J}^{*M} \frac{\partial \varphi_L(R\epsilon)}{\partial R}\bigg|_{R=R_0} - \right. \tag{17}$$

$$\left. \frac{\partial \bar{\Phi}_{0J}^{*M}}{\partial R}\bigg|_{R=R_0} \varphi_L(R_0\epsilon) \right\} \chi_\alpha \sum_m' C(LJJ;m\,M-m)\, Y_L^m\, \Psi_{J\sigma}^{M-m}$$

Introducing for $\varphi_L(\epsilon_0 R)$ the WKB approximation

$$\varphi_L(\epsilon_0 R) = \left(\frac{2M}{\pi\hbar^2}\right)^{1/2} \frac{1}{R_0 q_L^{1/2}} \cdot \frac{1}{2} \exp\left(-\int_R^{R_u} q_L\, dR\right) \tag{18}$$

$$q_L = \left[\frac{2M}{\hbar^2} \left(\frac{2(Z-2)e^2}{R} + \frac{\hbar^2}{2M} \frac{L(L+1)}{R^2} - \epsilon_0 \right) \right]^{1/2}$$

$$q_L(R_u) = 0$$

and defining a function $G_{JJ\sigma L}(R)$

$$G_{JJ\sigma L}(R) = \left[\binom{N}{2}\binom{Z}{2}\right]^{1/2} R_0^{3/2} \cdot \int d\xi_\alpha \, d\xi_K \, d\Omega_R \left(\overset{*}{\Phi}{}_{0J}^{M} \right.$$

$$\left. \cdot \, \chi_\alpha \sum_m C(L\,J\,J;\, m\; M-m)\, Y_L^m \, \Psi_{J\sigma}^{M-m} \right) \tag{19}$$

the expression for the decay constant is brought into the following form:

$$\lambda = \frac{1}{2\hbar} \sum_{JL\sigma} \exp\left(-2\int_{R_0}^{R_w} q_L \, dR\right) \cdot \frac{\hbar^2}{2MR_0^2} \frac{\left(R_0 q_L(R_0) - 1\right)^2}{2 \cdot R_0 q_L(R_0)} \cdot \tag{20}$$

$$\left| G_{JJL\sigma}(R_0) + \frac{R_0}{1 - R_0 q_L(R_0)} \cdot \left.\frac{\partial G_{JJ\sigma L}}{\partial R}\right|_{R=R_0} \right|^2$$

$$= \frac{1}{\hbar} \sum_{JL\sigma} P_L(\epsilon_0) \cdot \delta^2_{JJ\sigma L}$$

where,

$$P_L = \exp\left(-2\int_{R_0}^{R_w} q_L \, dR\right)$$

$$\delta^2_{JJ\sigma L} = \frac{\hbar^2}{2MR_0^2} \frac{\left(R_0 q_L - 1\right)^2}{2R_0 q_L} \left| G_{JJ\sigma L}(R_0) + \frac{R_0}{1 - R_0 q_L} \left.\frac{\partial G_{JJ\sigma L}}{\partial R}\right|_{R=R_0} \right|^2$$

P_L is the well known barrier penetrability as already derived by Gamow in 1928 and $\delta^2_{JJ\sigma L}$ is the reduced width that accounts for the influence of the nuclear structure on α decay.

For the connection of our treatment with that of other authors we refer to reference (1).

We should also mention that there is no difficulty in introducing a more sophisticated potential V(R). One has just to replace everywhere the Coulomb potential by V(R).

Application of the theory to α-emitters, in the region of Pb^{208}

The theory developed in the preceeding chapter will now be applied to α-decays in the region of Pb^{208}. We have reasons to hope that for these α-emitters the nuclear wave functions may be approximated by some sort of shell model wave functions.

The details of calculating the function $G_{J j \sigma L}(R_o)$ and hence the reduced width $\delta^2_{J j \sigma L}$ will be given in an appendix. At the present time we only list and discuss the assumptions and compare the results with experimental data.

a. Ground state transitions of even-even and even-odd nuclei.

We assume that the ground state wave functions of even-even and even-odd nuclei have the following properties:

1. Even numbers of protons or neutrons are coupled pair-wise to angular momentum zero. In an even-odd nucleus therefore the last odd particle is responsible for the total angular momentum. (The explicit form of the wave function is given in the appendix). These are the wave functions with seniority 0 (even-even) and 1 (odd-even).

2a. There is no configurational mixing and the single particle states involved as well as the order of their filling are taken from the spectra of nuclei with one nucleon more or less than Pb^{208}, or

2b. configurational mixing is taken into account on the basis of the so called "quasi particle model".[5]

3. For the radial wave functions, harmonic oscillator wave functions are used. That means the single-particle wave functions $\Phi_{n\ell j}$ are

$$\Phi_{n\ell j} = \left[\frac{2\cdot n!\,\alpha^{3/2}}{(n+\ell+\frac{1}{2})!}\right]^{1/2} (\sqrt{\alpha}\,r)^{\ell}\, L_n^{\ell+\frac{1}{2}}(\alpha r^2)\, e^{-\frac{\alpha}{2}r^2} \sum_{\mu} C(\ell\tfrac{1}{2}j;\mu\, m-\mu)\, Y_\ell^{\mu}\!\left(\frac{\mathbf{r}}{r}\right) \chi_{\frac{1}{2}}^{m-\mu}(s) \tag{21}$$

with,

$$L_n^{\ell+\frac{1}{2}} = \sum_k \binom{n+\ell+\frac{1}{2}}{n-k} \frac{(-1)^k}{k!} (\alpha r^2)^k \; ; \quad \alpha = \frac{m\omega}{\hbar}$$

5. S. T. Belyaev, Kgl. Danske Videnskab. Selskab Mat.-fys. Medd. 31, No. 11 (1958)
B. R. Mottelson, Lectures at the University of California (1959).

Furthermore we assume for the α-particle wave function χ_α a Gaussian type wave function[1]

$$\chi_\alpha = \chi_0^0(12)\,\chi_0^0(34)\, e^{-\frac{\beta}{2}(\xi_1^2+\xi_2^2+\xi_3^2)} \left[\frac{2\beta^{3/2}}{\frac{1}{2}!\,4\pi}\right]^{3/2} \tag{22}$$

$$\xi_1 = \frac{1}{\sqrt{2}}(\mathbf{r}_1-\mathbf{r}_2) \qquad \xi_2 = \frac{1}{\sqrt{2}}(\mathbf{r}_3-\mathbf{r}_4) \qquad \xi_3 = \frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2-\mathbf{r}_3-\mathbf{r}_4)$$

where $\chi_0^0(ik)$ is a singlet spin function of particles i and k. For the constants α, β and R_0 we are going to use the values:

$$\alpha = 0.175\ f^{-2}$$
$$\beta = 0.625\ f^{-2}$$
$$R_0 = 9\ f$$

These values were chosen so that the RMS radii of the nuclei agree with the values given by electron scattering experiments.[6] But we shall only calculate relative transition probabilities and therefore the choice of these parameters is not too critical.

With the above assumptions (1, 2a, 3) we get for the reduced widths $\delta^2_{Jj\sigma L}$ (where J is the angular momentum of the parent nucleus; j, the angular momentum of the daughter nucleus; and L, the angular momentum of the outgoing α-particle):

Even-even nuclei:

$$\delta^2_{000} = \frac{1}{16}\, N_1\,(2j_1+3-N_1)\, N_3\,(2j_3+3-N_3)\, R^2_{\substack{n_1\ell_1 n_1\ell_1\\ n_3\ell_3 n_3\ell_3}}(R_0) \tag{23}$$

N_1 is the number of protons in the unfilled subshell with quantum numbers $n_1 \ell_1 j_1$ in the parent nucleus. N_3 is the number of neutrons in the unfilled subshell with quantum numbers $n_3 \ell_3 j_3$ in the parent nucleus. R involves the radial parts of the wave functions and is given below:

6. R. Hofstadter, Rev. Modern Phys. 28, 3, 214 (1956).

Even-odd nuclei:

$$\delta^2_{J_1J_1 0} = \frac{1}{16}(N_1-1)(2J_1+2-N_1)N_3(2J_3+3-N_3)R^2_{\substack{n_1\ell_1 n_1\ell_1\\ n_3\ell_3 n_3\ell_3}}(R_0)$$

$$\delta^2_{J_1J_1L} = \frac{1}{4}(N_1-1)(2J_1+2-N_1)N_3(2J_3+3-N_3)\left[\frac{2J_1+1}{2J_1-1}\cdot \right. \tag{24}$$

$$\left. C(J_1J_1L;\tfrac{1}{2}-\tfrac{1}{2})\right]^2 R^2_{\substack{n_1\ell_1 n_1\ell_1\\ n_3\ell_3 n_3\ell_3}}(R_0) \quad ; \quad L \neq 0$$

$$\delta^2_{J_2J_1L} = \frac{1}{4}N_1 C^2(J_1J_2L;\tfrac{1}{2}-\tfrac{1}{2})N_3(2J_3+3-N_3)\cdot$$

$$R^2_{\substack{n_1\ell_1 n_2\ell_2\\ n_3\ell_3 n_3\ell_3}}(R_0)$$

N_1 is again the number of nucleons in the subshell n_1, ℓ_1, J_1. In the first two cases ($\delta^2_{J_1J_1 0}$ $\delta^2_{J_1J_1L}$) N_1 is odd. In the third case ($\delta^2_{J_2J_1L}$) N_1 is even and there is one more particle in the state n_2, ℓ_2, J_2 in the parent nucleus. $R_{\substack{n_1\ell_1 n_2\ell_2\\ n_3\ell_3 n_4\ell_4}}(R_0)$ has the following form:

$$R_{\substack{n_1\ell_1 n_2\ell_2\\ n_3\ell_3 n_4\ell_4}}(R_0) = \left[n_1!\,(n_1+\ell_1+\tfrac{1}{2})!\;n_2!\,(n_2+\ell_2+\tfrac{1}{2})!\,\cdot\right.$$

$$\left. n_3!\,(n_3+\ell_3+\tfrac{1}{2})!\;n_4!\,(n_4+\ell_4+\tfrac{1}{2})!\right]^{-\frac{1}{2}}\left(\frac{2\alpha}{\alpha+\beta}\right)^{N+\frac{L}{2}}$$

$$\frac{N!}{2^{2N+L}}\cdot\left[\frac{\hbar^2}{MR_0^2}\cdot\frac{(R_0 q_L(R_0)-1)^2}{R_0 q_L(R_0)}\cdot(\sqrt{\alpha}R_0)^3\cdot\left(\frac{2\sqrt{\alpha\beta}}{\alpha+\beta}\right)^9\left(\tfrac{1}{2}!\right)^3\right]^{\frac{1}{2}} \tag{25}$$

$$\sum_{s=0}^{N} B_s\left(\frac{\beta-\alpha}{2\alpha}\right)^{N-s}\frac{(-1)^s s!}{N!}4^{N-s}\left(2(\alpha+\beta)R_0\right)^{\frac{L}{2}}e^{-2\alpha R_0^2}K_s^{L+\frac{1}{2}}$$

where,

$$K_\varsigma^{L+\frac{1}{2}} = \left[\left(1 + \frac{4\alpha R_0^2 - L}{R_0 q_L - 1}\right) L_\varsigma^{L+\frac{1}{2}}(2(\alpha+\beta)R_0^2) - \frac{4(\alpha+\beta)R_0^2}{R_0 q_L - 1} L_\varsigma^{L+\frac{1}{2}\,\prime}(2(\alpha+\beta)R_0^2) \right]$$

$$2N+L = 2(n_1+n_2+n_3+n_4) + \ell_1+\ell_2+\ell_3+\ell_4$$

where the prime means differentiation with respect to the argument $2(\alpha+\beta)R^2$. The coefficients B_ς are given in Eq. (A8).

With the help of these formulas and using assumption (2a) and the configurations listed in Table 1, we have calculated the reduced widths that are compared to the experimental data[2] in Fig. 2,3,4. In these diagrams the experimental and calculated reduced widths are plotted versus the mass number for the even-even Po isotopes, the even-odd At isotopes and the even-odd Po isotopes separately.

The reduced width of Po^{210} is taken as a standard and set equal to the experimental reduced width. We hope that in doing this, we minimize the ambiguities introduced by the choice of the radial wave functions and the set of parameters α, β and R_o.*

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*It may be shown that varying the parameters within reasonable limits affects only the absolute magnitude of the radial part of the reduced width $R^2_{\substack{n_1\ell_1 n_2\ell_2 \\ n_3\ell_3 n_4\ell_4}}(R_0)$, but leaves nearly unaffected the relative magnitudes we are mainly interested in. This result indicates also that it is sometimes allowed to use a very handy approximation for the radial part $R^2_{\substack{n_1\ell_1 n_2\ell_2 \\ n_3\ell_3 n_4\ell_4}}(R_0)$ which arises if one sets $\beta \gg \alpha$. Then

$$R_{\substack{n_1\ell_1 n_2\ell_2 \\ n_3\ell_3 n_4\ell_4}}(R_0) \sim \left[\frac{n_1!\, n_2!\, n_3!\, n_4!}{(n_1+\ell_1+\frac{1}{2})!\,(n_2+\ell_2+\frac{1}{2})!\,(n_3+\ell_3+\frac{1}{2})!\,(n_4+\ell_4+\frac{1}{2})!}\right]^{1/2}$$

$$e^{-2\alpha R_0^2} \left(\sqrt{\alpha}\, R_0\right)^{\ell_1+\ell_2+\ell_3+\ell_4} L_{n_1}^{\ell_1+\frac{1}{2}}(\alpha R_0^2)\, L_{n_2}^{\ell_2+\frac{1}{2}}(\alpha R_0^2)\, L_{n_3}^{\ell_3+\frac{1}{2}}(\alpha R_0^2)\, L_{n_4}^{\ell_4+\frac{1}{2}}(\alpha R_0^2)$$

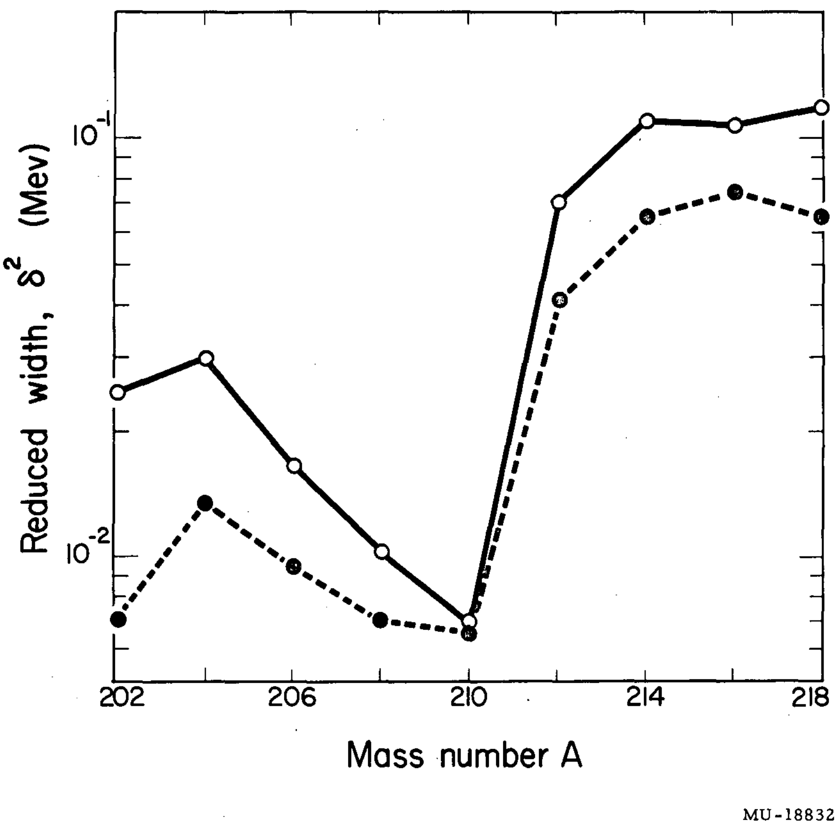
a very useful expression if one wants results quickly. The expression so obtained is identical with what one would get using the formulas proposed by Toelhock and Brussaard (P. T. Brussaard and H. A. **Tolhoek**, Physica 24, 233 (1958).) But one has to be very careful in using this expression because it favors too much the formation of α-particles from single particle states with high angular momenta.

Table 1. Shell model configuration for nuclei in the region of Pb^{208}

Element	Proton config.	Neutron configuration	Element	Proton config.	Neutron configuration
Po^{202}	$(1h_{9/2})^2_0$	$(p_{1/2})^{-2}(f_{5/2})^{-4}(p_{3/2})^{-2}$	Pb^{198}	Closed shell	$(p_{1/2})^{-2}(f_{5/2})^{-6}_0(p_{3/2})^{-2}_0$
Po^{204}	"	$(p_{1/2})^{-2}_0(f_{5/2})^{-4}_0$	Pb^{200}	"	$(p_{1/2})^{-2}_0(f_{5/2})^{-4}_0(p_{3/2})^{-2}_0$
Po^{206}	"	$(p_{1/2})^{-2}_0(f_{5/2})^{-2}_0$	Pb^{202}	"	$(p_{1/2})^{-2}_0(f_{5/2})^{-4}_0$
Po^{208}	"	$(p_{1/2})^{-2}_0$	Pb^{204}	"	$(p_{1/2})^{-2}_0(f_{5/2})^{-2}_0$
Po^{210}	"	Closed shell	Pb^{206}	"	$(p_{1/2})^{-2}_0$
Po^{212}	"	$(2g_{9/2})^2_0$	Pb^{208}	"	Closed shell
Po^{214}	"	$(2g_{9/2})^4_0$	Pb^{210}	"	$(2g_{9/2})^2_0$
Po^{216}	"	$(2g_{9/2})^6_0$	Pb^{212}	"	$(2g_{9/2})^4_0$
Po^{218}	"	$(2g_{9/2})^8_0$	Pb^{214}	"	$(2g_{9/2})^6_0$
At^{203}	$(1h_{9/2})^3_{9/2}$	$(p_{1/2})^{-2}_0(f_{5/2})^{-4}_0(p_{3/2})^{-2}_0$	Bi^{199}	$1h_{9/2}$	$(p_{1/2})^{-2}_0(f_{5/2})^{-6}_0(p_{3/2})^{-2}_0$
At^{205}	"	$(p_{1/2})^{-2}_0(f_{5/2})^{-4}_0$	Bi^{201}	"	$(p_{1/2})^{-2}_0(f_{5/2})^{-4}_0(p_{3/2})^{-2}_0$
At^{207}	"	$(p_{1/2})^{-2}_0(f_{5/2})^{-2}_0$	Bi^{203}	"	$(p_{1/2})^{-2}_0(f_{5/2})^{-4}_0$
At^{209}	"	$(p_{1/2})^{-2}_0$	Bi^{205}	"	$(p_{1/2})^{-2}_0(f_{5/2})^{-2}_0$
At^{211}	"	Closed shell	Bi^{207}	"	$(p_{1/2})^{-2}_0$
At^{213}	"	$(2g_{9/2})^2_0$	Bi^{209}	"	Closed shell
At^{215}	"	$(2g_{9/2})^4_0$	Bi^{211}	"	$(2g_{9/2})^2_0$

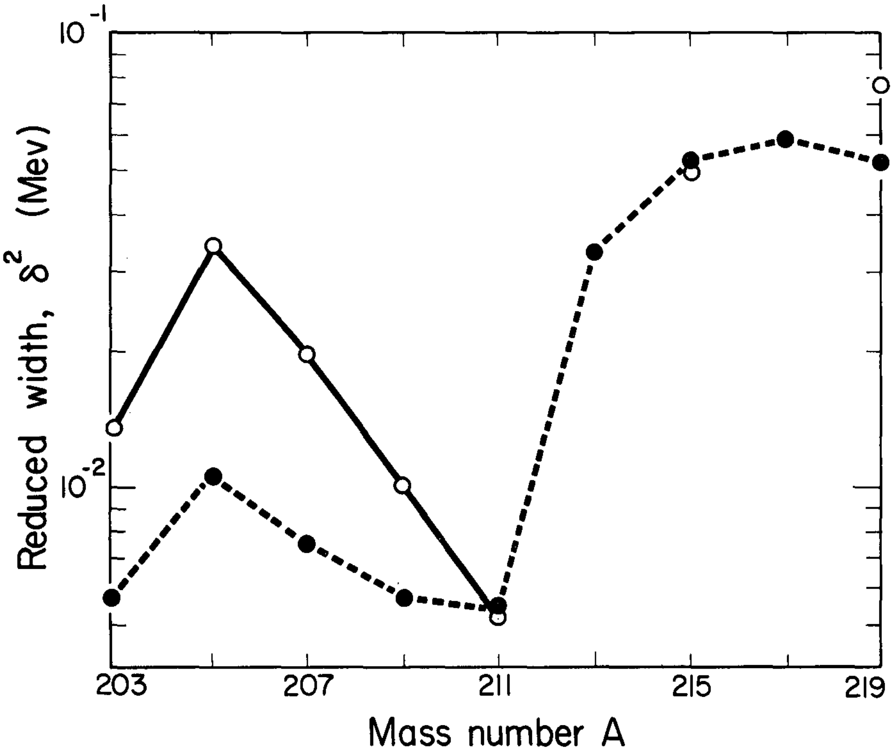
Table 1 (cont'd.)

Element	Proton config.	Neutron configuration	Element	Proton config.	Neutron configuration
At^{217}	$(1h_{9/2})^{3}_{9/2}$	$(2g_{9/2})^{6}_{0}$	Bi^{213}	$1h_{9/2}$	$(2g_{9/2})^{4}_{0}$
At^{219}	"	$(2g_{9/2})^{8}_{0}$	Bi^{215}	"	$(2g_{9/2})^{6}_{0}$
Po^{203}	$(1h_{9/2})^{2}_{0}$	$(3p_{1/2})^{-2}_{0}(f_{5/2})^{-3}_{5/2}(p_{3/2})^{-2}_{0}$	Pb^{199}	Closed shell	$(3p_{1/2})^{-2}(2f_{5/2})^{-5}_{5/2}(p_{3/2})^{-2}_{0}$
Po^{205}	"	$(3p_{1/2})^{-2}_{0}(f_{5/2})^{-3}_{5/2}$	Pb^{201}	"	$(3p_{1/2})^{-2}(2f_{5/2})^{-3}_{5/2}(p_{3/2})^{-2}_{0}$
Po^{207}	"	$(3p_{1/2})^{-2}_{0}(f_{5/2})^{-1}_{5/2}$	Pb^{203}	"	$(3p_{1/2})^{-2}(2f_{5/2})^{-3}_{5/2}$
Po^{209}	"	$(3p_{1/2})^{-1}_{1/2}$	Pb^{205}	"	$(3p_{1/2})^{-2}_{0}(2f_{5/2})^{-1}_{5/2}$
Po^{211}	"	$(2g_{9/2})_{9/2}$	Pb^{207}	"	$(3p_{1/2})^{-1}_{1/2}$
Po^{213}	"	$(2g_{9/2})^{3}_{9/2}$	Pb^{209}	"	$(2g_{9/2})_{9/2}$
Po^{215}	"	$(2g_{9/2})^{5}_{9/2}$	Pb^{211}	"	$(2g_{9/2})^{3}_{9/2}$
Po^{217}	"	$(2g_{9/2})^{7}_{9/2}$	Pb^{213}	"	$(2g_{9/2})^{5}_{9/2}$



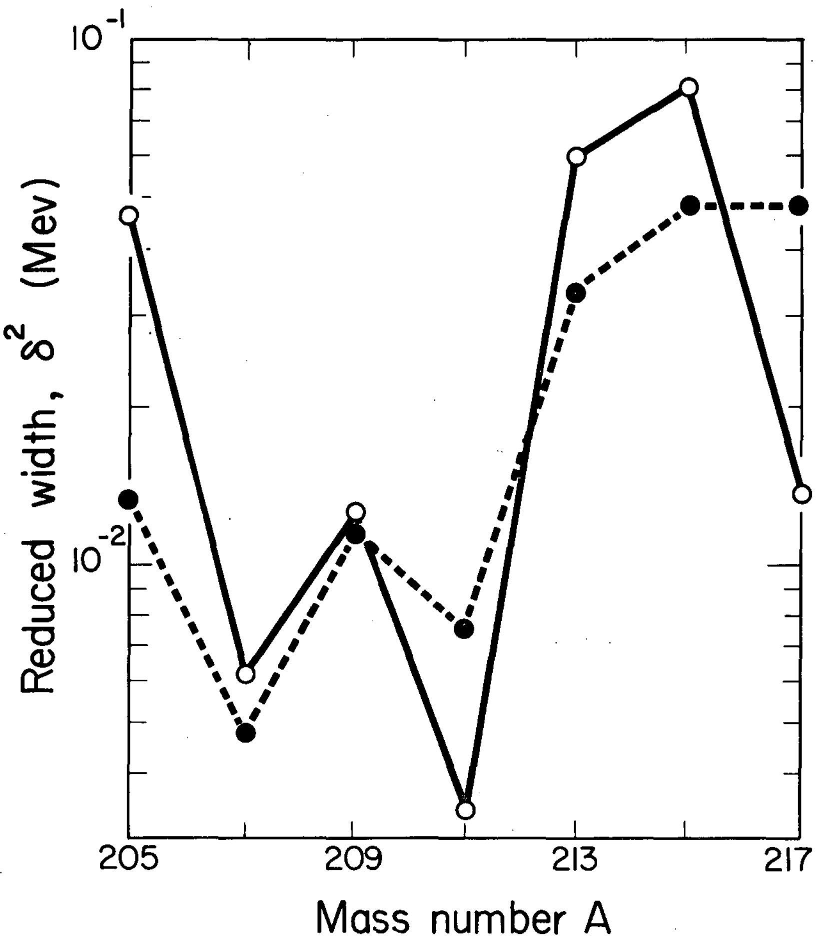
Reduced width, δ^2 (Mev)
10^{-1}
10^{-2}
202
206
210
214
218
Mass number A
MU-18832

Fig. 2.



MU-18831

Fig. 3.



MU-18833

Fig. 4.

The comparison of experimental and calculated reduced widths in Fig. 2, 3, and 4 shows clearly that taking into account the nuclear structure, even in a very crude approximation, gives results that agree well with the general features of the experimental data. Especially the behavior of the reduced width when crossing the neutron number 126 is well reproduced.

It seems to us, that one no longer needs to introduce a sudden jump of the nuclear radius at the double magic nucleus Pb^{208}. The increase of the reduced width when going from N = 126 to N = 128 is quite naturally brought forth by the change in the single particle states involved in the decay. Particles in the $2g_{9/2}$ subshell for instance are favored over particles in the $3p_{1/2}$ and $2f_{5/2}$ subshell in forming an α-particle first because of their higher angular momentum (compare Eq. 23) and second because the $2g_{9/2}$ level belongs to the next major shell and the radial part of the wave function is therefore also greater at the edge of the nucleus, which effects the quantity $R(n_1 \ell_1\, n_2 \ell_2\, n_3 \ell_3\, n_4 \ell_4\, ;\, R_0)$ (compare Eq. 23 and 25).

However, the fact that most of the reduced widths become too small as soon as one moves away from closed shells indicates that there might be a special enhancement due to configurational mixing, in particular for even-even nuclei. For, configurational mixing under the influence of a short range attractive force tends to bring the particles close together, an effect which increases the overlap with the α-particle wave function in particular as long as one deals with the configurational mixing of identical particles.

To get some information about the influence of configurational mixing we have used wave functions given by the so-called "quasi particle model" which is described in detail in reference 5.

These wave functions are of the following type:

$$\Phi_0^0 = \prod_J \prod_{m_J} \left(U_J + V_J \, a^+_{m_J} a^+_{-m_J} \right) |0\rangle$$

for even-even nuclei (26)

and

$$\Phi_J^M = a^+_{MJ} \prod_J \prod_{m_J \neq M_J} \left(U_J + V_J \, a^+_{m_J} a^+_{-m_J} \right) |0\rangle$$

for even odd nuclei with angular momentum J

The wave functions are normalized when

$$U_j^2 + V_j^2 = 1$$

The operator a^{+}_{mj} acting on the vacuum state $|0>$ creates a particle in a state Φ_j^m

The constants U_j and V_j are determined so that the energy has a minimum and the average value of the particle number $\langle \Phi | N_{op.} | \Phi \rangle$ is equal to a given value N.

We form with these wave functions

$$\langle \Phi_{N=A} | \Phi_{N=A-4} \chi_\alpha \rangle$$

and get for the reduced width in terms of the constants U_j V_j, even-even nuclei,

$$\delta^2_{000} = \Big[\sum_{J_1 J_3} \frac{2J_1+1}{2} V_{J_1} U'_{J_1} \left[V_{J_1} V'_{J_1} + U_{J_1} U'_{J_1} \right]^{\frac{2J_1-1}{2}} \tag{27}$$

$$\frac{2J_3+1}{2} V_{J_3} U'_{J_3} \left[V_{J_3} V'_{J_3} + U_{J_3} U'_{J_3} \right]^{\frac{2J_3-1}{2}} R_{\substack{n_1 \ell_1 n_1 \ell_1 \\ n_3 \ell_3 n_3 \ell_3}} (R_0) \Big]^2$$

The primes refer to the daughter nucleus. The indizes 1 and 3 are defined in Eq. 23.

Even-odd nuclei,

$$\delta^2_{JJL} = \Big[V_J U'_J \left[V_J V'_J + U_J U'_J \right]^{\frac{2J-1}{2}} \left[V_J V'_J + U_J U'_J \right]^{\frac{2J-1}{2}}$$

$$C(jJL; \tfrac{1}{2} -\tfrac{1}{2}) \sqrt{2j+1} \sum_{J_3} \frac{2J_3+1}{2} V_{J_3} U'_{J_3} \cdot \tag{28}$$

$$\left[V_{J_3} V'_{J_3} + U_{J_3} U'_{J_3} \right]^{\frac{2J_3-1}{2}} \cdot R_{\substack{n_J \ell_J n_j \ell_j \\ n_3 \ell_3 n_3 \ell_3}} (R_0) \Big]^2$$

$$\delta_{JJL}^{2} = \left[\sum_{J_3}' \frac{2J_3+1}{2} V_{J_3} U'_{J_3} \left[V_{J_3} V'_{J_3} + U_{J_3} U'_{J_3} \right]^{\frac{2J_3-1}{2}} \right.$$

$$\left(\left(\frac{2J-1}{2} \delta_{0L} + (2J+1) C(JJL; \tfrac{1}{2} -\tfrac{1}{2}) (1-\delta_{0L}) \right) V_J U'_J \cdot \right. \qquad (29)$$

$$\left[V_J V'_J + U_J U'_J \right]^{\frac{2J-3}{2}} \cdot R_{\substack{n_J l_J n_J l_J \\ n_3 l_3 n_3 l_3}} (R_0) + \delta_{0L} \cdot$$

$$\left. \left. \sum_{J_1 \neq J} \frac{2J_1+1}{2} U'_{J_1} V_{J_1} \left[V_{J_1} V'_{J_1} + U_{J_1} U'_{J_1} \right]^{\frac{2J_1-1}{2}} R_{\substack{n_1 l_1 n_1 l_1 \\ n_3 l_3 n_3 l_3}} (R_0) \right) \right]^{2}$$

The primes refer to the daughter nucleus.

The results obtained with the help of (28) for the even-even Po isotopes and the single particle states listed in Table 2 are compared to experimental data in Fig. 5. Again the reduced width of Po^{210} has been used as a standard.

It is very encouraging to see that the agreement with experimental data is now better, in particular for nuclei which are already rather far away from closed shells. We shall not discuss in detail the assumptions made in the "quasi particle model" but conclude with some remarks about assumption (3) (single-particle wave function and α-particle wave function).

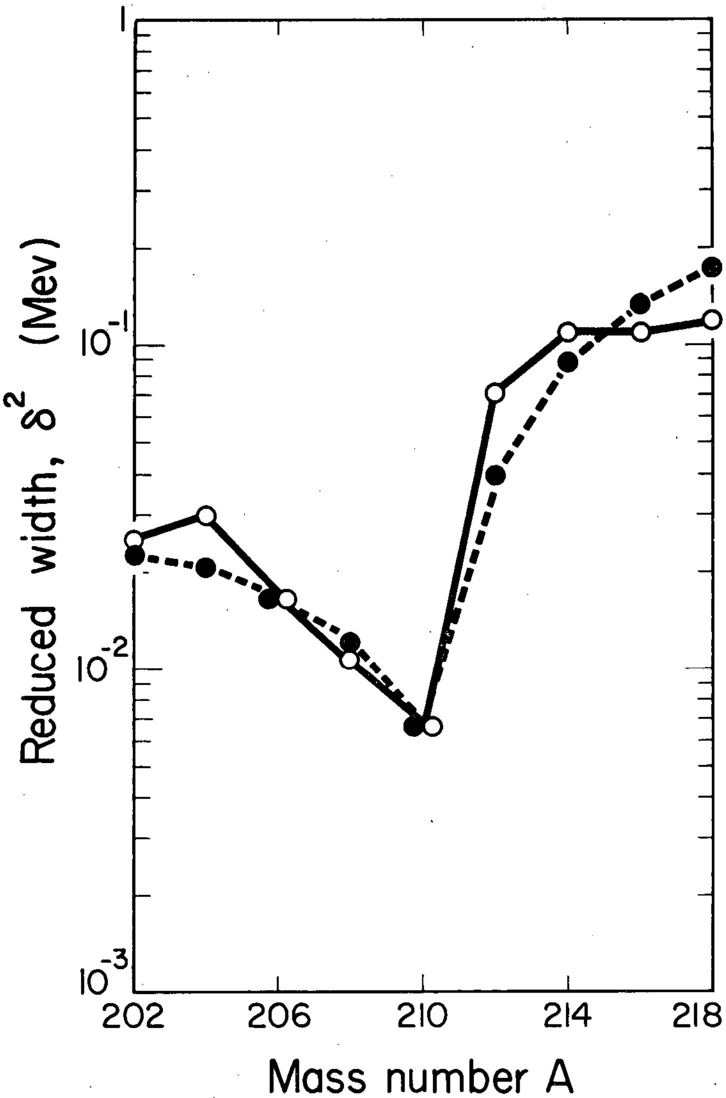

Reduced width, δ^2 (Mev)
1
10^{-1}
10^{-2}
10^{-3}
202
206
210
214
218
Mass number A
MU-18834

Fig. 5.

As far as the use of oscillator wave functions as single particle wave functions is concerned this is mainly justified by our lack of knowledge about the real single particle wave functions. The fact that the "radial part" of the reduced width varies only slightly when changing the parameters indicates that the approximation used as well as the parameters α and β are not too unreasonably chosen.

Concerning the α-particle wave function we feel quite sure, that Eq. (22) provides a good approximation to the actual wave function. There are experimental[6] as well as some theoretical[7] reasons for this feeling. High energy electron scattering experiments[6] show clearly that a Gaussian type charge density is an extremely good approximation to the real charge density.

Table 2. Single-particle levels in Pb^{207}, Pb^{209} and Bi^{209}

Pb^{207}		Pb^{209}		Bi^{209}	
Energy (Mev)		Energy (Mev)		Energy (Mev)	
0.000	$3p_{1/2}$	0.000	$2g_{9/2}$	0.000	$1h_{9/2}$
0.51	$2f_{5/2}$	0.75	$1i_{11/2}$	0.91	$2f_{7/2}$
0.89	$3p_{3/2}$	1.56	$3d_{5/2}$	(1.63	$1i_{13/2}$)
1.63	$1i_{13/2}$	(2.1	$2g_{7/2}$)	( ) means the assignment is not certain.	
2.34	$2f_{7/2}$	(1.7	$1j_{15/2}$)		

b. Fine structure of the α-decays. $Po^{210} \to Pb^{206}$; $Po^{211} \to Pb^{207}$; $Bi^{210} \to Tl^{206}$; $Bi^{211} \to Tl^{207}$; $Bi^{212} \to Tl^{208}$.

It has been shown already[1] that the fine structure of α-decays, where the parent nucleus has only few nucleons outside closed shells, provides a very sensitive test of any theory of α-decay. Also the assumptions about the nuclear wave functions are tested.

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7. H. J. Mang and W. Wild, Zs. f. Phys. 154, 182 (1959).

The decays of Po^{210}, Po^{211}**, Bi^{210}, Bi^{211}, and Bi^{212} were chosen for the following reasons: The number of particles outside closed shells is still so small, that one can hope to perform configurational mixing calculations. There are enough and accurate measurements of intensities of α-lines leading to excited states of the daughter nucleus and also some α-γ angular correlation[8] measurements.

The decay schemes[9,10,11] are given in Figs. 6, 7, 8, and 9. The nuclear wave functions including the possibility of configurational mixing and the formulas for the reduced widths are given in the appendix. (A 16 .... A 29).

In Table 3 are summarized the experimental data[8,9,10,11] and some preliminary calculated results.

**This decay has been discussed earlier, but we include it again for the sake of completeness.

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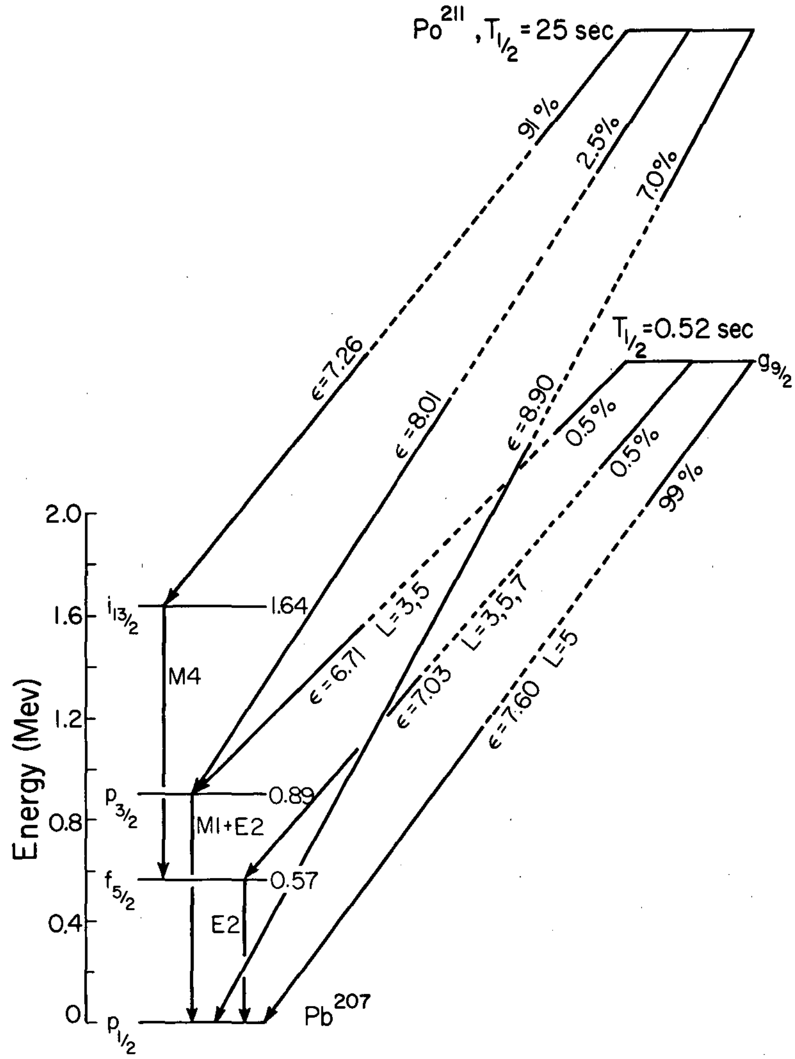
8. S. Singer, Thesis, University of Illinois, 1957.

9. W. Walen and S. Rosenblum, private communications with I. Perlman.

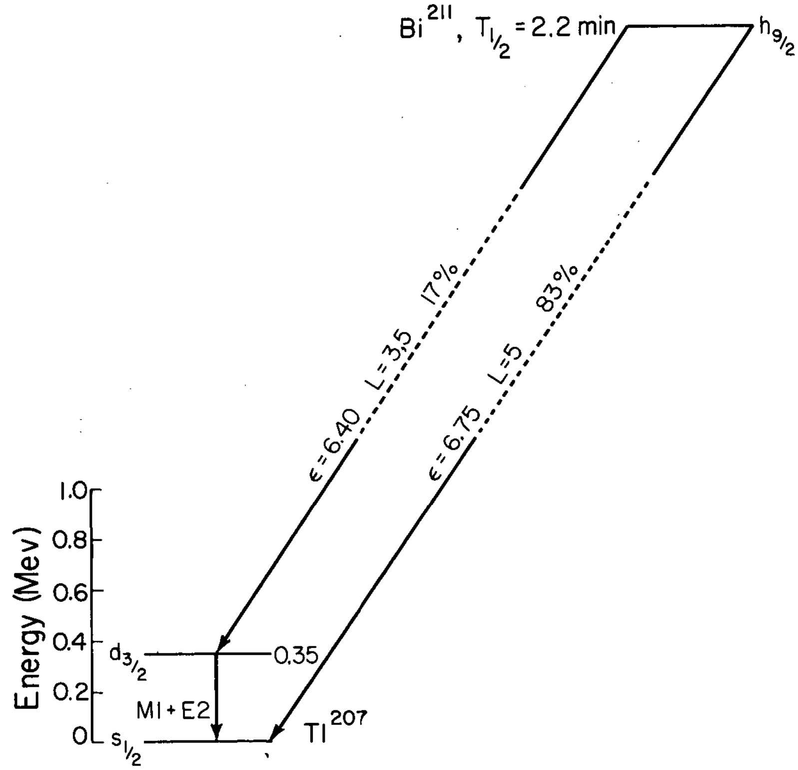
10. W. Jentschke, A. C. Juveland, and G. H. Kinsey, Phys. Rev. 96, 231 (1954).

11. I. Perlman and J. O. Rasmussen, Handbuch der Physik, Vol. XLII, and the references quoted there.



MU-18829

Fig. 6



MU-18830

Fig. 7.

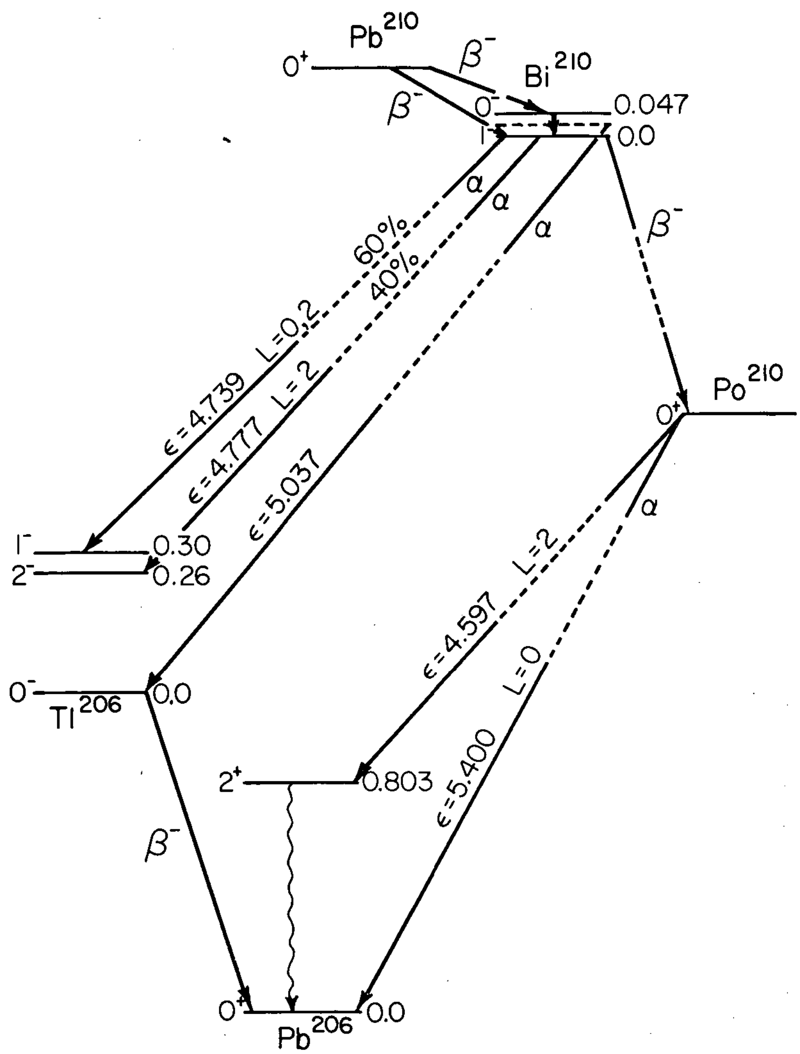
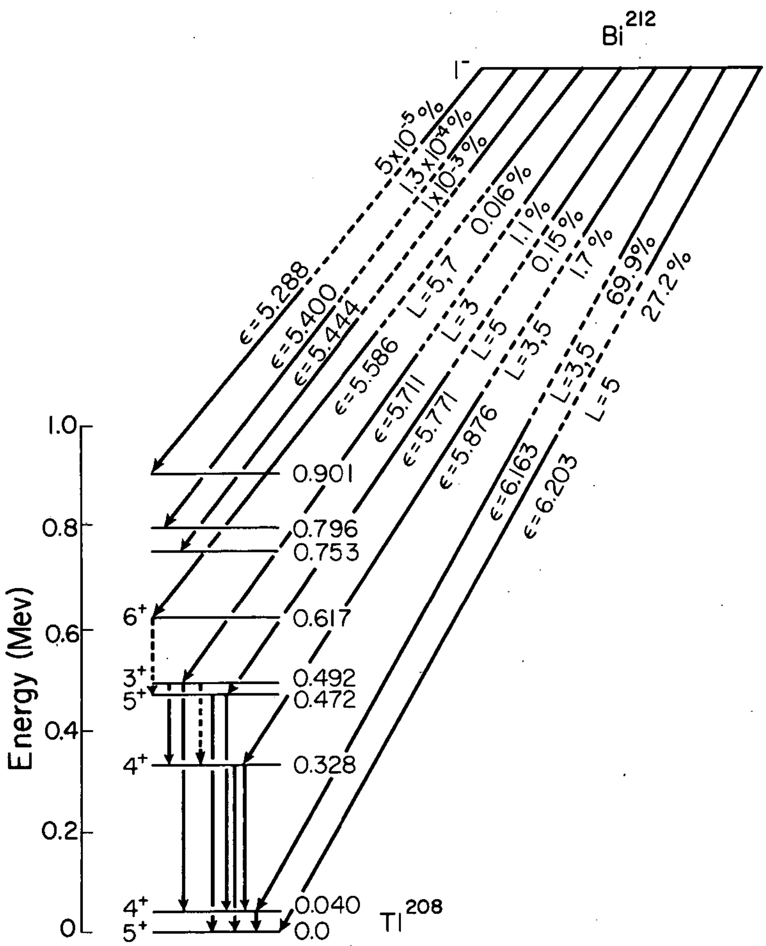

Pb^{210}
0$^+$
β^-
β^-
Bi^{210}
0$^-$
0.047
1$^-$
0.0
α
α
α
60%
40%
β^-
ϵ=4.739 L=0,2
ϵ=4.777 L=2
ϵ=5.037
Po^{210}
0$^+$
α
1$^-$ 0.30
2$^-$ 0.26
L=2
ϵ=4.597
L=0
ϵ=5.400
0$^-$ 0.0
Tl^{206}
2$^+$ 0.803
β^-
0$^+$ 0.0
Pb^{206}
MU-18828

Fig. 8.



MU-18827

Fig. 9.

Table 3

Parent nucleus	Spin and parity of the parent nucleus	Daughter nucleus	Spin and parity of the daughter nucleus	Energy of the state of the daughter nucleus (Mev)	Experimental reduced width (10^{-2} Mev)	Calculated reduced width (10^{-2} Mev)	Amplitudes of partial waves experimental values in parentheses	Remarks
Po^{210}	0+	Pb^{206}	0+	0.000	0.676	0.676		
		Pb^{206}	2+	0.803	0.931	1.9		
Po^{211}	9/2+	Pb^{207}	1/2-	0.000	0.286	0.23	L=3 L=5 L=7	The first set of amplitudes belongs to the $5/2^-$ the second to the $7/2^-$ state.
			5/2-	0.570	0.022	0.064	1.00 0.87 0.57	
			3/2-	0.890	0.350	0.30	(1.00) (0.96) (0.55)	
	((25/2+))		1/2-	0.000	0.020		L=3 L=5	The Po^{210} ground state transition is used as a standard.
			3/2-	0.890	0.038		1.00 0.35	
			13/2+	1.634	0.154		(1.00) (0.15)	
Bi^{211}	9/2-	Tl^{207}	1/2+	0.000	0.275	0.440		Po^{210} is used as a standard
			3/2+	0.350	0.258	0.310		
Bi^{210}	1-	Tl^{206}	1-	0.300	0.028	0.03		Po^{210} is used as a standard. The 8- state is assumed to be $(1h_{9/2}\ 2g_{9/2})_{8-}$
	((8-))		0-	0.000	0.0022	0.006		
Bi^{212}	1-	Tl^{206}	5+	0.000	0.166	0.166		The ground state transition is used as a standard for the transition to excited states.
			4+	0.040	0.126	0.150		
			(4+)	0.328	0.058	0.036		
			(5+)	0.472	0.125	0.020		
			(3+)	0.492	0.221	0.080		
			(6+)	0.617	0.068			

We discuss the results for each decay separately.

$\underline{Po^{210}}$

Fair agreement is obtained with the experimental data when the wave functions given by True and Ford[12] and Newby[13] are used.

$\underline{Po^{211}}$

The transition to the $f_{5/2}$ state is hindered as it should be. The results for the amplitudes of the partial waves agree well with the values derived from α-γ angular correlations.

The results for Po^{210} and Po^{211} seem to indicate that transitions involving the $2f_{5/2}$ state are still more hindered. (The first excited state in Pb^{206} is mainly $p_{1/2}^{-1}\ f_{5/2}^{-1}$.)

The experimentally reduced widths for the transitions from the isomeric state of Po^{211} have been included in the table to show that the assumption $J_{isom} = 25/2$ leads to quite normal reduced widths.

$\underline{Bi^{211}}$

The reduced width of the ground state transition comes out too large by a factor of 1.6 but the intensity of the transition to the first excited state relative to the intensity of the ground state transition agrees well with experiment. Unfortunately, the α-γ angular correlation is very sensitive to small E2 admixtures to the M1 transition ($3/2 \rightarrow 1/2$). Therefore we have no reliable information about the amplitudes of the partial waves.

$\underline{Bi^{210}}$

Quite good agreement is obtained for the transition from the 1^- state in Bi^{210} to the 1^- state in Tl^{206}. We have not calculated any reduced width for the transition to the state assigned 2^- in Tl^{206}. If this state is a 2^- state it is certainly of a highly mixed nature because it cannot arise from the lowest configuration $s_{1/2}^{-1}\ p_{1/2}^{-1}$ but has to be brought down near to the 1^- state which consists mostly of the configuration $(s_{1/2}^{-1}\ p_{1/2}^{-1})_{1^-}$. The assignment 8^- for the

12. W. W. True and K. W. Ford, Phys. Rev. 109, 1675 (1958).
13. N. Newby, Thesis, Indiana University (1958).

isomeric state of Bi^{210} is highly tentative and no conclusion can be drawn unless more information about this state is available.

Bi^{212}

The reduced widths are given relative to the reduced widths of the ground state transition. The agreement with the experimental data is as good as one can expect assuming a pure configuration $1h_{9/2}\,(2g_{9/2})^2_0\,1i_{11/2}$ 1- for the ground state of Bi^{212} and pure configurations $(3s^{-1}_{1/2}2g_{9/2})_{4+,5+}$ $(2d^{-1}_{3/2}\,2g_{9/2})_{3^+,4^+,5^+,6^+}$ for the states in Tl^{208}. If one compares the reduced width of the ground state transition with the reduced width of Po^{210} using pure configuration throughout one gets

$$\frac{\delta^2\ Bi^{212}}{\delta^2\ Po^{210}} = 0.133$$

The experimental value is 0.245.

The amplitudes of the partial waves are rather sensitive to configurational mixing — much more than the relative intensities as may be shown with the help of Eq. A 29. Therefore we do not give numerical values.

We may conclude that it seems to be possible to understand the α-decays in the region of Pb^{208} on the basis of the nuclear shell model with the refinement of configurational mixing for particles outside of closed shells. We hope to present soon more detailed calculations where configurational mixing has been taken into account for the parent and daughter nucleus of all the decays we have discussed above.

α-decay of deformed nuclei

To treat the α-decay of deformed nuclei, we have to include in our calculations the quadrupole interaction between the α-particle and the daughter nucleus.[14] That means the potential V(R) has to be replaced by

$$V(R,\theta_i) = V(R) + V_{quadrupole} \tag{30}$$

Furthermore, the surface S which was defined by $|\mathcal{R}| = R_o$ Eq. (16) may now be no longer spherical but is defined by

$$|\mathcal{R}'| = R_o\left(1 + \sqrt{\tfrac{5}{4\pi}}\, P_2^0(\cos\vartheta')\right) \tag{31}$$

14. A. Bohr, P. O Froman and B. R. Mottelson, Mat. Fys. Medd. Dan. Vid. Selsk. 29, No. 10 (1955).

These modifications lead to the following expression for the decay constant

$$\lambda = \frac{2\pi}{\hbar}\binom{N}{2}\binom{Z}{2}\sum_{J\sigma L}\left|\frac{\hbar^2}{2M}\int\left\{\overset{*}{\Phi}{}^{M}_{0J}(1\ldots A)\ \frac{\partial}{\partial n}\ \bar{\Phi}^{MK}_{JJ\sigma L\epsilon} - \frac{\partial}{\partial n}\overset{*}{\Phi}{}^{M}_{0J}(1\ldots A)\cdot\bar{\Phi}^{MK}_{JJ\sigma L\epsilon}\right\} d\xi_K\, d\xi_\alpha\, dS\right|^2 \tag{32}$$

Where $\frac{\partial}{\partial n}$ means the derivative normal to the surface S defined in Eq. (31).

The final state wave function $\Phi^{MK}_{JJ\sigma L\epsilon}$ is defined as having the following asymptotic behavior:

$$\begin{aligned}\Phi^{MK}_{JJ\sigma L\epsilon} \xrightarrow[R\to\infty]{} &\sum_m G(LJJ;\, m\ M-m)\ \varphi_L(R\epsilon)\ Y_L^m\left(\frac{\mathcal{R}}{R}\right)\\ &\sqrt{\frac{2J+1}{2}}\left\{D^J_{M-m,K}\,\chi_{K,\sigma} + (-1)^{J-q(\sigma)}\, D^J_{M-m,-K}\,\chi_{-K,\sigma}\right\}\cdot\chi_\alpha = \chi_\alpha\,\varphi_L(R\epsilon)\cdot\sqrt{\frac{2J+1}{2J+1}}\\ &\sum_\mu G(LJJ;\,\mu K)\sqrt{\frac{2J+1}{2}}\left\{D^J_{M,K+\mu}\,\chi_{K,\sigma}\, Y_L^\mu\left(\frac{\mathcal{R}'}{R}\right) + (-1)^{J-L-q(\sigma)}\, D^J_{M,-K-\mu}\,\chi_{-K\sigma}\, Y_L^{-\mu}\left(\frac{\mathcal{R}'}{R}\right)\right\}\end{aligned} \tag{33}$$

The unprimed coordinates refer to the "space-fixed system" and the primed coordinates to the "body-fixed system".

Each term in the sum on J L is the transition probability to a final state defined by the angular momentum of the daughter nucleus J and the α-particle L. ϵ is the sum of the α-particle energy and the rotational energy.

The phase[20] $(-1)^{q(\sigma)}$ is defined in the following way,

$$\chi_{K\sigma} = \sum_{J_i}' \prod_i A_{J_i}(\sigma)\, \Phi^{K_i}_{J_i}(i) \tag{34}$$

$$(-1)^{q(\sigma)} \chi_{-K\sigma} = \sum_{J_i}' \prod_i A_{J_i}(\sigma)\, \Phi^{-K_i}_{J_i}(i)\, (-1)^{J_i}$$

$$K = \sum_i K_i$$

To make the expression for the decay constant somewhat handier for computation we introduce another set of functions which fulfill the following boundary condition at the surface S:

$$\Psi^{M\ K\mu}_{J\ L\sigma\epsilon} = \sqrt{\frac{2J+1}{2}} \Big\{ D^{J}_{M\ K+\mu}\, Y^{\mu}_{L}\left(\frac{\mathbf{R}'}{R}\right) \chi_{K,\sigma}$$

$$+ (-1)^{J-L-q(\sigma)}\, D^{J}_{M\ -K-\mu}\, Y^{-\mu}_{L}\, \chi_{-K\sigma} \Big\} f^{L\mu}_{L\mu\epsilon}(R)\, \chi_{\alpha} \tag{35}$$

and behave asymptotically for R -> ∞ as follows:

$$\Psi^{MK\mu}_{JL\sigma\epsilon} \xrightarrow[R\to\infty]{} \chi_\alpha \sum_{J'L'\mu'} B^{L\mu}_{L'J'}(JK\sigma\epsilon;\, S) \sqrt{\frac{2J'+1}{2J+1}}$$

$$C(L'J'J;\ \mu'K) \sqrt{\frac{2J+1}{2}} \left\{ D^{J}_{M\ K+\mu'}\, Y^{\mu'}_{L'}\left(\tfrac{\mathfrak{R}'}{R}\right) \chi_{K\sigma} \right. \tag{36}$$

$$\left. + (-1)^{J-L-q(\sigma)} D^{J}_{M\ -K-\mu'}\, Y^{-\mu'}_{L'}\left(\tfrac{\mathfrak{R}'}{R}\right) \chi_{-K\sigma} \right\} \varphi_{L',\epsilon}(R)$$

where the functions $\varphi_{L'\epsilon}$ are Coulomb-wave functions.

There exists a unitary transformation connecting the two sets of wave functions.

$$\Psi^{MK\mu}_{JL\sigma\epsilon} = \sum_{L'J'} B^{L\mu}_{L'J'}\ \Phi^{MK}_{JJ'L'\sigma\epsilon}$$

$$\Phi^{MK}_{JJL\sigma\epsilon} = \sum_{L'\mu} B^{L'\mu}_{LJ}\ \Psi^{MK\mu}_{JL'\sigma\epsilon} \tag{37}$$

Introducing (35) and (37) into the expression for the decay constant (32) we get,

$$\lambda = \frac{2\pi}{\hbar}\binom{N}{2}\binom{Z}{2}\sum_{J L \sigma}\Bigg|\frac{\hbar^2}{2M}\int d\xi_\alpha\, d\xi_K\, dS \sum_{L'}$$

$$\Big\{\frac{\partial}{\partial n}\,\overset{*}{\chi}{}^{(1..A)}_{K_0}\cdot\chi_\alpha\Big[Y_{L'}^{K_0-K}\,\chi_{K\sigma}\, f^{L',K_0-K}_{L',K_0-K,\epsilon}$$

$$B^{L',K_0-K}_{L,J} + (-1)^{J-L'-q(\sigma)}\, Y_{L'}^{K_0+K}\,\chi_{-K\sigma}$$

$$f^{L',K_0+K}_{L',K_0+K\,\epsilon}\, B^{L',-K_0-K}_{L,J}\Big] - \overset{*}{\chi}{}^{(1...A)}_{K_0}\frac{\partial}{\partial n}\cdot\chi_\alpha \tag{38}$$

$$\Big[Y_{L'}^{K_0-K}\,\chi^{\sigma}_{K}\, f^{L',K_0-K}_{L'\,K_0-K\,\epsilon}\, B^{L'\,K_0-K}_{L\,J} +$$

$$(-1)^{J-L'-q(\sigma)}\, Y_{L'}^{K_0+K}\,\chi_{-K\sigma}\, f^{L',K_0+K}_{L',K_0+K\,\epsilon}\,\cdot$$

$$B^{L',-K_0-K}_{L,J}\Big]\Big\}\Bigg|^2$$

We have again split the problem into two parts. The "external problem" to calculate the coefficients $B^{L'\mu}_{Lj}$ and the functions $f^{L'\mu}_{L'\mu\epsilon}$, and the "internal problem" to perform the integration on ξ_K and ξ_α.

The functions $f^{Lk}_{Lk'\epsilon}$ fulfill the boundary conditions

$$f^{L\,k}_{L'k'\epsilon} \xrightarrow[R\to\infty]{} \sum_{J} B^{L'k'}_{LJ}\sqrt{\frac{2j+1}{2J+1}} \tag{39}$$

$$\cdot\, C(LjJ;\, k\, K)\cdot \varphi_{L\epsilon}(R)$$

$$f^{L\,k}_{L'k'\epsilon} = \delta_{LL'}\,\delta_{kk'}\,f^{L'k'}_{L'k'\epsilon} \tag{40}$$

and an analogous condition for the derivatives at the surface S.

$$\sum_{LJ}{}' \left| B^{L'k'}_{LJ} \right|^2 = 1 \tag{41}$$

and the following differential equations:

$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dR^2} + V(R) + \frac{\hbar^2}{2M}\frac{L(L+1)}{R^2} - \epsilon \right] \cdot R\, f^{L\,k}_{L'k'\epsilon} \tag{42}$$

$$= -\sum_{L''}{}' V^{k}_{L''L'}(R) \cdot R\, f^{L''k}_{L'k'\epsilon} - \sum_{k}{}' A^{J}_{KL'k''k} R\, f^{L\,k''}_{L'k'\epsilon}$$

where ,

$$A^{J}_{KL'k'k} = \sum_{J}{}' T_{rot}(JK)\,(-1)^{k-k'}\, C(JL'J;\, K+k'-k') \cdot C(JL'J;\, K+k-k)$$

and $V^{\mu}_{LL''}$ (R) are the matrix elements of the quadrupole interaction.

We should mention that our functions $f_{L'k'\epsilon}^{l'\mu'}$ are equivalent to the functions $g_{Lk\epsilon}$ defined by Frőman.[18]

To solve the differential equations (42) one can either integrate the system of coupled differential equations numerically[15,16,17] or use a three-dimensional W K B method.[18,19]

To make things easier from now on, we shall return to a spherical surface S. To justify this we remember, that we have derived the expression for the decay constant using the equations (14, (15), and (16). The conditions for the validity of these equations were that for $R < R_o$ the Hamiltonian H_o Eq. (14) was equal to the exact Hamiltonian H and that for $R > R_o$ the Hamiltonian H acting on the final-state wave function reproduced this wave function Eq. (16). But these two conditions can be fulfilled for a non-spherical nucleus too, if we are only careful in using approximations for the wave function of the parent nucleus. One has to be sure that this wave function is a good approximation up to $R = R_o$ when $\vartheta' = 90^o$ which is the direction of the minor axis of the nuclear ellipsoid.

15. J. O. Rasmussen and B. Segall, Phys. Rev. 103, 1928 (1956).
16. E. M. Pennington and M. A. Preston, Can. J. Phys. 36, 944 (1958).
17. R. R. Chasman and J. O. Rasmussen, UCRL-8632, Phys. Rev. (in press).
18. P. O. Frőman, Mat. Fys. Skr. Dan. Vid. Selsk., Vol. 1, No. 3 (1957).
19. V. G. Nosow, Dokl. Akad. Nauk. SSSR 112, 414 (1957).

Deformed even-even α-emitters

To describe α-decays of even-even nuclei we shall approximate the intrinsic wave functions $\chi_{K_O}\, \chi_K$ by determinants of Nilsson-single-particle wave functions.[20] Furthermore, we know that $K_o = K = 0$ for the lowest intrinsic state and $\mathcal{J} = 0$ for the ground state of the parent nucleus.

It follows

$$\lambda = \frac{2\pi}{\hbar} \sum_L \left| \frac{\hbar^2}{2M} \sum_{L'} B_{LL'}^{L'0} \cdot R_0^{1/2} \right. \tag{43}$$

$$\left. \left\{ \frac{\partial G_{L'}}{\partial R} f_{L'0\epsilon}^{L'0} - G_{L'} \cdot \frac{\partial f_{L'0\epsilon}^{L'0}}{\partial R} \right\}_{R=R_0} \right|^2$$

where

$$G_{L'}(R) = R_0^{3/2} \cdot \left[\binom{N}{2}\binom{Z}{2}\right]^{1/2} \int d\xi_\alpha\, d\xi_K\, d\Omega_{R'}\, \chi_0^* \chi_\alpha \chi_{0\sigma} Y_{L'}^0$$

and the functions $f_{L'0\epsilon}^{L0}$ are solutions of the somewhat simpler system of differential equations than (42).

$$\left[-\frac{\hbar^2}{2M} \frac{d^2}{dR^2} + V(R) + \frac{\hbar^2}{2M} \frac{L(L+1)}{R^2} - \epsilon + \frac{\hbar^2}{2\Theta} L(L+1) \right] R f_{L'0\epsilon}^{L0}$$

$$= - \sum_{L''} V_{LL''}^{0}(R) \cdot R f_{L'0\epsilon}^{L''0} \tag{44}$$

20. S. G. Nilsson, Dan. Mat. Fys. Medd. 29, No. 16 (1955).

If now the Nilsson states with 3-component of the angular momentum Ω_p and Ω_n are those occupied in the parent nucleus and unoccupied in the daughter nucleus, we get for $G_{L'}$

$$G_{L'}(R) = \int A\left(\overset{*}{\Phi}{}^{\Omega_p}_{(1)} \overset{*}{\Phi}{}^{-\Omega_p}_{(2)}\right) A\left(\overset{*}{\Phi}{}^{\Omega_n}_{(3)} \overset{*}{\Phi}{}^{-\Omega_n}_{(4)}\right) \cdot \chi_\alpha Y^0_{L'}\left(\frac{\mathbf{R}'}{R}\right) d\xi_\alpha \, d\Omega_{\mathbf{R}'} \tag{45}$$

which may be evaluated with the help of Eqs. (A8) and (A32). If it is necessary to include configurational mixing, we use a linear combination of terms like that given by Eq. (45).

Favored decays of deformed even-odd nuclei

A favored decay is characterized by $K_0 = K$; $\mathcal{J} = K_0$; $j = K, K+1 \ldots$ We shall further assume[14,18] that the odd particle is in the same Nilsson orbit in the parent and in the daughter nucleus. It follows,

$$\lambda = \frac{2\pi}{\hbar} \sum_{J L}' \left| \frac{\hbar^2}{2M} \sum_{L'} \mathcal{B}^{L'0}_{LJ} R_0^{1/2} \cdot \left\{ \frac{\partial G_{L'}}{\partial R} f^{L'0}_{L'0\epsilon} - G_{L'} \frac{\partial f^{L'0}_{L'0\epsilon}}{\partial R} \right\}_{R=R_0} \right|^2 \tag{46}$$

where $G_{L'}$ is given by Eq. (45). But the differential equation for the function $f^{L0}_{L'0\epsilon}$ are now different.

$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dR^2} + V(R) + \frac{\hbar^2}{2M}\frac{L(L+1)}{R^2} - \epsilon\right] R f^{L0}_{L'0\epsilon} = -\sum_{L''}' V^0_{LL''}(R) \cdot R f^{L''0}_{L'0\epsilon} - \sum_{\mu}' A_{KL0\mu} f^{L\mu}_{L'0\epsilon} \tag{47}$$

If one neglects the non-diagonal terms connecting $\mu=0$ and μ on the left-hand side of Eq. (47) one gets

$$B_{LJ}^{L'0} = B_{LL}^{L'0} \, C(JLJ; K0) \tag{48}$$

One may now as an approximation use the $B_{LL}^{L'0}$ from the neighboring even-even nucleus and get definite predictions for the intensity ratios of transitions to different members of a rotational band.[14,18] The validity of this approximation has been discussed by Chasman and Rasmussen.[17] We refer to their paper for details. In general, the agreement with the experimental data is quite good.[14,18]

Unfavored decays of deformed even-odd nuclei

Unfavored decays are characterized by $K_0 \neq K$ and we shall assume the K values to be the z-projections of angular momentum of the odd particle in the parent and daughter nucleus and the remainder of particles as paired in "Nilsson orbits".

It follows,

$$\lambda = \frac{2\pi}{\hbar} \sum_{J\,L} \Bigg| \frac{\hbar^2}{2M} \sum_{L'} \Big\{ B_{LJ}^{L'\,K_0-K} \cdot \left[\frac{\partial G_{K_0-K,\,L'}}{\partial R} f_{L'\,K_0-K\,\epsilon}^{L'\,K_0-K} - G_{K_0-K,\,L'} \cdot \frac{\partial f_{L'\,K_0-K\,\epsilon}^{L'\,K_0-K}}{\partial R} \right] + (-1)^{J-L-j} B_{LJ}^{L',-K_0-K} \cdot \left[\frac{\partial G_{K_0+K,\,L'}}{\partial R} f_{L',K_0+K\,\epsilon}^{L',K_0+K} - G_{K_0+K,\,L'} \cdot \frac{\partial f_{L'\,K_0+K\,\epsilon}^{L'\,K_0+K}}{\partial R} \right] \Big\}_{R=R_0} \Bigg|^2 \tag{49}$$

where

$$G_{K_0-K,L'}(R) = \int A\left(\overset{*}{\Phi}{}^{K_0}_{(1)}\ \overset{*}{\Phi}{}^{-K}_{(2)\sigma}\right) A\left(\overset{*}{\Phi}{}^{\Omega}_{(3)}\ \overset{*}{\Phi}{}^{-\Omega}_{(4)}\right) \chi_\alpha Y_{L'}^{K_0-K}\, d\xi_\alpha\, d\Omega_{R'} \tag{50}$$

$$G_{K_0+K,L'}(R) = (-1)^{J-q(\sigma)+1} \int A\left(\overset{*}{\Phi}{}^{K_0}_{(1)}\ \overset{*}{\Phi}{}^{K}_{(2)\sigma}\right) A\left(\overset{*}{\Phi}{}^{\Omega}_{(3)}\ \overset{*}{\Phi}{}^{-\Omega}_{(4)}\right) \chi_\alpha Y_{L'}^{K_0+K}\, d\xi_\alpha\, d\Omega_{R'}$$

The function $f^{Lk}_{L'k'\epsilon}$ are now solutions of Eq. (42).

If we neglect again the non-diagonal terms (k,k') in the differential Eq. (42) we get

$$B^{L',K_0-K}_{LJ} = (-1)^{L-K_0+K}\, C(JLJ;\ K_0\ K-K_0)\, B^{L',K_0-K}_{L,L} \tag{51}$$

$$B^{L',-K_0-K}_{LJ} = (-1)^{L+K_0+K}\, C(JLJ;\ -K_0\ K_0+K)\, B^{L',-K_0-K}_{L,L}$$

This may lead to "alternating intensities to successive rotational states" when $L > K + K_0$ because of the sign change of the Clebsch-Gordan Coefficients when J is set equal to K, K+1 ... for the members of a rotational band as was first pointed out by F. Asaro[21] in the case of the Cm^{243} α-decay to the ground band.

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21. F. Asaro, private communication.

Acknowledgements

I wish to thank Professor I. Perlman and J. O. Rasmussen for the hospitality extended to me at the Lawrence Radiation Laboratory and for many helpful discussions and for making available to me a computer program to reduce experimental data. Invaluable was the help of N. K. Glendenning who set up a computer program to solve the equations of the quasi particle model and of R. Diamond who worked out a computer program to calculate Laguerre Polynomials for me.

I also wish to thank the Deutschen Forschungsgemeschaft for a grant.

Appendix

The reduced width is essentially determined by the integral

$$G_{JJL} = R_0^{\frac{3}{2}} \left[\binom{N}{2}\binom{Z}{2}\right]^{\frac{1}{2}} \cdot \int d\xi_\alpha \, d\xi_K \, d\Omega_R \left\{ \overset{*}{\chi}_\alpha(1234) \right.$$

$$\left. \sum_m C(L\, J\, \mathcal{J};\, m\ M-m)\ \overset{*}{Y}{}_L^m\left(\tfrac{R}{R}\right) \overset{*}{\Psi}{}_J^{M-m}(5,\ldots A)\ \Phi_{\mathcal{J}}^M(1\ldots A) \right\} \qquad \text{(A1)}$$

The calculation of this integral will be performed in two steps, each of which involves different assumptions about the nuclear wave functions.

First, we expand the integral

$$\int d\xi_K \left(\overset{*}{\Psi}{}_J^{M-m}\ \Phi_{\mathcal{J}}^M \right) \left[\binom{N}{2}\binom{Z}{2}\right]^{\frac{1}{2}}$$

in terms of single-particle wave functions.

$$\left[\binom{N}{2}\binom{Z}{2}\right]^{\frac{1}{2}} \int d\xi_K \left(\overset{*}{\Psi}{}_J^{M-m}(5,\ldots A)\ \Phi_{\mathcal{J}}^M(1,\ldots A) \right) =$$

$$\sum_{\substack{n_1 n_2 n_3 n_4\, l_1 l_2 l_3 l_4\, J_1 J_2 J_3 J_4 \\ L\, J_P\, J_N\, m_1\, m_3\, m_P}} b\begin{pmatrix} n_1\, n_2\, n_3\, n_4\, l_1\, l_2\, l_3\, l_4 \\ J_1\, J_2\, J_3\, J_4\, J_P\, J_N\, L \end{pmatrix} \cdot \qquad \text{(A2)}$$

$$C(L\, J\, \mathcal{J};\ m\ \ M-m)\ C(J_1 J_2 J_P;\, m_1\ \ m_P-m_1) \cdot$$

$$C(J_3 J_4 J_N;\, m_3\ \ m-m_P-m_3)\ C(J_P J_N L;\, m_P\ \ m-m_P)$$

$$A\left(\Phi_{J_1 l_1 n_1}^{m_1}(1)\ \ \Phi_{J_2 l_2 n_2}^{m_P-m_1}(2) \right) A\left(\Phi_{J_3 l_3 n_3}^{m_3}(3)\ \ \Phi_{J_4 n_4 l_4}^{m-m_P-m_3}(4) \right)$$

where A means an antisymmetrization and normalization operator. For the single-particle wave functions we take the harmonic oscillator functions defined in Eq. (21). As long as we sum over a complete set of quantum numbers $(n_i \ell_i j_i)$ the expansion (A2) means no approximation.

In a first step we calculate,

$$g^{n_1 \ell_1 j_1 \; n_2 \ell_2 j_2}_{n_3 \ell_3 j_3 \; n_4 \ell_4 j_4 \; j_P j_N L} = \sum_{m_1 m_3 m_P} C(j_P j_N L; m_P \; m-m_P)$$

$$C(j_1 j_2 j_P; m_1 \; m_P - m_1) \quad C(j_3 j_4 j_N; m_3 \; m-m_P-m_3) \qquad \text{(A3)}$$

$$\int d\vec{\xi}_1^{\,3} \, d\vec{\xi}_2^{\,3} \, d\vec{\xi}_3^{\,3} \, d\Omega_{\mathcal{R}} \left(\overset{*}{\chi}_\alpha(1234) \; \overset{*}{Y}{}_L^m\left(\frac{\mathcal{R}}{R}\right) \cdot \right.$$

$$\left. A\left(\Phi^{m_1}_{n_1 \ell_1 j_1}(1) \; \Phi^{m_P - m_1}_{n_2 \ell_2 j_2}(2)\right) A\left(\Phi^{m_3}_{n_3 \ell_3 j_3}(3) \; \Phi^{m-m_P-m_3}_{n_4 \ell_4 j_4}(4)\right)\right)$$

For χ_α we use the α-particle wave function given by Eq. (22).

We transform now the single particle wave functions from jj to ℓs coupling and take the inner product with the two singlet spin functions $\chi^0_0(12)$ and $\chi^0_0(34)$ contained in χ_α. Using the standard Racah technique[22] we get

$$g^{n_1 \ell_1 j_1 \; n_2 \ell_2 j_2}_{n_3 \ell_3 j_3 \; n_4 \ell_4 j_4 \; j_P j_N L} = \left[(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1) \right]^{\frac{1}{2}}$$

$$\frac{(-1)^{j_1+j_3+\ell_2+\ell_4+j_P+j_N+1}}{2} \; W(\ell_1 j_1 \ell_2 j_2; \tfrac{1}{2} j_P) \; W(\ell_3 j_3 \ell_4 j_4; \tfrac{1}{2} j_N) \qquad \text{(A4)}$$

22. M. E. Rose, Elementary Theory of Angular Momentum, John Wiley and Sons, New York, 1957.

$$\int d\xi_1^3\, d\xi_2^3\, d\xi_3^3\, d\Omega_R \left(\frac{2\beta^{3/2}}{4\pi\,\frac{1}{2}!}\right)^{3/2} \cdot e^{-\frac{\beta}{2}(\xi_1^2+\xi_2^2+\xi_3^2)} \cdot R_0^{\,3/2}$$

$$\sum_{m_1 m_3 m_p} C(J_P J_N L;\, m_P\; m-m_P)\; C(\ell_1 \ell_2 J_P;\, m_1\; m_P-m_1)\; C(\ell_3 \ell_4 J_N;\, m_3\; m-m_P-m_3)$$

$$S\left(\Phi_{n_1\ell_1}^{m_1}(1)\; \Phi_{n_2\ell_2}^{m_p-m_1}(2)\right) S\left(\Phi_{n_3\ell_3}^{m_3}(3)\; \Phi_{n_4\ell_4}^{m-m_p-m_3}(4)\right)$$

where S means a symmetrization and normalization operator and $\Phi_{n\ell}^{m}$ is defined as

$$\Phi_{n\ell}^{m} = \left[\frac{2\,n!\,\alpha^{3/2}}{(n+\ell+\frac{1}{2})!}\right]^{1/2} e^{-\frac{\alpha}{2}r^2} (\sqrt{\alpha}\,r)^{\ell} \cdot L_n^{\ell+\frac{1}{2}}(\alpha r^2)\; Y_\ell^{m}\left(\frac{\mathbf{r}}{r}\right)$$

Next we introduce the coordinates ξ_1 ξ_2 ξ_3 and R as defined in Eq. (22) into the product of the four single particle wave functions. Using the invariance of an oscillator potential under the transformation*
$\kappa_1\ \kappa_2\ \kappa_3\ \kappa_4 \longrightarrow \xi_1\ \xi_2\ \xi_3$ R we get,

$$\sum_{m_1 m_3 m_p} C(J_P J_N L;\, m_P\; m-m_P)\; C(\ell_1 \ell_2 J_P;\, m_1\; m_P-m_1)\; C(\ell_3 \ell_4 J_N\; m_3\; m-m_P-m_3)$$

$$S\left(\Phi_{n_1\ell_1}^{m_1}(\kappa_1)\, \Phi_{n_2\ell_2}^{m_p-m_1}(\kappa_2)\right)\; S\left(\Phi_{n_3\ell_3}^{m_3}(\kappa_3)\, \Phi_{n_4\ell_4}^{m-m_p-m_3}(\kappa_4)\right) =$$

$$= \sum_{\nu_1\nu_2\nu_3 n\,\lambda_1\lambda_2\lambda_3\lambda_4\lambda\Lambda} B^{\,n_1 n_2 n_3 n_4\, \ell_1 \ell_2 \ell_3 \ell_4\, J_P J_N L}_{\,\nu_1\nu_2\nu_3 n\, \lambda_1\lambda_2\lambda_3\lambda_4\, \lambda\, \Lambda} \qquad \text{(A5)}$$

$$\sum_{\mu_1 \mu_3 \mu} C(\lambda \Lambda L;\, \mu\; m-\mu)\; C(\lambda_1 \lambda_2\, \lambda;\, \mu_1\; \mu-\mu_1)\; C(\lambda_3 \lambda_4 \Lambda;\, \mu_3\; m-\mu-\mu_3)$$

$$\Phi_{\nu_1\lambda_1}^{\mu_1}(\xi_1)\; \Phi_{\nu_2\lambda_2}^{\mu-\mu_1}(\xi_2)\; \Phi_{\nu_3\lambda_3}^{\mu_3}(\xi_3)\; \Phi_{n\,\lambda_4}^{m-\mu-\mu_3}(2R)$$

*

$$\kappa_1^2+\kappa_2^2+\kappa_3^2+\kappa_4^2 = \xi_1^2+\xi_2^2+\xi_3^2+4R^2$$

where $2(v_1 + v_2 + v_3 + n) + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 2(n_1 + n_2 + n_3 + n_4) + \ell_1 + \ell_2 + \ell_3 + \ell_4$, and $\lambda_1\ \lambda_2$ take only even values.

Integrating now on $\xi_1\ \xi_2\ \xi_3$ and Ω_R we get,

$$g_{\substack{n_1 \ell_1 j_1\ n_2 \ell_2 j_2 \\ n_3 \ell_3 j_3\ n_4 \ell_4 j_4}\ J_P J_N L} = \left[(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)\right]^{1/2} \cdot$$

$$\frac{(-1)^{j_1+j_3+\ell_2+\ell_4+J_P+J_N+1}}{2} W(\ell_1 j_1\ \ell_2 j_2 ; \tfrac{1}{2} J_P)\, W(\ell_3 j_3\ \ell_4 j_4 ; \tfrac{1}{2} J_N)$$

$$\left(\tfrac{1}{2}!\right)^{-3/2} \left(\frac{2\sqrt{\alpha\beta}}{\alpha+\beta}\right)^{9/2} \left(\sqrt{\alpha}\, R_0\right)^{3/2} e^{-2\alpha R^2} \left(\sqrt{2\alpha}\, R\right)^{L} \tag{A6}$$

$$\sum_{n=0}^{N} T(n\, L\, J_P J_N\ n_1 n_2 n_3 n_4\ \ell_1 \ell_2 \ell_3 \ell_4) \left(\frac{\beta-\alpha}{\beta+\alpha}\right)^{N-n} L_n^{L+\frac{1}{2}}(2\alpha R^2)$$

where ,

$$T(n\, L\, J_P J_N\ n_1 n_2 n_3 n_4\ \ell_1 \ell_2 \ell_3 \ell_4) =$$

$$= \sum_{v_1 v_2 v_3} B^{n_1 n_2 n_3 n_4\ \ell_1 \ell_2 \ell_3 \ell_4\ J_P J_N L}_{v_1 v_2 v_3\ n\ \ 0 0 0 L\ \ 0 L} \left[\frac{(v_1+\frac{1}{2})!\ (v_2+\frac{1}{2})!\ (v_3+\frac{1}{2})!}{v_1!\ \ v_2!\ \ v_3!}\right]$$

and

$$2(v_1+v_2+v_3) + 2n + L = 2(n_1+n_2+n_3+n_4) + \ell_1+\ell_2+\ell_3+\ell_4$$

$$2N + L = 2(n_1+n_2+n_3+n_4) + \ell_1+\ell_2+\ell_3+\ell_4$$

To calculate the coefficients $T(n L J_P J_N \; n_1 n_2 n_3 n_4 \; l_1 l_2 l_3 l_4)$ we go back to Eq. (A5) and set

$$\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \vec{x}$$

which is equivalent to

$$\xi_1 = \xi_2 = \xi_3 = \sigma \qquad \mathcal{R} = \vec{x}$$

Then we expand both sides of Eq. (A5) in powers of x and get a system of linear equations for the T coefficients which may be solved easily. The result is

$$T(n L J_P J_N \; n_1 n_2 n_3 n_4 \; l_1 l_2 l_3 l_4) =$$

$$\sum_{\varrho=n}^{N} (-1)^n \frac{\varrho!}{2^{2\varrho+L}} \binom{\varrho+L+\frac{1}{2}}{\varrho-n} A_\varrho (L J_P J_N \; n_1 n_2 n_3 n_4 \; l_1 l_2 l_3 l_4) \tag{A7}$$

where,

$$A_\varrho (L J_P J_N \; n_1 n_2 n_3 n_4 \; l_1 l_2 l_3 l_4) = \left[\frac{(2l_1+1)(2l_2+1)(2l_3+1)(2l_4+1)}{2L+1}\right]^{\frac{1}{2}} \cdot$$

$$C(l_1 l_2 J_P; 00)\; C(l_3 l_4 J_N; 00)\; C(J_P J_N L; 00) \cdot \left(\frac{\frac{1}{2}!}{\sqrt{2}}\right)^3 \cdot 4$$

$$\left[\frac{n_1! \quad n_2! \quad n_3! \quad n_4!}{(n_1+l_1+\frac{1}{2})!\,(n_2+l_2+\frac{1}{2})!\,(n_3+l_3+\frac{1}{2})!\,(n_4+l_4+\frac{1}{2})!}\right]^{\frac{1}{2}} \sum_{\nu_1 \nu_2 \nu_3 \nu_4} \binom{n_1+l_1+\frac{1}{2}}{n_1-\nu_1}$$

$$\binom{n_2+\ell_2+\frac{1}{2}}{n_2-\nu_2}\binom{n_3+\ell_3+\frac{1}{2}}{n_3-\nu_3}\binom{n_4+\ell_4+\frac{1}{2}}{n_4-\nu_4}\frac{(-1)^{\nu_1+\nu_2+\nu_3+\nu_4}}{\nu_1!\,\nu_2!\,\nu_3!\,\nu_4!}\Big[(2-\delta_{J_1J_2}\cdot\delta_{\ell_1\ell_2}\cdot\delta_{n_1n_2})\cdot$$

$$(2-\delta_{J_3J_4}\,\delta_{\ell_3\ell_4}\cdot\delta_{n_3n_4})\Big]^{1/2}$$

and

$$2\rho = 2(\nu_1+\nu_2+\nu_3+\nu_4)+\ell_1+\ell_2+\ell_3+\ell_4-L$$

We introduce the above expression into Eq. (A6) and find that we are able to sum over n. The final result is

$$g_{\substack{n_1\ell_1J_1,\,n_2\ell_2J_2\\ n_3\ell_3J_3\,n_4\ell_4J_4}}\,J_PJ_NL = (-1)^{J_1+J_3+\ell_2+\ell_4+L+1}\,C(J_PJ_NL;00)\,\cdot$$

$$\left[\frac{(2J_1+1)(2J_2+1)(2J_3+1)(2J_4+1)}{2L+1}\cdot\frac{(2-\delta_{J_1J_2}\delta_{\ell_1\ell_2}\,\delta_{n_1n_2})\,(2-\delta_{J_3J_4}\delta_{\ell_3\ell_4}\,\delta_{n_3n_4})}{4}\right]^{1/2}\cdot$$

$$\frac{1+(-1)^{\ell_1+\ell_2-J_P}}{2}\cdot\frac{1+(-1)^{\ell_3+\ell_4-J_N}}{2}\quad C(J_1J_2J_P;\tfrac{1}{2}\,-\tfrac{1}{2})\,C(J_3J_4J_N;\tfrac{1}{2}\,-\tfrac{1}{2})\,\cdot \qquad \text{(A8)}$$

$$\left[\frac{2\cdot(\frac{1}{2}!)^3}{n_1!\,(n_1+\ell_1+\frac{1}{2})!\;n_2!\,(n_2+\ell_2+\frac{1}{2})!\;n_3!\,(n_3+\ell_3+\frac{1}{2})!\;n_4!\,(n_4+\ell_4+\frac{1}{2})!}\right]^{1/2}$$

$$\left(\frac{2\sqrt{\alpha\beta}}{\alpha+\beta}\right)^{9/2}\left(\frac{2\alpha}{\alpha+\beta}\right)^{N+\frac{L}{2}}(\sqrt{\alpha}\,R_0)^{3/2}\quad e^{-2\alpha R^2}\,(\sqrt{2(\alpha+\beta)}\,R)^L$$

$$\sum_{\rho} B_\rho^L(n_1n_2n_3n_4\,\ell_1\ell_2\ell_3\ell_4)\,\frac{(-1)^\rho\,\rho!}{2^{2\rho+L}}\left(\frac{\beta-\alpha}{2\alpha}\right)^{N-\rho}L_\rho^{L+\frac{1}{2}}(2(\alpha+\beta)R^2)$$

$$B_{\varsigma}^{L}(n_1 n_2 n_3 n_4\, l_1 l_2 l_3 l_4) = n_1!\, n_2!\, n_3!\, n_4!\,.$$

$$\sum_{\nu_1 \nu_2 \nu_3 \nu_4} \binom{n_1+l_1+\frac{1}{2}}{n_1-\nu_1}\binom{n_2+l_2+\frac{1}{2}}{n_2-\nu_2}\binom{n_3+l_3+\frac{1}{2}}{n_3-\nu_3}\binom{n_4+l_4+\frac{1}{2}}{n_4-\nu_4}\frac{(-1)^{\nu_1+\nu_2+\nu_3+\nu_4}}{\nu_1!\,\nu_2!\,\nu_3!\,\nu_4!}$$

where the summation is restricted by,

$$2\varsigma = 2(\nu_1+\nu_2+\nu_3+\nu_4) + l_1 + l_2 + l_3 + l_4 - L$$

We should mention at this point that until now only the assumptions about the α-particle wave function have been used. All the information about the nuclear wave functions is contained in the still undetermined coefficients $b\begin{pmatrix} n_1 n_2 n_3 n_4 & l_1 l_2 l_3 l_4 \\ j_1 j_2 j_3 j_4 & J_P J_N L \end{pmatrix}$.

The next step in this development will, of course, be to assume that the nuclear wave functions may be reasonably approximated by products of single particle wave functions. Nevertheless, these single particle wave functions need not be harmonic oscillator wave functions. Because of the relatively small size of the α-particle it is sufficient to approximate the radial part R(r) of the actual single particle wave functions in the neighborhood of the point $r_1 = r_2 = r_3 = r_4 = R_0$ in the following manner.

$$e^{\frac{\alpha}{2} r_i^2} R_{n_i}(r_i) = \sum_{\nu_i=0}^{1} a_{\nu_i} (\alpha r_i^2)^{\nu_i} \tag{A9}$$

$$R_0 - R_\alpha \le r_i \le R_0 + R_\alpha$$

where R_α is a sort of α-particle radius. The constant α is arbitrary but has to be the same for all four radial wave functions involved in the calculations. The coefficients B_ρ^L of Eq. (A8) are then defined by the following equation.

$$B_\rho^L = \sum_{\nu_1 \nu_2 \nu_3 \nu_4} a_{\nu_1} \, a_{\nu_2} \, a_{\nu_3} \, a_{\nu_4} \tag{A10}$$

$$2\rho = 2(\nu_1 + \nu_2 + \nu_3 + \nu_4) + \ell_1 + \ell_2 + \ell_3 + \ell_4 - L$$

But actually we do not know enough about the radial dependence of single-particle wave functions. Therefore we shall from now on assume that harmonic oscillator wave functions are good approximations to the radial parts of the single-particle wave functions. Under this assumption we shall calculate the integral $\left[\binom{N}{2}\binom{Z}{2}\right]^{1/2} \int d\xi_K \left(\Phi_J^{M} \, \Psi_J^{M-m}\right)$

for several cases of interest.

The first case is that treated in §2 of this paper - even numbers of particles are "paired" to give total angular momentum zero. That means states with seniority 0. The wave functions of the parent and daughter nucleus are as follows (only that part of the wave function is given that refers to unfilled subshells):

Even-even nuclei

$$\Phi_0^0(\text{parent}) = \left[\binom{\frac{2j_1+1}{2}}{\frac{N_1}{2}} \binom{\frac{2j_3+1}{2}}{\frac{N_3}{2}} N_1! \; N_2! \right]^{-1/2} \cdot$$

$$\sum_{0<m_1<m_2 \ldots < m_{\frac{N_1}{2}}} (-1)^{j_1 - m_1, \ldots, j_1 - m_{\frac{N_1}{2}}} \, \mathrm{Det} \left\{ \Phi_{j_1 \ell_1 n_1}^{m_1} \, \Phi_{j_1 \ell_1 n_1}^{-m_1} \cdots \Phi_{j_1 \ell_1 n_1}^{-m_{\frac{N_1}{2}}} \right\} \cdot$$

$$\sum_{0<\mu_1<\ldots<\mu_{\frac{N_3}{2}}} (-1)^{J_3-\mu_1 \cdots\cdots J_3-\mu_{\frac{N_3}{2}}} \mathrm{Det}\left\{\Phi^{\mu_1}_{J_3 \ell_3 n_3} \cdots\cdots \Phi^{-\mu_{\frac{N_3}{2}}}_{J_3 \ell_3 n_3}\right\} \tag{A11}$$

$$\Psi^{0}_{0}(\text{daughter}) = \left[\binom{\frac{2J_1+1}{2}}{\frac{N_1-2}{2}}\binom{\frac{2J_3+1}{2}}{\frac{N_3-2}{2}}(N_1-2)!\,(N_3-2)!\right]^{-\frac{1}{2}} \cdot$$

$$\sum_{0<m_2<\ldots<m_{\frac{N_1}{2}}} (-1)^{J_1-m_2 \cdots J_1-m_{\frac{N_1}{2}}} \mathrm{Det}\left\{\Phi^{m_2}_{J_1 \ell_1 n_1} \cdots \Phi^{-m_{\frac{N_1}{2}}}_{J_1 \ell_1 n_1}\right\}$$

$$\sum_{0<\mu_2<\ldots<\mu_{\frac{N_3}{2}}} (-1)^{J_3-\mu_2 \cdots J_3-\mu_{\frac{N_3}{2}}} \mathrm{Det}\left\{\Phi^{\mu_2}_{J_3 \ell_3 n_3} \cdots\cdots \Phi^{-\mu_{\frac{N_3}{2}}}_{J_3 \ell_3 n_3}\right\}$$

The subscript 1 refers to protons and 3 to neutrons. N is the number of particles in the subshell with quantum numbers $J \ell n$ in the parent nucleus.

Even-odd nuclei

$$\Phi_{J_1}^{M}(\text{parent}) = \left[\binom{\frac{2J_1-1}{2}}{\frac{N_1-1}{2}}\binom{\frac{2J_3+1}{2}}{\frac{N_3}{2}} N_1!\; N_3!\right]^{-1/2} \cdot$$

$$\sum_{\substack{0<m_1\ldots<m_{\frac{N_1-1}{2}} \\ \neq M}} (-1)^{J_1-m_1\cdots J_1-m_{\frac{N_1-1}{2}}} \mathrm{Det}\left\{\Phi_{J_1 l_1 n_1}^{M}\,\Phi_{J_1 l_1 n_1}^{-m_1}\cdots\Phi_{J_1 l_1 n_1}^{-m_{\frac{N_1-1}{2}}}\right\} \tag{A12}$$

$$\sum_{0<\mu_1<\ldots<\mu_{\frac{N_3}{2}}} (-1)^{J_3-\mu_1\cdots J_3-\mu_{\frac{N_3}{2}}} \mathrm{Det}\left\{\Phi_{J_3 l_3 n_3}^{\mu_1}\cdots\Phi_{J_3 l_3 n_3}^{-\mu_{\frac{N_3}{2}}}\right\}$$

$$\Psi_{J_1}^{M-m}(\text{daughter}) = \left[\binom{\frac{2J_1-1}{2}}{\frac{N_1-3}{2}}\binom{\frac{2J_3+1}{2}}{\frac{N_3-2}{2}}(N_1-2)!\;(N_3-2)!\right]^{-1/2} \cdot$$

$$\sum_{\substack{0<m_2<\ldots m_{\frac{N_1-1}{2}} \\ \neq M-m}} (-1)^{J_1-m_2\cdots J_1-m_{\frac{N_1-1}{2}}} \mathrm{Det}\left\{\Phi_{J_1 l_1 n_1}^{M-m}\,\Phi_{J_1 l_1 n_1}^{-m_2}\cdots\Phi_{J_1 l_1 n_1}^{-m_{\frac{N_1-1}{2}}}\right\}$$

$$\sum_{0<\mu_2<\ldots\mu_{\frac{N_3}{2}}} (-1)^{J_3-\mu_2\cdots J_3-\mu_{\frac{N_3}{2}}} \mathrm{Det}\left\{\Phi_{J_3 l_3 n_3}^{\mu_2}\cdots\Phi_{J_3 l_3 n_3}^{-\mu_{\frac{N_3}{2}}}\right\}$$

N_1 is now odd and N_3 is even. $J\,l\,n$ have the same meaning as in (A11).

There is also the possibility that the "odd"-particle in the parent nucleus is in a single particle state different from the state $J_1\,l_1\,n_1$

$$\bar{\Phi}_{J_2}^{M}(\text{parent}) = \left[\binom{\frac{2J_1+1}{2}}{\frac{N_1-1}{2}}\binom{\frac{2J_3+1}{2}}{\frac{N_3}{2}} N_1!\ N_3!\right]^{-1/2} \cdot$$

$$\sum_{0<m_1<\ldots<m_{\frac{N_1-1}{2}}} (-1)^{J_1-m_1 \ldots J_1-m_{\frac{N_1-1}{2}}} \mathrm{Det}\left\{\bar{\Phi}_{J_2 l_2 n_2}^{M}\ \bar{\Phi}_{J_1 l_1 n_1}^{m_1} \ldots \bar{\Phi}_{J_1 l_1 n_1}^{-m_{\frac{N_1-1}{2}}}\right\} \tag{A13}$$

$$\sum_{0<\mu_1\ldots<\mu_{\frac{N_3}{2}}} (-1)^{J_3-\mu_1 \ldots J_3-\mu_{\frac{N_3}{2}}} \mathrm{Det}\left\{\bar{\Phi}_{J_3 l_3 n_3}^{\mu_1} \ldots \bar{\Phi}_{J_3 l_3 n_3}^{-\mu_{\frac{N_3}{2}}}\right\}$$

There are now two possibilities for the daughter nucleus:

a. The "odd"-particle is in the state $J_2\, l_2\, n_2$

b. The "odd"-particle is in the state J_1, l_1, n_1 in which case the wave function is already given in Eq. (A12).

$$\Psi_{J_2}^{M-m}(\text{daughter}) = \left[\binom{\frac{2J_1+1}{2}}{\frac{N_1-3}{2}}\binom{\frac{2J_3+1}{2}}{\frac{N_3-2}{2}}(N_1-2)!\ (N_3-2)!\right]^{-\frac{1}{2}}$$

$$\sum_{0<m_2<\ldots\, m_{\frac{N_1-1}{2}}} (-1)^{J_1-m_2 \ldots J_1-m_{\frac{N_1-1}{2}}} \mathrm{Det}\left\{\bar{\Phi}_{J_2 l_2 n_2}^{M-m}\ \bar{\Phi}_{J_1 l_1 n_1}^{m_2} \ldots \bar{\Phi}_{J_1 l_1 n_1}^{-m_{\frac{N_1-1}{2}}}\right\} \tag{A14}$$

$$\sum_{0<\mu_2<\mu_3\ldots<\mu_{\frac{N_3}{2}}} (-1)^{J_3-\mu_2 \ldots J_3-\mu_{\frac{N_3}{2}}} \mathrm{Det}\left\{\Phi_{J_3 l_3 n_3}^{\mu_2} \ldots \bar{\Phi}_{J_3 l_3 n_3}^{-\mu_{\frac{N_3}{2}}}\right\}$$

To perform the integrations on the coordinates of particles 5... A we expand the determinant wave functions for N_1 and N_3 particles (parent nucleus) in subdeterminants of 2 and N_1-2 (N_3-2) rows, so that particles 1,2 and 3,4 are together in the subdeterminants of two rows.

With the help of these expansions one gets immediately

$$\left[\binom{N}{2}\binom{Z}{2}\right]^{1/2} \int d\xi_K \, \overset{*}{\Phi}{}^{0}_{0} \, \Psi^{0}_{0} = \Phi^{(12)}_{j_1 j_1 0} \, \Phi^{(34)}_{j_3 j_3 0} \cdot \left[\frac{N_1 (2j_1+3-N_1) N_3 (2j_3+3-N_3)}{4(2j_1+1)(2j_3+1)}\right]^{1/2} \tag{A15}$$

$$\left[\binom{N}{2}\binom{Z}{2}\right]^{1/2} \int \overset{*}{\Phi}{}^{M}_{j_1} \Psi^{M-m}_{j_1} \, d\xi_K = \left\{ \Phi^{(12)}_{j_1 j_1 0} \cdot \delta_{0,m} - 2 \sum_{\substack{L \geq 2 \\ \text{even}}}' C(L j_1 j_1; m \; M-m) \frac{[(2j_1+1)(2L+1)]^{1/2}}{2j_1-1} \cdot \Phi^{m\,(12)}_{j_1 j_1 L} \right\} \Phi^{(34)}_{j_3 j_3 0} \cdot \left[\frac{(N_1-1)(2j_1+2-N_1) N_3 (2j_3+3-N_3)}{4(2j_1+1)(2j_3+1)}\right]^{1/2}$$

$$\left[\binom{N}{2}\binom{Z}{2}\right]^{1/2} \int \overset{*}{\Phi}{}^{M}_{j_2} \Psi^{M-m}_{j_2} \, d\xi_K = \delta_{0,m} \, \Phi^{(12)}_{j_1 j_1 0} \, \Phi^{(3,4)}_{j_3 j_3 0} \left[\frac{(N_1-1)(2j_1+4-N_1) N_3 (2j_3+3-N_3)}{4 \cdot (2j_1+1)(2j_3+1)}\right]^{1/2}$$

$$\left[\binom{N}{2}\binom{Z}{2}\right]^{1/2} \int \Phi_{J_2}^{M} \Psi_{J_1}^{M-m} d\xi_K = \sqrt{2L+1}\; C(L J_1 J_2 ; m\, M-m)$$

$$\left\{ \Phi_{J_1 J_2 L}^{m}(12) - \Phi_{J_1 J_2 L}^{m}(21) \right\} \frac{1}{\sqrt{2}} \cdot \Phi_{J_3 J_3 0}(34) \left[\frac{N_3\,(2J_3+3-N_3)}{2 \cdot 2J_3+1} \cdot \frac{N_1-1}{(2J_1+1)(2J_2+1)} \right]^{1/2}$$

where

$$\Phi_{JJ'L}^{m}(ik) = \sum_{\mu}{}' C(JJ'L; \mu\, m-\mu)\; \Phi_{J}^{\mu}(i)\; \Phi_{J'}^{m-\mu}(k)$$

From (A8) and the results just given one derives now easily the expressions for the reduced widths δ^2_{JJL} given in Eq. (23, 24).

We shall now evaluate the reduced width for the cases treated in § 3. The treatment will be sufficiently general to include configurational mixing of any type for the nucleons outside closed shells.

1. Two identical nucleons outside closed shells. (Example Po^{210})

$$\Phi_J^M(\text{parent}) = \sum_{J_1 \ell_1 n_1 J_2 \ell_2 n_2} a(J_1 \ell_1 n_1, J_2 \ell_2 n_2) \cdot$$

$$\sum_{m_1}{}' C(J_1 J_2 J;\ m_1,\ M-m_1) \cdot (N!\ Z!)^{-\frac{1}{2}} \cdot$$

$$\text{Det}\left\{ \Phi_{J_1 \ell_1 n_1}^{m_1}\ \Phi_{J_2 \ell_2 n_2}^{M-m_1} \cdot \text{ closed shell } \ldots \right\}_Z$$

$$\text{Det}\left\{ \ldots \text{ closed shell } \ldots \ldots \ldots \right\}_N \qquad \text{(A16)}$$

$$\Psi_J^{M-m}(\text{daughter}) = \sum_{J_3 \ell_3 n_3\ J_4 \ell_4 n_4}{}' a(J_3 \ell_3 n_3\ J_4 \ell_4 n_4) \cdot$$

$$\sum_{m_3} C(J_3 J_4 J;\ m_3\ M-m-m_3)\ \left[(N-2)!(Z-2)!\right] \cdot$$

$$\text{Det}\left\{ \ldots \Phi_{J_3}^{-m_3}\ \Phi_{J_4}^{-M+m+m_3} \text{ missing from a closed shell} \ldots \right\}_{N-2}$$

$$\text{Det}\left\{ \ldots \text{ closed shell } \ldots \ldots \ldots \right\}_{Z-2}$$

Z and N are the proton and neutron number of the parent nucleus.

We expand the determinants in the wave function of the parent nucleus as we did earlier (A14) and get,

$$\left[\binom{N}{2}\binom{Z}{2}\right]^{\frac{1}{2}} \int \Phi_{J}^{*M} \Psi_{J}^{M-m} d\xi_K = \sum_{m_1, m_3} (-1)^{J_3+J_4-M+m}$$

$$C(J_1 J_2 J; m_1\ M-m_1)\ C(J_3 J_4 J; m_3\ M-m-m_3)$$

$$A\left(\Phi^{m_1}_{J_1 \ell_1 n_1}(1)\ \Phi^{M-m_1}_{J_2 n_2 \ell_2}(2)\right) A\left(\Phi^{-m_3}_{J_3 \ell_3 n_3}(3)\ \Phi^{-M+m+m_3}_{J_4 \ell_4 n_4}(4)\right) \tag{A17}$$

Comparing this expression with Eq. (A2) one finds,

$$b\begin{pmatrix} n_1 n_2 n_3 n_4 & \ell_1 \ell_2 \ell_3 \ell_4 \\ J_1 J_2 J_3 J_4 & J_p J_N L \end{pmatrix} = \delta_{J_p J}\, \delta_{J_N J} (-1)^{J+J-L} \sqrt{\frac{2L+1}{2J+1}} \tag{A18}$$

From this we get for the reduced width δ^2_{JJL} including configurational mixing,

$$\delta^2_{JJL} = \left(\sum_{J_1 \ell_1 n_1\, J_2 \ell_2 n_2\, J_3 \ell_3 n_3\, J_4 \ell_4 n_4} a(J_1 \ell_1 n_1\, J_2 \ell_2 n_2) \right.$$

$$\left. a(J_3 \ell_3 n_3\, J_4 \ell_4 n_4) \cdot \delta_{JJL}(J_1 \ell_1 n_1\, J_2 \ell_2 n_2\, J_3 \ell_3 n_3\, J_4 \ell_4 n_4) \right)^2 \tag{A19}$$

$$\delta_{JjL}(j_1 l_1 n_1\, j_2 l_2 n_2\, j_3 l_3 n_3\, j_4 l_4 n_4) = (-1)^{j_1+j_3+l_2+l_4+J+j+1}$$

$$\left[\frac{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}{2J+1}\;\frac{(2-\delta_{j_1j_2}\delta_{l_1l_2}\delta_{n_1n_2})(2-\delta_{j_3j_4}\delta_{l_3l_4}\delta_{n_3n_4})}{4}\right]^{1/2}$$

$$\frac{1+(-1)^{l_1+l_2-J}}{2}\;\frac{1+(-1)^{l_3+l_4-j}}{2}\;C(JjL;00)$$

$$C(j_1j_2J;\tfrac{1}{2}\,-\tfrac{1}{2})\;C(j_3j_4j;\tfrac{1}{2}\,-\tfrac{1}{2})\;R^{n_1l_1\,n_2l_2}_{n_3l_3\,n_4l_4}(R_0)$$

2. Two neutrons and two protons outside closed shells (Example Po^{212}). In this case we shall only consider $J = 0$, $j = 0$, $L = 0$. The daughter nucleus is the double magic Pb^{208}. A straightforward calculation yields,

$$\delta^2_{000} = \left[\sum_{j_1l_1n_1\,j_2l_2n_2\,j_3l_3n_3\,j_4l_4n_4\,\lambda} (-1)^{j_1+j_3+l_2+l_4+1+\lambda}\right. \qquad \text{(A20}$$

$$a(j_1l_1n_1\,j_2l_2n_2\,j_3l_3n_3\,j_4l_4n_4\,\lambda)\;\frac{1+(-1)^{l_1+l_2-\lambda}}{2}$$

$$\frac{1+(-1)^{\ell_3+\ell_4-\lambda}}{2} \quad C(j_1 j_2 \lambda;\ \tfrac{1}{2}\ -\tfrac{1}{2})\ C(j_3 j_4 \lambda;\ \tfrac{1}{2}\ -\tfrac{1}{2})$$

$$\left[\frac{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}{2\lambda+1}\ \frac{(2-\delta_{j_1 j_2}\,\delta_{\ell_1\ell_2}\,\delta_{n_1 n_2})(2-\delta_{j_3 j_4}\,\delta_{\ell_3\ell_4}\,\delta_{n_3 n_4})}{4}\right]^{1/2}$$

$$R_{\substack{n_1\ell_1 n_2\ell_2\\ n_3\ell_3 n_4\ell_4}}(R_0)$$

where λ is the angular momentum to which the two neutrons and two protons respectively are coupled.

3. Three nucleons outside closed shells (Examples Po^{211}, Bi^{211}).

$$\Phi_J^M(\text{parent}) = \sum_{j_1\ell_1 n_1\ j_2\ell_2 n_2\ j_3\ell_3 n_3\ J_P} a_J\,(j_1\ell_1 n_1\ j_2\ell_2 n_2\ j_3\ell_3 n_3\ J_P)$$

$$\sum_{m_1\ m_3} C(j_1 j_2 J_P;\ m_1\ M-m_3-m_1)\ C(j_3 J_P J;\ m_3\ M-m_3) \tag{A21}$$

$$(N!\,Z!)^{-1/2}\ \mathrm{Det}\left\{\Phi_{j_1\ell_1 n_1}^{m_1}\ \Phi_{j_2\ell_2 n_2}^{M-m_3-m_1}\ \ldots\text{closed shell}\ldots\right\}_Z$$

$$\mathrm{Det}\left\{\Phi_{j_3}^{m_3}\ldots\text{closed shell}\ \ldots\ldots\ldots\right\}_N$$

$$\Psi_J^{M-m}(\text{daughter}) = \left[(N-2)!\,(Z-2)!\right]^{-1/2} \operatorname{Det}\{\text{closed shell}\}_{Z-2}$$

$$\operatorname{Det}\{\Phi_J^{-M+m} \text{ missing from a closed shell} \ldots\}_{N-2}$$

It follows,

$$\delta^2_{JJL} = \left(\sum_{j_1 l_1 n_1\, j_2 l_2 n_2\, j_3 l_3 n_3\, J_P} a_J(j_1 l_1 n_1\, j_2 l_2 n_2\, j_3 l_3 n_3\, J_P) \; \delta_{JJL}(j_1 l_1 n_1\, j_2 l_2 n_2\, j_3 l_3 n_3\, J_P) \right)^2 \tag{A22}$$

$$\delta_{JJL}(j_1 l_1 n_1\, j_2 l_2 n_2\, j_3 l_3 n_3\, J_P) = (-1)^{j_1+j_3+l_2+l_4+1}$$

$$\frac{1+(-1)^{l_1+l_2-J_P}}{2} \quad \frac{1+(-1)^{l_1+l_2+l_3+l_4-L}}{2} \quad C(jJL;\tfrac{1}{2}\,-\tfrac{1}{2})$$

$$C(j_1 j_2 J_P;\tfrac{1}{2}\,-\tfrac{1}{2})\; C(j_3 J J_P;\tfrac{1}{2}\,-\tfrac{1}{2}) \left[\frac{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}{2J_P+1}\right.\cdot$$

$$\left.\frac{2-\delta_{j_1 j_2}\,\delta_{l_1 l_2}\,\delta_{n_1 n_2}}{2}\right]^{1/2} \quad R^{n_1 l_1\, n_2 l_2}_{n_3 l_3\, n_4 l_4}(R_0)$$

where

$J \equiv j_4$

4. Two non-identical nucleons outside a closed shell (Example Bi^{210})

$$\Phi_J^M(\text{parent}) = \sum_{j_1 l_1 n_1\, j_3 l_3 n_3} a_J(j_1 l_1 n_1\, j_3 l_3 n_3)\,(N!\,Z!)^{-\frac{1}{2}}$$

$$\sum_{m_1} C(j_1 j_3 J;\, m_1\, M-m_1)\ \mathrm{Det}\left\{\Phi^{m}_{j_1 l_1 n_1}\cdot \text{closed shell}\ldots\right\}_Z \cdot \tag{A23}$$

$$\cdot\,\mathrm{Det}\left\{\Phi^{M-m_1}_{j_3 l_3 n_3}\cdot \text{closed shell}\ \ldots\right\}_N$$

$$\Psi_J^{M-m}(\text{daughter}) = \sum_{j_2 l_2 n_2\, j_4 l_4 n_4} a_J(j_2 l_2 n_2\, j_4 l_4 n_4)$$

$$\left[(N-2)!\,(Z-2)!\right]^{-\frac{1}{2}} \sum_{m_2} C(j_2 j_4 J;\, m_2\, M-m-m_2)$$

$$\mathrm{Det}\left\{\bar{\Phi}^{-m_2}_{j_2}\ \text{missing from a closed shell}\ldots\right\}_{Z-2}$$

$$\mathrm{Det}\left\{\bar{\Phi}^{-M+m_2+m}_{j_4}\ \text{missing from a closed shell}\right\}_{N-2}$$

We get for $\left[\binom{N}{2}\binom{Z}{2}\right]^{1/2} \int \Phi_J^M \overset{*}{\Psi}{}_J^{M-m} \, d\xi_K$

$$\left[\binom{N}{2}\binom{Z}{2}\right]^{1/2} \int \Phi_J^M \overset{*}{\Psi}{}_J^{M-m} \, d\xi_K = \sum_{J_1 \ell_1 n_1 J_2 \ell_2 n_2 \; J_3 \ell_3 n_3 \; J_4 \ell_4 n_4} (-1)^{J_2+J_4-M+m} \tag{A24}$$

$$a_J(J_1 \ell_1 n_1 \, J_3 \ell_3 n_3) \; a_J(J_2 \ell_2 n_2 \, J_4 \ell_4 n_4) \cdot$$

$$C(J_1 J_3 J; m_1 \; M-m_1) \; C(J_2 J_4 J; m_2 \; M-m-m_2)$$

$$A\left(\Phi^{m_1}_{J_1 \ell_1 n_1}{}^{(1)} \; \Phi^{-m_2}_{J_2 \ell_2 n_2}{}^{(2)}\right) \; A\left(\Phi^{M-m_1}_{J_3 \ell_3 n_3}{}^{(3)} \; \Phi^{-M+m+m_2}_{J_4 \ell_4 n_4}{}^{(4)}\right)$$

It follows for the coefficients $b\begin{pmatrix} n_1 n_2 n_3 n_4 & \ell_1 \ell_2 \ell_3 \ell_4 \\ J_1 J_2 J_3 J_4 & J_P J_N L \end{pmatrix}$

$$C(L J J; m \; M-m) \, b\begin{pmatrix} n_1 n_2 n_3 n_4 & \ell_1 \ell_2 \ell_3 \ell_4 \\ J_1 J_2 J_3 J_4 & J_P J_N L \end{pmatrix} =$$

$$a_J(J_1 \ell_1 n_1 \, J_3 \ell_3 n_3) \; a_J(J_2 \ell_2 n_2 \, J_4 \ell_4 n_4) \cdot \sum_{m_1, m_2} (-1)^{J_2+J_4-M+m} \tag{A25}$$

$$C(J_1 J_3 J; m_1 \; M-m_1) \; C(J_2 J_4 J; m_2 \; M-m-m_2) \; C(J_1 J_2 J_P; m_1 \; -m_2)$$

$$C(J_3 J_4 J_N; M-m_1 \; -M+m+m_2) \; C(J_P J_N L; m_1-m_2 \; m-m_1+m_2)$$

We introduce $b\begin{pmatrix} n_1 n_2 n_3 n_4 \ell_1 \ell_2 \ell_3 \ell_4 \\ j_1 j_2 j_3 j_3 \; J_P J_N L \end{pmatrix}$

and $g^{\,n_1 \ell_1 j_1,\, n_2 \ell_2 j_2}_{\,n_3 \ell_3 j_3,\, n_4 \ell_4 j_4}\; J_P J_N L$

into (A2) and find that it is possible to sum over J_P and J_N

The result for the reduced width is,

$$\delta^2_{JJL} = \Bigg[\sum_{j_1 \ell_1 n_1\; j_2 \ell_2 n_2\; j_3 \ell_3 n_3\; j_4 \ell_4 n_4} a_J (j_1 \ell_1 n_1\, j_3 \ell_3 n_3)$$

$$a_J (j_2 \ell_2 n_2\; j_4 \ell_4 n_4)\; (-1)^{j_1 + j_2 + j_3 + j_4 + \ell_2 + \ell_4 + J + \frac{1}{2}} \tag{A26}$$

$$\left[\frac{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}{4 \cdot (2J+1)} \right] \Big\{ C(j_1 j_3 J;\, \tfrac{1}{2}\, \tfrac{1}{2})\, C(j_2 j_4 J;\, \tfrac{1}{2}\, \tfrac{1}{2}) \cdot$$

$$C(J J L;\, 1\; -1) + (-1)^{j_3 + j_4 + \ell_3 + \ell_4}\, C(j_1 j_3 J;\, \tfrac{1}{2}\, -\tfrac{1}{2}) \cdot$$

$$C(j_2 j_4 J;\, \tfrac{1}{2}\, -\tfrac{1}{2})\; C(J J L;\, 0\; 0) \Big\}\; \frac{1 + (-1)^{\ell_1 + \ell_2 + \ell_3 + \ell_4 - L}}{2}$$

$$R_{\substack{n_1 \ell_1 n_2 \ell_2 \\ n_3 \ell_3 n_4 \ell_4}} (R_0) \Bigg]^2$$

5. Four nucleons outside closed shells. Three of them identical. (Example Bi^{212} and At^{212})

$$\Phi_J^M(\text{parent}) = \sum_{j_1 \ell_1 n_1\, j_2 \ell_2 n_2\, j_3 \ell_3 n_3} a_J(j_1 \ell_1 n_1\; j_3 \ell_3 n_3\; j_4 \ell_4 n_4)$$

$$\sum_{m_1;\, m_4>0} C(j_1 j_3 J; m_1\; M-m_1)\; \frac{(-1)^{j_4-m_4}}{\sqrt{2j_4+1}} \left[Z!\,N!\right]^{-1/2}$$

$$\left[\frac{2j_3-1+2\cdot\delta_{j_3j_4}\,\delta_{\ell_3\ell_4}\,\delta_{n_3n_4}}{2j_3-1}\right]^{1/2} \mathrm{Det}\left\{\Phi^{m_1}_{j_1\ell_1 n_1}\cdot\right.$$

$$\left.\text{closed shell}\;..\right\}_Z \cdot \mathrm{Det}\left\{\Phi^{M-m_1}_{j_3\ell_3n_3}\,\Phi^{m_4}_{j_4\ell_4n_4}\,\Phi^{-m_4}_{j_4\ell_4n_4}\cdot\right. \qquad \text{(A27)}$$

$$\left.\text{closed shell}\;..\right\}_N$$

$$\Psi_J^{M-m}(\text{dangles}) = \sum_{j_2\ell_2n_2\, j_6\ell_6n_6} a_J(j_2\ell_2n_2\; j_6\ell_6n_6)$$

$$\sum_{m_2} C(j_2 j_6 J;\; m_2\; M-m-m_2)\; \left[(Z-2)!\,(N-2)!\right]^{-1/2}\cdot$$

$$\mathrm{Det}\left\{\Phi_{j_2}^{-m_2}\;\text{missing from a closed shell}\;..\right\}_{Z-2}$$

$$\mathrm{Det}\left\{\Phi_{j_6}^{-M-m-m_2}\cdot\;\text{closed shell}\;.\;.\;.\;.\;.\right\}_{N-2}$$

We have assumed that two of the three identical nucleons are coupled to angular momentum zero.

We get,

$$\left[\binom{N}{2}\binom{Z}{2}\right]^{1/2} \int \Phi_J^{M} \overset{*}{\Psi}_J^{M-m} d\xi_K = \sum_{j_1 l_1 n_1 j_3 l_3 n_3 j_4 l_4 n_4 j_2 l_2 n_2 j_6 l_6 n_6}'$$

$$a_J(j_1 l_1 n_1\ j_3 l_3 n_3\ j_4 l_4 n_4)\ a_J(j_2 l_2 n_2\ j_6 l_6 n_6)$$

$$\sum_{m_1 m_2 m_4}' \frac{(-1)^{j_4 - m_4 - j_2 + m_2}}{\sqrt{2j_4+1}} C(j_1 j_3 J; m_1\ M-m_1)\ C(j_2 j_6 J;\ m_2\ M-m-m_2) \tag{A28}$$

$$A\left(\Phi_{j_1 l_1 n_1}^{m_1\ (1)}\ \Phi_{j_2 l_2 n_2}^{-m_2\ (2)}\right)\left[\frac{2j_3 - 1 + 2\cdot \delta_{j_3 j_4}\delta_{l_3 l_4}\delta_{n_3 n_4}}{2j_3 - 1}\right]^{1/2}$$

$$\Bigg\{ \delta_{j_6 j_3}\delta_{l_6 l_3}\delta_{n_6 n_3}\delta_{M-m-m_2;\,M-m_1}\ \Phi_{j_4 l_4 n_4}^{m_4\ (3)}\ \Phi_{j_4 l_4 n_4}^{-m_4\ (4)}$$

$$+\ \delta_{j_6 j_4}\delta_{l_6 l_4}\delta_{n_6 n_4}\delta_{M-m-m_2;\,m_4}\Big[\Phi_{j_4 l_4 n_4}^{-m_4\ (3)}\ \Phi_{j_3 l_3 n_3}^{M-m_1\ (4)}$$

$$-\ \Phi_{j_3 l_3 n_3}^{M-m_1\ (3)}\ \Phi_{j_4 l_4 n_4}^{-m_4\ (4)}\Big]\Bigg\}$$

and from this we get for the reduced width,

$$\delta^2_{JjL} = \sum_{j_1 l_1 n_1\, j_2 l_2 n_2\, j_3 l_3 n_3\, j_4 l_4 n_4\, j_6 l_6 n_4} a_J(j_1 l_1 n_1\, j_3 l_3 n_3\, j_4 l_4 n_4)$$

$$a_j(j_2 l_2 n_2\, j_6 l_6 n_6) \Big\{ \delta_{j_6 j_3}\, \delta_{l_6 l_3}\, \delta_{n_6 n_3}\, \frac{1+(-1)^{l_1+l_2-L}}{2}$$

$$\left[\frac{(2j_1+1)(2j_2+1)(2j_4+1)(2j+1)}{2}\right]^{1/2} W(J j_1\, j j_2\, ;\, j_3\, L)(-1)^{j_1+j_2+j_3+\frac{1}{2}+l_2+l_4+L+j}$$

$$C(j_1 j_2 L\, ;\, \tfrac{1}{2}\, -\tfrac{1}{2})\ \mathrm{I\!R}^{\,n_1 l_1\, n_2 l_2}_{\,n_4 l_4\, n_4 l_4}\,(R_0) \quad +$$

$$\delta_{j_6 j_4}\, \delta_{l_6 l_4}\, \delta_{n_6 n_4}\, (-1)^{j_1+j_2+j_3+j_4+J+l_2+l_4} \tag{A29}$$

$$\frac{1+(-1)^{l_1+l_2+l_3+l_4-L}}{2} \left[\frac{(2j_1+1)(2j_2+1)(2j_3+1)}{2(2J+1)}\right]^{1/2} \cdot$$

$$[\, C(j_1 j_3 J; \tfrac{1}{2}\, \tfrac{1}{2})\, C(j_2 j_4 J\, ;\, \tfrac{1}{2}\, \tfrac{1}{2})\, C(J j L;\ 1\, -1) \ +$$

$$(-1)^{j_3+j_4+l_3+l_4}\, C(j_1 j_3 J; \tfrac{1}{2}\, -\tfrac{1}{2})\, C(j_2 j_4 J\, ;\, \tfrac{1}{2}\, -\tfrac{1}{2}) \cdot$$

$$C(J j L\, ;\, 0\, 0)\,]\ \mathrm{I\!R}^{\,n_1 l_1\, n_2 l_2}_{\,n_3 l_3\, n_4 l_4}\,(R_0) \Big\} \cdot$$

$$\left[\frac{2j_3-1+2\cdot\delta_{j_3 j_4}\,\delta_{l_3 l_4}\,\delta_{n_3 n_4}}{2j_3-1}\right]^{1/2}$$

Deformed nuclei

We shall give the expressions for the two representations now in use for the single particle wave function in a deformed potential.

1. Nilsson's representation (isotropic three dimensional harmonic oscillator functions in spherical polar coordinates).

$$\Phi_{\Omega}(\mathbf{r}\, s) = \sum_{\ell \Lambda n} a^{\Omega}_{\ell \Lambda n} \, \Phi^{\Lambda}_{n\ell}(\mathbf{r}) \, \chi^{\Omega-\Lambda}_{\frac{1}{2}}(s)$$

$$\Phi^{\Lambda}_{n\ell} = \left[\frac{2 \cdot n! \, \alpha^{3/2}}{(n+\ell+\frac{1}{2})!}\right] e^{-\frac{\alpha}{2} r^2} (\sqrt{\alpha}\, r)^{\ell} \, L_n^{\ell+\frac{1}{2}}(\alpha r^2) \, Y^{\Lambda}_{\ell} \tag{A30}$$

Rewriting Φ_{Ω} slightly,

$$\Phi_{\Omega} = \sum_{J\ell n \Lambda} a^{\Omega}_{\ell \Lambda n} \, C(\ell \tfrac{1}{2} J; \Lambda\, \Omega-\Lambda) \, \Phi^{\Omega}_{J\ell n}$$

$$= \sum_{J\ell n} d^{\Omega}_{J\ell n} \, \Phi^{\Omega}_{J\ell n} \tag{A31}$$

where

$$d_{J\ell n} = \sum_{\Lambda=\Omega-\frac{1}{2}}^{\Omega+\frac{1}{2}} a^{\Omega}_{\ell \Lambda n} \, C(\ell \tfrac{1}{2} J; \Lambda\, \Omega-\Lambda)$$

we get for $g_{\Omega_1\Omega_2\Omega_3\Omega_4}$,

$$g_{\Omega_1\Omega_2\Omega_3\Omega_4}(\mathcal{R}) = \sum_{j_1 l_1 n_1\, j_2 l_2 n_2\, j_3 l_3 n_3\, j_4 l_4 n_4\, j_p j_N L} d^{\Omega_1}_{j_1 l_1 n_1}\, d^{\Omega_2}_{j_2 l_2 n_2}\, d^{\Omega_3}_{j_3 l_3 n_3}\, d^{\Omega_4}_{j_4 l_4 n_4}\, C(j_1 j_2 j_p\, \Omega_1 \Omega_2)$$

$$C(j_3 j_4 j_N;\, \Omega_3 \Omega_4)\; C(j_p j_N L;\, \Omega_1+\Omega_2\;\; \Omega_3+\Omega_4) \tag{A32}$$

$$g_{\substack{n_1 l_1 j_1\, n_2 l_2 j_2 \\ n_3 l_3 j_3\, n_4 l_4 j_4}\, j_p j_N L}(R)\; Y_L^{\Omega_1+\Omega_2+\Omega_3+\Omega_4}(\mathcal{R})$$

$$\left[(2-\delta_{j_1 j_2}\,\delta_{l_1 l_2}\,\delta_{n_1 n_2})(2-\delta_{j_3 j_4}\,\delta_{l_3 l_4}\,\delta_{n_3 n_4})\right]^{-\frac{1}{2}} \cdot 2$$

2. Asymptotic representation (three dimensional harmonic oscillator functions separated in cylindrical coordinates).

$$\Phi_\Omega = \sum_{\nu\mu\Lambda} f^{\Omega}_{\nu\mu\Lambda}\, \Phi^{\Lambda}_{\nu\mu}(\mathbf{r})\, \chi^{\Omega-\Lambda}_{\frac{1}{2}}(s)$$

$$\Phi^{\Lambda}_{\nu\mu} = \left[\frac{\nu!\;\alpha\cdot\alpha'^{\frac{1}{2}}}{(\nu+|\Lambda|)!\;2^{\mu}\,\mu!\;\pi^{\frac{3}{2}}}\right]^{\frac{1}{2}} e^{-\left(\frac{\alpha r^2}{2}+\frac{\alpha' z^2}{2}\right)} \cdot \tag{A33}$$

$$(\sqrt{\alpha}\, r)^{|\Lambda|}\, L^{|\Lambda|}_{\nu}(\alpha r^2)\; H_\mu(\sqrt{\alpha'}\, z)\, e^{-i\Lambda\varphi}$$

where

$$H_\mu(\sqrt{\alpha'}\, z) = \sum_k \frac{(-)^k\,\mu!}{k!\,(\mu-2k)!}\,(2\sqrt{\alpha'}\, z)^{\mu-2k}$$

$$\mathbf{r} = \{r,\varphi, z\}$$

A straightforward calculation which uses the technique described in detail for the case of spherical coordinates yields,

$$g_{\Omega_1 \Omega_2 \Omega_3 \Omega_4} = \sum_{\nu_1 \nu_2 \nu_3 \nu_4 \mu_1 \mu_2 \mu_3 \mu_4 \, s_1 \, s_3} f^{\Omega_1}_{\nu_1 \mu_1 \Lambda_1} f^{\Omega_2}_{\nu_2 \mu_2 \Lambda_2} f^{\Omega_3}_{\nu_3 \mu_3 \Lambda_3} f^{\Omega_4}_{\nu_4 \mu_4 \Lambda_4} \delta_{\Lambda_1 \, \Omega_1 - s_1} \delta_{\Lambda_2 \, \Omega_2 + s_1} \delta_{\Lambda_3 \, \Omega_3 - s_3} \delta_{\Lambda_4 \, \Omega_4 + s_3} (-1)^{1 - s_1 - s_3} \, g_{\substack{\nu_1 \Lambda_1 \mu_1 \, \nu_2 \Lambda_2 \mu_2 \\ \nu_3 \Lambda_3 \mu_3 \, \nu_4 \Lambda_4 \mu_4}} (R_0') \tag{A34}$$

$$g_{\substack{\nu_1 \Lambda_1 \mu_1 \, \nu_2 \Lambda_2 \mu_2 \\ \nu_3 \Lambda_3 \mu_3 \, \nu_4 \Lambda_4 \mu_4}} = \left[\frac{2\alpha'}{\alpha'+\beta}\right]^{\frac{\mu_1+\mu_2+\mu_3+\mu_4}{2}} \left[\frac{2\alpha}{\alpha+\beta}\right]^{N+\frac{|\Lambda|}{2}} \left(\frac{2\sqrt{\alpha'\beta}}{\alpha'+\beta}\right)^{3/2} \left(\frac{2\sqrt{\alpha\beta}}{\alpha+\beta}\right)^{3} \sqrt{\alpha}\,\alpha'^{1/2} R_0^{3/2} \left[2^{\mu_1+\mu_2+\mu_3+\mu_4} \mu_1!\,\mu_2!\,\mu_3!\,\mu_4!\right]^{-1/2} \left[\nu_1!\,\nu_2!\,\nu_3!\,\nu_4!\,(\nu_1+|\Lambda_1|)!\,(\nu_2+|\Lambda_2|)!\,(\nu_3+|\Lambda_3|)!\right.$$

$$(\nu_4 + |\Lambda_4|)\,]^{-\frac{1}{2}} \left(\frac{\alpha'}{\pi}\right)^{\frac{1}{4}} (2\alpha)^{\frac{1}{2}} \sum_{\tau} \frac{A_\tau}{4^\tau} \left(\frac{\beta-\alpha}{2\alpha'}\right)^{\frac{\mu_1+\mu_2+\mu_3+\mu_4-\tau}{2}}$$

$$H_\tau\left(\sqrt{2(\alpha'+\beta)}\, Z\right) \cdot \sum_{\varrho} \frac{B_\varrho\, \varrho!\, (-1)^\varrho}{2^{2\varrho+|\Lambda|}} \left(\frac{\beta-\alpha}{2\alpha}\right)^{N-\varrho}$$

$$L_\varrho^{|\Lambda|}\left(2(\alpha+\beta)r^2\right) \cdot \left(\sqrt{2(\alpha+\beta)}\, r\right)^{|\Lambda|} \cdot \frac{1}{\sqrt{2\pi}} e^{-i\Lambda\varphi}$$

where

$$\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4$$

$$2N = 2(\nu_1 + \nu_2 + \nu_3 + \nu_4) + |\Lambda_1| + |\Lambda_2| + |\Lambda_3| + |\Lambda_4| - \Lambda$$

$$\mathcal{R}' \equiv \{ r;\ \varphi;\ Z \}$$

$$B_\varrho = \nu_1!\,\nu_2!\,\nu_3!\,\nu_4! \sum_{k_1 k_2 k_3 k_4} \binom{\nu_1+|\Lambda_1|}{\nu_1-k_1}\binom{\nu_2+|\Lambda_2|}{\nu_2-k_2}\binom{\nu_3+|\Lambda_3|}{\nu_3-k_3}\binom{\nu_4+|\Lambda_4|}{\nu_4-k_4} \frac{(-1)^{k_1+k_2+k_3+k_4}}{k_1!\,k_2!\,k_3!\,k_4!}$$

$$2\varrho = 2(k_1 + k_2 + k_3 + k_4) + |\Lambda_1| + |\Lambda_2| + |\Lambda_3| + |\Lambda_4| - \Lambda$$

and A_τ is displaced by

$$H_{\mu_1}(z)\, H_{\mu_2}(z)\, H_{\mu_3}(z)\, H_{\mu_4}(z) = \sum_\tau A_\tau\, Z^\tau$$