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Methods of Mathematical Struggle

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by

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ABSTRACT OF THE DISSERTATION

Methods of Mathematical Struggle

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Mathematicians frequently report feeling an intense kind of internal effort when doing their research. There's some reason to believe that this experience of struggle is integral to developing skill and familiarity with novel mathematical ways of thinking. Yet surprisingly little is known about the structure of this experience as mathematicians know it in their research, in part because of the scarcity of studies examining mathematicians in the midst of working on problems that are novel to them. In order to help contribute to our understanding of this domain, I examine the practices of three pairs of mathematicians working together on some problem from their research. Using a framework of embodied

cognition to enact a phenomenological analysis, I note three ways in which we can see mathematicians structuring their experiences of struggle: *proximal inhibition*, *invoking the continuum*, and the use of *transformation mappings*. I propose that these three phenomenological constructs combine to offer us insight into the nature of mathematical practice and of mathematical objects as they are experienced.

Chapter 1

Overview

1.1 Mathematical Struggle

There's a collection of riddles based on the challenge of crossing a river with some constraints. Usually the trick with them is to catch ourselves making a key unnecessary assumption in our effort to solve them. Consider this problem as an example:

> *Cannibals will attack and eat missionaries whenever the missionaries are outnumbered, but otherwise the cannibals are friendly and cooperative. Suppose three cannibals and three missionaries come across a river they need to cross. There's a small boat available, but it can only hold at most two. How can the six get across using the boat without any missionaries getting eaten?*

Most people who try this puzzle notice that there's trouble after a few round-trips with the boat and get stuck. Since someone has to bring the boat back after delivering someone on the far side, most people quickly realize that you can't leave a missionary on the far side (since there would be three cannibals to two missionaries once the boat returns). And the second trip can't involve a missionary since that would get a missionary eaten (either on the far shore if a missionary and a cannibal go, or on the starting side if two missionaries go). So the second round trip has to have two cannibals go and one return. But then what? You could have two missionaries cross, but who would go back? No matter whether you send a cannibal or a missionary back, someone gets eaten!

As with most river puzzles, the trick is to notice where we've made an unwarranted assumption. In this case, the erroneous assumption is that the boat should never return with two passengers. The solution sequence goes like so:

1. Two cannibals cross, one returns.
2. Two cannibals cross, one returns.
3. Two missionaries cross, a missionary and a cannibal return. (This is the tricky part!)
4. Two missionaries cross, one cannibal returns. (So now the three missionaries are safely on the far side.)
5. Two cannibals cross, one returns.
6. Two cannibals cross.

It's actually not too hard to demonstrate that this is the only possible solution, too. At every stage of the puzzle, there are at most five possible moves: a missionary crosses, a cannibal crosses, a missionary and a cannibal cross, two missionaries cross, or two cannibals cross. We can exhaust the possible moves at every stage of the puzzle to see that each one falls into one of four categories: it either reverses a move just made, forces itself to be reversed in the next move (e.g., if we had started by ferrying one cannibal across, it would have had to return with the boat with no net effect), immediately results in a missionary getting eaten, or advances progress. The one and only exception is with the very first move, which could have involved a missionary and a cannibal instead of two cannibals — but in that case we'd be forced to bring the missionary back, resulting in the same overall effect. Therefore, we could have solved this problem simply by asking ourselves at each stage which of the possible moves does not result in an imminent reversal or immediate devouring of a missionary. Thus a once confounding puzzle requiring a burst of insight to solve has been reduced to a sequence determined uniquely by a simple algorithm.

This puzzle exposes three stages of reasoning that are worth noting. In the first stage, there's a sense of getting stuck; our first impulse leads us to a point where we cannot see how to proceed. In the second stage, there's a recognition of a key detail about the puzzle that suddenly makes our confusion and struggle vanish. In the third stage, we return to the puzzle with our transformed perception and re-present the solution in a way that's delightfully clear, almost obvious in retrospect, and largely devoid of any hint of the struggle that we had experienced to arrive at our insight.

These stages share a striking similarity with the process we see research mathematicians go through (Burton, 2004; Davis & Hersh, 1981; Hadamard, 1949; Pólya, 1945; Thurston, 1994). First they engage with a problem of some sort, following their intuition and skill until they reach a point where they're stuck. Then ideally at some point they achieve a breakthrough (although it's not always as dramatic and clear a dissolving of the question as in the river puzzle). Finally, they describe the original topic with the benefit of their newfound insight, although usually without highlighting the cognitive effort it took them to gain said insight. Generally speaking, it's this final, relatively polished description of the problem and its solution that gets published in academic journals of mathematics (Burton & Morgan, 2000; Thurston, 1994).

This insight gives us a first glimpse into a fascinating phenomenon about mathematics. The parts of math that mathematicians consider to be the most interesting, worthwhile, and challenging (Burton, 2004; Hadamard, 1949; Thurston,

1994) are actually hidden from view in a sense. This isn't strictly necessary, but it does seem to be a dominant trend. The sense of exploration, the encounter with an apparent paradox, the struggle to overcome it, and the delight in penetrating the confusion with insight are all well-known to and valued by mathematicians, but the polished solutions to the original puzzles are generally silent about this (Burton & Morgan, 2000).

I remember a recurring experience from several math classes I took that reflects on this phenomenon. At times, the professor would put a proof up on the board and walk us through the logic of it. Some professors would go to great lengths to make the logic seem sensible while others would be satisfied to convince us that each line followed from the previous ones. But in both cases, I found myself not infrequently caught in an odd position: I could agree that each step of the proof was true and that the proof demonstrated what it set out to demonstrate, but I didn't feel like I really "got" the proof. There was an intangible something about the proof that I just didn't quite perceive. But later on I could talk to classmates and give it some consideration, and often something would just "click." Suddenly the whole flow of the proof would make sense; I could comfortably distinguish between lines that employed some key insight and lines that were just machinery. Then, and only then, would I feel like I really understood the proof.

I think the analogy to the river problem is striking. If we were to give someone the algorithm for solving the river problem right after explaining the problem, they probably wouldn't have a chance to appreciate the difficulty of the problem or the insight of the solution. But if that person were to then apply the algorithm to the original problem and carefully consider what happens in each step, they might notice the key secret that's buried in that algorithm: two people must return at some point. At that point the nature of the problem and the cleverness of the solution both become available.

Mathematicians notably report a similar experience when reading others' published mathematical works (Burton, 2004; Thurston, 1994). There's a kind of effort involved in penetrating what the paper is really about, as though there's an extra something they seek that's beyond the truth of the statements given. In fact, without that seemingly intangible extra something, others' results are often useless to them. It's as though they need to experience the struggle on some level in order to feel that they actually understand the mathematics.

What all this seems to point toward, then, is the suggestion that the first phase of reasoning — the encounter of and struggle with a difficult puzzle — is central to mathematical work in a way that no one entirely understands. Mathematicians like Hadamard (1949), Thurston (1994), and Pólya (1945) have gone to considerable lengths to describe what seems to constitute this experience of struggle when they reflect on cases they and others like them have experienced, and what they report is not an irrelevant source of insight. But just as we're rarely aware of how priming encourages us to buy more when a store sign says "10 for $10" than when it says "$1 each" (Gladwell, 2005; Marcus, 2008; Wansink et al., 1998), it turns out that there's a great deal to the way this kind of struggle in mathematical thinking occurs that's generally below conscious awareness even for those who are doing the struggling. Mathematicians clearly know *that* there is effort, but precisely *what constitutes* that effort seems to be only partially available to their conscious minds (Hadamard, 1949; Pronin, 2009).

How, then, might we learn what the nature of this struggle is? And what would such an exploration buy us as math education researchers?

1.2 Phenomenology

1.2.1 The Case of Schneider

The philosopher Maurice Merleau-Ponty (1962) described the case of a World War I veteran by the name of Johann Schneider. Schneider was a patient of doctors Adhemar Gelb and Kurt Goldstein in the early 20th century due to brain injury he had sustained during the War. As a result of the injury, he had great difficulty performing some seemingly simple tasks such as touching his nose when instructed to do so or describing where his arms or legs were with respect to his torso with his eyes shut. Yet despite this, Schneider could readily scratch an itch on his hand or blow his nose using a handkerchief he pulled from his pocket. How are we to make sense of this paradox in which a man cannot find his nose when asked to do so but can easily do so when the need strikes him?

Merleau-Ponty addressed this challenge by examining the world as experienced by Schneider. Of course, since Merleau-Pointy is not Schneider, he had to perform this examination by considering Schneider's behavior as the behavior of an experiencing subject, which is to say of someone that experiences the world in a particular way. In a sense, Merleau-Ponty attempted to relate to Schneider as an experiencing being, and in making the attempt discovered possibly how Schneider's experiences differ from those he was more familiar with. Consider the following commentary:

> Clearly the patient finds in his body only an amorphous mass into which actual movement alone introduces divisions and links. In looking to his body to perform the movement for him he is like a speaker who cannot utter a word without following a text written beforehand. The patient himself neither seeks nor finds his movement, but moves his body about until the movement comes. The order given is not meaningless to him, since he recognizes the inadequacy of his first attempts, and also since, if a fortuitous gesture produces the required movement, he is aware of it and can immediately turn his piece of good fortune to account. But if the order has an *intellectual significance* for him and not a *motor*

> one, it does not communicate anything to him as a mobile subject; he may well find in the shape of a movement performed an illustration of the order given, but he can never convert the thought of a movement into actual movement. (Merleau-Ponty, 1962, p. 110, emphasis in the original)

In providing this account, Merleau-Ponty has given us some insight into the world as experienced by Schneider. Merleau-Ponty extended this account to theorize a kind of "motor intentionality," a kind of embodied anticipation of movement that is normally present in most people but is precisely what is missing in Schneider that causes his curious difficulties. This idea emerged from Merleau-Ponty's efforts to understand Schneider's point of view, which in turn helps to make us aware of elements of how we interact with the world that we simply may never have noticed before. For instance, Schneider's case seems to imply that there's a difference between a "concrete movement" like reaching forward to pick up a glass and an "abstract movement" like sticking our hand forward. In the first case, we use our hand as the medium through which we enact the intended effect of lifting the glass; the arm, hand, and indeed our entire body may be "invisible" to us because it is part of who we are in that instance of interacting with the glass as an object (Gallagher & Zahavi, 2008). In the second case, the goal is to have our arm extended forward, and thus the arm has become the object to be influenced rather than part of us as subjects trying to interact with some object. We might say that the body as used in "concrete" movements is *lived* (Gallagher & Zahavi, 2008; Merleau-Ponty, 1962) whereas the body as used in "abstract" movements is an anatomical object.

It's also worth noting that there continues to be discussion emerging from this one example Merleau-Ponty provided many decades ago. For instance, Jensen (2009) examines and attempts to elucidate the logic involved in Merleau-Ponty's account, and Gallagher & Zahavi (2008) expand on the idea of the lived body and how it applies to modern technology. Over half a century of philosophy and science has been impacted as a result of just one careful phenomenological examination of one man's experiences. The sincere effort to grasp the world as Schneider grasps it continues to teach us something about life as it is experienced that can complement and enrich the analysis of neurological and psychometric results.

1.2.2 The Study of Experience

Phenomenology, then, is a branch of philosophy that examines the structure of experience as it presents itself (Gallagher & Zahavi, 2008; Moustakas, 1994). A naturalist account of vision might describe the process of seeing in terms of light passing through the lens of the eye to strike the retina, stimulating optic nerves which run to the visual cortex, and so on; by way of contrast, a phenomenological account does not ask about the *mechanisms* of vision but instead focuses on what constitutes the experience of vision for us as seeing subjects. For instance, we can understand how brain damage might cause someone to be incapable of processing visual movement, but that tells us nothing about what it is like to actually see the world without motion (Nagel, 1974). It's certainly reasonable to expect the two perspectives to agree, but they are different. Phenomenologist want to explore the nature of the experience in ways that are not reducible to biological mechanisms.

The key advantage of such a focus is that it allows us to expand and evolve our perspective. In making a sincere effort to glimpse the world through a particular subject's eyes, we certainly learn a great deal about that other person, but we also learn more about ourselves and how we relate to the world. It broadens our vision of life and teaches us a deeper understanding of the human condition. In addition to learning something about how the other person experiences whatever the phenomenon under consideration is, we also come to understand what about our own experiences can be shared and what can be different from those of others. That makes our own experiences more meaningful and better understood, providing us with potentially powerful insight into the nature of the specific phenomenon at hand.

This approach introduces a different notion of generalization than one normally sees from clinical interviews (Clement, 2000; Ginsburg, 1997), think-aloud protocols (Ericsson & Simon, 1993), teaching experiments (Cobb, 2000; Steffe & Thompson, 2000), or quantitative statistical methods (Anderson, 2001). Merleau-Ponty's examination of the case of Schneider wasn't primarily valuable due to analyzing a syndrome that afflicts some number of patients, nor does it specify the neurological damage causing Schneider difficulties. Instead, it provides us with a way of grasping aspects of the human condition in novel ways. It generalizes by changing our perception of ourselves and the world. Once we know what to look for, we can see this difference between the body as lived and the body as an anatomical object in those around us and in our own movements. This is a general pattern in phenomenological analysis: we find that it generalizes not because we necessarily expect the analysis to transfer to another case study, but because it inspires a sense of familiarity when we spontaneously see an echo of the phenomenon elsewhere.

Applying phenomenological methods to mathematical struggle, then, can help enrich our understanding of the phenomenon and expand our ability to perceive aspects of mathematical practice that simply weren't available to us before. This can encourage new lines of research and open up potentially fruitful discussion on novel aspects of math teaching, learning, and doing.

Yet there are many subjects whose experience of mathematical struggle we could examine. Whom shall we pick, and why?

1.3 The Struggles of Research Mathematicians

There is good reason to choose any of a number of different subjects to investigate this phenomenon. An examination of K–12 students' struggles in math classes could tell us a great deal about where difficulty occurs and how at least some students currently try to address or avoid it. Applied statisticians working in marketing could provide us with some valuable insights about the interaction of mathematics and business challenges. These and other options are certainly worth considering.

The present study focuses on the struggles of research mathematicians as they engage in mathematical problems that are current and challenging to them. There are several reasons for this choice. First, it is sometimes suggested that students should engage with mathematics in a way that more closely mimics the engagement style of mathematicians (Brown, 1993; Brown et al., 1989; Cuoco et al., 1996; Lampert, 1990; Lave & Wenger, 1991; Schoenfeld, 1992; Stylianou, 2002; Wilkerson-Jerde & Wilensky, 2011; Yong & Orrison, 2008). However, as I'll illustrate in chapter 2, there are surprisingly few studies that watch mathematicians in the act of practicing their research. Mathematicians frequently report that struggle is a very common feature of their research (Aczel, 1996; Boyer, 1968; Burton, 2004; Burton & Morgan, 2000; Hadamard, 1949; Halmos, 1985; Thurston, 1994), so if we want a more rich understanding of mathematicians' research experiences then performing a phenomenological analysis on their experiences of struggle seems like a promising avenue.

Second, there seems to be a significant difference between how mathematicians and non-mathematicians view this kind of struggle. Mathematicians seem to expect it as part of their work and will stay with it for hours, days, and even sometimes years at a time (Aczel, 1996; Boyer, 1968; Hadamard, 1949). Yet many students seem to view their struggles with mathematical problems as a sign of ignorance or intellectual weakness (Hiebert & Grouws, 2007; Schoenfeld, 1985). Countless times I've heard people tell me, "I've always found math challenging" — which strikes me as odd, because I don't know of anyone who *doesn't* feel that way. Somehow, mathematicians seem to have developed more of a willingness to stay with this kind of mental difficulty than it seems many — perhaps most — non-mathematicians have. Exploring mathematicians' experiences of struggle and the ways in which they navigate them could give us a great deal of insight into this phenomenon.

Finally, I find mathematicians' practice fascinating in its own right. Mathematicians have developed a truly amazing amount of skill with one of the most sophisticated displays of human intelligence our species has yet created. It is a marvel that we are even capable of mathematics in the first place let alone able to conjure geometries that defy our physical intuitions and imagine abstractions that push the boundary of our minds. In the course of investigating mathematicians' ways of researching, I have found my own sense of what constitutes mathematical practice transforming. I have a deeper appreciation for the discipline of mathematics, for the remarkable work that mathematicians do, and the nobility of our joint endeavor to share the beauty and power of mathematics with our students. I'm fairly confident that I'm not alone in this sentiment.

1.4 The Present Study

This study is a phenomenological analysis of the kinds of struggle that research mathematicians encounter when doing their research. In other words, I've examined several mathematicians' experiences with this kind of intellectual challenge and given a series of case studies that, in a fashion similar to Merleau-Ponty's account of Schneider, seem to reveal previously unnoticed elements of how the experience is structured.

In order to gain something new from this analysis that wasn't already in mathematicians' self-reports (e.g., Hadamard, 1949), it was necessary to observe what mathematicians actually *do* when encountering struggle. In order to get behavior that I could observe and later analyze, it was necessary to find situations where mathematicians would work together in front of a blackboard to tackle a problem together. Therefore, all the subjects observed in this study were working in *pairs*.

1.4.1 Research Question & Overview

With this context in mind, then, here is the research question driving the present study: *What are some ways in which mathematicians structure their experiences of struggle while working in pairs in person on a current problem?*

In chapter 2 I'll describe how this work is situated in the literature. Chapter 3 will detail the methodology used in this study to answer the research question, including an overview of related methodological literature. Chapters 4, 5, and 6 present several case studies each as a means of answering the research question above. Chapters 7 & 8 will summarize the results, outline the limitations of the study, and suggest directions for future research.

Chapter 2

Literature Review

The literature relevant to the present study falls into two general categories: work reflecting the practice, mentality, and experiences of mathematicians; and writings pertinent to the theoretical framework used in this study. I will discuss each in turn and then discuss how this background reflects on the research question.

2.1 Mathematicians

The math education research community has had a growing interest, particularly in the last decade, in exploring the skills, practices, and attitudes of mathematicians. There seem to be two main motives behind this. Some researchers view mathematicians as experts in mathematical problem-solving, and therefore the people whose abilities and methods we should investigate so that we can better understand what we want students to emulate. For instance, within the context of justifying exploring mathematicians' approaches to proof, Weber & Mejia-Ramos (2011) say:

> If goals of mathematics instruction include having students (1) engage in the same types of proof-related activities that mathematicians do, (2) behave like mathematicians in these activities, and (3) adopt mathematicians beliefs regarding proof, then it is necessary to have an accu-

> rate understanding of the proof-related activities that mathematicians engage in, how mathematicians perform them, and what their beliefs about proof actually are. (Weber & Mejia-Ramos, 2011, p. 330)

The other main thread of research about mathematicians is more ethnographical. These studies are aimed at exploring the social dynamics and conventions of mathematical writing, seminar presentations, and collaboration within math departments, among other facets of the mathematical experience. These studies attempt to draw attention to subtle aspects of mathematicians' work that are often surprising to outsiders, such as the immense amount of creativity mathematicians consistently report is essential for their discipline (Burton, 2004; Davis & Hersh, 1981; Devlin, 2000; Hadamard, 1949; Halmos, 1985; Thurston, 1994). Such investigations often challenge conventional ideas about what mathematics *is*.

Each of these two branches — mathematicians' problem-solving approaches and expositions of the mathematical community of practice (Lave & Wenger, 1991) — deserves its own focus. I'll address each in turn.

Before going into each of these, though, I should mention in passing the work that's been done on impasse resolution and repair theory (Blumberg et al., 2008; VanLehn, 1988, 1990, 1999; VanLehn et al., 1998). This research program has been exploring in what ways students' encounters with difficulty (i.e., the reaching of an "impasse") influence those students' learning. This has produced some fascinating results that relate to mathematical struggle in students, and as such it would be necessary to account for this subfield in any studies that extend the present work to students. However, as this dissertation focuses on *mathematicians'* encounters with struggle while working on *novel* mathematics, I shall leave a more thorough review of literature on students' struggles as they relate to learning to a future research endeavor.

2.1.1 Problem-solving

In his famous book *How to Solve It*, the mathematician George Pólya (1945) outlined the overall process of problem-solving as mathematicians seem to think of it. Specifically, he suggested a four-step process by which one solves a mathematical problem:

1. First, one makes an effort to understand the problem, reading it over carefully and noticing the relevant components, possibly drawing a picture or diagram to capture important relationships. Pólya suggested that students should understand the problem well enough to be able to restate it in their own words.

2. Then one devises a plan for solving the problem. Here Pólya admits to there being a plethora of approaches, and that one learns which approaches are appropriate largely via experience and practice. However, he does list several such as guess-and-check and looking for symmetry.

3. Next, one carries out one's plan. Provided that one has the technical know-how (e.g., you have the algebraic manipulation skills to follow through on a strategy of solving an equation), Pólya seems to view this largely as a matter of perseverance.

4. Finally, one reviews what one has done and reflects on what went well and what didn't. This encourages any lessons learned from the problem at hand to be more available in future problem-solving situations where those techniques or insights might apply.

However, even if Pólya's descriptions might resonate with mathematicians' experiences, they don't seem to be a helpful medium for *learning* problem-solving Livingston (1999); Schoenfeld (1987). Alan Schoenfeld relates a relevant personal experience:

> Excited by my readings [about Pólya] in the early 1970s, I sought out some problem-solving experts, mathematics faculty who coached students for the Putnam exam or for various Olympiads. Their verdict was unanimous and unequivocal: Pólya was of no use for budding young problem-solvers. Students don't learn to solve problems by reading Pólya's books, they said. In their experience, students learned to solve problems by (starting with raw talent and) solving lots of problems. (Schoenfeld, 1987)

Out of a desire to work out what *can* help students become better problem-solvers, Schoenfeld (1985) developed a taxonomy of four key components of problem-solving ability:

- *Resources*, which are mathematical understandings that a problem-solver has available to them. For instance, familiarity with the rules for manipulating algebraic symbols would be a resource. Resources can be formal (such as knowing the ε-δ definition of continuity) or informal (such as knowing that a continuous function won't send two close-together points to radically different locations in the range).

- *Heuristics*, which are strategies for problem-solving such as working backwards from the solution or drawing a picture.

- *Control*, which is the degree to which a problem-solver is able to call upon resources. For instance, a student might know how to do integration by parts, but whether they notice that they can use this resource on a specific problem is a matter of what Schoenfeld calls control.

- *Beliefs* about mathematics and what it means to practice this. For instance, Schoenfeld suggests that students often have a belief that math problems should be relatively quick to solve and thus persistence isn't seen as a mathematical problem-solving virtue.

This framework shaped a great deal of the subsequent research on problem-solving (Carlson & Bloom, 2005; Lester, 1994). Viewing mathematicians as "expert problem-solvers," it can seem natural to ask what resources, heuristics, methods of control, and beliefs they have. Numerous researchers expand this to include affect and metacognition, often associating affect with beliefs and metacognition with control (e.g., Carlson, 1999b; Carlson & Bloom, 2005; Debellis & Goldin, 2006; Schoenfeld, 1992). I should note that this is significantly different from the theoretical perspective of the present study, as I'll explain in section 2.2. However, because of the history of this topic, it's helpful to describe the literature on mathematicians' problem-solving skills using this language and general theoretical framework.

Unsurprisingly, mathematicians display a rich and complex array of resources. They seem to be able and willing to think about the structure of mathematical objects rather than just their definitions (Sinclaire & Talbaghi, 2010; Weber & Alcock, 2004; Wilkerson-Jerde & Wilensky, 2011). For instance, in contrast to undergraduate math students, mathematicians are more likely to attend to properties such as that the set $\mathbb{Z}$ of integers has a single generator as a group while the set $\mathbb{Q}$ of rational numbers does not.[1]

We also find that mathematicians seem to involve metacognition deeply in their practice (Carlson, 1999a,b; Carlson & Bloom, 2005; Schoenfeld, 1992; Stylianou, 2002). Carlson & Bloom (2005) summarize many of these effects in their Problem-Solving cycle (Figure 2.1), which they derived from coding the efforts of mathematicians to solve several unusual problems related to calculus, geometry, and combinatorics. Once a mathematician had gotten an overall sense of what the problem was asking for (Orienting), they would plan out a possible approach for moving forward (possibly on a sub-goal), execute their plan, and then evaluate how

[1] There's an overview of what "group" and "generator" mean in this context in section 5.3.1 on page 145.

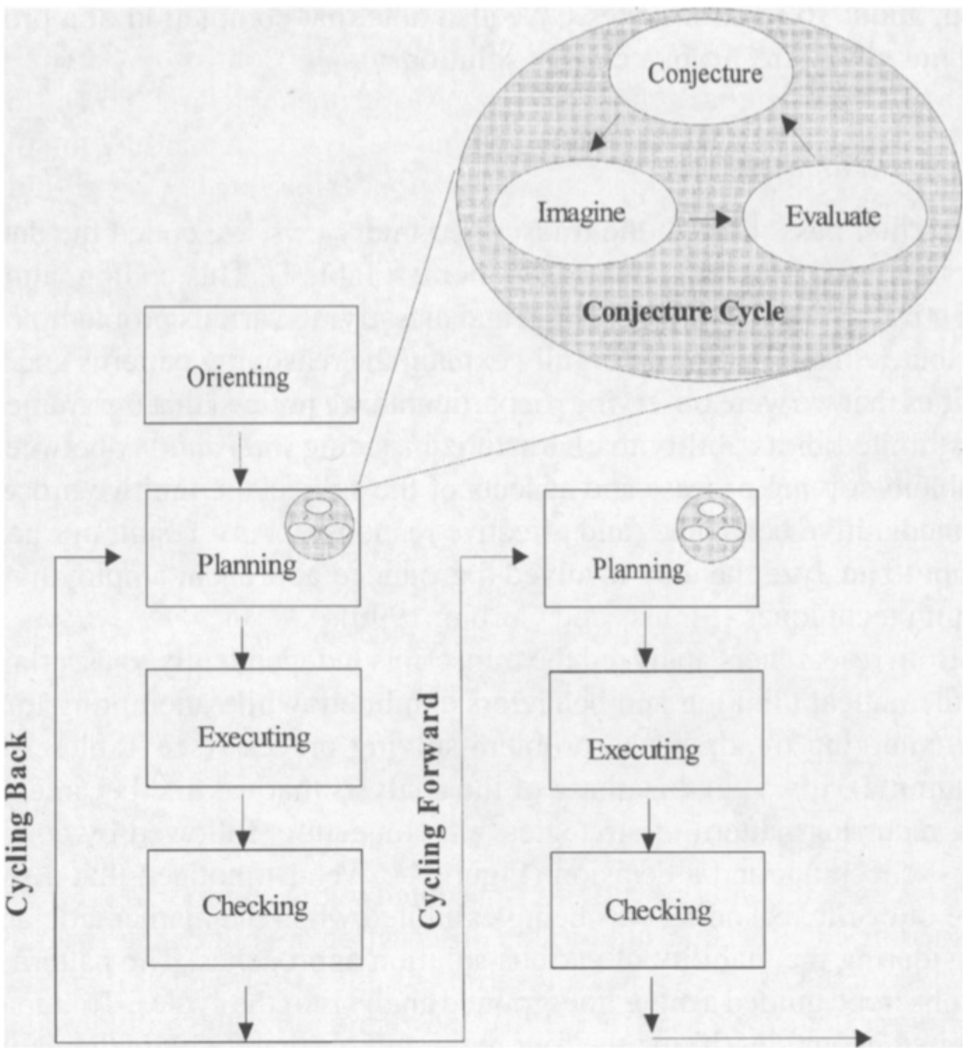

Figure 2.1: Carlson & Bloom's (2005) Problem-Solving Cycle

effective it had been at achieving their original goal. If it proved inadequate, they would cycle back and start planning anew. If it *was* successful, they would cycle *forward* to start planning the next stage. Within the "planning" portion there was a sub-cycle in which they'd form a conjecture, imagine cases in which they thought their conjecture would hold, and verified whether it worked, producing further conjectures to help formulate their overall approach in solving the problem at hand. Throughout this process Carlson & Bloom report that the mathematicians would show signs of continual self-monitoring, reevaluating whether their current efforts were moving them forward in a way that was likely to be fruitful.

Examinations of mathematicians' affect (relating to Schoenfeld's (1985) dimension of "belief") as they engage in problem-solving also prove illuminating. Far from a cold, passionless analysis of the truth or falsity of statements, we find that mathematicians express a range of emotional and aesthetic attitudes toward their problem-solving endeavors (Carlson, 1999a,b; Carlson & Bloom, 2005; Debellis & Goldin, 2006; Hemmi, 2010). They express frustration during times of struggle, confidence when asserting something sure and familiar to them, confusion when something they thought would work behaved unexpectedly, and relief and delight when making a breakthrough.

Carlson & Bloom's sub-cycle of conjecture-imagine-verify bespeaks a perhaps surprising finding noted elsewhere: mathematicians seem to employ heuristics of *inductive* reasoning (Hadamard, 1949; Lakatos, 1976; Weber, 2008; Wilkerson-Jerde & Wilensky, 2011). For instance, one of Weber's (2008) subjects was presented with the assertion that n is not a perfect square given that $n \equiv 3 \pmod 4$. He responded like so: "I'm using examples to see what, where the proof is coming from. So 5^2 is 25 and that's 1 mod 4, 36 is 0 mod 4, 49 is 1 mod 4, 64 is 0 mod 4. I'm thinking that, ah! So it is. . . 24 times 24, that's 0 mod 4. So a perfect square has to be 1 mod 4, doesn't it? n^2 equals 1 mod 4 or 0 mod 4. Alright" (p. 443).

As Weber points out, this is what Harel & Sowder (1998) refer to as an *empirical proof scheme*, and in classrooms it's typically considered to be a flawed approach to producing or verifying proofs, along with authoritarian proof schemes (i.e., those based on reference to who made the claim rather than how the claim was made) — which we *also* find mathematicians using at times. It seems as though mathematicians are sometimes satisfied with an assertion as long as it seems *plausible* to them and/or comes from a credible source rather than because they necessarily have a full proof in mind.

It should be mentioned that after about 1990, the research on problem-solving moved away from the heuristics-based model Schoenfeld (1985) had initially described. Heuristics and beliefs were recognized as intertwined, and Schoenfeld (1992) added a fifth category of "practice" due to the work that had been done in the years since he had first proposed the framework. Today the focus seems to have shifted toward teaching through problem-solving and placing the emphasis on "modeling" (Lester & Kehle, 2003; Lester & Lambdin, 2004). However, the considerable majority of the work done on exploring the problem-solving behavior of *mathematicians* occurred within the earlier theoretical context. Therefore, that more recent literature will not be reviewed for the present study.

2.1.2 Mathematical Culture

The Euler characteristic

The brilliant mathematician Leonhard Euler observed an initially surprising pattern about polyhedra: if one were to count up the number f of faces, the number e of edges, and the number v of vertices (i.e., corners), then:

$$v - e + f = 2$$

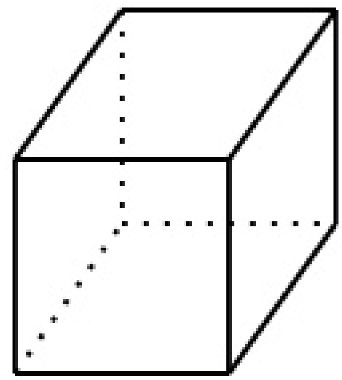

Figure 2.2: A cube

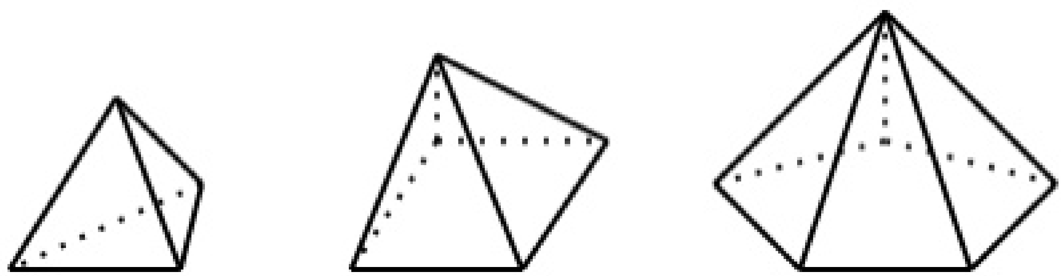

Figure 2.3: Pyramids — a triangular, square, and pentagonal pyramid, respectively

For instance, on a cube (Figure 2.2), there are six faces, twelve edges (four for the top square, four for the bottom square, and four lateral edges connecting the two squares), and eight vertices (four for each of the top and bottom squares) — and we have $6 - 12 + 8 = 2$. We can sort of see where this characteristic comes from in highly regular shapes like pyramids (Figure 2.3): by going counterclockwise as viewed from above, we can match up each edge with either a face or a vertex, and after the pairing we'll be left with a total of two features uncorrelated with edges (a vertex at the top and a face at the bottom in the case of pyramids, the top and bottom faces in the case of the cube).

The mathematician Augustin Cauchy proposed a proof of this, which is

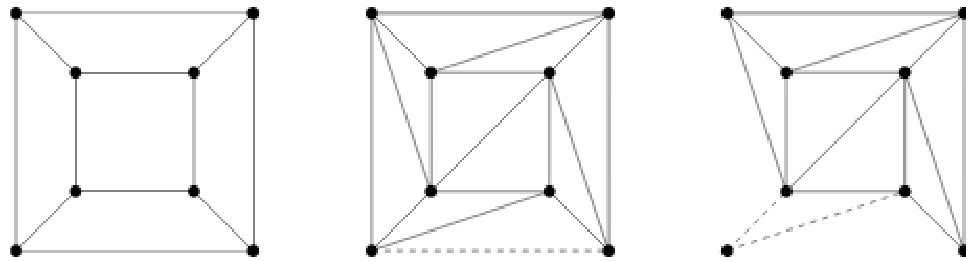

Figure 2.4: Cauchy's proof of the Euler characteristic for the cube (from http://tinyurl.com/cauchycube)

illustrated in the case of the cube in Figure 2.4. If we remove a face and flatten the shape out, we should end up with a collection of faces, vertices, and edges whose alternating sum is 1. We can freely add diagonal cuts across the faces remaining since every cut adds an edge but also adds a face, thus leaving the characteristic as 1. We can then remove one triangle from the periphery, which amounts to removing one edge and one face (again leaving the characteristic at 1). If we must remove *two* edges, then we also have to remove the vertex where those edges meet, again leaving the characteristic unchanged. Eventually we should be left with just one triangle, which has one face, three edges, and three vertices. We could, therefore, reverse this process to create the original polyhedron, demonstrating that its alternating sum must be 2.

The problem is that this turns out to be false. For instance, the alternating sum for the shape in Figure 2.5 turns out to be zero. This was reason for some concern, since there seemed to be a proof of a statement that turns out to be false. Lakatos (1976) refers to this phenomenon as *proof and refutation*, noting that part of mathematical discovery seems to involve coming across hidden assumptions.

A solution to this curiosity finally appeared when Henri Poincaré suggested generalizing the counting method. We can draw a collection of curvy triangles over a surface S in a particular way (e.g., if two triangles touch they must share

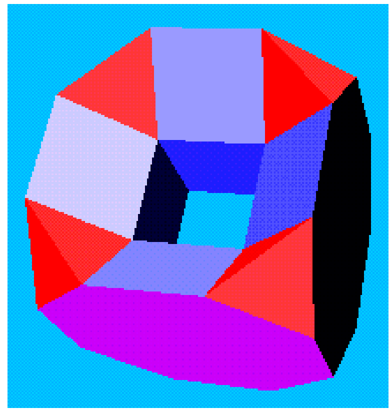

Figure 2.5: A polyhedron with Euler characteristic different than 2 (from http://tinyurl.com/toruslike)

vertices), and then we can define the (generalized) Euler characteristic $\chi(S)$ to be the alternating sum based on these triangles. This lets us talk about the Euler characteristic of a sphere, for instance: we can think of the equator as having three vertices and three edges between them, and then the top and bottom hemispheres are the faces of two different (curved) triangles.[2] In this case we can see that the Euler characteristic of a sphere is 2.

Rather than asking why so many polyhedra have an Euler characteristic of 2, it becomes sensible to ask the question in reverse: for which spaces S is $\chi(S) = 2$? It turns out that any space you can smoothly deform into being a sphere will have this property. That is, if the shape were a balloon, if inflating it would make it a ball then its characteristic is 2. This is true in particular for all convex polyhedra, which is why Cauchy's proof seems to work on so many familiar polyhedra but fails for the one in Figure 2.5.

The central value gained here, however, was not just an explanation of when the Euler characteristic is 2. Asking the question "backwards" like this opened the

[2] A picture, unfortunately, isn't very helpful here: it will simply show three dots on the equator of a sphere to represent the three vertices of each triangle.

pathway for what later became a major branch of mathematics known as topology. To a first approximation, topology is the study of what one can do to alter a shape without changing given characteristics of that shape. Today topology is a subfield of a size comparable to abstract algebra and analysis (the formal extensions of elementary algebra and calculus, respectively).

The cultural role of proof

The story of the Euler characteristic bespeaks an often surprising pattern in the culture of mathematical research. It would seem that with relatively rare exceptions, mathematicians do not read proofs in order to determine the proofs' correctness. Rather, they seem to read them in order to learn new techniques or to gain insight into why the theorem is true (Burton, 2004; Hemmi, 2010; Rav, 2007; Thurston, 1994; Weber & Mejia-Ramos, 2011). Several of the mathematicians in Weber & Mejia-Ramos's (2011) study mentioned that there simply isn't enough time for them to verify proofs line by line, and that frequently when reviewing papers submitted to academic journals of math they would examine the paper more based on whether the content seemed interesting than on whether each claim followed from those prior to it. "[Mathematicians] are usually not very good in checking formal correctness of proofs, but they are quite good at detecting potential weaknesses or flaws in proofs" (Thurston, 1994, p. 9).

Accenting this, it would seem that there's something of a discrepancy between the formal foundations of mathematics and mathematics-as-practiced (Fisher, 1973; Kleiner, 1991; Livingston, 1986; Steingart, 2012; Thurston, 1994). For instance, Gödel's Incompleteness Theorem throws into question the feasibility of ever having a consistent foundation for mathematical reasoning — and yet we find mathematicians essentially unmoved by this in practice. In practice, mathematicians seem to want to explore possibility and develop models and infrastructure to help them understand mathematical phenomena. The shaky foundations of their discipline seem to be an amusing afterthought rather than something that bears on how they think or what they do.

Occasionally some particular theorem looms large in the cultural awareness of the mathematics community, and at such times we can see what it is about proof that interests mathematicians as a collective. Andrew Weil's proof of Fermat's Last Theorem[3] received more scrutiny than most other theorems due to the history of the conjecture (Boyer, 1968; Kleiner, 1991), but the central value that the community found in Weil's contribution was the collection of methods he invented rather than a demonstration of this relatively isolated fact (Aczel, 1996; Thurston, 1994). By way of contrast, as of the 1970's the famed Poincaré Conjecture[4] had produced a flurry of intense work and was highly valued for its own sake, but it was considered an unfruitful ("unsuccessful"; Fisher, 1973, p. 1104) problem because the work that had been put into it failed to produce much in the way of interesting mathematics (Fisher, 1973). (This changed later: in the 1980's the idea of Ricci flow was introduced in part as a way of proving the conjecture, which finally happened in 2006.) We also find that sufficiently rich and successful theorems can also have the effect of *diminishing* subfields, as happened with the proof and subsequent compiling of the Classification Theorem in finite group theory[5] (Steingart, 2012) and with Thurston's (1994) work in a domain known as the "theory of foliations." Time and time again, we see that the communal interest in theorems seems to be driven by the *promise of techniques and insights to apply*

[3]In short, the theorem states that there are no triples of integers (x, y, z) for which $x^n + y^n = z^n$ for any integer $n > 2$. That is, although there are Pythagorean triples like $(3, 4, 5)$ (as $3^2 + 4^2 = 5^2$), there are no "super-Pythagorean" triples.

[4]Roughly speaking, this says that anything that topologically looks roughly like a 3-dimensional sphere in 4-space is, in fact, a 3-sphere (i.e., the set of solutions to $x^2+y^2+z^2+w^2 = 1$ viewed geometrically).

[5]To a first approximation, this theorem describes the complete set of the analogs to prime numbers in finite group theory. See section 5.3.1 on page 145 for a brief overview of the idea of groups.

elsewhere.

Part of *becoming* a mathematician, then, requires enculturation into these perspectives on proof (Burton, 2004; Geraniou, 2010; Hemmi, 2006). Math majors and graduate students have to wrestle with a seemingly paradoxical and implicit idea of what constitutes a proof and what the social and intellectual roles of proof are. Furthermore, different subfields of math have different standards for proof and express their ideas with different language and intellectual infrastructure, which can make communication across sub-disciplines challenging (Beswick et al., 2010; Burton, 2004; Thurston, 1994). Just as with joining any community of practice (Lave & Wenger, 1991), becoming a mathematician requires gaining familiarity with the nuances of *actual* practice, which can differ (sometimes dramatically) from what someone from outside the community might think the practices are.

Dovetailing the observations of how mathematicians engage in problem-solving, mathematicians also report powerful affective elements in their practice (Burton, 2004; Davis & Hersh, 1981; Devlin, 2000; Hadamard, 1949; Halmos, 1985; Poincaré, 1952). It's common for them to refer to proofs with language such as "beautiful," "elegant," or "ugly," and some of the more common criteria for judging the value of a published proof or area of study is whether it's "significant," "interesting," or "exciting" (or, on the negative side, "trivial"). Euphoria is a fairly common way of describing the emotional experience that accompanies a breakthrough on a problem they've been struggling with for some time.

Communication within the community

It's worthwhile to note, though, that the formal mathematical literature seems to be largely devoid of these references to affect (Burton, 2004; Burton & Morgan, 2000; Fisher, 1973; Rav, 2007). Feelings appear to be mixed on this matter: some mathematicians think of the rigor and formalism that the community expects as providing a helpful and productive constraint (much as writing down one's thoughts can help clarify them), while others lament the breakdown of communication and the effort it takes to figure out what the key ideas of a given paper are. One of Burton & Morgan's (2000) subjects articulates this last perspective fairly succinctly:

> I get annoyed with some of my collaborators and a lot of the papers I am sent, which are definition, theorem, lemma, proof. That seems to me to be appallingly bad. It is the sort of thing that no one is ever going to want to read. I think it is important to grab the reader from the opening sentence. Not "Let A be a class of algebras such that. . . ." Change it to "This paper opens a new chapter in duality theory." (Quoted in Burton & Morgan, 2000, p. 449)

A similar tone appears in colloquia, seminars, and other presentations-based environments, it would seem (Barany, 2010; Barany & MacKenzie, forthcoming; Burton, 2004; Greiffenhagen, 2008; Thurston, 1994). Many mathematicians seem to have a sincere desire to share their insights with others in the mathematical community. Yet there are considerable challenges to succeeding in this domain, such as the often unexpectedly large inferential distance between subfields and the tendency to present in more technical precision than necessary for what many audience members are looking for. Thurston (1994) describes this in a rather negative tone, giving the impression that virtually no one at mathematical colloquia get any benefit past the first five or so minutes of a presentation. Barany & MacKenzie (forthcoming), to offer some contrast, suggest that members of the audience "do not aim for a detailed working knowledge of the results being presented—this can take years to acquire (after which the talk would not have much to offer)—but rather comprehend the talk in the sense of *following the argument*, engaging with the talks conceptual narrative and technical and heuristic manipulations" (Barany & MacKenzie, forthcoming, p. 6, emphasis in the original).

Who is speaking or writing also plays a rather significant role. One of the reasons Hadamard (1949) conducted his exploration of the psychology of mathematical invention was because an earlier study that had explored that domain had neglected to reference anything about mathematicians' *failures*, which was something Hadamard felt that "only first-rate men would dare speak of" (p. 10). In a similar vein, Burton (2004) and Burton & Morgan (2000) noted that the statement "it is difficult to show that" takes on a very different character depending on whom the author is: a mathematician who has just recently received their Ph.D. is likely to be seen as showing their inexperience or lack of skill, whereas an established and respected member of the community with major contributions under their belt will tend to be perceived as acknowledging a difficulty that is *in the problem itself*.

Far from being an abstract description of truth, then, we find that communication in the field of professional mathematics is culturally mediated and nuanced — indeed, as we find in other communities of practice (Lave & Wenger, 1991).

Teaching attitudes & practices

When it comes to questions of teaching mathematics, it would seem that mathematicians show a complex — and at times paradoxical — range of attitudes, in terms of both stated beliefs and perspectives revealed through their actions. On one hand there's often a great deal of optimism about students' ability to get into the mathematics (Brown, 1993; Devlin, 2000; Hemmi, 2010; Nardi, 2007; Tsay et al., 2011; Weber, 2004). Oftentimes mathematicians in the role of college instructor will go to considerable lengths to clarify the "big ideas" of a proof, the rationale behind a method, or how they think about the behavior of some mathematical object. Although they seem to generally default to lecture-based teaching methods, their inclination is often to find a spin on the presentation style that will help to elucidate the ideas to the students more vividly. Getting students involved in asking and attempting to answer questions is often a pedagogical ideal and several mathematicians report having strategies for accomplishing this, although frequently with a sense of results being mixed (Burton, 2004; Nardi, 2007).

On the other hand, a certain sense of pointlessness to pedagogical methods sometimes arises (Fukawa-Connelly, 2012; Nardi, 2007; Thurston, 1994). At times, many mathematicians see students "learn anyway" (Nardi, 2007, p. 256) seemingly irrespective of the quality of teaching offered. It's as though sometimes it can seem that the ability to learn math is a trait of *individual students*, and the role of the professor is to model good mathematical reasoning so that those students who are willing and able to think about the material can have something to work with. However, I should be quick to add that this is certainly not a universal attitude, and many mathematicians who occasionally espouse such a view will also pour their effort and hope into clearer presentation styles and different classroom didactical patterns.

2.2 Theoretical Framework

In investigating the experiences of mathematicians, I employ analytic tools from phenomenology contextualized in a way that critically scrutinizes mind/body dualism. In chapter 3 I'll describe what those tools are and how I use them, but first I must explain their theoretical origin. I do so here.

2.2.1 Phenomenology

Temporality

In listening to Vivaldi's "Spring," I am struck with a brightness about the music. The strings create a rolling landscape of melody that seems to uplift and almost skip along. During a portion in the middle all but two the instruments pause, leaving the flittering sounds that remain to convey an impression that to me always calls to mind a pair of birds flying playfully together. Later the darker sounds of minor key convey the passage of a storm, and then a bright shift back to the growth and abundance of spring.

You and I might have different impressions of Vivaldi's piece. Perhaps you are a performing musician and can appreciate the technical details of how the music is performed much better than I can, or maybe you do not care for classical music. But one thing remains fairly certain: if we were to play the piece, you would probably not hear random isolated notes. You would hear *a melody*.

This deserves some close examination. When we hear someone plunk out the notes C, D, E on a piano, we hear one note at a time at any given moment. Yet our experience is not of three distinct notes without relation to one another; rather, there is a coherence to the experience that conveys the sense of a *progression* of notes. Upon the sounding of E, our sense of having heard C and D is still present with us in a way, shaping our experience of E. Furthermore, this isn't quite memory; remembering the first two notes of a melody is a fundamentally different kind of experience than is experiencing the third note right after the first two have just been played.

The philosopher Edmund Husserl referred to this phenomenon as *retention*, in contrast to recollection (Gallagher & Zahavi, 2008; Zahavi, 2003). Symmetric to retention is *protention*, which is an immediately felt anticipation of what is about to transpire. This is what provides room for surprise: we believe we know what is coming and then find our expectation confounded. It also gives us the ability to feel satisfaction in predictability, as when we delight in listening to a beloved piece of music and hear our expectations so precisely and pleasurably met. And much as retention is subtly but importantly different from memory, protention is distinct from imagination.

The point of such an analysis is to illustrate that our experience of temporal phenomena like music, speech, and dance cannot be a succession of moments like a string of beads. Instead, it seems that our experience is spread over an immediately felt past and an immediately anticipated future. As we examine our own experience we can become aware of this — but until it is pointed out to us or we think to reflect on it, it is an invisible feature of the structure of our experience rather than something to be considered and examined.

Husserl on psychologism and the foundations of science

This examination of temporality is a specific example of a result from a *phenomenological analysis*. Phenomenology is the study of phenomena, which is to say, experiences as they are presented. This is in contrast to reductionistic analyses, in which we might ask what *causes* retention and protention — e.g., what brain regions might be involved and how they might work together to present the impression of a coherent moment. It is the *impression itself* whose structure phenomenologists are interested in, regardless of how it might come into being "behind the scenes."

Phenomenology emerged in part out of Husserl's effort to address an idea known as *psychologism* (Husserl, 1901; Zahavi, 2003). Psychologism suggests that the rules of logic are the result of properties of psychology. For instance, modus ponens[6] is a valid rule of inference to us because our minds are structured in such a way as to find it compelling. Were our minds structured differently, we might find that we disagree with modus ponens in favor of some other rule.

Husserl's (1901) concern, in short, is that if logic is a subset of psychology, then with what shall we deduce the properties of psychology? This apparent incoherence in psychologism, Husserl argues, suggests that we cannot take a sub-

[6] This is the logical rule saying that if we know "A implies B" and we also know "A," then we may conclude "B."

jectivist approach to logic and deduction. Instead, we must justify the foundation of empirical sciences in the nature of the medium through which we are capable of empiricism in the first place: *conscious experience.*

Yet if we are to understand the sciences as emerging from experience, it becomes necessary to understand the structure of that experience (Gallagher & Zahavi, 2008; Zahavi, 2003). The structure and character of vision becomes key to reading measurements and observing experimental results, the fabric by which our sense of something being "an object" gives us insight into what it means to observe some thing at all, and the nature of temporality frames the way in which we're able to make sense of cause-and-effect, to name just a few examples.

Other-directed phenomenology

I should emphasize that, while phenomenology *can* focus primarily on the phenomenologist's experience, it needn't be so restricted. Max Scheler points out that our impressions of other people's emotional states are immediate. We do not recognize that someone is sad, for instance, because we watch their downward gaze, slouched body posture, and slight downward turn of the corner of the lips and from this deduce that they are sad. Rather, we see them and are struck with an impression of sadness, which is often mirrored in ourselves to some degree as part of our awareness of their emotional state. It is *after* this impression arises that we can consciously reflect on their body posture, tone of voice, and so on to work out how the impression might have come in the first place.

Just as in the case of examining our own experiences, there are always possible "errors" that we sometimes realize by further examination of their context. What we thought was sadness might turn out to be the other simply being lost in thought, and there is always the possibility for deception. Yet just as often we can be aware of another's emotional reactions before they are, such as when a friend shows us concern when we didn't consciously realize that we were bothered.

We saw this possibility for other-directed phenomenology in Merleau-Ponty's (1962) analysis of Schneider (section 1.2.1). We also see it in Noble et al.'s (2006) exploration of students' phenomenology of tool-use: by carefully examining students' behavior with an eye toward the structure of their experience, Noble et al. were able to illuminate how the tool, their use of the tool, and the "lived-in space" intertwined to define the way in which the process of drawing a circle unfolded for them.

2.2.2 Embodiment and Cartesian Dualism

One potential source for complication here is in implicitly viewing subjects' behavior as offering hints into their hidden, private mental processes. Much of the literature on gesture, for instance, talks about gesture as a window into the mind (Alibali et al., 1999; Alibali & Goldin-Meadow, 1993; Church & Goldin-Meadow, 1986; Goldin-Meadow, 2003; Hostetter et al., 2007; LeBaron & Koschmann, 2003; McNeill, 1992, 2005; Stone et al., 1991). In other words, mental and physical processes are being viewed as two fundamentally different kinds of activity, and when observing others we can access the former only by careful scrutiny of the latter — and even then, the strength of our conclusions depends on the strength of the correlation between mind and body.

However, there is reason to question this division. For instance, my ability to recognize a pencil as such comes bundled with it a whole realm of possibilities: I could walk around it and look at it from various angles, lift it and move it around, write with it, roll it across the table, and so on. Yet when I recognize the ability to snap it in half and use it as kindling for a fire, something subtle shifts in what that object is for me. Research on patients with lesions in the prefrontal cortex — the region of the brain primarily responsible for impulse control — suggests that this perception of affordance accompanies a neurological impulse to physically initiate the perceived-as-possible task (walking around the object, picking it up, etc.) (Aron et al., 2007). In other words, it seems as though we cannot separate our recognition of, or conception of, an object like a pencil from our body-felt sense of how we could interact with it and how we anticipate those interactions will affect what we experience.

In light of this, I join with others in rejecting the Cartesian mind/body dualism altogether (Gallagher & Zahavi, 2008; Nemirovsky & Ferrara, 2009; Nemirovsky et al., forthcoming; Noble et al., 2006; Roth & Thom, 2009; Thom & Roth, 2011). For instance, in observing a student solving a word problem in algebra by writing down a polynomial, I do not assume that the writing exhibits their thoughts; rather, their physical interactions with the algebraic symbols shape their experience of the mathematics. Were we to remove their use of the polynomial, their encounter with the problem would take on a different character. Furthermore, the inscriptions that they experience as "a polynomial equation" have their meaning *because of the interaction*, not because of any property of the inscriptions themselves. As in the pencil example, we cannot separate physical action, perception of affordance, and cognition in the student. They are all part of an intertwined whole.

Because this perspective intertwines cognition with physical action, it is sometimes referred to as "embodied cognition." However, we must be cautious with this term for two reasons. First, the phrase "embodied cognition" can seem to imply that there might be such a thing as *non*-embodied cognition. It's important to realize that this perspective is embodied cognition in the sense of viewing *all* cognition as embodied, which is to say, intertwined with actual or possible perceptuo-motor activity (Nemirovsky et al., forthcoming; Nemirovsky & Smith, 2011).

Second, there are many theoretical and empirical works that can fall under the umbrella of “embodied cognition” (Barsalou, 1999; Hutchins, 1999; Wilson, 2002). For instance, one account of abstract reasoning (including mathematical thinking) that comes out of cognitive science proposes that we apply conceptual metaphors or conceptual blends in order to build the abstractions from more basic components (Gibbs et al., 2004; Lakoff & Johnson, 2003; Lakoff & Núñez, 2000; Núñez, 2008; Núñez et al., 1999). As a specific example, Lakoff & Núñez (2000) suggest that complex numbers might emerge by combining the idea of the real number line (constructed from previous conceptual metaphors) with the sense of rotation involved in multiplying by -1, so that “half-multiplying” by -1 (i.e., multiplying by $\sqrt{-1}$) would result in rotating half-way from positive to negative on the real number line. In other words, they suggest that the fact that multiplication by i rotates numbers by 90° in the complex plane might be how the complex plane is *cognitively constructed* (in terms of conceptual metaphor) rather than being a result of its definition.

This theoretical perspective has methodological implications — but those will be explored in the next chapter.

2.3 Situating the Research Question

Recall the research question, first expressed on page 12: *What are some ways in which mathematicians structure their experiences of struggle while working in pairs in person on a current problem?* With the literature outlined in this chapter, there are three points to focus upon in this question:

- The structuring of experience
- The process of working in pairs in person
- Their working on a current problem

The structuring of experience refers to a phenomenological examination. The work done on mathematicians' problem-solving skills has typically focused on what mental constructs or strategies mathematicians bring to bear on problems. While this is valuable work and has given us a number of insights into mathematics as practiced at the frontier of mathematical knowledge, the implicit focus on mental structures has left the domain of *how struggle is experienced* largely unexplored. This is precisely the sort of domain for which phenomenology seems to be best suited.

The reason for watching mathematicians working in pairs in person is to encourage the mathematicians to enact some of the physical behaviors that help to constitute their sense of what the mathematical ideas are. Many mathematicians report working on mathematics while doing solo activities that to an outsider would seem totally mundane, such as taking a walk or driving (Burton, 2004; Hadamard, 1949; Poincaré, 1952). Such activities wouldn't afford us much to work with. But if we can watch mathematicians working under the constraint of needing to articulate their thinking with a mathematical partner, we significantly increase the likelihood of gaining insight into how they're thinking about the mathematics on a moment-to-moment basis.

Finally, we want to examine them working on a current problem because this is where the literature mentioned earlier in this chapter suggests we should see the kind of struggle that mathematicians engage in while working out new mathematics. While we could have chosen to look at mathematicians engaging in prepared problems as Carlson & Bloom (2005) do, doing so would distance us from our present aim of exploring struggle *as mathematicians experience it in the course of their own research.*

Chapter 3

Methodology

Because of the theoretical framework described in section 2.2, the methodology the present study calls for is different in some key ways from what one normally finds in math education research. In this chapter, then, I'll start by contextualizing my study with an overview of methodologies used to investigate mathematicians' practices. I'll then describe the literature for methods most closely related to those used in this dissertation. With that background I'll finally detail what was done to collect, process, and analyze the data to answer the research question.

3.1 Prior Methodological Trends

In terms of the methodologies used to investigate mathematicians' practices and culture described in chapter 2, the earliest and most abundant material comes from the reflections of the mathematicians themselves, whether self-prompted (Brown, 1993; Davis & Hersh, 1981; Devlin, 2000; Hadamard, 1949; Halmos, 1985; Poincaré, 1952; Pólya, 1945; Schoenfeld, 1992; Thurston, 1994) or based on researchers requesting such reflection from them (Burton, 2004; Edwards, 2010; Hadamard, 1949; Hemmi, 2006; Liljedahl, 2004; Sinclaire & Talbaghi, 2010). These range from Thurston's (1994) reflections on his experiences to Hadamard's

(1949) synthesis of his and his colleague's encounters with bursts of mathematical insight to Burton's (2004) exploration of trends resulting from interviews of numerous mathematicians.

Another body of literature explores the ways in which mathematicians solve challenges during an interview. Many of these provide the mathematician in question with a problem, paper, or argument from a generally accessible domain like calculus or group theory (Carlson, 1999a; Carlson & Bloom, 2005; DeFranco, 1996; Edwards, 2010; Stylianou, 2002; Weber, 2008; Weber & Alcock, 2004; Wilkerson-Jerde & Wilensky, 2011). The most common methodology employed in such studies is a think-aloud protocol (Ericsson & Simon, 1993), although some studies such as Edwards (2010) and Sinclaire & Talbaghi (2010) instead examine mathematicians' gestures as they explain their thinking.

A few studies emphasize the examination of mathematicians' published writings (Burton & Morgan, 2000; Fisher, 1973; Lakoff & Núñez, 2000; Livingston, 1986, 2006; Rav, 2007; Steingart, 2012). By and large, these are attempts to explore the way mathematical ideas and practices actually appear rather than working from schemata of "public" mathematics. The exceptions tend to put emphasis on social characteristics of publishing practice (Burton & Morgan, 2000; Steingart, 2012) or investigate formal mathematics in order to better understand the nature of mathematical cognition (Lakoff & Núñez, 2000).

Finally, a collection of works emerge from observations of mathematicians "in action" such as while teaching or giving presentations (Barany, 2010; Barany & MacKenzie, forthcoming; Fukawa-Connelly, 2012; Greiffenhagen, 2008; Greiffenhagen & Sharrock, 2011; Nemirovsky & Smith, forthcoming, 2011; Núñez, 2006; Tsay et al., 2011; Weber, 2004). These tend either to focus on embodied (often multimodal) phenomena such as gesture (Barany, 2010; Barany & MacKenzie, forthcoming; Greiffenhagen, 2008; Greiffenhagen & Sharrock, 2011; Nemirovsky &

Smith, forthcoming, 2011; Núñez, 2006) or to attend to didactical moves (Fukawa-Connelly, 2012; Tsay et al., 2011; Weber, 2004).

The study that comes closest to the present one in terms of methodology is Greiffenhagen & Sharrock (2011), who in their second case study examine the interplay between a Ph.D. student and his advisor. However, the doctoral student there had not yet determined his dissertation topic; instead, he was attempting to decide a dissertation topic with his advisor's guidance. The present study, by contrast, examines mathematicians working on problems at the frontier of mathematical understanding. Also, Greiffenhagen & Sharrock were focusing on the role of apparent fallibility or infallibility in both the creation ("back") and presentation ("front") of mathematics, rather than how their subjects were phenomenologically structuring their encounters with struggle.

The literature appears to lack an enthography of mathematicians working on problems that are current to the field. The present study offers an opening into this area of investigation.

3.2 Methodological Background

3.2.1 Phenomenological Methods

As explained in section 2.2.1, the aim of a phenomenological investigation is to recognize and capture (usually via description) the structure of a given experience "as-is." For instance, rather than asking how our ears can pick up the sound of music or how our brains recognize a melody, a phenomenologist would want to explore how it is that we structure our experience such that we hear a *melody* rather than just a sequence of notes.

The central challenge to phenomenology is in examining and articulating the structure of experience rather than simply writing down speculations. It's not enough, for instance, simply to jot down impressions and free associations while listening to Beethoven's symphonies. Doing so might be interesting, but it's unlikely to capture the *structure* of the experience.

Gallagher & Zahavi (2008) outline one solution to this, capturing elements that appear in most phenomenological methods based on Husserl's ideas (Husserl, 1901; Palmer, 1994; Zahavi, 2003):

- Performing an epoché (or equivalently, "bracketing" the topic in question)
- Phenomenological reduction
- Eidedic variation
- Intersubjective corroboration

The epoché is a shift away from presumptions about the experience. For instance, when we watch a bird in flight, we are usually tempted to assume that there really is a bird there and that this is why we see it. Bracketing this experience doesn't deny that there is a bird there, but it doesn't affirm it either. It is an intentional shift away from mentally asserting how it is that the experience arises or what it means, instead choosing to remain open and curious about the way the experience is presented. In doing this, we become capable of recognizing that we see certain colors and shapes, that there is also an impression of movement, and that these visual sensations together with the context immediately present us with the impression of a bird.

It should be emphasized that this is not a matter of introspecting on our impressions of our experiences. We do not ask ourselves how we think our impressions of the bird arise. The attention is directed at the experience itself. Were we to turn our attention instead to our reactions to the experience, the reactions would then become the experience that we would want to bracket and examine.

For instance, in watching the flying bird, we might feel a sense of freedom combined with a subtle longing. In bracketing the experience, we might note that the longing has a drawing-in character, but also a physically felt sense of being unable to comply with that draw. This is dramatically different from an introspective account, which might explore questions like why the experience arises in the first place or what thoughts or feelings we might associate with it.

> For example, how do we go about describing the experiential difference between tasting wine and tasting water, between hearing a foghorn and seeing the full moon, or between affirming and denying that the Eiffel Tower is taller than the Empire State Building? Do we do so by severing our intentional link with the world and by turning some spectral gaze inwards? No, rather, we discover these differences, and we analyse them descriptively by paying attention to how worldly objects and states of affairs appear to us. (Gallagher & Zahavi, 2008, p. 23)

It's worth noting that there are two steps in this process. The first is a relinquishment of presumptions about what a given experience means or how it arises. This is the epoché. The second is an examination within the mental framework of an epoché; it is here, for instance, that we note that our experience of the bird's flight has characteristics that distinguish it from an experience of, say, lifting a heavy box. This second piece is the *phenomenological reduction*.

Moustakas (1994) describes the phenomenological reduction in somewhat more detail than do Gallagher & Zahavi. In doing so he introduces a rigidity to the reduction that not all phenomenologists agree with, but it is still illustrative to consider his take on the topic as a way of better understanding at least one way in which it might work. Moustakas's approach includes three components of the reduction: horizontalization, delimiting horizons, and textural qualities.

Horizontalization is a natural extension of bracketing in which we view every component of the experience as having equal weight or value. For instance, in watching the flying bird, we do not place greater importance on any of the sight of the bird's movements, our anticipations about how the bird is about to move, or the slight feeling of effort and awkwardness in our neck that comes from looking upward. Different components of the experience might be more *salient*, but salience is considered an aspect of the experience to be examined rather than an indication of importance. Indeed, the very question of “importance” vanishes in a phenomenological examination, except to the extent that the sense of importance is itself subject to exploration.

As the bird swoops down toward a tree branch, we might feel some anticipation that the bird might change its movement and land. We probably would not anticipate that it would turn into an anvil and break the branch. This is something of an extreme illustration, but it highlights Moustakas's second point about the reduction: experiences have natural domains within which our examination is likely to apply. Husserl made reference to this via the term *“horizon”* (Husserl, 1901; Zahavi, 2003). This is intended via analogy: when looking over a landscape, we can clearly tell what is within it and can tell that there is a horizon dividing what we can see from that which is surely beyond it. Yet we would be hard-pressed to draw a pencil-thin line on the terrain where the horizon actually *is*. In a similar way, although there's a clear sense of “within” and also of “without” with respect to a phenomenon (e.g., we know what it's like to watch a bird in flight and can tell it's different from other experiences like eating a strawberry), we generally cannot define a clean and clear boundary. Yet there *is* a horizon, and part of the reduction involves outlining the extent of the “terrain” being examined even if we cannot always say exactly where one type of experience ends and another begins.

Moustakas's third component of the phenomenological reduction involves examining the “textural qualities” of the experience and articulating them. For instance, as I write this I can reach down and touch the surface of the table my keyboard is resting on. I notice that when I lightly brush it, my finger slides easily, with a hint of pulling as I move my finger parallel to the surface. There's a slight vibration as I do this, which causes a feeling similar to tickling to reach into the pads of my finger. When I press a little harder, the subtle stickiness dominates and there's a considerable increase in the effort I have to place in pulling my fingers along the surface of the table. The tickle-like feeling is almost entirely gone, except for a highly dimmed version around the edge of the greatest pressure. The transition is rather brisk: it seems to take very little increase in pressure to create a strong prominence of stickiness over smoothness.

In doing this, I continually bracket the experience, repeatedly setting aside thoughts such as the physics of where the vibration comes from or how my nerves cause the ticklish sensation to seem to reach inside my fingers. Instead, I reorient myself on the experience itself, and reorient and re-open as many time as needed. In this particular example I gave relatively little evidence of horizontalization or delimiting horizons (which a more in-depth analysis might have done), but I did attend to the matter of textural qualities: how it felt, what qualities presented themselves to me, how the qualities seemed to relate to one another, etc.

Gallagher & Zahavi's (2008) third component of phenomenological methods — *eidedic variation* — corresponds to what Moustakas (1994) refers to as "imaginative variation." The central idea is to vary the experience to see what keeps it "the same" (and in what ways) and what causes it to appear to change. For instance, would a melody retain its melodic nature if it were to lose its affective response and become just sounds? What if pitches seemed to go left/right instead of up/down? What if the pitches didn't seem to have a directionality at all?

This approach — varying the experience with imagination — emerged from Husserl's idea of experiences as having essences (Zahavi, 2003). The idea is that there is something core to the experience of a melody that makes it melodic, or something key to watching a flying bird that makes that experience what it is. By varying things that are attached to but not actually part of that essence, we can uncover this essence.

However, not all phenomenologists agree with this vision of Husserl's. Merleau-Ponty (1962), for instance, derived his idea of motor intentionality (explored in section 1.2.1) by examining the experiences of someone who appeared to lack it. He did not attempt to imaginatively vary the aspects that were *not* motor intentionality, but instead did practically the opposite as a way of bringing our attention to a facet of our daily experience that we typically are not aware of. We also see Nemirovsky (2005) illustrate the difference between a location (e.g., "9110 Judicial Drive") and a place ("my home"), but in the spirit of exploring how places become familiar to us and what that familiarity is experienced as rather than in an attempt to articulate some essence of place-hood. Neither Merleau-Ponty nor Nemirovsky are attempting to extract an "essence," but are instead drawing our attention to aspects of experience that we seem to all share.

However, the different approaches to phenomenology do appear to share an interest in exploring the *structure* of experience rather than simply giving reports of individuals' perspectives.

> ... phenomenology is not interested in understanding the world according to Gallagher, or the world according to Zahavi, or the world according to you; it's interested in understanding *how it is possible* for *anyone* to experience a world. (Gallagher & Zahavi, 2008, p. 26, emphasis in the original)

The fourth and final component of phenomenological methodology that Gallagher & Zahavi reference is *intersubjective corroboration*. The idea here is that because we can communicate with others about our phenomenological observations, we needn't let the first-person focus of phenomenology totally isolate us in our exploration of experience. Both you and I could explore the textural differences between, say, visualizing a tiger and remembering an image of a tiger seen in a video; and, having done so, we can compare our notes to explore the similarities and differences that appear to be between our encounters with these phenomena.

Similarly, because phenomenology aims to articulate the structure of conscious experience as-is, we should expect that a sufficiently rich account is one that others will relate to, illuminating some nuance of their own phenomena they had perhaps not consciously reflected on before. We find this in Merleau-Ponty's account of motor intentionality, for instance: although we have perhaps always experienced a difference between artificial and instrumental movements, many people do not notice this aspect of their experiences until it is illustrated, such as in the case of Schneider.

As with eidedic variation, not all phenomenologists emphasize the importance of intersubjective corroboration. For instance, Moustakas (1994) makes no such reference in his book on phenomenological methods. Max Scheler (2007) argues that we actually do not have the privileged access to our own experiences that we seem to, implying that intersubjective corroboration is unnecessary. However, phenomenology is culturally relevant in part because of the belief that it is possible to communicate phenomenological insights to others. Even if this is not taken as a component of the *methodology*, it would appear that all phenomenologists hold regard for the *possibility* of corroborating their observations with others.

At this juncture I should reiterate a point made earlier (page 32 in section 2.2.1): phenomenological methods can apply to *others'* experiences, not just those of the phenomenologist. When approaching the experiences of others in a phenomenological manner, the challenge is to immerse ourselves in the other person's life, while at the same time being open to question assumptions that we take for granted in our own life. Merleau-Ponty did this to explore Schneider's experiences in detail, and Nemirovsky & Ferrara (2009) display a similar spirit in helping us to understand students' encounters with ideas in trigonometry. Using language introduced earlier, this process entails bracketing one's preconceptions of the other's and of one's own experiences. A rich and nuanced understanding of the other's experiences is something we pursue in everyday life with more or less success. The work of phenomenology is to enrich these common practices by lifting usual barriers, which take the form of stereotypes and pre-judgments about others and ourselves.

Connection with Research Question

Recall again the research question: *What are some ways in which mathematicians structure their experiences of struggle while working in pairs in person on a current problem?* The value of a phenomenological analysis is in its ability to capture and articulate the ways in which people seem to structure their experiences. Since the aim here is to explore the texture of mathematical struggle as members of a particular population experience it, phenomenological methods seem like a natural choice.

Notice in particular that this is distinct from asking what techniques mathematicians seem to use when they encounter struggle. Learning how it feels to hit difficulty in research mathematics, and what arises naturally in response to this difficulty, may give us insight into the experience of *being a mathematician*, which is potentially powerful due to its ability to transform our way of experiencing related phenomena. This might enrich our understanding of what it is like to implement different strategies, but a phenomenological analysis is unlikely to articulate those strategies for us.

The appropriateness, then, of this methodology in the present study is due to the focus on *the structure of experience*. The term "structure" here could use some more elaboration, however, which I'll address below in discussing multimodal microanalyses.

3.2.2 Multimodal Microanalysis

One last domain in the literature that's pertinent to the present methodology is the idea of a *multimodal microanalysis*. The term itself isn't universally used — for instance, Greiffenhagen (2008) refers to the process as a "video ethnography" — but the core ideas driving the methodological approach seem to be fairly stable across the different names.

In this context, a microanalysis is a careful examination of the moment-by-moment behavior of the subject. For instance, a microanalysis of someone explaining how they think about eigenvectors (cf. Sinclaire & Talbaghi, 2010) might examine the details of how their hands move on a timescale of seconds. Erickson (2004), referring to the process as a "microethnography," examines people's speaking cadences and places the transcript on a musical score in order to capture pitch, duration, volume, and so on.

Although some gesture analyses are examples of microanalyses (e.g., Edwards, 2010), those that attempt classification schemes often are not. For instance, McNeill (1992; 2005) suggests classifying gestures by whether they direct attention (deixis), emphasize speech (beat), represent by making hand shape and/or movement similar to the referent (icon), or refer to something by shaping and/or moving the hands in analogy to the referent (metaphor). Such categorization of gestures has zoomed out from a moment-by-moment examination of how each individual gesture occurs, seeking to blur the differences between hand movements in order to emphasize families of similarity. This can be incredibly useful, but methodologically it should be distinguished from a microanalysis.

The concept of multimodality dovetails with the phenomenological idea of horizontalization (Moustakas, 1994), although not all researchers who advocate multimodal approaches intend the connection (e.g., Thom & Roth, 2011). The idea is that instead of privileging speech (such as in analyses of transcripts), hand movements (as the research on gesture has traditionally tended to), or any other one semiotic modality (Roth & Thom, 2009) in isolation to others, we can gain a more holistic insight about the subject by honoring all such modalities as meaningful and deserving of our attention (Nemirovsky et al., forthcoming; Nemirovsky & Smith, 2011, forthcoming).

A multimodal microanalysis, then, is a zoomed-in examination of a subject's movements, speech, verbal cadence, eye gaze, body posture, tool use, and so on in a way that attends to how these many different factors are all connected. This results in an analysis that could potentially be not as in-depth as Erickson's (2004) placing a transcript on a musical score, but the attention to the person as a whole offers a means that seems perhaps more natural for gaining insight into the subject's structuring of experience.

Connection with Research Question

As was emphasized on page 46, a key aspect of the research question is the exploration of the *structure of experience* with regard to mathematical struggle. Part of what is meant by "structure" here is included in the phenomenological reduction: what are the characteristics of the experience of struggle in research mathematics as mathematicians encounter it? Yet as outlined in section 2.2.2, the perspective being adopted here views the body's seen-as-sensible actions as inextricable from the sense of meaning. That is to say, many aspects of what one physically does in terms of gesture, speech, eye gaze, and so on while engaging in mathematical reasoning are parts of the way in which one experiences that reasoning. In this spirit, then, we can talk about the seen-as-mathematically-sensible interactions that mathematicians have with one another and their environment while encountering mathematical struggle as *structuring* their experience.

However, in saying this we must be careful not to think that a catalog of physical behaviors such as a taxonomy of gestures would suffice. It's not the physical behavior on its own that matters, but rather the sense that such behavior is sensible in the context in question. It is therefore essential to outline not just what mathematicians *do* in order to understand how they structure their experiences of struggle, but in what sense what they do is meaningful.

A multimodal microanalysis allows us to carefully examine a wide array of behaviors the mathematicians in this study engage in while encountering difficulty in their research. This becomes the foundation upon which a phenomenological analysis in an embodied cognitive theoretical framework stands. This helps the phenomenological analysis to incorporate as many nuances of activity as we can pragmatically track.

3.3 Data Collection

3.3.1 Participant Selection

The participants in the present study are pairs of mathematicians at a large southwestern research university in the United States.The choice to observed pairs was informed by the need to encourage clear but naturally occurring displays of struggle. It was judged that ABD doctoral students working with their advisors would be a close proxy to observing mathematicians working together in terms of skill level, and additionally meetings between doctoral students and their advisors were also anticipated to be reasonably predictable and thus recordable.

Mathematicians were contacted for another study (cf. Nemirovsky & Smith, 2011, forthcoming) and were asked if they had ABD doctoral students whom they thought would find being observed in meetings acceptable. This resulted in the three pairs examined in this study:[1]

[1] Names given here are pseudonyms.

- Joseph is a topologist with interests in mathematical physics. Working with him is Bill, his doctoral student. The topics they discuss in the episodes examined here are in an effort to flesh out Bill's dissertation as he approaches his defense.
- Matt is an algebraic geometer who has contributed to string theory. His working partner is Bart, a German algebraic geometer with whom Matt had just finished a paper. Bart was visiting so that the two of them could choose a new problem and make progress on it in-person. (It was fortuitous that when Matt was approached, this visit of Bart's was already planned and imminent.)
- Fay is a graph theorist who specializes in probabilistic graphs. As with Joseph & Bill, Fay's graduate student Martha works with her on Martha's dissertation as it nears completion. The two of them were also in the process of writing a paper on Martha's dissertation material.

3.3.2 Session Observation

After coordinating with a given pair, I reserved a room with a blackboard at their mathematics department during a time when they were planning to meet. Two cameras were set up in the two corners of the room opposite the blackboard, pointed generally toward the center. This allowed for video capture of multiple angles of participants' behavior.

I asked the two participants to write on the blackboard rather than on paper when they wanted to write something for one another, but to otherwise treat the meeting like any other of theirs. The six participants reported generally not feeling disturbed by the shift in setting; indeed, Matt & Bart claimed that this was very much like the way that they always work anyway.

Each pair was observed three times this way. For Joseph & Bill, the first two sessions were a week apart and the third was a few months later. Matt & Bart met continually throughout one week while Bart visited, so all three sessions occurred that week (Monday, Wednesday, and Friday). Fay & Martha's observations occurred evenly over a period of two months.

3.3.3 Video Synthesis

Once the video data were collected, they were compressed and then merged between the two videos. The compression used the program "Handbrake" to convert from an MTS file format to an M4V one. The merger of the two videos in parallel used the program "QT Coffee" and synchronized the videos by matching sound and gesture patterns.[2]

The resulting MOV video shows the images from both cameras side-by-side with the two sound tracks overlapped. This way it's possible to watch the video in a stereoscopic way, catching nearly all the interactions with the blackboard even as the mathematicians stand in front of it to write on it. The overlapping sound tracks result in more stable sound: if camera 1 picks up a word that camera 2 barely registers due to the direction a mathematician is facing when they speak, the speech won't be relevantly more difficult to hear in the resulting video than if both cameras had picked up the sound.

This process resulted in nine videos total: 1 video per observation $\times$ 3 observations per pair $\times$ 3 pairs.

[2]If you listen to two merged videos that are almost, but not quite, synchronized, it sounds like the subjects are speaking in an echo chamber. QT Coffee allows you to make small tweaks to the merger parameters to practically eliminate this effect, yielding a close-to-perfect synchronization of the two videos.

3.4 Data Analysis

3.4.1 Developing Mathematical Background

In order to immerse myself in a phenomenological analysis, I had to be able to follow the mathematical content for each episode in enough detail that I could understand both the broad strokes of the pair's efforts throughout the episode as well as the logic of key mathematical choices at specific times. This required me to build a significant amount of mathematical background. This was approachable in part because I needed to develop only enough background to *follow* the mathematics rather than be able to *participate* in it — something like how it's usually easier to learn how to read a foreign language than it is to learn how to write in it.

Prior to the observations, I had performed an informal interview (Bernard, 1988) with one member from each of the pairs (Joseph, Matt, and Fay) about a paper they had published in order to collect data for other studies (Nemirovsky & Smith, 2011, forthcoming). These interviews were video-recorded with two cameras in much the same way described above for the pairs. Preparing for these interviews required me to read the papers in question and reference graduate math textbooks and Wikipedia in order to develop enough of an understanding of the material to ask pertinent questions. The interviews themselves focused on fleshing out the reasoning in the paper as well as the mathematicians' motives for investigating those areas.

The preparation for and engagement in these interviews prepared me with some background that was relevant to what each pair was doing in each episode. During the video recording, I took note of any key points that confused me and asked the pair about it after they had concluded their session. I then created the merged MOV file as described earlier and reviewed the episode, making a sketch of the mathematical moves made throughout the episode. This required me to specify the mathematics in enough detail to notice where the gaps in my understanding were. At this stage I also created an index of key moments that seemed to be pivotal in terms of mathematical struggle.

Once I had the index and the sketch of mathematical reasoning for all three episodes, I interviewed each mathematician from the pair separately. These interviews focused on filling the gaps in my understanding of the mathematics as well as commenting on why they thought they had made specific mathematical moves. I showed them selected clips of the merged videos as needed to illustrate what I was asking them about.

It should be emphasized that the intent behind asking the mathematicians to reflect on why they made certain mathematical moves was not because their reflections are authoritative. When I review videos my family took of our travels through Europe two decades ago, I can make educated guesses about what I was thinking at various moments, but the only advantage I have over anyone else in doing so is in my somewhat better ability to empathize with my past self; my memories offer informed hypotheses about the experiences of the person in the video, but memories are highly fallible and subject to influence by one's present perspective Burgess & Shallice (1996); Kahneman (2011); Kahneman & Tversky (1973); Scheler (2007). In the same way, the central benefit of having mathematicians reflect on videos of themselves is in their ability to produce highly educated guesses about what the person in the video is thinking and feeling, and why. However, beyond having those informed hypotheses, the mathematicians in these one-on-one interviews have no more (or less) access to the thinking of the people in the video than I do (Scheler, 2007).

I used these video-recorded interviews and repeated viewings of the original pair-observation videos, as well as any other references the mathematicians suggested I look at and what I could find on my own in textbooks and via the internet, to develop a more complete technical description of the mathematics involved. When I encountered difficulty that I couldn't solve via my own research and reasoning, I consulted the mathematicians either by email or in person. Portions of the resulting mathematical description — specifically, the portions needed for understanding the clips that are examined here for analysis — are included in the results chapters.

3.4.2 Selecting Clips

While developing my understanding of the mathematical background, I also repeatedly reviewed the merged videos of pair observations in order to internalize the flow of the session as a whole. Each time I gained new insight into the mathematics in a way that connected or clarified parts that hadn't entirely made sense to me before, I reviewed the videos again to watch for what, if any, of my impressions of the mathematics changed while doing this. During each viewing I would write notes describing my impressions of the overall mathematical flow, particularly moments of struggle and shifts in strategy.

My familiarity with the videos, combined with my notes, gave me solid hypothesis about which selections were likely to be fruitful for further analysis. I presented my selections and tentative ideas for analysis to others[3] and considered their feedback in selecting which clips to proceed with.

3.4.3 Generating Multimodal Microanalyses

Once the clips were selected, they were subjected to a multimodal microanalysis. The selections were annotated in Elan[4], which allows for frame-by-frame

[3] The primary person to kindly provide me with this first step toward intersubjective corroboration was my advisor, Ricardo Nemirovsky.

[4] http://www.lat-mpi.eu/tools/elan/

creation of sub-annotations. This allowed each clip to be subdivided by gesture components (e.g., when someone moves their hand from their side to the space in front of them).

Once the subdivision had been created, I took screenshots of the video and drew arrows and other annotations on the image to indicate what movements (gestures, changes in eye gaze, inscriptions, leg movements, body position changes, etc.) had occurred since the previous screenshot.[5] I then produced a transcript for each sub-annotation, resulting in a correspondence between transcript and image that typically matched on a scale of a fraction of a second.

With all this material produced, it was synthesized into a tabular format using TEX. This produced a stable PDF that allows for cross-platform viewing of the resulting microanalysis. This consists of a table with three cells per row: a row number, an annotated image, and the corresponding transcript. The microanalyses for all clips referenced in the present work are in the appendix (starting on page 256).

3.4.4 Phenomenological Analysis

The microanalysis provided me with a way of viewing the video in detail without letting the movements distract me, but also without me missing the relevance of the movements. Having watched the videos many times, the microanalyses also provided me with an index that reminded me quite vividly of what the video looked and sounded like when played. Also, the images have associated with them a timestamp, so whenever there was any question in my mind about what was going on or what the spirit was of a given phrase or movement, I could quickly and easily go to that portion of the video and view it.

With this tool in hand, then, I engaged in a phenomenological analysis. This

[5]The program used for this was Snagit: http://www.techsmith.com/snagit.html

roughly follows the four steps that Gallagher & Zahavi (2008) describe minus the eidedic variation (so bracketing, phenomenological reduction, and intersubjective corroboration), although the process was more detailed and less linear than this.

I began by carefully looking over the microanalysis while bracketing what I observed. This consisted of being open to whatever impressions arose and writing them down, with a sincere attempt to avoid coming to conclusions. For instance, at this stage I might note that "Fay is standing away from the board with her arms crossed and with her brow slightly furrowed," but rather than asserting that it's because she's probably bothered by what she's looking at I simply make note that *my impression* of her being bothered arose in me as I looked at this section of the clip. The goal here was to immerse myself in the clip *as it is presented*, minimizing the influence of my interpretations by noting that they are my interpretations rather than claiming them to be hypotheses about what is "really going on." However, my impressions are not irrelevant; indeed, they become key in later stages of the phenomenological analysis, so recording them (along with a timestamp of when they were recorded) is a key step.

I cycled through a clip repeatedly, starting my written commentary anew each time, to allow relevant themes to emerge. I would allow for at least one day between cycles, and normally I would cycle through a different clip in the interim. This proved to be useful as a way of preventing my earlier impressions of a clip from suppressing other potentially relevant impressions. For instance, Fay furrowing her brow and crossing her arms might simply be a sign of her being deep in thought rather than being particularly troubled by what she's looking at on the blackboard. Taking time between such journaling helped me to distance myself from the prior impression so that others have room to arise naturally.

This process allowed certain themes to emerge. For instance, it became evident that mathematicians' use of proximity (in the sense of how far away they were from what seemed to be the center of their attention, such as an inscription or an imagined object in the air they were forming via gesture) was a central theme in how they structured their experiences with struggle. As these themes emerged, I shared them with others[6] along with the corresponding clips as a first pass at intersubjective corroboration (and also as a test of my ability to convey the mathematics needed to contextualize the clip in question). Based on such input, I focused on a few key themes and selected a few more clips for multimodal microanalysis based on their promise in providing insight into those phenomena.

The write-up of the results is, itself, part of the phenomenological analysis. The articulation of the mathematicians' apparent experiences, along with the way they structure those experiences, helps to clarify its form. It is also a necessary medium for intersubjective corroboration. I found I needed to write and rewrite the description of each clip multiple times in order to develop a sufficiently compelling account that others found illustrative of the phenomena in question.

It was also necessary to present the material in seminars and at conferences in order to get feedback from the audience about the degree to which they were able to perceive something new in the clips as a result of the analysis. This, too, involved repeated cycles of preparing the presentation, presenting it, and carefully considering the audience's input until the analysis reached a respectable point of saturation.

3.4.5 Themes and the Research Question

This process resulted in three themes:

- *Proximal inhibition*, which explores an aspect of how proximity is used in the encounter with struggle.

[6] Again, I'm indebted to my advisor Ricardo Nemirovsky for his insightful input in this process.

- *Invoking the continuum*, which illustrates a particular use of a type of diagram we might be inclined to label as "geometric."

- *Transformation mappings*, which seem to capture a key element of how mathematicians experience the use of the blackboard itself.

These three themes in turn weave together to offer insights into the nature of mathematical practice itself. The next few chapters will detail each of these themes, as well as their interweaving, in some detail.

However, the research question[7] focuses on the *structure of experience of struggle*. The relevance of these themes in mathematical practice is that they help to contextualize the experiential structuring. Struggle is a key part of mathematicians' practices, and as such the phenomenological analysis of this study came to touch on some themes that extend beyond the original domain of interest. Yet the focus in this study is on the encounters with difficulty rather than the whole range of practices in mathematicians' research.

[7]For ease of reference, here is the research question again: "What are some ways in which mathematicians structure their experiences of struggle while working in pairs in person on a current problem?"

Chapter 4

Proximal Inhibition

In short, proximal inhibition is a structuring of struggle in which the mathematician seems to be pushed back from the blackboard's diagrams or inhibited in his ability to approach them. The inhibition results not from any physical inability to get closer, but instead comes from a sense that such a physical movement would not afford anything for the mathematics. However, the mathematician wants to "zoom in" to the particular aspect that is meaningful. This clash between the desire to modify the key part(s) of the diagrams and the inhibition from doing so due to not perceiving exactly where or how to make the key adjustment creates a palpable tension that an observer can often feel.

I'll illustrate this phenomenon with three clips. Then with these three examples in mind, I'll explicate the components of proximal inhibition as they seem to appear in general.

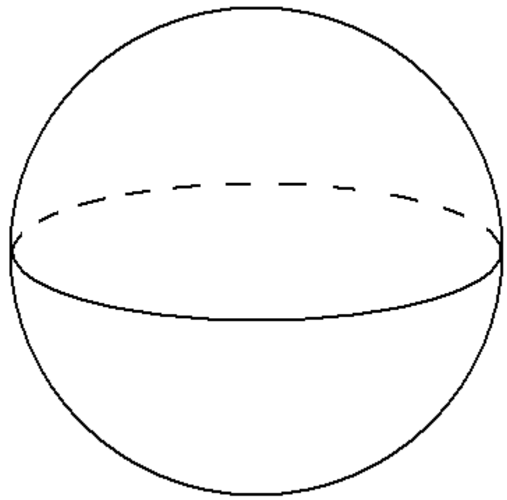

Figure 4.1: A sphere

4.1 Clip #1: Odd Scalars

4.1.1 Mathematical Context

Homology

The content was in an area known as "odd Khovanov homology." Homology in this context is a technique that lets us describe some subtle geometric features using algebra. For instance, one application of homology — called "singular homology" — lets us count the "holes" of different dimensions in a geometric object. Consider a sphere, for instance (Figure 4.1). No matter where we place a loop on this sphere, we can slide it around along the sphere so that it's all on one side. At that point we can shrink it down to a point (Figure 4.2). We can say, then, that there are no one-dimensional holes on the sphere — or more properly said, the first singular homology group is the trivial group.

By way of contrast, consider a torus (i.e., a "doughnut" — Figure 4.3). It turns out that the first homology group of the torus is $\mathbb{Z} \oplus \mathbb{Z}$, which is to say, two

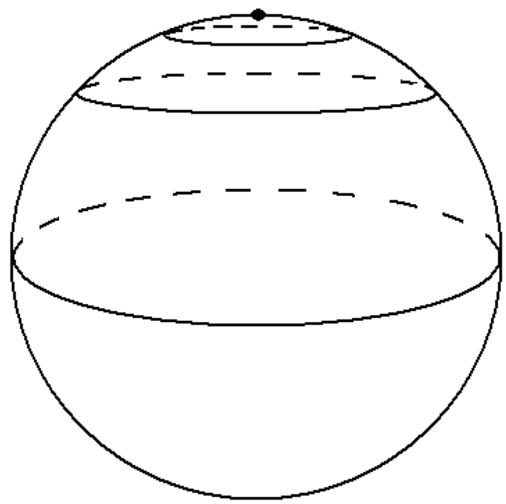

Figure 4.2: A loop on a sphere shrinking to become a single point

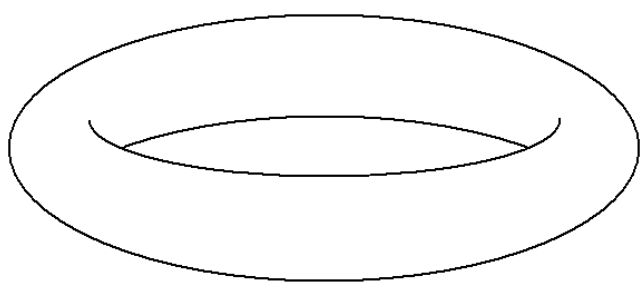

Figure 4.3: A torus

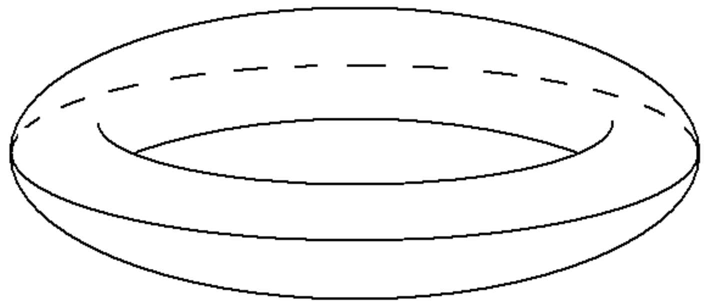

Figure 4.4: A loop going the long way around a torus

copies of the integers. This tells us that there are actually two one-dimensional holes in the torus. We can sort of see one right away as the hole in the middle of the torus: if we put a loop the long way around the torus, there's no way to slide it along the surface so that it can contract to a point (Figure 4.4). This loop acts like the 1 for one of the copies of the integers in the homology group. So a loop that goes twice around would be 2, one that goes thrice around would be 3, and so on. And similarly, a loop going around in the opposite direction would be -1. (We consider two loops to be the same for this purpose if you can slide one along the toruss surface to become the other.)

The other hole is actually inside the torus (Figure 4.5). This is like tying a string around an inner tube: no matter how you try to slide the string around, it won't just fall off the inner tube. Then *this* loop acts like the 1 for the *other* copy of the integers in the homology group. Since there isn't a third copy of the integers in the first homology group of the torus, we know that weve noticed all the two-dimensional holes now.

This second copy of the integers would vanish, though, if we were to fill the "tube" of the torus. At that point the second loop would have somewhere to go to collapse to a point. So the first singular homology group of the *solid* torus is just

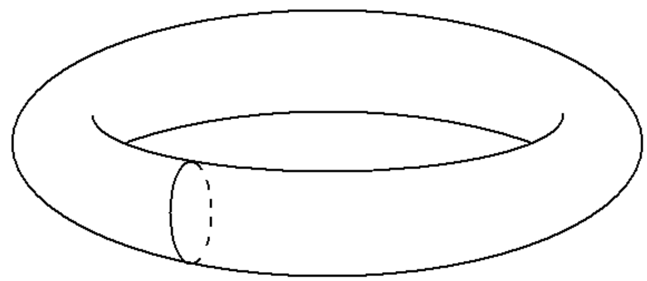

Figure 4.5: A loop going the short way around a torus

$\mathbb{Z}$. The fact that it's missing a copy of the integers that's present in the normal (i.e., hollow) torus tells us something about how the hole structures of solid and hollow tori compare. In particular, since the solid torus, the circle, and the annulus all have the same first singular homology group, we know that the one-dimensional hole structure for all these different geometric objects must be the same and must, in turn, be different from that of the hollow torus in a particular way.

Khovanov Homology

Singular homology is just one sort of homology, namely the kind that reveals certain kinds of holes. The kind of homology of interest to Joseph and Bill was known as *Khovanov* homology, which can give hints about subtle features of *knots*.

A knot, in a mathematical context, is a way of twisting up a circle in three-dimensional space. Mathematicians distinguish between different kinds of knots based on whether it's possible to "wiggle" one knot into another one without making the circle cross itself. For instance, a trefoil is viewed as being a different knot than an "unknot" (i.e., a knot that can be wiggled into the shape of a perfectly normal, flat circle), but a circle and a circle with a kink in it are both considered the same knot (namely the "unknot"). (See Figure 4.6.)

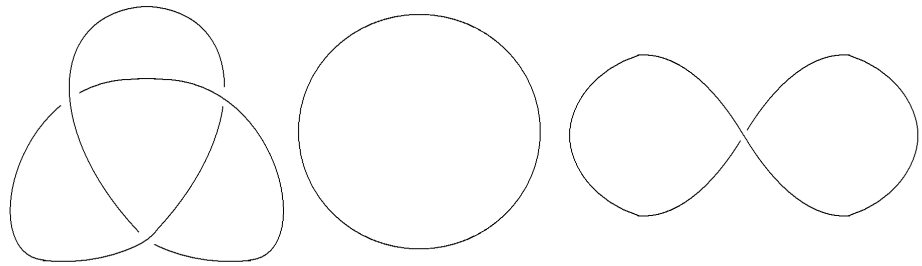

Figure 4.6: Three knots: a trefoil, a circle, and a figure-8 unknot

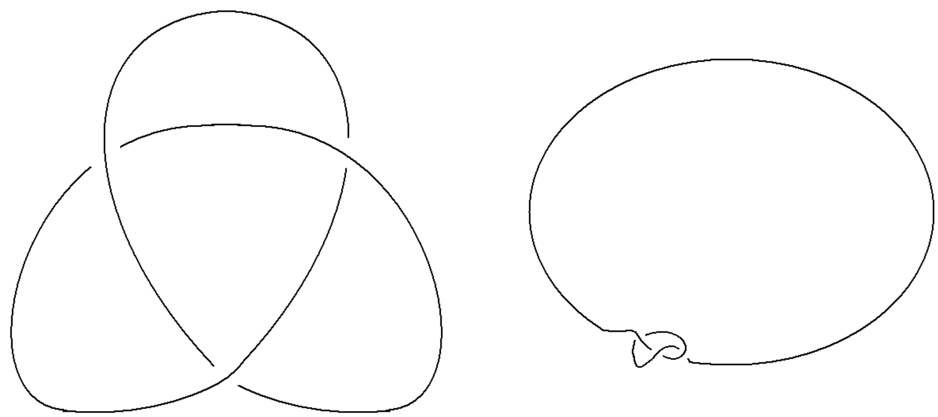

Figure 4.7: A trefoil in two different shapes

These ideas are meant to model creating physical knots with physical string. The idea is that two knots are different if you can't get from one to the other without cutting the loop of string and then tying it back together again. However, if you can wiggle one knot into another, in some sense you'd want to consider them the same knot. For instance, a trefoil is the same as a simple kink-knot since you can get the latter by just pulling one of the loops of the trefoil out (Figure 4.7).

In the same way that singular homology tells us about the number of certain kinds of holes, Khovanov homology tells us something about the structure of a given sort of knot. Among other things, we can tell whether two knots are different

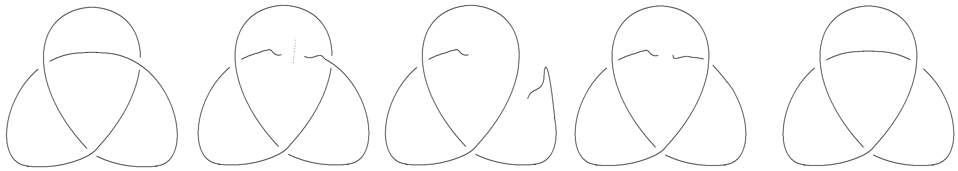

Figure 4.8: Untying a trefoil with a cut and a splice

based on whether their Khovanov homology groups are different, and we can learn something about *how* the knots are different based on the algebraic differences in their homologies. (This is akin to how, in singular homology, we could tell how the solid and hollow tori compared to one another based on how their singular homology groups differed.)

Khovanov homology works in part by looking at how you would transform one knot into another. You can think of this as playing a movie: you might start with a trefoil, cut it, move the cut pieces to the other side of another part of the string, and then splice the ends together (Figure 4.8).

For technical reasons, though, Joseph and Bill don't model this as cutting the circle and then putting it back together again. Instead, they model it by thinking of pinching the circle together into a figure-8 of a sort, splitting the figure-8 into two separate circles at the pinch point, rearranging them, and then splicing them back together (Figure 4.9).

If we think of time like a spatial dimension, then the collection of "frames" for the movie will create a kind of cylinder – technically called a *cobordism* — that connects the two knotted-up circles. Each cross-section of the cobordism would be one frame of the movie, so to speak. In the particular case shown in Figure 4.10, there's a hole in the cobordism because the trefoil has to be cut, moved, and spliced back together again in order to untie it. As we move left to right, the "cut" occurs the moment we encounter the hole (at what Joseph and Bill call a "split"), and the "splice" occurs the moment the hole closes (at what they call a "merge").

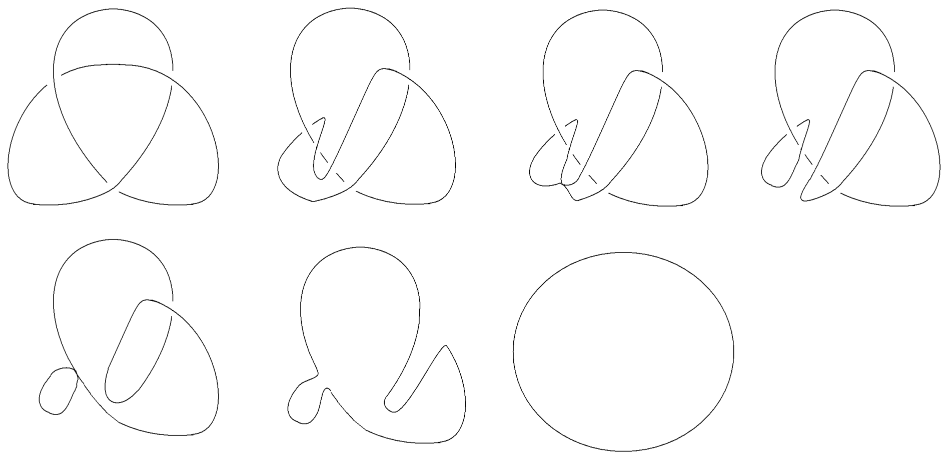

Figure 4.9: Splitting and merging (instead of cutting & splicing)

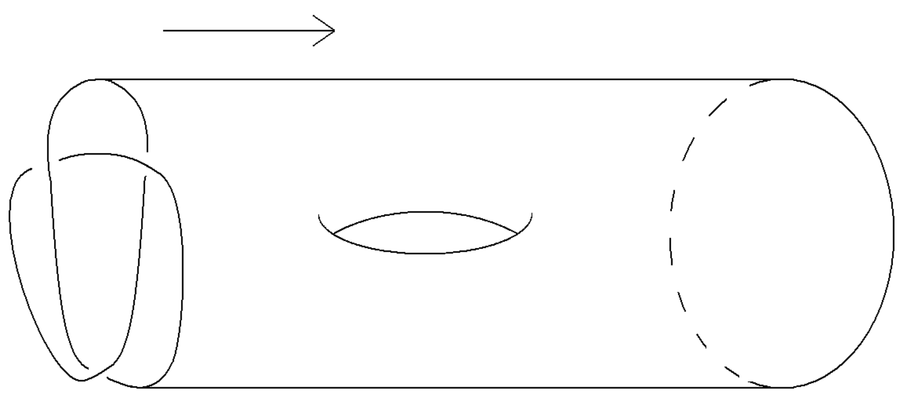

Figure 4.10: A cobordism showing the untying of a trefoil

So in a sense, Khovanov homology counts how many such holes at a minimum must appear in the process of untying the knot in question.

It turns out that in order to work out the Khovanov homology groups, we have to use what's called a *Frobenius algebra*. Part of the description of this algebra involves the following two formulas:

$$\begin{aligned} \Delta(1) &= x \otimes x \\ \Delta(x) &= x \otimes 1 + 1 \otimes x \end{aligned}$$

Roughly speaking, is describing the algebraic side of what happens at a split in a cobordism. What was once one circle becomes two at a split, which the algebra describes using the tensor notation (i.e., the "$\otimes$" symbol).

Odd Khovanov Homology

Some knots that are more complicated than the trefoil require more than one cut-and-splice to get untied. In those cases, the order of the cuts and splices doesn't matter in terms of Khovanov homology. (See Figure 4.11.)

Joseph and Bill were exploring what happens when they modify the Khovanov homology so that it does distinguish between different cut-and-splice sequences. This is akin to how we might decide to modify singular homology so that it detects different *orientations*. No matter how we slide a Cartesian axis around on a sphere, we can't swap the x and y axes, which means we can think of there as being two orientations of the sphere: normal and "swapped" (Figure 4.12).

In trying to do something similar with Khovanov homology, Joseph and Bill were working with what they called *odd Khovanov homology*. This required them to modify the Frobenius algebra so that was no longer commutative (or, more properly speaking, cocommutative), which they would need to do by inserting what they called "odd scalars." In the video clip I'll describe shortly, Joseph specifically

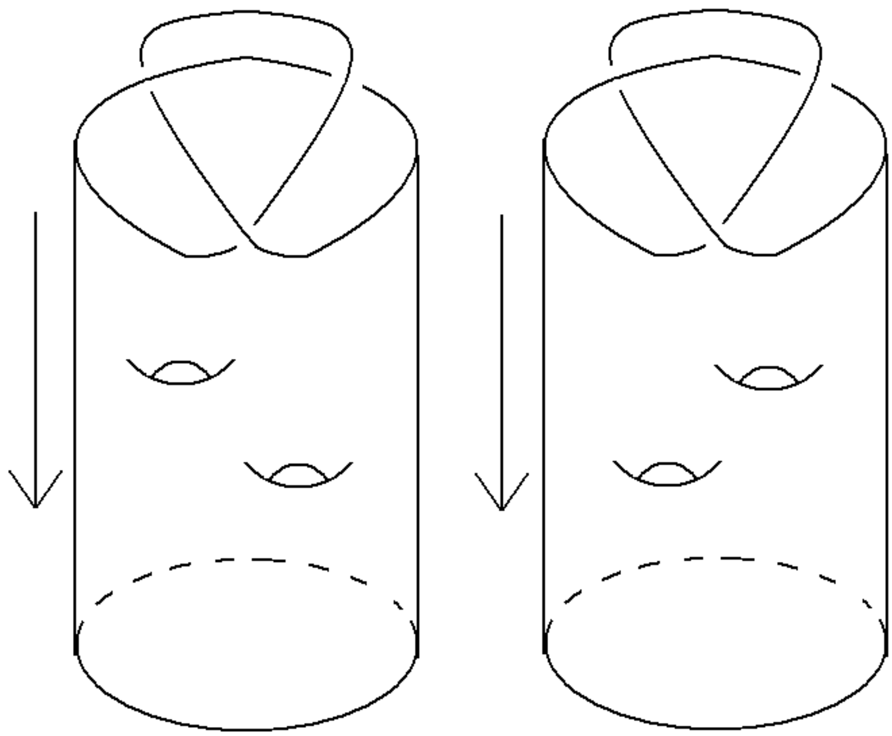

Figure 4.11: Two slightly different cobordisms that don't look different to (normal) Khovanov homology

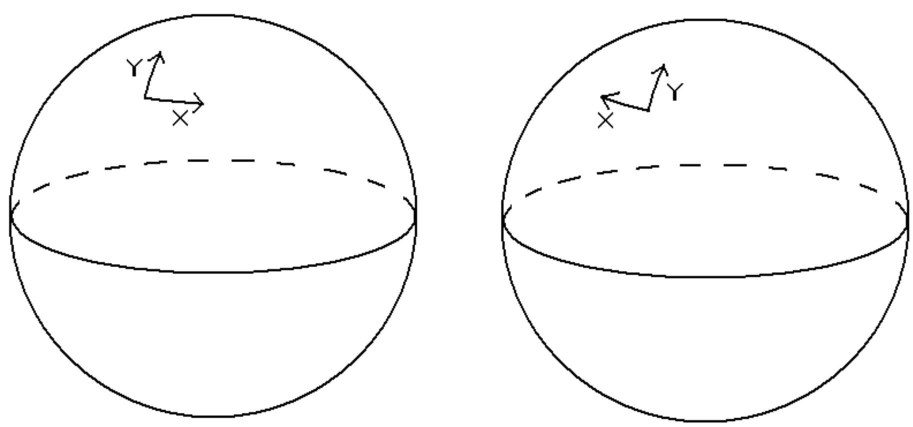

Figure 4.12: Two different orientations for the sphere

points out that "there are odd scalars which are called 'one."' This would be similar to them noticing that they forgot to put a negative-one in the second term:

$$\Delta'(x) = x \otimes 1 + (-1)(1 \otimes x) = x \otimes 1 - 1 \otimes x$$

Fixing this oversight would make it clear that swapping the roles of x and 1 would give a different formula:

$$x \otimes 1 - 1 \otimes x \neq 1 \otimes x - x \otimes 1$$

Odd scalars aren't exactly the same as negative numbers, but their presence modifies the Frobenius algebra in some comparable ways.

4.1.2 Video Description

As I describe this clip, I'll cite row numbers. Those row numbers refer to the microanalysis in Table A.1 of the appendix.[1]

Setup

After about 25 minutes of trying to think of odd Khovanov homology in one way, Joseph erases the board and proposes instead thinking about modifying Δ by adding odd scalars. This seems to demarcate a shift in strategy. He then writes a sequence of equations on the board (Figure 4.13):

$$\begin{aligned} x^2 &= 0 \\ \Delta(1) &= x \otimes 1 - 1 \otimes x \\ \Delta(x) &= x \otimes x \\ \epsilon(1) &= 0 \\ \epsilon(x) &= 1 \end{aligned}$$

[1]If you are viewing this document on a computer and you have the file "tables.pdf" stored in the same folder as this document, you can click on the words "Table A.1" in order to open the microanalysis in a separate window. This will allow you to read the text in one window while referencing the video in another rather than having to skip back and forth.

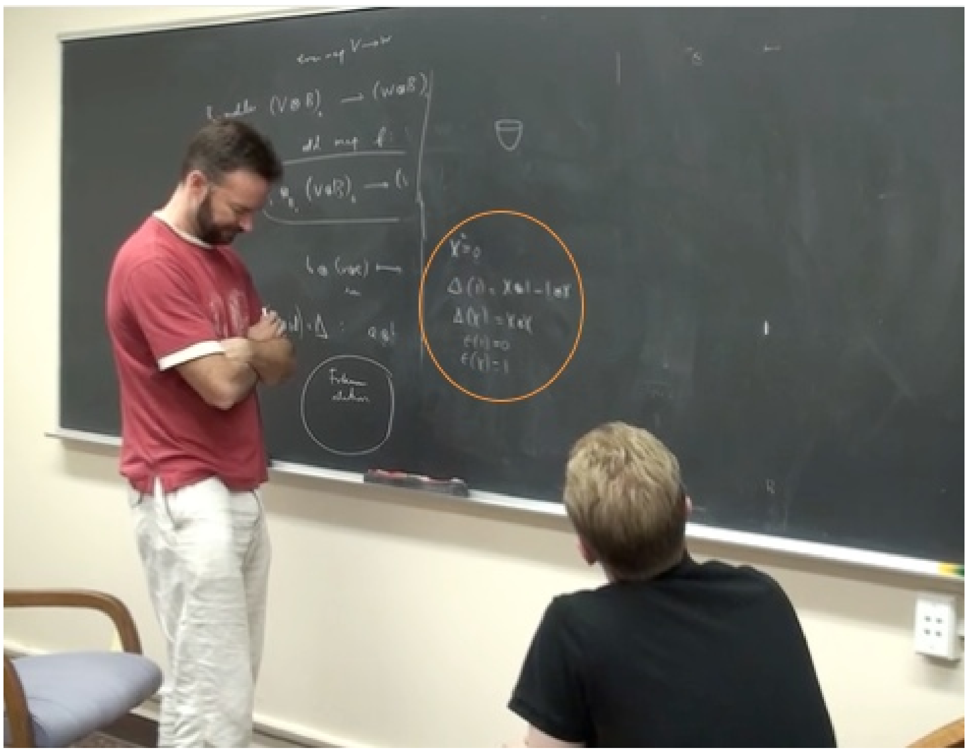

Figure 4.13: Joseph writes some equations on the board

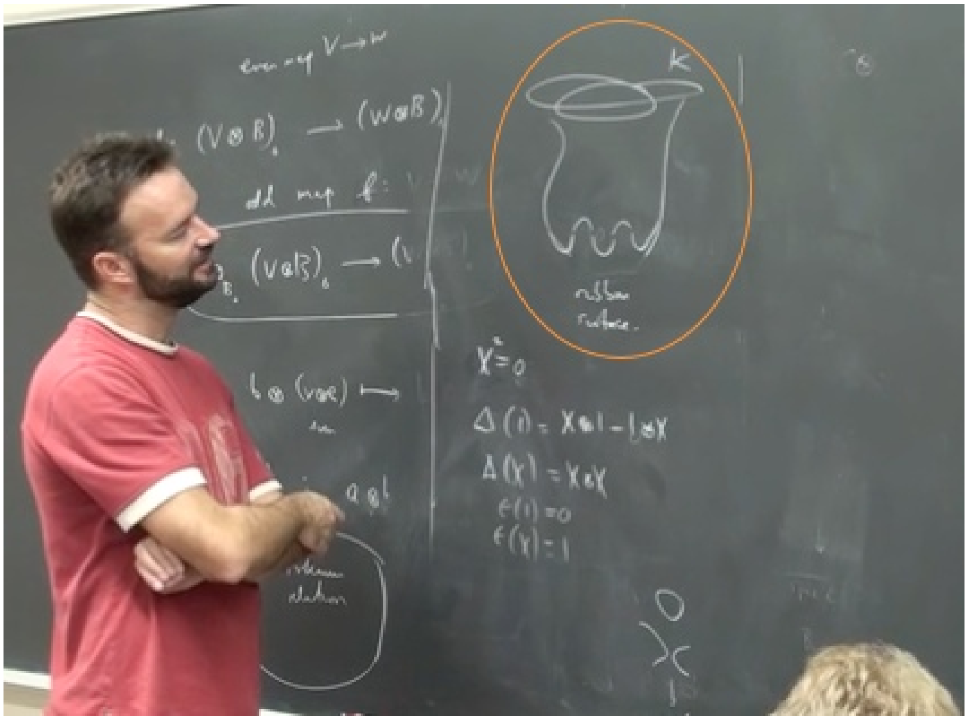

Figure 4.14: Contemplating a ribbon surface

(The "ϵ" symbol refers to the part of the Frobenius algebra that corresponds to "capping off" in the cobordism, which in terms of knots means that a circle shrinks to a point and vanishes. It ends up not playing an explicit role in the rest of the clip. Similarly, the fact that $x^2 = 0$ ends up not arising as particularly important for the following analysis.)

Joseph then spends the next two minutes reminding himself and Bill about the mathematical context, namely that they think this description of a Frobenius algebra will, in fact, result in the odd Khovanov homology they're trying to develop. In order to work out whether this is true, he draws a cobordism. (In this case the cobordism is referred to as a "ribbon surface." The difference is that instead of thinking of going from one knot to another, we think of creating circles and then merging them together in some knotted-up way. The diagram in Figure 4.14 should be thought of with "forward in time" being upward.)

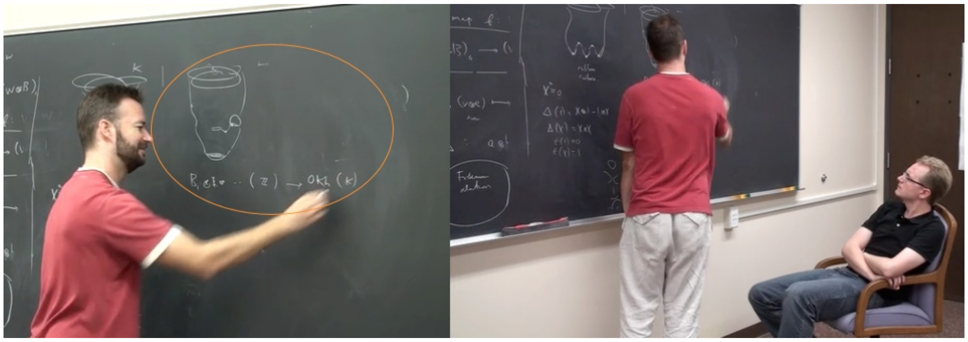

Figure 4.15: Adding "caps" to navigate confusion

Once he does this, though, he realizes there's a problem: the thinking that goes into relating the cobordism to the algebra will result in the *normal* Frobenius algebra. They'd end up ignoring the formulas Joseph had just written down. This confuses Joseph because the normal Frobenius algebra yields the normal (non-odd) Khovanov homology even though they're thinking about the cobordism in terms of *odd* Khovanov homology. The "odd" part just gets ignored. He tries to reconcile this by thinking about a cobordism that has "caps," which in the movie-like view would be seen as a part of a knot splitting off as an unknot and then shrinking to a point and vanishing (Figure 4.15).

Proximal inhibition demonstrated

But even after relating this to some algebra of homology (shown below the ribbon surface in Figure 4.15), he still seems flummoxed and sits down, scooting the chair a little closer to the board but leaning away from it (row 1).

This is a moment where we can see a strong hint of what I'm calling "proximal inhibition." Joseph physically distanced himself from the board for some reason. It seems unlikely that he's leaning against the wooden arm of the chair purely for reasons of comfort, especially when we consider that Bill is also leaning away from the board but to the opposite side. It's much more likely that Joseph created this distance as a physical enactment of his current relationship to the mathematics depicted on the board.

Think of how people react when someone stands too close to them in conversation: the person who feels a kind of invasion into his or her personal space will tend to lean away from the intruder and might even take a step back. Similarly, when we're working with a difficult object in our hands like an object we're trying to repair or a 3D puzzle, we might tilt our heads away and a bit to the side and even push the object just a little bit away from us as we furrow our brow when we're confused by it. This kind of distancing behavior is a visible aspect of our discomfort with the situation in question. Yet if we were totally disinterested in the situation — if we aren't engaged in a conversation with the person invading our space or if we lose interest in the object we're working with — we don't lean away while still fixing our attention on it. Instead, we walk away from it or at least turn our attention (and often our torsos) to something else.

In the same way, it seems likely that Joseph's distancing behavior here illustrates two pieces in parallel. On the one hand, he sits down and leans away because that drops the intensity of the discomfort with the situation. It might also act as a kind of "zooming out" for him so that he can see the situation from a wider, more holistic point of view. On the other hand, he's still engaged with it, trying to resolve this conundrum. As I'll illustrate later, this desire to engage seems to be at least in part an inhibited impulse to move closer to the material on the blackboard.

Twenty seconds later, Joseph starts suggesting that it might be possible to resolve this by having all the "odd" parts of the odd Khovanov homology cancel out to give zero when considering the role of the Frobenius algebra (row 16). Notice how Joseph remains leaned back in order to reach forward with his left hand. There's no mechanical need for him to lean back like this. Quite the contrary, it would have been just as easy for him to let the weight of his left arm pull his torso a bit closer to the board, and doing so would probably have made it visually clearer which specific part of the board he was trying to indicate. This suggests once again that he's leaning back due to his relationship with the mathematics. In this particular moment, he's trying to suggest a result that would be convenient and might help to resolve the paradox vexing him, but he's not sure it's true or that its being true would really resolve the issue. This leaning-back aspect of his movement is accented by, for instance, the way he leans forward slightly when he later withdraws his hand from another gesture (rows 28–30).

After he suggests the above, Joseph utters a sentence fragment ("So I think, I mean, I, I dont quite understand how this can be possible, but assuming that the...") and then falls into a lengthy pause (rows 31–44). In the middle of this pause, he reaches up and touches his face — first his nose (row 37), then brushing his mouth (row 38), and finally moving to his chin to stroke his beard (row 39). As we'll see in several of the case studies here, this touching of the mouth can act as a signal that the mathematician has just thought of something but is "holding the tongue" for a moment, presumably to let the new idea gel a bit before saying anything.

At the end of the pause, Joseph rolls his chair forward and leans forward to indicate the algebra he had just written below the rightmost cobordism (rows 45–52). This gesture contrasts rather dramatically with the one he had given less than a minute earlier (row 28). Here he's leaning into the board, as though much of his discomfort with the material has lifted. He says, "You know, I, I sort of want to argue that these just give the zero map. That's...." This is in reference to the same point made earlier (namely wanting the "odd" part of the homology to cancel to zero in terms of the "odd" Frobenius algebra). His body language indicates some degree of increased comfort, as though he's more confident of the point he's making. However, as his speech trails off ("That's…"), he leans back again and crosses his left arm over his torso as though to recreate his sense of comfortable distance from the convoluted issue while still remaining engaged with it (rows 53–54).

Three movement-free seconds later he says, "But I don't know whether that makes sense." He juts his chin forward and points with his left hand while slightly rolling his shoulders more toward the board (row 59). This makes it seem somewhat more likely that at the moment Joseph first moved in toward the board, he was actually recognizing this concern about "whether [getting the zero map] makes sense." We can see that he's still keeping his distance from the board, but he's sort of "pushing" toward it, like he's trying to overcome his sense of resistance or inhibition. This is reminiscent of how a shy person might simultaneously feel a desire to talk to a particular person and a hesitation to do so, resulting in a kind of forward/back yo-yo movement. The principle difference here, though, is that Joseph's hesitation seems to come from not knowing how to approach the board in a meaningful way. It's not clear to him exactly where on the board the confusion lies.

Joseph spends the next several seconds trying to articulate his concern to Bill from this head-forward, pointing position (rows 60–69). To paraphrase, Joseph worries that computing the odd Khovanov homology without adding odd scalars into the formulas written for the Frobenius algebra might somehow not "make sense." He has some trouble expressing exactly why it might not make sense, though. As he struggles to describe what the problem is, his sentence trails off ("We just write these things down and sort of forget about the, uh…"; rows 69–72) and he drapes his arm across his torso again and leans back a little, even tilting his head slightly away from the board (rows 70–71).

After freezing for a few seconds he quietly says, "Forget about the grading.

Well, then, then, um. . . ” (rows 77–79). While speaking, he rights his head (toward the board) and rocks the chair forward (row 77), but on “um” he rocks his chair back again (row 79) as though the momentum of what he had been saying had been helping him oppose his resistance to approaching the board, but once the momentum faltered he had to fall back to navigate his discomfort with the problem at hand.

After another few seconds of essentially motionless silence (rows 80–84), Joseph suddenly brings his hands forward as though holding something invisible and tilts his head away from the board slightly (rows 85–86). He seems to be reiterating the problem he has been trying to articulate, namely this challenge of reconciling the fact that they’re viewing a knot as having an odd Khovanov homology but that the homology doesn’t seem to follow from the Frobenius algebra they want to use. This moment where he seems to be holding the entire concept as a whole in his hands seems to be an effort to rebuild momentum, to keep pressing against this inability to “zoom in” to the board in a mathematically meaningful way. We can see it as a kind of momentum-building because in the same breath, without pausing after the previous sentence, he says, “I mean, I think. . . So. . . ” (rows 87–89). As he says this, he lifts his left hand to point his palm toward the board while leaning his head dramatically away from the board. At this moment, the pressure between two drives — the need for distance from the difficult situation on the one hand and the desire to “zoom in” to the as-yet undiscovered point of difficulty on the other — is quite palpable. He’s probably using his left hand to indicate the region of the blackboard where he has written the formulas for the Frobenius algebra, but the fact that he chose to rotate his arm the way he has seems to indicate a sort of pushing against resistance. It’s almost as though his hand is pushing against an invisible wall blocking his access to the board. However, since the “wall” is an internally felt need for distance, the resistance it offers seems to repel his head and shoulders even as his hand reaches forward.

This whole sequence has shown many rather vivid examples of proximal inhibition in action. However, I think we can gain some significant insight by seeing what happens when the inhibition lifts, as it does in the next moment of the video. In the same breath in which Joseph speaks while holding his hand up, he says, "Somehow deeply in here there are odd scalars which are called 'one'" (rows 90–93). On "deeply" he rapidly leans forward and reaches with his right hand to a specific point in the formulas for the Frobenius algebra (row 90). The specific point he's touching is just to the right of the equal sign in the equation $\Delta(x) = x \otimes x$. What's critical to notice here is that as soon as he sees where the right place to "zoom in" on is, his whole body is abruptly drawn in. It seems almost like what happens if two people are arm-wrestling and one person suddenly relaxes his or her arm. Once the inhibition to moving toward the board vanishes, all that's left is the desire to move in — or more properly said, the physically felt impulse to move in that we subjectively experience as noticing the affordance of approach. There's no constraint left. This is presumably why we see Joseph rapidly launching himself toward the board rather than slowly leaning in to make his point.

4.2 Clip #2: Confusion in Frobenius Algebra

The second instance I'll show of proximal inhibition comes from the same pair of mathematicians (Joseph and Bill) as in the first clip described in Section 4.1. Bill had sent an email asking another mathematician, James, for help with the trouble he and Joseph were having with the Frobenius algebra (see Section 4.1.1). In this clip, Bill offered an overview of James' response, which prompted Joseph to detail the algebra involved. This led to some further difficulty that illustrates proximal inhibition.

This clip includes a few instances that one might interpret as proximal inhibition but stand a good chance of being something else. I'll point those out as we go.

4.2.1 Mathematical Context

This was a continuation of the work Joseph and Bill were doing in clip #1, so you can find most of the relevant mathematical ideas in Section 4.1.1. I'll talk about just two other facets of the mathematics here: the relationship between the operations in the Frobenius algebra and the cobordisms, and the 't' parameter.

Frobenius operations and cobordisms

There are four kinds of events that can change what a knot is:

1. A circle can appear out of nowhere. That is, a point can appear and then expand out into a circle. (Figure 4.16)

2. Two different knots can touch and merge together. (Figure 4.17)

3. A knot can squeeze two points together and split off a part of itself. (Figure 4.18)

4. A circle can shrink to a point and vanish. (Figure 4.19)

Each of these looks like a specific feature of a cobordism:

1. The appearance of a circle looks like a *cup* (Figure 4.20).

2. The merging of two knots looks like a *merger* of two "tubes" (Figure 4.21).

3. The splitting of a knot into two looks like a *split* of one "tube" into two (Figure 4.22).

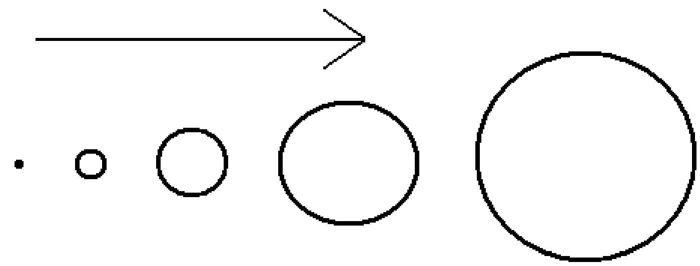

Figure 4.16: A circle appears — sometimes called a "birth"

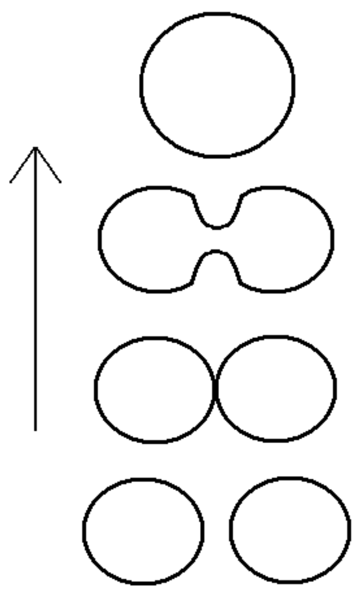

Figure 4.17: Two knots merge together

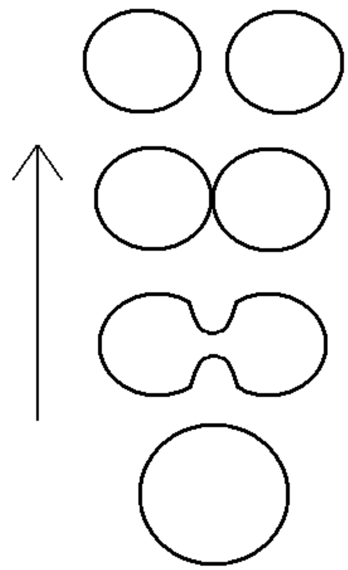

Figure 4.18: A circle splits into two

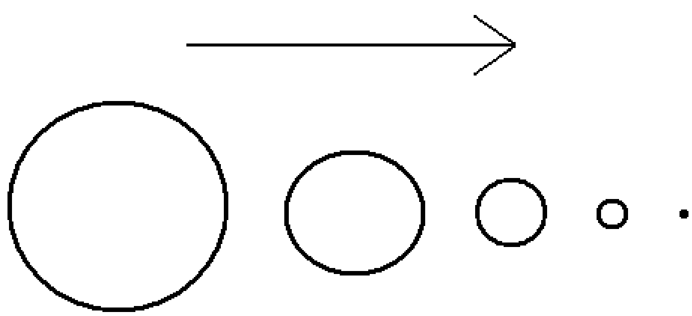

Figure 4.19: A circle vanishes — sometimes called a “death”

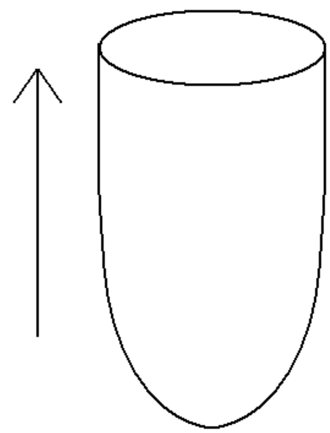

Figure 4.20: A cup in a cobordism

4. The vanishing of a circle looks like a *cap* (Figure 4.23).

In order to record these possibilities, the Frobenius algebra has four operations:

1. The "birth" or "cup" corresponds to a choice of a particular element x of the Frobenius algebra.

2. The merger corresponds to multiplication. Specifically, it takes elements of the form $a \otimes b$ and "crunches" them down into a non-tensored form.

3. The splitting corresponds to comultiplication, which Joseph and Bill usually write using the symbol 'Δ'. This adds a tensor ('$\otimes$') to the input, as in the equation $\Delta(x) = x \otimes x$.

4. The "death" or "cap" corresponds to a function ϵ that turns elements of the Frobenius algebra into rational numbers.

In normal Khovanov homology, the Frobenius algebra is fixed. That is to say, these four operations are known. This means that a specific drawing of a cobordism acts something like the graph of a function: you can tell exactly what

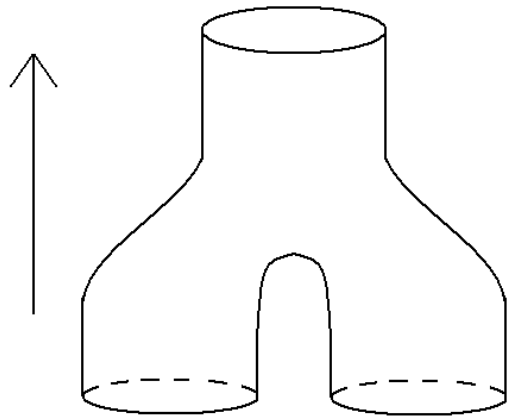

Figure 4.21: A merger in a cobordism

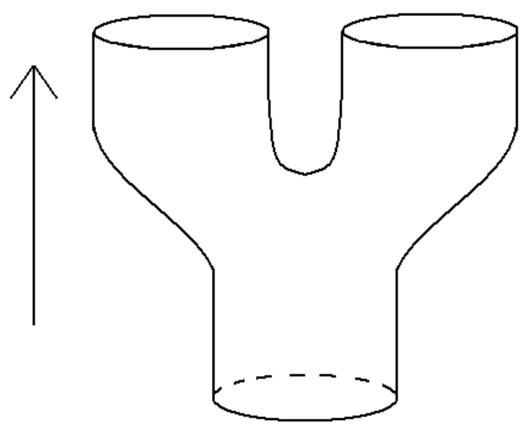

Figure 4.22: A splitting in a cobordism

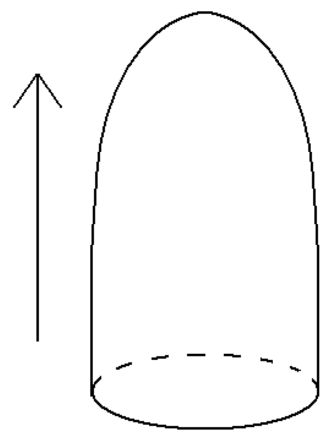

Figure 4.23: A cap in a cobordism

the cobordism represents in terms of how it acts on any given input. Think of it this way: We can push elements of the Frobenius algebra through the "tubes" of the cobordism, and the various junctures will alter the input in various ways. It's sort of like a factory that affects just elements of this algebra.

In normal (non-odd) Khovanov homology, if there are multiple parallel "tubes" of the cobordism then it doesn't matter where different splits and merges occur. (This is another way of stating something I said back in Section 4.1.1: the order in which cuts and splices are done doesn't matter for normal Khovanov homology.) However, it *does* matter with *odd* Khovanov homology. The main challenge Joseph and Bill are dealing with in this clip is the question of what effect swapping the order of Δ and ϵ has on the result. One direction in particular — doing Δ first and ϵ second — is particularly confusing to them.

The 't' parameter

In normal (non-odd) Khovanov homology, the comultiplication Δ acts on the chosen element x by $\Delta(x) = x \otimes x$. It's then extended linearly, which means that $\Delta(x \otimes x) = \Delta(x) \otimes \Delta(x) = (x \otimes x) \otimes (x \otimes x)$. However, as Joseph had pointed out the previous week (see Section 4.1.2), this approach doesn't work with *odd* Khovanov homology. They need "odd scalars" tossed in. In order to account for this, Joseph and Bill agreed that they needed to modify their formula for Δ in the "odd" case:

$$\Delta(x) = x \otimes x + t(1 \otimes 1)$$

In this case, t is an "odd scalar." They hope that by adding the second term there (the $t(1 \otimes 1)$) they can correct for the concern Joseph had raised earlier.

4.2.2 Video Description

The row numbers cited in this clip refer to the microanalysis in Table A.2 of the appendix.[2] However, sometimes I'll refer to rows from the previous analysis, in which case I'll specify which table the other row references are meant to point toward.

Joseph and Bill spent a few minutes discussing the material James had sent them in response to Bill's email question. Joseph then suggested that they try just working out the computations involved in swapping Δ and ϵ. He started by drawing two pictures, one for each order of Δ and ϵ (Figure 4.24). In each case, "forward in time" is to the right. The pair on top in Figure 4.24 shows doing ϵ before doing Δ (i.e., capping off before splitting). The bottom pair shows just the opposite: first do a split (Δ), then cap off (ϵ). The straight tubes just indicate "do nothing."

Joseph then leaps up and starts writing down the algebraic effect of each of these patterns. First he writes the algebraic sequence for the top pair based on an input of $x \otimes x$ (rows 1–3) and then steps back slightly from the board (rows 4–5).

[2] As with the first video, you can open the microanalysis in a separate window if you're viewing this on your computer and you have the file "tables.pdf" in the same folder as this document. Just click on the words "Table A.2."

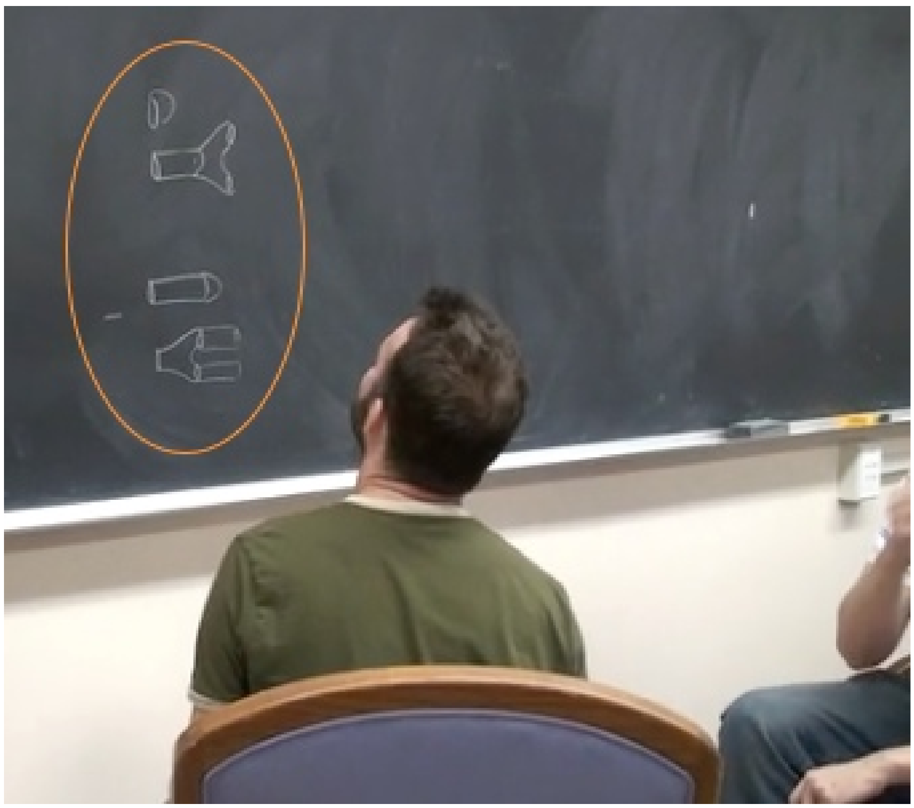

Figure 4.24: Two cobordisms on the board

The sequence he has written is as follows:

$$\begin{array}{ccccccc} x & & & & x & & 1 \\ \otimes & \mapsto & & \mapsto & \otimes & +\,t & \otimes \\ x & & x & & x & & 1 \end{array}$$

For the sake of clarity, I'll rewrite it like so:

$$x \otimes x \mapsto x \mapsto (x \otimes x) + t(1 \otimes 1)$$

The reason Joseph wrote this somewhat vertically was so that the top part of each tensor ('$\otimes$') would correspond to the top "tube" of the drawn cobordism, and likewise the bottom tensor factor to the bottom part of the cobordism. So the first step involves capping off the top part, making the top factor vanish; and then the second step applies Δ to what remains.

Joseph then starts trying to write out the algebra for the bottom cobordism pair. However, he gets stuck midway through. Since $\Delta(x) = (x \otimes x) + t(1 \otimes 1)$ and he's applying Δ only to the bottom factor of the tensor, the result of the first step should be $x \otimes x \mapsto x \otimes ((x \otimes x) + t(1 \otimes 1))$. He gets as far as writing the $x \otimes x$ in the output of Δ (rows 6–15) and then laughs and pulls back (row 16) while saying, "I don't understand."

It's worth noticing that this actually does *not* seem to depict proximal inhibition. Joseph doesn't seem "held back" in any clear sense. Instead, he's pulling away for a moment to glance up at the formula he had just used in the top algebraic sequence to remember what Δ does with x. In fact, even while still finishing speaking the word "understand," Joseph leans back in to finish writing the new sequence (row 17). Therefore, this particular proximal shift is unlikely to be an example of proximal inhibition.

He gets as far as writing the following (row18):

$$\begin{matrix} x \\ \otimes \\ x \end{matrix} \mapsto \left(\begin{matrix} x \\ \otimes \\ x \\ \otimes \\ x \end{matrix} \; \right)$$

$$\begin{matrix} x \\ \otimes \\ x \end{matrix} \mapsto \begin{matrix} x \\ \otimes \\ \left(\begin{matrix} x \\ \otimes \\ x \end{matrix} + t \begin{matrix} 1 \\ \otimes \\ 1 \end{matrix} \right) \end{matrix}$$

The trouble here is that the next step (capping off in the top factor) seems to send that top x to a rational number, which makes that factor of the tensor vanish. The net effect would be exactly the same as in the first case. But since this is *odd* Khovanov homology, the result should be the *negative* of the first case. That is, if the top sequence results in $\Delta(x)$, then the bottom sequence should result in $-\Delta(x)$. Joseph is confused at this point because this simply isn't working out as he had hoped it might. *This* causes him to step back in a display of proximal inhibition (row 19). He looks back and forth between the cobordism drawings, the algebra he had just written, and the algebra he had written for the top pair of cobordisms (rows 20–33). This goes on for a few seconds with very little speech ("Uh.... And then...").

In the next moment Joseph tilts his head to the side (row 34). We saw this before in Section 4.1; see Table A.1, row 53 for example. There it seemed to serve as an aid to creating distance between Joseph and the board, whereas here it seems to indicate only puzzlement. It's as though he's changing the angle of his head in order to see the symbols differently in a physically literal way. It's very unlikely that he believes that tilting his head like this will actually matter — but much as we are usually unaware of bouncing our toes when we're happy, it seems likely that Joseph's desire for a "different angle" on this topic is being expressed rather literally. This phenomenon isn't universal in the cases of proximal inhibition I've observed, but it's quite common and seems to be a reasonably reliable indicator of *some* sort of struggle.

After righting his head, Joseph then goes on to scratch his neck for 24 seconds (rows 39–57). This is a classic example of what I'll call a *self-soothing behavior* (Navarro, 2008). While it's certainly possible that Joseph has unusually itchy skin under his beard, that doesn't immediately suggest why he waited at least seven minutes before rubbing there and then did so quite casually for nearly half a minute. Instead, it seems much more likely that he was using the sensitive parts of his face and neck to help distract him from the discomfort he felt with the situation at the time, although he probably wasn't aware of the gesture at all let alone of its purpose when it occurred. It's as though by creating sensations in sensitive areas of the head and neck, Joseph is trying to make the discomfort of not being able to meaningfully approach the algebra a little less intense. That in turn might make it easier for him to think about the problem at hand.

We saw this behavior in clip #1 when Joseph touched his nose (Table A.1, row 37) and chin (Table A.1, row 39). (I'm excluding his touching his mouth in Table A.1, row 38 since there it seems to serve as a kind of "holding back words" and might not be soothing behavior.) In fact, in those same Figures we can see Bill doing the same self-soothing: he has his left hand splayed across his jaw, chin, and lips. It's actually notable by its *absence* in Bill throughout clip #2 up to this point.

Four seconds after Joseph starts rubbing his neck and jaw, Bill sits forward in his chair and then redirects his movement to the side while lifting his hand (rows 43–45). He then says, "Yeah, so I think where we're losing something right in these is the tensors" (rows 49–53). He starts out by pointing his finger forward but keeps his hand close to his chest. He then shoots his hand forward and quickly taps the board twice at one of the tensor symbols ('$\otimes$') upon saying "something right in these is" (rows 51–52). Immediately after the second tap, though, Bill's hand springs back to touch his face (row 53). He starts by holding his nose but then slowly moves his hand down his face to his chin (row 55). There his hand stays for 6 seconds (rows 55–56).

Since Bill had been sitting back fairly calmly throughout most of this clip, it's not clear whether his sudden movement forward in row 43 is a kind of collapse in proximal inhibition. However, his bodily approach here clearly speaks of his desire to engage with the material, which is one of the two key components of proximal inhibition. His sudden deflection (rows 44–45) suggests a hesitation in his ability to do so; however, this could just be him quickly checking that the point he's about to make is one he wants to make, or him taking a moment to articulate it in his mind. Even the soothing behavior he exhibits (rows 53–56) could be due to *social* discomfort: he might feel concerned about whether Joseph thinks Bill's contribution was relevant. Therefore, while it's certainly possible this sequence shows Bill experiencing proximal inhibition, there isn't enough evidence for us to privilege that hypothesis over others.

Next Bill lowers his hand (row 57), and then one second later Joseph ends his 24 seconds of jaw-and-neck touching by crossing his arms in front of his torso, leaning back, and tilting his head (rows 58–60). Notice how Joseph's feet are trying to be closer to the board than the rest of him. We can tell this from the angle between his hips and upper torso. He even crosses his arms over his upper torso as though to create a barrier between his trunk and the material that's bothering him. This shows the clash between the desire to move in on the one hand and the inability to do so in a mathematically meaningful way on the other that defines proximal inhibition. We even see him tilting his head to the side, which I've mentioned before as an indication of mathematical struggle. Bill, too, seems to be engaged in the push/pull nature of proximal inhibition: he's focused intensely on the board and is even trying to lean forward, but at the same time he's leaning to the side as though simultaneously trying to lean away from the board. In fact, after seven seconds of sitting semi-forward like this, Bill leans back while connecting his hands together across his belly (row 63), as though to create a barrier much like Joseph has done. It seems as though the "push back" side of proximal inhibition is dominant in this moment, and it's possible that Bill has actually disengaged from that particular struggle.

In later interviews, Joseph and Bill explained that they had to abandon this direction of thinking. They decided that the confusion here was too great for them to spend any more time on the matter. Accordingly, they never succeeded in collapsing this instance of proximal inhibition the way Joseph was able to in clip #1 (see Table A.1, row 90). They stayed with this struggle for another 2.5 seconds, and then Bill suggested a change in notation that they both immediately engaged with instead of continuing to push through what Joseph had written.

4.3 Clip #3: Splicing Varieties Together

Joseph and Bill often show very clear instances of proximal inhibition, but the phenomenon occurs in many situations across many mathematicians. I'll illustrate with one more case, this time taking from a different pair — Matt & Bart. At the time of this clip, Matt was a research mathematician in algebraic geometry at a large southwestern university, and Bart was a visiting professor from western continental Europe. Matt and Bart had worked together on a paper that had just been published at the time of this video. Bart had come over to Matt's research institution so that the two of them could work together to decide where the next place to explore should be.

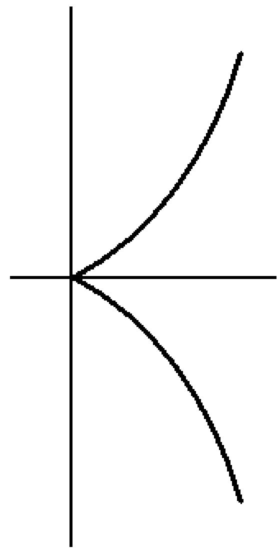

Figure 4.25: The (real) graph of $y^2 = x^3$

4.3.1 Mathematical Context

Matt & Bart are working on a problem in algebraic geometry having to do with *splicing varieties together*. A *variety* is basically the graph of an algebraic equation.[3] For instance, a parabola can be considered a variety since it's the graph of an equation of the form $y = ax^2 + bx + c$. Another example is shown in Figure 4.25, which depicts the graph of the algebraic equation $y^2 = x^3$ (known as a *cusp*). It's perfectly reasonable to have as many variables as you like involved in a variety; for instance, the expression $5x^3 + 2x^2 - 3y^2 + 18z^5 - 4z + 7w^5 = 0$ might have a solution set, and if so its solution set mapped out in Euclidean space would be (roughly speaking) a variety. However, since there are four variables involved, it would require at least four dimensions to depict what this variety might look like. Most varieties involve too many dimensions to draw.

[3]Properly speaking, what I'm describing is a type of *algebraic set*. An algebraic set is a solution to a system of algebraic equations. A variety is an *irreducible* algebraic set, meaning in essence that it's not the union of two distinct algebraic sets. But this "varieties are graphs" idea is close enough for our purposes.

One question that arises is, what happens as we continuously slide from one kind of variety to another? In normal Euclidean geometry, we can use a parameter t to describe smoothly moving from one point in space to another like so:

$$t(a,b,c) + (1-t)(a',b',c')$$

The addition here happens entrywise: $(a,b,c)+(x,y,z) = (a+x, b+y, c+z)$. As t takes on values from zero to one, the point that the expression above refers to moves smoothly from (a',b',c') to (a,b,c). We can use this same trick to convey what it means to move smoothly between two varieties:

$$t\sum x_n^5 + (1-t)\left(\sum x_n^3 \times \sum x_n^2\right) = 0$$

Each sum (i.e., $\sum x_n^m$) represents an algebraic expression that corresponds to a variety. So as t moves from zero to one, we move smoothly from the variety described by $\sum x_n^3 \times \sum x_n^2 = 0$ to the one described by $\sum x_n^5 = 0$. That is, t parameterizes a whole *family* of varieties the same way in Euclidean geometry it can parameterize a whole family of points along a straight line.

Of particular interest to Matt and Bart is what happens to *curves* on these varieties. A curve on a variety, in this context, refers to the intersection of the variety in question with a variety produced via a linear equation. For instance, in Euclidean 3-space, we could describe the curves on the sphere[4] as the set of all possible circles on the sphere since a curve would be the intersection of the sphere with a plane[5]. However, this description implicitly uses the *real* numbers $\mathbb{R}$, and from the perspective of an algebraist the real numbers have a significant shortcoming: they aren't algebraically complete. That is, it's possible to write

[4]We can describe the sphere by the equation $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = r^2$ where r is the sphere's radius and (x_0, y_0, z_0) is the sphere's center.

[5]Planes in $\mathbb{R}^3$ come from linear equations of the form $ax+by+cz=d$ where a, b, c, and d are constants. The intersection of this with a sphere gives us the ability to substitute one of the variables — say, z — in the equation for the sphere, and the result is the equation of a circle. It's basic algebra, although the formula looks a little messy so I'll omit it here.

polynomials with coefficients in $\mathbb{R}$ such that the polynomial doesn't have any roots in $\mathbb{R}$. ($x^2 + 1$ is the most typical example of this.) The set of all complex numbers $\mathbb{C}$, on the other hand, *is* algebraically complete. So, when interpreting algebraic equations like $y^2 = x^3$, algebraic geometers often expand the context such that the variables (in this case x and y) are considered *complex*. For instance, $(x, y) = (i, e^{\pi i/3})$ can be considered a solution to $y^2 = x^3$ even though it's not a point on the cusp in $\mathbb{R}^2$ depicted in Figure 4.25.

For technical reasons that aren't important in our analysis, Matt and Bart find it helpful to take this one step further and look at the variables as elements of *complex projective space* $\mathbb{CP}^n$. (Most commonly $n = 2$.) I'll elaborate more on what this means in section 5.3.1 on page 143. The key part that *is* important for our purposes is that this move makes the number of possible curves finite. That is, it makes sense to ask how many curves there are in total on, say, the projective parabola and the answer will be a finite number.

It turns out that this number of curves doesn't change as you smoothly transition from one variety to another using the t parameter described earlier. This is called the *Gromov-Witten invariant* and was a result that had been established about a decade before the video observations of Matt and Bart. Part of the challenge that Matt & Bart were facing was how to splice together several different varieties at once such that you'd need more than one parameter t to transition across the splicing. They aren't entirely sure how to do this; that's part of the content of their struggle. But during the clip we'll examine here, they're trying to reason through how they might make an analogous extension of the "one-parameter" argument to the "multiple parameters" situation. The difficulty, in short, is that when splicing n varieties together for $n > 2$, the curves on all the varieties can interact in complicated ways. It's therefore unclear to Matt & Bart what exactly needs to be modified in order for this all-at-once splicing to work.

4.3.2 Video Description

This analysis refers to Table A.3. Matt is wearing a white sweater and Bart is wearing a dark jacket and glasses with a dark frame.

At the start of this clip, Matt & Bart are examining details of a diagram on the blackboard trying to make sense of it. On row 1 you can see most of this diagram immediately around Matt's right shoulder in the left-hand image. This diagram is meant to describe how different curves on different varieties interact when the varieties are spliced together.[6] The basic idea for doing this came from the decade-old result looking at splicing just two varieties at a time together.

After Bart suggests that two different objects on the diagram refer to the same thing (rows 2–6), both mathematicians seem to search the whole board for clues as to how to think about whether that statement is true and how to proceed from that point (rows 7–10). In row 11 we see classic embodied displays of struggle: Bart steps back and tilts his head, and Matt reaches up to scratch his temple and then (row 12) drop his hand to his chin.

Here we have both of the key ingredients for proximal inhibition: both Matt and Bart feel a desire to move in meaningfully toward the board but are inhibited from doing so due to not perceiving where the meaning lies. This manifests most obviously with Bart as he leans back and tilts his head in a way reminiscent of many of Joseph's movements. There are hints of this phenomenon in Matt as well: he has covered his torso with his right arm as though to form a barrier between himself and the blackboard, and his posture has collapsed slightly. However, the palpable sense of *resistance* or inhibition that characterizes proximal inhibition

[6]Properly speaking, these various functions are recording information about the *tangency* of the curves. For instance, a curve whose points are of the form (x, x^4) has a tangency of 4 at the origin. It's possible to record tangency information about a curve via a function $u_p : P_p \to \mathbb{N}$ from an algebraic structure P_p closely related to the monoid to the natural counting numbers $\mathbb{N}$. However, the proper definition of tangency and how all the tangencies of the different curves relate turns out not to be terribly important for our analysis.

isn't clearly here in Matt. This illustrates an important caveat in the logic of watching for proximal inhibition: seeing it certainly seems to signal a particular way of structuring the experience of struggle, but it doesn't always appear in challenging situations when it might seem appropriate.

Bart then zips in and annotates the diagram (rows 13–15). However, it's important to notice that this actually is *not* a collapse of proximal inhibition like that described at the end of section 4.1.2. This wasn't a matter of Bart suddenly gaining insight into the mathematical situation. Instead, the annotations Bart is adding here are labels for some of the arrows. For instance, one part of the diagram read at the beginning like so:

$$P_\eta \to Q$$

In row 13, Bart adds a label to the arrow, like so:

$$P_\eta \xrightarrow{\varphi_\eta} Q$$

From earlier, it was clear that both Bart and Matt knew quite well what label belonged there. It didn't provide any deep insight for Bart to write φ_η there. It's possible that Bart felt the need to write that label in order to make it easier to remember which label goes there. However, it seems at least as plausible that Bart did this because he felt drawn toward the board and noticed something he *could* approach the board to add. Notice how he immediately turns away from the board and walks away (rows 16–17) as soon as he has written this annotation, seeming to emphasize the fact that his writing φ_η bought him only a moment of relief from the friction between wanting to approach the problem at the board meaningfully and being uncertain of how to do so. We even see him lean his upper torso away from the blackboard and brace his arm on a chair behind him once he turns around (rows 18–20).

It's worth noticing here that Bart does *not* put a barrier between his torso and the blackboard. This seems to be a stylistic difference in dealing with the discomfort involved in proximal inhibition. Bart seems to prefer getting physical distance, whereas here Matt seems to prefer folding his arms over his torso. These patterns seem to be fairly consistent for each mathematician across the various clips observed; for instance, Bart generally seems to prefer getting distance over torso-shielding whereas Matt's preference seems generally reversed. This suggests that there may be a kind of stability in expression of proximal inhibition for each individual even though the variance between individual styles of expression might be considerable.

However, this tendency is not strict. We see in rows 14–17 how Matt looks at Bart annotating the diagram, lowers his left arm to form crossed arms across his solar plexus, and leans back from the board. It seems as though he was checking what Bart was doing to determine whether Bart had had some kind of insight and was writing it up on the board. This bespeaks the same kind of "pressure" back from the board that we saw in previous cases of proximal inhibition. So even though we see Matt defaulting to something of a shielding strategy for dealing with the discomfort involved in this struggle, we still see him responding to the need for distance from the board.

At row 22 we see Bart moving his left hand up to adjust his glasses while leaning forward slightly. This seems to be his expression of the beginning of an insight into the problem at hand. Four seconds later he tilts his head to the side and moves his hand to his mouth as Matt looks at him (row 23). Both the head-tilt and the hand touching the mouth are signals we've seen before: the head-tilt (paired with fixedness of attention) seems to bespeak an attempt to navigate struggle by getting a new perspective, and the hand touching the lips can indicate an effort to "hold back words" as the mathematician checks both whether the insight just seen is correct and how to express it. Yet when Bart doesn't speak for a few seconds, Matt starts to suggest something while initiating a hand movement toward the board (row 24). Notice that this is consistent with the idea that Matt knows on some level what some of Bart's gesture mean. Matt is probably trying to offer a possible way forward given that their current direction of thought might be too unproductive for the effort they're putting into it.

A fraction of a second after Matt starts speaking, Bart starts leaning forward (row 24). He hasn't yet taken his hand away from his mouth, so it's reasonable to suggest that he hasn't quite finished thinking through what he wants to say. Yet he apparently wants to make sure it gets said and would rather move toward the board slightly prematurely than have the thought lost. In the very next moment he interrupts Matt, rights his head, and throws his left hand forward away from his lips (row 25). Notice in particular the words he says here: "Oh, of course it is! It's zero! Uh, I mean that's the kernel…" (rows 25–27). A *kernel* of a function is the set of all values that get sent to zero; for instance, the function $f(x, y) = x - y$ has a kernel consisting of all ordered pairs of the form (a, a) (i.e., all the points on the Cartesian plane along the line $y = x$). Saying "it's zero" is actually pretty imprecise as a way of describing why two different parts of the blackboard diagram are in fact equal when neither one is zero. However, in this case Bart is talking about the *difference* between the two objects in question. For instance, it turns out that $(1, 2) + (3, 3)$ gets sent to the same value under f as $(1, 2)$ does (namely -1) because $(3, 3)$ gets sent to zero under f. Bart's comment here would be akin to saying that $(1, 2) + (3, 3)$ and $(1, 2)$ are the same because "it's zero." So there's a degree of imprecision here in how Bart first spoke, which matches the fact that he started moving his torso toward the board before he felt ready to remove his hand from his mouth. This is also why we see him needing to correct himself a moment later.

Row 25 most vividly shows the moment of collapse of proximal inhibition. Bart is no longer just moving toward the board, but the words fly out with his hand. In response, Matt deflects the movement he had started with his left hand in row 24 so as to touch his ear (row 26) and return to the crossed-arms position (row 27). His attention has gone to the board where Bart is about to describe his thoughts. From that point through most of the rest of the clip, Matt slowly drifts toward the board (rows 27–31). This is not the "zooming in" movement we saw with Joseph earlier in this chapter or with Bart in rows 25–27. However, there's still a sense of the proximal inhibition having been lifted, albeit in a way Matt has not yet seen. This highlights a key social dynamic for proximal inhibition: some of the loss of inhibition can transfer socially even before all parties involved have a sense of how to move in meaningfully. There's a sense here of Matt moving in to be involved in the revealing of the insight. (We can be fairly confident that Matt isn't moving in here just to see the board better because throughout this and other clips, he has been quite comfortable referring to specific parts of the board from a distance greater than that shown in row 26.)

4.4 Chapter Summary

Proximal inhibition consists of two phenomenological elements, namely:

- a desire to move in toward the blackboard to clarify a point, paired with
- an inability to do so due to not perceiving precisely where to move and/or what to say.

This phenomenon occurs in some instances of mathematical struggle. Here I've focused on three clips to examine a few examples and non-examples.

I've also made an effort to highlight three embodied phenomena that, while not directly related to proximal inhibition specifically, are sometimes seen in fairly specific contexts in many forms of mathematical struggle including proximal inhibition. First, there are *self-soothing gestures* such as touching the neck or face.

These seem to serve at least two purposes: they help to decrease stress when faced with internal difficulty or struggle (Navarro, 2008), and they help to distract one's senses from one's environment beyond the skin. The latter effect may aid concentration on challenging problems like those of our subjects here.

Second is *touching the lips with the hand*, especially after a long pause (often more than 10 seconds). This gesture signals that the mathematician is about to speak, as though he or she has just had a key insight but is holding words back in order to check the idea for veracity and the phrasing for accuracy. The converse does not necessarily hold, though: we don't find that mathematicians almost always touch their lips upon having an insight.

Third, we see that some mathematicians at times *tilt their heads to the side* when tackling a challenging problem related to material on the blackboard. In the same way that the presence of the lip-touching gesture tells us more than its absence does, this head-tilt seems to be more powerful as a positive signal of struggle than does its absence as an indicator of ease.

Notice that these various embodied phenomena do more than signal to an onlooker that the subject is experiencing mathematical struggle; they also give us insight into what the structure of the experience is for the person in question. For instance, imagine that we see someone working on a problem in complex analysis and are writing a few diagrams and equations on a board. They then step back and cross their arms over their belly. Thirty seconds later they tilt their head to the side, and then around ten seconds after that they reach up and rub the side of their neck with one hand while keeping the other crossed over their torso. Suddenly the hand rubbing their neck zips to their mouth and all movement stops. Three seconds later they drop their hand and start walking toward the board.

Being aware of the embodied patterns described in this chapter gives us a reasonably vivid sense of what's going on here. This person structured the problem on the board and then stepped back and crossed their arms in a display of proximal inhibition. The head-tilt likely indicated an effort to think of the problem differently, and the rubbing of the neck served to help reduce some of the internal discomfort that comes from cognitive struggle. The flash of insight occurred at the moment the hand zipped to the mouth, and the proximal inhibition lifted a few moments later as demonstrated by their stepping forward.

This is just one possible sequence of many. By having encountered instances of embodied patterns of navigating struggle such as in this chapter, our perception of this kind of behavior changes and we see more than we did before. The next two chapters will explore what a similar type of embodied phenomenological analysis can reveal to us about mathematicians' use of the blackboard itself.

Chapter 5

Invoking the Continuum

Many people have observed how there seems to be a kind of natural division in mathematical practice between algebraic and geometric reasoning. For instance, Hadamard (1949) describes how he and many other mathematicians use a great deal of conscious mental imagery to think about mathematics, whereas many of their colleagues report that they see nothing when thinking about math problems. Burton (2004) indicates how these non-visualizers seem to be quite aware that some people use mental imagery, whereas those who report mental imagery sometimes indicate that they think *all* mathematicians reason this way. In my own experience as a teacher, I've noticed that I often have to work out which of a more algebraic or geometric approach is likely to work for the student who comes to me asking for clarification during office hours.

The question of what exactly distinguishes algebraic and geometric ways of thinking or representing seems intuitively obvious at first pass, but it actually turns out to be pretty complicated. At first we might want to say that a mathematical representation is geometric if it "looks like" the object it's supposed to represent, whereas it's algebraic if it represents arbitrarily. For instance, a drawing of a sphere is meant to represent a sphere by looking like one, whereas the algebraic formula

$x^2 + y^2 + z^2 = 1$ represents a sphere by indicating quantities and relationships between quantities in a way we could have in principle represented any other way (e.g., the equation $\alpha^2 + \beta^2 + \gamma^2 = e^{2\pi i}$ would work just as well as long as we relabel the Cartesian axes of three-space). But this turns out to be problematic. Just as I'm free to alter x to become, say, α in the algebraic expression, I could also draw the sphere using a different color of ink. Thus the arbitrariness of the algebra *does* have an analog in the geometric situation. And according to the set-theoretic definition of geometric objects like the sphere, the equation form actually visually appears more similar to what a sphere "really is" than does the relatively arbitrary representation we use by drawing something that looks like a ball.

So instead of trying to come up with a definition that properly captures our intuition about the division between algebraic and geometric methods, I'll highlight just one aspect of reasoning that seems to be important for what one might be inclined to label as "geometric thinking." That aspect is *the invocation of the continuum.* That is to say, one thing that makes the drawing of the sphere different from the equation of the sphere is that making small alterations to the drawing slightly changes key properties of the mathematical object to which it refers. For instance, if you redraw the sphere so that it's a little bit smaller, that means its radius is a little bit shorter. However, if you redraw the *equation* a little bit smaller, that doesn't change anything at all about its meaning. Tiny movements matter in the drawing but not in the equation.

This might not capture everything of interest in the distinction between algebraic and geometric reasoning. However, it *does* seem to highlight one aspect that mathematicians implement rather frequently. Most of the mathematicians I've investigated invoke the continuum as a method for navigating mathematical struggle. However, the role that the objects within the continuum of the blackboard play differs significantly depending on *how the continuum was invoked.* In this chapter, I'll spell out the major nuances that emerged as salient from the work of the mathematician pairs I observed.

5.1 Clip #1: Inverted Pyramid

For the opening clip of this chapter, I'll return to Matt & Bart, whom we met in section 4.3 (page 90). The focus for Matt & Bart is algebraic: when the objects they're referring to have a natural geometric interpretation, more often than not the objects involve too many dimensions to be visualized, and the way this pair chooses to deal with this is by focusing mainly on the equations and functions involved. Yet Bart still elects to invoke the continuum here to organize his thinking and encourage himself and Matt to cease making a particular sort of error.

5.1.1 Mathematical Context

Most of the mathematical context for this clip was given in section 4.3.1: Matt and Bart are thinking in terms of splicing varieties together. The new element we'll need for this clip is the idea of a *monoid structure* attached to certain varieties.

Monoids

A *monoid* is a set of mathematical objects together with an operation such that the set (a) is closed under the operation, (b) is associative under the operation, and (c) has an identity element with respect to the operation. For instance, the set $\mathbb{N} = \{1, 2, 3, \cdots\}$ of natural numbers is a monoid under multiplication: it's closed (meaning you can multiply any two natural numbers and you'll get another natural number) and associative (meaning $n \times (m \times p) = (n \times m) \times p$ for any natural numbers n, m, and p), and multiplying any natural number by the natural

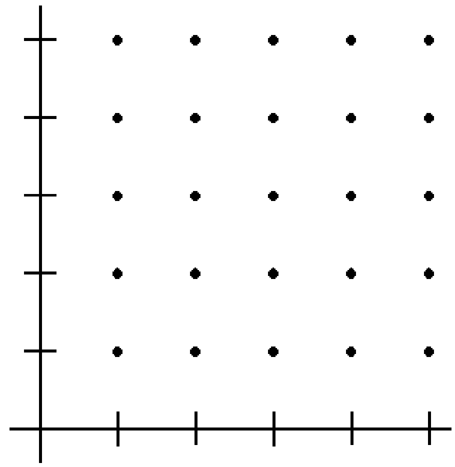

Figure 5.1: A geometric depiction of a monoid

number 1 results in no change. The set $[1, \infty)$ of real numbers no less than 1 under multiplication is another example of a monoid. Yet another example is the set $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ of integers under addition.[1]

We can see a geometric example of a monoid in Figure 5.1. The elements of the monoid are the crossing points of the grid — essentially any point on the Cartesian plane whose coordinates are of the form (m, n) where m and n are non-negative integers. The operation is vector addition: if we draw an arrow from the origin (i.e., $(0, 0)$) to each of the two points being added together, then the sum will be the point you get by translating one of the arrows so that it starts where the other arrow ends. This has the net effect of adding coordinates: $(a, b) + (c, d) = (a + c, b + d)$. But it's worth noticing that we can capture this monoid structure with a quick picture.

The picture Bart uses in this clip is related, but is more like that in Figure 5.2. The idea here is that you can get to any element of the monoid by adding

[1] If you're familiar with the idea of a *group*, you can think of monoids as a generalization of groups where we don't require the existence of inverses. Therefore all groups are monoids.

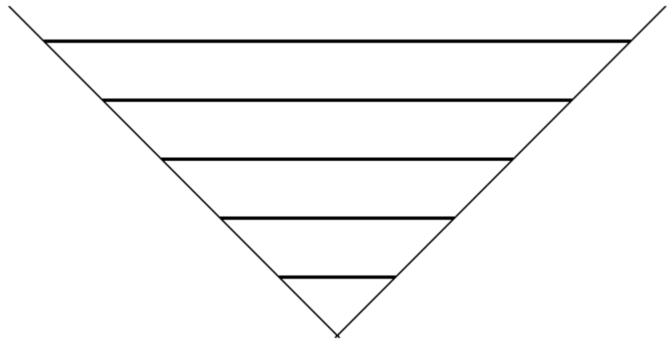

Figure 5.2: A different depiction of the same monoid from Figure 5.1

together multiple copies of just two vectors, namely $(1, 0)$ and $(0, 1)$. (These are thus referred to as *generators* of the monoid.) For instance, $(2, 3) = (1, 0)+(1, 0)+(0, 1)+(0, 1)+(0, 1)$. Each addition of a generator causes us to go up the "ladder" by one "rung," so the horizontal lines in the middle are meant to depict the internal dynamic of the monoid.

Monoids on varieties

It turns out that it's possible to assign a monoid structure to a variety in a way similar to how we assigned a monoid structure to the first quadrant of the Cartesian plane above. However, for a general variety V, it turns out to be necessary to pick a point p on V in order to generate the monoid structure. To highlight this, Matt & Bart refer to the monoid generated by the choice of a point p on a given variety as P_p.[2] The point p allows Matt & Bart to focus on one of the (finitely many) curves passing through p in order to create a function $u_p : P_p \to \mathbb{N}$. That is, elements of the monoid P_p are given correspondence to natural numbers.

[2]This isn't exactly true. P_p is a quotient of a particular monoid structure with respect to p. However, the difference turns out to be both very complicated and unimportant for the purpose of following what Matt & Bart are doing in this clip.

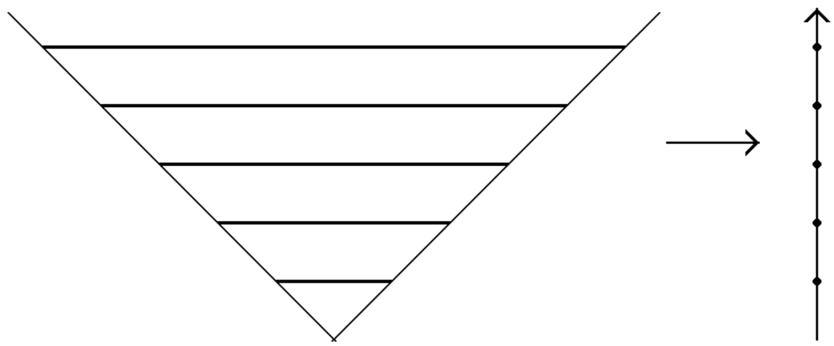

Figure 5.3: The map $u_p : P_p \to \mathbb{N}$ represented graphically

If we represent P_p as in Figure 5.2, then we can see the function u_p as being a compression to a vertical line of counting numbers (Figure 5.3).

One of the challenges Matt and Bart face when splicing varieties together is to track what happens to the monoid structures of each of the varieties being spliced. In order to work this out for splicing just two varieties at a time, it turns out to be helpful to pick points on the two varieties that will end up being fused together in the splicing. To be more specific, suppose our two varieties are V and V', and we're picking points $p \in V$ and $p' \in V'$. Then we'll rename the corresponding varieties P_p and $P_{p'}$ to keep track of our choice of points and curves. When we splice these two varieties together, the points p and p' will fuse together to form a new point q (Figure 5.4). Matt & Bart refer to the monoid that corresponds to the resulting spliced variety as P_q.

In section 4.3.2 we saw Matt and Bart using an algebraic diagram that

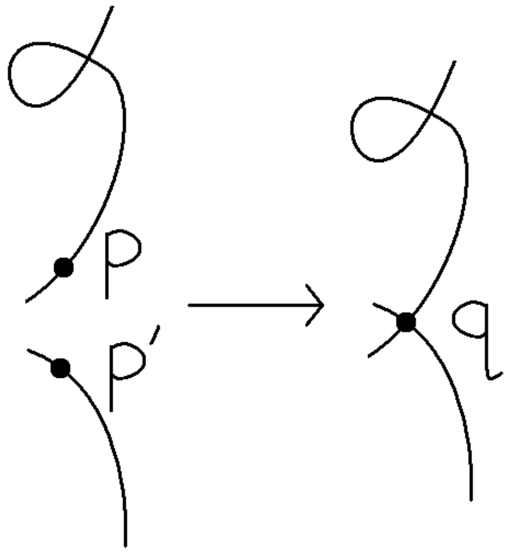

Figure 5.4: Two curves fuse together at q

looked something like this:

$$\begin{array}{ccccc} & & P_\eta \overset{\varphi_\eta}{\to} Q & & \\ & \nearrow & & \nwarrow & \\ P_q & \to & Q \oplus_{\mathbb{N}} \mathbb{N}^2 & \subset & Q \oplus Q \\ & \searrow & & \swarrow & \\ & & P_{\eta'} \overset{\varphi_{\eta'}}{\to} Q & & \end{array}$$

This is meant to capture what exactly happens to the monoid structure during the splicing. Q here is a monoid, and the functions φ_η and $\varphi_{\eta'}$ are meant to take the place of the maps u_p and $u_{p'}$ respectively.[3] So, the top and bottom rows represent the monoid structures of the two varieties being spliced together, and the middle row is the same for the monoid structure of the resulting spliced variety. The arrows going between the rows represent functions that relate the rows to one another in a way consistent with the splicing. The set inclusion (i.e., $Q \oplus_{\mathbb{N}} \mathbb{N}^2 \subset Q \oplus Q$) points

[3]This also isn't exactly true. There is, again, a difference emerging from the fact that we're actually taking *quotients* of the spaces involved. However, as before, we can basically ignore this for the purposes of the present analysis.

out that there's a clear way to send elements of the monoid for P_q to each of P_η and $P_{\eta'}$ based on a kind of projection.

5.1.2 Video Description

The row numbers cited in this section refer to the microanalysis in Table A.4 of the appendix.

Build-up to geometric shift

At the beginning of this clip, Bart (on the left) and Matt (on the right) have spent 4 minutes and 19 seconds sitting across from one another discussing how to approach this question of splicing multiple varieties together at once. The material currently on the board came from an earlier attempt that was almost purely algebraic. They had been trying to approach the problem as a whole, looking at the collection of all the varieties involved and trying to describe the result of splicing them all together at once.

Bart suggests that they focus instead on the question of "how to move *some* curve" (rows 4–7), by which he means that they should consider splicing just two varieties at a time via tracking curves on each of them. He then jumps up and prepares to erase the board (rows 10–12). But then he pulls his hand away from the board and says, "What we *do* have..." (rows 13–15). On "do," he makes a jabbing movement with the eraser as though to punctuate his point (rows 14–15). But after this he pauses and stares at the board for a few seconds without moving, and then erases most of the left-hand half of the board (rows 17–20), pausing before erasing one last portion near the middle (row 21).

Both the erasing and the pauses here are significant indicators of Bart's thinking. He chose to erase the part of the board with lots of algebraic reference to an approach they were partially abandoning. The diagrams to the left of that algebra remains untouched. At no point throughout the rest of this clip do either Bart or Matt refer to those diagrams on the leftmost edge of the board, but later algebraic expressions *do* replicate some of the material Bart has just now erased. Thus it seems likely that this act of erasing is meant to signal a *shift in approach*. Bart and Matt have agreed to try a new strategy in response to their earlier struggles.

Part of what signals this action as meaningful is how Bart did *not* simply jump up and mechanically erase parts of the board to create space to write. Instead, he got up, prepared to erase, and then *aborted the movement* in order to make a point ("What we *do* have…"). He *then* erased sections of the board, pausing briefly (row 21) in order to consider whether he would need the algebraic map he was about to erase. Thus this is not just a clearing of space on the blackboard, but a *calculated* clearing of space, demarcating a shift in strategy.

You can see in row 21 how Matt is about to speak. Right before then he reached up to touch his mouth (row 18), temple (row 19), and nose (row 20) in sequence before dropping his hand down to speak (row 21). This is yet another instance of the mouth-touching "I'm formulating something I want to say" signal mentioned earlier. As mentioned before, this gesture seems to indicate that the mathematician has an insight that he or she thinks is pertinent but wants to take a moment to make sure the idea makes sense or is properly thought out. At this moment, Matt is referring to the idea that if we look two curves each on a different variety, we can examine what happens around those curves as we splice the two varieties together in a way such that those two curves meet at a single point (rows 22–24). This defines their mathematical approach throughout the rest of this clip.

At just the moment Matt finishes speaking, Bart raises his hand to his forehead and glasses while bowing his head (row 24). He thinks in this position for two seconds, at which point Matt starts to speak. That prompts Bart to interrupt with “Yeah, exactly” while touching his mouth with his hand (rows 25–26). This seems to mark the moment when Bart and Matt are on the same page, so to speak. Now they’re both thinking about approaching the splicing of varieties together by focusing on what happens as two paths, one on each variety, get spliced together at a point. It’s only after this moment of acknowledged agreement that Bart lowers his left hand, as though he no longer needs the self-soothing that hand appears to provide.

Bart prepares to draw what they’re describing on the board but then withdraws his hand while thinking about it (rows 32–33). He pauses all movement for five seconds at this point. He then walks up to the board and erases some faint leftover markings using his fingers (rows 35–36) and then paces slowly as though deep in thought (rows 37–40). Notice also how Matt has turned his face away from Bart and the board while touching his chin, despite the fact that his torso is still oriented solidly toward both Bart and the board (row 36). This pattern of seemingly absent-minded body stimulation — Bart’s pacing and Matt’s chin-touching — combined with the turning away of the gaze seems to be a telltale signal of careful thought.

Geometric shift

Bart abruptly turns toward the board, starts speaking, and prepares to write on the board (rows 41–42). But at that moment he breaks his speech and then starts over as he moves his hand to the right (row 43). This seems to indicate that what Bart is about to put on the board here is meant as something non-central, like a guideline to thinking rather than the main work. As we’ll see in a moment, it’s reminiscent of how a student who’s working on a difficult trigonometry problem might quickly sketch a unit circle off to the side in order to keep the algebraic relationships straight: while the drawing helps to guide the thinking,

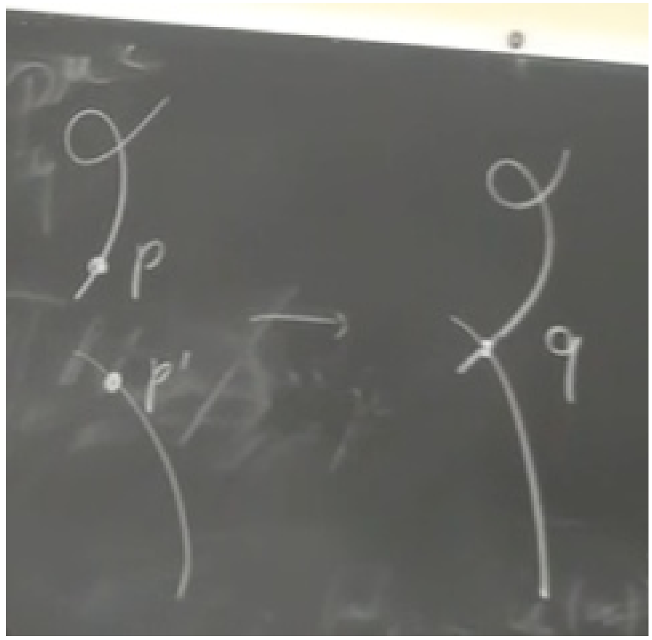

Figure 5.5: The points p and p' merge into point q

in some sense the algebra is more central and is what will make it into the final presentation of a solution.

As Bart pauses after saying “we want to,” he draws two curves with points p and p' on them, a right arrow, and a picture showing the two curves joined at a point q (rows 43–45; Figure 5.5 depicts Bart’s drawing). This is a considerable shift in representation methods. Although we can see some seemingly geometric depictions on the left-hand side of the board (in, say, row 45), those are more like graphs than drawings: they indicate relationships between algebraic objects. Slight movements of the inscriptions don’t change the meaning the way slight movements of the graph of $y = 2x$ would make the graph refer to something entirely different. And while the diagrams on the right are indeed geometric (technically referred to as “tropical diagrams”), they aren’t part of the particular reasoning pathway that had Bart and Matt write algebraic statements on the left-hand half of the board where Bart later erased them. Thus we can deduce that there is something about the specifically *geometric* representation that is this pair of paths merging into one that Bart anticipates as being useful. (Notice that Matt still hasn't looked up at the board at this point.)

It's worth noting, though, that these paths have an arbitrary character to them. Although the top path with the point p on it in Figure 5.5 has a loop in it, there actually wouldn't be anything different about its having no loop, or two loops. It has the shape it does because it represents a *generic* path. This is much the same way as someone who is trying to think about triangles in general might visualize one specific triangle (say, a 30-60-90 triangle) with which to reason. Many of Euclid's proofs draw a specific triangle but intend the triangle to be generic in some sense with variable side lengths and angle measures. We still call such proofs "geometric," and for our purposes they still fit the phenomenon we're interested in: they invoke the continuum. Small adjustments in the shape of the triangle might not affect the *reasoning*, but they do actually *mean* something in terms of which triangle is depicted. In the same way, the paths that Bart has drawn here seem to represent variable paths that we might twist or loop or otherwise modify. Such modifications have meaning in that it would change one specific path to another — but it doesn't affect the role that the *representation* has in his reasoning.

Contrast this with an expression like $x^2 - 8x + 15$. This can stand in for a whole family of quadratic expressions — for instance, any of the form $x^2 + (a + b)x + ab$. So if we were to use this particular expression to reason about such quadratic expressions in the abstract, we would have made this example generic while still being specific. Yet it still would not invoke the continuum: if I were to move the 15 slightly upward, or if I were to make the 2 a little bit smaller, it would not change *which* example from the family of quadratic expressions it is.[4]

[4]It might occur to some readers that were I to make the 2 sufficiently *larger* so that x^2 became $x2$, it would mean something different. However, it doesn't clearly mean anything to enlarge the 2 by half as much, or a tenth as much. This is somewhat analogous to the ε-δ definition of continuity: something counts as an invocation of the continuum for a given mathematician if

Bart steps back, checks that Matt agrees with the drawing, and touches his chin (rows 46–47). This begins a five-second-long staring at the board in which it seems that this new drawing is helping to guide Bart's and Matt's thoughts as they wrestle with the exact method of splicing the varieties in question together.

Confusion in the algebra

After the pause, Bart jumps forward and starts writing down functions at the top of the board (rows 48–52). Notice that this is in the same place on the board where he had originally been about to draw the path-splicing image. This seems to indicate that the algebra he's writing there may be more central to the point of his analysis here. As of row 52, Bart has written this:

$$u_p : \quad P_p \to \mathbb{Z}$$
$$u_{p'} : \quad P_{p'} \to$$

Here, P_p and $P_{p'}$ are meant to be varieties containing the points p and p' respectively as well as their corresponding paths. The u maps indicate the monoid structure that is part of each variety. Recall that part of the question of what happens in this splicing is precisely what happens to the monoid structure. So part of what Bart and Matt need to work out is how these u maps behave and interact after splicing P_p and $P_{p'}$ together. In this case Bart made a minor mistake that's about to be corrected: he had written $\mathbb{Z}$ (the integers) instead of $\mathbb{N}$ (the natural numbers, i.e., the positive integers) as the codomain for u_p. This presumably came from Bart defaulting to thinking about groups instead of monoids (since group structure is much more common in abstract algebra than is monoid

and only if arbitrarily small changes in the notation correspond to *arbitrarily small* changes in meaning for the mathematician in question.

structure). The resulting correction (rows 53–54) makes the algebra look like so:

$$u_p : \quad P_p \to \mathbb{N}$$

$$u_{p'} : \quad P_{p'} \to \mathbb{N}$$

At this point Matt suggests that P_p and $P_{p'}$ are actually the same "because p and p' have to attach to the same thing." Bart thinks for a few moments and then agrees, writing an equal sign between P_p and $P_{p'}$. This turns out to be an error, which is key to what follows: the relationship between P_p, $P_{p'}$, and P_q (which has not been mentioned as yet in this clip) is something that Matt and Bart straighten out only after invoking the continuum again later on.

Then Bart moves his hand near the board and then pulls it away to turn to Matt and say, "So now we now have the whole diagram that we drew a hundred times" (rows 62–65). The diagram they're referring to is shown in section 5.1.1. It refers to the process of splicing just two varieties together. Bart is suggesting here that they can use essentially the same diagram in order to explore what happens specifically to two varieties in a splicing of many others at once. It's important to note, however, that the diagram to which Bart is referring does *not* invoke the continuum at all: it uses letters to represent varieties and monoids, and labeled arrows to indicate functions going from one type of mathematical object to another. Small movements in this diagram do not reflect changes in the concepts to which it refers.

Bart then starts to draw the diagram. The leftmost part of this diagram should be P_q (again, not yet referenced in this clip), so he writes "P" and then glances up to check the relationships indicated by the monoid maps at the top of the board (row 65). Those u maps are very closely involved with the diagram Bart is trying to draw, so it's unsurprising that he'd embody a connection between the two by keeping his hand and body oriented on the diagram-in-progress while focusing his attention on the u maps. Bart seems to have a sense that he needs to navigate between those monoid maps above and the diagram-in-progress below, as though doing so binds their meanings together (rows 65–68). This is the first time that P_q is referenced here.

At this point, the mistake Matt and Bart had made becomes more salient. Bart recognizes that if $P_p = P_{p'}$, then by the same reasoning they're both equal to P_q. So he laughingly says, "So let's just call it 'P'" while he zips up to edit the line below the u maps (row 68). So now the material at the top of the board reads:

$$\begin{array}{rll} u_p : & P_p & \to \mathbb{N} \\ & \| & \\ u_{p'} : & P_{p'} & \to \mathbb{N} \\ & P_q & = P \end{array}$$

Yet at this point Matt objects (rows 70–73). Notice here how Matt does *not* touch his face before speaking, as though to interject his objection as rapidly as possible as soon as he saw reason to believe there was a problem. However, immediately thereafter he touches his nose as he formulates how to express his objection (row 72). This kind of abrupt change in face-touching patterns arises at times as a kind of, "I need a moment to think about this" signal. Bart seems to take the strength of Matt's initial objection as good reason to literally step back and reconsider the situation, with his attention fixated first on the u maps and then on the paths to the right of the u maps.

Then after a moment Bart suddenly sees what Matt saw, tosses his hand up in the air in revelation, and goes forward to erase the equation $P_q = P$ (rows 74–75). Notice how Bart seems to realize this at the same moment that Matt figures out how to express what he had wanted to express. Thus we see Matt pointing to the error at the same time that Bart tosses his right hand in the air (row 74). Matt goes back to touching his own head as Bart erases the equation (row 75), but in this case it would seem that he does this in order to address a sense that there is something else wrong with this scenario as they've been thinking of it. He then notices the mistake made before: P_p and $P_{p'}$ are in fact *different* varieties (rows 76–78).

At this point Bart recognizes what Matt is saying and replies, "They are different components" (rows 79–81)[5]. After a moment of thinking at the board, Bart walks toward Matt while making a chopping movement and saying, "They are both faces of P_q" (rows 83–88), as though to drill this point into his own mind while communicating the point to Matt.

The sense we get while watching this recognition of error is that it's not enough for Bart that he recognizes the mistake. There was a reason the mistake was made in the first place. Without emphasizing a different way of thinking, that old reasoning method that caused them to conclude that P_p is the same as $P_{p'}$ might resurface. Thus Bart has to emphasize the reasoning that seems to correct the error: P_p and $P_{p'}$ (the original two varieties) are faces of P_q (the resultant spliced variety). This interpretation gets reinforced throughout the remainder of the clip.

The next move Bart makes is to replace the equation $P_q = P$ that he had just erased with $P_p, P_{p'} \subset P_q$ (row 89). So now the equation set looks like so:

$$\begin{aligned} u_p : &\quad P_p \to \mathbb{N} \\ u_{p'} : &\quad P_{p'} \to \mathbb{N} \\ P_p, P_{p'} &\quad \subset P_q \end{aligned}$$

Then he stands back, glances at Matt, and then looks back to the board while Matt says, "And then what I think we want is that, uh…" (rows 90–91). But

[5]I should note here that I've seen both of the European mathematicians in this study (Bart as well as Joseph) use their middle fingers as pointers like is shown in row 80. I encountered the same phenomenon in my graduate studies in math when I had Russian teachers. There doesn't seem to be any intent to be offensive. It seems as though it's simply an effort to use the longest finger for pointing.

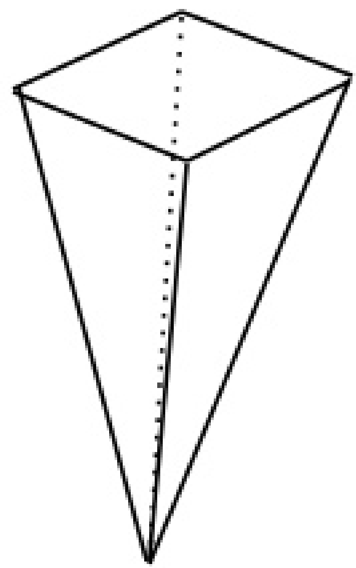

Figure 5.6: A depiction of Bart's drawing of an inverted pyramid

Bart interrupts, which in turn seems to cause Matt to reach up and ruffle his hair in a self-soothing gesture (rows 91–92). Bart's manner and the hesitant way he speaks, paired with the need he feels to interrupt Matt with his not-yet-carefully-phrased hesitation, seems so suggest that Bart is tentatively holding to a new way of viewing the situation that he's still having some difficulty with. It's as though Matt's input threatened to derail Bart's efforts to work through his new perception of the situation, encouraging him to interject his conversational placeholder.

Appearance of the inverted pyramid

After a two-second pause, Bart speaks as though he suddenly sees how to proceed (row 93). It's at this moment we see him invoke the continuum again, this time in the form of capturing the relationships that make his insight sensible. He draws an inverted pyramid like that shown in Figure 5.6 (rows 94–101). This is a marked shift from non-continuum-based representations (e.g., the u maps and the P-lettering for the varieties) to a visibly continuum-based one: whereas small variations in where u_p is written won't change the mathematical meaning of the map $u_p : P_p \to \mathbb{N}$, small variations in the angles at which the lateral edges of the inverted pyramid meet will definitely change which pyramid is being represented, even if only slightly.

As with the paths drawn earlier (Figure 5.5 on page 111), this pyramid is a generic particular: it takes the place of any other of a number of possible pyramids that could have been drawn in its place. It's worth noticing that the shape this pyramid represents probably won't look like a pyramid at all. Most varieties with any interesting properties will have at least four real dimensions[6], and most varieties are too complicated to model with something as flat as the lateral side of a pyramid. So the fact that he's trying to model the relationship between P_p, $P_{p'}$, and P_q with a pyramid like this indicates a use of the continuum that is intended only to capture *relationships between* mathematical entities rather than to capture how they would literally look if we could somehow see the spliced spaces.

This is reminiscent of how we draw a unit circle to indicate the family of solutions to the equation $x^2 + y^2 = 1$. At first glance we might want to say that the circle really is what the solution set looks like. However, that's not literally the case. The solution set is really an infinite collection of ordered pairs of numbers — one for x and one for y in each pair. The circle emerges because we invoke the continuum in order to organize the relationships between these numbers: we think of the x coordinate as indicating how far to the right we move, and the y coordinate as indicating how far up we move. Had we instead interpreted x as distance from

[6]Properly speaking, what I should say here is that most interesting varieties involve more than one complex variable. If there are n complex variables, then we can model the resulting variety with $2n$-dimensional Euclidean space. If there were some way to visualize four dimensions, then, we would be able to see the whole variety of an equation like $y^2 = x^3$ rather than just the two-dimensional slice that looks at only the real component of each of x and y.

the origin and y as a measure of how far counterclockwise we had rotated (i.e., if the equation had instead been of the form $r^2 + \theta^2 = 1$), then the solution set would "look like" an infinity sign. Thus invoking the continuum doesn't necessarily show us what a mathematical object "really looks like," but it *can* give us a spatial way of keeping track of the relationships between different mathematical entities.

The emphasis here is on the word "spatial" rather than "relationship." Someone who is sophisticated at algebraic reasoning can recognize $y = x^2$ as a description of a relationship between quantities x & y even though it's explicitly not an invocation of the continuum. Indeed, some people such as Devlin (2000) argue that mathematics is essentially about nothing but relationships. What continua seem to afford for Bart here is a way of involving his spatial intuitions in representing and thinking about the relationships. For instance, it's not literally the case that the monoid P_q has faces at all; instead, by envisioning it as an inverted pyramid, Bart brings his spatial sense of "behind," "next to," movement, and so on to bear on how P_p, $P_{p'}$, and P_q are related in a way that the algebraic representations simply didn't seem to him to afford.

As Bart draws the inverted pyramid, Matt describes a point about the behavior of the monoid structure of P_q and how it relates to those of P_p and $P_{p'}$ (rows 93–105). During this time, Bart continues to direct his attention toward the inverted pyramid, making minor cosmetic corrections such as closing gaps that his first attempt to quickly sketch the pyramid had left (rows 104–106). These kinds of minor changes actually do not seem to be important in terms of *changing* the diagram per se; rather, they're correcting imperfections in key attributes that the diagram is meant to capture, which Bart fixes as part of his continual effort to focus on it as a way of organizing his thoughts about the interrelationships between each of the varieties involved in this problem. It's unlikely that he felt relevantly confused by the fact that one of the lateral edges he had drawn didn't quite reach

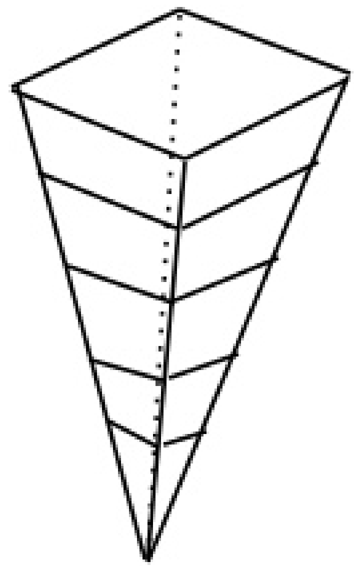

Figure 5.7: Bart's drawing of the monoid structure for P_q

all the way down to the vertex, so it seems reasonable to conclude that this kind of minor perfecting is primarily a focusing tool, like the student who fills in the corners of a triangle while trying to work out a geometric proof.

Bart replies to Matt's comment after two seconds by saying, "Yeah, but it's *facets*, right? Maybe that's important. It's still facets" (rows 107–108). Bart's primary aim here seems to be on correcting a way of thinking that seems to be getting him confused, and which he seems to think that this tool of the inverted pyramid can help him with. He even goes up to the board and writes the word "facets" on it to the right of the line "$P_p, P_{p'} \subset P_q$" (row 109). Notice how vividly the spatial nature of the continuum is shaping Bart's way of thinking and talking about this clarification. He follows this up by drawing horizontal lines on the faces of the inverted pyramid in a depiction of the joint monoid structure of P_q (rows 111–112). The resulting drawing looks like Figure 5.7.

Bart takes a few moments to try to label the faces of the pyramid (rows 113–118). He then recognizes an error: he had mistakenly labeled one of the faces

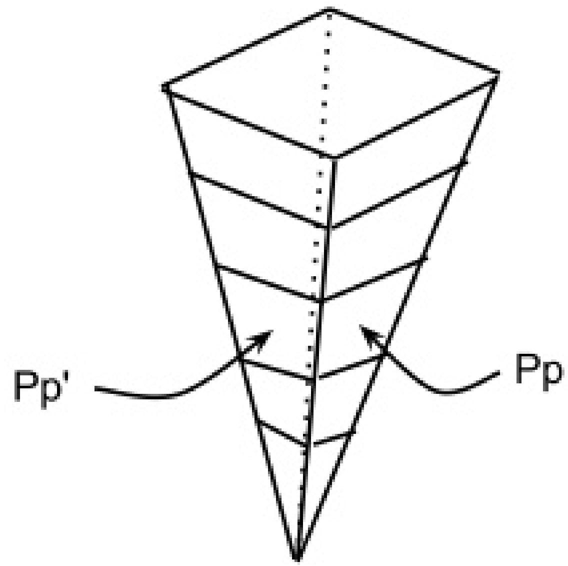

Figure 5.8: The pyramid after getting its lateral faces labeled

P_q when really the whole thing is P_q (rows 118–119). After correcting it (rows 120–121), he ends up with the image shown in Figure 5.8. Bart finds that he needs to indicate that the whole pyramid is P_q by writing "P_q" on the top while lamenting, "Now I'm confused all the time" (rows 120–122). The combination of his expressed concern and his act of labeling seems to suggest that he is intending this invocation of the continuum to guide his thinking enough to show him the way through his confusion.

Summary

It's worth emphasizing that the main domain of interest for Matt and Bart was basically algebraic. Although they were working with ideas originally inspired by geometry, the kind of algebraic geometry they were trying to navigate was too different from common intuitions about space and shape for visual approximations to be of much use in directly depicting the varieties they were examining. Thus the notational default for these two algebraic geometers was something whose representations — usually algebraic symbols — generally tended not to invoke the continuum.

However, we still see cases here where the continuum is invoked in order to keep relationships organized, presumably via spatial intuitions that aren't available in non-continuous notations. In the case of the paths, there was a generic character to them even though they were particular instances as they were drawn: drawing different representatives would not have changed the relationships that those drawings were meant to capture. The same was true of the inverted pyramid, only it also had an additional characteristic: in most cases, even though the objects they're interested in (i.e., varieties) have components that can be visualized (the way it's possible to visualize the two-dimensional part of a four-dimensional sphere as a circle), they generally are *not* going to look like pyramids or faces of pyramids. Yet they still found value in capturing the relationships between various aspects of the varieties within the continuum using this shape. In particular, they were able to capture the fact that the two varieties they were starting out with were being spliced together (along with others) in order to create the larger variety at the end; and they were also able to capture the monoid structure of each variety as well as display the central question that they were trying to explore about the algebraic structure of the resulting spliced space.

So this switching to a medium in which they were invoking the continuum seemed to be an effort to involve spatial intuitions in the perception and organization of the monoids' relationships. This was something they were finding difficult to manage in the non-continuous-notation context, and indeed they found it difficult to keep consistent in the continuum-based context as well. Yet they (especially Bart) persevered anyway out of a sense that this would act as a kind of map out of their confusion-based struggle.

5.2 Clip #2: Finger Moves

This clip centers on Joseph and Bill, the two mathematicians who were working on odd Khovanov homology in section 4.1. In contrast to Bart's and Matt's work, Joseph and Bill work almost entirely with diagrams that invoke the continuum. One effect of this is that those invocations of the continuum take on a different character than did the inverted pyramid Bart drew, with more of an immediate sense that the drawings are what the shapes "look like." The contrast is a bit like that between a circle viewed as a solution set to $x^2 + y^2 = 1$ on the one hand and it viewed as the intersection of a plane with a right circular cone on the other.

5.2.1 Mathematical Context

The problem Joseph & Bill are working on in this clip is distantly related to the Khovanov homology problem described in sections 4.1.1 and 4.2.1, but by this stage they had largely given up on odd Khovanov homology. Instead, they were exploring how to extend the ideas of knot theory into ribbon knots. This requires us to explore two different domains: knot diagrams and Reidemeister moves, and how that translates to ribbon knots. I'll also need to explain one of the diagrams that they use in the second half of the clip, referencing one of the Reidemeister moves.

Knot diagrams & Reidemeister moves

Mathematicians want to consider two knots to be "the same" if they differ purely by wiggling or sliding the "string" around without having to make any cuts or splices. For instance, we want to consider two knots to be "the same" if they differ only by a single twist at some point (Figure 5.9). Similarly, we wouldn't

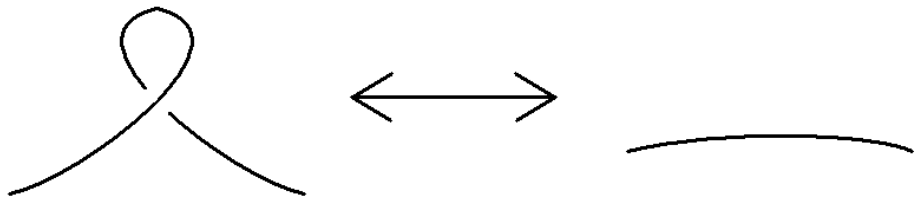

Figure 5.9: Reidemeister 1 (ignoring a kink)

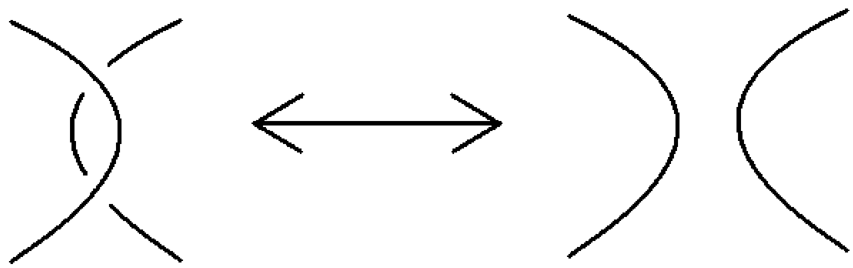

Figure 5.10: Reidemeister 2 (ignoring overlap)

want to consider two knots to be different if two sections of the string happen to overlap without being linked together (Figure 5.10), nor would we want to call them "different" if we were just to slide one portion of string from one side where two other portions cross to the other side of a crossing (Figure 5.11).

This turns out to be a little tricky to formalize when talking about strings. However, if we restrict ourselves to talking about *pictures* of knots (like Figures 5.9, 5.10, and 5.11), we can formalize the rules around how we can manipulate the *pictures*. These pictures are referred to as *knot diagrams*, and the rules for manipulating these pictures are called the *Reidemeister moves*. Figures 5.9, 5.10, and 5.11 are referred to as Reidemeister 1, Reidemeister 2, and Reidemeister 3, respectively.

For a quick example, Figure 5.12 shows how we can transform a circle (an

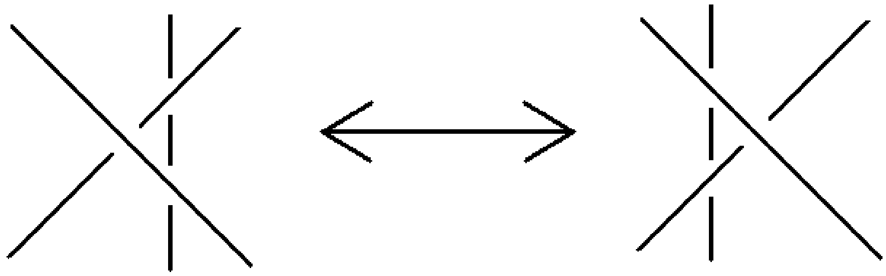

Figure 5.11: Reidemeister 3 (ignoring sliding portions)

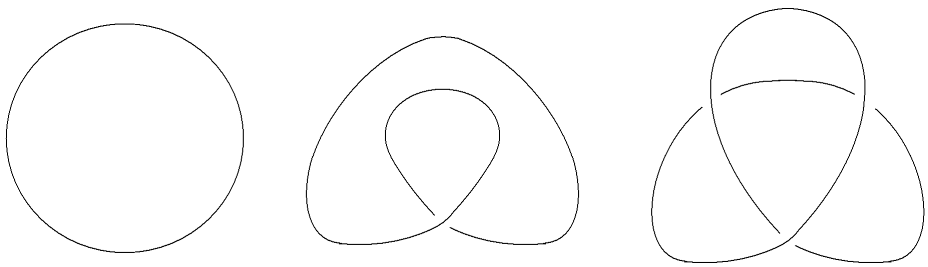

Figure 5.12: Turning a circle into a pseudo-trefoil

"unknot") into something that looks very close to a trefoil - but if you were to actually make this pseudo-trefoil out of string, you could easily stretch it back out into a circle. So as you'd intuitively expect, mathematicians generally want to consider all the diagrams shown in Figure 5.12 as referring to "the same knot" in some sense.

Ribbon knots

Normally, knot theory looks at ways of tangling up a one-dimensional loop in three-space. However, another fair question is what happens when you widen the string into a ribbon so that it's (locally) two-dimensional. In many cases the ribbon knot will look the same, such as in Figure 5.13. However, it's also possible to create a new kind of knot in which the ribbon goes through its own center

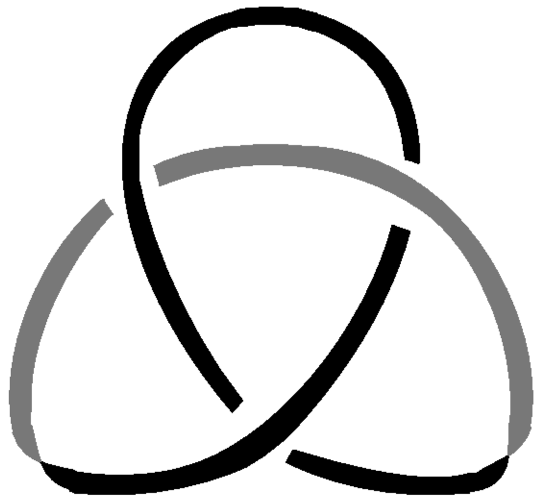

Figure 5.13: A ribbon knot that's equivalent to a normal knot

(Figure 5.14). Since it's always possible to knot a ribbon the way you would knot a string, the collection of ribbon knots includes the collection of normal knots as a subset. So, working out the rules for manipulating ribbon knots should include the Reidemeister moves as a special case.

When trying to switch from talking about knots to talking about knot diagrams, we could imagine just laying the knot down flat on a piece of paper and tracing the string, leaving small gaps to indicate what crosses over what. With ribbon knots, however, a similar trick ends up creating many, many more cases of equivalence than with knots. In fact, the three Reidemeister moves seem to capture all the differences between knot diagrams that we want to describe as being of the same knot. But with a ribbon, we could end up flattening a hood into a fold (Figure 5.15). We certainly would want to consider a ribbon knot to be the same whether it displayed this kind of 'V' pattern even though there's no analog in normal knots. Similarly, a "bubble" in the middle of a ribbon (Figure 5.16) shouldn't make a difference as to which knot it is, so the corresponding ring that comes from flattening out such a shape shouldn't matter.

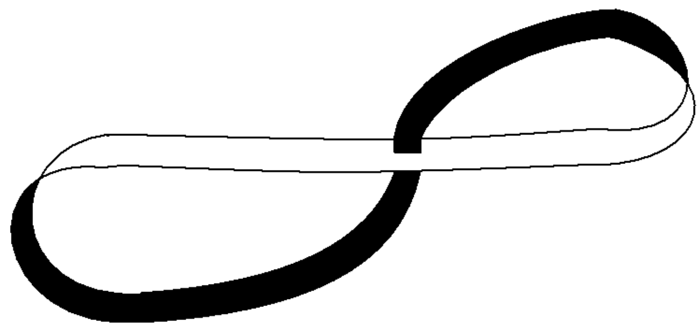

Figure 5.14: A ribbon knot that's different from a normal knot

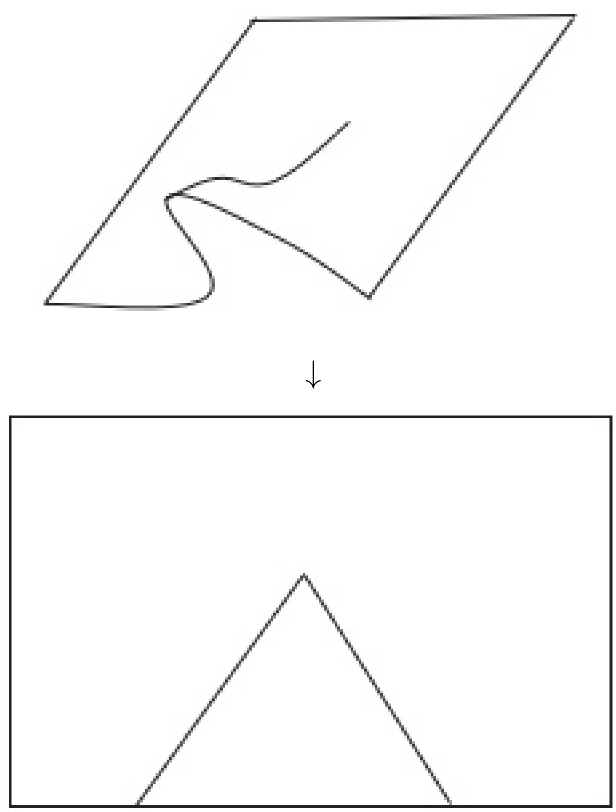

Figure 5.15: Flattening a hood into a 'V' shape

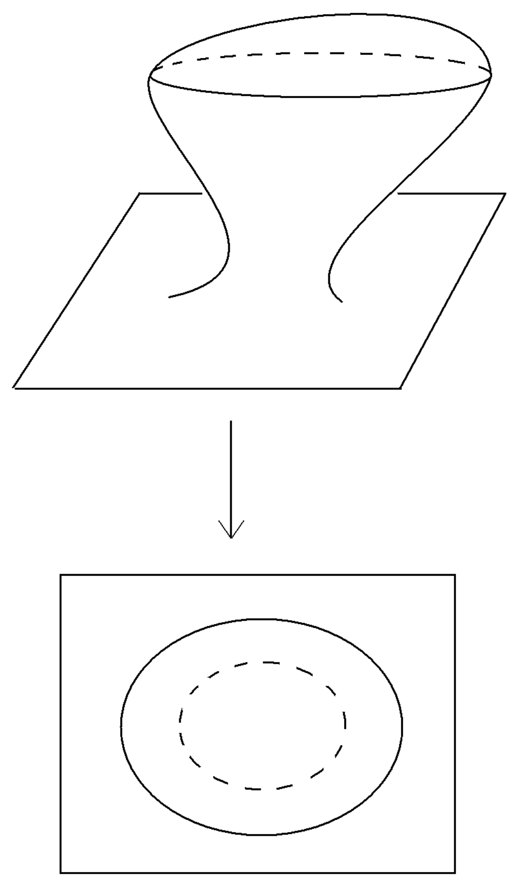

Figure 5.16: A “bubble” flattening into a ring of sorts

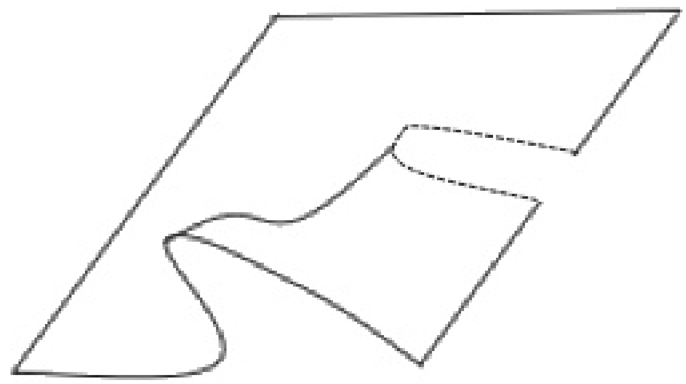

Figure 5.17: A "finger move" pushing the boundary into the hood's base

Finger moves

A large amount of the challenge that Joseph & Bill face in this clip is coming up with a way to account for *all* such fold patterns rather than thinking of some of them haphazardly. The main idea driving this session was Joseph's observation that they could push the boundary of the ribbon up to key points (what they referred to as *singularities*) but that they couldn't really do this for similar singularities when the ribbon cuts through itself (like in Figure 5.14).

For example, in considering the phenomenon show in Figure 5.15, Joseph points out that it might be possible to imagine pushing one's finger into the side of the ribbon up to the base of the hood (Figure 5.17). Doing this makes it so that when we flatten out the hood into a 'V', the point where the two sides of the fold meet is no longer there (Figure 5.18).

The question driving most of the use of continuum-based drawings here is whether this "finger moves" approach will work for *all* such singularities. (The answer turns out to be "no," but that doesn't turn out to matter for the purposes of the present study.)

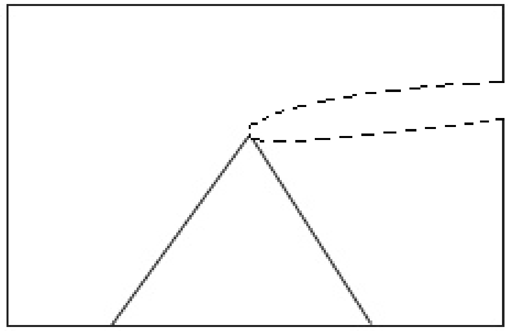

Figure 5.18: The "finger move" of Figure 5.17, flattened out

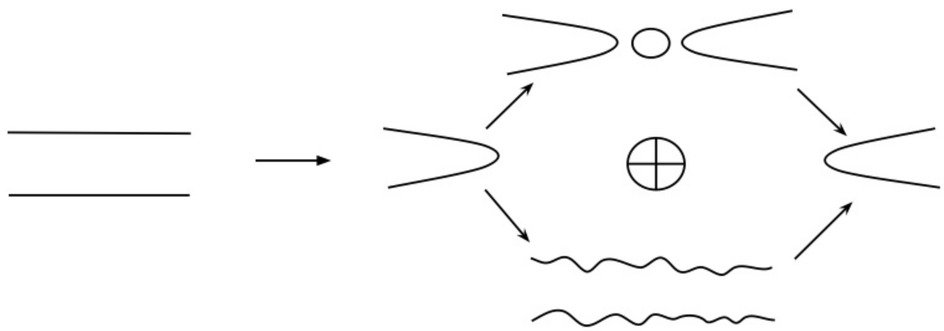

Figure 5.19: A (sideways) diagram depicting the algebra of Reidemeister 2

The Reidemeister 2 diagram

A second portion of the analysis considers what happens when Joseph & Bill switch to thinking in terms of a particular diagram depicted in Figure 5.19. The diagram they actually mean to refer to is that shown in Figure 5.20.

This diagram references the connection between the Reidemeister moves and homology (see section 4.1.1 on page 60). We won't need the details of this connection until Chapter 6 (see section 6.1.1); instead, I'll just sketch roughly what this diagram refers to in relatively broad strokes since that will turn out to be good enough for our purposes in this clip.

Stage 1 of the diagram in Figure 5.20 shows that we're dealing with two

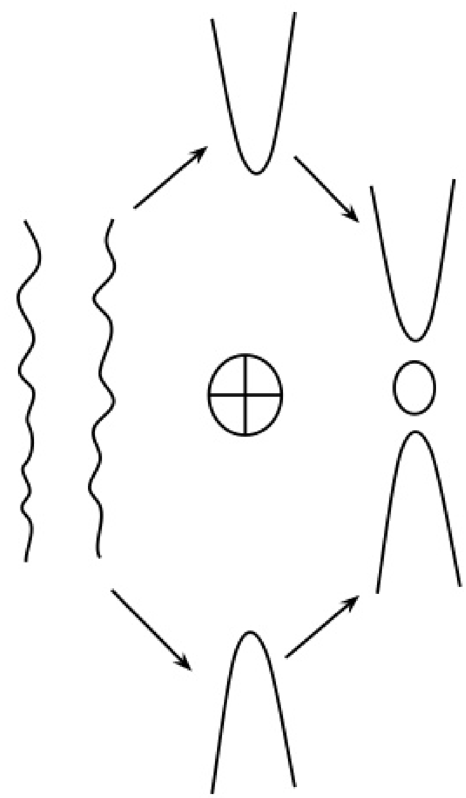

Figure 5.20: Figure 5.19 corrected

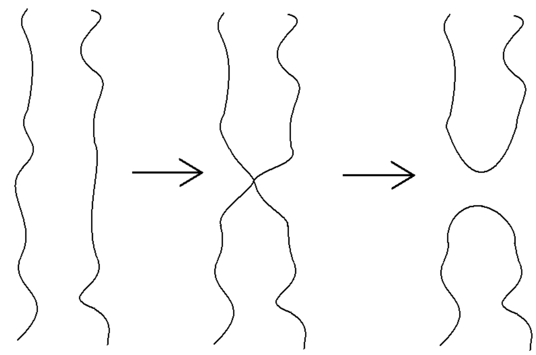

Figure 5.21: Moving from stage 1 to stage 2 of the diagram in Figure 5.20

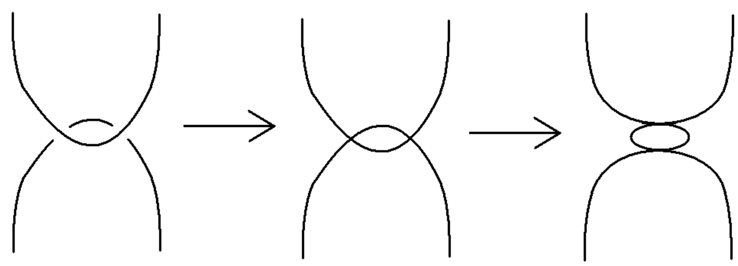

Figure 5.22: Loops on top of one another

loops or portions of string (roughly speaking). We can pinch these two together in the middle, merge them, and separate them into two loops (Figure 5.21). This results in a "sum" of two loops. If we read the diagram of Figure 5.20 in columns, we can see this "sum" by the direct sum ('$\oplus$') symbol between the two loops.

Remember that Reidemeister 2 says in effect that if one loop is lying on top of another, you can pull them apart — or conversely, you can take two separate loops and put one on top of the other. If we do that with two loops, we end up with a ring in the middle (Figure 5.22). If we were to separate the three loops we see there, we'd get stage 3 of Figure 5.20.

This overall sequence — pinching two strings into two loops, then overlapping them and separating them into a circle and two loops — describes roughly the process involved with connecting Reidemeister 2 to the Khovanov homology of knots. (Recall that we want the Khovanov homology of two knot diagrams that differ by a Reidemeister 2 move to be the same.)

5.2.2 Video Description

The analysis for the first part of this section refers to Table A.5.

When this session begins, Joseph and Bill had just spent fifteen minutes sitting across from one another while discussing ribbon knots. They've been focusing on a paper sitting at a desk next to them. The paper describes ways of computing the usual Khovanov homology for a knot (see Section 4.1.1). They've been trying to work out how to extend the reasoning to work for ribbon surfaces.

Setting up the problem

Joseph is trying to explain to Bill that there are some kinds of singularities that can occur with ribbon surfaces but not with normal knots. He struggles to convey his intent through gestures (rows 4–7), but he decides that this is less effective than working at the board and thus jumps up to draw a picture of an example of what he means (rows 8–14). The shape he draws is the hood depicted in Figure 5.15 on page 127, although he refers to it as a "blanket" (rows 15–16). He describes the folding process that would occur if one were to try to flatten the hood down (rows 16–21), eventually describing the 'V' shape that would be three sheets thick (rows 23–25).

At this point he indicates how this particular folding crease meets at the top in the shape of a kind of mathematical object known as a *cusp* (rows 25–26). An example of where this shape occurs is in the graph of the equation $x^2 = y^3$ (Figure 5.23; see also Figure 4.25). In saying that this "is what we normally call a 'cusp'" (rows 39–40), Joseph is describing how sharply the two sides of the 'V'

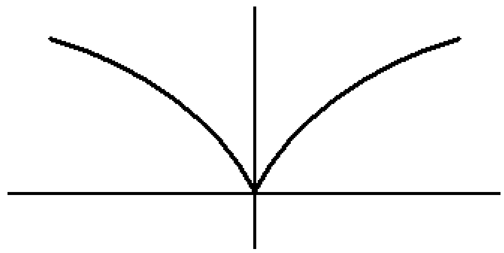

Figure 5.23: A cusp — the graph of $x^2 = y^3$

come together. It's interesting to notice how, in preparing to make this point, he engages in a similar process of perfecting the invocation of the continuum (row 31) that we saw Bart doing in section 5.1.2. We also see Bill briefly engage in eye-shielding behavior (rows 35–37) in response to Joseph sharing that he felt the need to work out that the meeting point of the two sides of the 'V' is indeed a cusp.

Joseph emphasizes how "nothing funny is happening there," pointing at the spot on the three-dimensional hood that corresponds to the cusp in the resulting folded hood (rows 44–45). This is true specifically of the ribbon knot itself, not what we could fairly call the ribbon knot diagram containing the 'V' shape. So the singularity appears only in the flattening out of the ribbon, and "nothing funny is happening" in the part of the ribbon corresponding to the singularity point.

It's worth noticing how richly Joseph is using the continuum here. A slight movement in how the hood is shaped would result in a meaningful change in where the cusp ends up and what the curves leading up to the cusp are shaped like. Had Joseph indicated a point just a few millimeters to the right, he would have been specifying a mathematically different area. This is in marked contrast to pointing at the 'x' in the equation "$x^2 = y^3$" where a slight shift from the center of the 'x' to one of its sides doesn't really change the fact that you're still indicating that particular x. Joseph's use of these diagrams is rather like how you or I might use a map: yes, there are some circular blots indicating key locations and indicating one part of that blot or another doesn't really change much, but by and large slight movements of our fingers will result in us pointing to significantly different locations. This seems to be key to how Joseph is viewing this particular challenge in describing kinks in ribbon knots.

After this point Joseph steps back for a moment and says, "So anyway, so there's guys like that as well that can happen" (row 45). Here he's noting that this is one more type of singularity that they'll have to account for in developing Reidemeister-like moves for ribbon knot diagrams. In stepping back like this, he's disengaging from his hands-on interactions with the continuous diagrams on the board, presumably because he reached a stage of completion in conveying the original point that brought him to the board in the first place.

Thus far we've seen Joseph invoke the continuum in order to lay the groundwork for later reasoning. Just as with Bart's continuum drawings in Section 5.1, Joseph's figures are generic particulars: pulling the hood a little farther to the left would have changed which example he was considering but would not have changed anything relevant to what he was trying to describe. But there's also a naïve sense in which we want to say that Joseph's drawings really, honestly do look like the shapes he's talking about: unlike using a circle to represent the solution set to $x^2 + y^2 = 1$, the objects Joseph and Bill are working with really could be modeled fairly accurate with *physical ribbons*. We can imagine pinching a portion of the edge of an elastic ribbon and folding it to the side, and the resulting three-layer-thick region around the fold would have essentially the shape of a 'V' like that in Figure 5.15. So while Joseph (like Bart earlier) seems to be invoking the continuum in order to capture some relationships, he seems to do so with something more like a *cartographical map* than, say, a Venn diagram.

Describing the finger moves

After about four seconds of thought, Joseph gets drawn in toward the board (row 47) in a pattern that weakly shares some characteristics of what happens when proximal inhibition is lifted. He has had a flash of insight about how to approach this problem — namely the use of finger moves. Notice how, even though Joseph is trying to talk to Bill, he keeps everything but his head and eyes oriented on the continuum-based drawing at the board. He even chooses to indicate the drawing with his hand that's holding the chalk, even though doing so makes it awkward for him to turn and look at Bill. Furthermore, his speech doesn't refer directly to the diagram much less any one part of it. This seems to suggest that his thinking about this situation is closely tied to his sense of what the diagram on the board affords: he can interact with it in some way in order to address this problem. That is to say, his thinking about this situation seems to be bound up in this particular invocation of the continuum.

In order to explain his idea, Joseph first notes that if the ribbon surface is closed[7] then it must have a boundary somewhere (rows 49–51). Joseph's hand gesture in rows 49–50 involves repeatedly and rapidly chopping both hands in synchrony as though to indicate a clean cut on either side of the continuum-based drawing at the board. This has a rich kind of interaction with the continuum-based nature of the drawing: there's an implication that the drawing should be thought of as continuing beyond what is shown, but Joseph's hand gestures emphasize that this continuing does not go on forever. The object can be contained between his hands in a sense. Yet it's understood that this boundary would change in some mathematically relevant way if it were moved slightly (although because Joseph's

[7]It's worthwhile to be aware that the term "closed" has special meaning for mathematicians. In topology it often refers to the idea that there's a crisp edge of the set, so to speak. Roughly speaking, if a country were "closed" in the mathematical sense, then it would include the boundary markers (e.g., rivers) that separate its territory from those of its neighbors rather than the boundary markers being owned by neither.

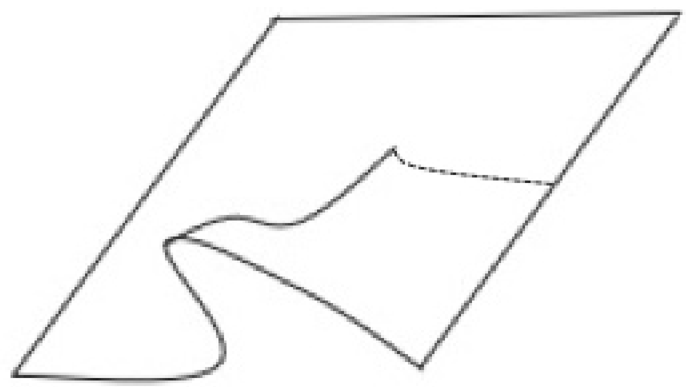

Figure 5.24: Joseph's drawing of an arc on the ribbon

example is a generic particular, such slight movement wouldn't affect the point he's making). Contrast this with those algebraic expressions that have a sense of continuing on beyond what's shown: it doesn't make immediate sense what it would mean to *slightly* vary where you stop. For instance, stopping "$1+x+x^2+\cdots$" at, say, x^8 creates a kind of clear boundary, but there isn't an obvious way of stopping ever so slightly before or after x^8: you can't make the stopping point halfway between x^8 and x^9.

Joseph's next move (rows 52–55) is to "draw an arc" from the base of the hood to the imagined boundary of the ribbon (Figure 5.24). Then right away he emphasizes that they're looking at the boundary of the ribbon (rows 56–57) and draws a kind of loop around the arc and erases the arc with his finger, yielding Figure 5.25 (rows 59–60). Because of the mathematical context, we know that Joseph is showing a kind of before-and-after pair of images meant to describe a smooth movement: the arc is sort of like a very, very fine cut in the ribbon, and the "finger move" involves pushing a finger through the middle of the elastic fabric along the cut in order to clearly separate the two sides. Again, this richly uses the continuum: although we can show a similar before-and-after sequence for, say,

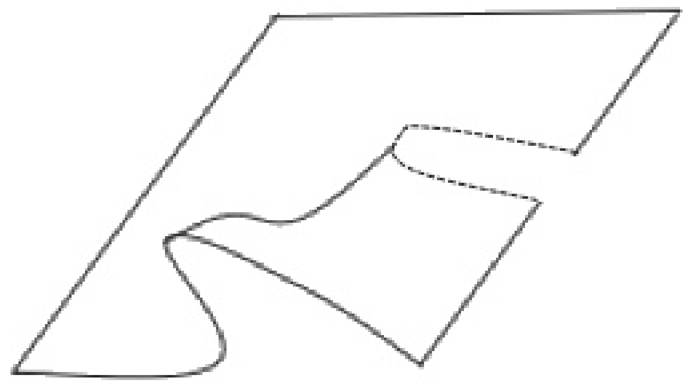

Figure 5.25: Joseph's drawing of a "finger move" after erasing the arc

expanding a factored quadratic expression, there isn't really a way to imagine what it looks like as we smoothly transition from factored form to expanded form.

One could argue that here Joseph is using the continuum-based diagram in order to communicate clearly with Bill — and I think that's part of what's motivating him. But the fact that Joseph stepped back, crossed his arms in front of his torso, shielded his eyes from Bill's direction, and just stared at the drawing (rows 69–71) seems to indicate that Joseph is also using the drawing as something to fix his attention on as he navigates his uncertainty about how to address this type of singularity. Rather than teaching Bill the way he might teach a student during office hours, Joseph seems to be sharing his thinking basically as it comes to him.

Partially dropping use of the continuum

We now jump ahead 29 minutes. The row references for the remainder of this discussion will be with respect to Table A.6.

Joseph and Bill have illustrated more of their thinking about these finger moves on the board (row 1). So far everything they've drawn there has been an

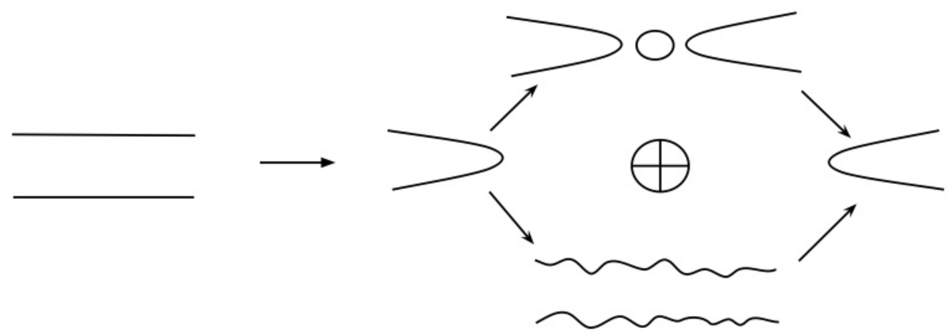

Figure 5.26: Joseph's drawing of the diagram for the chain complex for Reidemeister 2

invocation of the continuum. However, at this point the pair of them are trying to work out the ribbon-surface analogy of the second Reidemeister move for knots. In order to do this, Joseph jumps up and attempts to draw the diagram they use to describe the homology involved in the second Reidemeister move for normal knots (rows 4–12). This results in the picture shown in Figure 5.26.

This diagram is a heterogeneous mix of continuum-based components and non-continuum-based ones. Each of the ends of the arrows is a generic-particular drawing that shows some portion of a knot (or several knots as in the topmost drawing), and those implicitly invoke the continuum. However, the overall layout of the diagram does *not* invoke the continuum: small movements in the location of the diagrams or the positioning of the arrows wouldn't change the meaning of any part of the diagram at all. The direct sum symbol ('$\oplus$') in the middle is also explicitly algebraic: it indicates that the algebraic structures corresponding to the top and bottom knot drawings should be added using the direct sum. So the central ideas and relationships involved in this diagram are the *non-continuum-based* ones; it's quite conceivable that relatively little would have been lost had the continuum-based drawings in this diagram been replaced with verbal descriptions.

It would seem that Joseph drew the diagram rotated 90 degrees counterclockwise, which results in the lower-left and upper-right arrows being reversed. Once Joseph sits down, Bill attempts to emphasize this point by saying, "I think that's turned on its side" and tilting his head sideways to indicate which way the diagram should be read (row 14). Later we'll see this detail cause some confusion, indicating that Joseph might not have understood Bill's point here.

Thereafter Bill returns to his earlier sitting posture (row 17). The pair of them stare at this new diagram for six seconds, and then Bill speaks up. He tries to depict the analogy to ribbon surfaces using his hands while saying, "Since they're the same…" (rows 17–19). At this moment he's moving his right hand so that it's sliding along the palm of his left hand back and forth along the direction of his right hand's fingers. He pauses for six seconds while concentrating on the diagram, does a faster version of the same sliding gesture, and then abandons the gesture in favor of indicating the diagram and saying, "Yeah, I mean, you can deal with that top circle, right?" (rows 19–20). Bill is referring to the circle in the middle of the topmost continuum-based drawing of the new diagram. He follows up by sweeping his extended hand to the side and saying, "Clear away both of those, and then there's just some sort of ways that we bootstrap" (rows 21–24). The process he's describing are manipulations of knots of ribbon surfaces. This all happens within the continuum containing the topmost knot of the Reidemeister diagram (Figure 5.26).

Bill's next move is to point his index finger and indicate the transition represented by the upper-lefthand arrow (row 24). Then after a one-second pause he opens his hand again and says in a semi-startled way, "Oh! No, that keeps them the same!" (row 25). So in this moment here we can see an interplay between the continuum-based thinking of manipulating ribbon surfaces on the one hand and the non-continuum-based description of how each of those continua are connected. In an almost perfect reversal of the situation we saw with Bart's use of an inverted pyramid in order to capture algebraic relationships in a continuum (section 5.1.2 on page 117), here Joseph and Bill seem to be using an algebraic description in order to capture relationships that we're to interpret as occurring within a continuum.

Summary

There are several details about invoking continua that arose from this case study:

- There was a sense in which most of the continuous drawings used here seemed as though they "look like" the objects to which they refer. (Note that this is a statement about the phenomenology rather than the physical notations: it's in Joseph's & Bill's *perception* of the drawings, rather than in the drawings themselves, that the sense of "looks like" arises.) This framing allowed the invocations of the continuum to be central to the reasoning here and even take over the central working areas of the blackboard. (This is in direct contrast to Bart's choices for where to draw his invocations of the continuum in section 5.1.2.)

- Virtually every invocation was a generic-particular. This isn't strictly necessary; for instance, they could have examined just one particular hood-folding pattern, worked through the finger-movement argument, and then noted that the argument generalizes to other hoods as well. Instead, the emphasis was on treating the drawing as actually *being* all cases like it at once. This appears to be a robust pattern in continuum-use amongst the mathematicians studied here.

- We saw the animation of an invocation in such a way so as to suggest that it extended, continuously, beyond what was drawn. This seems to be a character of continua-as-experienced: even though the continuous drawings themselves seem to take up a relatively specific region of the board, implicit in their drawing is the idea that the continuum itself extends arbitrarily in all directions. This fact sometimes appears in mathematicians' gestural interactions with the drawings.

- A sequence of edits to the drawings (Figures 5.24 and 5.25) conveyed a *dynamic movement* whose nature can be understood only by knowing that the drawings — and what happens between them — are embedded in the same continuum.
- The switch to a non-continuous diagram (Figure 5.26) seems to serve a similar (if reversed) purpose here as switching *to* continua did for Bart in the previous clip: it changes the available affordances in order to offer a different and potentially fruitful way of perceiving the relationships of interest. The fact that the opposite direction (continua to non) served a similar purpose seems like it was probably a result of the "looks like" sense involved in Joseph's & Bill's understanding of the drawings of ribbon knots.

These phenomena help to paint a fairly vivid picture of the role invocations have in mathematical struggle. However, there are still a few more nuances that are worth developing our awareness about; we'll explore those in the remaining clip.

5.3 Clip #3: Tropical Diagrams & the $\mathbb{CP}^2$ Triangle

The final clip for this chapter returns to Matt & Bart. They have moved on from thinking about splicing varieties to exploring a nuance of what they call *tropical diagrams*. Tropical diagrams encode certain aspects of the algebra of *complex projective plane* in a continuum-invoking way. We also find that the complex projective plane gets a representation as what appears to be a simple triangle. The timbres of these two invocations of the continuum are significantly different in ways that turn out to be informative for the nature of how continua relate to struggle.

5.3.1 Mathematical Context

The two main threads I'll need to explain for this clip are (a) tropical diagrams and (b) the representation of the complex projective plane $\mathbb{CP}^2$ as a triangle. To describe tropical diagrams, I'll need to explain what a complex torus is and then what the tropical vertex group is, at least in broad strokes. For the triangle representation, I'll start by explaining *real* projective space and then show how that extends to the complex projective plane.

Complex torus

Recall that a *torus* is a shape that looks quite a lot like a doughnut (Figure 4.3). It turns out that we can specify each location by saying how far *along* the tube and how far *around* the tube we have to go from some predetermined starting point to get to the destination point. This is in much the same way that we can specify any point on the Cartesian plane by saying how far *over* and how far *up* we have to go from the origin to get there. In order to do this, we can develop a coordinate system for the torus by focusing on two different circles: a circle that rings the tube, and a circle that goes around the doughnut hole. Those two circles will cross at some point, which we can call the origin. Then we can specify how far along each of the circles you want to travel in order to get to a given point on the torus. For instance, (π, π) could be used to describe a point on the outmost rim of the torus on the opposite side from the circle that rims the tube. In this sense, mathematicians often model a torus as a Cartesian product $S^1 \times S^1$ of two

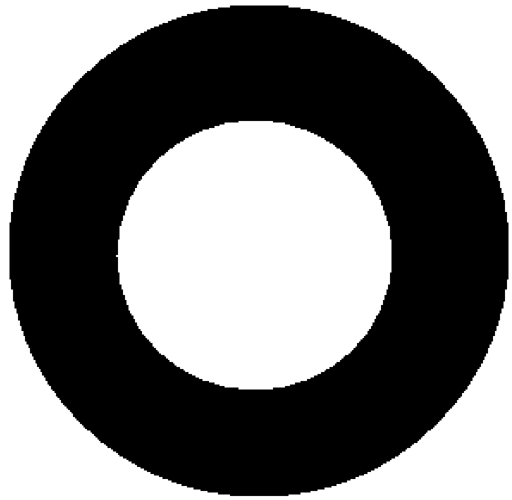

Figure 5.27: An annulus

circles.

Topologists commonly think of the main feature of a circle S^1 as being that it has a hole. It shares many of the same properties of other one-holed flat objects such as an annulus (Figure 5.27). We could imagine stretching the annulus out arbitrarily far and squeezing its inner circle arbitrarily inward such that we have an entire plane minus a single point. We can think of this as the Euclidean plane $\mathbb{R}^2$ minus the origin.[8] However, algebraists (including algebraic geometers) generally prefer to work with *complex* numbers $\mathbb{C}$ rather than real numbers $\mathbb{R}$ due to the former being algebraically complete (see section 4.3.1). So rather than referring to the punctured Euclidean plane, Matt & Bart prefer to use the punctured *complex* plane $\mathbb{C}^*$.

With this adjustment, then, we can refer to the *complex torus* $\mathbb{C}^* \times \mathbb{C}^* =$

[8]Properly speaking, an annulus and a punctured plane do differ in a topologically relevant way: the annulus is *closed* whereas the punctured plane is *open*. This is relevant because there isn't actually a way to stretch out a closed annulus to become an open punctured plane. Fortunately, this difference turns out to be irrelevant for pretty much everything we're going to discuss here.

$(\mathbb{C}^*)^2$.[9] This is essentially an algebraic trick that lets us describe the coordinate of a "torus" as (z, w) where z and w are nonzero complex numbers. For instance, $(3 + 2i, -3i)$ is a valid point on the complex torus.

Groups, automorphisms, and generators

The complex torus has a group structure, meaning that it's a monoid (see section 5.1.1) such that every element has a corresponding partner (an *inverse*) that, when combined with the monoid operation, results in the monoid's identity. An example of a possible group structure on the complex torus would be pointwise multiplication: $(x, y) \times (z, w) = (xz, yw)$. (This is *not* the group structure that Matt & Bart use, but I'll describe the idea of the tropical vertex group in terms of this group structure since it'll convey the essential idea well enough.) Another example of a group is the set $\mathbb{Z}$ of integers under addition.

An *automorphism* of a group is a way of rearranging the elements of the group in a way that respects the group operation. For instance, one valid automorphism of $\mathbb{Z}$ sends n to $-n$. If we apply this automorphism to each integer in the true equation $3 - 5 = -2$, then we still end up with a true equation: $-3 + 5 = 2$. Somewhat more formally, we say that an automorphism ϕ on a group G is a function such that, if g and h are elements of G, then $\phi(gh) = \phi(g)\phi(h)$.

It turns out that the set $Aut(G)$ of all automorphisms on a group G is *also* a group if we take the group operation to be function composition. For instance, it turns out that there are only two automorphisms on $\mathbb{Z}$: the one described earlier (which we'll call ψ), and the trivial one ι that sends n to n. That is, $Aut(\mathbb{Z}) = \{\iota, \psi\}$. We have $\psi \circ \psi = \iota$ since $-(-n) = n$, and ι composed with anything is the original function. So, $Aut(\mathbb{Z})$ has a group structure under $\circ$.

[9] There's another way to define the complex torus that seems to be more common: take the quotient of $\mathbb{C}^2$ by the lattice $\Gamma = \{(a + bi, c + di) \ : \ a, b, c, d \in \mathbb{Z}\}$. This makes it easier to see that the complex torus is, as a topological space, equivalent to the 4-torus $T^4 = (S^1)^4$. However, Matt and Bart apparently prefer the $(\mathbb{C}^*)^2$ formulation.

A more complicated example would be $Aut(\mathbb{Z} \times \mathbb{Z})$. It turns out that this has eight elements:

1. $(m, n) \mapsto (m, n)$
2. $(m, n) \mapsto (-m, n)$
3. $(m, n) \mapsto (m, -n)$
4. $(m, n) \mapsto (-m, -n)$
5. $(m, n) \mapsto (n, m)$
6. $(m, n) \mapsto (-n, m)$
7. $(m, n) \mapsto (n, -m)$
8. $(m, n) \mapsto (-n, -m)$

Notice that we can create every one of these functions by composing functions 2, 3, and 5. For the sake of a specific example, let's say that 2, 3, and 5 are f, g, and h, respectively. Then 7 can be thought of as the function $h \circ f$, or as the function $g \circ h$. 4 is $f \circ g$ or $g \circ f$. Because of this, we can say that f, g, and h *generate* the group $Aut(\mathbb{Z} \times \mathbb{Z})$, which is to say that you can get every element of this automorphism group by various combinations of these three. In short, a collection S of elements of a group G *generate* G if every element of G comes from some combination of elements of S.

Tropical vertex group

It turns out that the space $Aut((\mathbb{C}^*)^2)$ is very small the same way $Aut(\mathbb{Z}\times\mathbb{Z})$ is. Therefore, Matt and Bart chose to explore a related but more complicated space. Specifically, that space is a subspace of this one:

$$Aut(\mathbb{C}[x^{\pm 1}, y^{\pm 1}][[t]])$$

The notation $\mathbb{C}[x,y]$ indicates the set of all polynomials with two variables x & y whose coefficients are complex numbers. For instance, $3x^5 - 4ix^2 + (2-3i)x^3y^2 - (5-i)y$ is an element of $\mathbb{C}[x,y]$. $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ is the same, only we allow for the multiplicative inverses of x and y. In a sense, we can think of $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ as representing the set of varieties that can exist on the complex torus, thinking of x and y as variables that can each take on values in $\mathbb{C}^*$.

The "$[[t]]$" part means that we're considering all polynomials in t whose coefficients are from $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$; for instance, if f, g, and h are from $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$, then $ft^3 + gt^2 + ht$ is an element of $\mathbb{C}[x^{\pm 1}, y^{\pm 1}][[t]]$. More generally, an element of $\mathbb{C}[x^{\pm 1}, y^{\pm 1}][[t]]$ has the form $\sum f_n t^n$ where $f_i \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ and $f_i = 0$ for all values of i larger than some maximum. (That is, there aren't any "power series" in t.) Adding this formal "t" parameter[10] creates a sufficiently complicated structure in the automorphism group that Matt & Bart considered it interesting and worth studying.

The *tropical vertex group* $\mathbb{V}$ is a special subset of $Aut(\mathbb{C}[x^{\pm 1}, y^{\pm 1}][[t]])$ defined by a particular set of generators. To describe the set of generators, I'll describe how one could construct any given one of these generators. We'd start by picking a pair of integers $(a,b) \in \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$ such that $(a,b) \neq (0,0)$ (i.e., they're not both zero). Then pick a function $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}][[t]]$ with the following form:

$$f_{(a,b),g}(x,y,t) = 1 + tx^a y^b \cdot g(x^a y^b, t)$$

Here, $g \in \mathbb{C}[z][[t]]$. That is to say, g is a polynomial in the formal parameter t whose coefficients are complex polynomials in one variable z. So g has the general form:

$$g(z,t) = \sum_{n=0}^{\infty} h_n(z) t^n$$

[10] This is a different t than was used in section 4.3.1 to describe the process of transitioning from one variety to another. Matt & Bart never used "t" in the latter sense in the clips examined in the present work.

where h_n is a polynomial with complex coefficients for every n and with $h_n = 0$ for all sufficiently large n (so that g is finite in length).

The choice of those two (f and (a, b)) completely determines an automorphism $\theta_{(a,b),f}$ on $\mathbb{C}[x^{\pm 1}, y^{\pm 1}][[t]]$. This is because we can use f to describe how the automorphism will act on each of our three parameters:

$$\begin{aligned} \theta_{(a,b),f}(x) &= \frac{x}{(f(x,y,t))^b} \\ \theta_{(a,b),f}(y) &= y(f(x,y,t))^a \\ \theta_{(a,b),f}(t) &= t \end{aligned}$$

(In short: $x \mapsto xf^{-b}$, $y \mapsto yf^a$, and $t \mapsto t$.) This defines $\theta_{(a,b),f}$ entirely.[11] For instance:

$$\theta_{(a,b),f}(3x + 2xy - xy^2t) = 3(xf^{-b}) + 2(xf^{-b})(yf^a) - (xf^{-b})(yf^a)^2 t$$

The tropical vertex group $\mathbb{V}$ is then defined to be the subgroup of the automorphism group $Aut(\mathbb{C}[x^{\pm 1}, y^{\pm 1}][[t]])$ that's generated by the set of all such automorphisms $\theta_{(a,b),f}$ as we vary our choices of a, b, and f.

Tropical diagrams

It turns out that there's a graphical way to describe elements of the tropical vertex group — though not obviously as a subgroup of $Aut(\mathbb{C}[x^{\pm 1}, y^{\pm 1}][[t]])$. The pictures we get from doing this are called *tropical diagrams*.

These pictures occur in the Cartesian plane. In order to represent an element $\theta_{(a,b),f}$ of $\mathbb{V}$'s generating set, we draw either a line through the origin or a ray from the origin (whichever we prefer) that passes through the point (a, b).We

[11] I'm glossing over a detail here, which is that $\theta_{(a,b),f}$ is actually a *ring* homomorphism. That is, there are two operations (addition and multiplication) in the set $\mathbb{C}[x^{\pm 1}, y^{\pm 1}][[t]]$, and $\theta_{(a,b),f}$ interacts nicely with both of them in the same way a group automorphism interacts nicely with the group operation. Roughly speaking, this just means we can apply $\theta_{(a,b),f}$ to each of x, y, and t without worrying about what's immediately around those variables.

then say that any path not passing through the origin in $\mathbb{R}^2$ represents an element of $\mathbb{V}$ via two rules:

1. Each time the path crosses a line or a ray corresponding to an automorphism, we view that as indicating the composition of that particular automorphism or its inverse.

2. The choice of whether it's the automorphism or its inverse that's composed depends on the direction in which the path crosses the line or ray. That choice is determined upon the drawing of the line or ray.

I'll walk through three examples in order to illustrate how these tropical diagrams work.

Example #1 Consider two generators θ_1 and θ_2 of $\mathbb{V}$. θ_1 has $(a, b) = (1, 0)$ and $f_1(x, y, t) = 1 + tx$.[12] This we choose to represent with a line, which must therefore be a horizontal line so that it passes through the point $(1, 0)$. We'll let θ_2 have $(a, b) = (0, 1)$ and pick $f_2(x, y, t) = 1 + ty$.[13] We'll also represent this with a line, which therefore must be a vertical line.

This gives us the picture in Figure 5.28. Notice that although Figure 5.28 *looks* like a Cartesian axis, it's not. Each line represents an element of the generating set for $\mathbb{V}$. It just so happens that because of our choices of elements of $\mathbb{Z}^2$ for each automorphism, we ended up with a horizontal line and a vertical line. Also notice that we've chosen just two elements of the generating set to display here; we're free to pick as many or as few as we like since we're just trying to depict *one* element of $\mathbb{V}$.

[12] This fits the form described earlier by letting $g(z, t) = 1$. Thus $f_1(x, y, t) = 1 + tx^a y^b \cdot g(x^a y^b, t) = 1 + tx^1 y^0 \cdot 1$.

[13] This is actually almost the same choice of f. We use exactly the same g, namely $g(z, t) = 1$, so the difference is entirely accounted for in the new choice of (a, b).

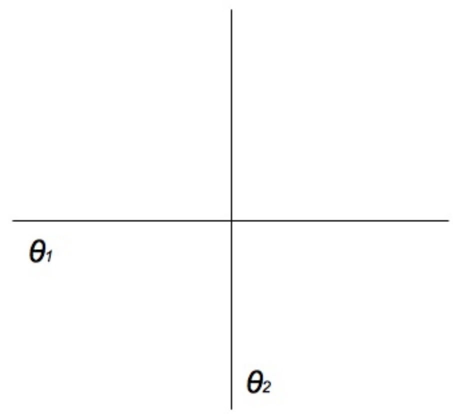

Figure 5.28: Two lines representing two generators of $\mathbb{V}$

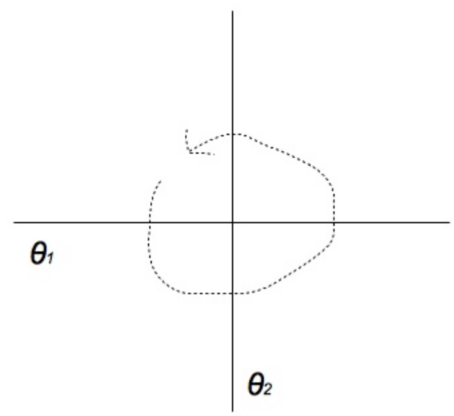

Figure 5.29: A tropical diagram of an element of $\mathbb{V}$

We'll specify that "down" is non-inverse composition for the θ_1 line and that "right" is non-inverse for θ_2. Having done that, we can add a path to indicate an element of $\mathbb{V}$ via a counterclockwise loop around the origin starting from the upper-lefthand region (Figure 5.29). (Really, any path that doesn't pass through the origin is acceptable as a means of representing some element of $\mathbb{V}$. It doesn't have to be a loop. This just happens to be a useful example.) This path corresponds to applying θ_1 first, then θ_2, then the *inverse* of θ_1, and then the *inverse* of θ_2. That is, this element is $\phi = \theta_2^{-1} \circ \theta_1^{-1} \circ \theta_2 \circ \theta_1$.

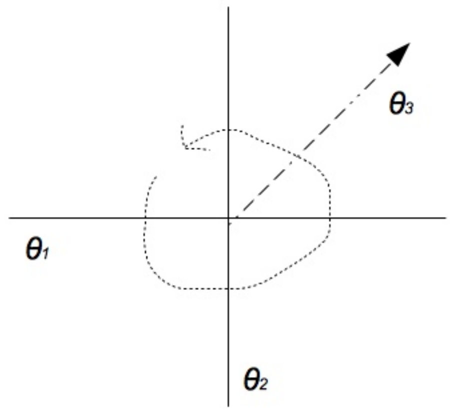

Figure 5.30: Adding a ray to make this element the identity automorphism

We actually have enough information that we could, potentially, work out exactly what ϕ is. We end up with the following:

$$\begin{aligned}
\phi(x) &= \frac{x + tx^2 + 2t^2x^2y + txy + t^3x^2y^2}{1 + tx + t^2xy + ty} \\
\phi(y) &= \frac{y(1 + 2tx + 2t^2xy + ty + t^2x^2 + 2t^3x^2y + t^4x^2y^2)}{(1 + tx + t^2xy)(1 + tx + t^2xy + ty)} \\
\phi(t) &= t
\end{aligned}$$

One thing this shows is that ϕ is not the identity (since, if it were, $\phi(x)$ would be x and $\phi(y)$ would be y). It "turns out to be interesting" (to quote Matt from a one-on-one interview) to ask what we would have to add in, and where, in order to modify the composition so that it *does* come out to be the identity. We can express the answer in terms of the tropical diagram pretty readily (Figure 5.30): we add a ray from the origin with slope 1 (say, with $(a, b) = (1, 1)$) and associate with it the function $f_3(x, y, t) = 1 + t^2xy$. What this says, then, is that if we call the new automorphism θ_3, then $\theta_2^{-1} \circ \theta_3 \circ \theta_1^{-1} \circ \theta_2 \circ \theta_1$ is equivalent to the identity in the automorphism group.

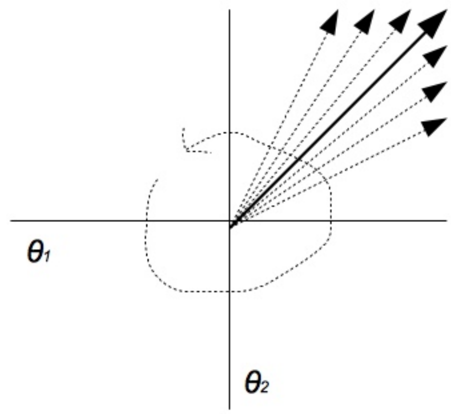

Figure 5.31: Bunching infinitely many rays arbitrarily close to the one of slope 1

Example #2 Now we'll use the same values for (a, b) for each of θ_1 & θ_2, but we'll modify their corresponding functions like so:

$$\begin{aligned} f_1(x, y, t) &= (1 + tx)^2 \\ f_2(x, y, t) &= (1 + ty)^2 \end{aligned}$$

This makes the initial picture look the same (Figure 5.28). However, in this case a loop like we did before (Figure 5.29) would not produce the same element of $\mathbb{V}$ as it did in example #1. The composition would end up looking quite different. That said, we can still ask the same question as before: what rays and/or lines can we add in order to turn this element into the identity map? It turns out that in this case, we have to add an *infinite* number of rays of slopes $\frac{n}{n+1}$ and $\frac{n+1}{n}$ for every natural number n (so slopes 1/2, 2/3, 3/4, $\cdots$; and also slopes 2, 3/2, 4/3, $\cdots$). Visually this looks like rays that get "bunched up" against the ray of slope 1 (Figure 5.31). Each one of these rays needs to have a function attached to them. Exactly what those functions are isn't terribly important for what comes next, but it will be worthwhile to know that the function associated with the ray passing through $(1, 1)$ is:

$$f_{(1,1)}(x, y, t) = \frac{1}{(1 - t^2xy)^4} = \frac{(1 + t^2xy)^4}{(1 - t^4x^2y^2)^{2\cdot 2}}$$

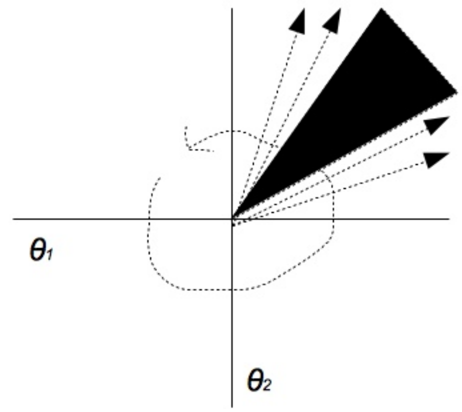

Figure 5.32: A dense cone of rays

Example #3 Finally, let's look at what happens when we *cube* the functions from example #1. Expressed algebraically:

$$\begin{aligned} f_1(x,y,t) &= (1+tx)^3 \\ f_2(x,y,t) &= (1+ty)^3 \end{aligned}$$

We consider the same loop as before (Figure 5.29). Only this time, instead of getting rays bunching up against a single line (as in Example #2), we get rays bunching up on either side of a *dense*[14] *cone* within which every ray of rational slope is present (Figure 5.32). The function corresponding to the ray with slope 1 (in the very center of the cone) has the following form:

$$f_{(1,1)}(x,y,t) = \left(\sum_{n=0}^{\infty} \frac{1}{3n+1} \binom{4n}{n} (t^2xy)^n \right)^9$$

Mark points and modularized spaces In all the examples I've given here, all the rays and lines have been centered on the origin. In some tropical diagrams, however, there might be multiple "origins" that are connected with line segments (Figure 5.33). The basic idea is the same, though: we can represent an element of

[14]The term "dense" here refers to the fact that there aren't any "gaps" between the rays of rational slope even though the ones of irrational slope aren't there. This comes from the fact that every real number can be approximated as well as we like with rational numbers.

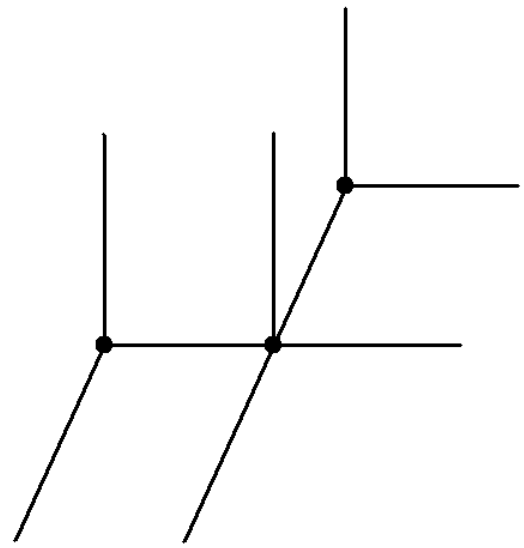

Figure 5.33: A tropical diagram with multiple mark points

$\mathbb{V}$ with a path that meanders anywhere other than through any of the "origins," which they refer to as *mark points*. This allows Matt & Bart to get around the fact that there are some elements of $\mathbb{V}$ that cannot be represented with the "one mark point only" version. For instance, $\theta_{(a,b),f} \circ \theta_{(c,d),g} \circ \theta_{(a,b),h}$ can't be represented without multiple mark points if $f \neq h$ because the two automorphisms with the same choice of (a, b) would have the same slope, meaning that whichever one is represented with a ray could not be "activated" without "activating" the other one. If we offset where the two otherwise overlapping automorphism lines are centered, though, we needn't encounter this problem.

Matt & Bart used these kinds of tropical diagrams to construct what they referred to as *modularized spaces*. These spaces come from gluing together "cells" that are copies of Euclidean space. For instance, the tropical diagram in Figure 5.33 has four things that can vary: the location of the center mark point (i.e., the one not attached to any rays) and the lengths of the three line segments connecting that center point to each of the mark points acting as bases of the rays. The central mark point can go anywhere in the Euclidean plane $\mathbb{R}^2$, and the lengths of the three line segments just have to each be nonnegative. Therefore the space across which Figure 5.33 can vary is $\mathbb{R}^2 \times \mathbb{R}^3_{\geq 0}$. This space forms one cell of the modularized space.

When creating topological spaces out of cells, topologists will often treat copies of Euclidean space like disks of various dimensions. For instance, we can treat $\mathbb{R}^2$ like a disk of radius $\pi/2$ by representing points of $\mathbb{R}^2$ in polar coordinates (r, θ) and then using the function $(r, \theta) \mapsto (\arctan(r), \theta)$. In a similar way, it's possible to contract $\mathbb{R}^3$ into a ball of radius $\pi/2$, and in general the same trick will turn $\mathbb{R}^n$ into an n-dimensional "ball" of radius $\pi/2$. It's possible, then, to think of a sphere as consisting of a point, a copy of $\mathbb{R}^1$, and two copies of $\mathbb{R}^2$: turn all the $\mathbb{R}^n$ into "balls," glue both ends of the interval that was $\mathbb{R}^1$ to the point (thus making a circle that'll be the sphere's equator), and sew each disk to this new circle along each disk's boundary (so one disk becomes the northern hemisphere and the other the southern one). This describes the sphere — a kind of topological space — in terms of a *cellular structure*.

Modularized spaces get built similarly, but the cells are copies of $\mathbb{R}^{2m} \times (\mathbb{R}^n_{\geq 0})^m$ when there are m central mark points and n lines connecting the central mark points to each other and the origins of the rays. Figure 5.33 had $m = 1$ (just one central mark point) and $n = 3$ (for the three lines connected to the central mark point). One particular case Matt and Bart consider is when several of the lengths are zero, in which case the diagram has lines doubled up on one another (Figure 5.34). Although technically the rays would be right on top of one another and thus not visible like in Figure 5.34, Matt and Bart still consider it important to draw these distinctly so that they keep track of the fact that there *are* multiple there. It's critical since a path crossing that spot would refer to the composition of *multiple* automorphisms.

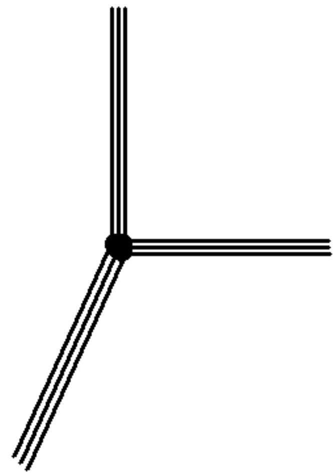

Figure 5.34: Figure 5.33 with the variable lengths set to zero

Projective planes and homogeneous coordinates

One way to think of the *real* projective plane $\mathbb{RP}^2$ is as the collection of all lines through the origin in three-space. Thus every nonzero point in $\mathbb{R}^3$ defines a unique element of $\mathbb{RP}^2$ since only one line through the origin passes through any given nonzero point in three-space. In fact, one way to think of an element $\ell \in \mathbb{RP}^2$ that passes through a point $(x, y, z) \neq (0, 0, 0)$ is as the set of points that differ by a constant:

$$\ell = \{(\lambda x, \lambda y, \lambda z) \mid \lambda \in \mathbb{R}\}$$

Thinking this way, we can represent the right-hand side of this equation using *homogeneous coordinates*:

$$\ell = (x : y : z)$$

Notice that this means that the choice (x, y, z) is not unique — which makes sense since ℓ passes through many different nonzero points.

We can do a similar trick with the *complex* projective plane $\mathbb{CP}^2$, only we have to do it via analogy. $\mathbb{CP}^2$ is the collection of all *complex* lines passing through the origin in *complex* three-space $\mathbb{C}^3$. It turns out that every nonzero point in $\mathbb{C}^3$

has *exactly one* element of $\mathbb{CP}^2$ passing through it and, analogously to $\mathbb{RP}^2$, the whole collection of points that constitute $\mathbb{CP}^2$ differ only by a (complex) constant. That is to say, if $\rho \in \mathbb{CP}^2$ passes through $(x, y, z) \in \mathbb{C}^3$, then:

$$\rho = \{(\lambda x, \lambda y, \lambda z) \mid \lambda \in \mathbb{C}\} = (x : y : z)$$

It's worth noticing that the same notation gets used for homogeneous coordinates in $\mathbb{CP}^2$ as in $\mathbb{RP}^2$, even though there's a critical difference between them (namely that the constants in the former are from $\mathbb{C}$ and in the latter are from $\mathbb{R}$). So it's important to keep track of context in order to properly interpret homogeneous coordinates.

Representing projective planes with triangles

Because $\mathbb{R}^3$ is a vector space, we can pick a few specific points (i.e., our basis) and describe any point in terms of those few specific points:

$$(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$$

The trouble with doing something algebraically analogous in $\mathbb{RP}^2$ is that it's not immediately clear what it means to add lines together.[15] However, we can still do something roughly close by thinking geometrically in terms of how "close" lines are to one another.

Let me illustrate with a few examples. On the xy-plane (i.e., the plane in $\mathbb{R}^3$ corresponding to the first two coordinate slots), let's imagine we're pivoting the line $(1 : 0 : 0)$ (i.e., the x-axis) about the origin until it lies atop the y-axis (i.e., becomes $(0 : 1 : 0)$). There are two directions we could go in to do this: we could pivot the line so that it sweeps through the first and third quadrants (Figure 5.35), or we could pivot it to sweep through the second and fourth quadrants

[15] At first pass it might seem like we could do this pointwise: $(a : b : c) + (d : e : f) = (a + d : b + e : c + f)$. Unfortunately, this turns out to be inconsistent since $(a : b : c) = (\lambda a : \lambda b : \lambda c)$ for nonzero λ.

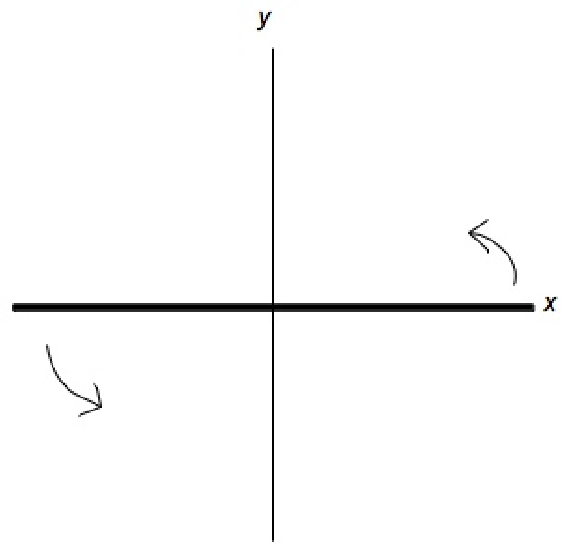

Figure 5.35: Rotating the x-axis counterclockwise

(Figure 5.36). Correspondingly, there are two lines in $\mathbb{RP}^2$ that are "halfway" between the x- and y-axes, namely the diagonal line with slope 1 (i.e., $(1 : 1 : 0)$) and the diagonal line with slope -1 (i.e., $(1 : -1 : 0)$). This means we can depict all the lines in the xy-plane while being true to how "close" lines are to one another by placing them on a lens (Figure 5.37).

The same kind of reasoning works for lines in the xz-plane and the yz-plane. However, if we try to put all of these shapes together to form $\mathbb{RP}^2$, we run into a problem of not having enough spatial dimensions to put these lenses together meaningfully. We can get around this problem provided that we're okay with losing some information. One choice of information to lose (the one that Matt & Bart chose) is the *dierction* of travel from one axis to another. This amounts to smashing the top and bottom arcs of the lens from Figure 5.37 together so that it now forms a single line (Figure 5.38). Now the center point represents *both* $(1 : 1 : 0)$ *and* $(1 : -1 : 0)$. This creates some ambiguity, but it lets us depict all of $\mathbb{RP}^2$ with a single triangle (Figure 5.39). So now the base represents all lines in the xy-plane,

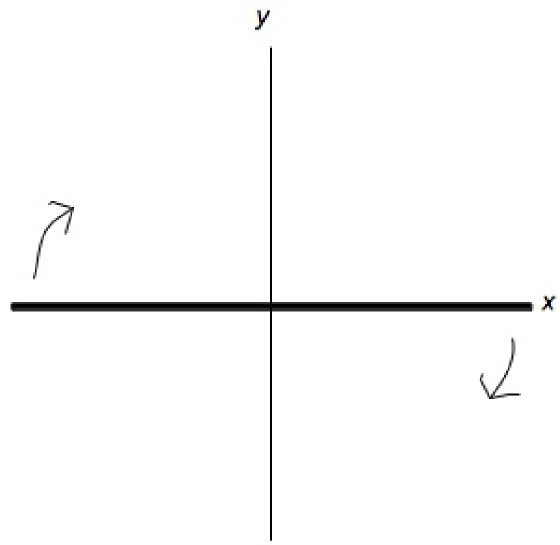

Figure 5.36: Rotating the x-axis clockwise

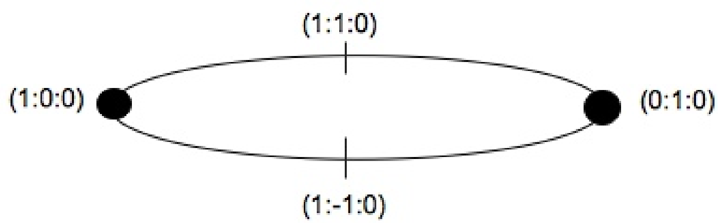

Figure 5.37: A lens depicting all possible lines in $\mathbb{RP}^2$ that are "directly between" $(1:0:0)$ and $(0:1:0)$

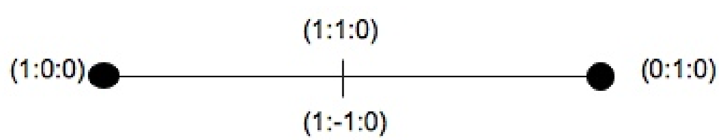

Figure 5.38: The lens from Figure 5.37 with one "direction" ignored

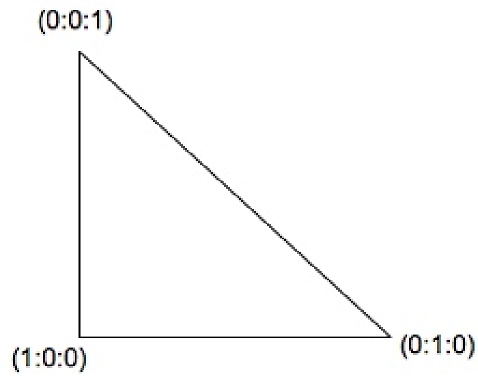

Figure 5.39: A triangle representing all of $\mathbb{RP}^2$

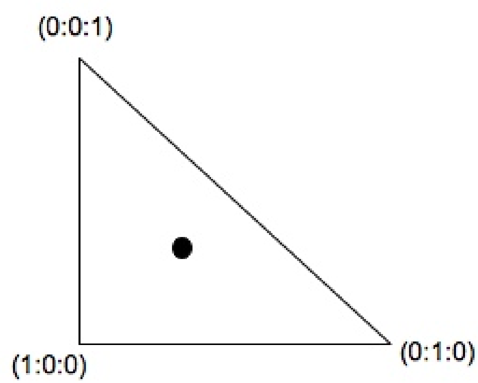

Figure 5.40: The centroid, which represents all of $(\pm 1 : \pm 1 : \pm 1)$ in $\mathbb{RP}^2$

the vertical side represents those in the xz-plane, and the hypotenuse[16] represents those in the yz-plane. Or said more simply, each edge represents the region of $\mathbb{RP}^2$ where one homogeneous coordinate is zero. Then the region in the middle indicates all the lines that are *not* in any of those three planes. In particular, the four three-dimensionally diagonal lines (i.e., $(1 : 1 : 1)$, $(-1 : 1 : 1)$, $(1 : -1 : 1)$, and $(1 : 1 : -1)$) would all be represented as the centroid (Figure 5.40).

It turns out we can do the same thing with $\mathbb{CP}^2$. It's much harder to

[16]It's not actually important that this triangle be a right triangle. In fact, it's not even important that the sides be straight. For ease of reference, though, I'll continue using right triangles.

understand what's going on geometrically, but in effect that just means that each edge of the triangle obfuscates more details about $\mathbb{CP}^2$ than the same diagram does for $\mathbb{RP}^2$. For instance, instead of there being just two ways to get from $(1 : 0 : 0)$ to $(0 : 1 : 0)$ while leaving the last homogeneous coordinate zero, there are *infinitely* many. The drawing for $\mathbb{CP}^2$ that corresponds to Figure 5.37 would actually depict a squished *sphere* rather than a squished circle.[17] As long as we're willing to account for this degree of information loss, we can use diagrams for $\mathbb{CP}^2$ that look for all intents and purposes just like triangles.

Relating tropical diagrams to $\mathbb{CP}^2$ triangles

There's a close connection between the tropical diagrams and the triangles that get used to represent the complex projective plane. I'll illustrate the relationship by elaborating upon the earlier examples given for the tropical diagrams.

Returning to example #1 In example #1 (see page 149), recall that $f_1 = (1 + tx)^1$ and $f_2 = (1 + ty)^1$. That 1 in the exponent of each means that we're considering an arbitrary point on the interior of each of two different edges of the triangle representing $\mathbb{CP}^2$ (Figure 5.41). In other words, we're picking two arbitrary complex lines through the origin in $\mathbb{C}^3$, each with exactly one homogeneous coordinate being zero, but not the same coordinate between the two.

So within this context, we can ask the following question: How many algebraic varieties pass through the edges of the triangle at only those two points while being maximally tangent to the hypotenuse? Or if we were to phrase this question in terms of $\mathbb{CP}^2$ instead of in terms of its representation as a triangle, the question would be: How many collections of lines that form algebraic varieties will include

[17] Why, you might ask? Because if you were to tie both ends of the real line together at a point, you'd end up with a circle — but if you were to tie the whole edge of the complex plane together at a single point, you'd end up with a sphere.

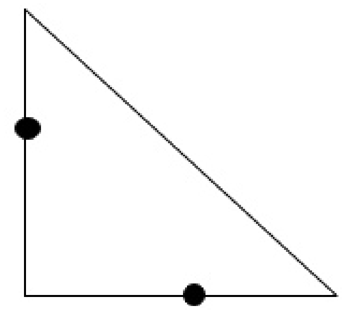

Figure 5.41: One point on each "leg" of $\mathbb{CP}^2$ is chosen

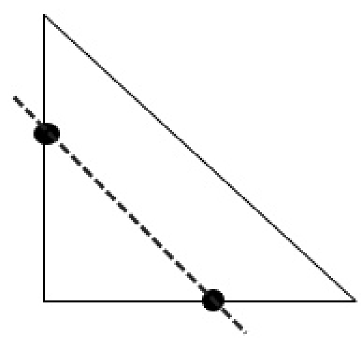

Figure 5.42: The variety (collection of lines) passing through the two given points (complex lines)

the two lines we started with while being as tangent as possible to the remaining axis that excludes our two chosen lines?

It turns out that this question is answered by the formula of the function corresponding to the ray of slope 1 that makes the loop represent the identity automorphism in $\mathbb{V}$. Recall that the function in this case was $(1 + t^2xy)^1$. It turns out that the form of the formula in the middle means that the *only* varieties that can fit this scenario are lines. The exponent of 1 indicates that there is just one such line, depicted in Figure 5.42. This shouldn't be too surprising since it's pretty normal for there to be one unique line (as a variety) passing through two given

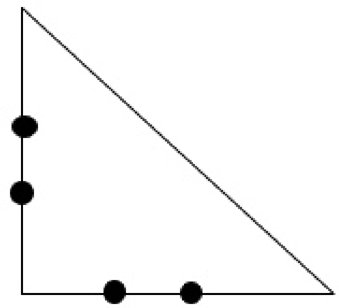

Figure 5.43: Two points per "leg" are now chosen

points. The condition of tangency ends up not being all that relevant in this case because lines (as varieties) either don't meet anywhere or meet transversely. The tricky part, really, is keeping track of the fact that "line" has two meanings in this context: a complex line (meaning an element of $\mathbb{CP}^2$) and a variety that we want to call a "line." The latter is a line in the sense of being the collection of solutions to an equation of the form $aX + bY + cZ = d$ where X, Y and Z are the coordinates of a complex line $(X : Y : Z)$ and a, b, c, d are complex numbers. In other words, it's a "line" because it's a collection of solutions to a linear equation. However, the "points" in this "line" are complex lines.

Returning to example #2 In the second example (page 151), $f_1 = (1 + tx)^2$ and $f_2 = (1 + ty)^2$. The 2 in the exponent for each of these indicates that we're concerned with *two* points on each of two sides of the triangle (Figure 5.43). The same question arises: How many varieties pass through these points while being maximally tangent to the hypotenuse? Again, this question gets answered by looking at the form of the function corresponding to the ray with slope 1:

$$f_{(1,1)}(x, y, t) = \frac{1}{(1 - t^2xy)^4} = \frac{(1 + t^2xy)^4}{(1 - t^4x^2y^2)^{2\cdot 2}}$$

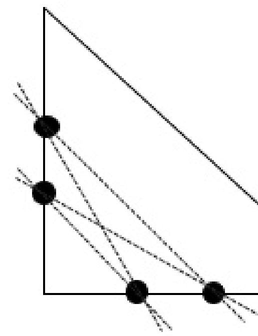

Figure 5.44: The four lines that fulfill the two-by-two restriction

As with example #1, we see that there's a factor on the top of the form $1 + t^2xy$. The exponent on this one — in this case 4 — indicates how many *lines* (in the sense of solutions to linear equations) there are between these. The fact that this number is four isn't too surprising (see Figure 5.44). In a similar way, there's a factor of the denominator of the form $(1 - t^4x^2y^2)^2$, which apparently refers to *conics*. (Whereas in $\mathbb{R}^2$ there are several families of conics (parabolas, hyperbolas, circles, etc.), there are very few families of conics in $\mathbb{CP}^2$. Restricting ourselves to nonlinear conics makes this designation fairly specific.) The fact that this factor is squared (so that it reads $(1 - t^4x^2y^2)^{2\cdot 2}$) indicates that there are *two* conics that pass through all four points while being maximally tangent to the hypotenuse (Figure 5.45).

Although it seems intuitively obvious as to why there are only four possible lines based on the triangle representation, it might not seem so clear why there are just two conics. I'll illustrate why this is the case via analogy. Here is the general form of a conic section in $\mathbb{R}^2$:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

Although there are technically six coefficients we want to work out, we can always divide both sides by one of the nonzero ones so that we're left with just five of

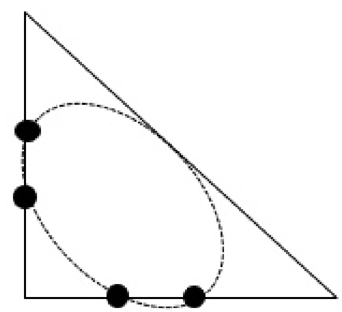

Figure 5.45: A conic passing through four points while being maximally tangent to the hypotenuse

them.

This means that if we specify that, say, $(x, y) = (2, 3)$, then we end up with one equation in five unknowns. If we specify that the conic section we're looking for passes through *four* such specific values for x and y, then we end up with *four* equations in five unknowns. This means that the space of all possible solutions is (at most) one-dimensional — which is to say, we should be able to specify what four of the unknowns are once we specify what one of them is.

The tangency condition gives us one more restriction. If the conic is to be tangent to, say, the line $y = 0$ (i.e., the x-axis), then to find the point of tangency we can substitute zero in for y to get $ax^2 + dx + f = 0$. The fact that it's *tangent to* this point rather than simply passing through it means that the discriminant must be zero, which is to say, $d^2 - 4af = 0$. This places a quadratic condition on our set of possible solutions, which forces there to be at most two solutions. Since we can always apply a rigid transformation to the plane such that the line that the conic section is tangent to is sent to the x-axis, this actually captures all the possible cases.

Something very similar happens in $\mathbb{CP}^2$. There are two main differences, though. First, we're working over $\mathbb{C}$ instead of $\mathbb{R}$, which grants algebraic completion. This means that instead of having *at most* two solutions, there will be *exactly* two solutions (up to multiplicity). Second, the general form of a conic in $\mathbb{CP}^2$ is a bit different. We can write the above formula for conics in $\mathbb{R}^2$ because we're thinking of points in $\mathbb{R}^2$ as (x, y). But in $\mathbb{CP}^2$, we think of "points" (i.e., lines) in the general form $(X : Y : Z)$. This means that as long as we take the axis of tangency to be where $Z = 0$, we can replace x and y with X and Y respectively and place Z wherever it needs to be so that the degree of each term is 2:

$$aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2 = 0$$

Otherwise the computations and arguments are basically the same: we divide through by one of the nonzero variables and then notice that we have four values for $(X : Y : Z)$, yielding four linear equations in five unknowns; and finally, the tangency condition imposes a quadratic restriction on the remaining possible set of solutions.

Recall, though, that we were able to derive this purely by considering the formula for the function corresponding to the ray of slope 1 in the tropical diagram. Although I haven't specified why this connection exists (and don't plan to for the sake of keeping this math description somewhat manageable in length[18]), the take-away message here is that there *is* this close association.

Returning to example #3 In example #3 (page 153), $f_1 = (1 + tx)^3$ and $f_2 = (1 + ty)^3$. This means we're dealing with *three* points on each of the legs of the triangle (Figure 5.46). The formula for the ray of slope 1 is a bit less

[18]There is a body of literature in algebraic geometry, spanning *at least* ten years, that goes into a great deal of depth about how this connection works. Apparently even currently practicing research mathematicians outside of algebraic geometry need to do a lot of reading before following the reasoning behind this connection in technical detail.

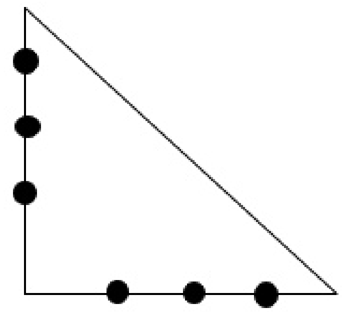

Figure 5.46: Three points are chosen on each of two axes of $\mathbb{CP}^2$

transparent in this case:

$$f_{(1,1)}(x,y,t) = \left(\sum_{n=0}^{\infty} \frac{1}{3n+1}\binom{4n}{n}(t^2xy)^n\right)^9$$

However, there's an interpretation of this situation that goes back to the same question as before: How many varieties are there that pass through these points and are maximally tangent to the hypotenuse? Unsurprisingly, there are nine lines (three choices for each edge, so 3×3). There are also 18 conics because of the counting method used in "Returning to example #2" (page 163): for two choices on each of the legs, there are two conics. This is equivalent to one choice on each leg (namely, the one the conic does *not* pass through). So there are $3 \times 3 = 9$ choices for points, and each choice yields two conics. These numbers *do* appear in the above formula, although much more subtly so than in the formulas for examples #1 & #2.

Summary

The key aspects of all this background we'll need is (a) a sense of how to see tropical diagrams, (b) an understanding of how to read the triangle as a depiction of $\mathbb{CP}^2$, and (c) a recognition that these two are closely associated. The main thrust of Matt & Bart's work in this clip is on figuring out how large n needs to be to fit a given modularized space (see page 153) into $\mathbb{R}^n$ (i.e., how “large” they need the ambient Euclidean space to be).

5.3.2 Video Description

This analysis makes reference to Table A.8.

Setup

At the beginning of this clip (row 1), we can see two images on the blackboard: what looks like a Cartesian axis for three-space $\mathbb{R}^3$, and what looks like two fuzzy axes for $\mathbb{R}^3$ connected by a fuzzy line. The “Cartesian axis for $\mathbb{R}^3$” drawing is actually a tropical diagram without a path drawn on it. The same is true of the drawing of two “axes” connected by a line. The fuzziness of the latter comes from the fact that there are *several* mark points (see page 153) clustered around two areas, and thus the rays coming off of them with the same slope are on top of one another. The multiple lines here are meant to keep track of the fact that there are multiple rays occupying the same space.

Matt drew the fuzzy diagram five minutes earlier (at the start of their meeting) in order to express a concern of his stemming from a conversation the two of them had started the day prior. The non-fuzzy diagram was Bart’s doing in his attempt to work out what restrictions should be present on Matt’s diagram (and therefore what the dimensions of the corresponding cell should be). Notice that both of these diagrams are invocations of the continuum, with Matt’s drawing being a generic particular (since the line segments have variable length) but Bart’s not. (Bart has, in effect, drawn Figure 5.34 on page 156.)

Creation of the $\mathbb{CP}^2$ triangle

In rows 2–11 we see Bart invoke the continuum again by creating the $\mathbb{CP}^2$ triangle. He says, "…you're talking about a situation where your mark points maps [sic] to the intersection of the strata." When saying "mark points," Bart fills in a dot at the right angle of the triangle he just created (row 14). This dot represents a single line in $\mathbb{CP}^2$ with exactly one nonzero homogeneous coordinate (e.g., $(1 : 0 : 0)$). He then starts to point at one of the legs of the triangle (rows 15–16) but then jumps over to point at the collection of arrows emitting above (row 17) and to the right of (row 18) one of the mark points on Matt's diagram ("These two strata, right?")

Roughly speaking, a *stratum* is a collection of rays emitting from a mark point. There's a sense in which these can be seen as corresponding to the legs of the $\mathbb{CP}^2$ triangle. Bart is suggesting that under the correspondence between the tropical diagram and the triangle, the lower-left mark point corresponds to one of the corners of the triangle and the collection of rays emitting from it map to the legs of the triangle. Notice in particular how he keeps his torso oriented to the board the whole time and, when pointing out "these two strata" (rows 17–19), he leans over to his right rather than moving his feet. He even immediately returns to the position where his body is centered back on the triangle (row 20) and re-engages with the triangle (row 21). In doing this, he's expressing that his thinking is *about the triangle* but that this thinking is inextricably bound up in Matt's diagram. He's continuing to navigate within the two continua invoked thus far.

Bart then notices that he's confused (rows 28–29). The trouble here is that the correspondence between the tropical diagram and the triangle isn't clear enough in Bart's mind to let him make sense of what would happen in the triangle were the lower-left mark point made to move along the ray going down and to the left. Initially he thinks he has made sense of it (rows 30–31), but he decides that he's mistaken (row 32). Notice how, at the same time, Matt feels a need to move away from the board, first just standing at a bit of a distance (rows 27–30) and then sitting down (rows 33–35). It seems as though Matt is too relaxed for this to be an instance of proximal inhibition: the component of displaying an embodied but inhibited drive to move closer to the board in a meaningful way seems to be missing. Yet he still seems to want to get a wider view even though he could presumably see the whole board perfectly well from his vantage point shown in row 26.

Contrast this with Bart's behavior. We see him touching the key mark point of the tropical diagram (e.g., rows 30 & 35) while keeping his eyes, torso, and feet oriented on the board. He also repeatedly moves his attention back and forth between the two continua (rows 36–40). This doesn't quite have the character of his explaining something to Matt; rather, he's actively using both continua in order to reason. That is, while Bart is demonstrably thinking *with* the diagrams, Matt seems to be thinking *about* them from afar at this point.

"Strata" versus "d_is" & "h_is"

Bart questioningly suggests that the behavior of the mark point as it moves downward in the tropical diagram implies that its corresponding point in the $\mathbb{CP}^2$ triangle should be in the interior (rows 40–43). Matt responds affirmatively (row 43), and Bart reacts to this by disengaging with the continua on the board (rows 45–46) and engaging more directly with Matt (rows 47–49). Bart's concern here is that if the mark point maps to the triangle's interior, then the strata can't map to the boundary of the triangle since the mapping is continuous. In Bart's mind, the strata are labeled by d_i (so one stratum is d_1, another d_2, etc.), so this failure for the strata to map to the boundary doesn't match a fact the pair of them had determined before this meeting, namely that all d_i are mapped to the boundary.

Bart's confusion gets compounded by Matt's reply: "We have d_is over there. I'm not talking about the d_is; I'm talking about just the simple strata" (rows 50–56). In this moment we can see how Bart, despite looking at the tropical diagram as Matt speaks, is not currently engaging with them (rows 52–54). Instead, he turns away from them (row 55) and walks up to the table to engage with one of the papers they're referencing (row 56). He reiterates that he's confused (row 57) and seems to be looking for some clarification on the relationship between strata and the d_is (rows 58–59).

It seems that Matt is *not* confused by this (rows 60–76). There's a distinction between the d_is (in the cells that make up the modularized space), the h_is (in the modularized space itself), and the strata (in the tropical diagram). Bart references the need to define "stratum" more precisely (rows 58–59), which Matt sees as a non-issue (rows 60–63). Instead, Matt seems to think that Bart is focused on the h_is (and, because of how the h_is are defined, the d_is as well) but wants him to focus on the strata instead (rows 71–76).

Trading control of the board

Bart sits down to examine the paper more closely (rows 78–79). Matt seems to take this as a signal to elaborate on what he has been talking about (rows 79–81), but just as he starts Bart seems to get the point he thinks Matt has been trying to convey (row 79). Bart here is rather insistent that Matt not continue the explanation, though clearly Matt *does* want to continue and takes a while to step back (row 82). Having so recognized Bart's emphatic request, though, Matt turns and sits down just as Bart jumps up to continue the train of thought (rows 83–85).

This illustrates a dynamic of using invocations of the continuum that we've seen before but was usually more subtle than in this instance: we do *not* see mathematicians both animating the same invocation at the same time. They might both be posed to engage with it, as in Joseph and Bill in, say, Table A.2 row 44. Yet we never see both mathematicians tracing parts of an invocation with their hand or using chalk or an eraser to modify the diagram *at the same time*. Instead, the mathematicians in this study use *social dynamics* to create an opening for their ability to animate and/or modify the invocations in question. In this case, Bart talks over Matt until Matt surrenders control of the board. It seems reasonable to suggest that Bart wasn't really *trying* to get Matt away from the board; rather, it's more likely that Bart just didn't want an idea he now understands to be explained to him. Yet the fact remains that Bart did not make any movement toward the board until *after* Matt backed away from it (rows 82–83). This pattern of negotiating and trading control of the board is highly robust across all three mathematicians studied in the present work.

I should be quick to add that this phenomenon is not restricted to invocations of the continuum. I'll return to this matter in the next chapter.

Moving the point in the $\mathbb{CP}^2$ triangle

Bart seems to experience a brief moment of hesitation here (rows 84–86). Even after standing up to engage with the board, he stops halfway there and leans back while haltingly trying to say what the next step is ("Because then, because then, but then I want... but then you want..."). He then concludes that "this is [the] usual kind of duality," at which point he engages with the tropical diagram (rows 86–89). The "usual kind of duality" refers to the correspondence between the behavior of the tropical diagram and the $\mathbb{CP}^2$ triangle.

Bart's recognition here draws him back to the triangle to his left (row 90) in order to reaffirm the point he had briefly mentioned earlier: the point on the triangle that he had drawn at the right angle should actually be in the interior (rows 91–93). It's worth emphasizing at this point just how incredibly rich this invocation of the continuum really is. A passive observer watching this portion of the clip with the sound off would see Bart erase a dot in the right angle of a triangle and draw another one near the triangle's centroid, much like Figure 5.40 (page 160). Because Bart had spent so much time looking back and forth between this triangle and the tropical diagram, such an observer might deduce that the two are related somehow. Yet this is significantly inadequate to explain what role the triangle plays in Bart's struggles here. The context of building cells that in turn build modularized spaces which relate to conics in $\mathbb{CP}^2$ would all be missing, despite the fact that this is the central focus of the struggle here. This illustrates the incredible importance of mathematical context and the rich expertise of the mathematicians involved to make such invocations relevant. A deeply intricate topic needn't yield a deeply intricate drawing.

Applying the body's multiplicity

Having made this adjustment in the triangle, Bart then pulls away from the board and expresses confusion again (rows 95–97). Specifically, he stoops and turns away from the board and moves away from it before turning back to it, all while holding the back of his head as though in a protective or self-isolating state. He then launches into engagement with the tropical diagram (row 98) in order to start deriving the relationship of the point he just drew in the triangle.

While checking with Matt, Bart keeps his feet and hips oriented on the tropical diagram even when associating the point he's making with the triangle (rows 98–102). Once he starts referencing "the geometric side," though, we see him shifting his torso toward the triangle (row 103). Through most of the rest of the clip we see him creating an ongoing association between the triangle and the tropical diagram (rows 104–112), much like he did earlier (see page 169). This time, however, he's attending to the *movement* of the point rather than just its location. After Matt mentions it, Bart focuses on what happens as the mark point moves downward (rows 107–111) and then concludes that the point in the interior of the triangle would move toward the hypotenuse (row 112).

Notice how, because both diagrams here are invocations of the continuum, the correspondence between them that Bart expresses by his back-and-forth of body and attention doesn't just mean that they're related. It allows for *covariation with respect to continuity*. That is, small movements in the tropical diagram are meaningful (to Bart), and that meaning transfers to small movements of the corresponding point in the triangle, which are also meaningful (to Bart). This is mediated by the *multiplicity* of Bart's body — that is, by the fact that he can engage (say) his torso and one hand with one continuous drawing and engage his other hand with another drawing. The covariation here is embedded in a sense of how two different parts of Bart's body (mainly his torso and right hand) behave with respect to one another.

Let me illustrate what I mean here with a relatively simple example. If we have two runners X and Y and we know that Y runs twice as quickly as X, we can represent this fact a couple of different ways. One way would be to write $y = 2x$ where y is Y's speed and x X's. This is a representation of covariation that doesn't involve the continuum and doesn't obviously involve the multiplicity of the body either.[19] Alternatively, I could represent X with my left hand and Y with my right, and apply the movement rule that whenever I move my left hand away from (or toward) me by some amount, I need to move my right hand away from (or toward) me by twice that amount. In doing this I'm applying the multiplicity of my body

[19] I note that I and many others, including the subjects of this study, actually *do* use the body's multiplicity for algebraic non-continuous inscriptions in some cases. See, for instance, Alibali & Nathan (2005) for an example of teachers doing this.

(although still not invoking the continuum in any inscription) to demonstrate the covariation of X's & Y's positions. If I were to do something similar but instead I use each hand to trace a drawn number line, then this becomes an animation of continua via my body's multiplicity.

In a similar way, Bart here is animating both invocations — the $\mathbb{CP}^2$ triangle and the tropical diagram — via his body's multiplicity. There are two differences worth noting, though. First, rather than making a point of moving his hands at the same time, he alternates. This would be somewhat akin to a teacher who alternates between tracing the distance X travels and tracing the corresponding distance Y travels. This is probably at least in part due to the fact that the movement in the $\mathbb{CP}^2$ triangle is actually quite complicated since nearly all points of the triangle actually correspond to multiple different elements (lines) in $\mathbb{CP}^2$.

Second, rather than animating two different continua with different parts of his body, Bart instead uses one part (his right hand) to animate one diagram at a time while he uses another (his torso) to remain engaged with the triangle. His doing this repeatedly, combined with what he is saying and the mathematical context, all demonstrate his sense of the covariation here. This seems to provide him with a way of navigating the mathematical situation by literally physically navigating the two continua with a sense of correspondence between them.

Summary

This case study required the most lengthy description of mathematical context for any in the present study thus far — and yet the behavior of the diagrams on the blackboard were no more complicated than those we saw in previous case studies in this chapter. We see, quite simply, Bart erasing a point on one part of a triangle and drawing a new point elsewhere in the same triangle. This highlights the incredible richness of invocations of the continuum: the sense in which small movements are meaningful requires a remarkably sophisticated understanding of why the diagram might be useful in the first place.

We also saw how Bart created a close association between his $\mathbb{CP}^2$ triangle and Matt's tropical diagram by repeatedly switching his attention between the two. Keeping in close proximity to both of them kept them active in a sense, but much of what he wanted to attend to was the way in which these two continua were *related*. In particular, because of how they were related, the meaningfulness of small movements in one transferred to meaningful small movements in the other. Bart seemed to envelop this via his application of the multiplicity of his body. This in turn allowed him a way of navigating his uncertainty by literally physically navigating each of two invocations of the continuum in a connected way.

Finally, this clip has a moment that vividly illustrates a social dynamic of negotiating interactions with the inscriptions on the board. Here, all inscriptions were invocations of the continuum, so at first glance this might seem like a property of continua. However, as I'll illustrate in the next chapter, this dynamic seems to be a key part of how mathematicians address the more general question of board use.

5.4 Chapter Summary

When we first move into a new house, it has a foreign character to it. Our feet aren't used to the height and placement of the steps, the sounds the front door makes as it swings open and latches closed are unfamiliar to us, and we might have a bit of trouble setting the stove burners to the right temperature if "medium" here is different than "medium" was at the last place we lived. But as we settle in and get used to the quirks and character of the new domicile, what was once foreign becomes familiar: we tug the door just so in order to make it latch, we know without having to think much exactly where the cups are, and our hands reach exactly where the light switches are just by our wanting to flick them. In short, our repeated local travel turns this once-new place into a dwelling (Nemirovsky, 2005).

Mathematicians seem to experience a similar sort of process with mathematics. Devlin (2000) even describes mathematics in terms of exploring a strange mansion filled with odd and fascinating art and furniture, lamenting that many math students never get an opportunity to *enter* let alone explore. We can catch a glimpse of this phenomenology by looking at our own experiences with math: the Cartesian plane started out as a foreign mathematical place (Nemirovsky, 2005), but with "local travel" (graphing equations, anticipating how function graphs change when we alter the original function slightly, etc.) we made this place more of a dwelling for us.

Invocations of the continuum seem to behave somewhat like cartographical maps for the mathematicians observed here. Their goal, it would seem, is to explore the mathematical terrain at hand — but in some cases they seem lost or like they're unsure of how to get from point A to point B. For instance, in the first clip we saw Matt & Bart getting confused because they made a wrong turn, so to speak, which prompted Bart to draw an inverted pyramid to illustrate the relationships between the involved monoids. This provided him with a new way of experiencing the mathematics they were exploring: since small movements would reflect small changes in meaning, the physical sense of placement and orientation became a tool he could bring to bear on thinking about the mathematical relationships that were challenging him. The local travel of the inverted pyramid has a different phenomenological character for Bart than does the local travel of, say, the inscribed function declaration $u_p : P_p \to \mathbb{N}$. Indeed, this different character seems to be precisely *why* Bart draws the pyramid in the first place: it affords different kinds of interaction.

There's also a strong hint here of these continua being transparent-in-use, much as Peirce describes in terms of Icon use (Nemirovsky & Smith, 2011; Peirce, 1991). For instance, when we see a photograph of a relative, it's quite natural for us to say "This is my cousin" rather than "This is a picture of my cousin." Similarly, when using a map to navigate a city, we might point at an intersection and say "We turn right here," essentially dropping the map/territory distinction in the practical use of the map even though we would never confuse the two upon reflection. In the same vein, we see Bart highlighting sides of the pyramid and saying ". . . it could be this and that," and we also see Joseph pointing at the base of the hood in his drawn ribbon to emphasize how there's "nothing funny going on" at that point. Much like with cartographical maps, the continua become stand-ins for the terrain the mathematicians are attempting to navigate.

It's worth reflecting on the different degrees to which invocations of the continuum seem, to the mathematicians, to look like that which they depict. The inverted pyramid from section 5.1 is quite dramatically *not* a look-alike for arbitrary spliced varieties, while Joseph's ribbon knot diagrams from section 5.2 — and even his projected diagrams (e.g., Figure 5.15) — seemed to be attempts to show *almost exactly* what the mathematical objects should look like. Yet this isn't a clear division. In the third clip (section 5.3) we find that Matt and Bart are using a tropical diagram, which is just about as literal in the "looks like" sense as is a classical drawing of the unit circle in the Cartesian plane, the main exception being the multiple rays drawn next to one another to represent that they're on top of one another. This Bart connected to the $\mathbb{CP}^2$ triangle, which at first pass dramatically does not look like what it represents — but in another way it *is* what $\mathbb{CP}^2$ "looks like" in the same way that a sphere can be drawn as a circle.[20] So rather

[20]This is actually a very close analogy: we could represent a sphere with a triangle in a similar way as we did for $\mathbb{RP}^2$ and $\mathbb{CP}^2$. The main difference is that we can visualize the sphere's

than being a dichotomy, we see something of a spectrum, along which different degrees of looks-like-ness appear as part of mathematicians' ways of experiencing the notation in question.

However, we do *not* see any particular differences in how continua of different degrees of "looking like" are treated for the purpose of navigating struggle. Some (such as the paths with the points p, p', and q from section 5.1 and the cobordism drawings from section 4.2) are used as references: they were not written down with the intent of being manipulated or transformed, but instead were drawn to keep track of relationships in a spatial way. Many of Joseph's & Bills drawings (especially from section 5.2) are drawn specifically to be altered as a way of animating them — but not all of them, as we can see in the continua embedded in the Reidemeister 2 diagram (Figure 5.26). Some continua are traced and, as a result, altered, such as the inverted pyramid from section 5.1 and the $\mathbb{CP}^2$ triangle from section 5.3. What's dominant here isn't whether the mathematicians can literally see the mathematical objects in their drawings; rather, what dominates is the *cartographical nature* of continua for them — the sense in which movements and positioning are richly meaningful as a means of navigating the mathematics.

Beyond this cartographical method of turning mathematical places into dwellings, we find that two properties of invocations of the continuum appear as significant. First, there is the notion of *generic particulars* — invocations that are specific by their nature, but are meant to represent any of a family of shapes like them. This is similar to how in elementary calculus it's common to draw an arbitrary smooth graph to illustrate the ideas of derivative and integral: we are, indeed, drawing the graph of a particular function, but in fact we as teachers often mean to indicate by the graph that *any* differentiable function would do. It's fairly common to use unnecessary flairs like an extra local maximum in order to accent

compression and distortion process occurring in 3-space, but we need more dimensions even to make sense of $\mathbb{RP}^2$ or $\mathbb{CP}^2$.

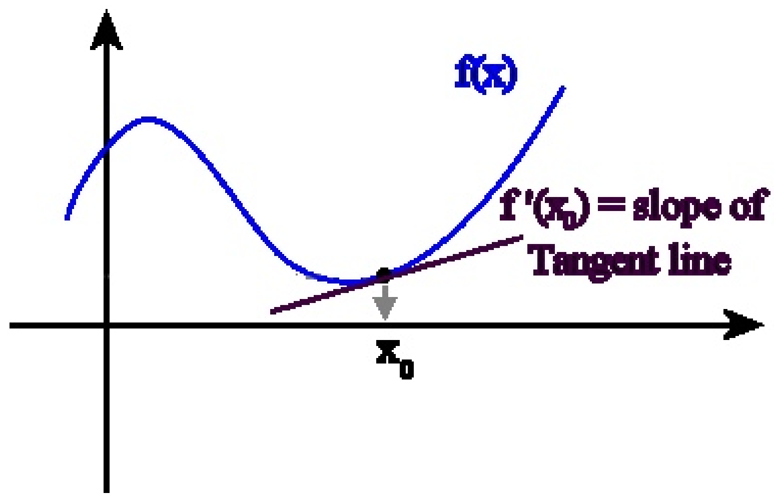

Figure 5.47: A generic-particular function graph with a local-maximum "flair" (from http://clas.sa.ucsb.edu/staff/lee/Tangent%20and%20Derivative.gif)

the generic-particular nature of the graph (Figure 5.47), just as we saw Bart add a loop to the path containing the point p (Figure 5.5) and saw Joseph draw a *curved* boundary to the ribbon surface in section 5.2. This addition of flairs doesn't always happen with generic particulars; we find none in the cobordism drawings Joseph creates in section 4.2, for instance. Yet it occurs often enough to be a characteristic worth noting.

Second, we find that the continua are implicitly boundless, but in a peculiar way. Both Joseph's ribbon knot drawings and Bart's inverted pyramid are to be understood as extending beyond the region drawn. However, there's no confusion involved in designating regions near the invocations for something else. For instance, even though Bart's inverted pyramid extends infinitely upward in principle, there doesn't seem to be any sense of conflict for Matt or Bart about there being notation on the board directly above the pyramid. The infinite expanse is contained in a particular (if only vaguely outlined) region of the board. This is an intriguing property of continua that will help to fit the material of this chapter with the results of the next.

In fact, a number of phenomena arose here that will receive more detailed examination in the next chapter. Bart's choice to invoke continua off to the side in section 5.1 but more centrally in section 5.3 raises the question of the *role of board centrality*. The phenomenon of *idle perfecting* of the invocations appears with non-continuous inscriptions as well, and also ties in proximal inhibition. And finally, the social dynamics involved in negotiating who it is that may activate and engage in the board's notation paint a picture that helps synthesize the phenomena discussed throughout the present work.

But before I can elaborate on these connections, I will need to illustrate one more phenomenological construct, which is the topic of chapter 6.

Chapter 6

Transformation Mappings

In the mid-19th century, the scientist Michael Faraday isolated what he called "bicarburet of hydrogen," which later received its current name of *benzene*. Benzene had a myriad of uses and thus became the center of study for chemists of the era. They worked out that its molecule must be made up of six carbon atoms and six hydrogen atoms (C_6H_6). Yet whereas other known hydrocarbons such as methane (CH_4) had a fairly clear atomic arrangement (Figure 6.1), benzene's structure was much more elusive. Most ways of arranging six carbons and hydrogens into a molecule don't result in something with benzene's known chemical properties.

Chemists struggled with this for nearly half a century. The revelation finally came when the chemist August Kekulé thought to arrange the carbon atoms in a *ring* (Figure 6.2). This gave them a way to understand the origin of many of benzene's properties in the same way they could understand methane's from its diagram. Notice, though, that the difficulty was not in their understanding of chemical structure (Lewis) diagrams like Figures 6.1 & 6.2. Rather, the challenge was in finding a *legitimate transformation* from "C_6H_6" to a diagram like Figure 6.1 that preserves benzene's properties.

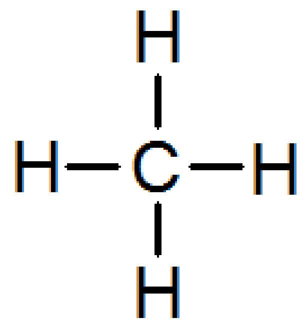

Figure 6.1: The molecular structure of methane (as a Lewis diagram)

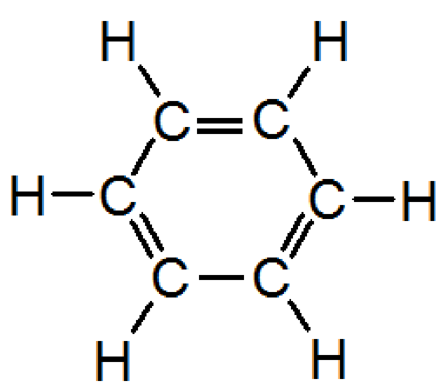

Figure 6.2: A Lewis diagram of benzene

This theme of finding legitimate transformations that seem to preserve something of interest to the individuals in question is central to the mathematicians in this study. Every pair in every observation covers the blackboard with inscriptions that are meant to illustrate these transformations. As such, the board becomes, for the mathematicians, a canvas upon which are painted implicit *transformation mappings* that illustrate the mathematical objects being explored.

The key defining feature here is the sense of something being preserved between transformations. When we rewrite $x^2 + 2x + 1$ as $(x + 1)^2$, there's an impression that we've simply expressed "the same thing" (namely some quantity) differently. Indeed, the sense that the transformation is legitimate is inseparable from this impression of having preserved something while factoring the original expression. Similarly, when we rewrite $y = x^2$ as a parabola drawn on a Cartesian plane, the transition seems to preserve the relationship between two variable quantities. Indeed, it's precisely this sense that "the relationship" is preserved that makes Cartesian graphs useful as an algebraic tool.

In this chapter I'll illustrate several ways in which transformation mappings — by which I mean networks of inscriptions that the mathematicians see as connected via legitimate (i.e., something-preserving) transformations — structure the subjects' encounters with struggle. In an analogous way to how we become familiar with a new object like a wallet by seeing "the same thing" from many different angles, mathematicians seem to gain familiarity with a mathematical object by seeing it as preserved across transformations. Many nuances of how they engage in these transformations, and what they do when they struggle to "see from a different angle" the objects they're interested in, will emerge as a result. This, in turn, will help to unify many of the findings of the present study into a wider framework of these mathematicians' experiences with struggle.

6.1 Clip #1: Chain Maps

6.1.1 Mathematical Context

The mathematical context for this clip is largely the same as that given in section 4.1.1. Joseph and Bill are still talking about odd Khovanov homology. They're explicitly referencing Reidemeister moves, which were explained in section 5.2.1. The main idea I'll need to elaborate upon here is that of a chain map.

Chain Maps

Mathematicians want the Khovanov homology (see section 4.1.1) of two different knot diagrams to be the same if the two diagrams differ only by Reidemeister moves. (That is, transformations of knot diagrams by Reidemeister moves are legitimate.) That way the Khovanov homology says something about the *knot* (i.e., that which is seen as preserved across Reidemeister moves) instead of saying something about how the knot happens to have been drawn. Doing this requires building the homology in a particular way that ignores these Reidemeister differences.

As a general rule, homologies come from *chains*. A chain in the mathematical sense is a sequence of algebraic objects that are connected by functions corresponding to the type of object such that going two steps down the chain results in being sent to "zero," whatever that means for the objects in question. So if the objects are groups (see section 5.3.1, page 145), the functions have to be homomorphisms (i.e., group-operation-preserving: $f(xy) = f(x)f(y)$), and the "zero" in this case is the group's identity. Here's an example of a chain:

$$0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \times \mathbb{Z} \xrightarrow{g} \mathbb{Z} \to 0$$

In this case, $\mathbb{Z}$ represents the integers under addition, and $\mathbb{Z} \times \mathbb{Z}$ indicates ordered pairs of integers. "0" refers to the group with one element.[1] To give a specific example, let's say the homomorphism f here sends n to $(n, 0)$ while g projects on the second coordinate (i.e., $g(a, b) = b$). Then $g \circ f$ always sends everything to zero: $g \circ f(n) = g(n, 0) = 0$.

Homologies in topology often come from creating chains by looking at geometric relationships and then essentially setting all the parts of the chain that refer to things we don't care about to zero. So to construct Khovanov homology, we would somehow create a chain that corresponds to a knot diagram of the knot we're interested in, and then set everything that would change by doing Reidemeister moves to zero. One way to do this is by creating a *second* chain based on a knot diagram that differs by, say, one Reidemeister 1 move and then checking that the way we built the two chains will result in the same homology. We can do this by constructing what's called a *chain map*[2].

To get the idea of a chain map, pretend that we have two chains, $\mathcal{G}$ and $\mathcal{H}$. $\mathcal{G}$ looks like this:

$$\cdots \to G_n \to G_{n+1} \to G_{n+2} \to \cdots$$

and $\mathcal{H}$ looks like this:

$$\cdots \to H_n \to H_{n+1} \to H_{n+2} \to \cdots$$

You can think of each G_i and H_i as being a group, like $\mathbb{Z} \times \mathbb{Z}$. A chain map $\phi : \mathcal{G} \to \mathcal{H}$ would be a whole sequence of functions $\phi_i : G_i \to H_i$. That is, ϕ is the

[1] If a group has only one element, then it doesn't matter what we call that element or what we call the group operation. For instance, the group could be zero under addition, or it could be 1 under multiplication, or it could be the identity function $\iota(x) = x$ on the real numbers under function composition. In every case, although the names change, the group structure remains the same: the group element combines with itself to produce itself. Thus, mathematicians commonly refer to *the* group with one element, or the "zero group."

[2] Mathematicians often use the term "map" to refer to functions that are special in some way. For instance, topologists often say something like "f is a map" to indicate that f is a continuous function. It's also fairly common to have homomorphisms called "maps" when the objects being considered are groups. There is in fact a relationship between this use of "map" and a cartographical map, but for our purposes we can ignore that and just read "chain map" as "a special function between chains."

stuff in the middle band here:

$$\begin{array}{ccccccccc} \cdots & \to & G_n & \to & G_{n+1} & \to & G_{n+2} & \to & \cdots \\ & & \downarrow \phi_n & & \downarrow \phi_{n+1} & & \downarrow \phi_{n+2} & & \\ \cdots & \to & H_n & \to & H_{n+1} & \to & H_{n+2} & \to & \cdots \end{array}$$

But chains aren't just a collection of groups or other algebraic objects. A chain includes the functions that, in our diagrams above, are going horizontally. (Metaphorically, a chain isn't just a collection of links, but a collection of links *connected in an order.*) So just like how we want linear transformations to preserve linearity (i.e., $f(v + w) = f(v) + f(w)$), we want chain maps to preserve something about the sequentiality of the chain. Mathematicians capture this desire by requiring that all the squares "commute." Let's focus on just one square to clarify what this means:

$$\begin{array}{ccccccc} \cdots & \to & G_n & \stackrel{f}{\to} & G_{n+1} & \to & \cdots \\ & & \downarrow \phi_n & & \downarrow \phi_{n+1} & & \\ \cdots & \to & H_n & \stackrel{g}{\to} & H_{n+1} & \to & \cdots \end{array}$$

If we imagine we're traveling around this square starting at G_n in the upper-lefthand corner, then we can either follow f to the right or ϕ_n down. If we follow ϕ_n down to H_n, then the only direction we can go that's with the arrows is along g to H_{n+1}. On the other hand, if we start out following f to G_{n+1}, we can still reach H_{n+1} by then taking ϕ_{n+1}. That is, we're comparing the functions $g \circ \phi_n$ and $\phi_{n+1} \circ f$. It's possible that these two functions are not equal — that is, we could end up in radically different parts of H_{n+1} depending on which path we take. But if ϕ is a chain map, then it doesn't matter which path we take: $g \circ \phi_n = \phi_{n+1} \circ f$. This requirement means that ϕ preserves something about how the algebraic objects in $\mathcal{G}$ are connected, much the same way a linear transformation preserves linearity.[3]

[3]In fact, this analogy runs quite deeply. This bespeaks a general pattern from a domain of mathematics called *category theory*, in which a category is the collection of all of a given kind of mathematical object together with "arrows" that preserve whatever mathematicians consider to be interesting about those objects. In the category of vector spaces, linear transformations

Chains that are "equivalent"

We know that two linear spaces are "equivalent" in some sense if there's a way we can go back and forth while preserving the linear structure of the space. This lets us say that Euclidean 3-space $\mathbb{R}^3$ and the collection of quadratic polynomials $\mathbb{R}[x]/\langle x^3 \rangle$ are "the same" in the sense that we can easily figure out the ordered grouping $(3, 5, -2)$ from the polynomial $3x^2+5x-2$ and vice versa, and this correspondence preserves the linear structure of both spaces. Although these two spaces are different, there's nothing about their *linear structure* that's different. So, as vector spaces, they're the same (i.e., *isomorphic*).

In the same way, we can say that two chains $\mathcal{G}$ and $\mathcal{H}$ are isomorphic if there's a way of going back and forth between them while preserving the chain structure. Chain maps preserve chain structure, so this gets formalized as saying that there are chain maps $\phi : \mathcal{G} \to \mathcal{H}$ and $\psi : \mathcal{H} \to \mathcal{G}$ such that $\psi \circ \phi$ and $\phi \circ \psi$ are both identity maps on their respective chains. At one point we'll see Joseph and Bill draw chain-map arrows going both ways, and this is why: in normal Khovanov homology, two chains that result from knot diagrams that differ by a Reidemeister move are isomorphic, and they're trying to explore what happens when they switch the context to *odd* Khovanov homology.

Using pictures in place of groups

Finally, I should note that topologists often use drawings of the geometric objects the groups in the chain come from in place of the groups. For instance, we can think of a circle as two points with two strings connecting those points (Figure 6.3). If we call the points p and q, then we can pretend that the symbols "p" and

are the "arrows." In the category of chains, chain maps are the "arrows." In group theory, homomorphisms are the "arrows." So from a categorical point of view, what we've just done is said that a chain map is what it has to be in order to make the collection of chains into a category.

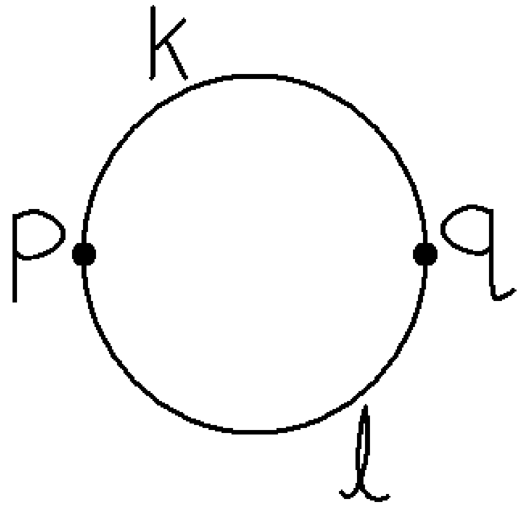

Figure 6.3: A circle as two points plus two line segments

"q" have algebraic meaning as variables and add or subtract copies of them. For instance, you could write $2p-3q$ the same way you could write $2x-3y$ for variables x and y. So all told, the zero-dimensional parts of this circle create a group that we can write as $\mathbb{Z}p \oplus \mathbb{Z}q := \{mp + nq \ : \ m, n \in \mathbb{Z}\}$. Similarly, we can call the two lines k and ℓ, in which case the group we'd get from that is $\mathbb{Z}k \oplus \mathbb{Z}\ell$, treating k and ℓ like variables too.

Because there aren't any pieces of higher dimension, the corresponding groups are all trivial, or "zero." So the chain[4] will look like this:

$$\cdots \to 0 \to 0 \to \mathbb{Z}k \oplus \mathbb{Z}\ell \xrightarrow{f} \mathbb{Z}p \oplus \mathbb{Z}q \to 0$$

Each of those arrows is a homomorphism. Most of them are zero, with the exception being f (from dimension 1 to dimension 0). There's a way to work out geometrically what f is, but for our purposes it's not important. What's important

[4]This chain is specifically for deriving *singular* homology — i.e., counting "holes" in the space. Khovanov homology focuses on different features than these for knot diagrams. The exact details aren't important for the analysis that follows. All that's needed is a recognition that the chain complexes we'll see in the video analysis come from geometric features.

is recognizing that we get a chain by attending to geometric features of whatever we're looking at.

It'd be perfectly acceptable to use drawings of the *original geometric features* rather than symbols representing the groups. Figure 6.4 shows what the same chain above might look like if we were to draw k and ℓ instead of writing "$\mathbb{Z}k \oplus \mathbb{Z}\ell$" and similarly drawing two points instead of showing the zero-dimensional group in the chain.

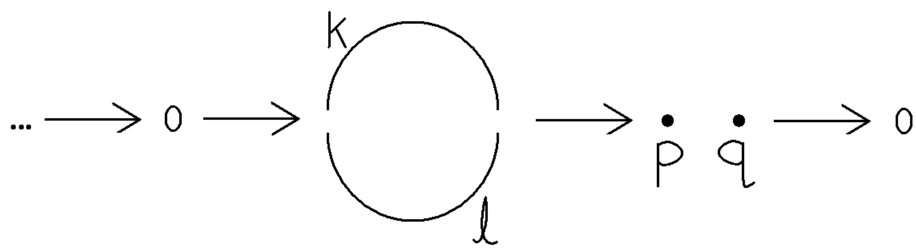

Figure 6.4: A chain depicted via continua

Summary

What Joseph and Bill are doing in this clip, then, is exploring what happens to the chain map in normal Khovanov homology when they apply it to chains built toward developing *odd* Khovanov homology. They're specifically looking at two chains corresponding to knot diagrams that differ by a Reidemeister 1 move. They *want* odd Khovanov homology to come out the same for knot diagrams that differ only by Reidemeister moves, and they're trying to make sense of how they might make that happen.

6.1.2 Video Description

This section references the gesture analysis found in Table A.9.

Preparing the chain map diagram

Seven minutes prior to the image shown in row 1, Joseph tried to tackle this question about invariance under Reidemeister moves by drawing a chain map on the board. We can see this chain map immediately above Bill's head in the rightmost image of row 1. This is a single square of two very simple chains, each ending in zero on either side. Each chain uses continuum-based drawings, each of which is a cross-section of a cobordism — or equivalently, a snapshot in time of the process of cutting and splicing knots into new configurations. (However, the overall depiction of the chain map does *not* invoke the continuum: slightly shifting the bottom-right continuous drawing slightly downward wouldn't change the meaning of the chain map at all for Joseph or Bill.)

Joseph erases the bottom row corresponding to one of the chains (rows 2–5). The diagrams that had been written there before turned out not to be as helpful as Joseph and Bill thought they'd be. The top row is the chain that emerges from a knot diagram with a twist in it, but the bottom row doesn't correspond to the same diagram with the loop un-twisted (i.e., after reversing the twist with Reidemeister 1; see Figure 5.9). Joseph's erasing that bottom board occurs as soon as they recognize the error.

This is a common theme we'll see throughout this chapter. The mathematicians are not concerned about keeping track of the history of their wrong turns, even as warning signs about what mistakes not to make. Instead, the inscriptions on the board must preserve the integrity of the overall transformation mapping — which is to say, all inscriptions that bespeak transformations must bespeak *legitimate* transformations. Things that are mistaken get erased right away rather than be left to corrupt the work on the board.

Joseph then writes a new chain (rows 6–11). It's worth noticing that both of these chains have a very simple structure:

$$0 \;\rightarrow\; A \;\rightarrow\; B \;\rightarrow\; 0$$

$$0 \;\rightarrow\; C \;\rightarrow\; 0 \;\rightarrow\; 0$$

(Henceforth I'll drop writing the zeros at either end since those aren't written in Joseph's drawing.) Here, A represents the structure implied by the curve with a circle separate from it, B represents the two line segments having been merged, and C is just the curve without the circle. Joseph drew a single dot to represent "zero" to the right of C (row 10). Each of these diagrams is meant to show a cross-section of a cobordism, or equivalently, a moment in time of the movie that plays from turning one collection of knots into another. (See section 4.1.1.) The single dot represents a loop shrinking to a point and then vanishing, which in the cobordism looks like a cap. The corresponding algebraic structure of this is zero. All of this means that the only nonzero horizontal function is that between A and B, and that function could be anything at all that preserves the structure of A and B in order to make it a chain: no matter what it does, going two steps will result in zero.[5]

Joseph continues his attempt to describe this transformation (i.e., the chain map) by drawing arrows between A and C and also drawing a vertical line below B (rows 12–19):

$$\begin{array}{ccc} A & \rightarrow & B \\ \uparrow\!\downarrow & & | \\ C & \rightarrow & 0 \end{array}$$

He says here, "We need to write down chain maps this way and this way." The idea is that since everything but C is zero in the bottom row, a chain map going

[5]Joseph later clarified in a follow-up interview that this simplicity comes from the fact that the Reidemeister 1 move is relatively simple. The pairs of chains corresponding to Reidemeister moves 2 and 3 are each quite a bit more complicated, and the "go two steps and land at zero" requirement plays a much more important role.

down would by necessity be zero everywhere except possibly in going from A to C. Since the functions in a chain map are structure-preserving, the function going up to B from zero would have to have B's zero as its entire range. Therefore, creating a chain map going down and another going up would require only that Joseph and Bill determine the functions going between A and C.

In particular, notice that the function going from A to C could be just about anything in order to make the chain map going down be, in fact, a chain map: whether we go $A \to B \to 0$ or $A \to C \to 0$, we'll end up at zero. (Joseph later points this out explicitly, on rows 167–170.) So the main direction of interest is going up from C to A, which makes the function going from A to B very important. Thus, we see Joseph mark the $A \to B$ function with an m and calling it a "multiplication" (rows 21–22):

$$\begin{array}{ccc} A & \stackrel{m}{\to} & B \\ \uparrow\downarrow & & | \\ C & \to & 0 \end{array}$$

This is a multiplication because in the Frobenius algebra one uses to build Khovanov homology, mergers of knots correspond to multiplication. The key question, then, is what the function from C to A must be.

Initial attempt to work out the chain maps

This involves a fair amount of effortful thinking for the both of them (rows 23–29). We see Bill touching his face in a self-soothing and/or self-isolating gesture (rows 23–25) and then using his pencil to point at something he's imagining that's presumably related to this situation while looking down and to his right (row 26). We also see Joseph essentially frozen, staring at the board during this whole segment; although it's possible he's simply spacing out, given the context and what comes next it seems more likely that he's concentrating heavily on his thoughts

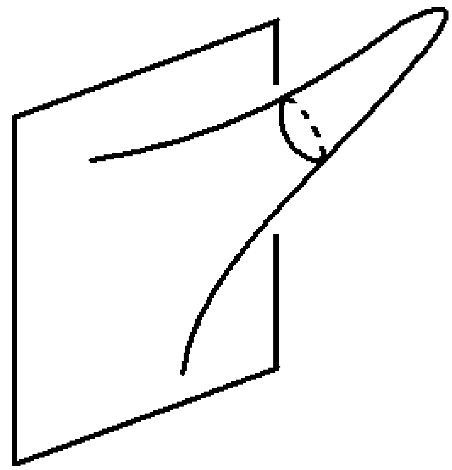

Figure 6.5: A sheet with a cap attached to it

about the diagram.

In row 29 Joseph says, “So what are the usual ones and why?” He’s referring to the “usual” chain maps involved in normal Khovanov homology for Reidemeister 1, and he’s hoping to make sense of what the chain maps should be in this odd Khovanov homology setting based on those. This prompts him to look away from the board (rows 30–31) presumably to try to remember those chain maps. We also see Bill engage with his notes in order to look up something that will help them recall what the maps were (rows 31–34).

Joseph notes that “in one direction you cap off with a torus, don’t you?” while glancing up at the board briefly (rows 33–35). This is in reference to a cobordism, where capping off refers to shrinking a knot to a point and then making it vanish. In the cobordism this might look like a sheet with cap attached to it (Figure 6.5). A torus, on the other hand, looks like a handle when attached to surfaces. So capping off with a torus would look like a sheet with a handle attached to it such that the handle is angled upward (Figure 6.6). If we think of Figure 6.6 with time going upward, then the movie consisting of watching the cross-sections

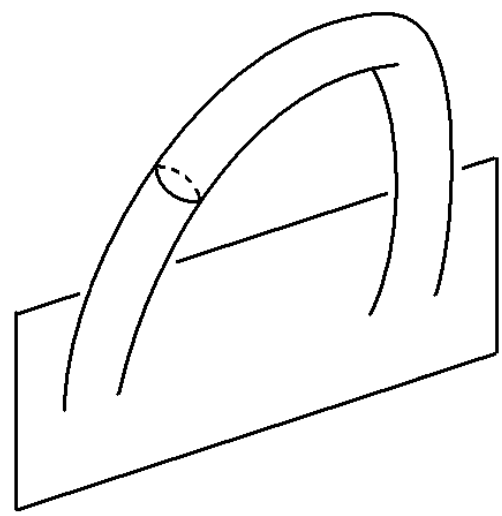

Figure 6.6: A sheet with a handle attached to it

go forward in time would look like Figure 6.7. Notice that the last two images of Figure 6.7 look like the drawings for A and C. What Joseph is saying, then, is that the capping off that happens when moving down the chain map from A to C is capping off with a torus (as opposed to capping off with a non-torus such as in Figure 6.5).

The point of working out the function $A \to C$ when what they want is the function $C \to A$ is that in normal Khovanov homology, these two compose to form the identity. That is, going $A \to C \to A$ sends everything in A to itself, and going $C \to A \to C$ sends everything in C to itself. Thus knowing what the function $A \to C$ does can give them a hint about what the function $C \to A$ must do. This comes from the fact that Joseph and Bill want A & C to be "the same" in some sense, and therefore they need to work out what the legitimate transformations between them are. Much as how knowing how to expand an algebraic expression can give insight into how to factor, Joseph and Bill are hoping to understand the transformation $C \to A$ by understanding the other direction first.

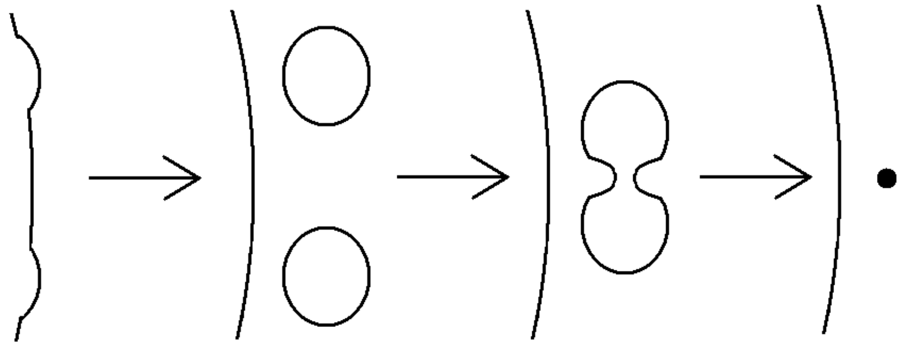

Figure 6.7: The "movie" of capping off with a torus

This turns out not to be enough for Joseph and Bill at first, and they spend a fair amount of time thinking about this and referring to their notes (rows 36–58). However, in rows 50–57 we see Bill notice something: he rapidly switches his attention between the notes in his lap and the blackboard and then points toward the board. Bill then reaches forward to indicate A (rows 58–63). In doing so, he keeps his gaze fixated on his notes. Bill's reach here is slow and careful, and he's checking and possibly re-checking his conclusion even as he moves to make it. This behavior bespeaks a kind of caution in expressing reasoning about material on the board that one doesn't see with, say, quick calculations on a piece of scratch paper. Bill *could*, in principle, share his thinking as soon as it occurs to him (e.g., "I wonder if that top row is a merge minus a torus") — but what we see instead is this checking and hesitation and careful reflection before he's willing to speak. This is reminiscent of how mathematicians' published papers often say little about where an idea came from or what mistakes they made in the course of coming to a conclusion (Burton & Morgan, 2000; Thurston, 1994). Yet here, in a way reminiscent of one of Greiffenhagen & Sharrock's (2011) points, we see this same sort of caution present in the process of developing the mathematical ideas themselves.

Figure 6.8: Joseph's drawing in row 75 of Table A.9

Notice also the social dynamic here: Joseph rolls up to the board and starts modifying the diagram, and as soon as Bill looks up and notices this he pulls back from the board (rows 61–66). Bill continues to make his point, but by gesturing in the space in front of him while he leans back instead of by pointing to parts of the diagram on the board (rows 68–71). This is another example of the phenomenon we saw in the case of Matt and Bart in chapter 5 where they were implicitly negotiating who had control of the board (see page 171). In this case Joseph is inscribing Bill's point on the board while checking what he needs to add to make it true (rows 71–75).

Figure 6.8 shows what Joseph has drawn in row 75. This is a transformation in which Joseph & Bill are trying to preserve the function between C and A. Much as how rewriting $y = x^2 + 2x$ as $y = (x+1)^2 - 1$ can make it easier for us to see where the vertex of the function's parabolic graph is, Joseph has written Figure 6.8 as a way of seeing the function in question from a different (and, he hopes, insightful) angle.

Correcting a mistake

After a bit of a pause (rows 81–83), Joseph's wrist zips to his mouth (row 84). This is the mouth-touching behavior mentioned in chapter 4. Accordingly, Joseph then drops his hand and says, "Wait a minute" (row 85). The moment his hand zips to his mouth, it seems that he noticed an inconsistency on the board: in Figure 6.8 either m isn't a merge and is thus misleadingly labeled, or the cup needs to be inverted to be a cap. Which it is depends on which direction they're going (from A to C or vice versa). This is why he laughingly says, "That's downwards isn't it? How are we... Which way are we reading..." (rows 86–90). The laughter here seems to serve as a way of dispelling tension caused by confusion and a sense that there's something wrong written on the board.

Once Joseph asks the question, Bill squeezes his eyes in concentration (row 87) and then specifies that they're trying to describe the function going from C up to A (rows 90–100). From this Joseph concludes that m must be referring to a split rather than a merge and immediately erases the m (rows 101–102). Again, we see this immediacy with which something that corrupts the overall transformation mapping gets erased, even though doing so erases the history of their thinking on the blackboard. It's far more important that the inscriptions refer only to *legitimate* transformations, as otherwise the different "views" being drawn on the board don't refer to "the same thing."

Because Joseph knows what m should actually be, he immediately writes the word "split" where m used to be (rows 103–105). This is a nuance of writing down transformations that appears repeatedly throughout every case we'll examine in this chapter. Because the board is used almost exclusively for transformation mappings[6], inscriptions take on a charge of significance. Joseph could have simply made a mental note that the m represented a split, but because it was written with the intent of standing for "merge" it had to be erased and replaced with

[6]In all the recorded observations of mathematicians in the midst of working that I've engaged in thus far — both for this research endeavor and related ones — I've seen only one exception to the use of the board for transformation mappings: sometimes mathematicians will write a note to themselves off to the side of their main workspace. For instance, they might write "Email Alex about supervector spaces" at the edge of the board. I've always seen these notes written at the edge of the board, and they've almost always been separated by a line dividing the note from the workspace. This doesn't happen in any of the clips used in the present study.

something written with a more accurate intention. Furthermore, rather than just writing "s," we see him write down the entire word "split." Arguably this could have been because the letter 's' looks a great deal like the numeral '5' and as such it's normal to avoid the use of the letter 's' in mathematics, but in areas closely related to Joseph's field of research it's quite common to ignore this convention.[7] Writing the whole word "split" *exactly on top of* where the m had been before seems to be a way of emphasizing that thinking in terms of a merge was just wrong and that this is a correction.

Also, notice how Bill snatches his hand back from the board and covers his mouth with his forearm as soon as Joseph points out the mistake (rows 102–104). While doing this, he apologizes twice, laughs, and acknowledges that Joseph's correction is right. This gesture is made even more stark by the fact that Bill drops his arm from around his face once Joseph has finished writing "split" on the board (row 106). It's as though Bill's discomfort comes largely from there having been something wrong *on the board* rather than just from his having suggested something erroneous. This bespeaks the importance to Bill of links in a link-map on the board having flawless integrity since a weak link would make the subsequent chain of reasoning fragile and confusing.

We also, again, see a trade-off here as to who may animate the diagram. In row 106 we can see how Bill starts his movement toward the board only once Joseph has withdrawn from it. This seems to be a case of Joseph relinquishing control of the diagram entirely on his own rather than because of any clear signal Bill gave that he wanted to engage with it.

[7] "s" (lower case) often gets used to express what's called a *section* of a projection mapping, and it also gets used to describe a path $s : I \to X$ in a topological space X. Other variable names are common, too — most notably σ and γ for sections and paths respectively. But the point is that "s" is actually a common variable name in topology and differential geometry, which are major parts of what Joseph researches.

Checking one direction for isomorphism

Jumping ahead about 40 seconds, we see Joseph using the "commuting square" property of chain maps (rows 141–143). In defining the chain map going upward, they already know that going from C to the right to zero and then up to B must land them at the zero of B. This means that going from C up to A and then over to B must *also* land them at the zero of B. They think the function $C \to A$ is described by Figure 6.8, so Joseph explores what happens to each of the split (rows 145–156) and the cup (rows 157–163) under the multiplication map $A \to B$. Since the two components cancel out, Joseph and Bill conclude that the description they've given for the function $C \to A$ does work.

It's worth noting how Joseph navigates between the two "views" the $C \to A \to B$ process: he traces the path in the chain map diagram (row 142) and then switches to pointing toward and navigating the cobordism-like drawings to the left (rows 143–148). He even uses a hop upward of his hand near the blackboard to mimic what the $A \to B$ step will look like in this "view" (row 147). The "sameness" here that makes the implicit transformation legitimate is described by the up-and-over path on the chain map and by the cobordism-like arithmetic in the left-hand diagrams.

Also, notice how this $C \to A \to B$ path is itself part of a larger object Joseph and Bill are trying to create, namely the chain map going upward. The chain map upward can be thought of as a transformation of the lower chain to the upper one; indeed, this is essentially what they're trying to construct. And even the chains themselves can be thought of as sequences of transformations (e.g., from A to B). The transformation expressed in the contrast between the chain map diagram and the cobordism-like drawings seems to be taking *paths within* the chain map diagram as that which is preserved. This suggests a considerable dynamism in these mathematicians' experience as to what constitutes "a mathematical object."

Joseph notes that they can't use the commuting-square property to work out what the function $A \to C$ should be since "you can put right anything for it to be a chain map" (rows 167–170). Bill suggests that it just be a matter of capping off the circle in A so that it vanishes to produce C (rows 172–180), which Joseph says seems reasonable (rows 182–185). Thus at this point, they have their candidate chain maps and just need to test what happens when they go back and forth (i.e., $A \to C \to A$ and $C \to A \to C$).

Joseph starts out by looking at $C \to A \to C$. He uses his finger to traverse the chain maps (rows 189–192) and then quickly reasons that the resulting composite function must be "minus the identity" (rows 199-201). It seems that this reasoning was too quick for Bill (rows 203–204), so Joseph pauses for a moment and labels the two key functions f and g (rows 208–212):

$$\begin{array}{ccc} A & \overset{m}{\to} & B \\ f \uparrow\downarrow g & & | \\ C & \to & 0 \end{array}$$

This allows him to clarify that he was claiming that $g \circ f = -Id$, which he writes on the board (rows 213–215). (Mathematically this means that going $C \to A \to C$ sends each element of C to its additive inverse. If C were the integers $\mathbb{Z}$, going up and down again would send, say, 3 to -3. This happens because they're dealing with *odd* Khovanov homology instead of normal Khovanov homology.) These labels serve as a way of signaling a transformation from the chain map path to the function equation. Thus, roughly speaking, the current transformation mapping has three nodes: the chain map diagram, the cobordism-like arithmetic, and the function equation — yet it's important to bear in mind that the chain map diagram in particular does not seem to consistently behave as though it represents "one thing" for these mathematicians.

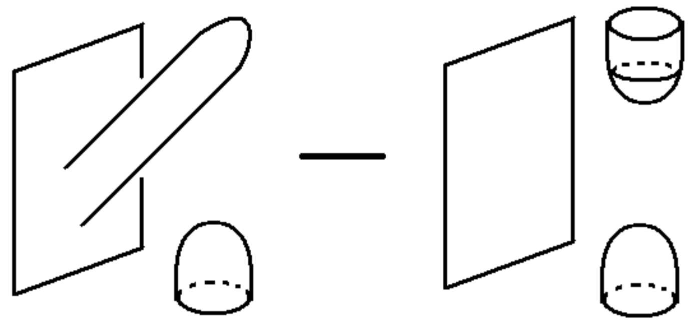

Figure 6.9: The last diagram drawn on the board (row 248 of Table A.9)

The opposite function loop

The remainder of the clip is focused on the other direction — which is to say, working out what $f \circ g$ is. This seems to be considerably more challenging for them (rows 219–224). Notice that in row 219, Joseph actually writes $f \circ g$ on the board. Given the mathematical context, there's relatively little ambiguity that this is the next thing they need to look at. Thus Joseph's writing this seems to be emphasizing, "This is the next view we need a legitimate transformation to." It's an attention-directing inscription, applying the nature of the board-in-use as a medium for transformation mappings to focus them on the next step. It's probably also a matter of Joseph knowing that he wants to write "$f \circ g =$ [something]" and he's writing down the part that he knows — although it's worth noting that he doesn't actually write down the equal sign.

Joseph then haltingly draws each component of this map in a form similar to that in Figure 6.8 (rows 226–248). The drawing he ends up with at the end of this clip is shown in Figure 6.9. Notice how Joseph picks a new space fairly far away from the $f \circ g$ inscription to draw this. That could be a matter of space, but this phenomenon is fairly common throughout the mathematicians observed here. It seems to have the character of a lemma: Joseph seems to intend for the transformation mapping to go between $f \circ g$ and this pictorial arithmetic drawing to justify whatever the final expression for the function equation should be.

Summary

This episode has within it most of the phenomena that we'll use to detail the role that transformation mappings can play in struggle navigation. One of the main points to notice here is Joseph's & Bill's reaction to error (page 197): nervousness, a need to release tension (through laughter or chuckling), and an impulse to correct it right away. Furthermore, the correction (erasing m and replacing it with the word "split") wasn't strictly necessary, but because it represented something that was mistaken *in its intent upon being written* it needed to be corrected. The correction itself was also made directly on top of where the original mistake had been; this might seem like it would have to be that way (since the m was part of an algebraic expression), but this pattern is sufficiently robust in situations where it *isn't* necessary that the necessity stands a good chance of not having been the central motive for Joseph in that moment. We'll see more examples of this later in this chapter.

There's also a question of what constitutes "a mathematical object." On one level, Joseph and Bill are trying to construct chain maps between chains that represent two knot diagrams that differ by one Reidemeister move. But in zooming in on the behavior of these chain maps, the objects become paths in the chain map diagram — or equivalently (after a path-preserving transformation), composite functions. Even then, though, the composite functions' formulas become the center of attention the same way that terms and variables become the center of our attention when completing the square on an expression like $x^2 + 6x - 3$. This nuance will become key when we reflect on the apparent contrast between created and discovered mathematics in mathematicians' phenomenology.

We also briefly saw the board being used for emphasis, as a means of directing attention. Joseph's writing $f \circ g$ was, strictly speaking, unnecessary — but it still constituted part of his effort to think about the mathematics. This reflects on moves such as his decision to roll up to the board and write down Bill's suggestion as Figure 6.8 (rows 60–75): by writing the pictorial expression down, he's implicitly agreeing that this is a legitimate transformation of the $C \to A$ path. This contrasts, for instance, with Joseph's simply changing the topic when he disagrees with Bill's suggestion in rows 125–141.

There were also a few specific examples of negotiating board control, first raised in chapter 5 (see page 171). This dynamic seems to emerge from a property of the transformation mappings together with social norms in mathematical communities of practice (Lave & Wenger, 1991) and the interpersonal dynamics of these two particular mathematicians. As we examine other examples of board control negotiation, we'll be able to see some attributes that underly this dynamic, which will in turn tell us something more about the role transformation mappings play in mathematical struggle.

6.2 Clip #2: Facets

For this clip we return to the scene with Matt & Bart already described in section 5.3. This time we'll focus on the interactions with the algebraic statements that resulted in Bart drawing the inverted pyramid.

6.2.1 Mathematical Context

There is nothing new to add about mathematical context beyond what's already described in section 5.3.1.

6.2.2 Video Description

This analysis again refers to Table A.4.

Regions of workspace

The material on the board at the beginning (row 1) is a collection of transformation mappings, several of which are separated by drawn dividers (the vertical line in the middle and the half-box in the lower-righthand corner). The left half of the board is mostly algebraic notation that Bart had written and is all part of one transformation mapping. The triangle and the three-pronged star on the far left were Matt's doing and referred to notation that was where Bart's formulas are now written. The diagrams on the upper-lefthand corner of the board (which you can barely see in row 1; see row 18 for a better view) are part of Bart's thinking that led him to write the algebra. The two regions to the right of the vertical line are Matt's doing and reference a way of thinking about this splicing problem from a more tropical-vertex-group point of view.

After they agree to shift strategies to focusing on just one pair of varieties at a time (rows 1–9), Bart jumps up and grabs the eraser as he says, "Basically this is not such a concern at this point" (rows 10–13). He then proceeds to erase his algebraic notation (rows 17–22), pausing for a moment to evaluate whether a particular square diagram (rather like a square of a chain map) is part of what should be erased (row 21). The transformation mapping Bart erases has no known flaw as far as the two of them are concerned, but they've determined that it's not something they should be thinking about right then. This, combined with the need for space on the board within which to write a new transformation mapping, brings Bart to erase this portion.

It's worthwhile to notice that Bart does *not* erase everything related to the original transformation mapping: he leaves the two diagrams in the upper-lefthand corner alone. This occurred earlier, too, which is why we see Matt's triangle and three-pronged star on the left even though the algebra to which it's connected was erased quite a bit earlier. This illustrates how, while it's totally unacceptable to have nodes of the transformation mapping that imply illegitimate transformations, it's perfectly fine to have fragments of earlier transformation mappings.

This also reveals a phenomenological element of the blackboard for these mathematicians, namely the idea of *regions of workspace* (Nemirovsky & Smith, 2011). Bart finds it helpful to erase the region of the board where the now-irrelevant algebra is but seems to implicitly view the column to the left of that region with the diagrams as separate. There's no need to erase them because he already has a region within which to work.

Notice also that while thinking about what to write, Bart moves in toward the board to use his hand to erase a part of the region he just cleared (rows 32–36). This is reminiscent of the idle perfecting phenomenon we saw in chapter 5 in which intense concentration on an invocation of the continuum sometimes encouraged the mathematician engaged with the drawing to make minor adjustments that didn't pertain directly to the reasoning. (See, e.g., the section starting on page 117.) It's normal for erasers to miss a few marks when used on a blackboard, and on occasion those marks, if left alone, might cause some minor visual confusion when something new is written. (For instance, a lingering mark might make a y look like an x.) Yet mathematicians are willing to ignore these imperfections in erasing up until those flaws threaten to cause some confusion, at which point they erase them. (E.g., noticing that a y looks like an x, a mathematician will likely erase the small mark that's threatening to cause confusion.) Bart's being drawn to make this small gesture of erasing, then, seems like it's probably a similar sort of perfecting-while-thinking behavior to what we saw in invocations of the continuum. It seems like a way of overcoming something that might have otherwise turned into proximal inhibition since he's able to engage in the board this way. This connection seems especially evident when we see how he seems to be actively repelled from the board once the imperfection has been erased (row 37).

After Bart paces (rows 37–40), he moves toward the board to prepare to write the functions describing the monoid structure of the two varieties being spliced (rows 41–42). But then we see his hand deflect to the side (row 43) in order to draw the paths depicted in Figure 5.5. Earlier when examining this move, we emphasized how the paths were an invocation of the continuum, and specifically how the paths drawn were generic particulars. Here, though, I'd like to emphasize *where* the paths were drawn. The intent here is to create a snapshot of the relationships between the key points p & p' on the two varieties about to be spliced together; the diagram of paths is not itself intended as a center for reasoning the way Joseph's cobordism-like drawings in section 6.1.2 were. There seems to be this implicit effort to designate the center of the working space for nodes of the transformation mapping that are actively involved in the effort to explore the mathematical ideas involved.

Emphasizing correction with erase-and-replace

Jumping ahead half a minute, we find Bart writing down the functions u_p and $u_{p'}$ (rows 49–54). Here again is what he has written on the board at this point:

$$\begin{array}{rccc} u_p : & P_p & \to & \mathbb{N} \\ & \| & & \\ u_{p'} : & P_{p'} & \to & \mathbb{N} \end{array}$$

Notice, again, how Bart initially wrote $\mathbb{Z}$ instead of $\mathbb{N}$ for the function u_p, but once he noticed the mistake he abandoned his in-process writing of $u_{p'}$ to erase the $\mathbb{Z}$ and replace it (rows 52–53).

Matt argued for the equality of P_p and $P_{p'}$ based on the idea that p and

p' both become q after the two paths are made to cross (rows 52–57). This is what later leads Bart to suggest that the monoid $P_{p'}$, which he's viewing as being the same as P_p, is also P_q (row 66). This turns out to be a mistake, but we can understand why this seemed sensible at the time: if p and p' are being viewed as already merged at q, then P_p and $P_{p'}$ are being viewed as already spliced to form (part of) P_q.

Bart wrote P_q immediately below $P_{p'}$:

$$\begin{array}{rccc} u_p : & P_p & \to & \mathbb{N} \\ & \| & & \\ u_{p'} : & P_{p'} & \to & \mathbb{N} \\ & P_q & & \end{array}$$

There's no equal sign written between $P_{p'}$ and P_q here. Instead, equality is implied by where P_q has been written. When suggesting that it just be called "P" (without subscripts), he appends "$= P$" to the last row (row 68):

$$\begin{array}{rccc} u_p : & P_p & \to & \mathbb{N} \\ & \| & & \\ u_{p'} : & P_{p'} & \to & \mathbb{N} \\ & P_q & = & P \end{array}$$

Almost immediately after Bart writes this, though, Matt objects (rows 70–72). Bart immediately backs away from the board (rows 72–73), heeding Matt's warning but not quite seeing the problem just yet. As soon as he does, though, he immediately zips in and erases the bottom row (the one reading $P_q = P$) (rows 74–77). So, we can see the urgency to erase expressions that imply illegitimate transformations manifests here as well.

There is a brief exception to this, though. The vertical equal sign between P_p and $P_{p'}$ is also mistaken, but Bart takes a few moments to get to erasing it. Instead, he spends a few seconds emphasizing what he sees as the correct way to describe the situation, even adding an emphasis gesture seemingly directed toward Matt (rows 79–81), and *then* erases the equal sign (rows 81–82). Yet while doing all of this he keeps his torso oriented on the board as though to stay engaged with adjusting the inscriptions there. This suggests that in that moment, he feels that it is more urgent to convey to Matt that he understands the correction than it is to actually make the correction on the board.

Bart takes a few moments to continue the emphasis after making the correction (rows 83–88) and then returns to the board to add a correction (row 89):

$$\begin{array}{rcll} u_p & : & P_p & \to \mathbb{N} \\ & & \| & \\ u_{p'} & : & P_{p'} & \to \mathbb{N} \\ P_p, P_{p'} & \subset & P_q & \end{array}$$

The subset symbol ('$\subset$') is being used here to indicate "are faces of." Notice how Bart wrote this in such a way that P_q ended up being written exactly where it was before, but this time it does *not* carry the connotation of equality to $P_{p'}$. Quite the opposite: the subset symbol indicates that P_p and $P_{p'}$ are both different than P_q since they constitute parts ("faces") of it. This erase-and-replace pattern seems to act as a way of emphasizing that P_q takes on a different role here even though it's written in the same place. Also, as we saw with Joseph erasing the m in Figure 6.8, what was important here was the fact that P_q had been written where it was *with the intent of* conveying equality with $P_{p'}$, so erasing it and rewriting it becomes important for Bart.

A written word in the working space

Jumping ahead about 20 seconds, Bart has drawn the inverted pyramid and is finishing perfecting it as he thinks about Matt's suggestions (row 106). Matt had just suggested a relationship between u_p, $u_{p'}$, and an unwritten function u_q.

Bart wants the focus to be on the fact that P_p and $P_{p'}$ are facets of P_q rather than just how the monoid structure of the three spaces relate (rows 107–108). In order to make this point, he writes the word "facets" on the board under the lower-righthand instantiation of $\mathbb{N}$ (row 109):

$$\begin{array}{rcccc} u_p & : & P_p & \to & \mathbb{N} \\ & & \| & & \\ u_{p'} & : & P_{p'} & \to & \mathbb{N} \\ P_p, P_{p'} & \subset & P_q & & \text{facets} \end{array}$$

As we saw in Joseph's writing "$f \circ g$," writing something on the board seems to emphasize it, presumably because of the insistence that what's on the board imply a legitimate transformation. The word "facets" isn't really a mathematical statement like an equation is or even like a topological diagram is, but it still plays an important role here: it amplifies the role of the inverted pyramid as a way of highlighting the importance and nature of the relationships between the three varieties involved here. (Note that the very phrasing of P_p and $P_{p'}$ as *facets* of P_q makes sense only in the context of this invocation of the continuum. The notation "$P_p, P_{p'} \subset P_q$" only implies that P_p and $P_{p'}$ are *subsets* of P_q rather than having some kind of spatial relation to one another or to how they contribute to P_q.)

Summary

In section 6.1 we saw Joseph erasing "m" from Figure 6.8 and writing on top of it with the word "split." We see something similar here, first in replacing $\mathbb{Z}$ with $\mathbb{N}$ but then more dramatically in the use of P_q as written below the function expressions for u_p & $u_{p'}$. Not only was it important to erase the mistake immediately and then write a correction directly on top, but again the *intent when writing* is important. This emphasizes the point that transformation mappings are not a property of the marks on the board, but are rather a property of *how the* *mathematicians experience them.* The first P_q had been written where it was to convey equality to $P_{p'}$, but the second time it was written in the same place as a correction but without any intent to imply equality with $P_{p'}$.

However, we find that there are subtleties to this impulse to erase. Bart finds it sufficiently important to convey in gesture and vocal emphasis that he understands the corrections being made that he abandons the correction process in the middle *twice* (once before erasing the vertical equal sign between P_p & $P_{p'}$, and once before writing $P_p, P_{p'} \subset P_q$). This suggests that the interpersonal dynamic between Matt & Bart makes it seem to Bart as though part of working together requires conveying mutual understanding of why a correction is necessary. We saw something similar with Joseph & Bill, but it didn't stand out as dramatically at the time: Joseph actually waits on making his correction of replacing "m" with "split" until Bill verifies that he recognizes the error too, giving credence to Joseph's concern (Table A.9, row 102).

We also saw Bart replace part of the mistaken expression ("$P_q = P$") with a *written word* ("facets"). This is particularly relevant because the word isn't actually the result of a transformation. Yet Bart still intends it as a correction on his earlier mistake. He's emphasizing the type of thinking that goes into keeping future transformations legitimate. Writing the word itself on the board provides this way of thinking with a kind of place of emphasis reminiscent of when Joseph write "$f \circ g$" as a means of guiding attention.

More generally, the role of placement emerges here as salient for transformation mappings. In principle, the nodes of a transformation mapping could be placed anywhere on the board; what's important is how mathematicians see the connections between the inscriptions on the board. Although a single node (like the inverted pyramid) might be an invocation of the continuum, the overall transformation mapping doesn't have the defining property: slightly moving where the pyramid had been drawn wouldn't change its meaning at all. Yet we find there *are* rules of placement: reference nodes are pushed to the periphery, and the central region seems to be reserved for active reasoning and navigation. Indeed, it's this reservation of the central space as a working space that seems to grant the written word "facets" such emphasis. Furthermore, even though positioning doesn't have the precision of meaning that defines an invocation of the continuum, we do find that positioning can imply meaning, as with the placement of "P_q" immediately below "$P_{p'}$" to encode the fact that they are being thought of as equal.

Finally, there was a brief moment in which Bart engaged with perfecting the board as a blank canvas in a way reminiscent of the idle perfecting he does with the inverted pyramid later (rows 102–107). I'll return to this matter in chapter 7, as it will help relate transformation mappings and proximal inhibition.

6.3 Clip #3: Erasing δ and graph-confining

Our final clip will examine a pair that has not had an opportunity to appear as yet: Fay and Martha. The two of them are working in a subfield known as *graph theory*, which explores what one can tell about how different nodes of some network are connected. The topic, working style, and interpersonal dynamics of this pair resulted in them never invoking the continuum in any of the three observations made of them, and also in no clearly discernible instances of proximal inhibition. Yet we find that their use of transformation mappings offers us insight in this domain.

6.3.1 Mathematical Context

An example of a graph from mathematical graph theory is a friendship graph from Facebook. This might work like so: you draw a node for each person

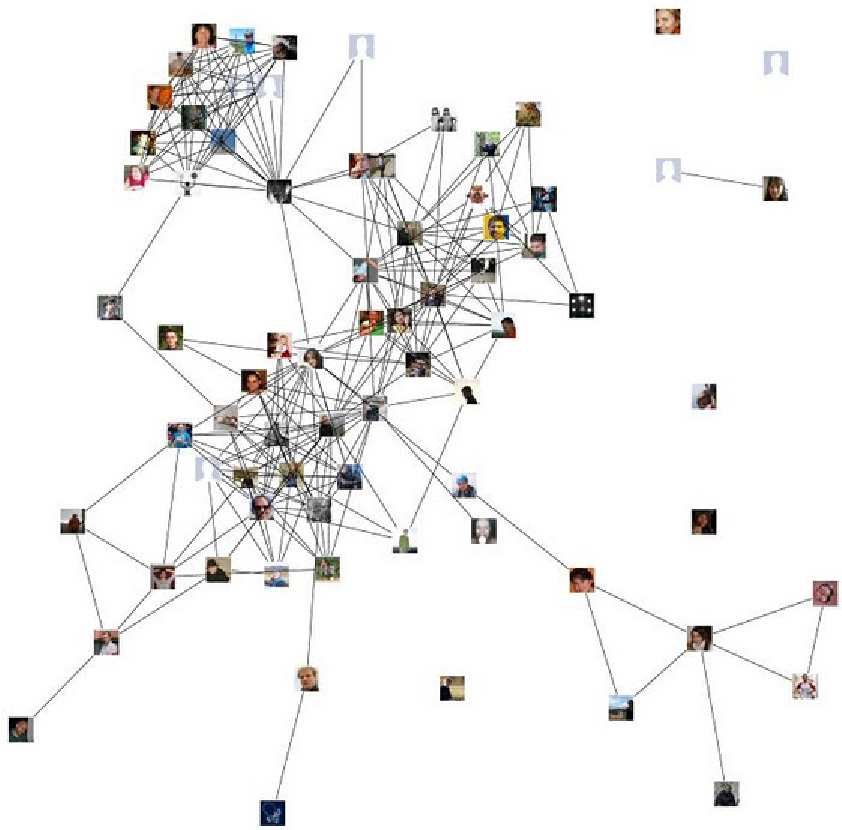

Figure 6.10: An example of a Facebook friendship graph (from http://tinyurl.com/examplefriendshipgraph)

who is "friended" on your Facebook profile and a node for each person who has "friended" one of your Facebook friends, and then draw connections between everyone who has "friended" one another. The result typically shows clusters of highly interconnected nodes with the clusters loosely tied to one another (Figure 6.10). In principle, we could create a similar graph for the entire internet by having a node for each webpage and an arrow going from node A to node B if website A has a hyperlink to website B. However, in practice this is a horribly unwieldy graph. Even if we set aside the problem of how to physically draw such a huge graph,

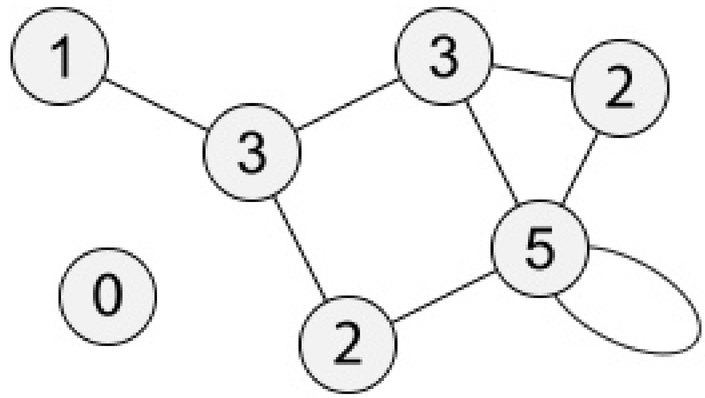

Figure 6.11: A graph with vertices labeled with their degrees (from http://tinyurl.com/wikigraph)

finding out what that graph *is* at a given moment would be quite a challenge, particularly since it changes all the time.

In order to examine ways of analyzing such "real-world" graphs, Fay & Martha decided to apply probabilistic techniques to measure various aspects of such graphs. In this episode, they were applying probabilistic techniques to something called the *Laplacian matrix* of the graph in question. To describe the Laplacian matrix, we'll need the context of knowing what the degree of a graph's vertex is and also what the adjacency matrix is.

Degree of a vertex

The graph shown in Figure 6.11 has each vertex labeled with its degree. Roughly speaking, the *degree* of a vertex is the number of edge endpoints that are plugged into the vertex. Notice that this means that loops are counted twice; for instance, the vertex labeled "5" has a degree of 5 because both ends of the loop attached to it count toward the degree.

One way to construct the Laplacian matrix of a graph involves creating

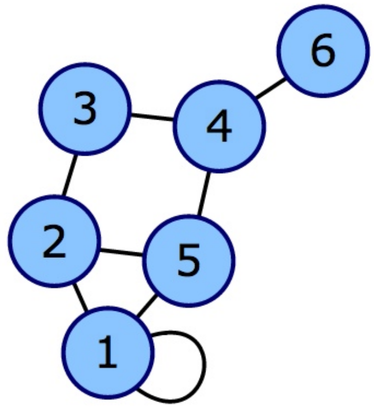

Figure 6.12: A graph with enumerated vertices (from http://tinyurl.com/wikigraph2)

what's called the *degree matrix* of said graph. For a graph with n vertices, this consists of an $n \times n$ diagonal matrix (i.e., a matrix whose nonzero entries are all along the diagonal) where the the entry on row i and column i is the degree of the i^{th} vertex. For instance, the degree matrix for the graph shown in Figure 6.12 is:

$$\begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We can read this as saying that vertex #1 has degree 4, vertex #2 has degree 3, and so on.

Adjacency matrix

Commonly in graph theory, it's helpful to encode the graph under consideration into matrix form. One way of doing this is with what's called an *adjacency* *matrix*. This is a matrix whose entry in the i^{th} row and j^{th} column is 1 if there's an edge going from vertex i to vertex j, and zero otherwise. For instance, the adjacency matrix for the graph shown in Figure 6.12 is:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Notice that this matrix is symmetric across the diagonal. That is, representing the entry on the i^{th} row and j^{th} column by a_{ij}, the matrix would look exactly the same if we were to swap each entry a_{ij} with the entry a_{ji}. This is always true of an *undirected* graph: if vertex i is connected to vertex j, then vertex j is connected to vertex i.

This isn't necessarily true of a *directed* graph in which the edges are arrows (implying you can go in only one direction). For instance, the graph shown in Figure 6.13 has the following adjacency matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix *would* be symmetric if every arrow in its graph had a corresponding arrow going in the opposite direction. The only arrow that wouldn't need a corresponding partner would be the loop on vertex #5 since going forward or backward along that arrow results in the same overall movement (i.e., from vertex #5 to vertex #5).

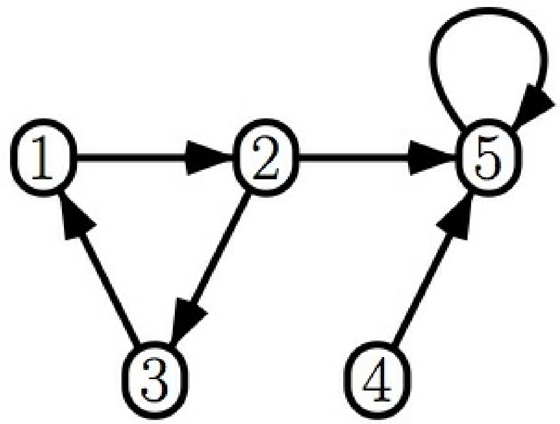

Figure 6.13: A directed graph (from http://tinyurl.com/wikigraph3)

Laplacian matrix

The *Laplacian matrix* L of an undirected graph can be constructed from the graph's adjacency matrix A and degree matrix D like so:

$$L = I - D^{-1/2} A D^{-1/2}$$

The matrix I is the $n \times n$ *identity matrix* consisting of 1s along the diagonal and 0s everywhere else. For instance, if $n = 3$, then:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The value that Fay & Martha get from considering the Laplacian is actually in its *eigenvalues*. The eigenvalues of a matrix are numbers that can be algorithmically derived from the matrix by solving a particular equation (namely $Ax = \lambda x$ and solving for λ). There's a geometric interpretation for eigenvalues, but that ends up not being used here. Instead, Fay and Martha are interested in what is called the *spectrum* (labeled "spec L") of the Laplacian matrix L. In essence, the spectrum is the set of eigenvalues where duplications are counted. For instance, $(0, 0, 1, 3)$ is a possible spectrum for a matrix.

It turns out that the number of a given eigenvalue in the Laplacian matrix tells us something important about the original graph. For instance, the number of times zero occurs as an eigenvalue of the Laplacian matrix ends up being the number of connected components of the original graph. ("Connected component" in this case means basically what it sounds like: if you can reach one vertex from another by traveling along edges, then those two are part of the same connected component — and otherwise they're part of different connected components. Multiple connected components look like separate graphs.)

Expectation

One way to get around the mess of real-world graphs is to treat each graph's properties as random variables. The example that's most central to this particular clip has to do with the degree of each vertex: since the graph is so unwieldy, the degree of any given vertex of the graph can be treated as though it's an unknown random number. The statistical trick, then, is to work out roughly what its distribution must look like. That's often very challenging, so a good approximation to this is working out how likely a given vertex is to deviate from its *expected* degree.

The word "expectation" has special meaning in probability theory. The *expected value* of a random variable is, roughly speaking, the value you generally expect it to take on average over many iterations. For instance, let's say that for a given coin we'll give heads the value of 1 and tails the value of 0. If the coin is fair, then the expected value is $\frac{1}{2}$ (since after many, many coin flips you expect that half of them will come up heads). On the other hand, if it's weighted so that you're three times more likely to get heads than tails, the expected value is $\frac{3}{4}$ (since on average you expect three out of four coin tosses to come out heads).

To give a different example, suppose we have a random variable X that can take on any integer value between zero and six. If X is uniformly distributed (i.e., every number is equally likely), then the expected value of X is 3.[8] Notice that unlike the case of the coin-flip, it's actually possible for X to take on its expected value. As a general rule, it's more likely that a given random variable ranging across consecutive integers can take on a value close to its expected value when the number of possible integers is large.

Deviation from expected degree

Fay & Martha treat the degree of each vertex as a random variable and then ask the question, "How likely is this random variable to differ significantly from its expected degree?" One way to approach this question is to ask what the probability is that the random variable will differ from its expected value by more than some specific amount. The particular formula Martha puts on the board is this:

$$p\left(|d_i - t_i| > \sqrt{\frac{3\ln(4n/\varepsilon)}{\delta}}\right) < \varepsilon$$

It turns out that this formula is wrong, so I won't explain all its details. I'll just highlight the ones that are important for understanding what happens in this clip. The p function is probability; for instance, if X is a random variable representing a coin flip as described earlier, then we could say "The probability of the coin coming up heads is $\frac{1}{2}$" by writing $p(X = 1) = \frac{1}{2}$. d_i and t_i both refer to the degree of the i^{th} vertex: d_i is its actual degree, and t_i is its expected degree. So $|d_i - t_i|$ is the (positive) difference between the actual and expected degree of vertex $\#i$.

[8] Whenever a discrete random variable is uniformly distributed, the expected value of that variable is just the familiar average: add up each possible value and the divide by the number of such values. So the expected value is $(0+1+2+3+4+5+6) \div 7 = 3$. This is in fact the same thing done for the fair coin: if each option of 1 and 0 is equally likely, then the expected value is the average of 1 and 0, which is $\frac{1}{2}$.

The value involving the square root turns out to be where the error is. The important thing to notice here is that it involves ε as a divisor. Typically ε is used to represent a *small* variable amount, so the argument of the natural logarithm here should be considered to be quite large. Roughly speaking, we could have written this inequality like so:

$$p(|d_i - t_i| > M_\varepsilon) < \varepsilon$$

Here M_ε is some large number depending on ε such that M_ε gets bigger as ε gets smaller. So, we can read this inequality as being a precise version of the claim that "The probability that the degree of vertex $\#i$ differs from its expected value by a large amount is small." That is, a given probability is small, and that given probability is that there's a large difference between the actual and expected values of the degree. What's missing in that verbal formulation is the fact that the words "large" and "small" here are connected in a precise way: if "small" means ε, then "large" means $\sqrt{\frac{3\ln(4n/\varepsilon)}{\delta}}$.

The last detail to be aware of here is that δ is supposed to be the *minimum* degree of the graph in question. The minimum degree of a graph is the degree of the vertex with the smallest degree. For instance, the minimum degree of the graph in Figure 6.11 is zero, and that of Figure 6.12 is 1 (because of vertex #6).

6.3.2 Video Description

This discussion uses the gesture analysis found in Table A.7. Fay is the one wearing the solid-colored long-sleeve shirt, and Martha is wearing a striped short-sleeve shirt and holding a paper in one hand throughout the clip.

The flawed inequality

The clip begins with Martha jotting down the inequality described at the end of section 6.3.1 (rows 1–4):

$$p\left(|d_i - t_i| > \sqrt{\frac{3\ln(4n/\varepsilon)}{\delta}}\right) < \varepsilon$$

She then writes "if $\delta > C \ln n$" above this inequality (row 12), referring to C as a constant and pointing out that they can make ε be as small as they like by making C sufficiently large (rows 14–18). Here she's outlining an idea she got from a paper she's holding in her hand.

Martha is presenting this idea because while it seems to bear on her work, this inequality seems flawed to her. In effect, the idea is to replace δ with $C \ln n$ — but, Martha points out, if you do that and make C large, then the expression in the square root becomes small (rows 22-29). This is a problem because it suggests that as you widen the range of possible values for $|d_i - t_i|$ to take on, you get a *smaller* probability. (This would be akin to saying that were you to flip a coin twice, the probability of getting heads on the first toss is less than the probability of getting a heads on the first toss and a tails on the second. It simply doesn't make sense since the second case is a special case of the first.)

Fay then steps forward and indicates that the expression involving the square root is wrong (rows 30–37). She takes the time to shake her head in response to Martha's confusion (rows 33–37), and *then* erases the part she sees as flawed (row 38). This, as in the previous two clips, demonstrates the impulse to erase an error on the board as quickly as possible — but it also shows, again, how this impulse is tempered by a desire that both mathematicians involved know what the error is.

Yet it's worth noting that Fay doesn't get Martha to the point of agreeing with her about the nature of the problem before erasing δ. Instead, Fay erases δ

by hand and then steps away from the board to work on articulating the problem (rows 41–50). Martha slowly follows her during this time, pouring over the paper in her hand in an effort to make sense of Fay's objection (rows 46–52). Then the two of them stand at a distance thinking about it until Fay calls for a shift in emphasis ("...let's forget about anything"; rows 63–66). This pattern seems to emerge from Fay knowing that there is a problem — and in particular that the δ shouldn't be written there — but not really knowing how to express it. Since it takes so much effort and time for her to articulate the problem, she feels the need to erase the error before reaching an understanding with Martha. This is compatible with the idea that the impulse to erase perceived errors on the board gives way to negotiation between the pair only because the negotiation seems more urgent: in this case, where the urgency isn't something Fay feels she can act upon, she rubs out the mistake as soon as the difficulty in communicating the problem presents itself to her.

In short, the issue is that the top inequality ($\delta > C \ln n$) lets them treat δ roughly like $\ln n$. ("So this [is] basically saying you know δ is like log n"; rows 87–92.) But if they could treat δ like $\ln n$, then they would already know that the probability they're considering is small. That would make the bound they're looking at irrelevant. ("...everything is beautiful. What's the problem?"; rows 98–101.) Thus the δ in the square root doesn't belong there.

Once Martha recognizes Fay's point (rows 104–105), she starts articulating her understanding of the problem (rows 107–110). It's worth emphasizing this point: Martha is already clearly aware that Fay knows that the formula is problematic, so this explanation cannot be in order to get Fay on board with the nature of the problem. It therefore seems most likely that Martha is detailing the problem she sees here because of some combination of (a) wanting to convey to Fay that she really does get what the problem is and (b) wanting to crystalize her own thinking about the mistake.

However, Fay doesn't seem terribly interested in listening to Martha explicate this. She jumps in to proclaim that the use of δ is "nonsense" (rows 110–111). In fact, we see this conversational pattern throughout the next several seconds (rows 112–129): Fay seems to be primarily interested in emphasizing the fact that the expression is flawed, not in whether Martha agrees since that has already been established. It's as though the delay in articulating the flaw here needs to be made up in emphasis.

Also, notice that there is a dynamic of board-use negotiation that emerges from this conversational pattern: when Fay cuts Martha off this way, Martha disengages from the board in mid-sentence (row 111). Since Martha wasn't using the board for her own reasoning but rather to communicate that she now understands Fay's point, but Fay conveyed in her interruption that this wasn't a concern for her, Martha relinquished control of the board. This is a somewhat different interpersonal dynamic than we observe in Matt & Bart, who at times seem to socially wrestle with one another for board control; or in Joseph & Bill, where we usually only see Bill taking control of the board during some moments when Joseph shows no particular interest in using it. It's certainly possible that Martha's pulling away from the board is akin to Bill's when Joseph moves toward the board, as though she anticipates Fay wanting to write — but in most other situations observed, it's clear that Fay commonly conveys her desire to write by *walking up to* the board rather than just by interrupting Martha.

Drawing a graph in outline

After she finishes expressing her disapproval of the square-root expression, Fay walks up to the board, picks up chalk, and draws a circle (rows 131–139). While doing this she says, "So in fact the reason I asked you to look at the big. . . .

So suppose now I have a lot of points." This circle is meant to represent the "lot of points" she has, meaning that the number of vertices for the graph in question is quite large (i.e., n is large). It's as though she intends Martha to imagine that there are "a lot of points" already drawn on the board and that her circle is intended to encapsulate all of these.

It's worth emphasizing that this is *not* an attempt to imply a legitimate transformation from the inequality they had been examining before to what Fay is drawing. Instead, the intent here is to *start afresh*. This node of a transformation mapping actually relates to the inscriptions we see at the far right of the board ("$e = I - D^{-1/2}AD^{-1/2}$," etc.). What is preserved here is more subtle than we've seen up to this point: the equations on the right are describing ways of constructing objects analogous to the Laplacian matrix of an arbitrary "large" graph, and the imagined graph that Fay is outlining here is being outlined in sufficient vagueness that the only properties it will have are those that relate to the spectrum of the Laplacian matrix. So, that which is being preserved here is a collection of *abstract properties of an arbitrary "large" graph.*

At this point she has not yet articulated her idea verbally. In short, the trick she's suggesting is to argue that there are very few vertices of very small degree. This means that they can work just with vertices whose degree is *not* small in order to determine the eigenvalues of the graph's Laplacian matrix, and then argue that the influence of the low-degree vertices is very small because there are so few of them and the eigenvalues describe global properties of the graph.[9]

The remainder of the clip involves Fay using highlighting (Goodwin, 1994) to articulate the idea. She shades in the region where the small-degree vertices of

[9]What's meant by "global properties" here is that the eigenvalues are derived from the *overall* structure of the graph and aren't affected strongly by small changes like adding or subtracting an edge. This is akin to how the median of a collection of numbers doesn't change much if you add or subtract a few numbers, so the median is a global property of the data set. This would be in contrast to the *sum* of the values, which will change quite significantly when you add or subtract more values.

the graph ("points") are intended to fall (row 168), emphasizes that there aren't many there by saying as much while tapping the board with her non-writing hand (row 174), and concludes by encircling the rest of the implied graph (rows 180–181). The key thing to note here is that, rather than engaging in legitimate transformations between nodes of the transformation mapping, she is modifying an already-extant node with legitimate transformations. In particular, it's fine to view all the vertices of small degree as collected in a corner of the graph; we're free to rearrange them however we please since the arrangement of vertices doesn't affect the graph at all. Furthermore, Fay is using the small area of the region holding the vertices of small degree as a way of representing the fact that there are relatively few of them. Most ways of arranging vertices won't distribute them uniformly, meaning that area won't vary linearly with the number of vertices. Thus Fay's representation of the small number of low-degree vertices as a small area asserts this structure on the imagined-as-present graph.

This behavior is reminiscent of Joseph's manipulation of the ribbon surface to demonstrate finger moves in section 5.2.2. For instance, he modified the "hood" of the ribbon by drawing an arc (Figure 5.24) and then erasing the arc to draw a "finger move" (Figure 5.25). The legitimate transformations there were not just between different nodes of the transformation mapping, but were actually *within* a particular node as it was modified to reflect a particular unfolding process. In this same way, Fay conveys rearranging the vertices so that the problematic ones are in a corner as a way of highlighting the fact that there are relatively few of them.

Summary

There are three insights to take away from the examination of this clip. First, we see here a nuance of the interplay between the impulse to rid the board of error and the desire to verify that both people involved understand why the error needs correcting. We see Fay immediately moving up to the board to touch the mistake once she sees and starts articulating the problem (row 26), but she hesitates long enough to give Martha a chance to recognize the problem. Yet when it becomes clear that Martha isn't going to see it right away, Fay erases the error before stepping back to work on expressing the problem. This suggests that, at least for Fay, there is a relatively small bound on the ability of the social dynamic to hold back the need to erase the error on the board.

Second, and closely related, we see that the effort to articulate the error probably isn't being made primarily to convey what the error is. When Martha understands Fay's point, she starts to express it in her own words — yet it's not with the tone of, "Is this right?" Instead, it's a pretty high-confidence statement. She even abandons the effort to continue when Fay interrupts her (rows 110–111). Similarly, Fay continues to criticize the error for nearly a quarter of a minute after Martha acknowledges the problem. This reinforces the point we saw with Matt & Bart when Bart abandoned his corrections on the board in order to emphasize *at Matt* what the correction was — seemingly not because Matt needed to understand, but because Bart needed Matt to know that *Bart* understood. This bespeaks a kind of *social metacognition* that's key in mathematicians' navigation of struggle: it's not just important that both mathematicians understand what's going on, but that when working together they need the other to know that they understand.

Finally, Fay's modification of the invisible graph shows how legitimate transformations can occur by modification of a single node rather than strictly as something seen going between inscriptions. As mentioned earlier, we saw this before with Joseph's description of finger moves on ribbon surfaces. This helps to emphasize the fact that transformation mappings are not embedded in the physical inscriptions per se, but are rather emergent from *the mathematicians' phenomenology of* the inscriptions.

6.4 Chapter Summary

The results we've seen in this chapter can help us interweave most of the phenomena discussed in the present work. This will receive detailed discussion in chapter 7. But before we move on, let us briefly revisit what we've seen so far about transformation mappings:

- Whether a node of the transformation mapping implies an illegitimate transformation for the mathematicians in question depends not just on what is written, but also upon *the intent when it was written.* We saw this both in Joseph's erasing "m" to replace it with the word "split" as well as in Bart's erasing P_q to later replace it with the same symbol. Indeed, the sense that something important was *not* preserved seems to have been intertwined with the impulse to erase the perceived error.

- The board also seems to be implicitly carved into regions with designated purpose. For Bart, for instance, the region immediately below the function statement for $u_{p'} : P_{p'} \to \mathbb{N}$ was being used to specify the relationship between P_q, $P_{p'}$, and P_p. The subsequent correction specifying that P_p & $P_{p'}$ were facets of P_q therefore also went in this region, implicitly emphasizing the correction over the error.

- Because the board seems to be designated primarily for transformation mappings, what is written there takes on a charge of significance, as in Joseph's writing "$f \circ g$" seemingly to direct focus to that next step. This applies to written words (e.g., "split" and "facets") as well even though they needn't necessarily represent the result of a legitimate transformation.

- The matter of what the "object" is that's being preserved between legitimate transformations appears to be a flexible matter and can change on a moment-to-moment basis depending on the needs of the mathematicians at that moment.

- There are at least two ways to convey a legitimate transformation. The most common one seen in these episodes is the creation of multiple nodes, akin to how we might write "$2x + 6$" immediately below "$2(x + 3)$" in order to indicate distribution. Yet we also see mathematicians sometimes modifying a single node in order to convey a process that is something-preserving.

With these points in mind, we will now summarize the results of this study.

Chapter 7

Summary

When reading the expression 25×11, we have many resources to bring to bear on it. We can run it through the standard multiplication algorithm to find out that it's 275. We can use distribution to think of it as $25 \times 10 + 25$. We can think of the 25 as referring to a quarter (25¢) and notice that this is one quarter shy of 12 quarters, which is \$3, so the total quantity is $300 - 25$. We can rearrange the primes to view it as 5×55 and think of that as $5 \times 50 + 5 \times 5$. In the end, we might be tempted to think that the quantity "really is" 275 — but the symbol "275" itself implies a kind of procedure too: add two hundreds, seven tens, and five ones. The sense that the quantity "really is" 275 presumably comes from great familiarity with this kind of notation (and in particular being repeatedly told when learning arithmetic that expressions like this are "answers" to arithmetic problems).

What makes all these different ways of viewing this quantity appropriate is the sense that in converting from one to another, we're performing legitimate transformations — which is to say, we're doing something that leaves "the quantity" unchanged. In much the same way that we think of pictures of the Eiffel Tower that are taken at different angles or at different times of day as being images of the same thing, these different "views" of 25×11 come across as being of something that is preserved across these transformations.

Just as we can never see the whole Eiffel Tower at once but can see it only from a given vantage point, we can never see the whole of 25×11. We can only ever catch glimpses of it. Each strategy for reinterpreting this quantity captures a different angle, and the more angles we perceive the more nuanced our understanding of this quantity becomes. Indeed, it's reasonable to suggest that our understanding of 25×11 is, in fact, our anticipations about what legitimate transformations the symbol "25×11" affords (Roth & Thom, 2009).

Notice that in the case of the Eiffel Tower, we have a collective sense that there really is an object in Paris that all these images, memories, descriptions, and so on are referring to. There's a strong impression that if the words "the Eiffel Tower" constitute a symbol, there really is a physical referent to which the symbol points. Analogously, those who are conversant with arithmetic have a sense that the symbol "25×11" has a referent, and that this referent is preserved across transformations such that "275" and "$300 - 25$" also refer to it in a way that "274" does not.

From a phenomenological point of view, what is of interest here is not the question about whether there really is such a quantity as "two hundred seventy-five" apart from its symbols and instantiations (e.g., as $2.75 in pennies). What is of interest is the *impression* that there is such a thing. Whether or not Platonic mathematical objects exist, there still seems to be some structure to the *experience of* mathematical objects as Platonic to which we need to attend if we want to understand mathematicians' experiences with struggle in their research. In other words, in bracketing mathematicians' experiences (see chapter 3, section 3.2.1), I am setting aside the question of what mathematical objects *are* and am instead exploring what mathematical objects *seem to be.*

As we saw in chapter 6 — indeed, as we've seen in all clips considered for this

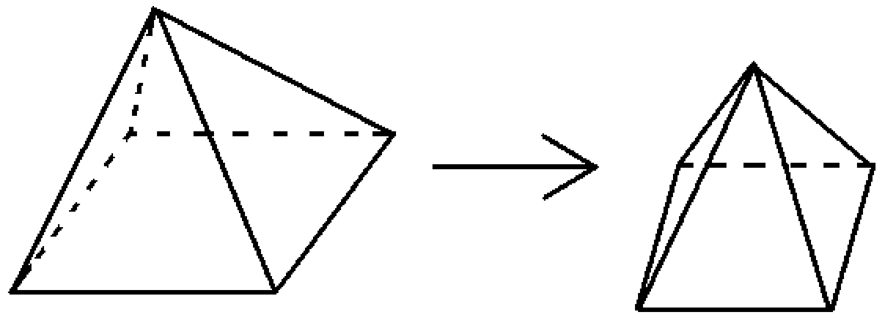

Figure 7.1: Rotating a pyramid to see exactly three faces

study — this perspectival shifting between expressions via legitimate transformations seems to structure mathematicians' attempts to navigate novel mathematics. Just as it might require effort to work out how to orient a model of an Egyptian pyramid such that we see exactly three faces (Figure 7.1), we see mathematicians working to "angle" the objects of their study in an insightful way. From Joseph's & Bill's attempts to make sense of odd Khovanov homology to Fay's graphical manipulation of implied low-degree vertices of a graph, the felt sense that there is something there that is being partially described by the inscriptions on the board defines the framework within which struggle even becomes possible for them.

7.1 Creation and Discovery

One effect of this insight is to help dissolve a subtle paradox in mathematical practice. On one hand, mathematics sometimes gives the impression of speaking to eternal truths. For instance, it seems like a fact about the world that $\sqrt{2}$ is irrational — and if it were to turn out that we actually *could* write it as a ratio of whole numbers, then we'd probably think of ourselves as having been *mistaken* as though there's a truth to compare our understanding to. On the other hand, mathematicians at times report a sense that they're *creating* mathematical objects or structures; indeed, Thurston (1994) argues that much of the value of his contributions in one field came from communicating a novel infrastructure he had created, and Hadamard (1949) emphasizes the question of mathematical *invention*, not discovery.

To illustrate a possible resolution, let's consider a familiar situation: the Cartesian plane. René Descartes' basic idea was to explicitly choose to view two related variables as each describing a kind of distance traveled from a starting point. We can reasonably say that Descartes created this correspondence.[1] We can also view this correspondence as being a definition of *which transformations are legitimate* between equations of two variables and graphs on the Cartesian plane. Thus, once Descartes created this correspondence, there was a correct answer to the question of what, say, $y = x^2$ looks like when graphed. Anything other than a particular upward-opening parabola centered on the origin would not be seen as representing the same relationship between x & y under this correspondence. Thus, once the rules of legitimacy for a transformation have been created, the implications of those transformations can — and, I suspect, usually do — take on the character of being discovered.

When we watch Joseph & Bill struggle with chain maps (section 6.1), we can see how this interplay of creation and discovery manifest in their phenomenology. Khovanov homology is a formalization of a transformation — namely, it's a rule for deriving certain algebraic structures from knot diagrams in a way that preserves "properties of the knots." *Odd* Khovanov homology is a modification

[1]It could be argued that there "really is" a correspondence in a Platonic sense between equations like $2x^2 + y^2 = 3$ and planar shapes like ellipses, and Descartes was actually discovering this correspondence, not creating it. However, mathematicians often are under the *impression* that they are creating entities or correspondences, and as we are attempting to explore mathematics *as experienced* it is this impression, rather than the ontology from which it comes, that we're interested in. For the sake of the example, I'll assume Descartes felt he was creating rather than discovering his Cartesian plane; even if I'm mistaken, the analogy should still work just fine.

of this transformation. Much as how equations in two variables look very different on the plane when we view the variables as polar coordinates[2], changing the homology to be odd Khovanov offers a rich domain for exploration — including the need to test whether this definition of a legitimate transformation is internally consistent. Joseph's & Bill's efforts to make sense of the chain maps involved in the Reidemeister moves reflects this exploration within a constructed context.

What we find, then, is that in practice mathematics is not a matter of creation *versus* discovery. Rather, creation is necessary to make discovery possible in the first place, and discovery is the point of bothering with creation at all. Creation — the definition and clarification of legitimate transformations — generates the sense of there being mathematical objects that inscriptions represent.[3] Within this context, each object presents itself in the mathematicians' experience as having properties that can be explored via these transformations.

It seems reasonably fair to say, then, that the phenomenology of mathematical struggle consists mostly of the interplay between (a) determining transformations that are internally consistent and have room to be interesting and (b) exploring what can be learned about mathematical objects-as-presented as they undergo legitimate transformations. This can take many, many forms: developing promising transformations (e.g., the Cartesian plane, tropical diagrams, the inverted pyramid), exploring whether a transformation is internally consistent (e.g., odd Khovanov homology), working through an unfamiliar transformation (e.g., section 4.2), knowing that a legitimate transformation to a new inscription type ("representation") should already exist but not being sure how to find it (e.g.,

[2]To give a specific example, if we view x as radians counterclockwise from straight right and y as distance from the origin, then $y = x^2$ looks like a rapidly expanding spiral rather than a parabola.

[3]Again, this is independent of the question of whether these objects "really exist." Whether they do or not, we somehow come under the impression that there really are things like "the number 275" and "a quadratic function." This phenomenology has a structure, and it would seem that the structure is made up out of legitimate transformations.

the Lewis diagram for benzene, the effort to remove singularities in ribbon knot diagrams that resulted in finger moves), and possibly others.

It's particularly helpful to notice how this interacts with the sense of what mathematical objects *are* on a moment-to-moment basis. Sometimes a chain is an object being transformed under a chain map, and sometimes it's a collection of mathematical objects connected by a sequence of transformations (section 6.1). In one moment a function describing the monoid structure of a spliced variety is a transformation, and the next it's an object that has components such as a kernel (section 4.3). This dynamism of object-ness makes a great deal of sense when we recognize that the impression of there being an object in the first place seems to come from the view of transformations as something-preserving: since the transformations can themselves be represented (e.g., labeling a linear transformation with a letter like "f" or "Λ") and since those representations can themselves undergo legitimate transformations (e.g., describing a linear transformation on a finite-dimensional linear space with a matrix), we can shift the phenomenological encounter with "a mathematical object" to bear on the process of granting phenomenological object-hood. In other words, this seems to offer a phenomenological reduction of the process of mathematical abstraction in a way that is compatible with the apparent fluidity of object-hood in mathematics.

7.2 Phenomenology of the Blackboard

This vision of the experience of mathematical struggle helps us to understand some quirks about board-use we observed in chapter 6. When the mathematicians walk into the room, they are faced with a blank canvas.[4] It is pregnant

[4]Of course, this isn't always true. Sometimes the previous residents of the room won't have erased the marks on the board, in which case I would imagine the marks are seen as basically unimportant. It's also possible that there are markings on the board left from the last time *those* mathematicians were there. However, these exceptions shouldn't distract from the main point,

with possibility and, for many, practically calls to be structured and given form. Yet as soon as they write upon it, the structure given splits it into *regions* (Nemirovsky & Smith, 2011). For instance, Bart's inverted pyramid implicitly gives the area immediately around it a three-dimensional structure — but this infinite expanse of continuous space simply does not continue beyond a somewhat ill-defined boundary. Inscriptions just inches away are comfortably seen as separate from this space.

What seems to be going on here is that the board is a place for inscribing "perspectives" of mathematical objects. Perspectives are kept mentally distinct on the board, and their positioning with respect to one another matters only if, and only to the extent that, they are drawn with the intent of positioning mattering. We see this, for instance, in Bart's placing P_q immediately below $P_{p'}$ to imply equality, but after erasing P_q and then writing it again it no longer conveys this implication because the equality was not intended in the second writing of P_q.

One trend seen in the mathematicians of this study is a tendency to cluster nodes of a common transformation mapping into one overall area of the board. We see this in how Joseph groups the notation for the chain maps (including the cobordism-based expressions) close together and also in how Bart erases one transformation mapping in order to prepare the board for another (as though the mapping region has already been specified and he's simply working with this implicit structuring of the blackboard). We also see this in its *contrast* when Fay draws her circular depiction of a graph a fair distance to the side of the inequality she has just soundly rejected as "nonsense" as though to emphasize that what she's drawing is significantly different from what that inequality refers to. Mind you, this last example also illustrates how this clustering phenomenon is a tendency rather than a firm rule: Fay doesn't feel that it's important to walk to the far side of

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and it's also simply a matter of fact that the boards actually *were* blank when the observed mathematicians entered the rooms in which they were recorded for the present study.

the board across Martha in order to draw her circle near the inscriptions to which it most relates. Yet with occasional exceptions like this one, the general trend seems to be to carve out regions of the blackboard for individual transformation mappings.

This also bears on the issue of errors. Since the board is seen as a space for expressing the results of legitimate transformations, as soon as something on the board is recognized as illegitimate (which is to say, as failing to preserve that which the mathematicians are trying to preserve), its recognition seems to include the drive to erase the misleading node. The presence of an error seems to be uncomfortable, which we see most dramatically in Joseph's & Bill's laughter and self-soothing reactions to recognizing a flaw (e.g., in the recognition that the "m" in Figure 6.8 needs to be a split rather than a merge). Furthermore, as an extension of the point about placement being relevant based on intent upon being written, what matters seems to be the intent upon writing what turns out to be an error rather than whether the inscription could be reinterpreted differently. Thus we see the "m" being erased and replaced with the word "split," and we see "P_q" being erased and replaced with "P_q" again.

The replacement phenomenon further illustrates this carving of blackboard space. Because the specific spot upon which the error had been written was designated for a particular sort of node of the transformation mapping, it calls for an appropriate substitution. That is, the correction must occur directly on top of where the error once was. This seems to help emphasize the correction in the mathematicians' minds while also providing them with a legitimate collection of perspectives on the mathematical properties they are exploring.

The use of single words in this context is particularly interesting, as it seems to ride on this phenomenological structuring of the blackboard as a working space for transformation mappings. Words like "split" and "facet" aren't themselves the result of any legitimate transformations, although in their respective contexts they do refer to such transformations. Yet we find them appearing as part of the erase-and-replace phenomenon. This seems to emphasize key aspects of *thinking about* the transformation mappings rather than being emphases of the results of the transformations. For instance, Bart's focus on thinking of P_p & $P_{p'}$ as *facets* of P_q offers a way to think about what the splicing of varieties is doing under this particular transformation from varieties to monoids. Calling them "facets" doesn't actually describe the technical nature of the relationship the way "parabola" describes a quadratic graph in a technical way[5], but it does help Matt & Bart think about how they might determine what a more detailed description should be.

This charge of significance doesn't just apply to corrections, though. The most blatant example of this we saw was in Joseph's writing down "$f \circ g$" to direct attention to a particular path in the chain map diagram, but we've seen it many times implicitly. Bart choosing to write down "$u_p : P_p \to \mathbb{N}$" and the related function $u_{p'}$ implicitly claims that these are important things to attend to — that these are relevant perspectives on some kind of mathematical object they're interested in. We also see this in its absence, as when Joseph chooses not to write down Bill's suggestion about the "sheet cup" and "sheet torus" (Table A.9, rows 125–131). Since the meat of the struggle is in navigating across legitimate transformations to get a better sense of how it's possible to experience a given mathematical object, what is written on the board takes on an implicit claim of being in service to this purpose.

[5]Saying that "parabola" is a technical description of the Cartesian graph of a quadratic function assumes we're defining a parabola to be a locus — specifically, the set of points equidistant from a given line (the directrix) and a given point (the focus).

7.3 The Role of Continua

This also offers a new spin on the the "algebra versus geometry" matter that comes up at times in mathematical practice. Algebraic manipulations seem to consist of legitimate transformations that leave a given (typically unknown or variable) quantity unchanged. For instance, we can rewrite $x^2 + 5x + 6$ as $(x + 2)(x + 3)$, and in saying that we're *rewriting* it we're implicitly declaring that there's a transformation that's legitimate by its preservation of the overall quantity to which these expressions refer. Those two expressions — "$x^2 + 5x + 6$" and "$(x + 2)(x + 3)$" — become two different nodes of a transformation mapping from which our impression of "this quantity" as a mathematical object emerges.

However, we find that these characters also appear in invocations of the continuum, which seem to capture a majority of what we intuitively want to label as "geometric." Just as we can transform $y = x^2 + 5x + 6$ into $y = (x + 2)(x + 3)$ legitimately (in its preservation of the variable quantity y), we can legitimately transform either of these algebraic equations into a Cartesian graph of a particular parabola. The Cartesian graph would then be a node in a transformation mapping, giving us another view of the mathematical entity we can typify as "the relationship" or "the covariation" between x and y. And we see this actually happening in mathematicians' practice, such as in Bart's depiction of the relationship between P_p, $P_{p'}$, and P_q by an inverted pyramid. Thus it would seem that in practice, invocations of the continuum constitute a type of node in a transformation mapping — a node of a phenomenologically different character than algebraic statements, for sure, but not one playing a fundamentally different role. In other words, we find that at least one key aspect of geometry is actually *embedded within* the sense of what is algebraic.

Why, then, is there the phenomenological impression of geometry being in *contrast* to algebra? Why, for instance, does it feel so incredibly different to go from $y = x^2+2x$ to a graph than it does to go from $y = x^2+2x$ to $y = x(x+2)$? My guess is this: invocations of the continuum bring our sense of spatial arrangement (and, sometimes, our sense of movement within space) to bear on the expression of that which is preserved across transformations. When we draw Cartesian axes to start defining a node of a transformation mapping, our familiarity with the axes bestows the region around them with the character of a planar continuum that it didn't have before. We see this phenomenon in Bart's creation of the inverted pyramid: there is *space* implicitly surrounding the pyramid, which is not an impression we get at all from, say, his function declarations (e.g., $u_p : P_p \to \mathbb{N}$). This integrates our sense of spatial arrangement and movement into the legitimacy of the transformation in a way that simply doesn't happen when factoring x out of $x^2 + 2x$. Certainly, positioning is relevant there — but not *continuous, spatially charged* positioning, and not in a way that is intrinsic to the transformation.

In chapter 5 we noticed that mathematicians treat invocations of the continuum with a kind of transparency. We can now see a possible reason why this is: these continua are nodes in a transformation mapping, and as such *help to constitute* the mathematicians' experience of there being a mathematical object that they're examining. We actually observed the same thing before, with Bart proclaiming that a particular portion of a monoid was part of the kernel of a map φ_η while pointing to inscriptions on the board (section 4.3). We also saw Joseph referring to the arrows representing parts of chain maps as *being* functions partially constituting chain maps (section 6.1).

We also noticed, in the case of Bart navigating between a tropical diagram and a $\mathbb{CP}^2$ triangle, how there's this attempt to show the relationship between two continua via the multiplicity of the body. We now have a more rich framework within which to embed this observation: the multiplicity of the body is one tool with which mathematicians can illustrate a legitimate transformation. When the two nodes that are being related via a legitimate transformation are both continuous, movement and positioning of the body can illustrate this transformation. However, *any* two nodes can, in principle, be linked by this kind of multiplicity. We see a non-continuous example of this in the way in which Bill interacts with both the board and his notes in Table A.9, rows 59–64: he's immersing himself in his notes while reaching forward, trying to offer new information via a legitimate transformation from his notes to the chain map on the board.

7.4 Node Activation and Social Dynamics

The nodes of transformation mappings can be activated and animated in the course of use (Nemirovsky & Smith, 2011). We can see, for instance, how Joseph gestures directly in front of his newly drawn ribbon surface to indicate how it's intended to be folded (Table A.5, rows 15–19): the positioning of these gestures immediately in front of the drawing seems to be an effort to animate it and convey in more detail what the ideas are behind it. We also see this in Martha's navigation of the inequality that is troubling her (Table A.7, rows 22–28), although in this case it's more a matter of her navigating the various inscriptions in a deictic way.

Although this kind of closeness to the board clearly isn't *necessary* for navigating struggle, it certainly seems preferable to the mathematicians in this study. Indeed, this is what makes proximal inhibition possible: there is a desire to engage with the nodes of the transformation mappings that gets frustrated by a lack of clarity about how to perform transformations on them that are at once legitimate and insightful. But we also see some work-arounds in these clips. For instance, Bart's idle perfecting of the inverted pyramid gives him a way of concentrating on the inscription and remaining intimately close to it despite being somewhat befuddled by the situation to which it refers ("Now I'm confused all the time"; Table A.4, rows 120–122). In ways like this, the mathematicians studied here repeatedly find ways to engage with the mathematical symbols and drawings on the board in a very hands-on sort of way.

This is a specific instance of a point first raised in section 2.2.2: we miss something important if we view abstraction as a purely mental phenomenon. These nodes of the transformation mappings cannot be fully understood as inscriptions that are seen from afar in a certain way. Rather, they call for certain kinds of action, and the range of seen-as-possible actions is richer and more detailed the more familiar the related legitimate transformations are to the mathematician in question.

It also seems that this call for hands-on interaction in symbol-use is intrinsically an activity for one person at a time, at least for the pairs observed here. Indeed, in all nine observations, there was only one instance of both mathematicians engaging in the *board* at the same time.[6] The rest of the time there was some kind of implicit negotiation as to who may control the board and, thus, engage with the nodes of the transformation mappings. This negotiation took on a different character with each pair based on the particulars of their interpersonal dynamic, but was still clearly present. It seems to carry a similar tone to how two people in a conversation will socially negotiate who is talking at a given moment; indeed, in a way, we can think of their interactions in these sessions as a sort of ongoing mathematical conversation where large portions of the "speech" happen through node activation.

One last part of this transformation-based conversation that appeared in the present work was a kind of social metacognition. We see again and again how, when it's discovered that an attempted transformation was actually illegitimate

[6]This clip was excluded since it was nearly impossible to see what the two mathematicians — Matt & Bart — were doing at the board. Based on the context and the inscriptions that appeared, it seems that they were effectively using different parts of the blackboard as personal scratch "paper."

(i.e., that which was supposed to be preserved in fact might not have been), it becomes important to convey the recognition of the error to the other mathematician. The wording here is important: the goal is to convey the *recognition*, not the error itself. We saw this in how Martha spelled out the problem in the argument that Fay had just discredited: clearly Fay knew what the problem was, so Martha's only reason for doing this was to convey her understanding of the error. Similarly, Bart emphatically described the correction that would have prevented the mistake of identifying P_p and $P_{p'}$ — but again, in a context in which Matt clearly didn't need that correction emphasized for his understanding. This need for social metacognition is so strong that it can even put on hold the impulse to erase nodes representing illegitimate transformations.

7.5 In Sum

All told, then, the present exploration of the phenomenology of mathematicians' struggles has brought us to recognize the structuring of struggle within the framework of legitimate transformations. In terms of the inscriptions on the blackboard, we can recognize these as transformation mappings, where each node provides the mathematicians with a particular vantage point on the object of their study. The struggle, in essence, is in determining and applying legitimate transformations that will bear on the topic in question. Some transformations invoke the continuum, bringing to bear a collection of affordances that are unique to continuous spaces. We also find that these nodes of the transformation mappings call for a kind of closeness when they are being actively used for reasoning, which provides the framework for proximal inhibition (when the desire to move in is actively present but being stymied by confusion about how to perform or verify a legitimate transformation).

It is promising at the end of a phenomenological investigation to see the structures explored synthesizing in this way. Mathematicians in the midst of doing research seem to feel that they are *doing mathematics*, as though it's some one thing that can be done. Developing several separate observations akin to Merleau-Ponty's (1962) "motor intentionality" can certainly provide us with a significant amount of insight — but seeing these different pieces combine to describe a whole suggests that we are detailing something key to the nature of the experience itself. We can hope, therefore, that this examination has given us a solid beginning into the examination of the phenomenology of mathematical struggle.

Chapter 8

Discussion

In this study we've seen three phenomenological constructs. Transformation mappings form the experiential backbone of the notation used on the blackboard, some of the nodes of the mapping take on the phenomenological character of being invocations of the continuum, and most nodes call for close proximal engagement which, in turn, can be frustrated and result in the pattern of proximal inhibition.

8.1 Implications

What, then, have we gained by these observations? First, as mentioned in chapter 1, we are more likely to notice relevant instantiations of these phenomena as they arise. For instance, after writing the chapter on proximal inhibition, I started noticing it in my students[1] at times when they were working on worksheets, such as by leaning back from the paper with a perplexed look and a tilt of the head. This was even clearer when they were working in small groups. Yet I did *not* notice it when watching them take exams. These are informal observations, emerging naturally in the moment, so it would be premature to form conclusions from this.

[1]These were college students taking a mathematics course sequence intended to prepare them for teaching at the elementary school level.

However, it does raise some questions: Is there something about worksheets that better allows for proximal inhibition as compared to exams? If so, why? To what extent does this phenomenological difference discourage students — whose progress is typically determined by exams — from exploring mathematics in a way akin to how mathematicians do?

I should also emphasize that these observations do not emerge from my watching my students carefully and asking whether and when I can see proximal inhibition in what they're doing. The process is more organic than that. Most of us would probably recognize a person who smiles nervously, bows their head, covers their eyes with their hand, and turns away slightly as expressing embarrassment; yet many people aren't conscious of this eye-shielding pattern until it's explicitly pointed out, at which point it leaps to our attention in places we wouldn't have thought to look for it such as deep concentration. In the same way, I notice when my students invoke the continuum without my really trying to notice, and I find it quite natural at that point to consider what the continuum affords for them and how it relates to questions of legitimacy in transformations.

However, it is still an open question whether these kinds of insights offer any pedagogical advantages. They might; it might turn out that teachers who have a deeper phenomenological insight into their students' encounters with mathematics can guide those students more effectively toward experiencing the material in a particular way. Yet it might also turn out to be mostly irrelevant for teachers to know this, much as a dancer might find little to no benefit in knowing how nerves cause muscles to tense despite the obvious relationship to the art. This domain needs more exploration; I'll describe some possible avenues for this in section 8.3.

What we *can* say, though, is that this provides insight that can be useful for *researchers*. It helps us to ask questions that might not have occurred to us to ask otherwise, such as with the possible phenomenological contrast between tests and worksheets. It offers us a different kind of insight into how mathematics can be, and is, experienced, which can inform us as to what some valuable research programs might be that we might not have considered otherwise. And it invites us to recognize subtleties about the mathematics itself, which can help us to determine what we would like to see our students capable of.

This raises a second point about what these phenomenological constructs offer us: insight into the nature of mathematics. It seems highly relevant, for instance, that the mathematicians in this study seem to develop a sense of object-ness about that which they're studying by the process of generating different inscriptions that they think of as outlining a something-preserving transformation. If this holds in general — for instance, if our sense of "the unit circle" as a mathematical object comes from seeing different expressions as being of "the same thing" — then this raises some interesting and potentially valuable questions: What determines whether a transformation can be seen as legitimate? How does the sense that something is being preserved arise in the first place? Is this arising of object-ness a separate experiential process from understanding the transformation itself, and if so, how might we typify that difference?

We can speculate on the answers to questions such as these, but in essence they are empirical questions. I'll describe some possible avenues for answering them in section 8.3.

8.2 Limitations of the Study

8.2.1 Specialized Population

Before describing future studies, though, I'll explicate some of the limitations of this study. First, as mentioned in section 1.2.2, phenomenological studies typically generalize by transforming our perceptions rather than by showing invariance across a representative sample. They're general in the sense that we see the world differently in situations other than the case studies examined. Yet all six mathematicians observed were from the same department, so it's quite plausible that we've seen just one relatively tiny piece of how mathematicians structure their encounters with struggle in their discipline. It would be somewhat surprising but not wholly implausible, for instance, to find that Russian mathematicians working in Moscow navigate struggle completely differently than we've seen in our subjects here.

In particular, Fay & Martha ended up providing vastly less material for the present study than did the other two pairs. Most of their interaction across all three observations consisted of Martha presenting a question or suggestion and Fay responding with explanations or suggestions. There were surprisingly few clear instances of struggle anywhere in their in-person interactions. It was clear that there was some sort of effort going on between sessions, but it seemed to occur individually and privately. It seems highly plausible, then, that the phenomenological constructs described in the present work might be peripheral or maybe even irrelevant to certain working styles. How the experience of struggle differs for some individuals in isolated practice as compared to when they're doing partnered work in front of a blackboard remains an essentially untouched question beyond mathematicians' after-the-fact reflections.

It's also quite plausible that by focusing on *mathematicians* as experts, I've left out other kinds of mathematical struggle that are navigated in the course of professional work. For instance, in what ways might physicists' encounters with mathematical difficulty differ from those of mathematicians? They might feel that the existence of the electromagnetic field is more literal than the existence of, say, the cyclic group with 12 elements. Is this actually the case, and if so, from where does this impression arise? We could also consider engineers' mathematical difficulties: how does their pragmatic aim with the math bear on challenges they encounter? Does legitimacy play a different role for them than it does for mathematicians, and if so, how? Having a collection of phenomenological constructs as they apply to some mathematicians' struggles can help to frame these questions, but it doesn't really answer them. It's important, therefore, to recognize that what we've gained here is some insight into *one* way of viewing what mathematics is, and that there are other perspectives on what math is that the present study simply does not touch on.

Also, this study did not consider students at any of the levels our field is typically interested in. As described earlier and as I'll elaborate upon in section 8.3, this study seems to raise some interesting questions about students' experiences that call for further examination — but nothing said here actually *shows* anything about students' experiences. Therefore it seems somewhat hazardous to make any pedagogical recommendations based on what has been described in the preceding chapters.

8.2.2 Observational Interference

Also, as with most ethnographic studies, there are complications based on the fact that the very effort to observe the mathematicians working might have interfered with their usual working methods. It's unrealistic for us to tell how profound such an influence might have been since we have nothing to compare it to.

One possible influence along these lines was the presence of two cameras and one to two observers (me and sometimes another camera operator) who weren't involved in the process. Across the $10\frac{1}{2}$ hours of video, there were four glances made at the camera during the course of the mathematicians' working, which seems to suggest that they generally ignored the cameras. Yet it seems unlikely that there was no influence at all on their working style, and at the end two pairs (Joseph & Bill and Matt & Bart) showed concern that what we had observed might have been confusing due to a lack of context.

Another possible influence was the somewhat unusual setting. I asked each pair to meet in a room large enough to set up cameras far enough away that a wide-angle shot would capture most of what they were doing. This ended up occurring in classrooms in the math department rather than in any faculty offices. Yet all three pairs reported that they usually met in an office[2]. I also asked them to use the blackboard instead of working on paper so that it was possible to observe what they were writing. This probably influenced the dynamics of how they usually would interact (although in the case of Matt & Bart, they joked that this didn't affect what they did at all except in providing a larger-than-usual blackboard).

This raises concerns of authenticity. How much does the medium of scratch paper contrast with that of the blackboard in terms of the role of inscription for mathematical struggle? How heavily do the mathematicians normally rely on the ability of a private blackboard in an office holding their prior inscriptions from their last meeting rather than being erased? In follow-up interviews, all six mathematicians indicated that they felt that the setting had not really affected the way they had worked — but it's quite possible that this was simply true *upon reflection* rather than actually having no relevant effect on their phenomenology at the time. This is a concern about which the present study's methods provide virtually no illumination.

8.3 Some Possible Future Research

With all that said, we still seem to gain from this study some insight into mathematicians' practices and, through that, a more detailed sense of what math-

[2]Joseph's for Joseph & Bill, Matt's for Matt & Bart, and Fay's for Fay & Martha.

ematics is. This suggests some promising further lines of inquiry.

8.3.1 Further Phenomenological Investigation of Struggle

Further explorations of related data

Some of these directions are suggested by the data from this study that were *not* analyzed for time considerations. For instance, in comparing early interviews for different studies (Nemirovsky & Smith, 2011, forthcoming), the observations made for this study, and the recorded follow-up interviews with each mathematician afterwards, there seems to be a shift in how the mathematicians articulate and gesture about the objects to which they're referring. In particular, familiarity seems to encourage them to treat the object as more of a self-contained thing that takes up less gestural space. It seems worthwhile to explore this phenomenon in more detail since it seems like it should bear on the process of making legitimate transformations pre-reflective (e.g., as in our immediate knowing that $2(x+3)$ can be converted into $x+6$ as part of our sense of what "$2(x+3)$" means; see Gallagher & Zahavi, 2008).

There's also a fascinating trend of *anthropomorphization* in much of the data mentioned as well as in the pilot data. You can see a hint of this in Joseph's reference to how the two sides of the folded hood of a ribbon knot meet as a cusp: "... in fact, these guys do actually meet like the equation $x^2 = y^3$..." (Table A.5, rows 36–39). This certainly isn't ubiquitous; we don't see Joseph or Bill clearly demonstrating this in the "odd scalars" material (Table A.1), for instance. This caused me to remember how, as a math graduate student, I would sometimes hear others in my program using intention-based language (e.g., "This guy doesn't like anything from this subspace and eats all of them" as a description of a map's kernel). This seems to be reminiscent of Devlin's (2000) argument that mathematical reasoning uses the same brain circuitry as we use for discussing other people, citing his own experience of working with well-known mathematical structures like the complex plane as meeting with "an old friend." It could be quite fruitful to explore how and when this kind of anthropomorphization shapes mathematicians' experience of mathematical objects, both in the already-collected data as well as in presentations such as colloquia and lectures.

Explorations of struggle in other populations

There also seems to be promise in applying phenomenological methods to investigate struggle in other populations. Two general classes seem likely to be fruitful: other technical professionals, and students. Professionals, as mentioned earlier, could give us a more nuanced understanding of how mathematics behaves in practice. How does the sense of object-hood differ for physicists, engineers, and economists as compared to professional mathematicians? How does this change the sense of what mathematics affords in terms of exploring the questions these professionals are interested in? Essentially the same methods used in this study could be brought to bear on professionals in a few different domains to flesh out these questions.

Exploring students' phenomenology might give us insight into the *development* of mathematicians' styles of structuring their struggle. How do first-year doctoral students differ from those who just passed their qualifying exams, in terms of how they navigate the experience of struggle? How does this differ from undergraduate math majors, or high school students? Exploring the phenomenology of students at these various levels might provide us with a rich understanding of how mathematical experience can unfold.

8.3.2 Explorations of Legitimacy

Longitudinal studies of students

Along similar lines, it could prove to be incredibly insightful to track how object-ness arises in a few specific case-studies of math students over the course of a term or longer. For instance, linear algebra on the surface seems to require students to develop familiarity with a wide range of transformations and abstractions, such as at one level treating vectors as objects and at another level treating entire vector *spaces* as objects in order to navigate the ideas of isomorphism and local linearity in multivariable calculus. This could prove to be a rich domain for exploring how mathematical objects come into being for students, and in what ways this appearance of object-ness is related to and different from familiarity with the procedures involved in going from one type of inscription to another (e.g., going from a geometric description of a linear transformation to its matrix form given a basis). Another potentially rich domain might be introduction to algebraic topology, which rapidly introduces new transformations and then treats those transformations as objects in further transformations.

Teaching experiments

It would also be potentially very valuable to learn to what extent attending to the sense of something being preserved as the origin of mathematical objects as experienced can help inform a curriculum. For instance, Smith & Thompson (2007) advocate something to this effect for algebra, focusing on the idea that *quantities* are what algebraic transformations preserve. Sowder et al. (2009) seem to put this idea to excellent effect in their math curriculum for prospective elementary school teachers, explicitly emphasizing the idea of quantities and then asking these prospective teachers to reason with the *quantities* rather than just the symbols representing the quantities. We see a similar idea in, for instance, Yerushalmy &

Schwartz's (1993) argument for framing algebra education around the theme of *functions* rather than rules.

This same idea could be applied to other curricula. In probability, say, we might think of preserving *information*. For instance, suppose we think it's equally likely that a given coin is (a) fair or (b) twice as likely to come up heads as tails. If we flip the coin and it comes up tails, that gives us new information that makes it less likely that the coin is biased. Specifically, whereas the probability of it being biased was $\frac{1}{2}$ before, after the coin-flip it's $\frac{2}{5}$.[3] If we can give mathematical structure to the idea of "information," we might be able to teach probability with an eye toward legitimate transformations.

This raises the possibility of running a series of teaching experiments (Steffe & Thompson, 2000) to explore what happens when different math courses have an emphasis on legitimacy. It might turn out that explicitly focusing on what is preserved might not significantly improve students' understanding — but if so, that would be incredibly valuable to know. It stands a fair chance of revealing nuances of how the sense of object-ness comes into being for students. It might also help us to better understand what kinds of "local travel" (Nemirovsky, 2005) are needed in different domains in order to develop the intuition of something-preservation.

8.4 In Conclusion

I would be remiss in omitting one final dimension that emerged from this study, however ephemeral. I found that in the course of developing the constructs of this study, I came to understand *my own* mathematical efforts in more detail.

[3] This comes from Bayes' theorem:

$$P(\text{biased}|\text{tails}) = \frac{P(\text{tails}|\text{biased})P(\text{biased})}{P(\text{tails}|\text{biased})P(\text{biased}) + P(\text{tails}|\text{fair})P(\text{fair})} = \frac{(\frac{1}{3})(\frac{1}{2})}{(\frac{1}{3})(\frac{1}{2}) + (\frac{1}{2})(\frac{1}{2})} = \frac{\frac{1}{6}}{\frac{5}{12}} = \frac{2}{5}$$

I remember doing many of the things I observed these mathematicians doing — but seeing it from another perspective cast it in a new light for me. I now feel as though I understand my own understanding better, that I have a more refined sense of how my mathematical thinking works. I also feel that I have a better sense of the nature and value of the kind of effort I would experience when trying to make sense of something that seems impossible.

Because of this, I find mathematics even *more beautiful* than I did before. This exploration has enriched my appreciation and sense of aesthetics for mathematical reasoning. It is also clear to me that this did not come simply from learning more mathematics; rather, it came from examining the process of doing mathematics in detail, with a kind of reverence for the ways in which masters of the art display their prowess.

This brings to mind a well-known quote from Richard Feynman:

> I have a friend who's an artist and has sometimes taken a view which I don't agree with very well. He'll hold up a flower and say "look how beautiful it is," and I'll agree. Then he says "I as an artist can see how beautiful this is but you as a scientist take this all apart and it becomes a dull thing," and I think that he's kind of nutty. First of all, the beauty that he sees is available to other people and to me too, I believe. Although I may not be quite as refined aesthetically as he is. . . I can appreciate the beauty of a flower. At the same time, I see much more about the flower than he sees. I could imagine the cells in there, the complicated actions inside, which also have a beauty. I mean it's not just beauty at this dimension, at one centimeter; there's also beauty at smaller dimensions, the inner structure, also the processes. The fact that the colors in the flower evolved in order to attract insects to pollinate it is interesting; it means that insects can see the color. It adds a question: does this aesthetic sense also exist in the lower forms? Why is it aesthetic? All kinds of interesting questions which the science knowledge only adds to the excitement, the mystery and the awe of a flower. It only adds. I don't understand how it subtracts. (Feynman, 1988)

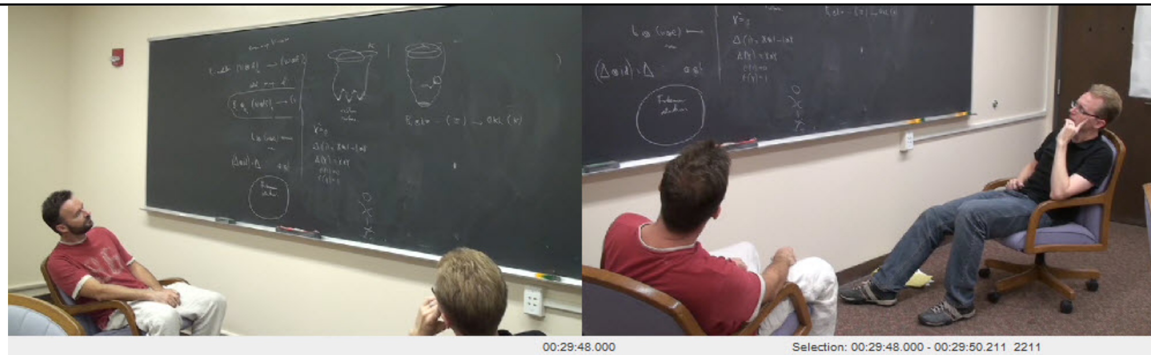
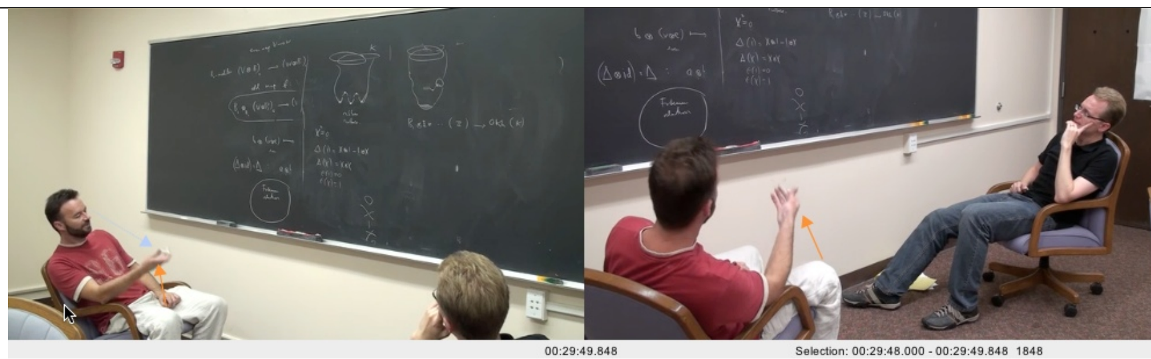
In a similar spirit, I feel that this exploration of the mathematical spirit only adds. It has been personally valuable and inspiring, something I am immensely pleased to have performed. I hope I've managed to share at least a glimpse of this delight with you as well.

Appendix A

Gesture analyses for selected clips

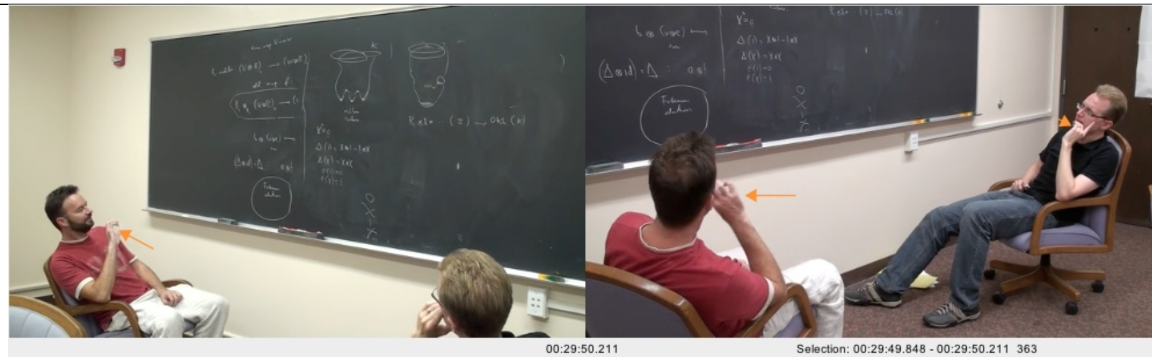
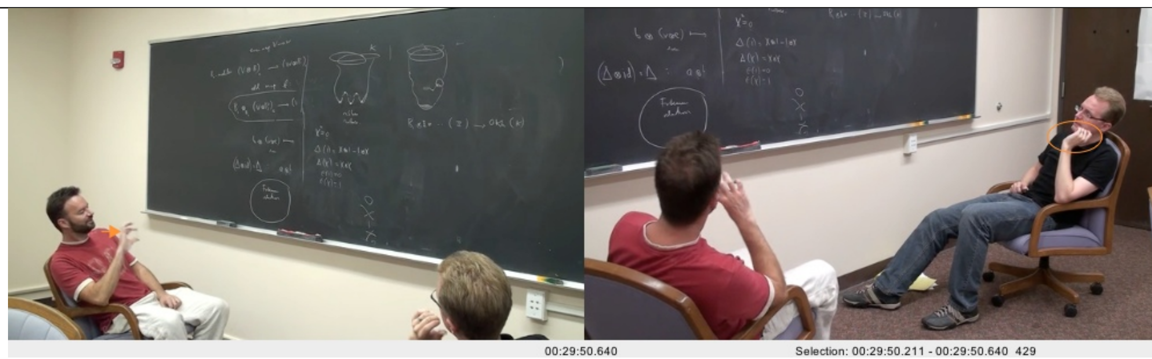
A.1 Joseph & Bill — "odd scalars"

Table A.1: Joseph & Bill — "odd scalars"

Row	Gesture & timestamp	Transcript
1		[N/A]
2		J: [pause] But think-

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
3		Yeah; I think
4		what I'm most in-

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

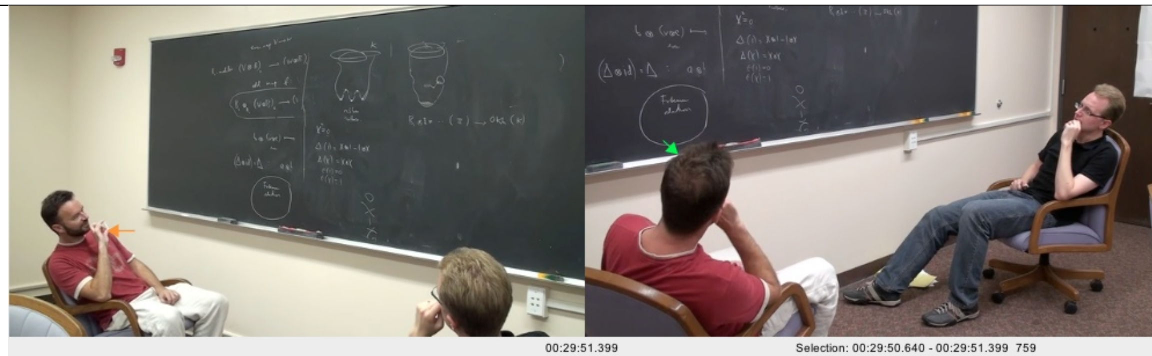
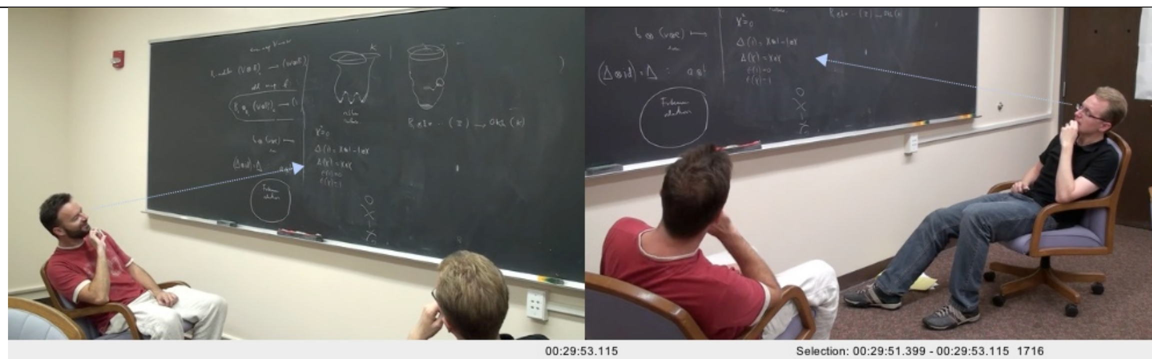
Row	Gesture & timestamp	Transcript
5		-terested in is setting
6		B_1 to be zero.

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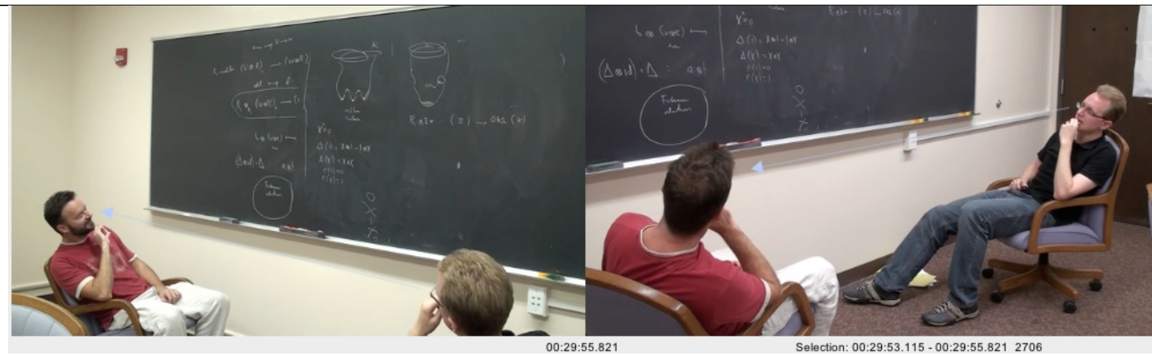
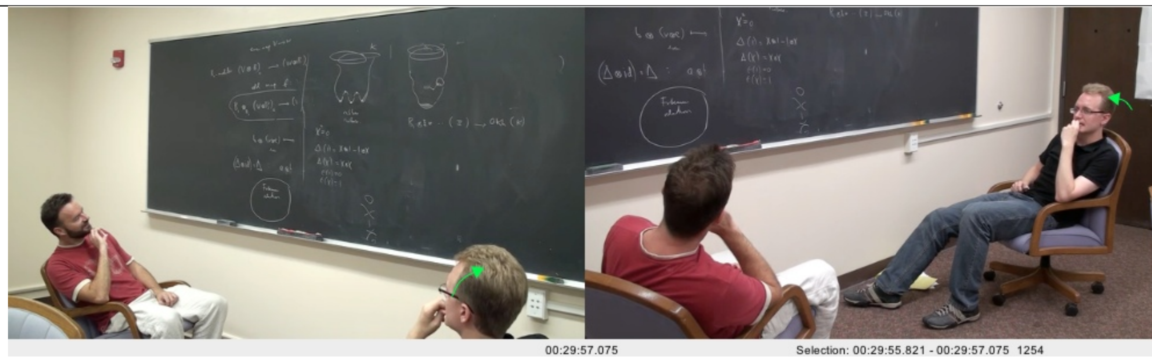
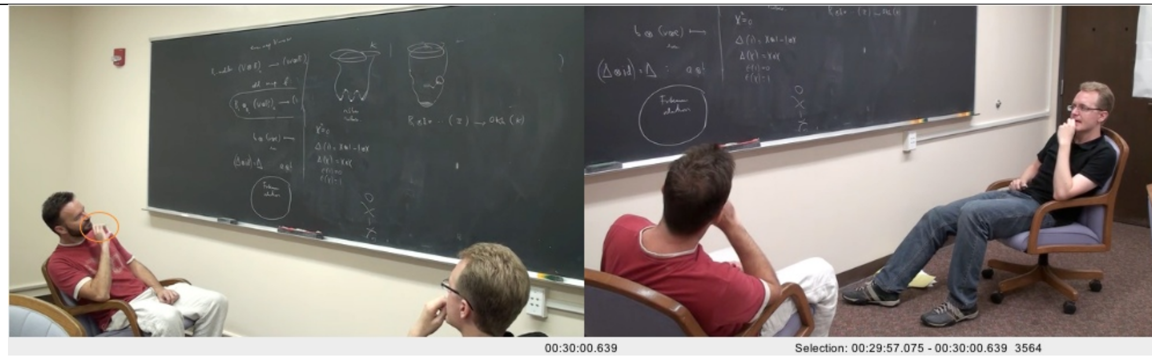
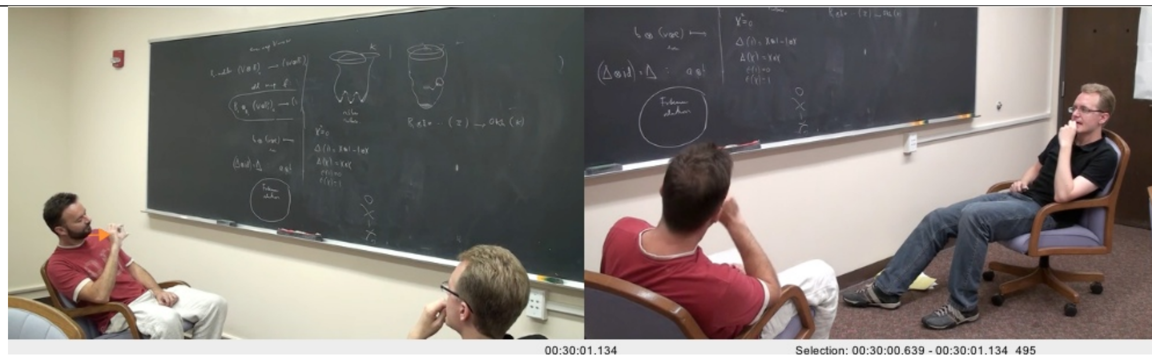
Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
7		[laugh] [pause] See,
8		B: Yeah. J: It's this

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
9		interesting thing; I mean the, the [pause] the, um,
10		[pause]

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

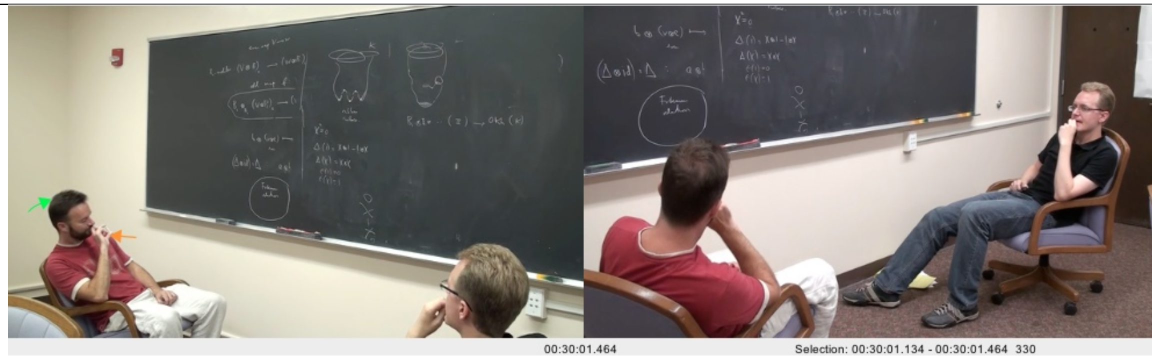
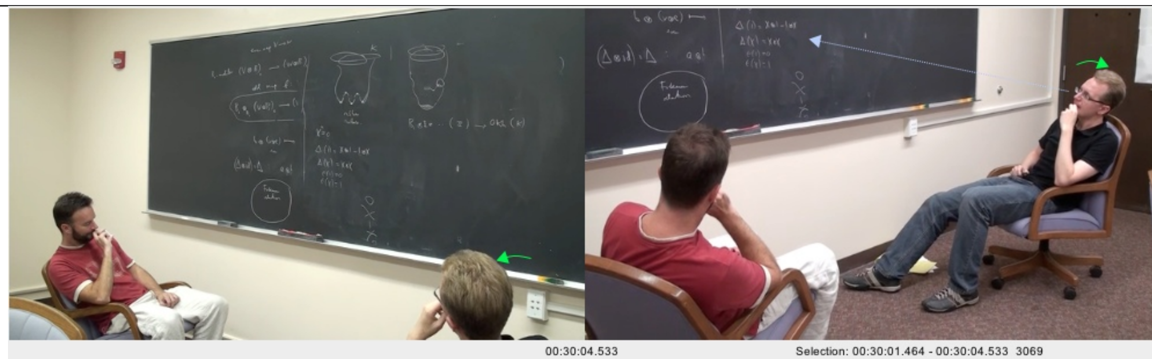
Row	Gesture & timestamp	Transcript
11		[pause]
12		[pause]

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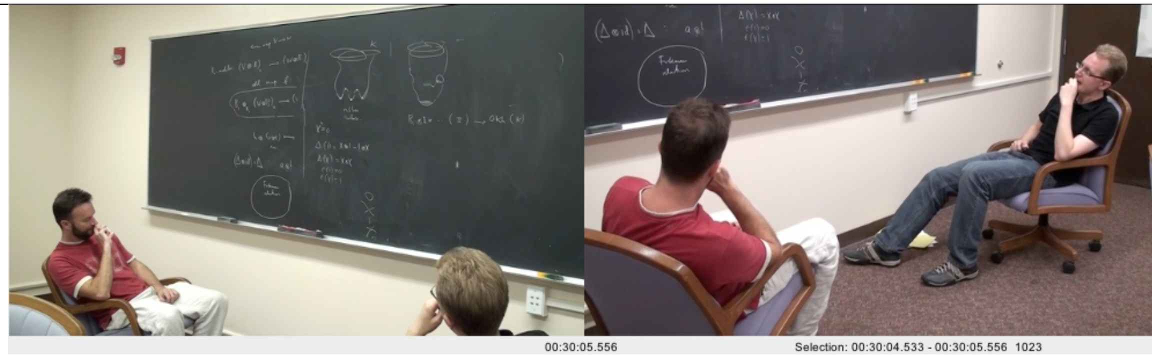
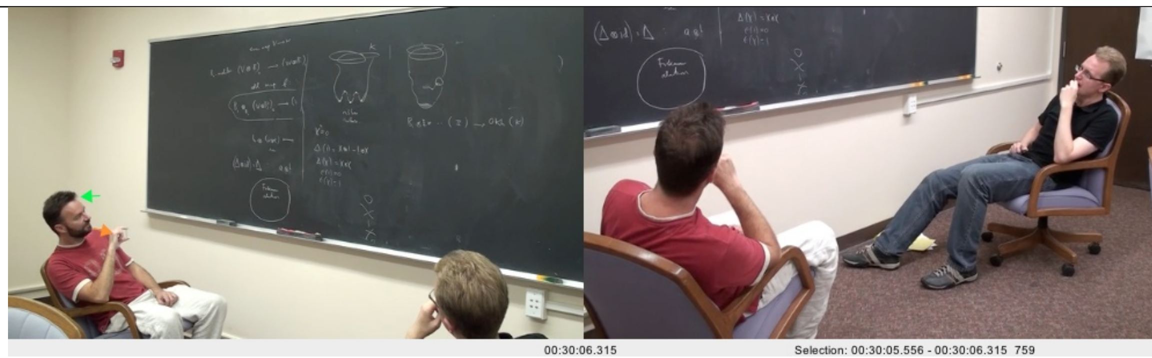
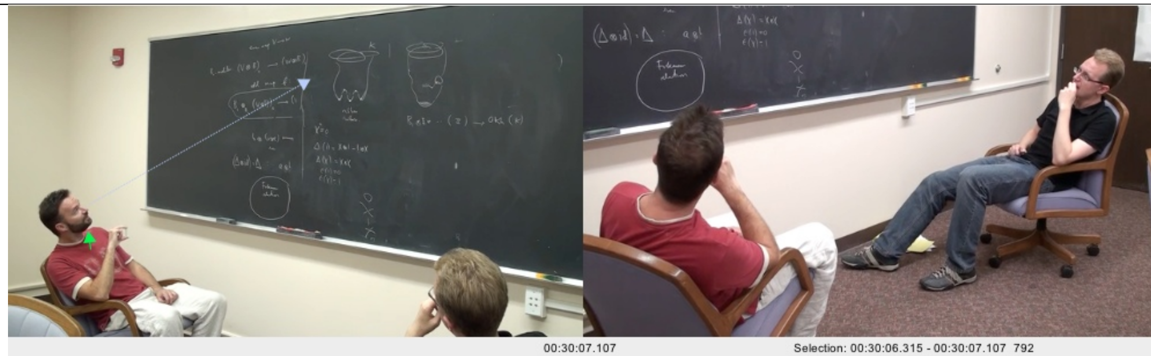
Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
13		[pause]
14		You know, so

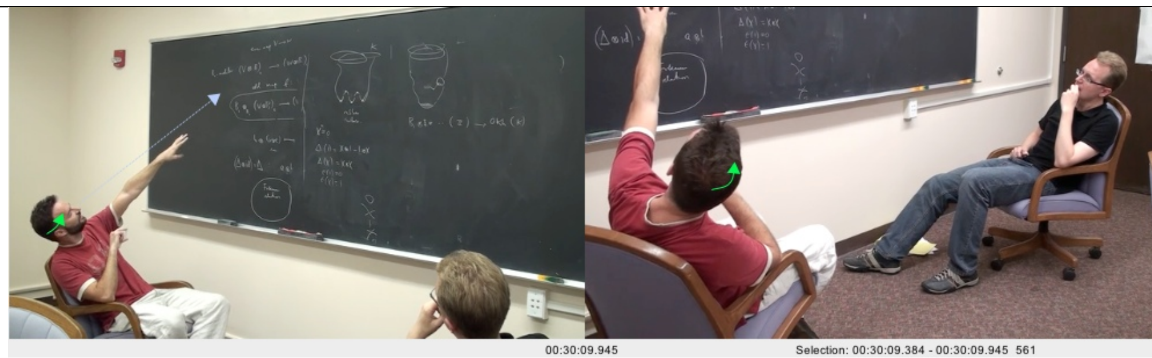
Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
15		a, a map between
16		supervector spaces is just a

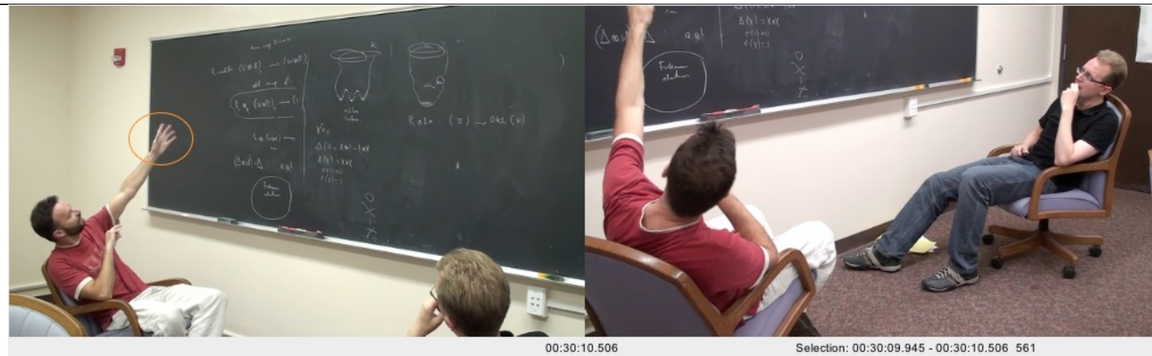
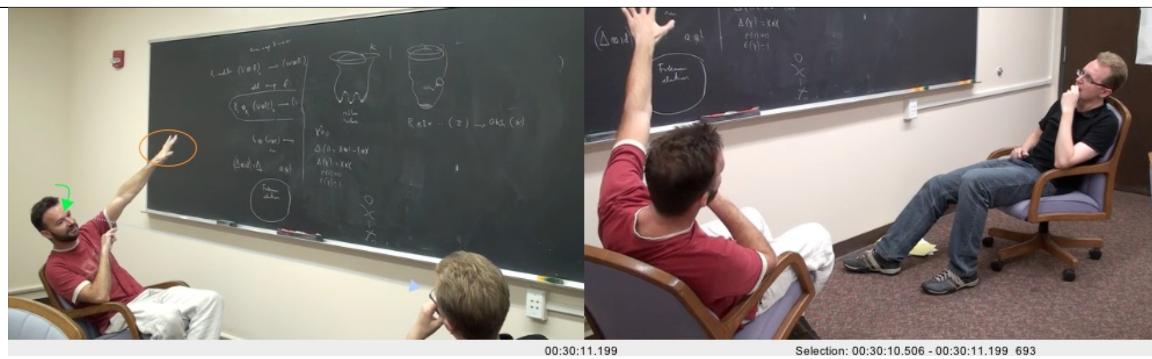
Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
17		functorial system
18		of things like

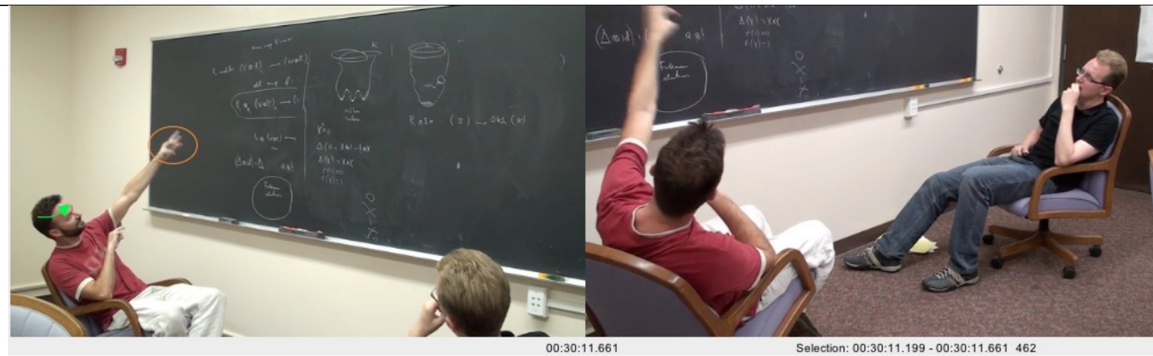
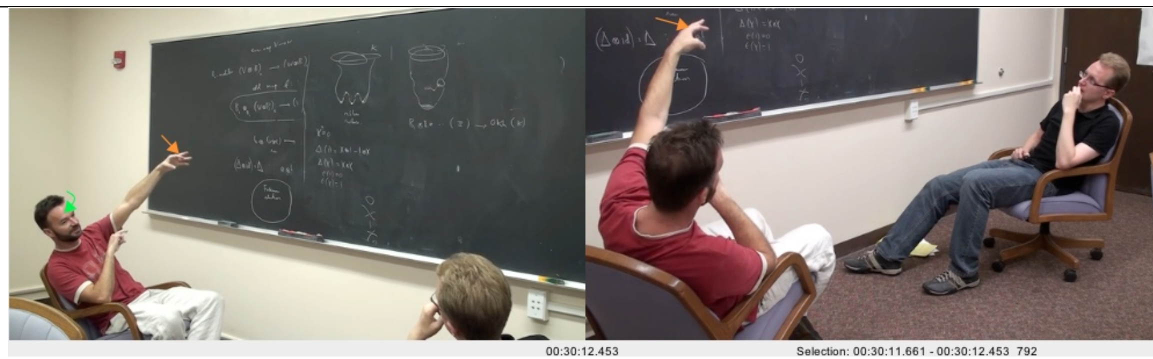
Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
19		this, true,
20		for any B

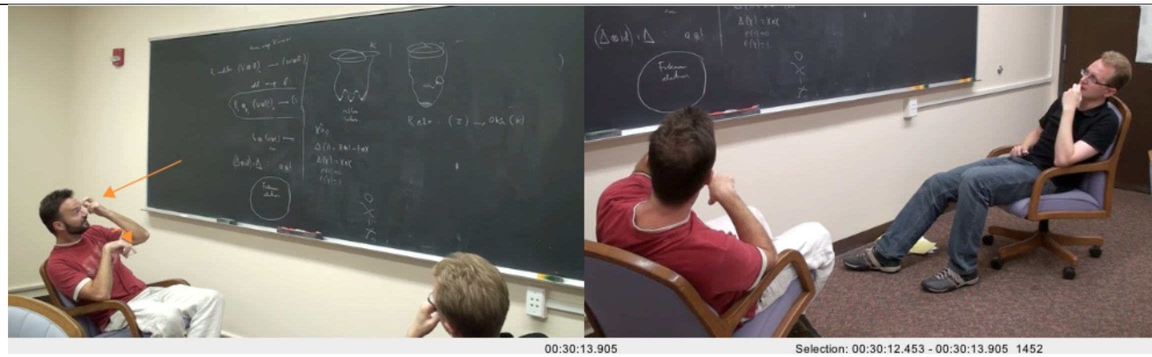
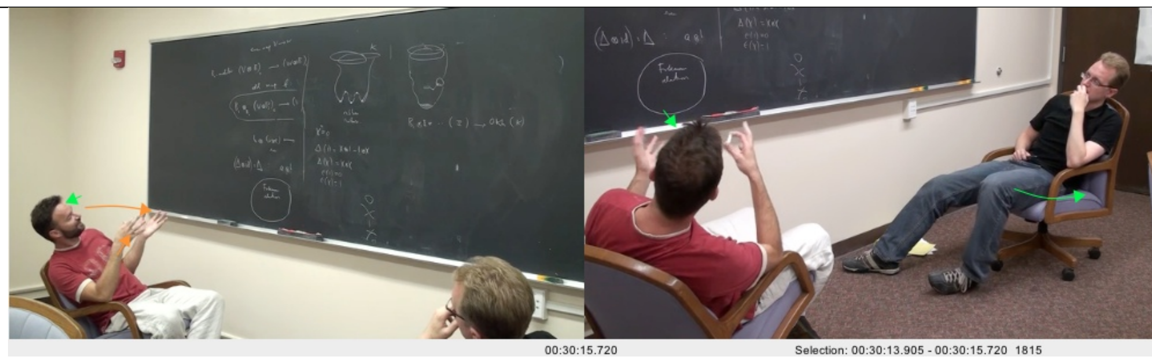
Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
21		there, there ex-
22		-ists something

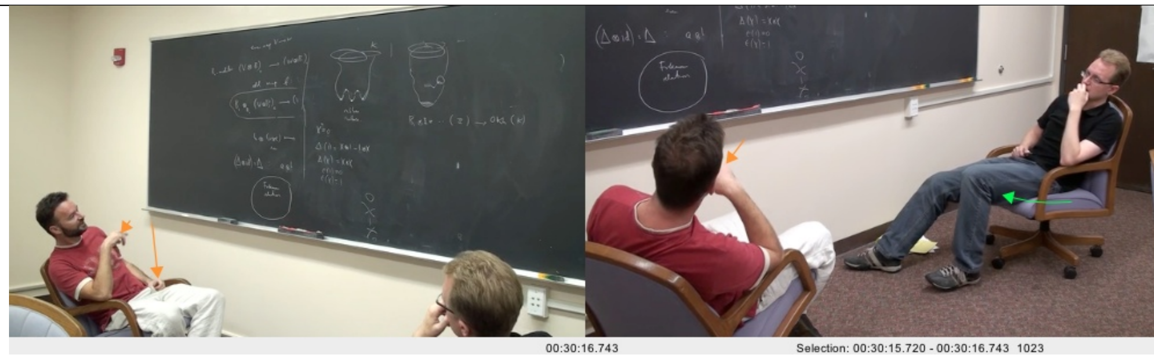
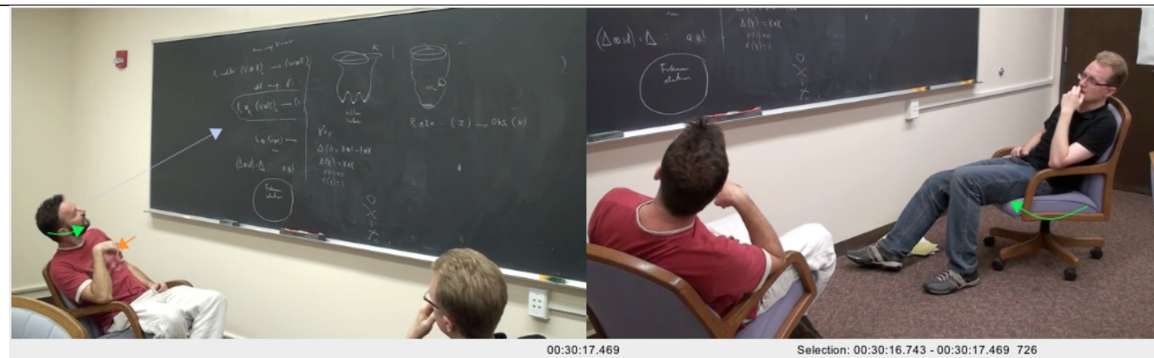
Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
23		like that. B: Yeah. J: It's the
24		the compatibility between different Bs

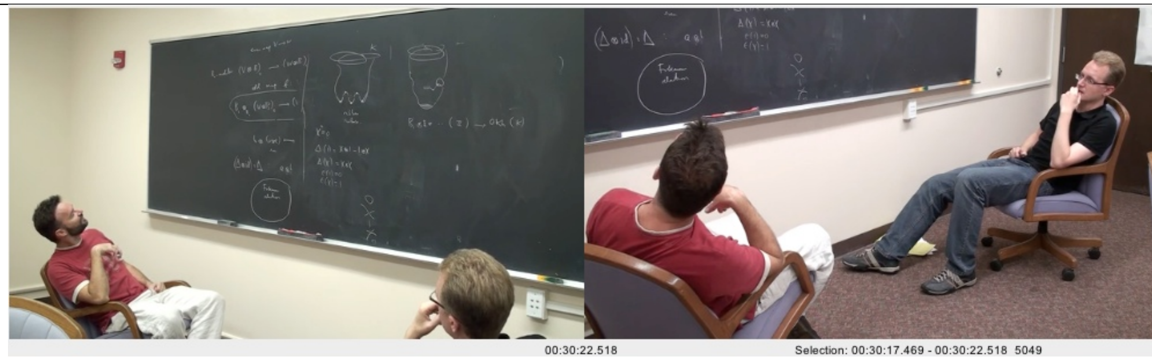
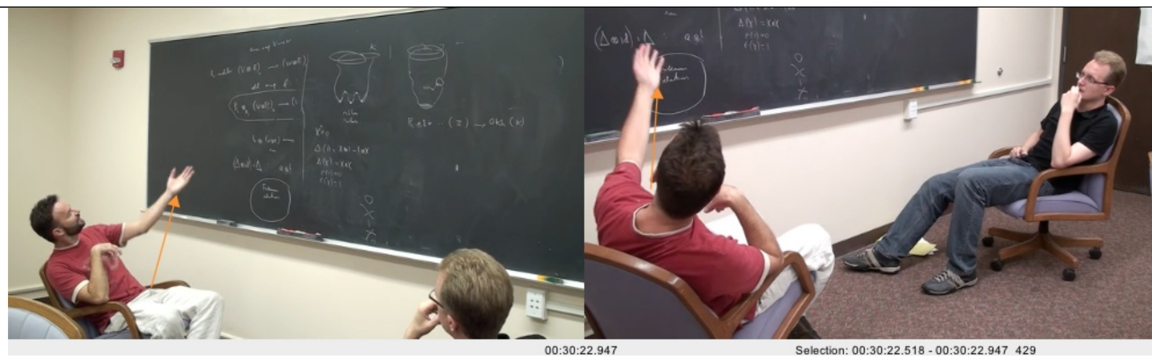
Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
25		which really gives us
26		the restriction.

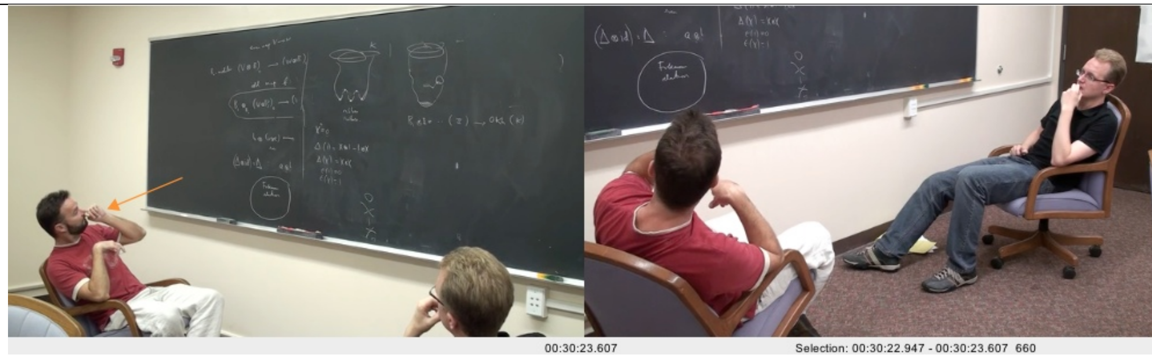
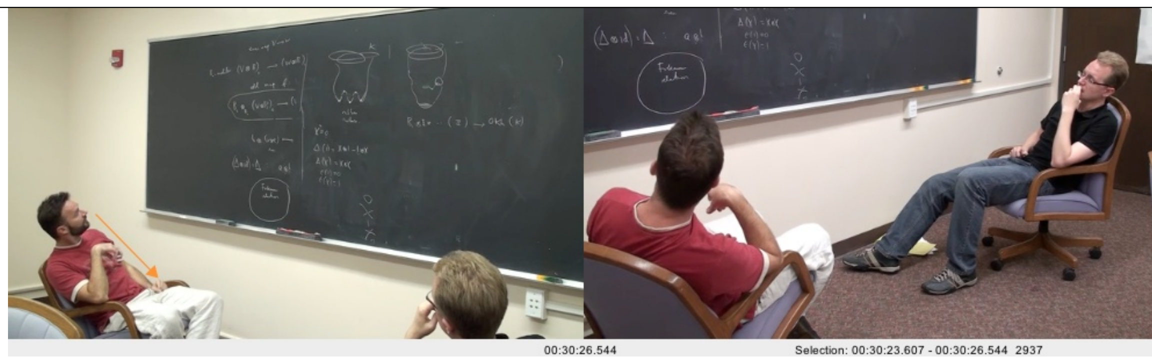
Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
27		But certainly, in particular we could just set B to be trivial. [chuckle] And then an odd map would
28		just be ze-

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
29		-ro.
30		B: Right. [pause]

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

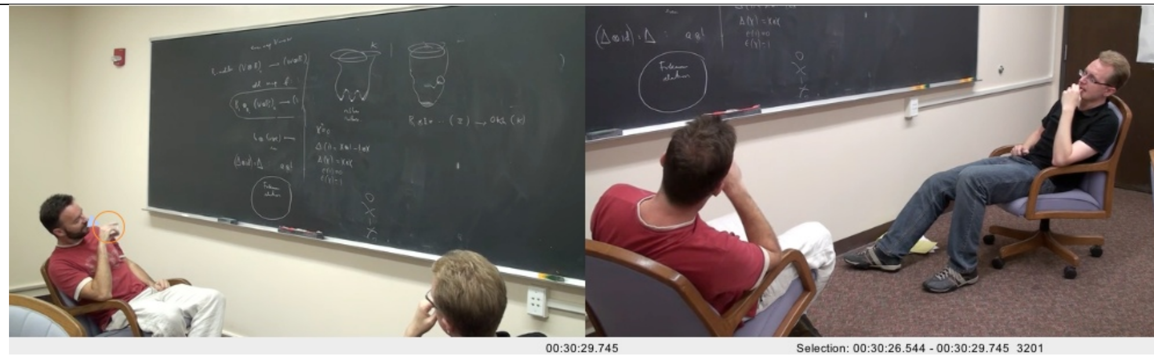
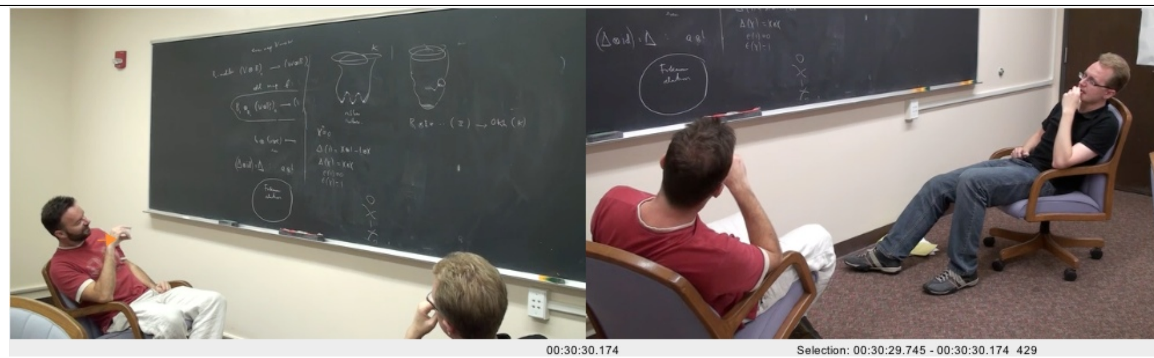
Row	Gesture & timestamp	Transcript
31		J: [pause] So, I think,
32		I mean, I

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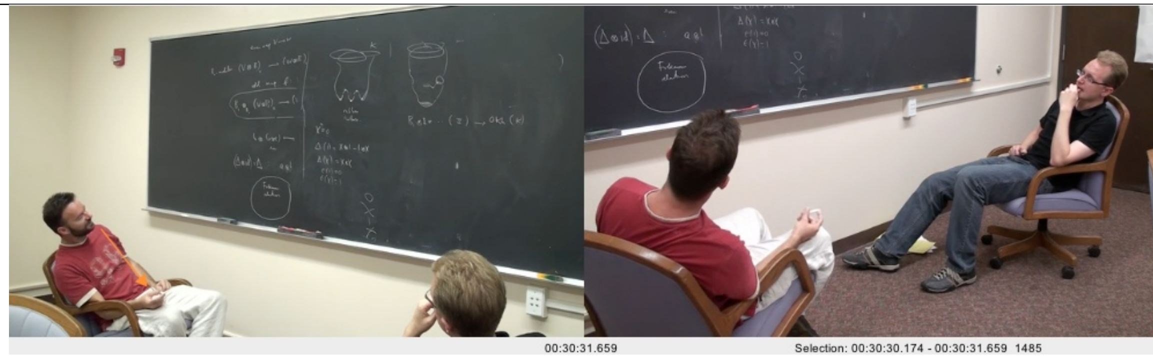
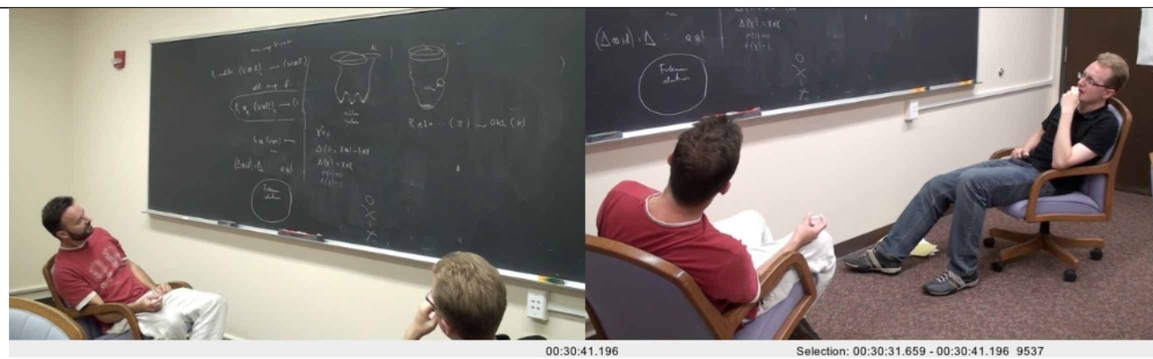
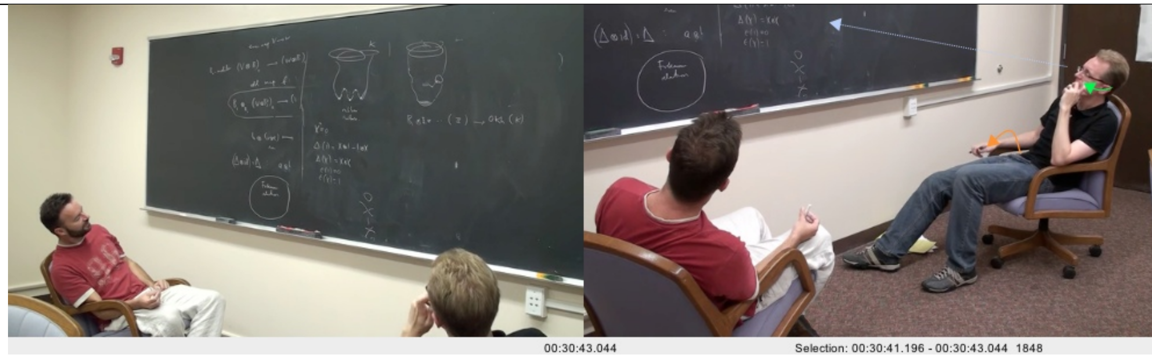
Table A.1 – “odd scalars” – *Continued from previous page*

Row	**Gesture & timestamp**	**Transcript**
33		I don’t quite
34		understand how this can be possible but [pause] assuming that the [long pause]

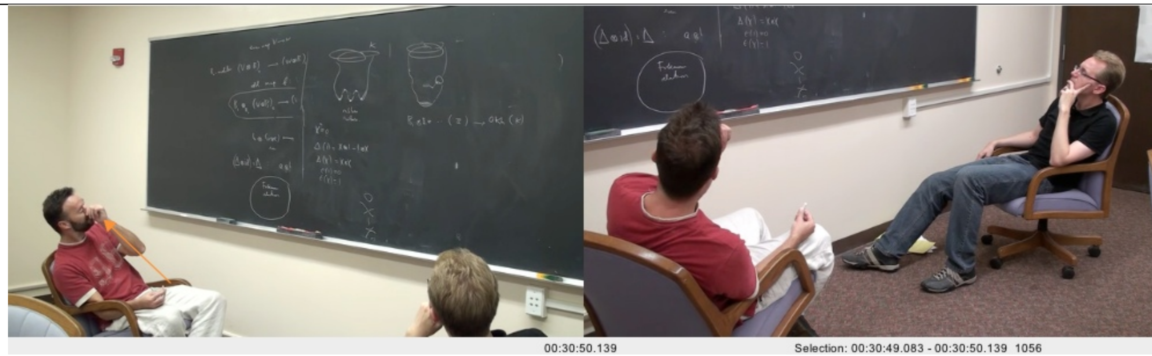
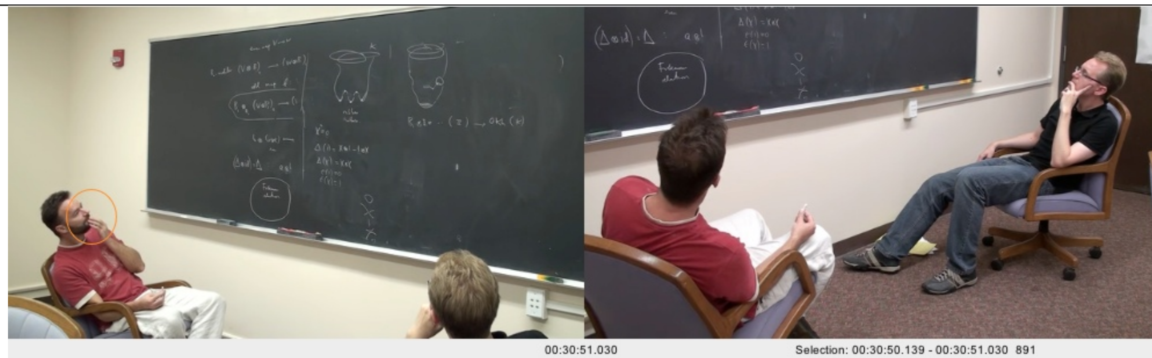
Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
35		[pause]
36		[pause]

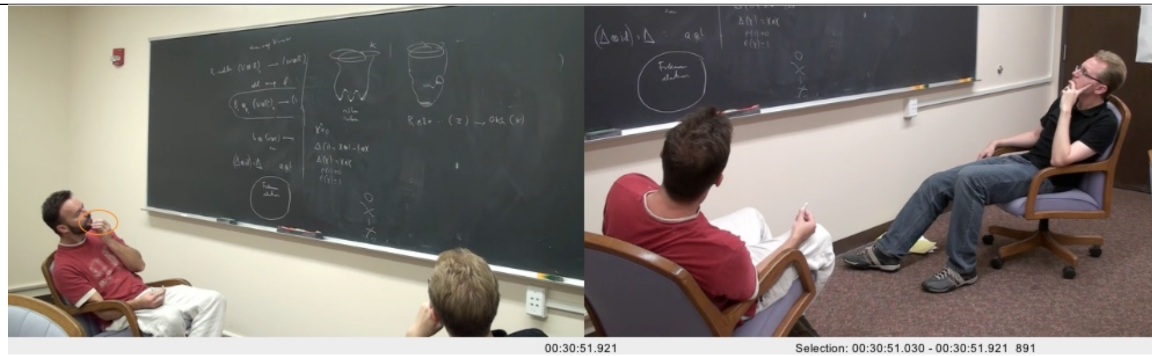
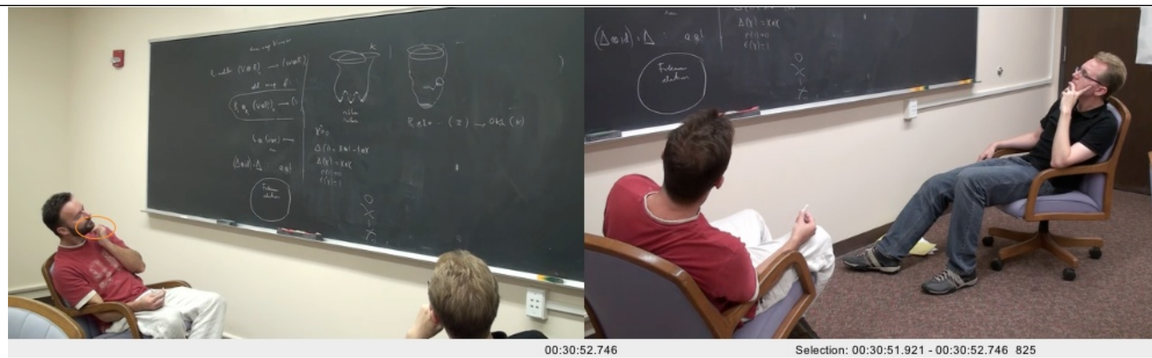
Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
37		[pause]
38		[pause]

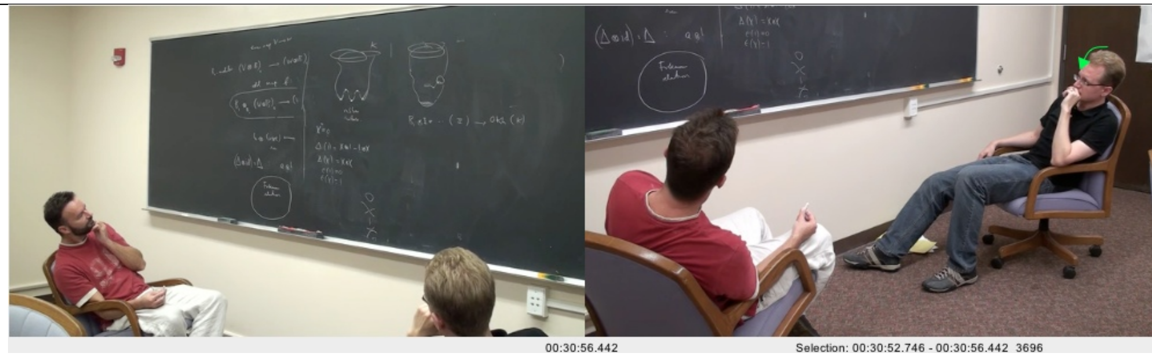
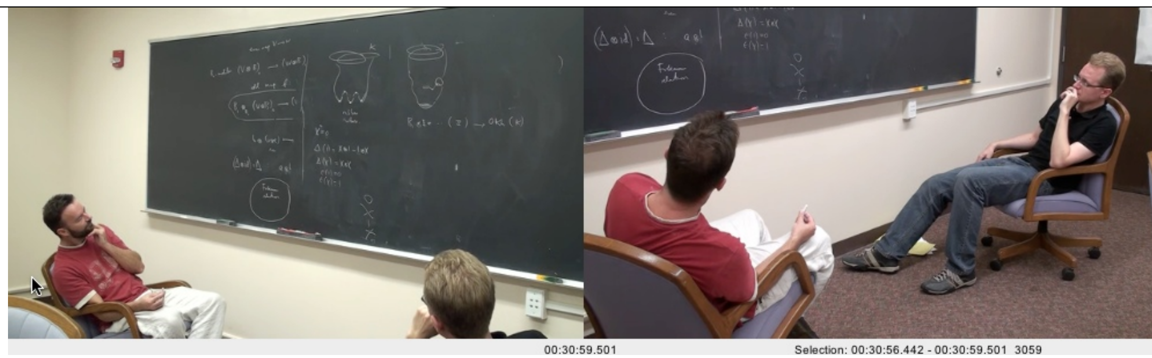
Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
39		[pause]
40		[pause]

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
41		[pause]
42		[pause]

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
43	 31:01.183	[pause]
44	 31:01.546	[pause]

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
45	 31:02.437	J: I- You know, I, I sort of
46	 31:03.295	want to argue that these

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
47	 31:03.724	just give
48	 31:03.988	the ze-

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
49	 31:04.153	-ro m-
50	 31:04.318	-ap

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
51	 31:04.615	that-
52	 31:04.780	-’s

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
53	 31:05.572	[inaudible]
54	 31:06.727	[pause]

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	**Gesture & timestamp**	**Transcript**
55	 31:07.486	[pause]
56	 31:07.981	[pause]

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
57	 31:09.103	But I don’t know whether that makes sen-
58	 31:09.598	-se; I, I

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
59	 31:10.357	think I’m, I’m worried
60	 31:11.017	that

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
61	 31:13.690	um, [pause] If we, if we
62	 31:16.000	compute the odd Khovanov homology in a sort of na-

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
63	 31:16.660	-ïve way without sti-
64	 31:17.254	-cking any odd sc-

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	**Gesture & timestamp**	**Transcript**
65	 31:17.848	-alars in to ma-
66	 31:18.607	-ke these thi-

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
67	 31:18.970	-ngs
68	 31:19.663	[pause] ma-

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
69	 31:20.719	-kes sense; we just write these th-
70	 31:21.280	-ings down and

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
71	 31:21.808	sort of f-
72	 31:28.078	-orget about the [pause] uh [pause]

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
73	 31:29.002	[pause]
74	 31:30.322	[pause]

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
75	 31:30.784	[pause]
76	 31:33.424	[pause]

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
77	 31:34.282	Forget about the grading; well the-
78	 31:35.173	-n, then

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
79	 31:35.899	um
80	 31:40.750	[pause]

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
81	 31:42.070	[pause]
82	 31:42.928	[pause]

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
83	 31:45.667	[pause]
84	 31:46.195	[pause]

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
85	 31:46.954	I ju- Um, I jus-
86	 31:47.317	-t afraid that

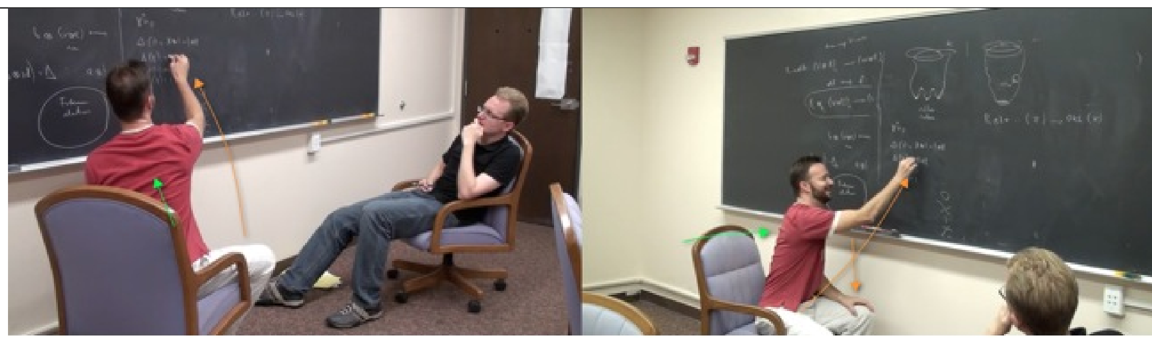
Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	**Gesture & timestamp**	**Transcript**
87	 31:48.241	it’s a bit inconsistent; I
88	 31:48.736	mean I think

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
89	 31:49.396	[pause] so-
90	 31:50.848	somehow deeply

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
91	 31:51.706	in here there are odd scal-
92	 31:52.234	-ars which are ca-

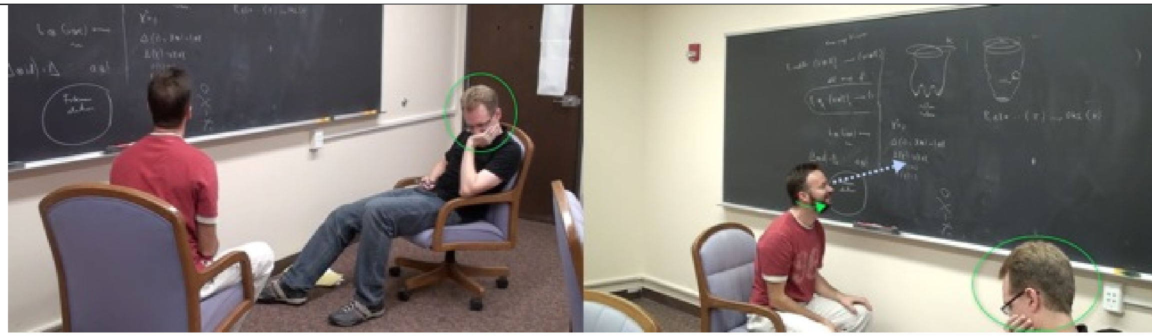
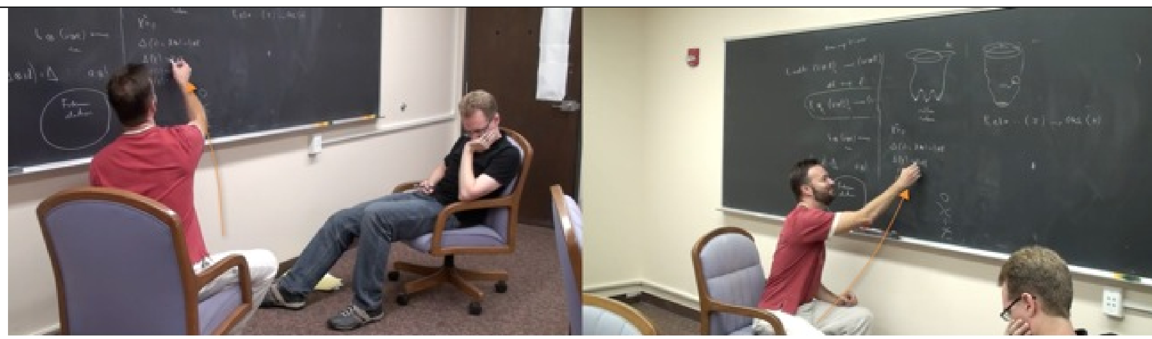
Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
93	 31:52.564	-lled “one”
94	 31:54.676	[laughs] It would sort of mean that the-

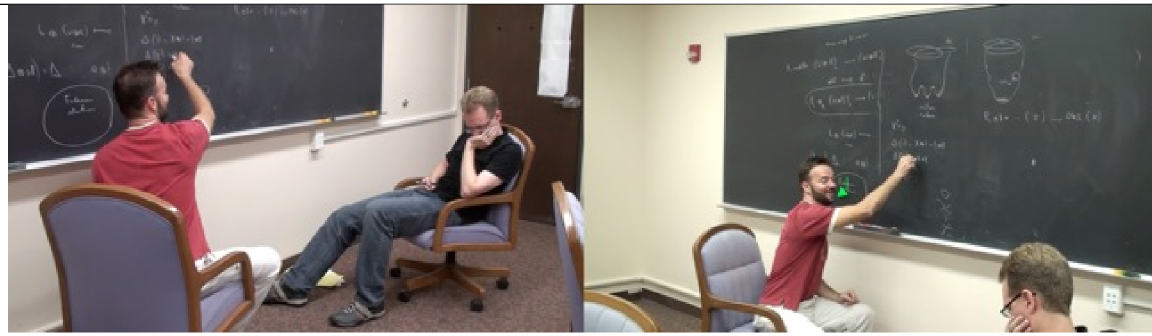
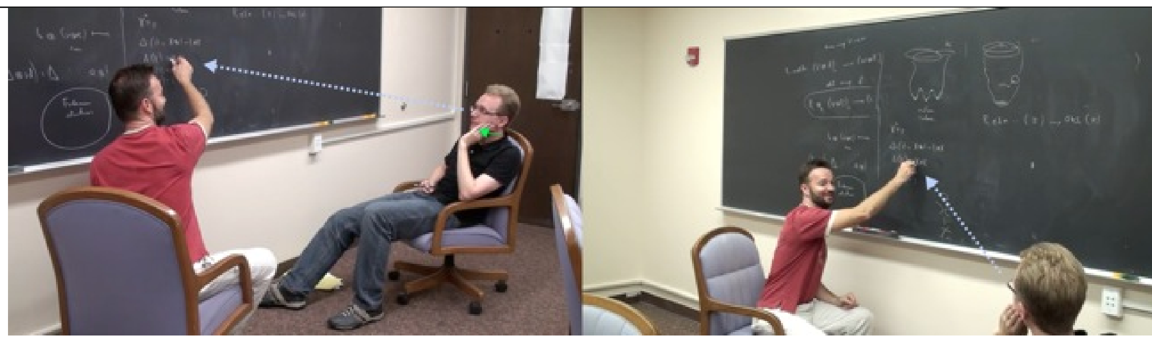
Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
95	 31:55.468	-y, i- it’s
96	 31:56.854	[pause] There probably should

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
97	 31:57.382	be a bunch of
98	 31:57.910	“A”s and “B”s a-

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
99	 31:59.197	-nd things here in the, in the cons-
100	 31:59.560	-truction

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
101	 32:00.286	of the whole Kho-vanov
102	 32:00.583	compl-

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
103	 32:01.210	-ex every time we
104	 32:01.441	see

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
105	 32:01.771	a
106	 32:02.332	a spli-

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
107	 32:03.586	-t, we really ought to h-
108	 32:04.411	-ave- We, we we ha-

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
109	 32:05.137	-ve this suppl-
110	 32:05.335	-y o-

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
111	 32:05.698	-f extra
112	 32:06.028	odd v-

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
113	 32:06.127	-ari-
114	 32:06.292	-ables

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
115	 32:06.457	b-
116	 32:06.655	-eing

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
117	 32:06.952	f-
118	 32:07.711	-ed in B: Yeah.

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
119	 32:08.206	J: to countera-
120	 32:10.945	-ct those. B: Uhu. J: So

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
121	 32:11.209	perhap-
122	 32:12.100	-s the correct way to thi-

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
123	 32:13.189	-nk about it is that the whole odd
124	 32:14.080	complex itself

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
125	 32:15.070	is defined over so-
126	 32:16.258	-rt of a, a tensor pr-

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
127	 32:17.149	-oduct of some inde-
128	 32:17.578	-terminant n-

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
129	 32:17.941	-umber of
130	 32:18.304	extr-

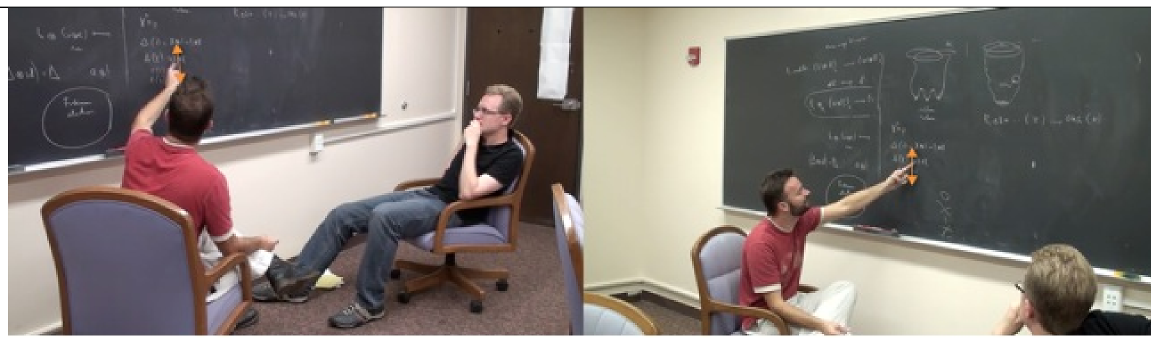
Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
131	 32:19.228	-a [pause]
132	 32:21.010	“B” “1”s there

Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
133	 32:22.297	and they really should be things he-
134	 32:23.650	-re, [inaudible] all the way through.

Continued on next page

Table A.1 – "odd scalars" – *Continued from previous page*

Row	Gesture & timestamp	Transcript
135	 32:25.036	[pause]
136	 32:25.630	[pause]

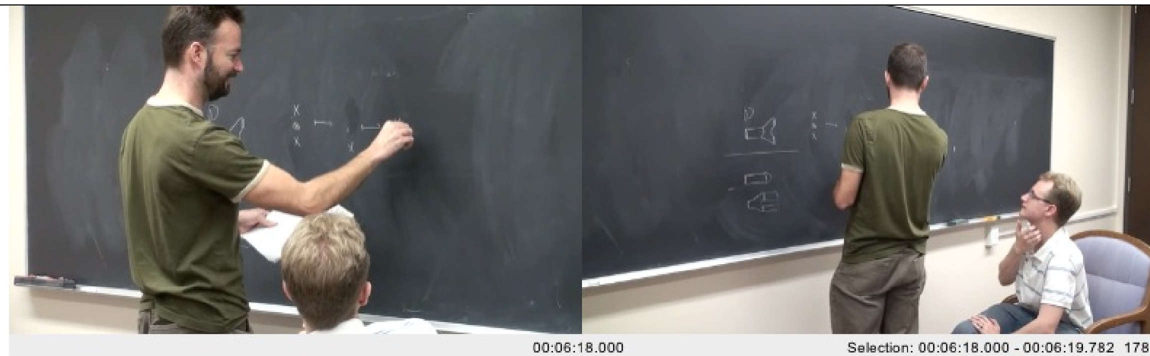
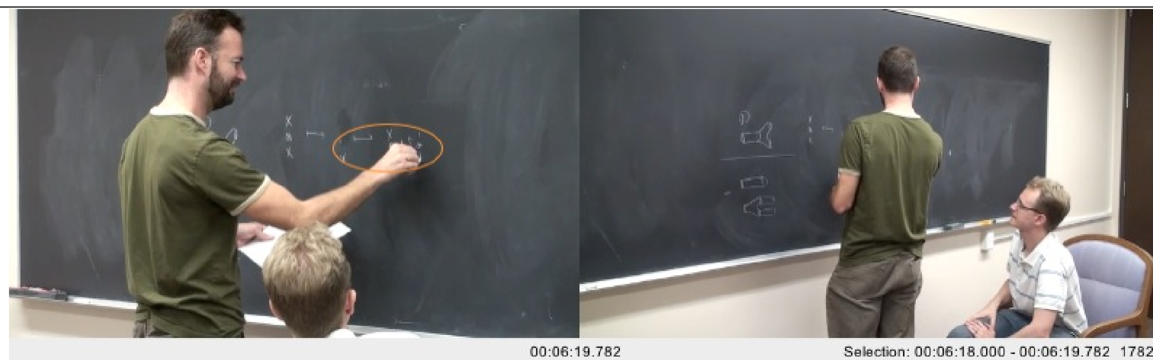
Continued on next page

Table A.1 – “odd scalars” – *Continued from previous page*

Row	Gesture & timestamp	Transcript
137	 32:27.429	[pause]

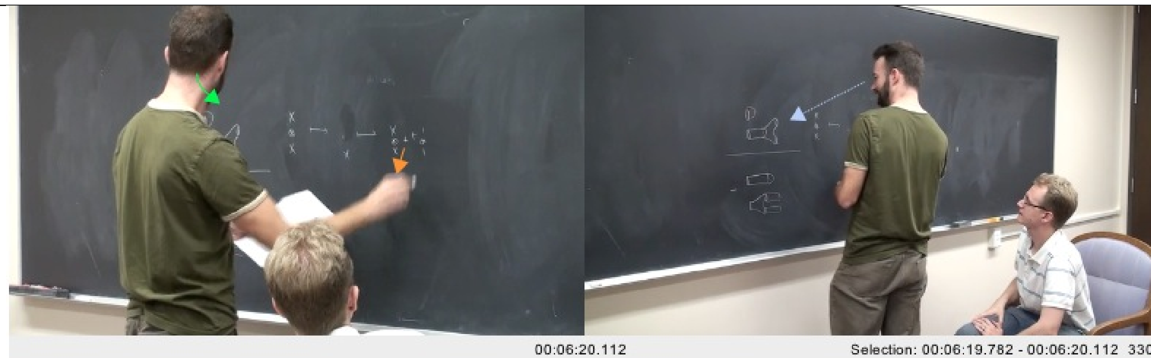
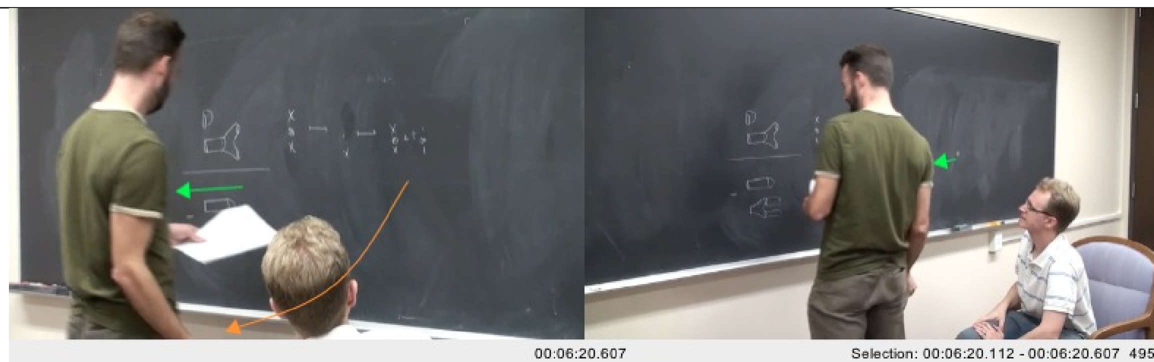
A.2 Joseph & Bill — tubes

Table A.2: Joseph & Bill — tubes

Row	Gesture & timestamp	Transcript
1		[N/A]
2		J: Right? Plus t one tensor one.

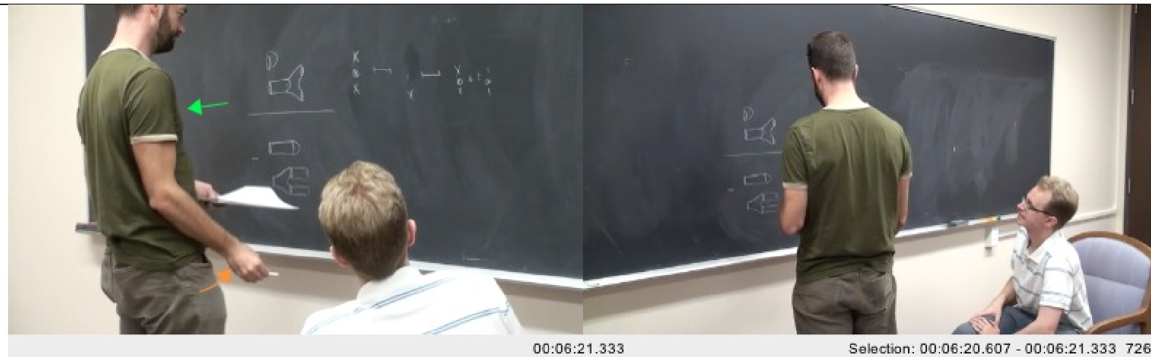
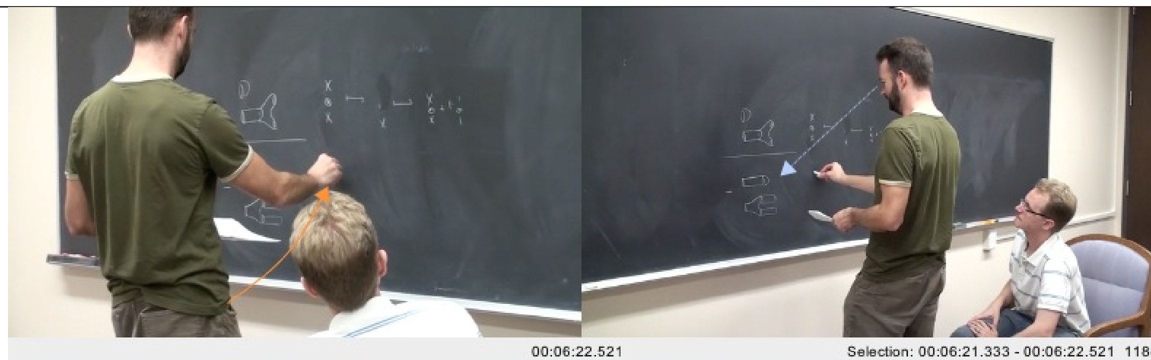
Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
3		B: Sure.
4		[pause]

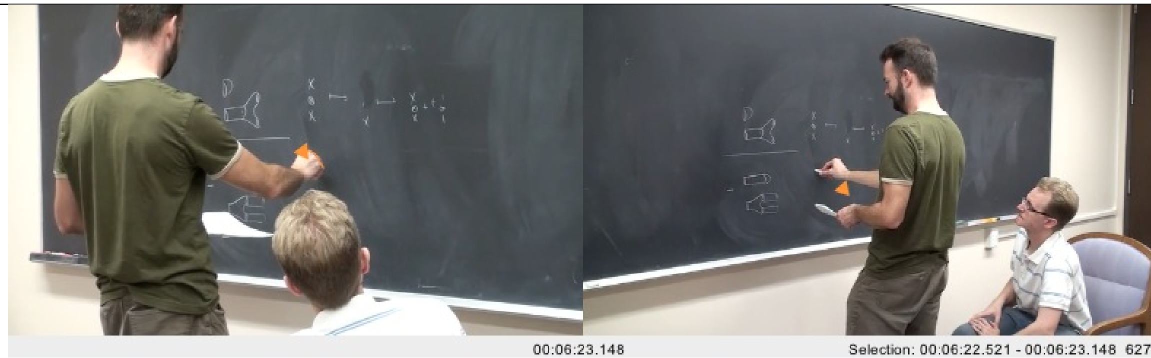
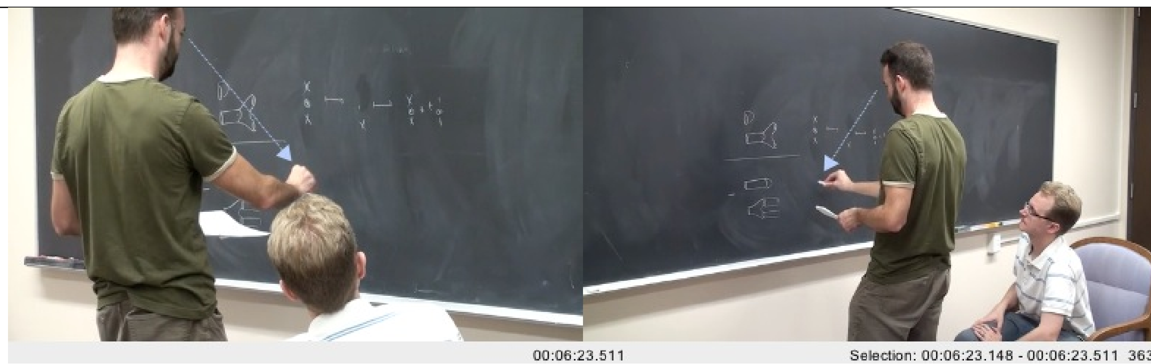
Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
5		J: Um,
6		[pause] whereas this

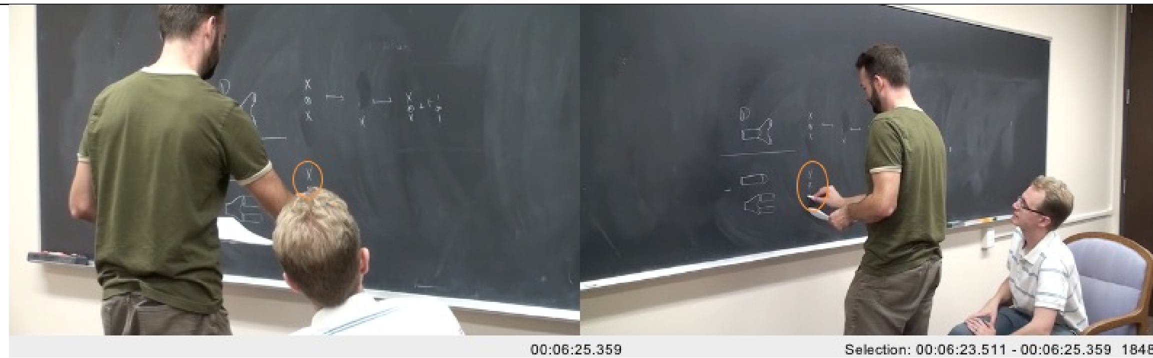
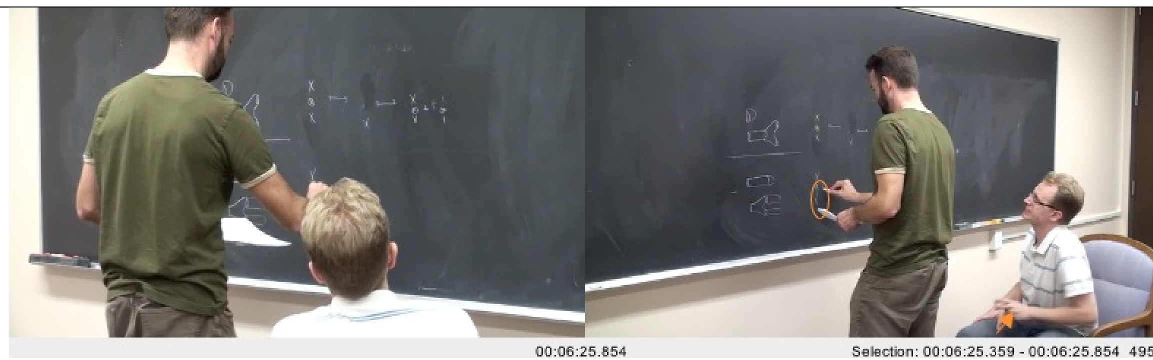
Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
7		one. . .
8		B: Mhm.

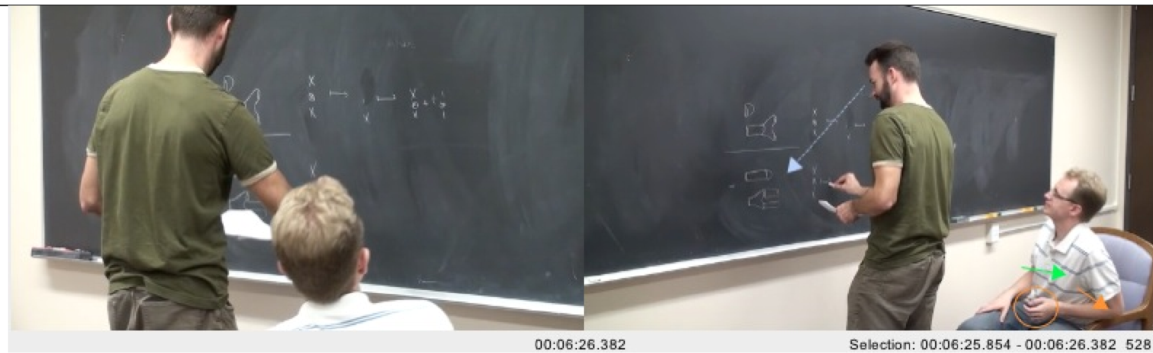
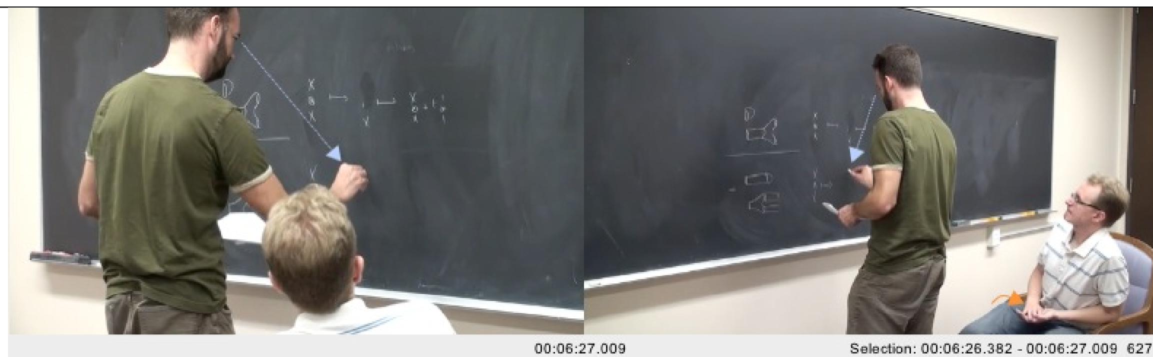
Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
9		[pause]
10		[pause]

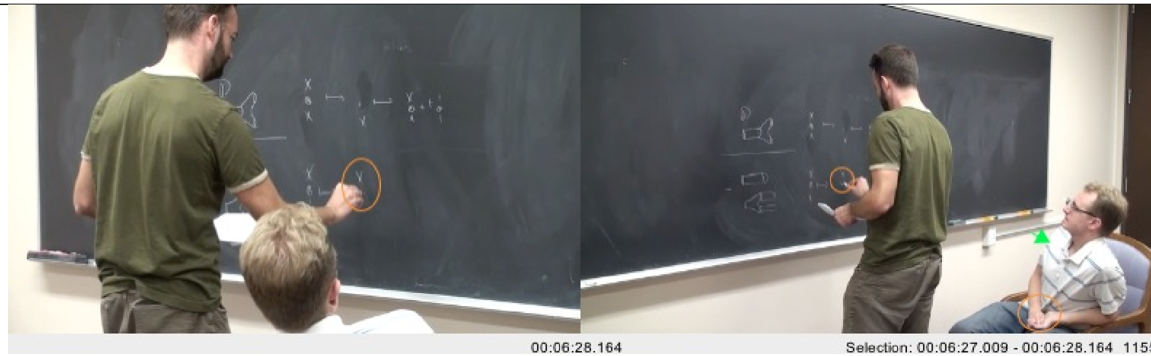
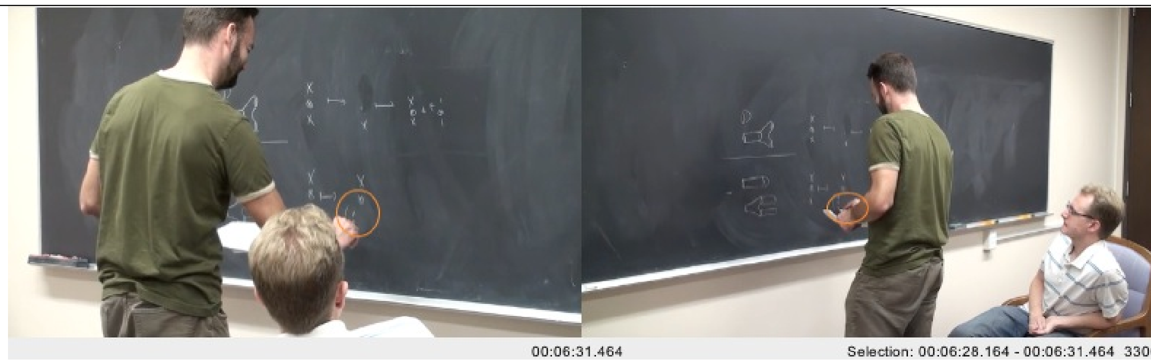
Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
11		J: We s-
12		-eem to...

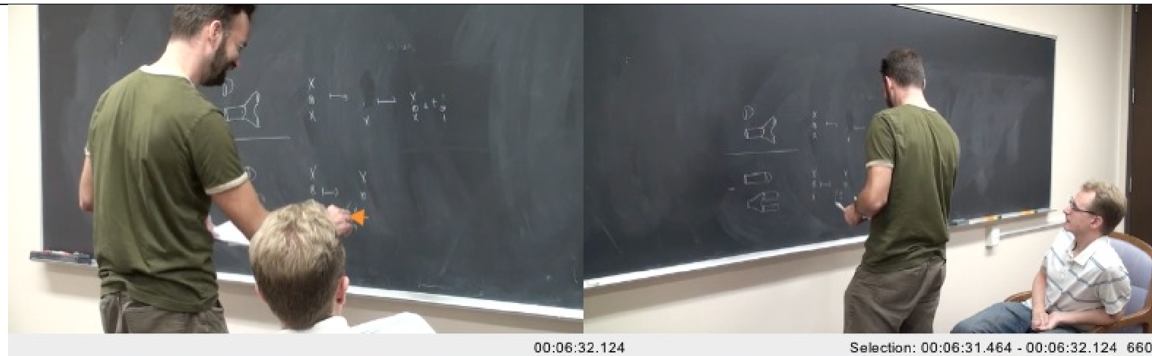
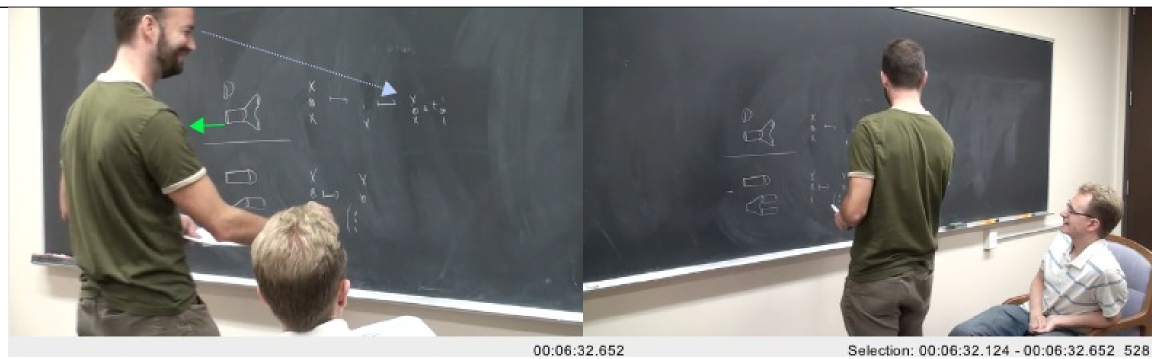
Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
13		[pause]
14		'N we have x tensor x. [one-breath laugh]

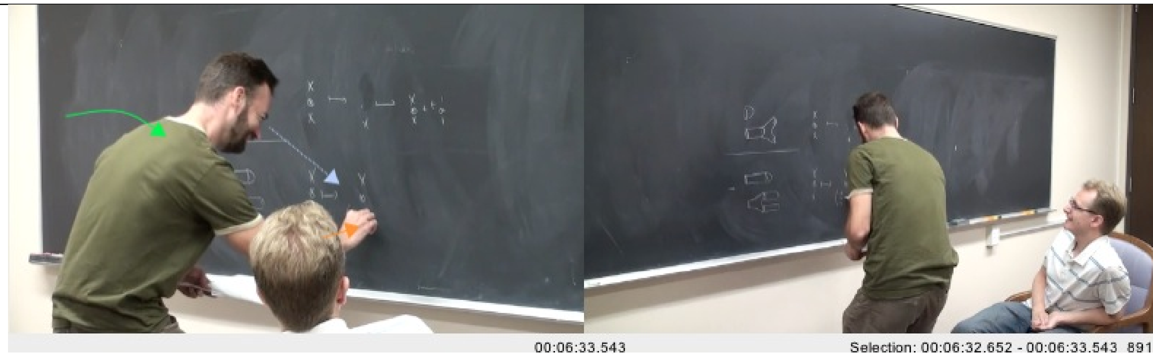
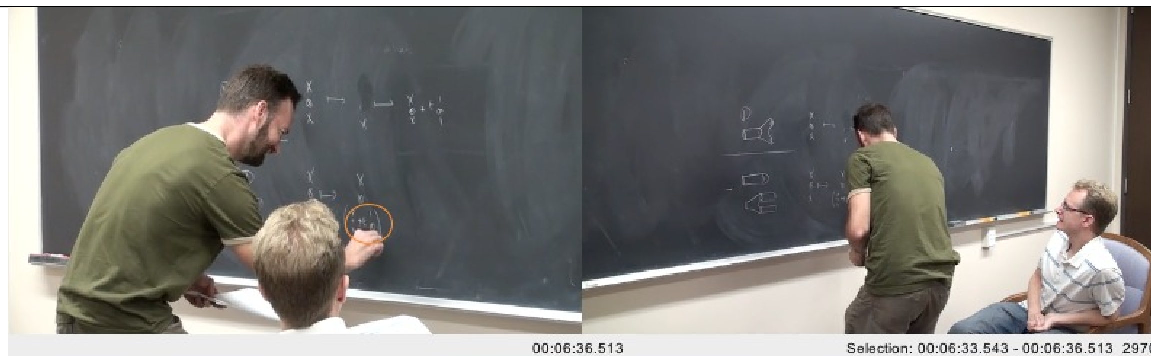
Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
15		I noti-
16		-ce I don't un-

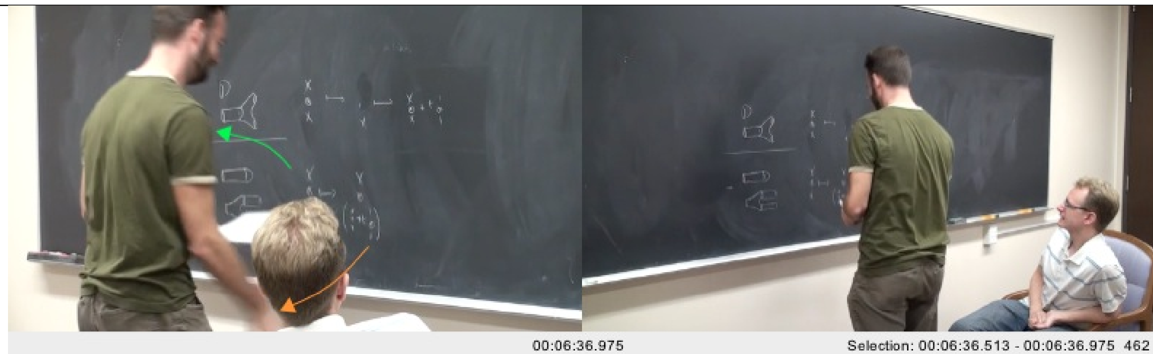
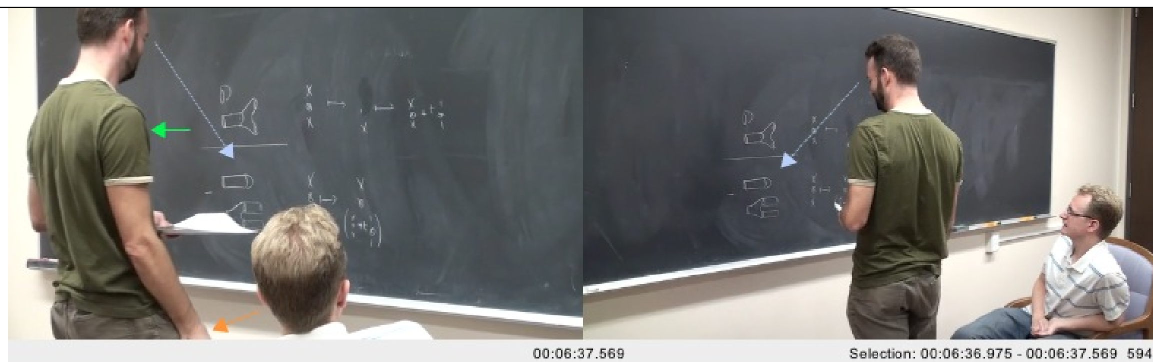
Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
17		-derstand.
18		B: [chuckles] J: Uh.

Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
19		[pause]
20		[pause]

Continued on next page

Table A.2 – tubes – *Continued from previous page*

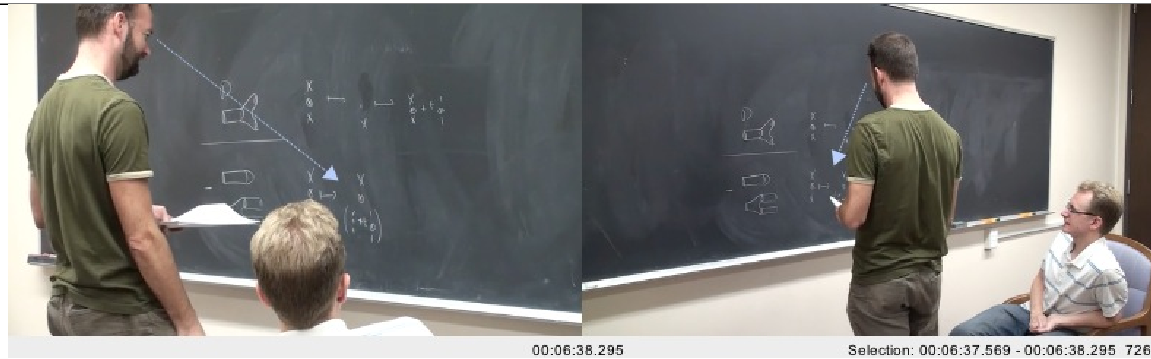
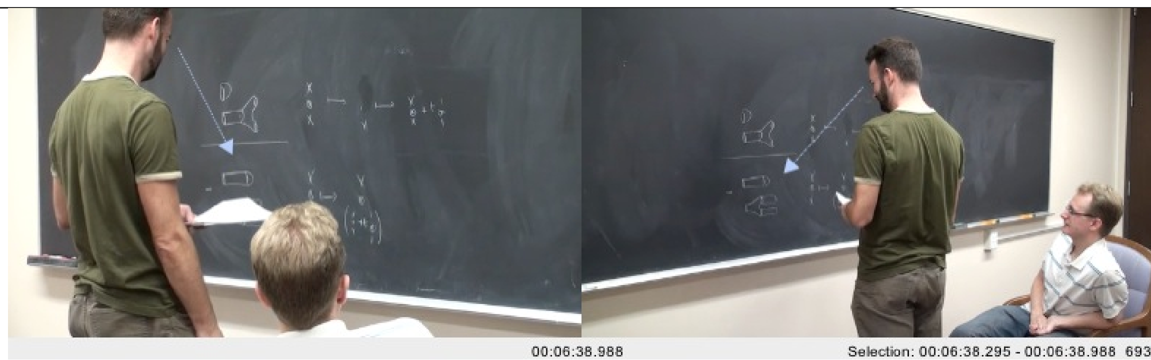
Row	Gesture & timestamp	Transcript
21		B: Yes. J: U-
22		-h [pause]

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Table A.2 – tubes – *Continued from previous page*

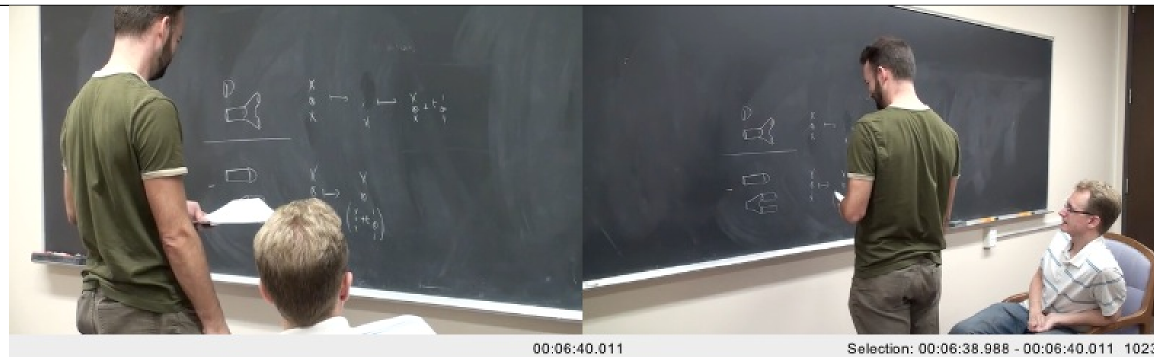
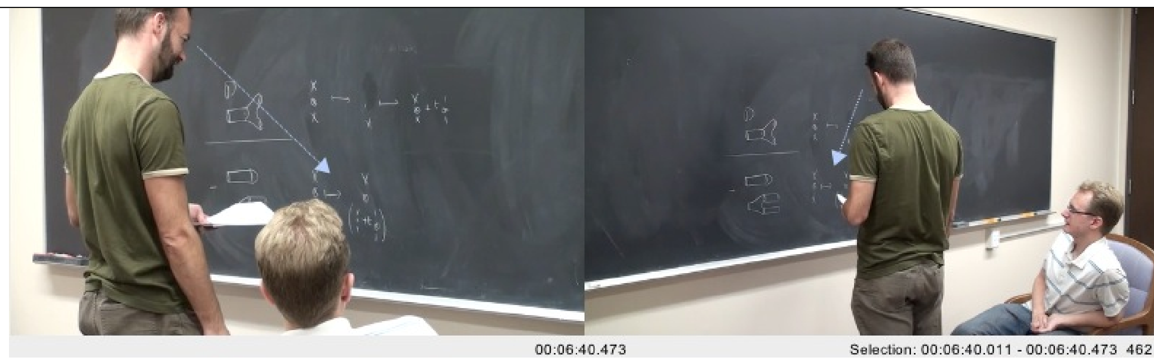
Row	Gesture & timestamp	Transcript
23		[pause]
24		[pause]

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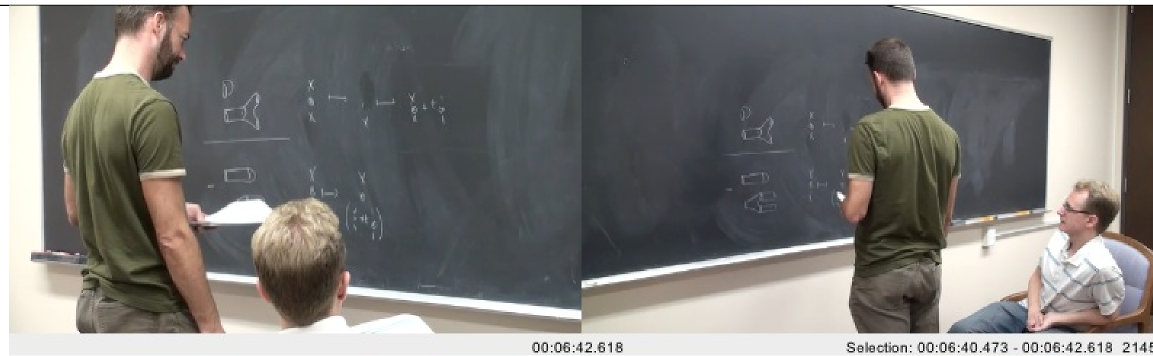
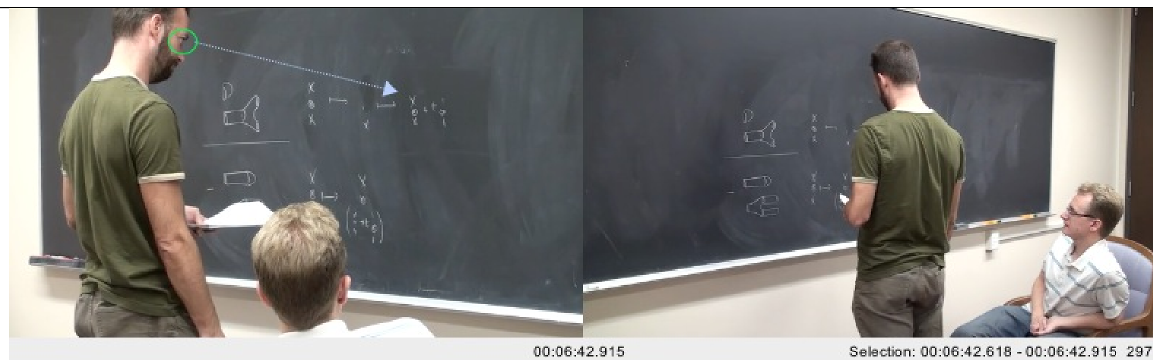
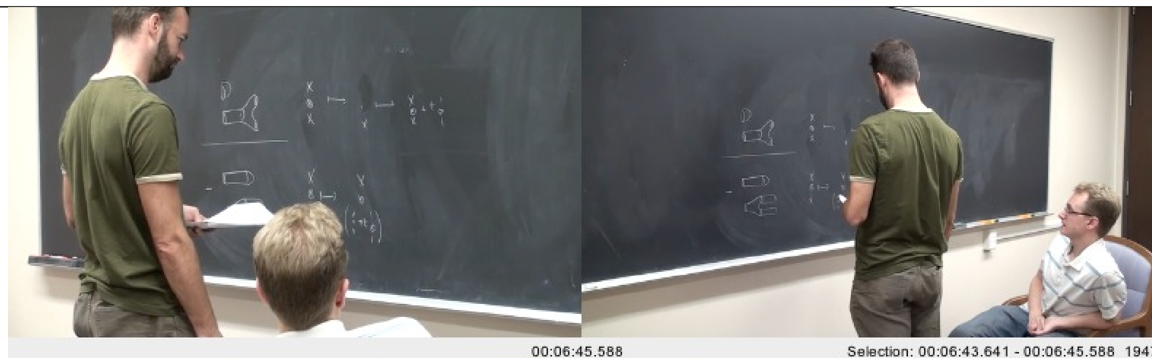
Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
25		And then [pause]
26		[pause]

Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
27		B: Cap 'em
28		out. [pause]

Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
29		[pause]
30		[pause]

Continued on next page

Table A.2 – tubes – *Continued from previous page*

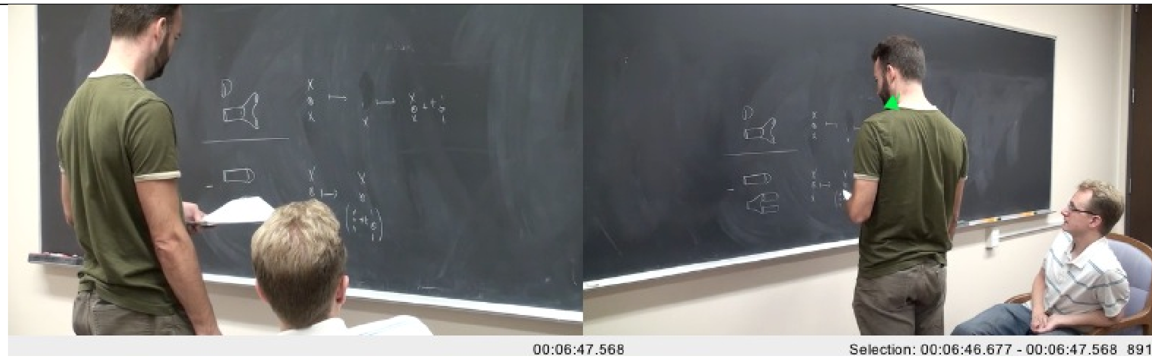
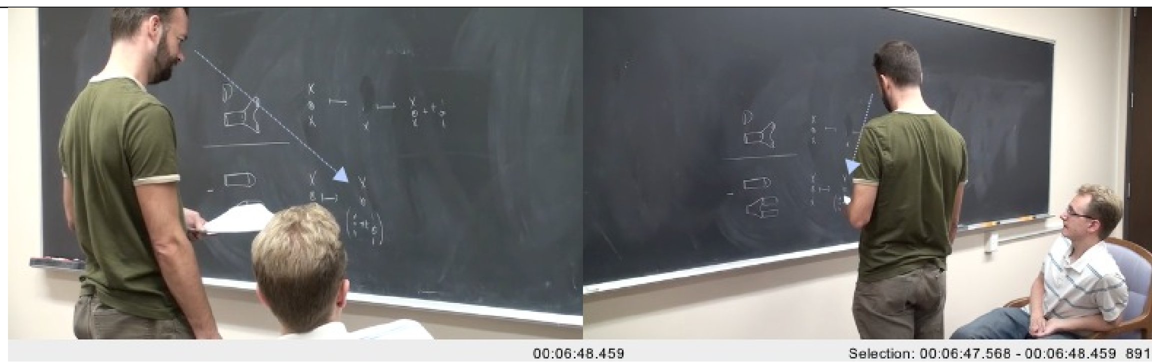
Row	Gesture & timestamp	Transcript
31		J: So why is this m-
32		-inus at all?

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Table A.2 – tubes – *Continued from previous page*

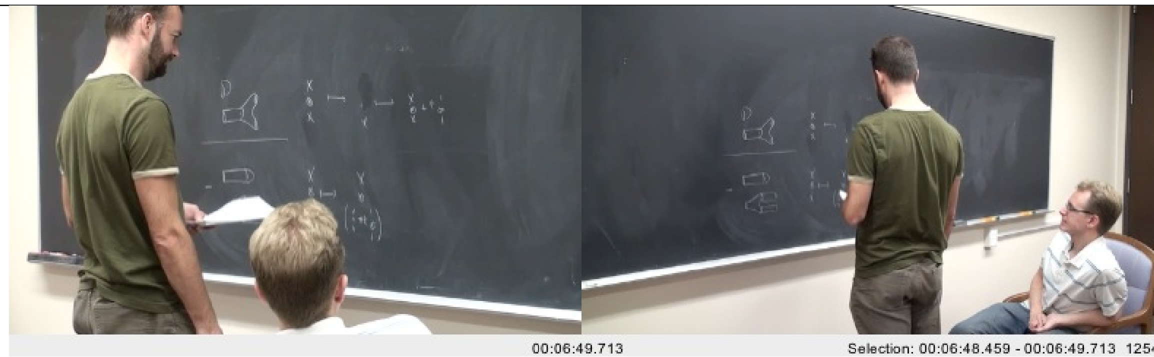
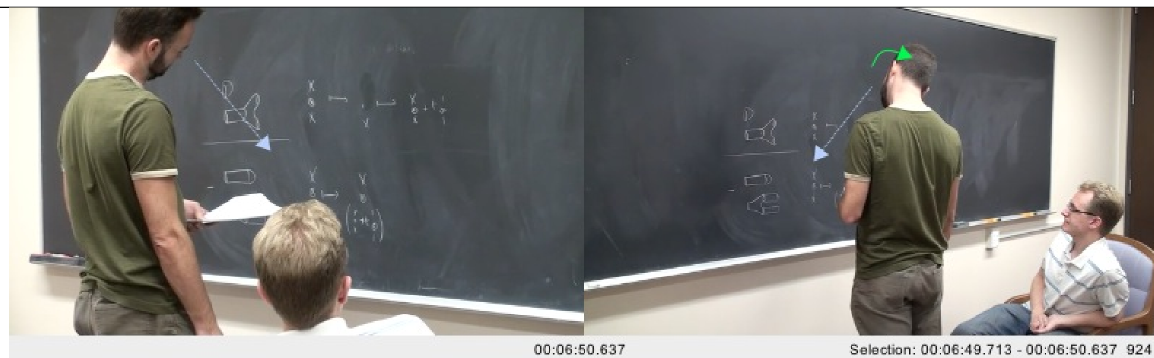
Row	Gesture & timestamp	Transcript
33		[pause]
34		[pause]

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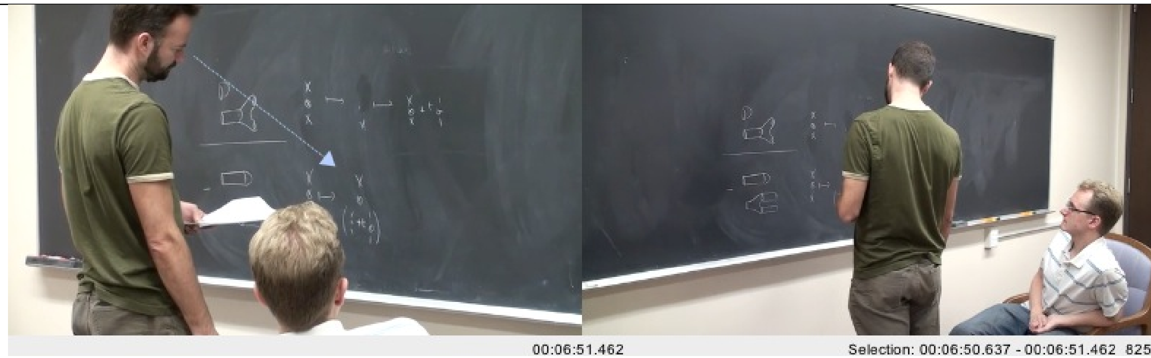
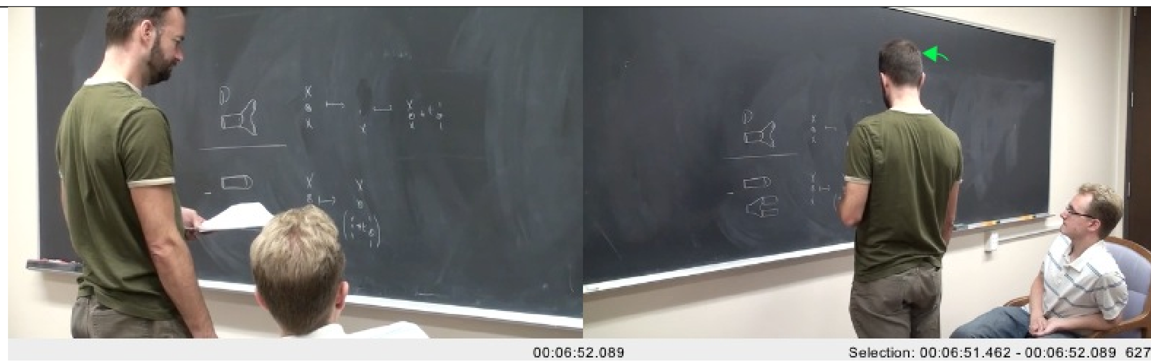
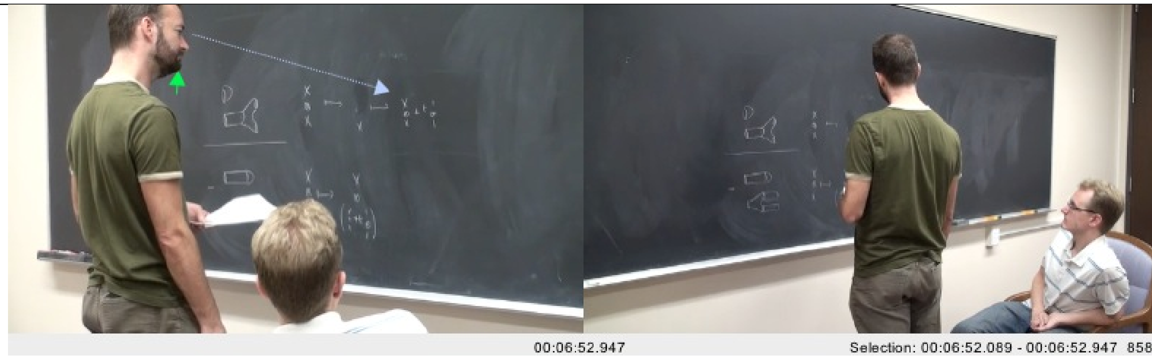
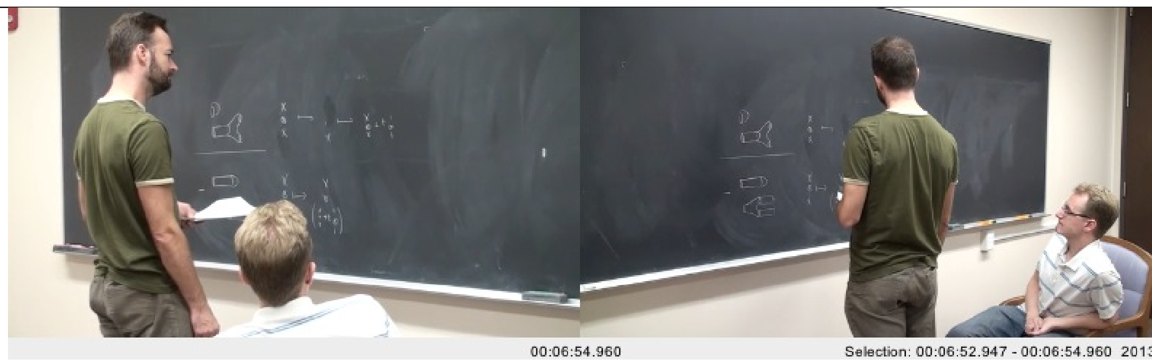
Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
35		[pause]
36		[pause]

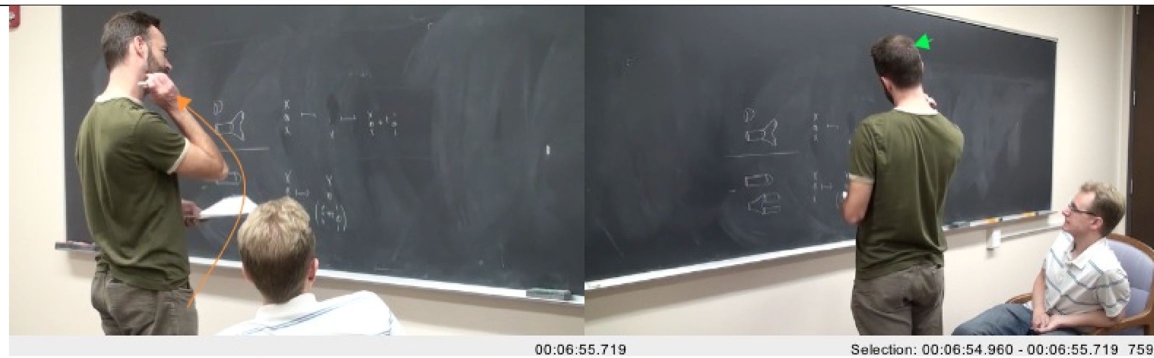
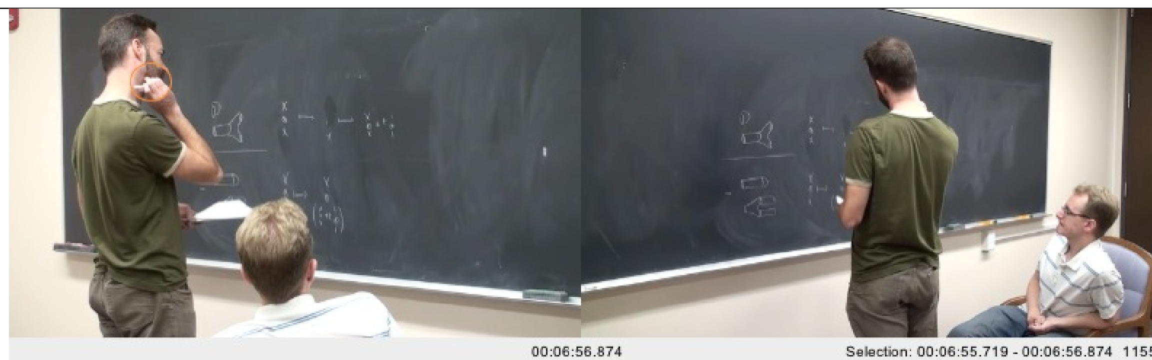
Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
37		[pause]
38		[pause]

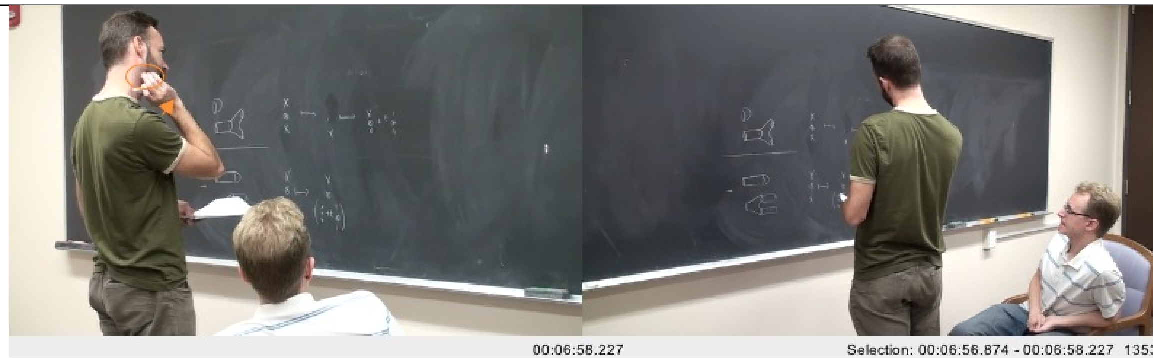
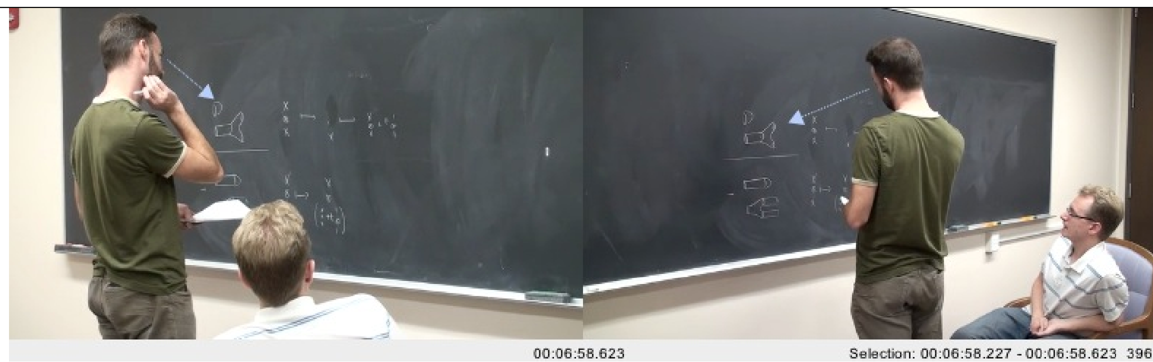
Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
39		[pause]
40		[pause]

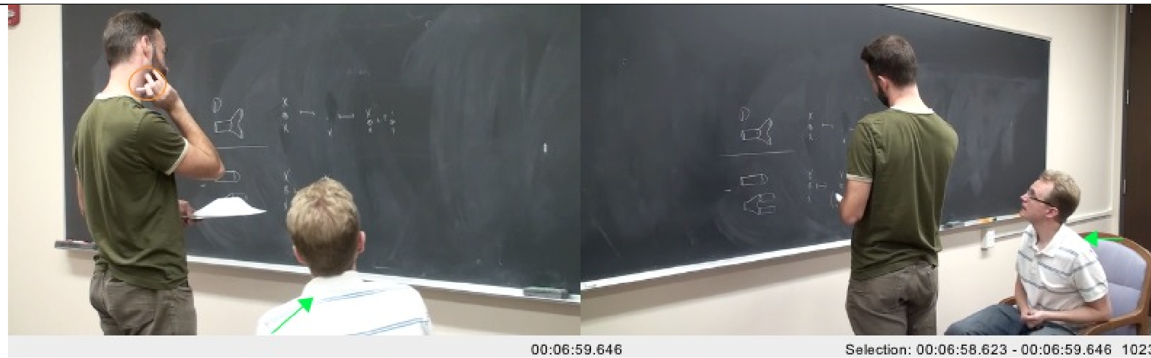
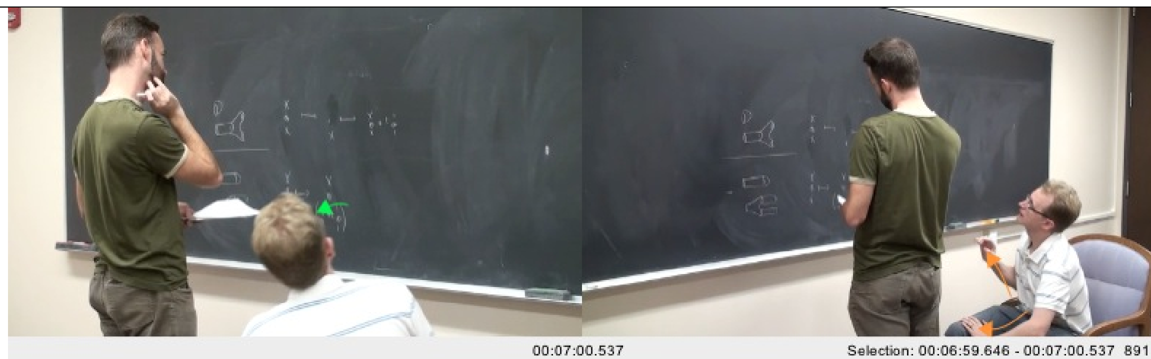
Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
41		[pause]
42		[pause]

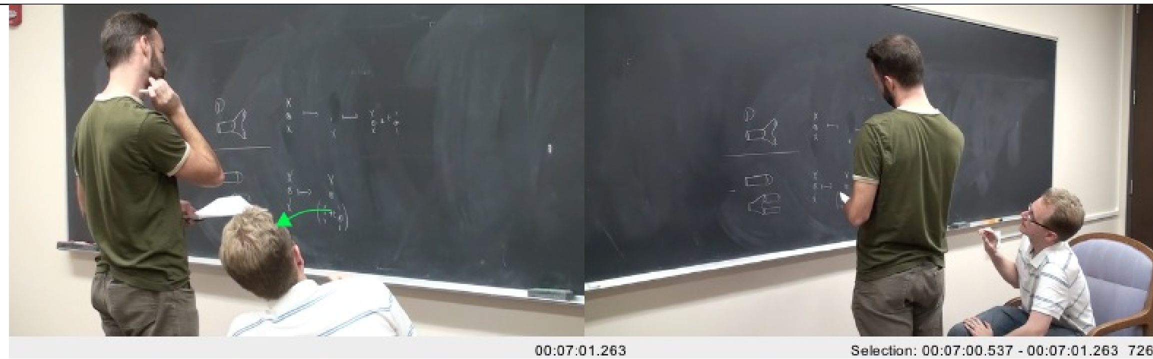
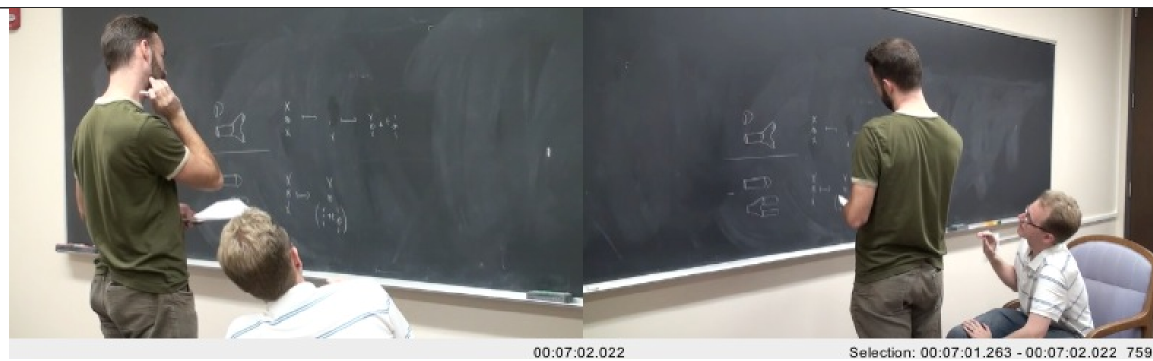
Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
43		[pause]
44		[pause]

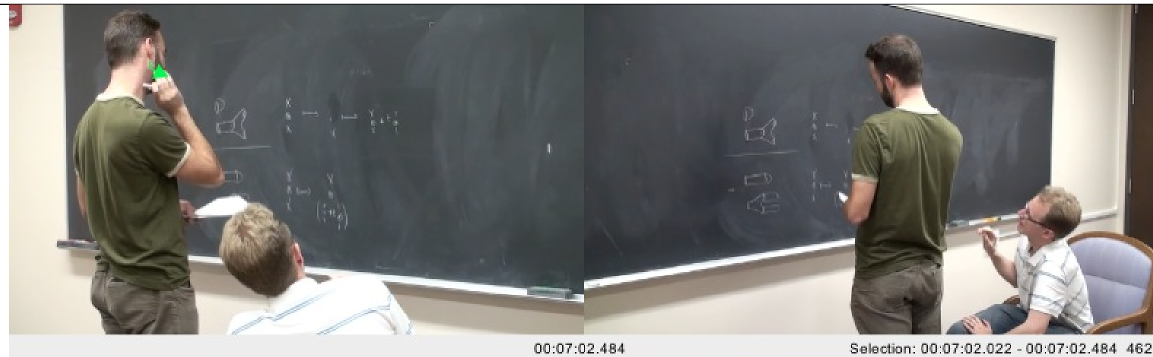
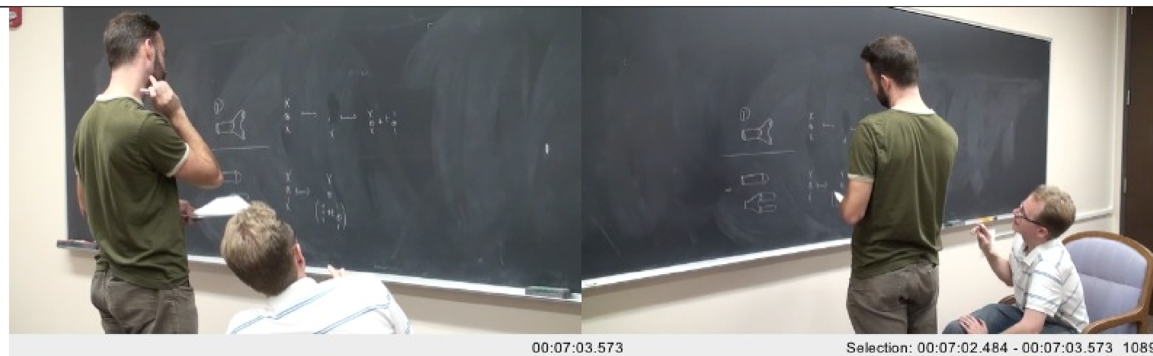
Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
45		[pause]
46		[pause]

Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
47		[pause]
48		[pause]

Continued on next page

Table A.2 – tubes – *Continued from previous page*

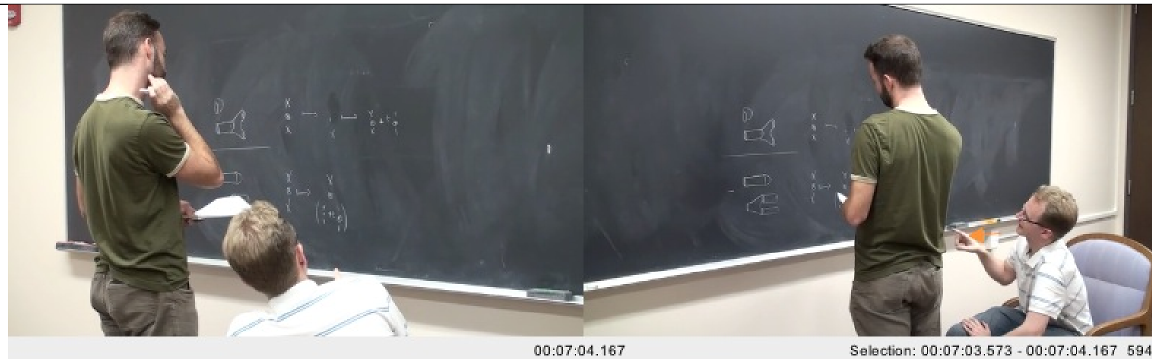
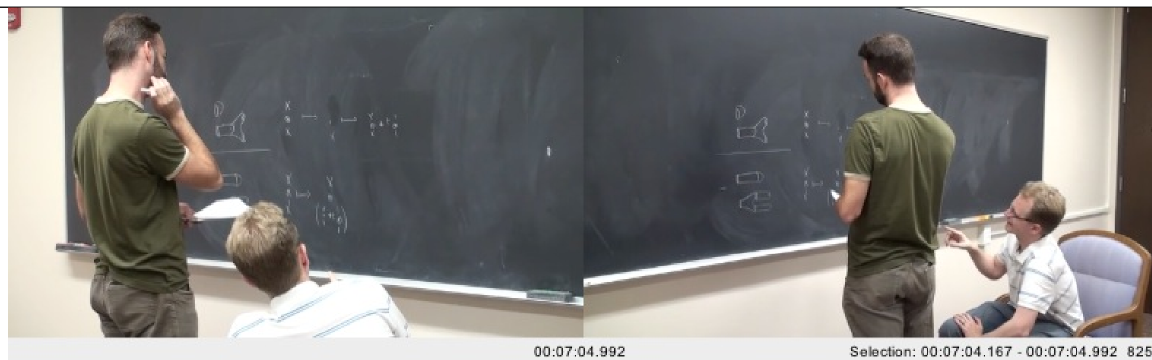
Row	Gesture & timestamp	Transcript
49		B: Yeah, so I think
50		where we’re losing

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Table A.2 – tubes – *Continued from previous page*

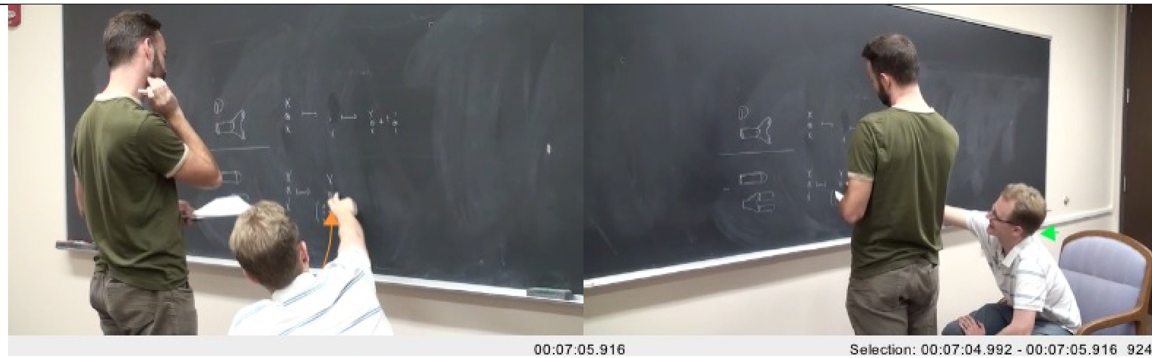
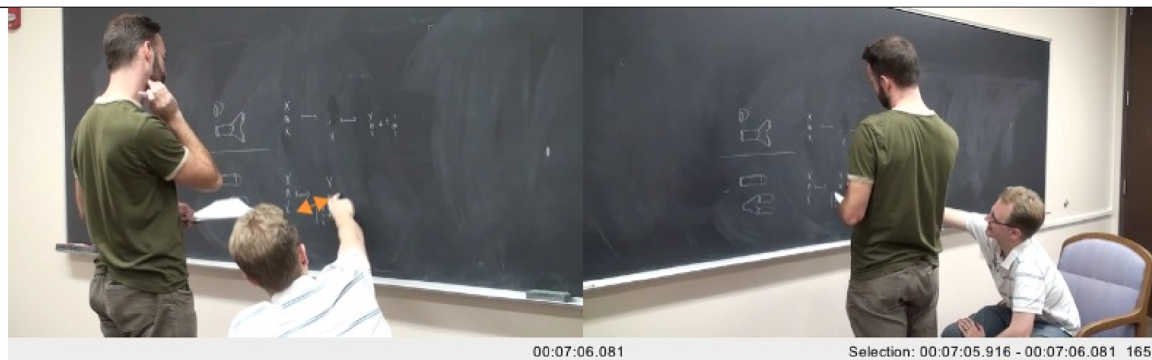
Row	Gesture & timestamp	Transcript
51		something right in these
52		is

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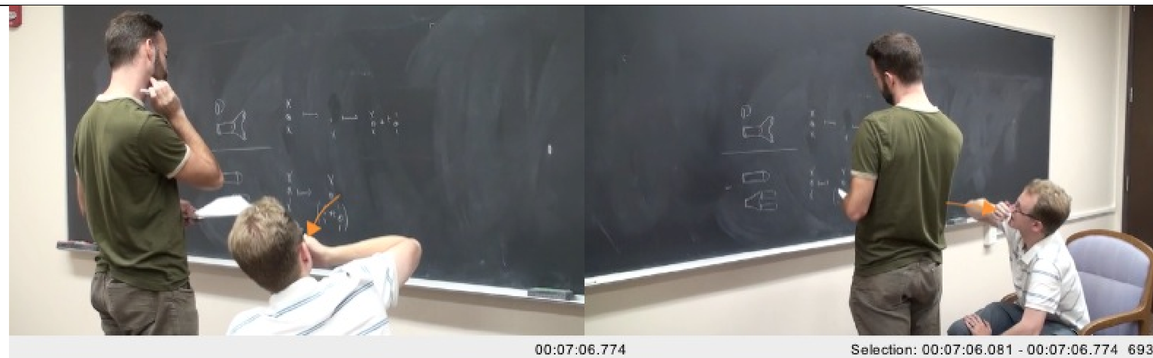
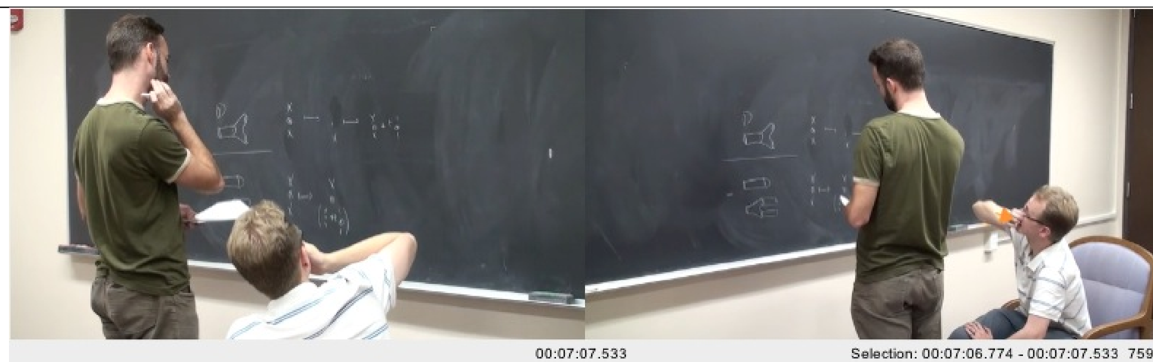
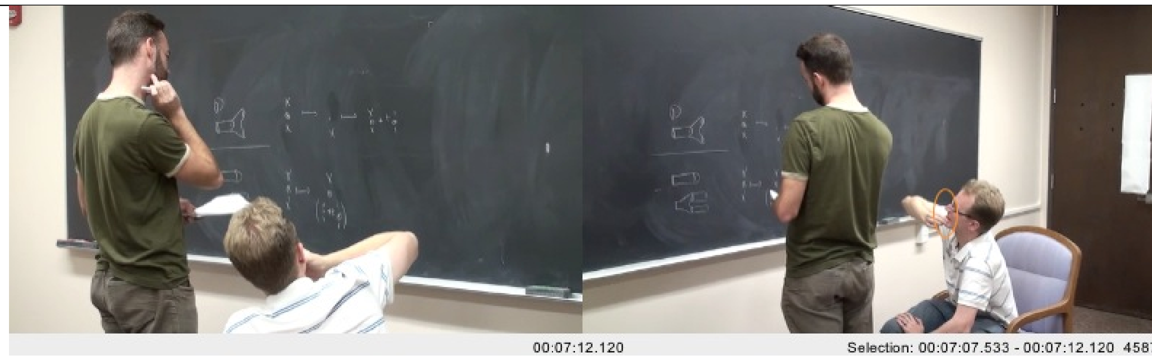
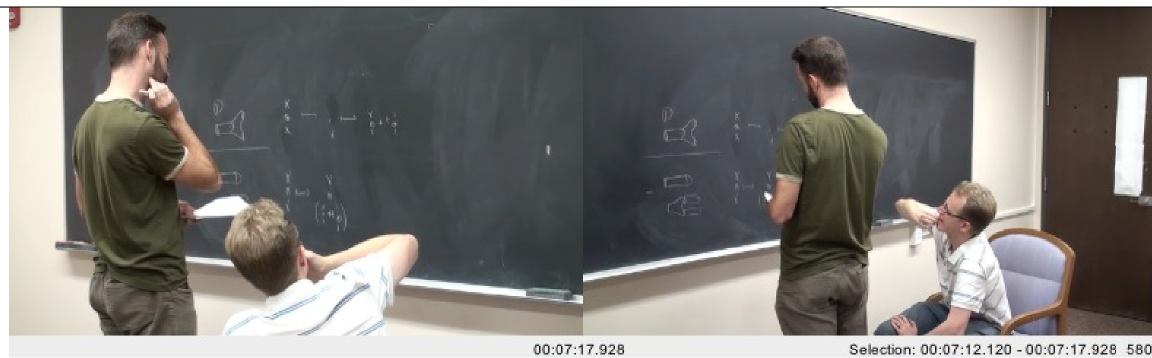
Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
53		the tensors.
54		Right?

Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
55		[pause]
56		[pause]

Continued on next page

Table A.2 – tubes – *Continued from previous page*

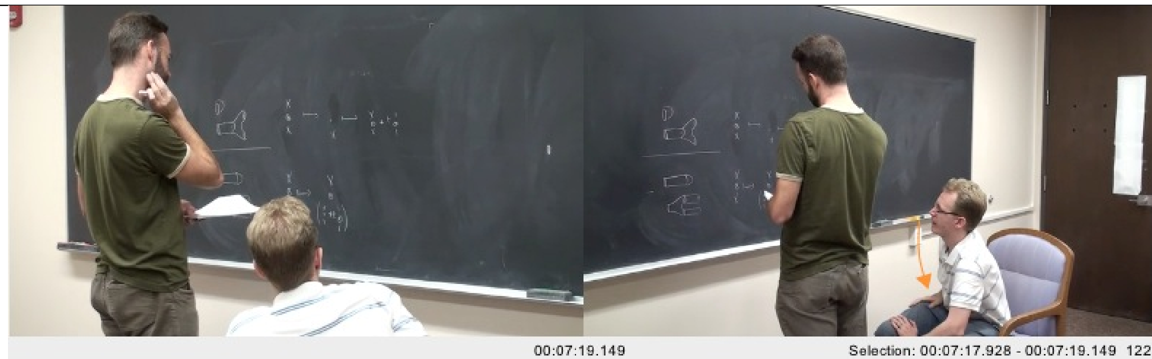
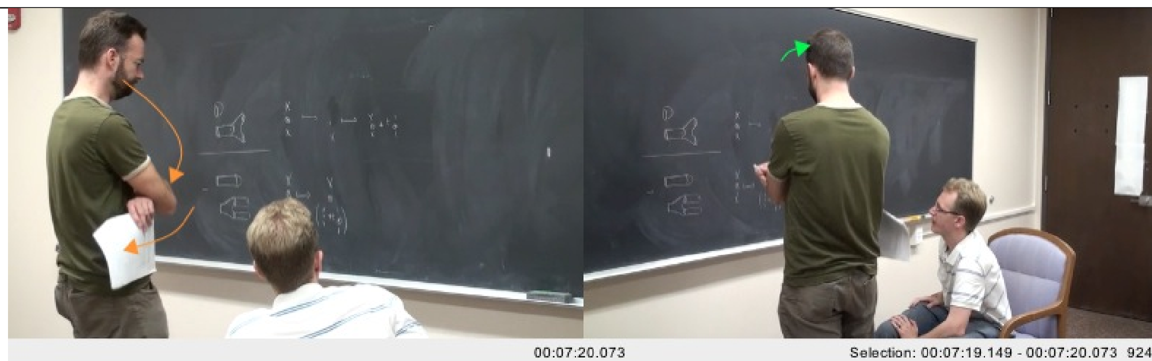
Row	Gesture & timestamp	Transcript
57		[pause]
58		J: [pause] Hm!

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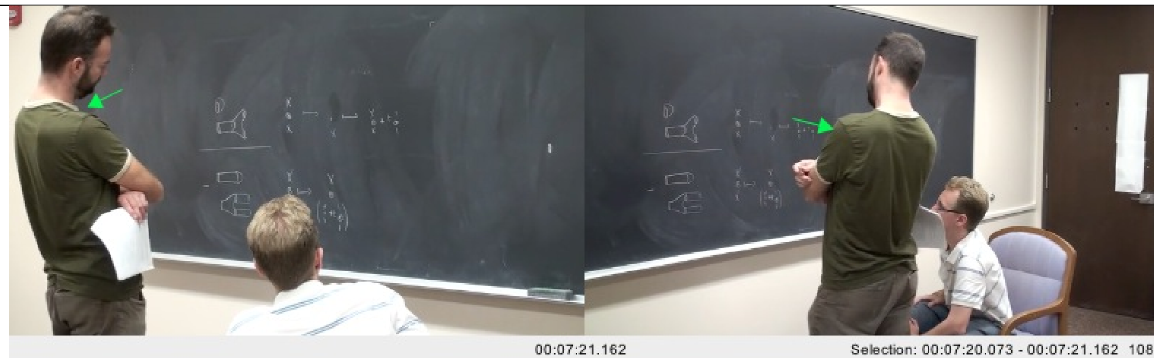
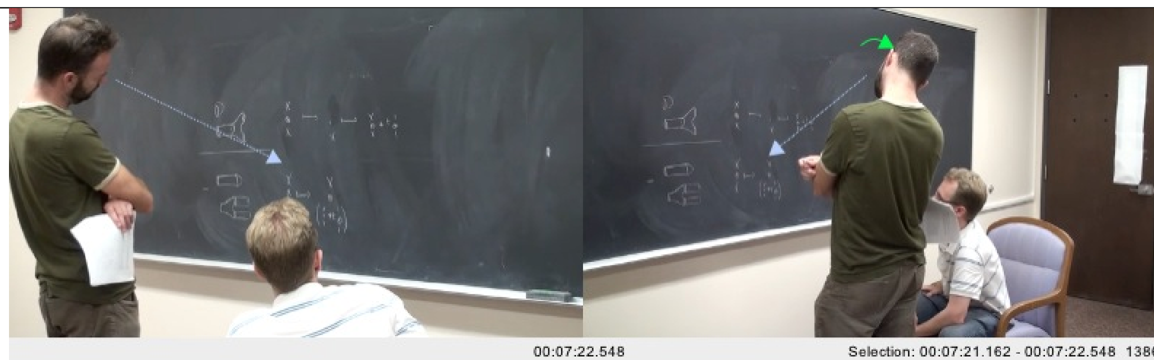
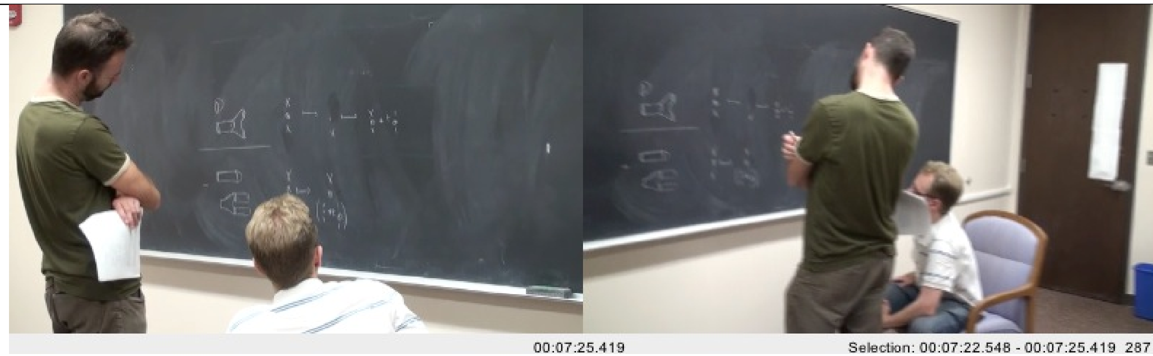
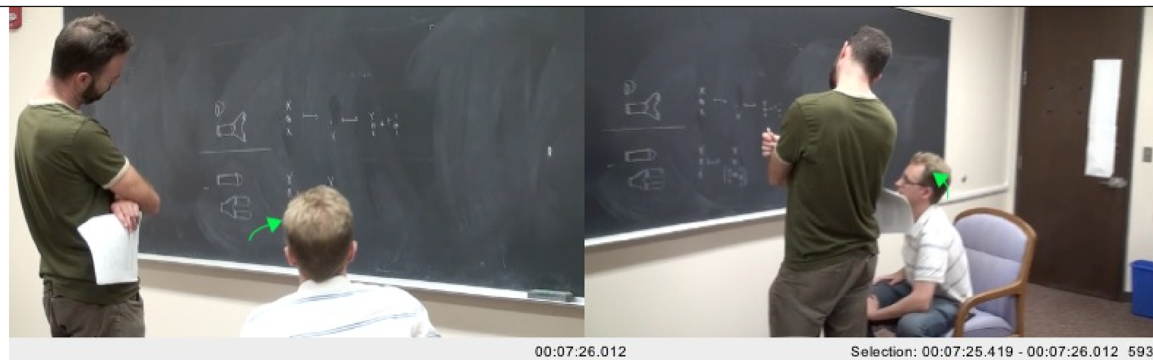
Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
59		[pause]
60		[pause]

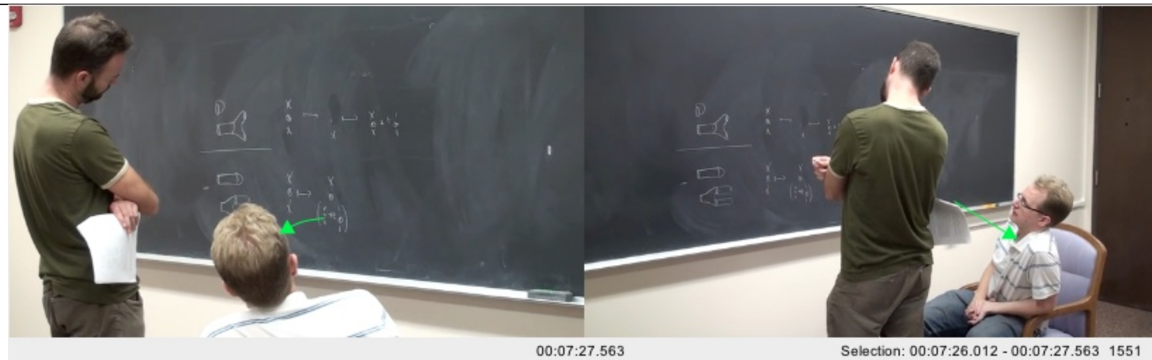
Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
61		[pause]
62		[pause]

Continued on next page

Table A.2 – tubes – *Continued from previous page*

Row	Gesture & timestamp	Transcript
63		[pause]

A.3 Matt & Bart — Bart pushed back

Table A.3: Matt & Bart — Bart pushed back

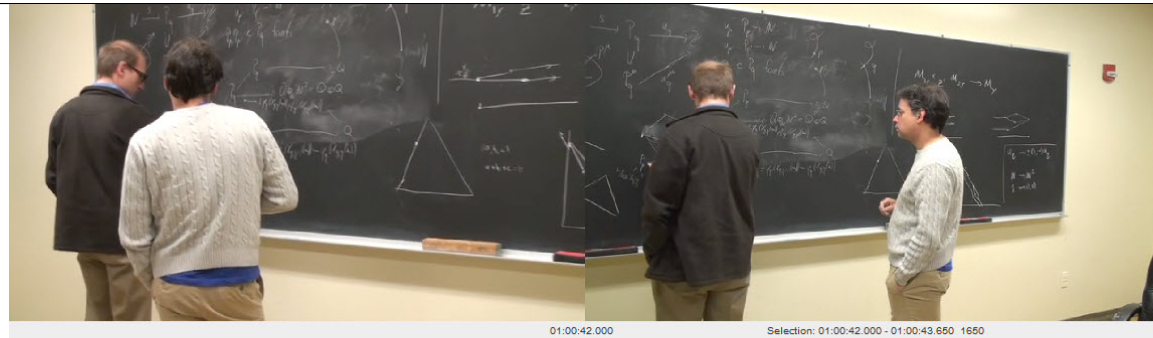
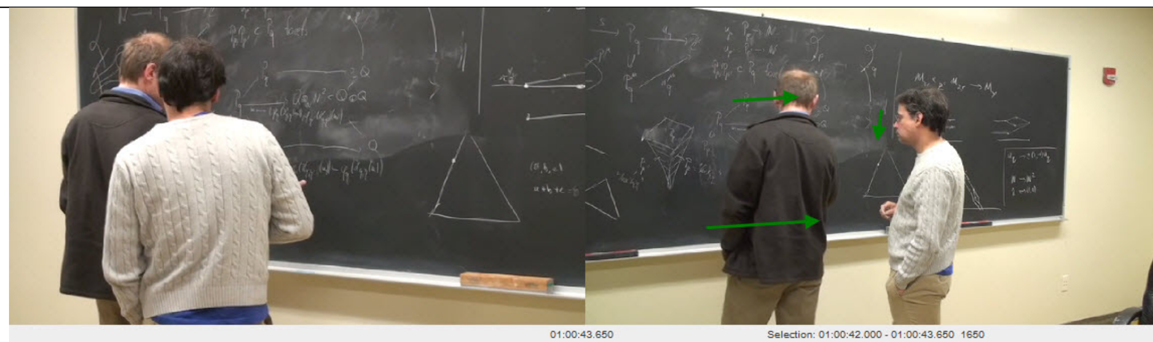
Row	Gesture & timestamp	Transcript
1		[N/A]
2		B: And these things

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Table A.3 – Bart pushed back – *Continued from previous page*

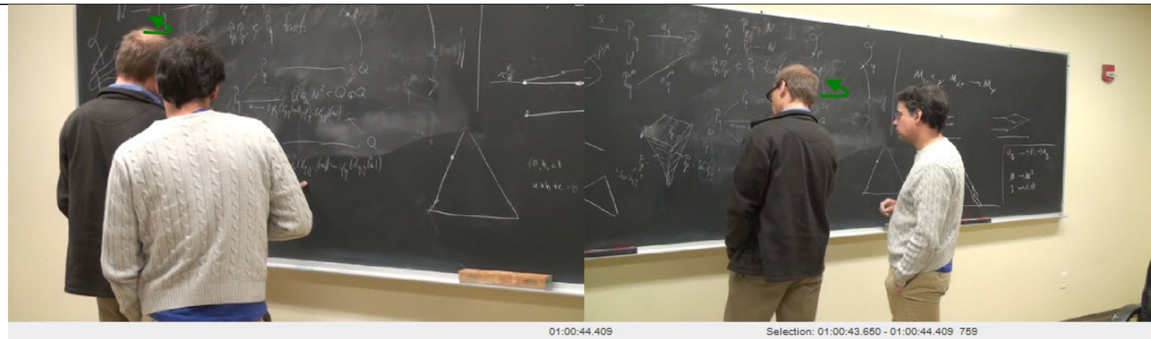
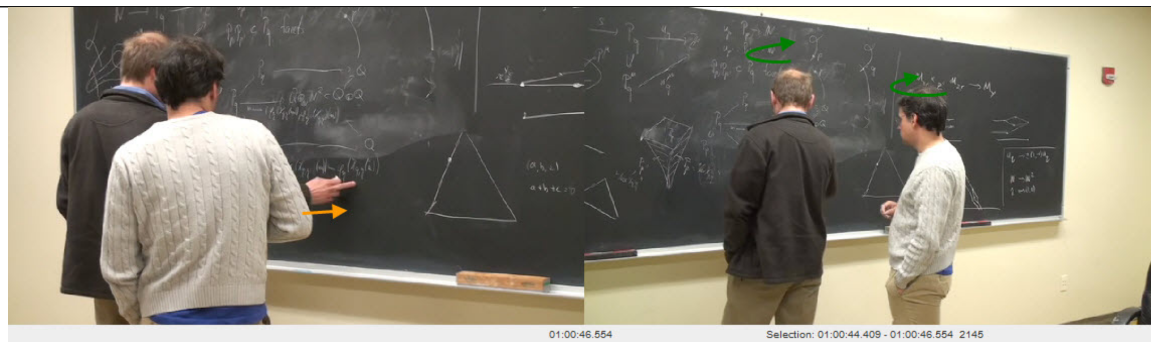
Row	Gesture & timestamp	Transcript
3		[pause]
4		[pause]

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Table A.3 – Bart pushed back – *Continued from previous page*

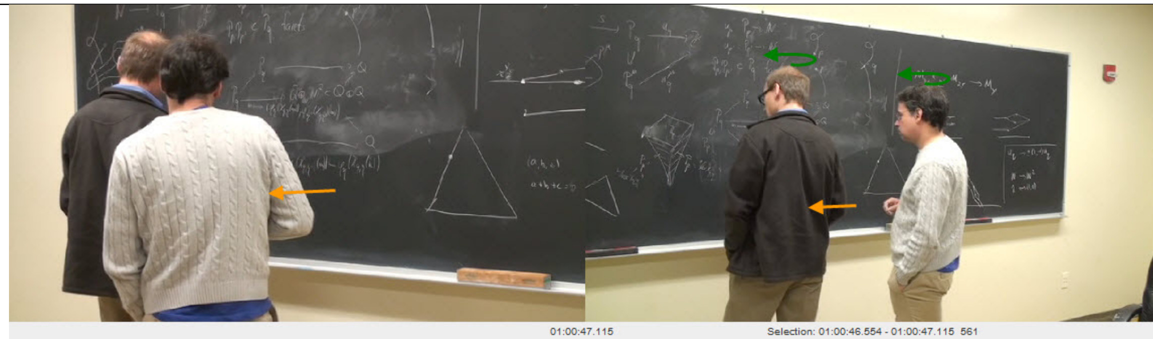
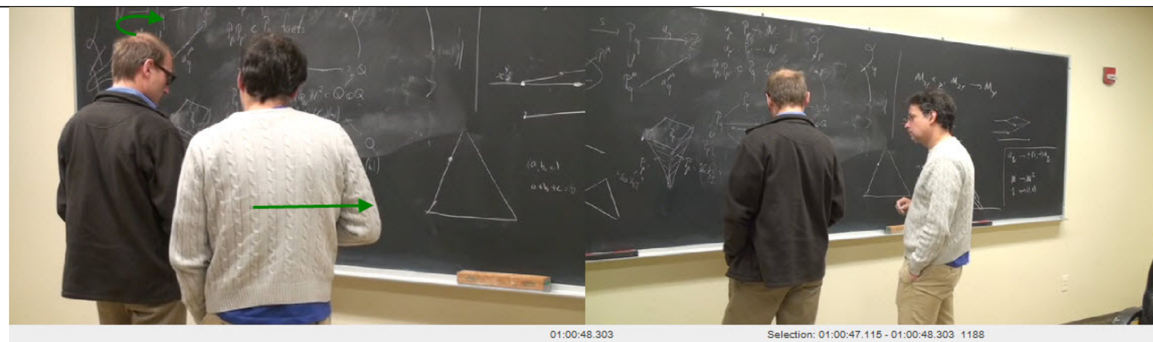
Row	Gesture & timestamp	Transcript
5		are the same,
6		right?

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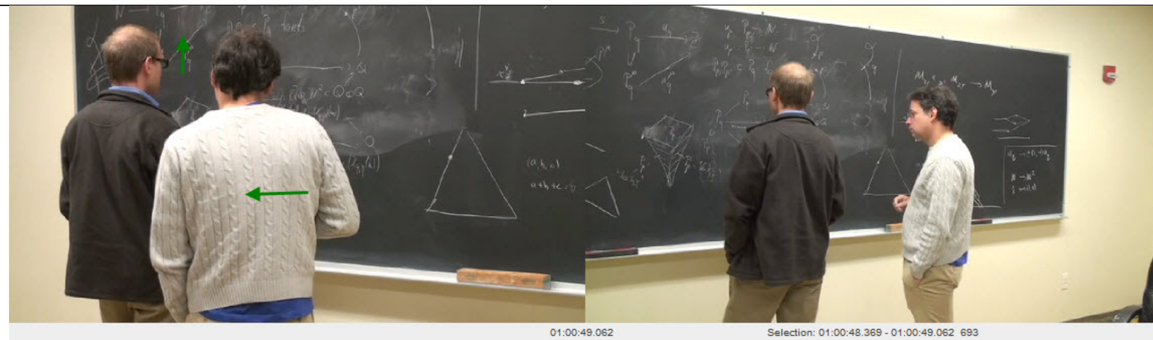
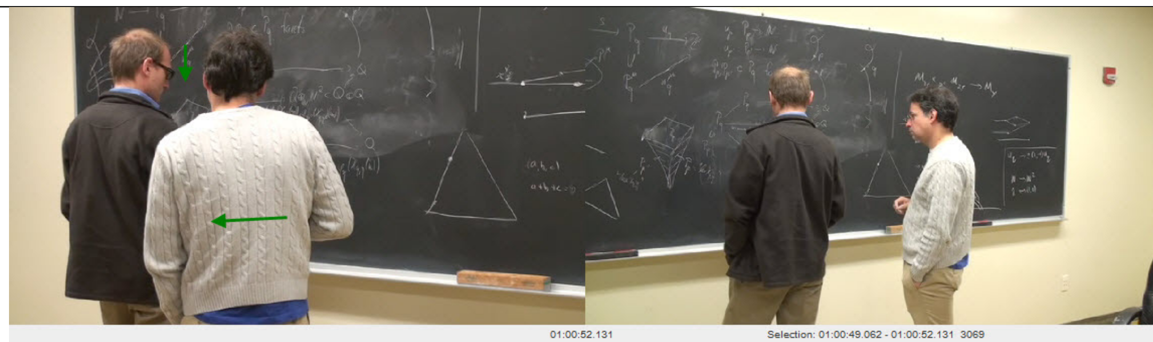
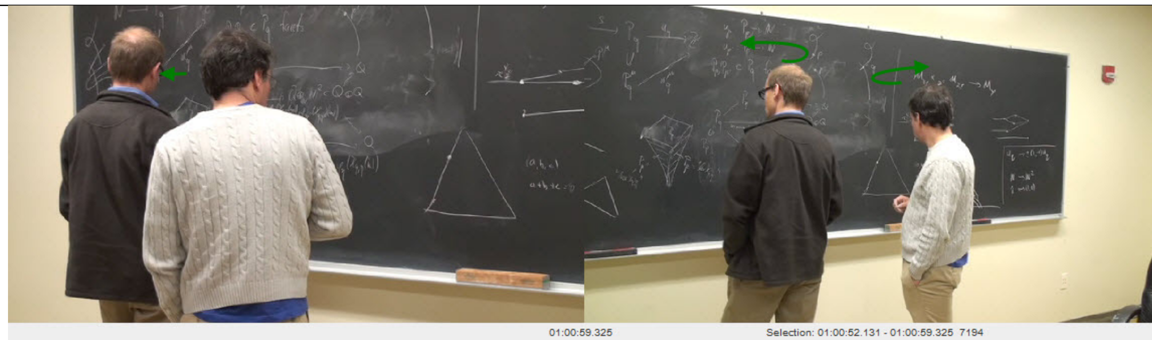
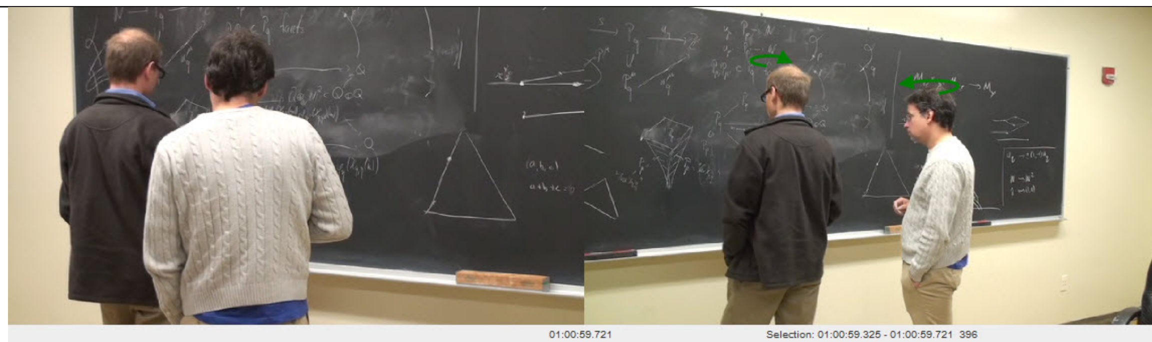
Table A.3 – Bart pushed back – *Continued from previous page*

Row	Gesture & timestamp	Transcript
7		[pause] Is that correct?
8		[pause]

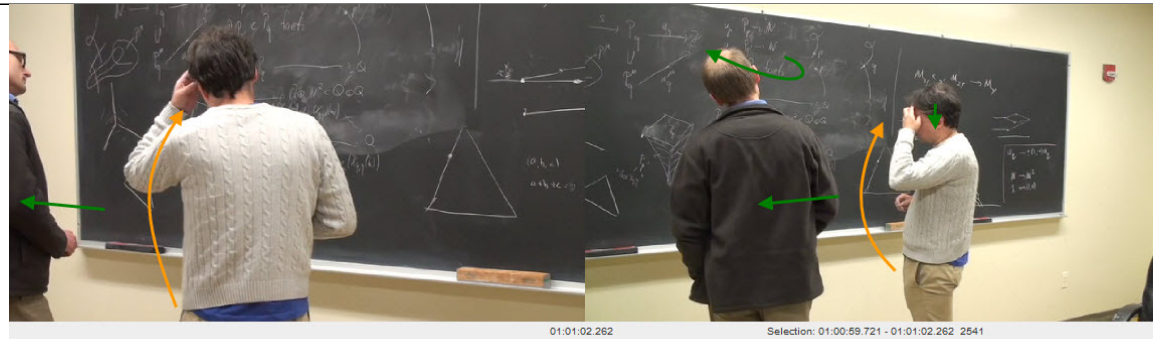
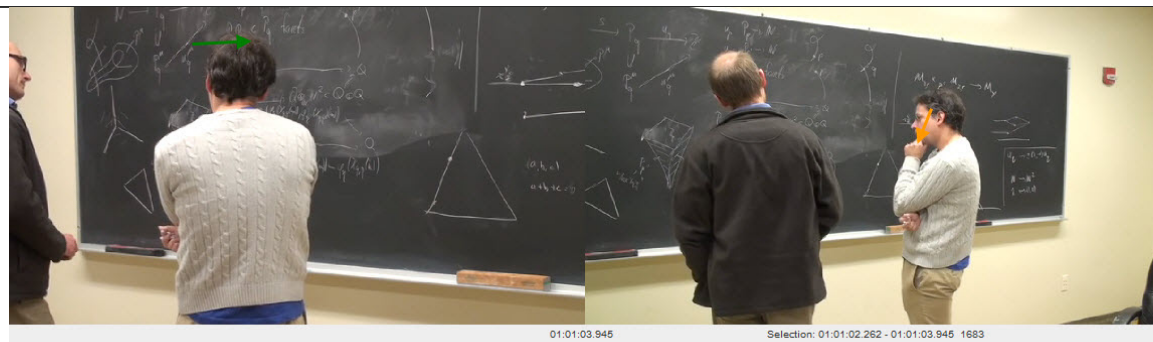
Continued on next page

Table A.3 – Bart pushed back – *Continued from previous page*

Row	Gesture & timestamp	Transcript
9		M: [inaudible murmuring]
10		[pause]

Continued on next page

Table A.3 – Bart pushed back – *Continued from previous page*

Row	Gesture & timestamp	Transcript
11		[inaudible murmuring] [pause]
12		[pause]

Continued on next page

Table A.3 – Bart pushed back – *Continued from previous page*

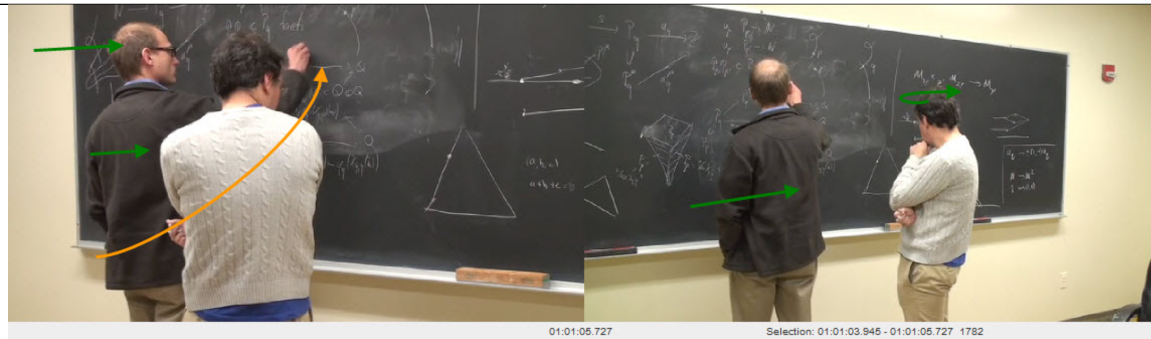
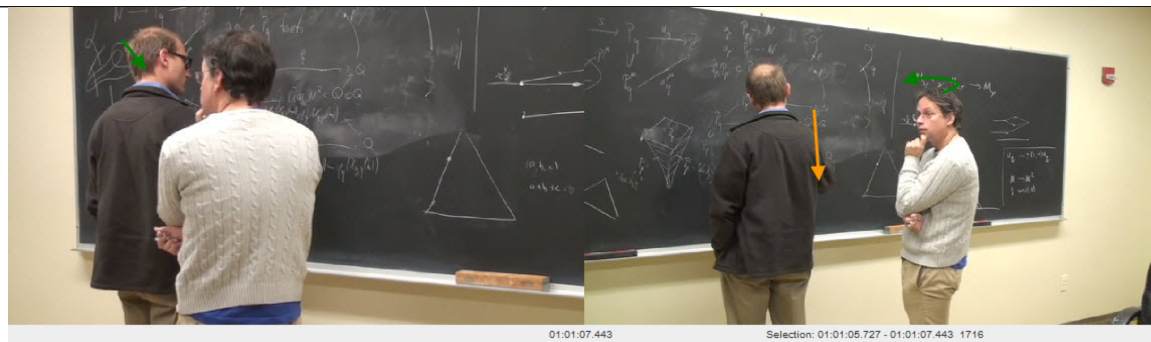
Row	Gesture & timestamp	Transcript
13		[pause]
14		[pause]

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Table A.3 – Bart pushed back – *Continued from previous page*

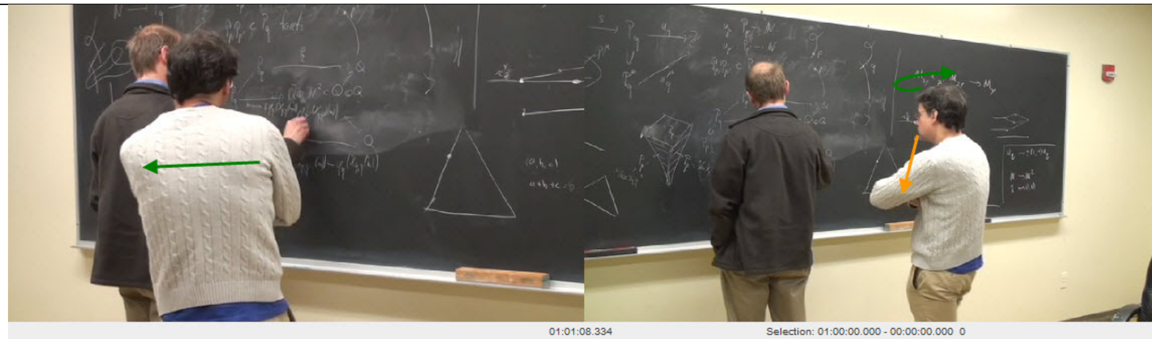
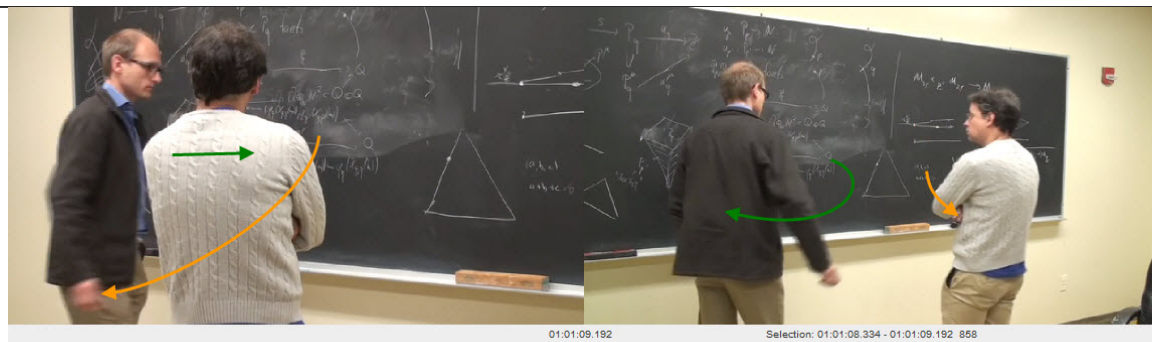
Row	Gesture & timestamp	Transcript
15		[pause]
16		[pause]

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Table A.3 – Bart pushed back – *Continued from previous page*

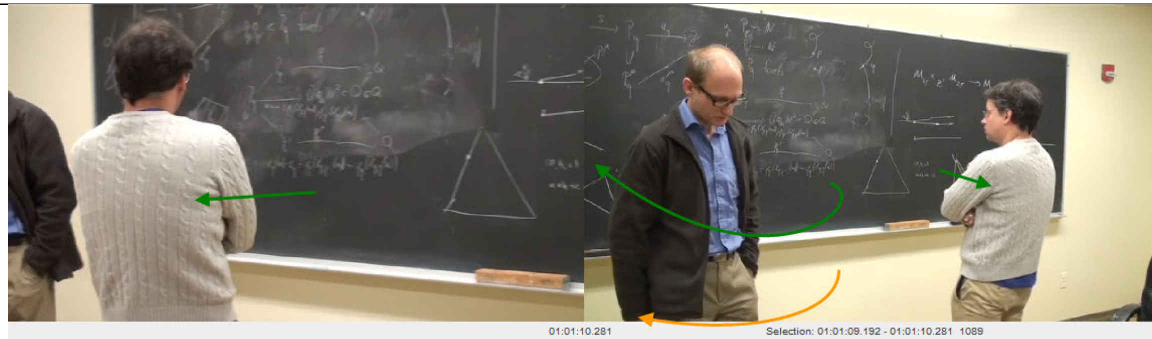
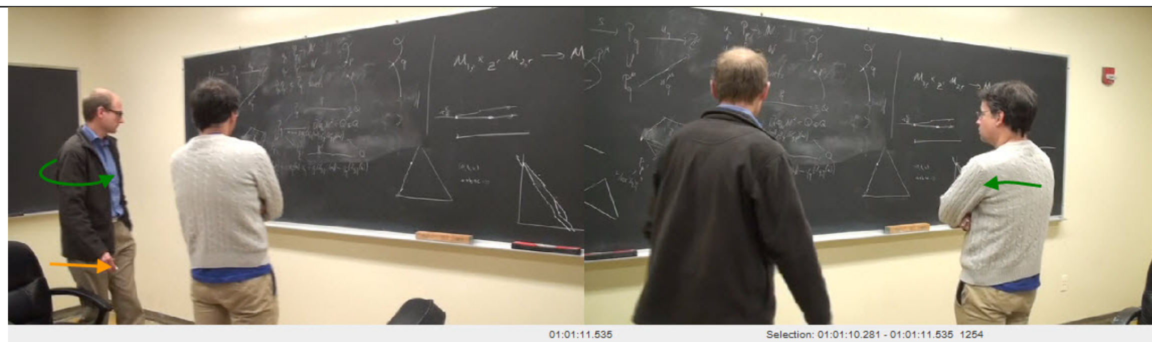
Row	Gesture & timestamp	Transcript
17		[pause]
18		B: True?

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Table A.3 – Bart pushed back – *Continued from previous page*

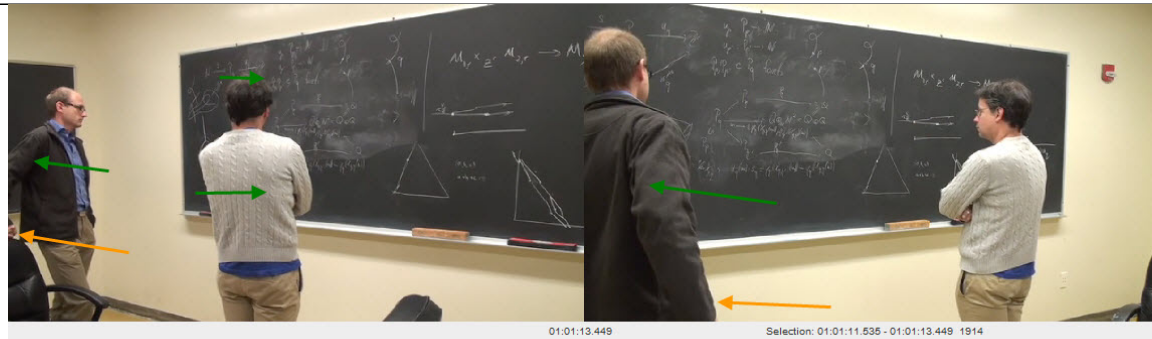
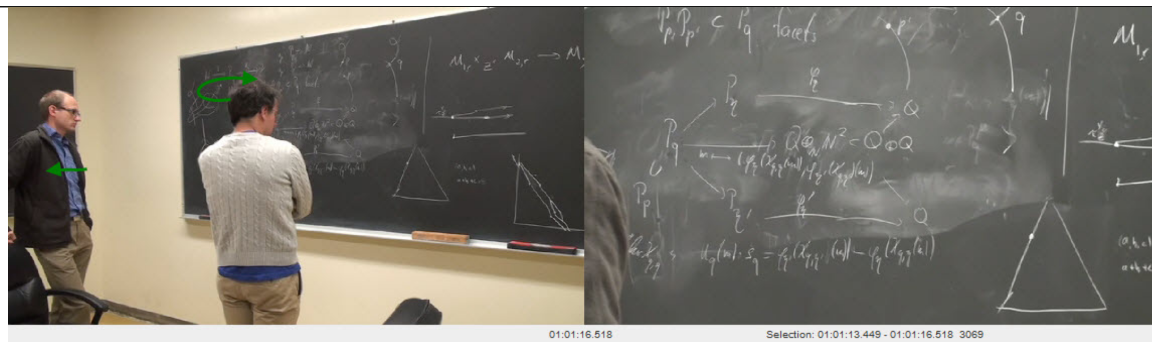
Row	Gesture & timestamp	Transcript
19		[pause]
20		[pause]

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Table A.3 – Bart pushed back – *Continued from previous page*

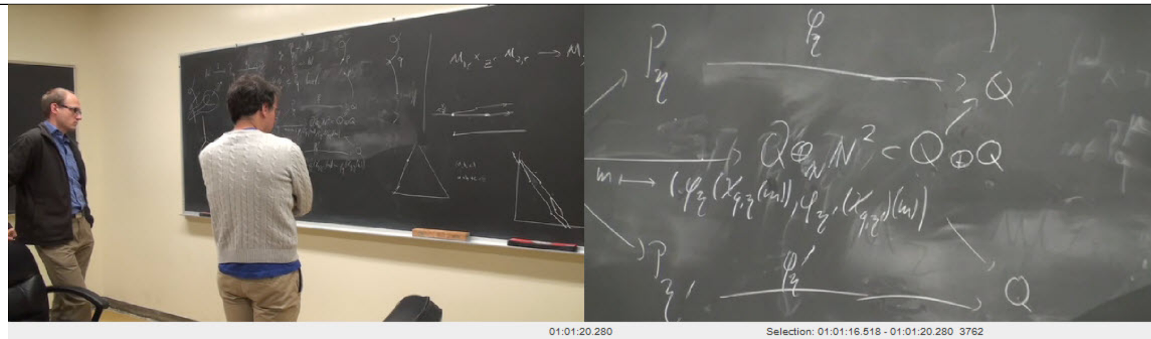
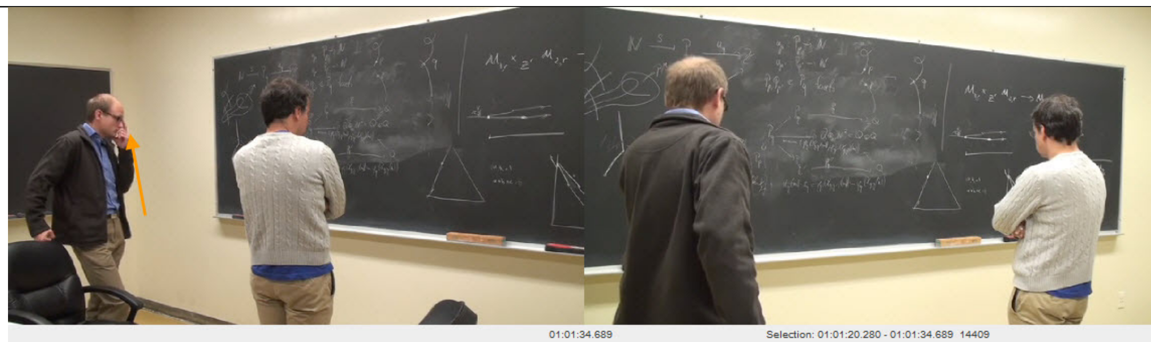
Row	Gesture & timestamp	Transcript
21		[pause]
22		[pause]

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Table A.3 – Bart pushed back – *Continued from previous page*

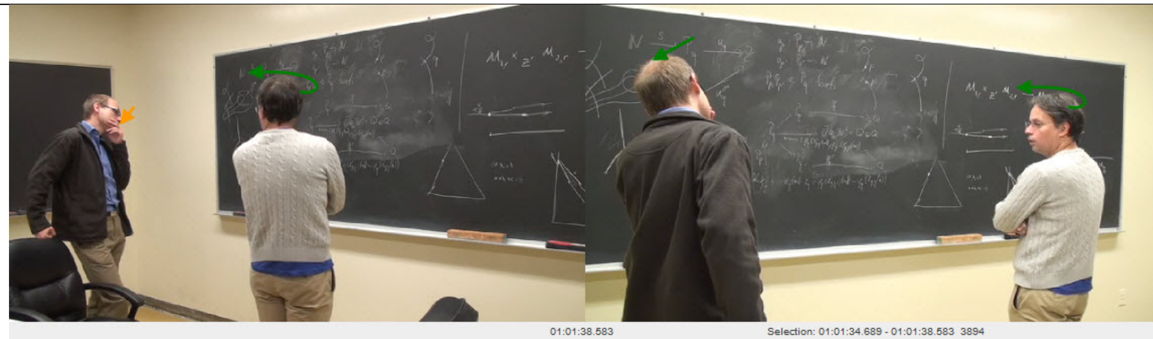
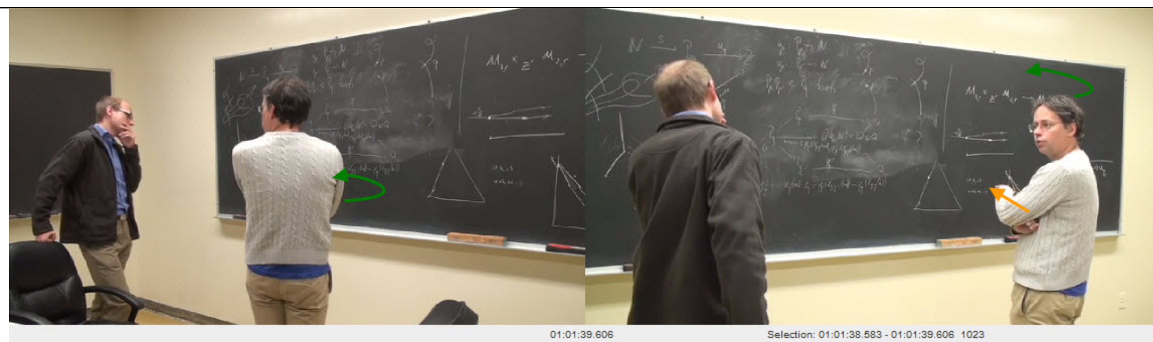
Row	Gesture & timestamp	Transcript
23		[pause]
24		M: Well, let me just show you

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Table A.3 – Bart pushed back – *Continued from previous page*

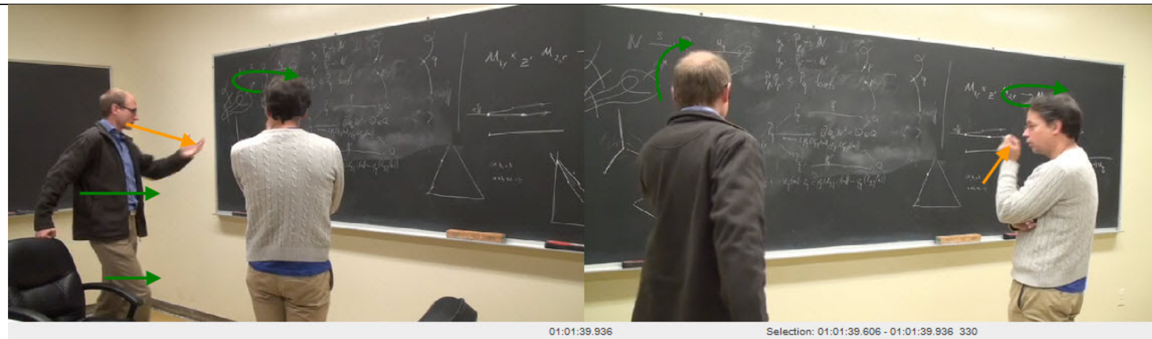
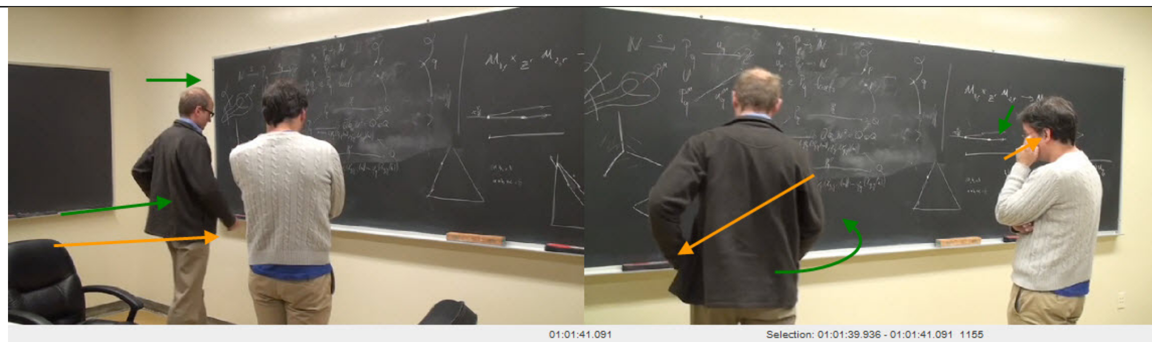
Row	Gesture & timestamp	Transcript
25		B: Oh of course it is, it's zero!
26		Uh,

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Table A.3 – Bart pushed back – *Continued from previous page*

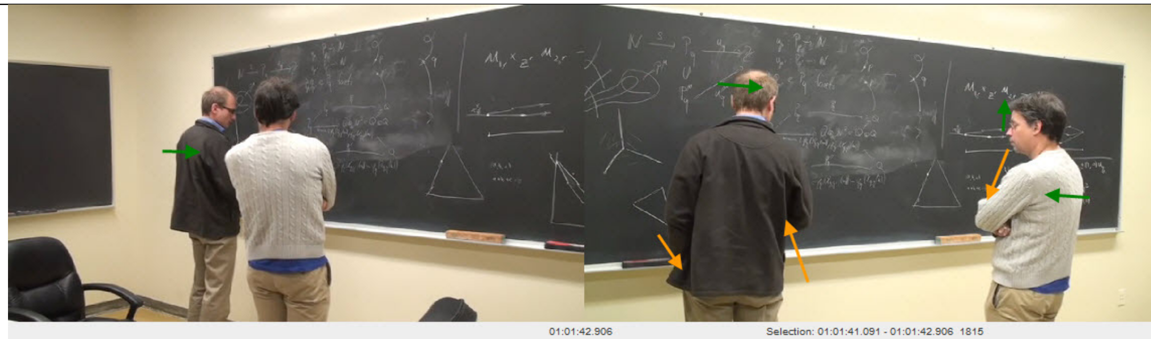
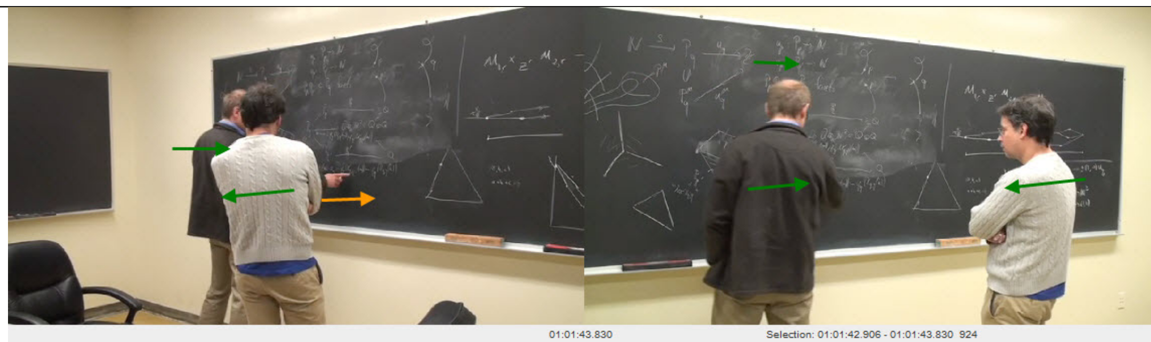
Row	Gesture & timestamp	Transcript
27		I mean that's the ker-nel
28		[pause] So these

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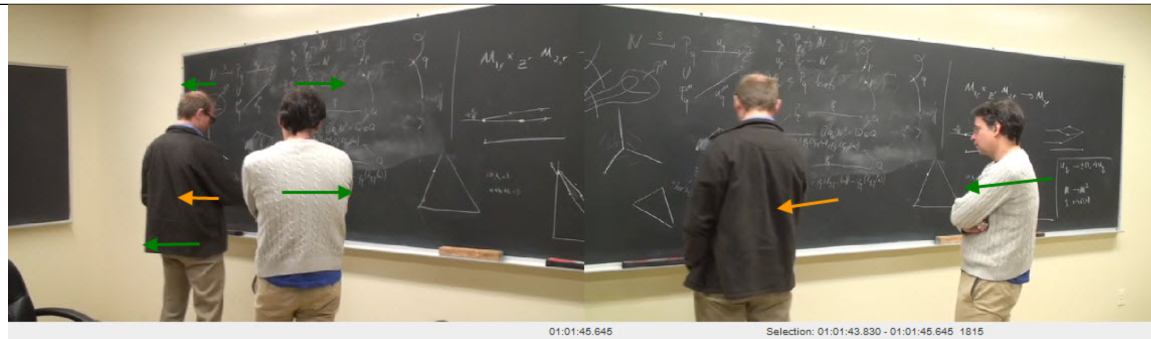
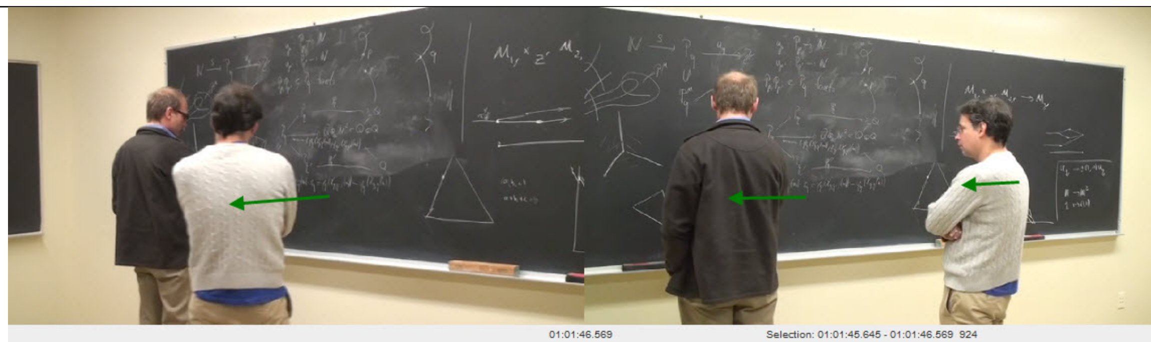
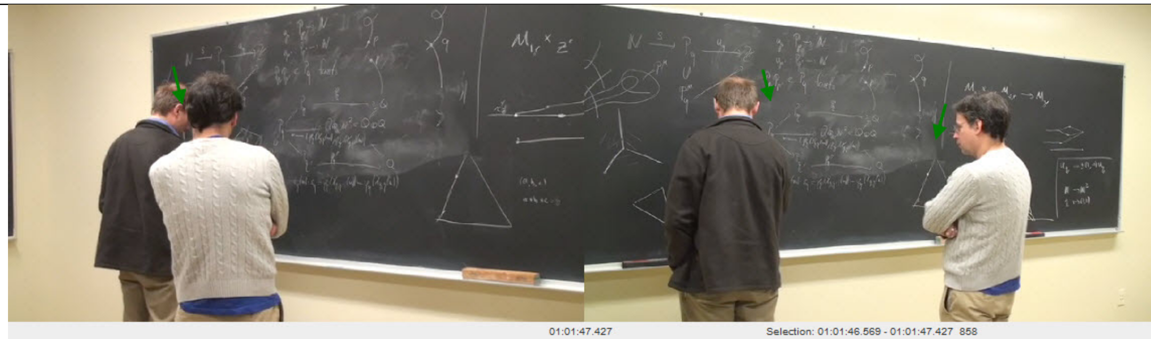
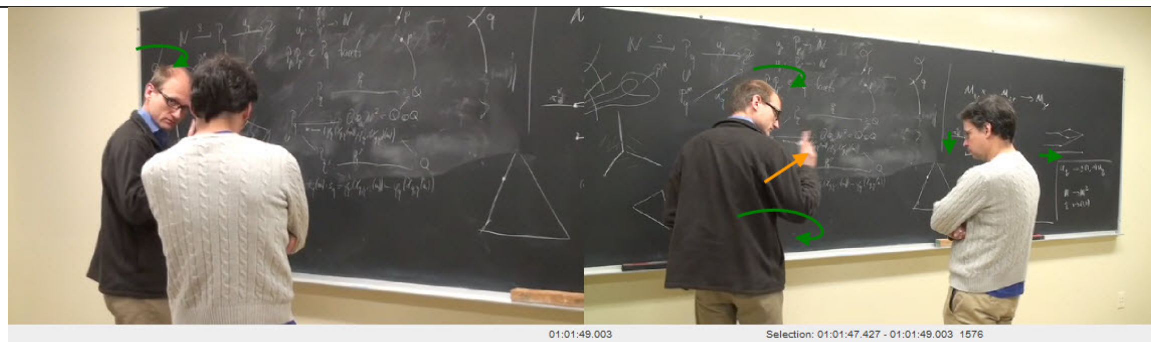
Table A.3 – Bart pushed back – *Continued from previous page*

Row	Gesture & timestamp	Transcript
29		things are zero.
30		[pause] So u_q va-

Continued on next page

Table A.3 – Bart pushed back – *Continued from previous page*

Row	Gesture & timestamp	Transcript
31		-nishes
32		on on this intersec- tion.

A.4 Matt & Bart — inverted pyramid

Table A.4: Matt & Bart — inverted pyramid

Row	Gesture & timestamp	Transcript
1		[N/A]
2		B: Uh,

Continued on next page

Table A.4 – inverted pyramid – *Continued from previous page*

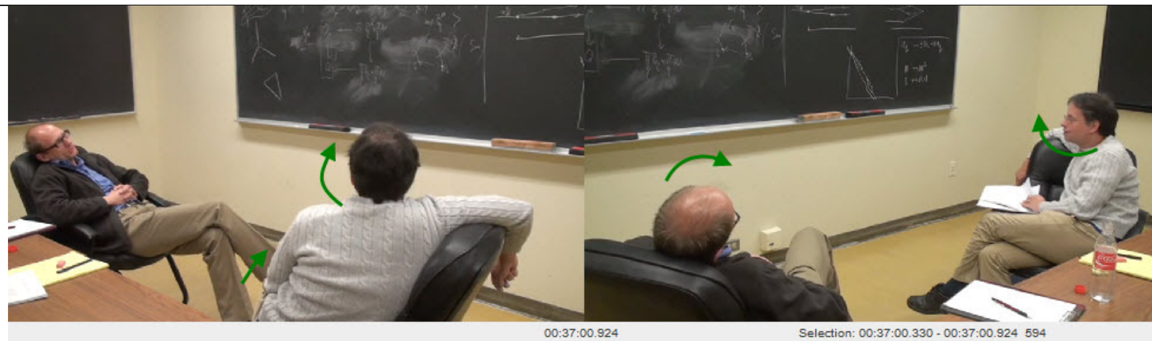
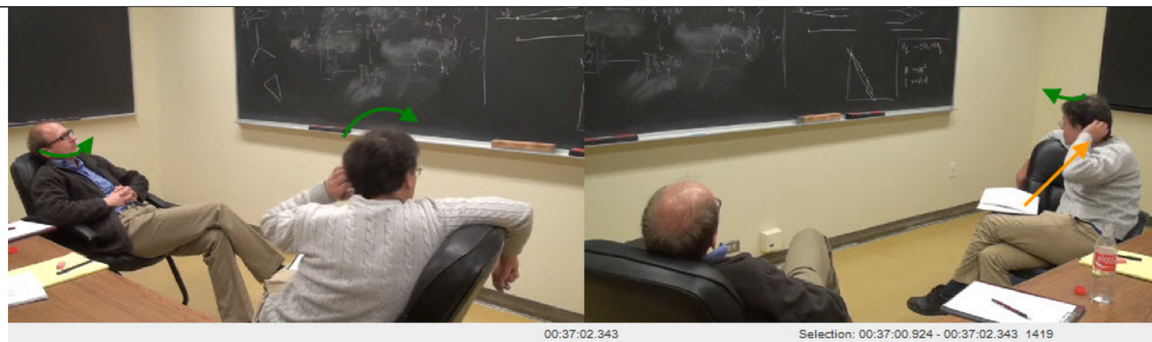
Row	Gesture & timestamp	Transcript
3		I mean,
4		whatever it is, I mean, le-

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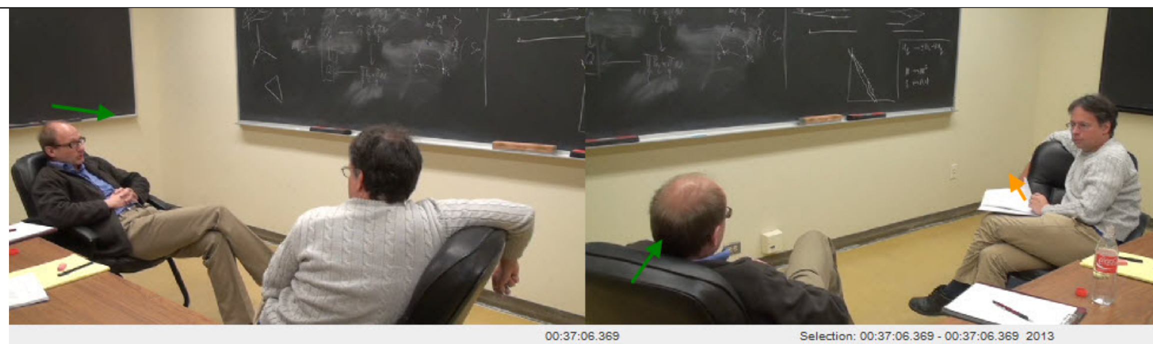
Table A.4 – inverted pyramid – *Continued from previous page*

Row	Gesture & timestamp	Transcript
5		-t's, let's maybe continue the dis-
6		-cussion on how to, how to move *some*

Continued on next page

Table A.4 – inverted pyramid – *Continued from previous page*

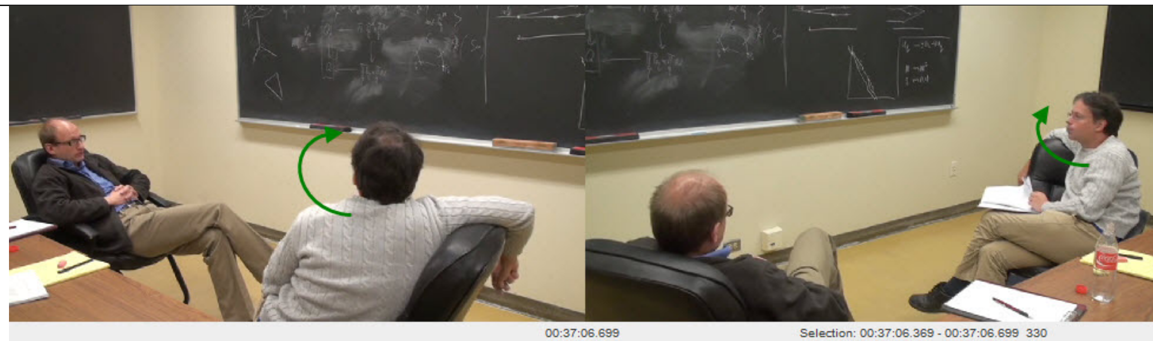
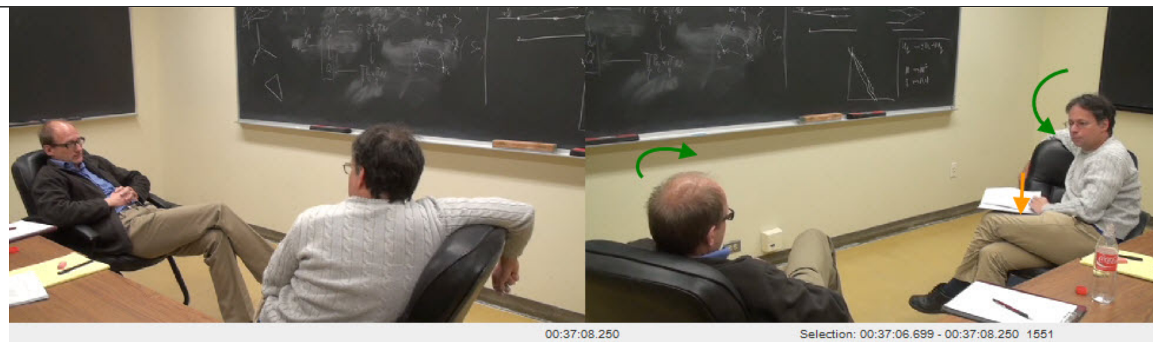
Row	Gesture & timestamp	Transcript
7		curve.
8		M: Mmh. [pause]

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Table A.4 – inverted pyramid – *Continued from previous page*

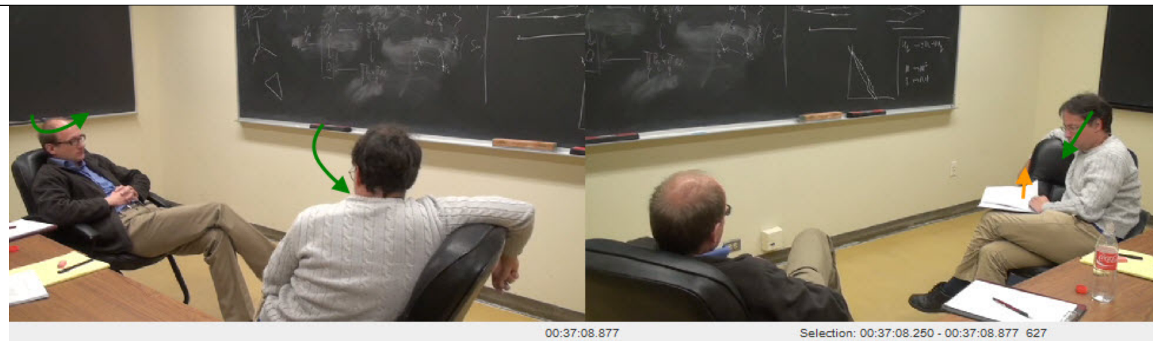
Row	Gesture & timestamp	Transcript
9		B: Right? M: Mmh.
10		B: So maybe you're right. Basically

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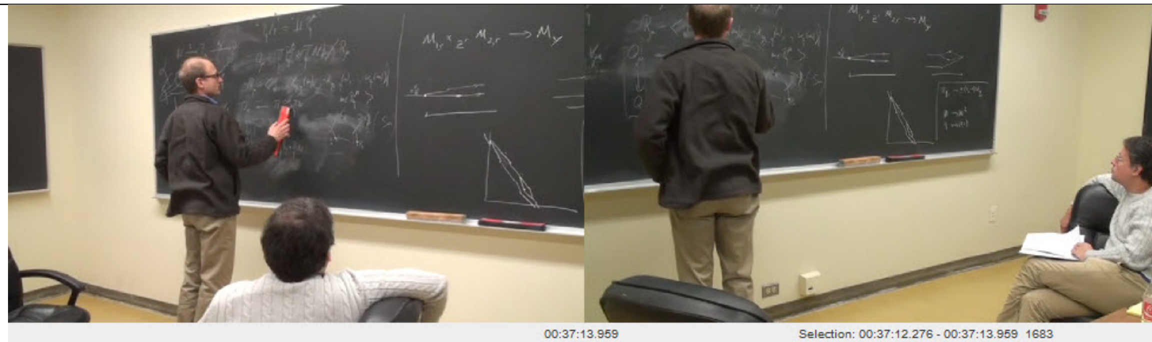
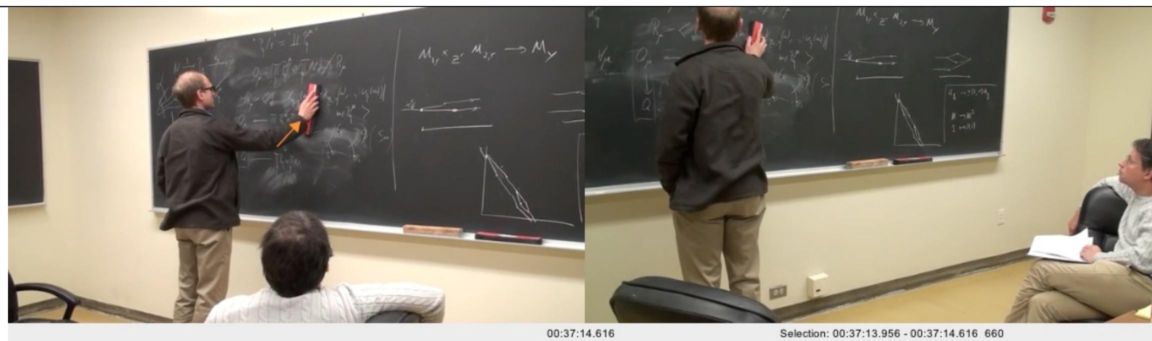
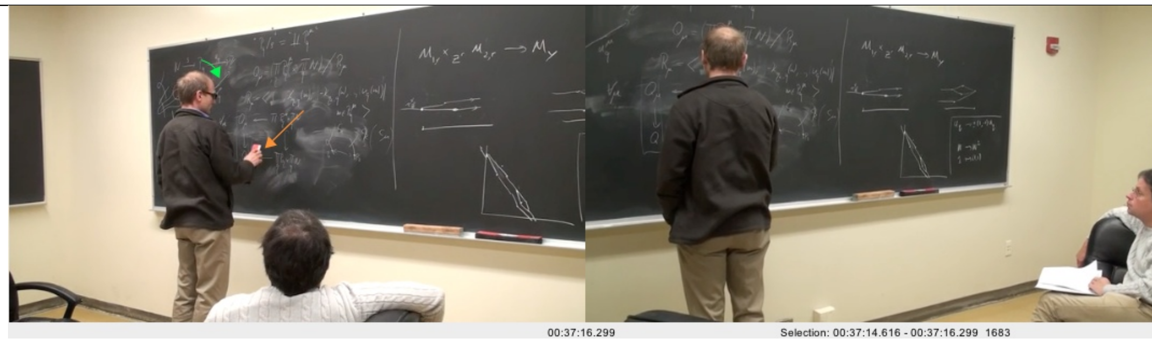
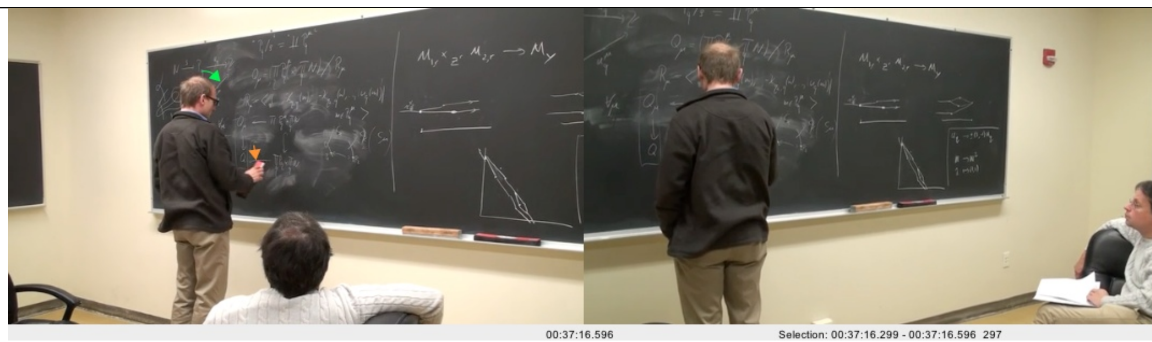
Table A.4 – inverted pyramid – *Continued from previous page*

Row	Gesture & timestamp	Transcript
11		this is not
12		such a con-

Continued on next page

Table A.4 – inverted pyramid – *Continued from previous page*

Row	Gesture & timestamp	Transcript
13		-cern at this point. What w-
14		-e *d*-

Continued on next page

Table A.4 – inverted pyramid – *Continued from previous page*

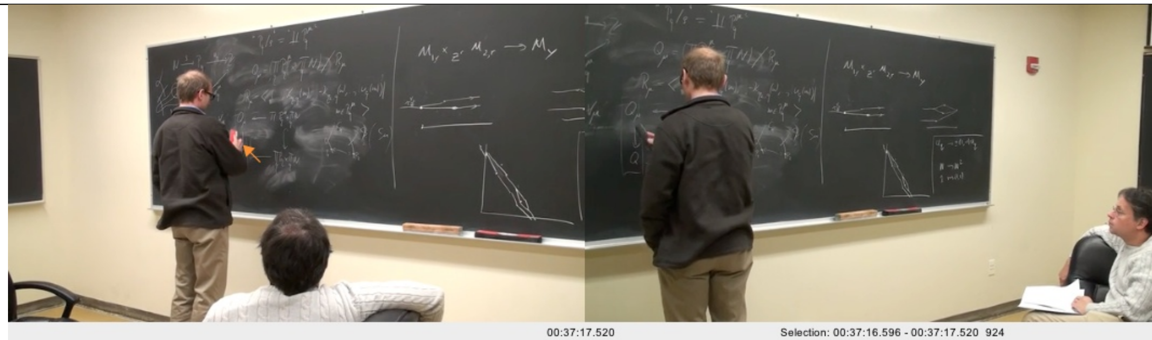
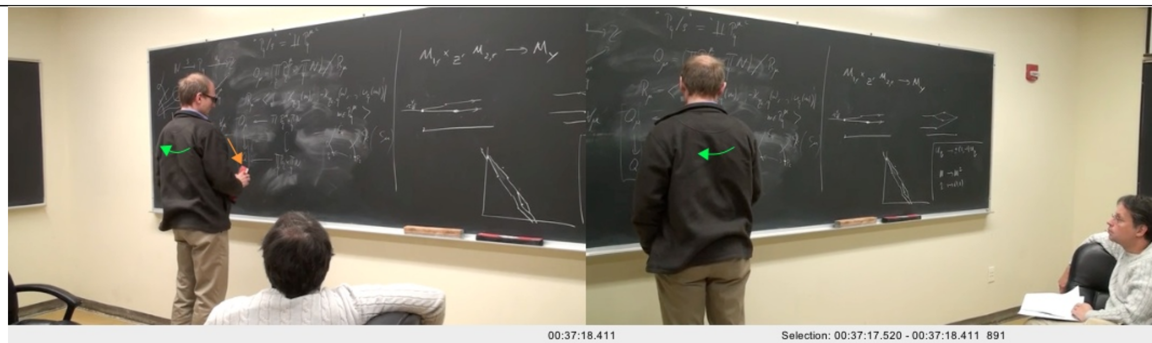
Row	Gesture & timestamp	Transcript
15		-*o* have. . .
16		[pause]

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Table A.4 – inverted pyramid – *Continued from previous page*

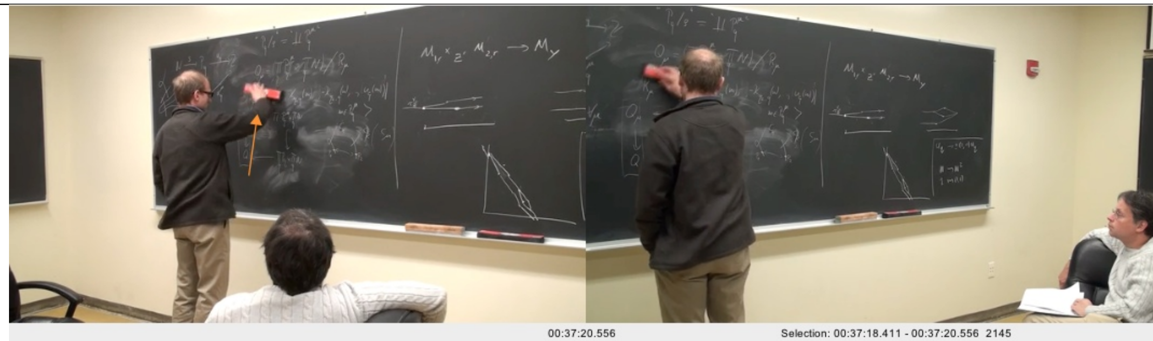
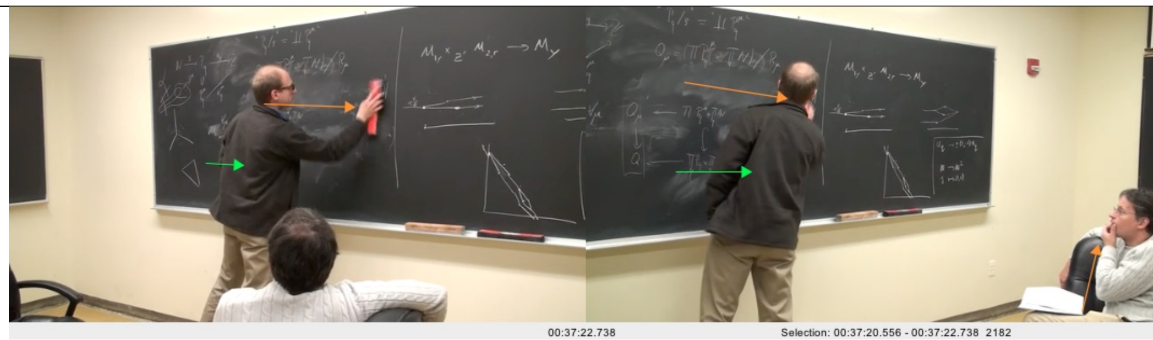
Row	Gesture & timestamp	Transcript
17		[pause]
18		[inaudible] Uh

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Table A.4 – inverted pyramid – *Continued from previous page*

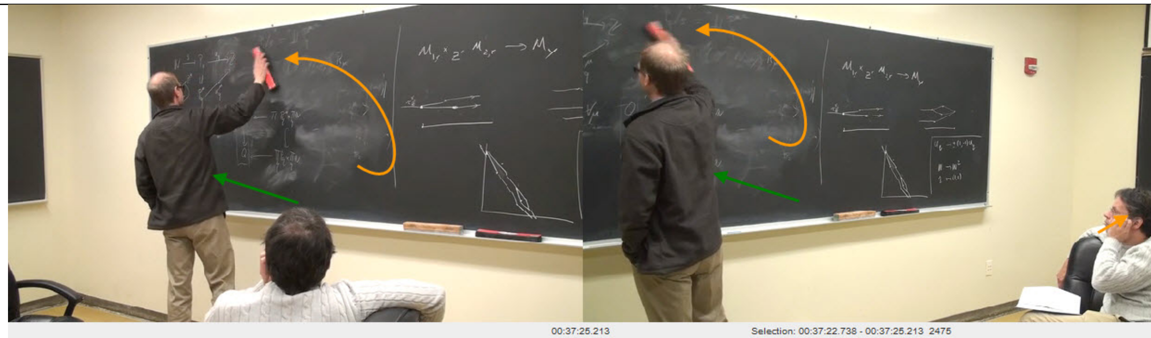
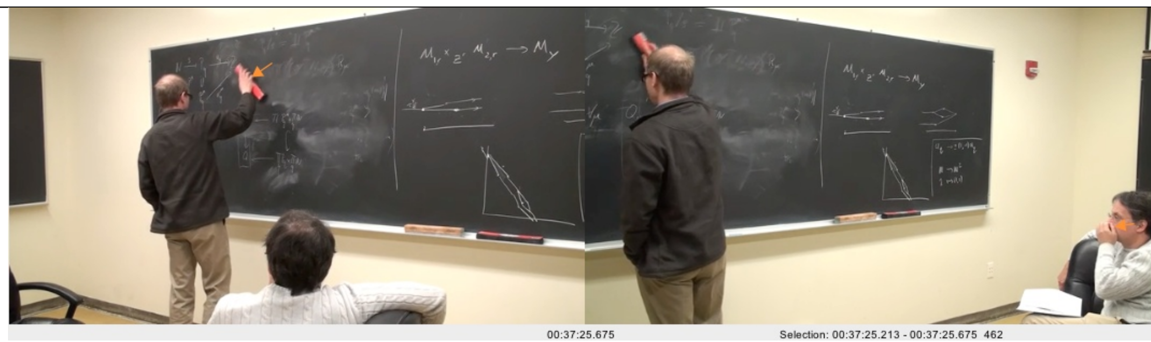
Row	Gesture & timestamp	Transcript
19		[pause]
20		[pause]

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Table A.4 – inverted pyramid – *Continued from previous page*

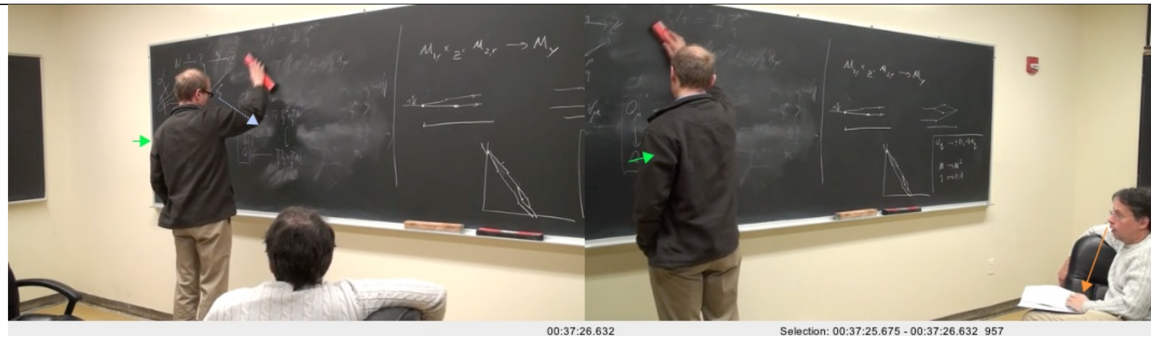
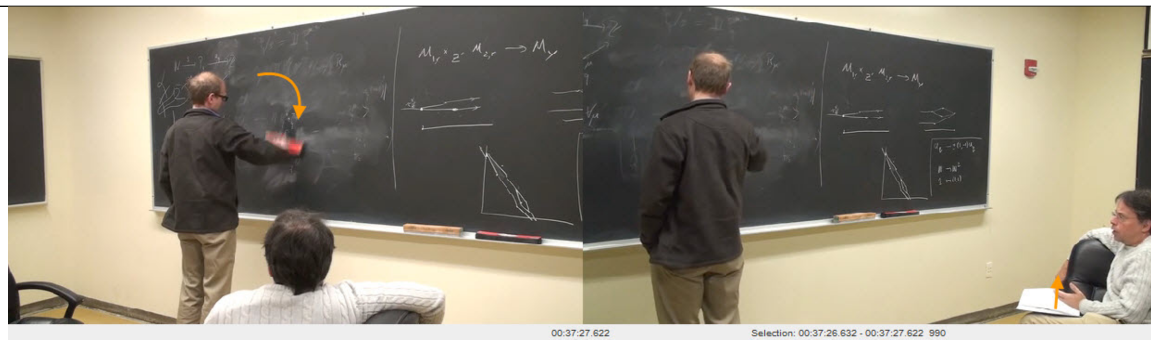
Row	Gesture & timestamp	Transcript
21		[pause]
22		M: Yeah, I mean, so I

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Table A.4 – inverted pyramid – *Continued from previous page*

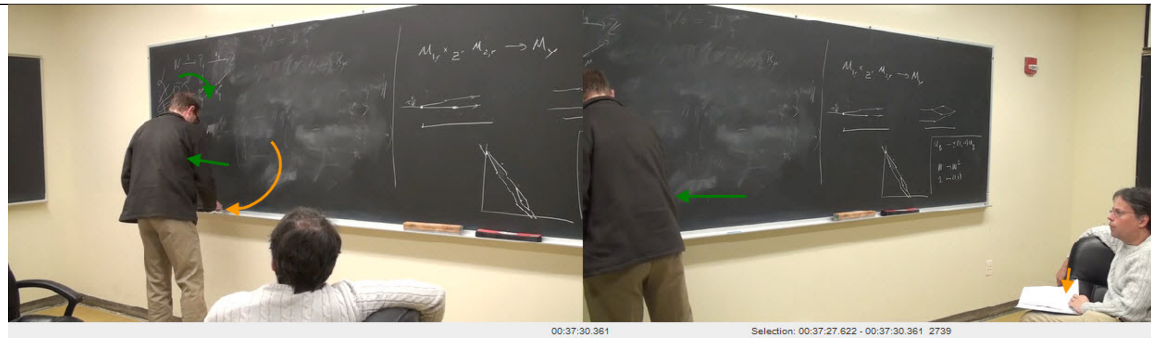
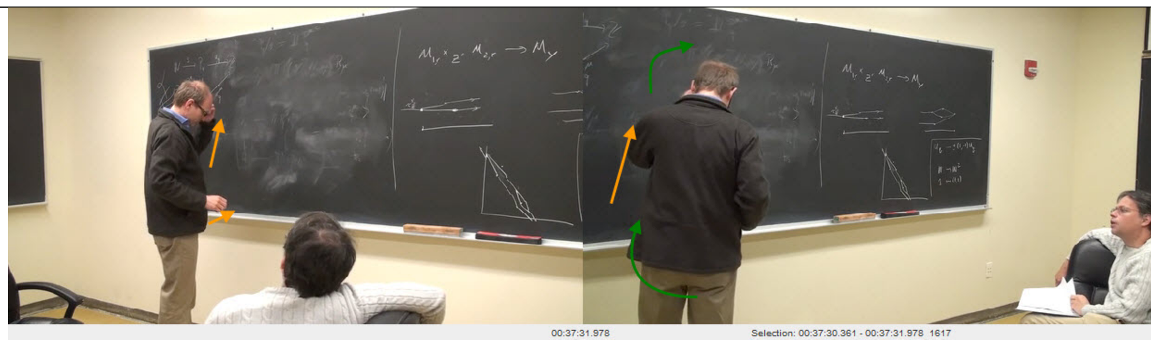
Row	Gesture & timestamp	Transcript
23		guess the first question is, can we at least get the curve at the local
24		[inaudible]?

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Table A.4 – inverted pyramid – *Continued from previous page*

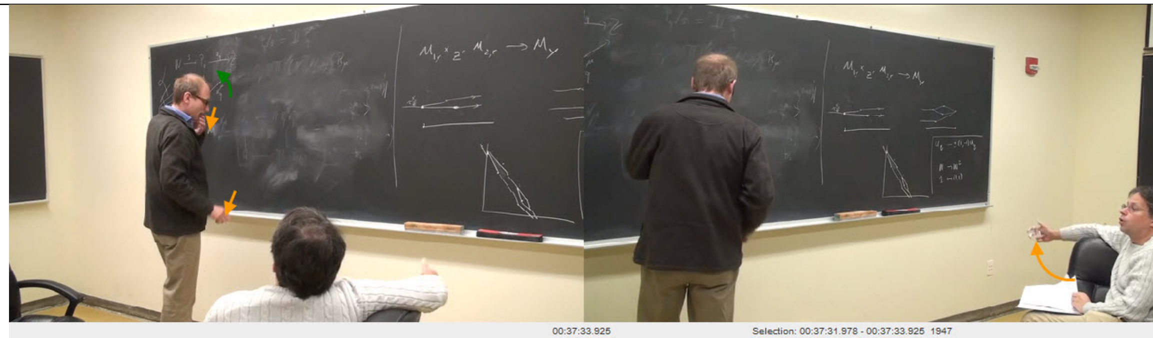
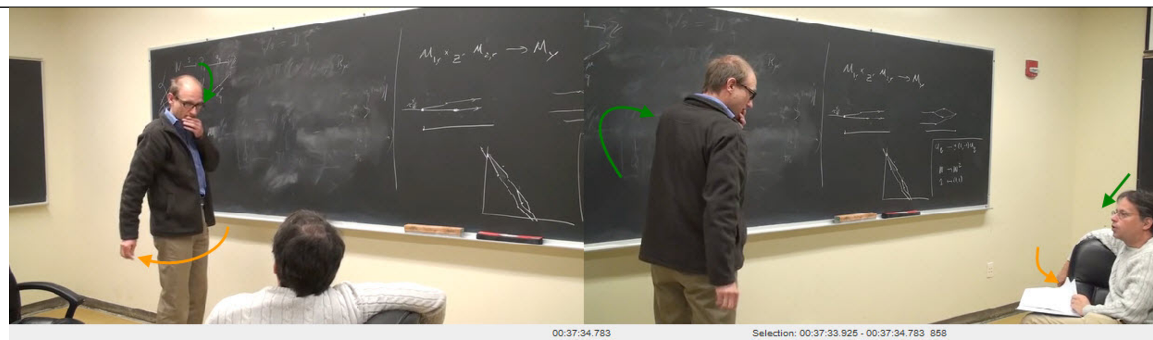
Row	**Gesture & timestamp**	**Transcript**
25		[pause] Yeah, so I. . . B: Yeah,
26		exact-

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Table A.4 – inverted pyramid – *Continued from previous page*

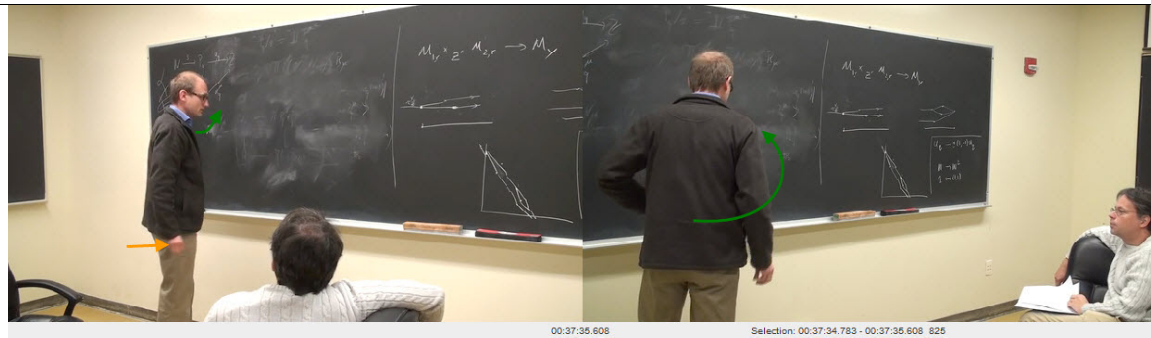
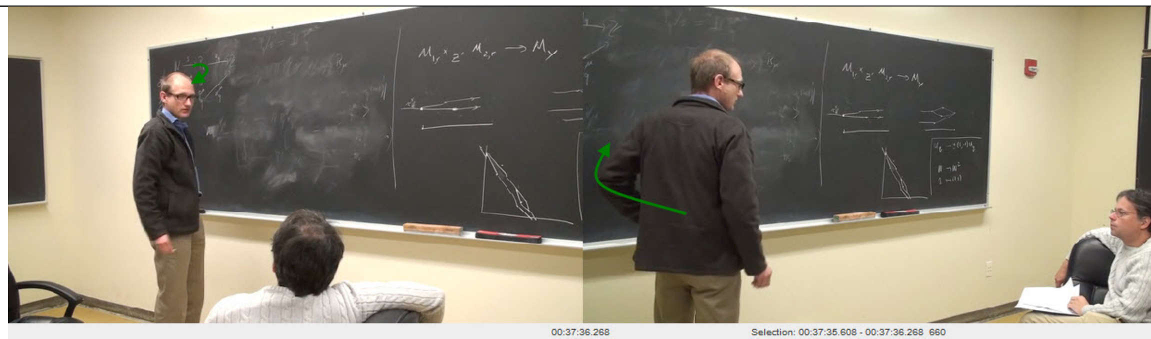
Row	Gesture & timestamp	Transcript
27		-ly so I'm,
28		I'm, uh,

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Table A.4 – inverted pyramid – *Continued from previous page*

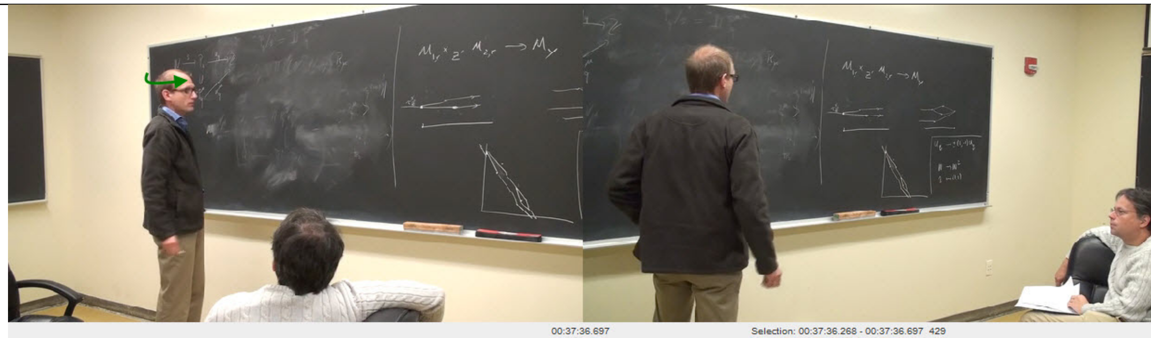
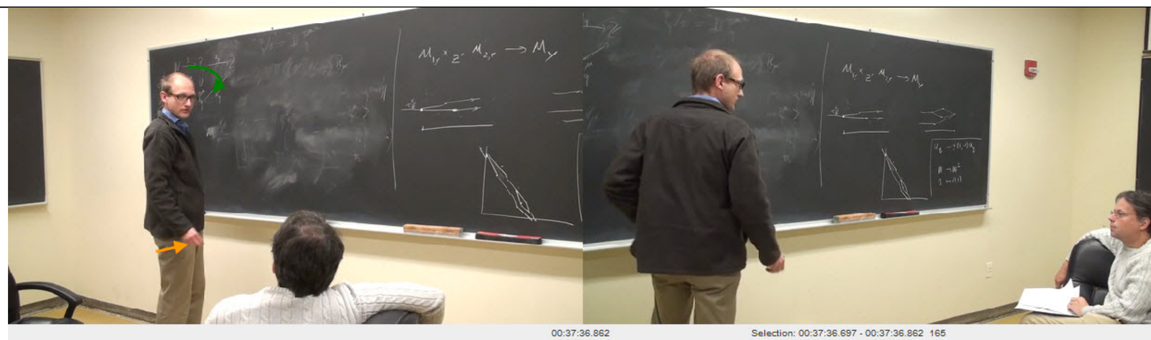
Row	Gesture & timestamp	Transcript
29		slightly
30		lo-

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Table A.4 – inverted pyramid – *Continued from previous page*

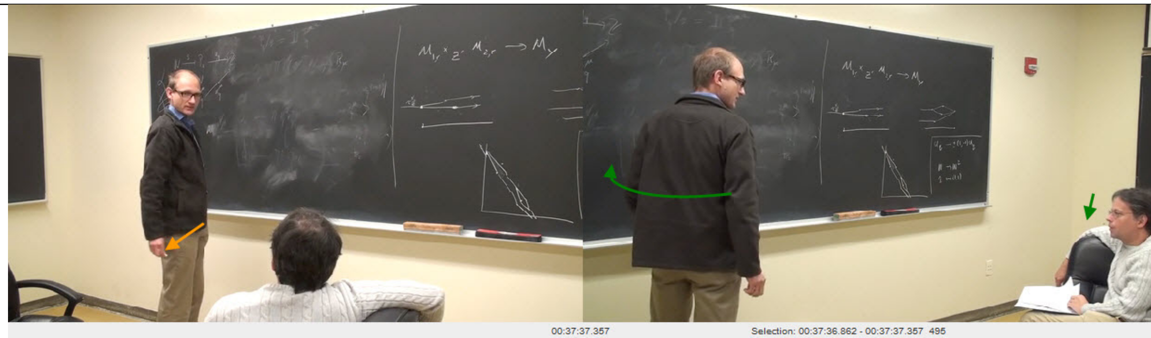
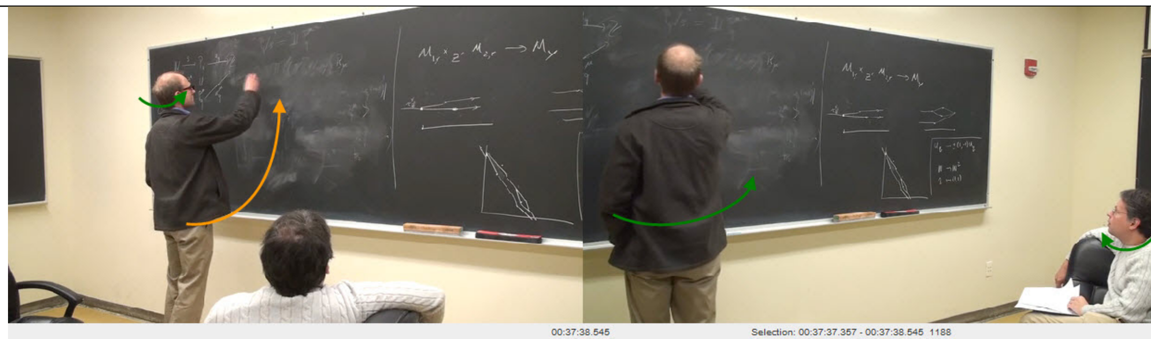
Row	Gesture & timestamp	Transcript
31		-st.
32		[pause] U-

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Table A.4 – inverted pyramid – *Continued from previous page*

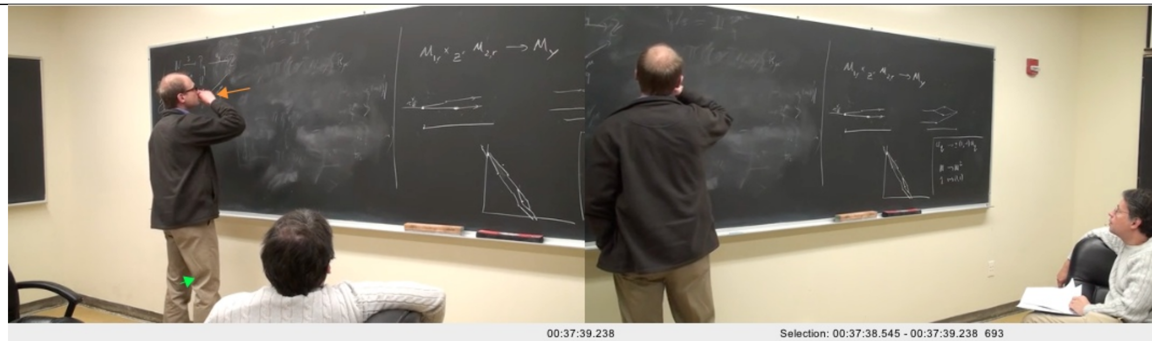
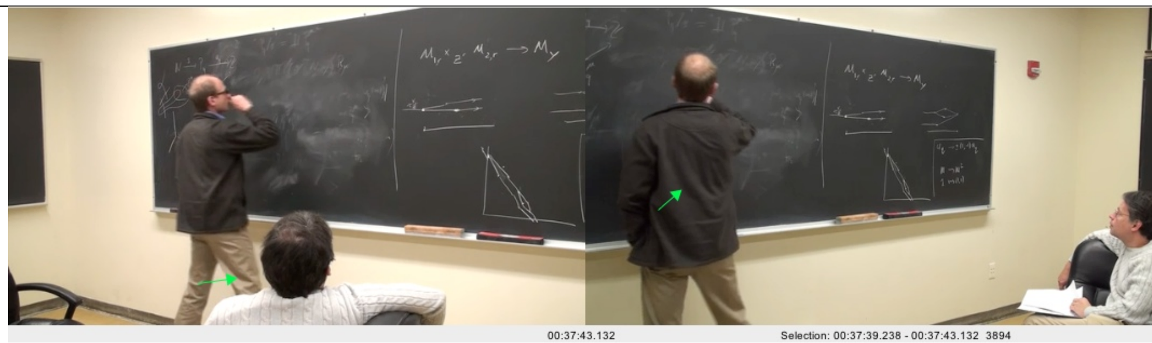
Row	Gesture & timestamp	Transcript
33		-m. . .
34		[pause]

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Table A.4 – inverted pyramid – *Continued from previous page*

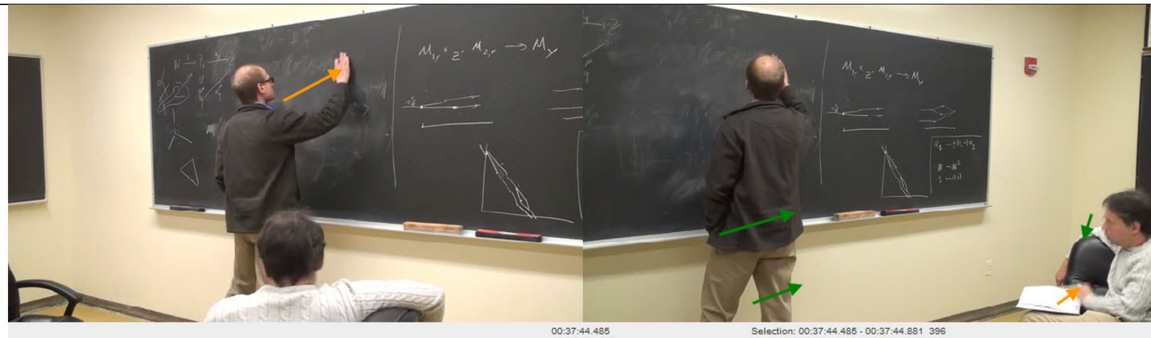
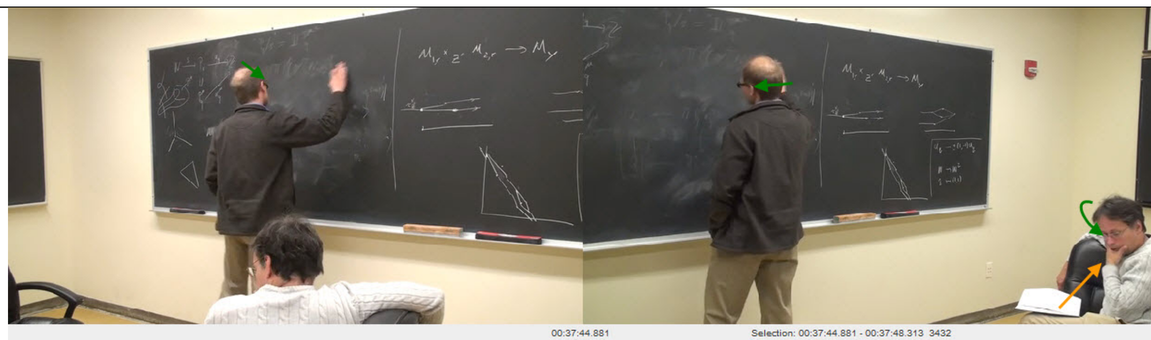
Row	Gesture & timestamp	Transcript
35		[pause] Okay. . .
36		[pause]

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Table A.4 – inverted pyramid – *Continued from previous page*

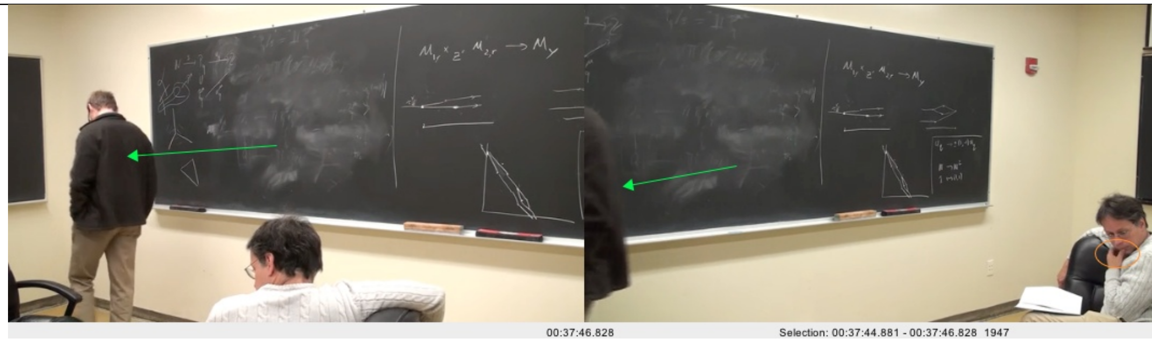
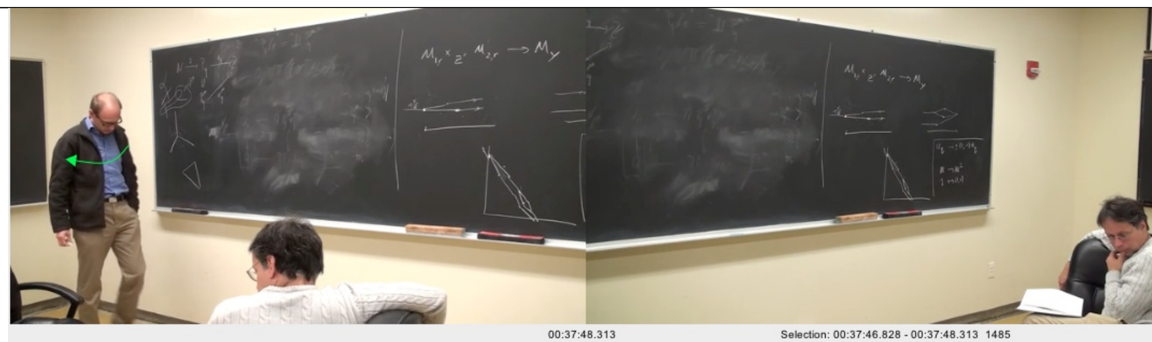
Row	Gesture & timestamp	Transcript
37		[pause]
38		[pause]

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Table A.4 – inverted pyramid – *Continued from previous page*

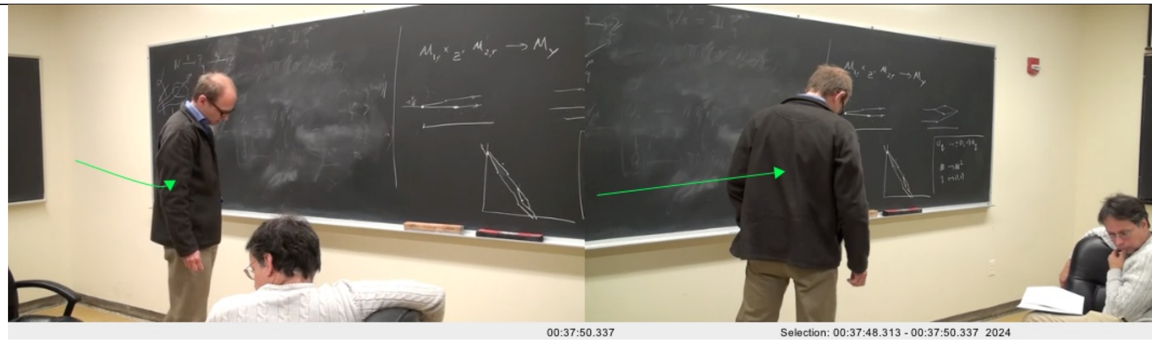
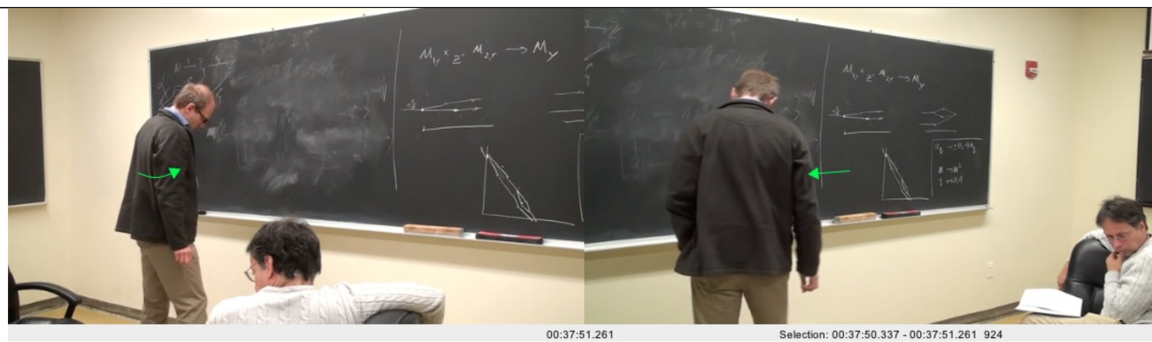
Row	Gesture & timestamp	Transcript
39		[pause]
40		[pause]

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Table A.4 – inverted pyramid – *Continued from previous page*

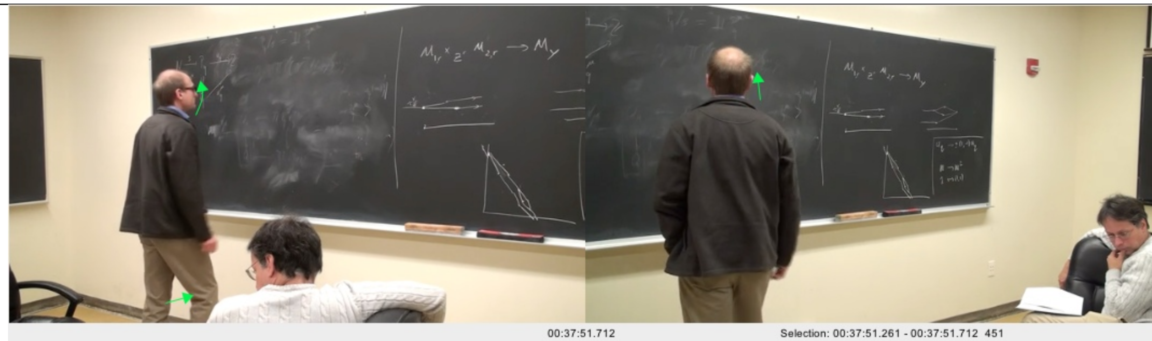
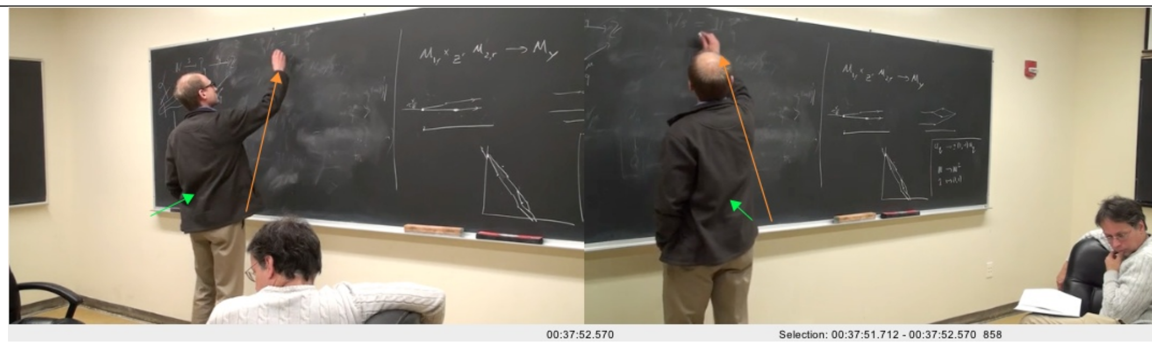
Row	Gesture & timestamp	Transcript
41		[pause]
42		So le-

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Table A.4 – inverted pyramid – *Continued from previous page*

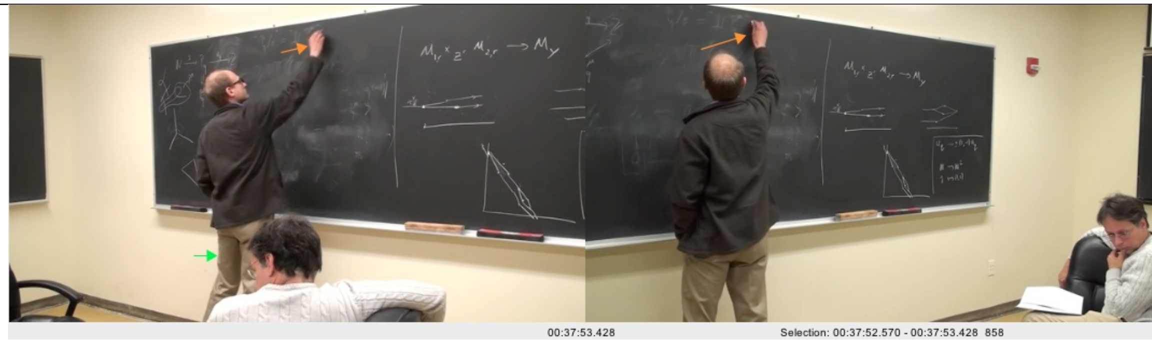
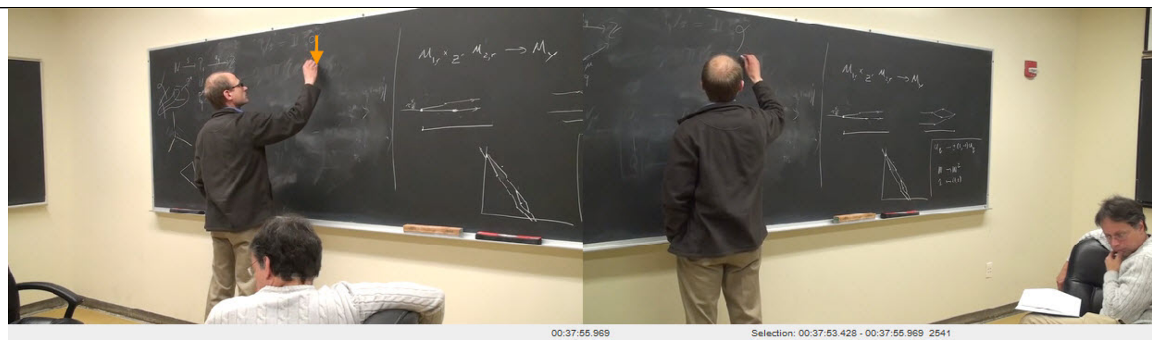
Row	Gesture & timestamp	Transcript
43		-t's, let's say we have
44		two curves that we want to...

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Table A.4 – inverted pyramid – *Continued from previous page*

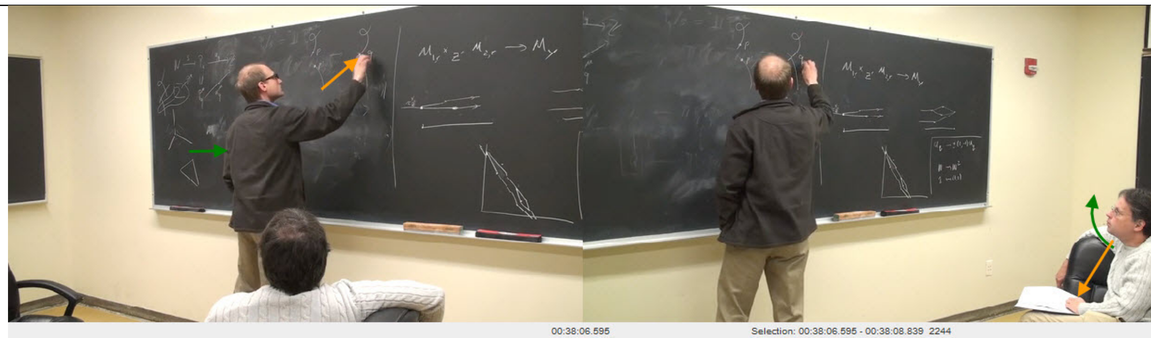
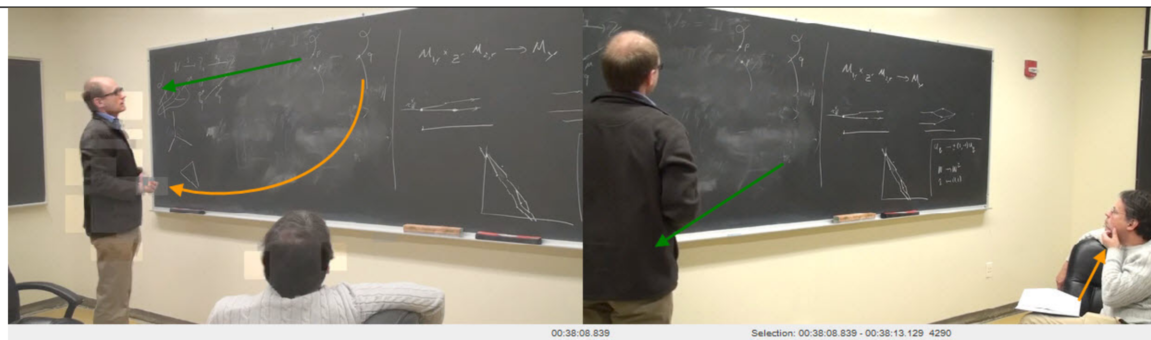
Row	Gesture & timestamp	Transcript
45		[pause]
46		Right? M: Mhm.

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Table A.4 – inverted pyramid – *Continued from previous page*

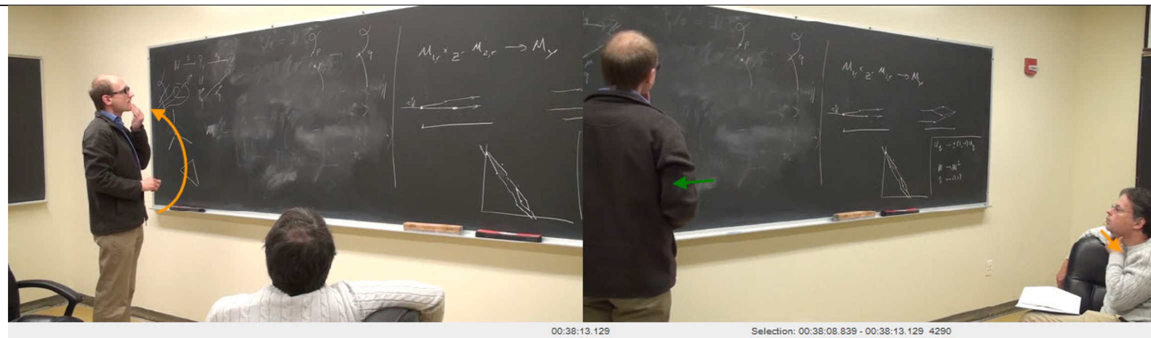
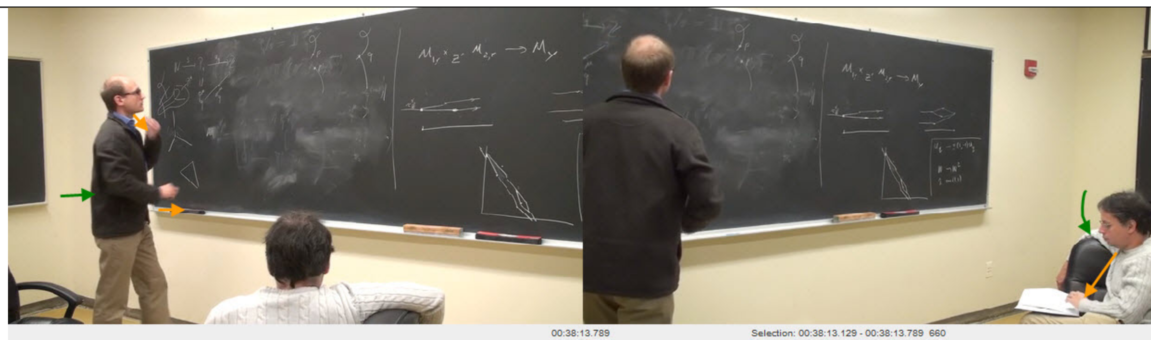
Row	Gesture & timestamp	Transcript
47		B: Um. [pause]
48		[pause] So

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Table A.4 – inverted pyramid – *Continued from previous page*

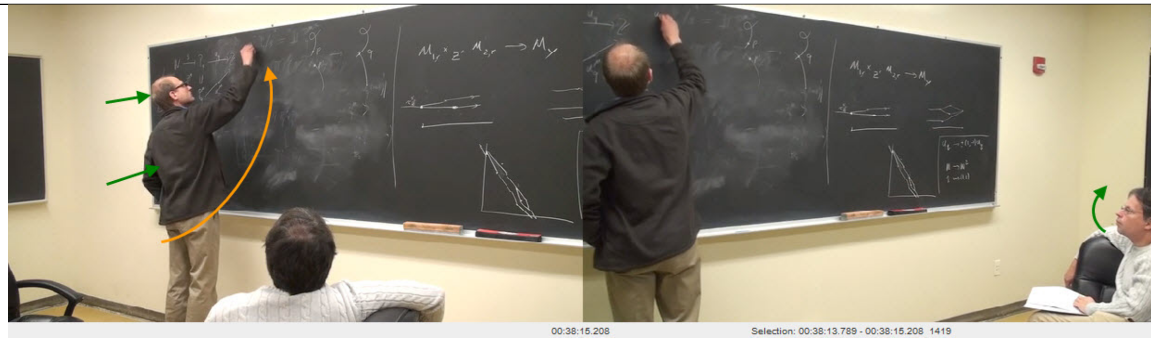
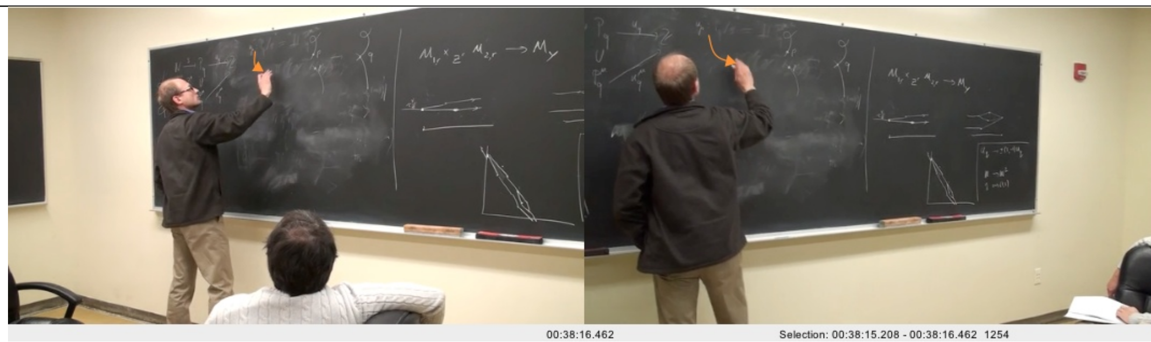
Row	Gesture & timestamp	Transcript
49		we have u_p
50		[pause]

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Table A.4 – inverted pyramid – *Continued from previous page*

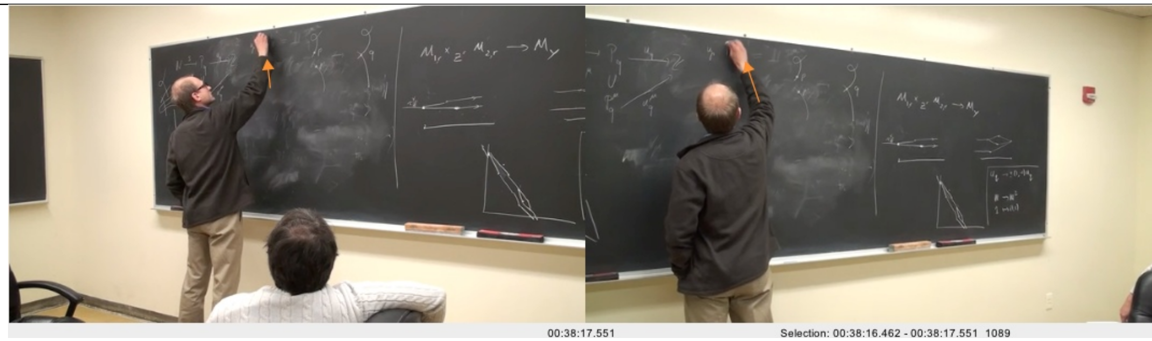
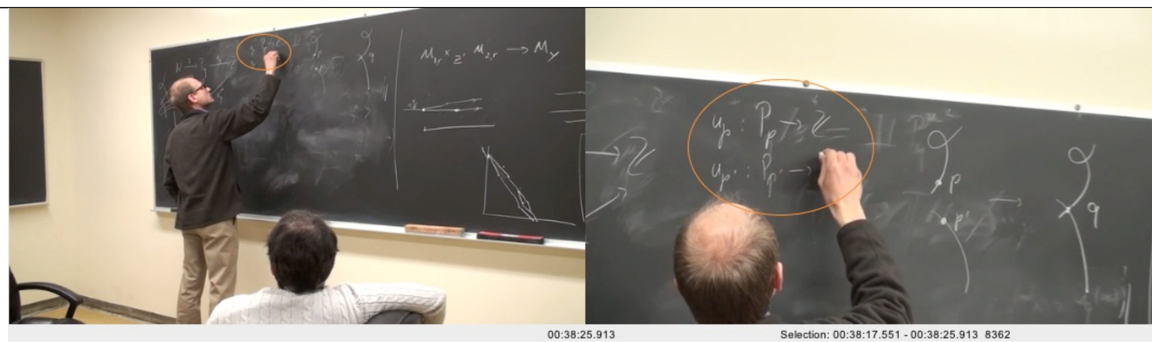
Row	Gesture & timestamp	Transcript
51		which is a map f-
52		-rom P_p to $\mathbb{Z}$ and similarly $u_{p'}$ ["'u' 'p' prime"] M: Those monoi-

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Table A.4 – inverted pyramid – *Continued from previous page*

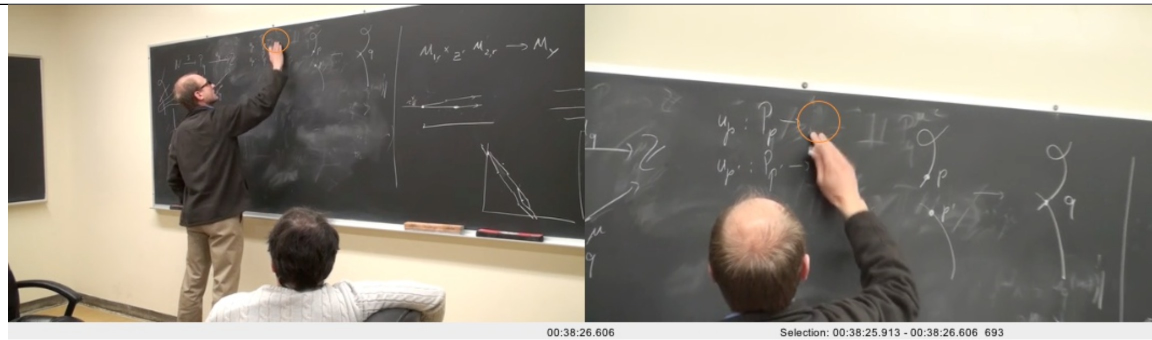
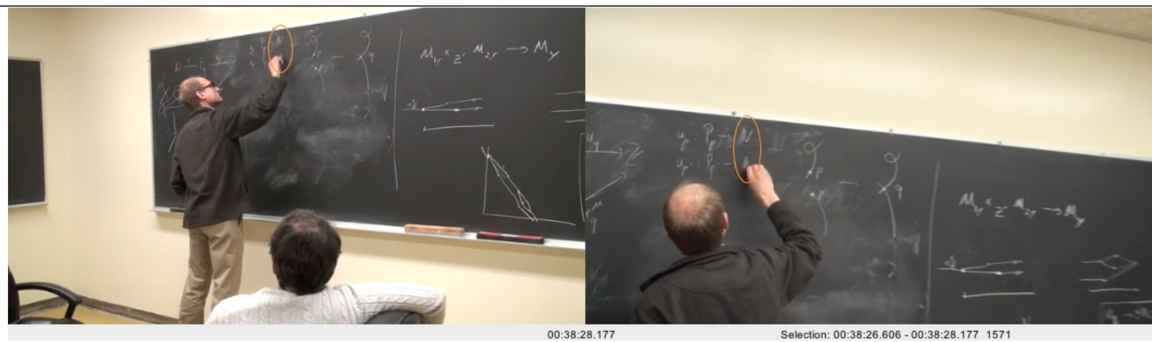
Row	Gesture & timestamp	Transcript
53		-ds should be. . . B: Uh,
54		ℕ, actually. M: And those

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Table A.4 – inverted pyramid – *Continued from previous page*

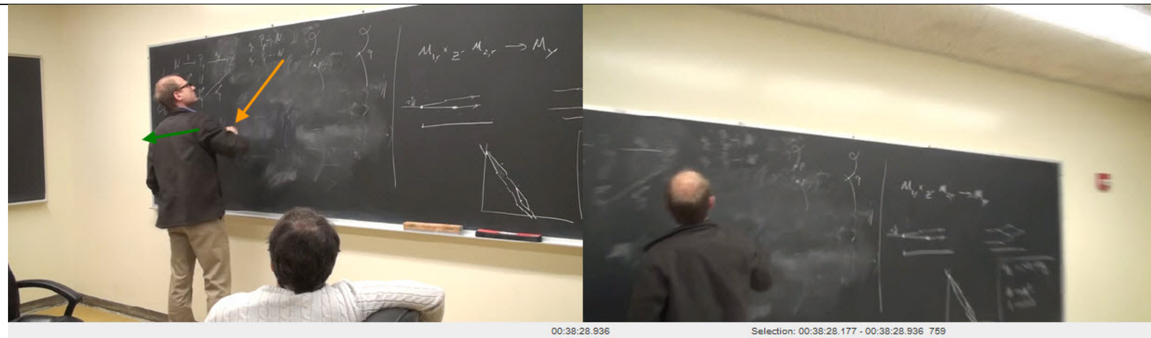
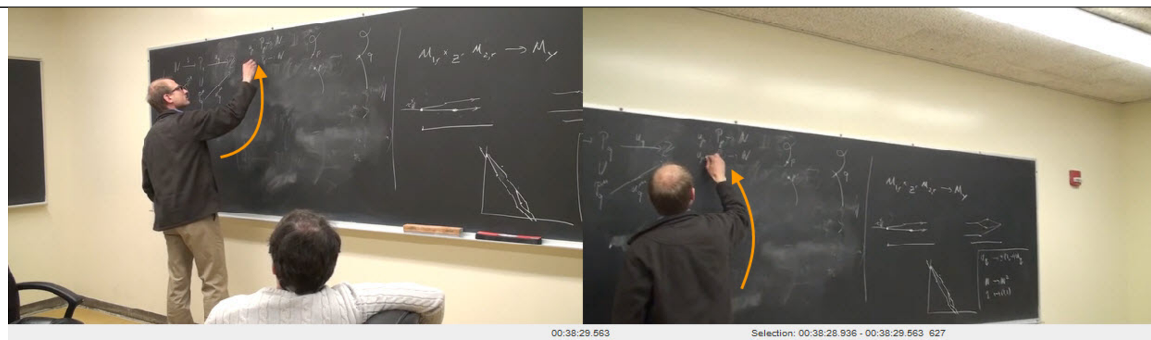
Row	Gesture & timestamp	Transcript
55		monoids are the same
56		because p and

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Table A.4 – inverted pyramid – *Continued from previous page*

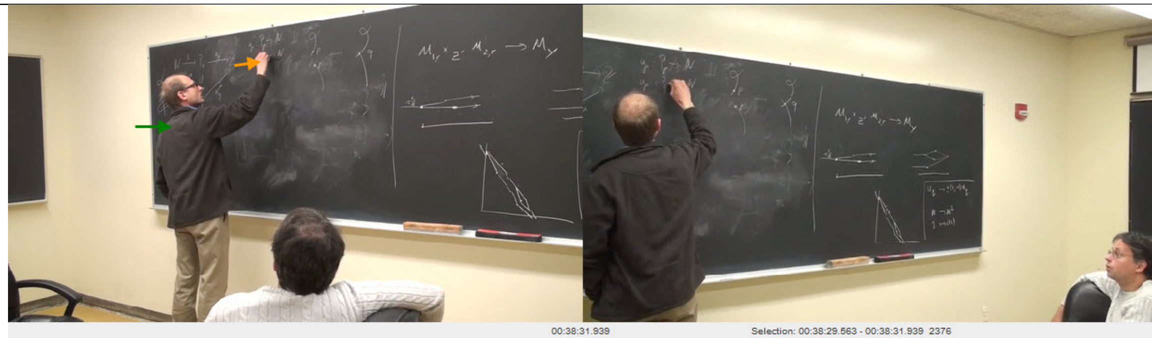
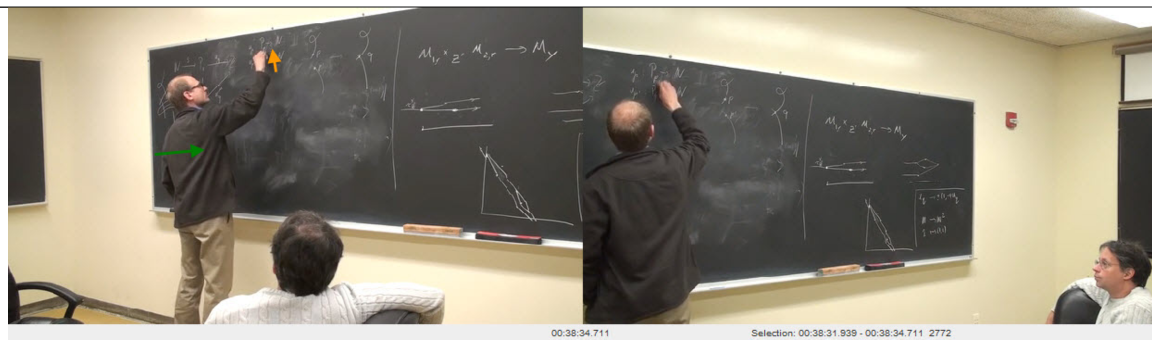
Row	**Gesture & timestamp**	**Transcript**
57		p' [“p prime”] have to be. . . B: Uh, M: . . . exactly the same thing.
58		B: Uh, exactly. That’s the same.

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Table A.4 – inverted pyramid – *Continued from previous page*

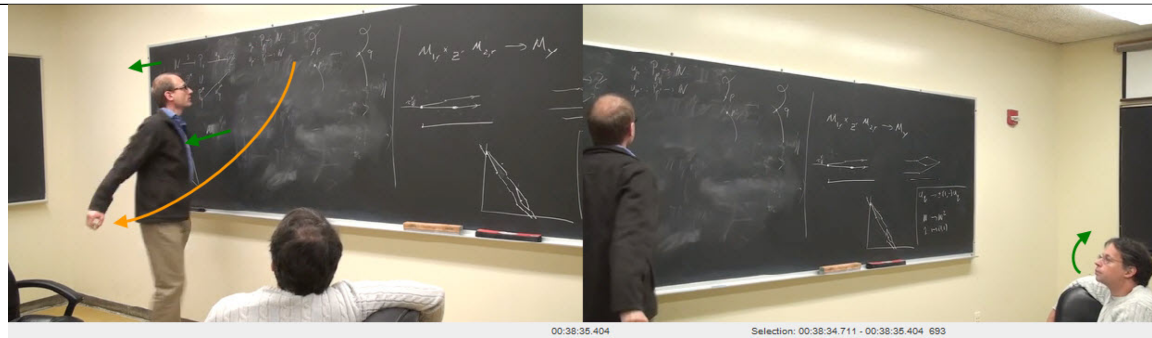
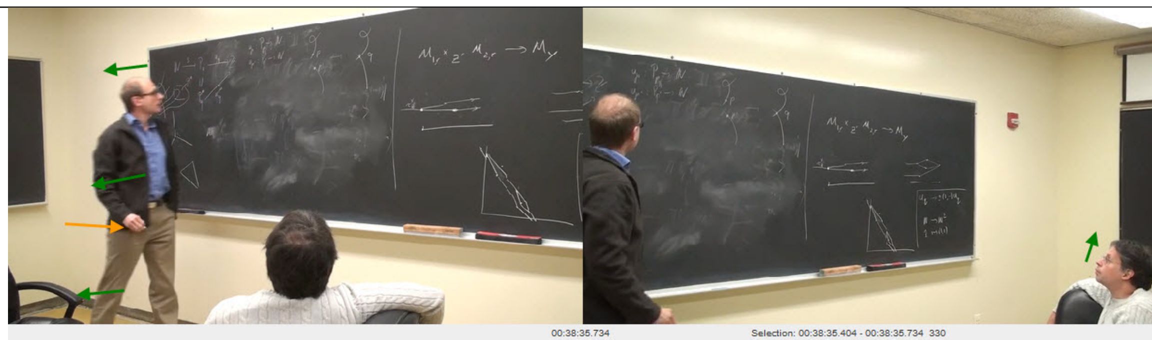
Row	Gesture & timestamp	Transcript
59		[pause]
60		[pause]

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Table A.4 – inverted pyramid – *Continued from previous page*

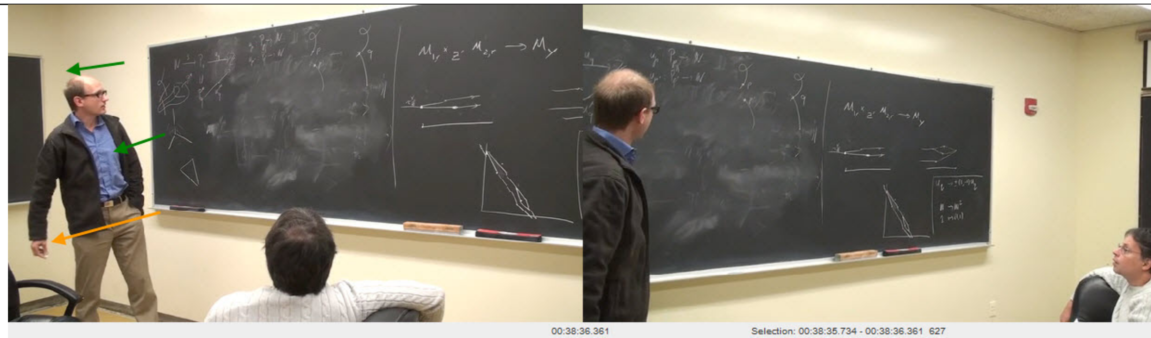
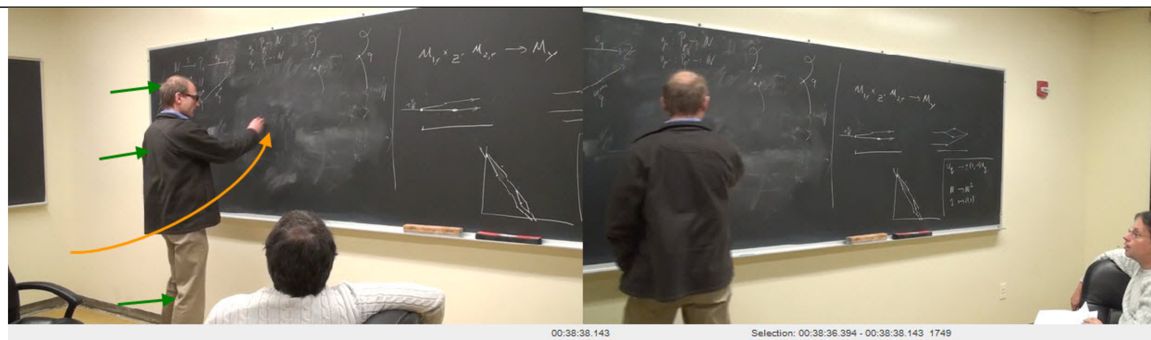
Row	Gesture & timestamp	Transcript
61		And then
62		we expect- So now we

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Table A.4 – inverted pyramid – *Continued from previous page*

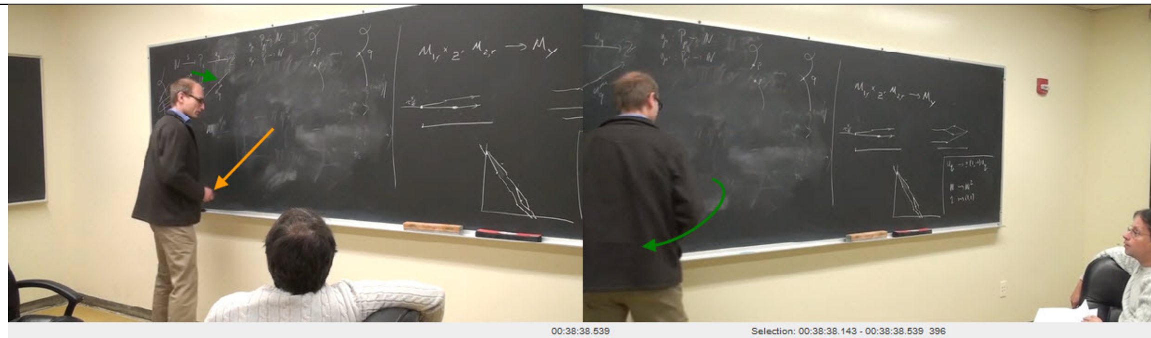
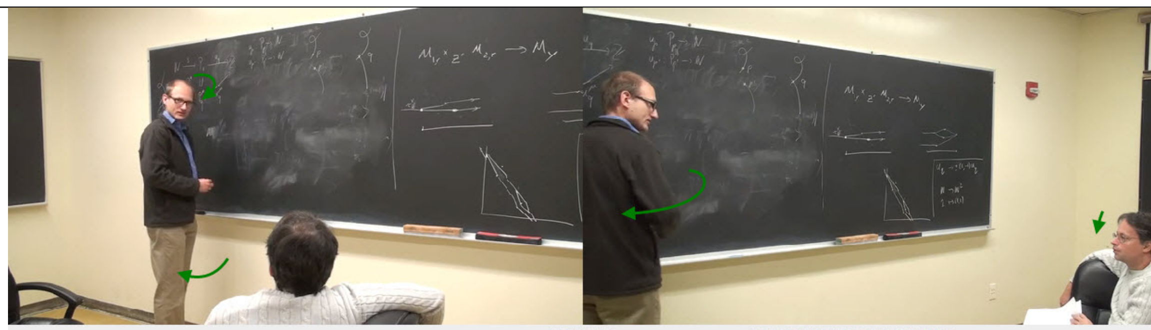
Row	Gesture & timestamp	Transcript
63		now have
64		the whole diagram that we, we drew a hundred

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Table A.4 – inverted pyramid – *Continued from previous page*

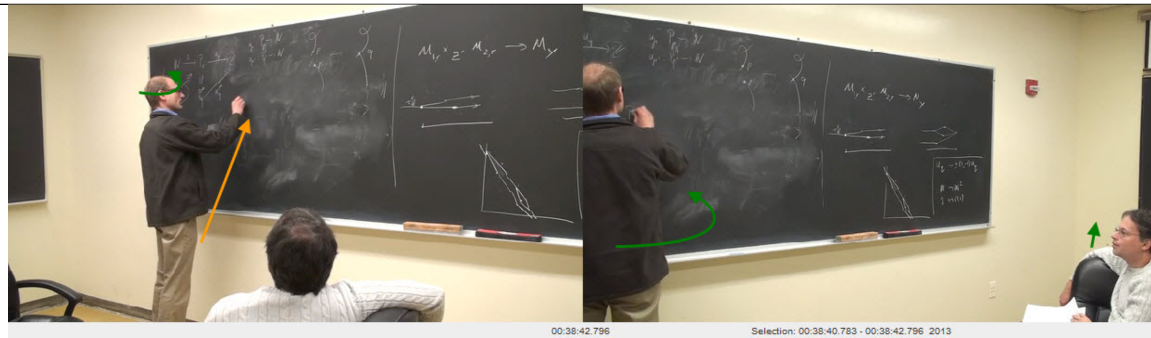
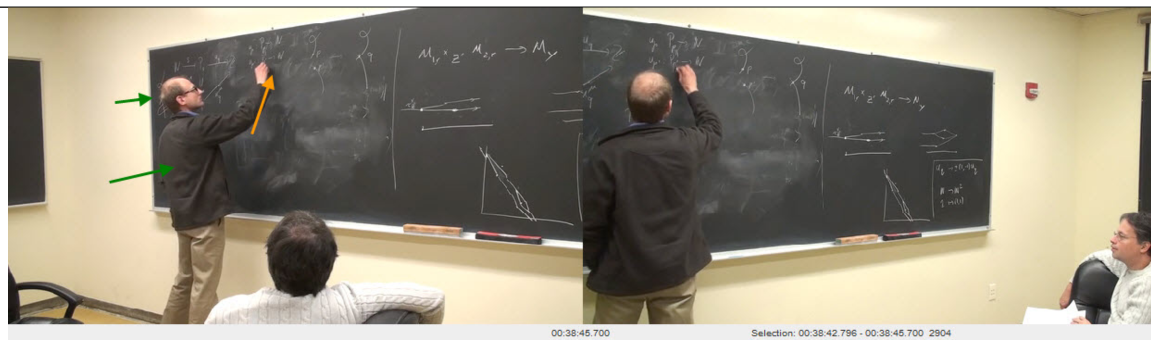
Row	Gesture & timestamp	Transcript
65		times. Um... We w-
66		-ant that the uh, and that will be the P_q.

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Table A.4 – inverted pyramid – *Continued from previous page*

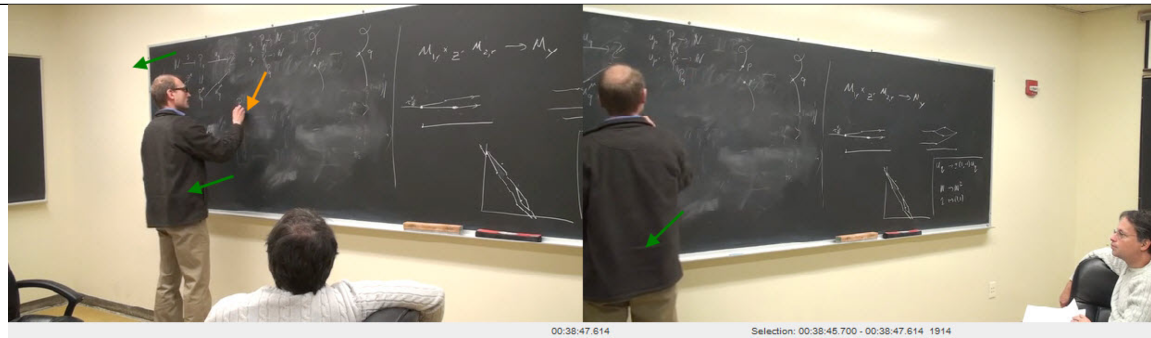
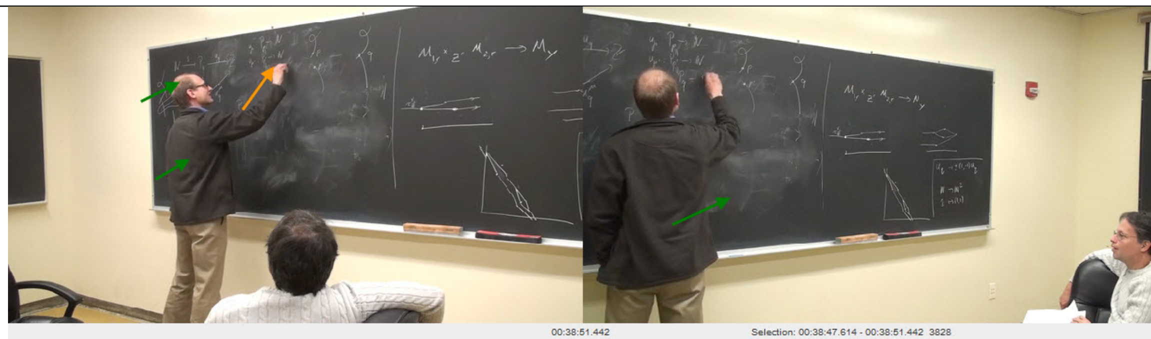
Row	Gesture & timestamp	Transcript
67		Right?
68		So let's just call it *P* [said laughingly] [pause]

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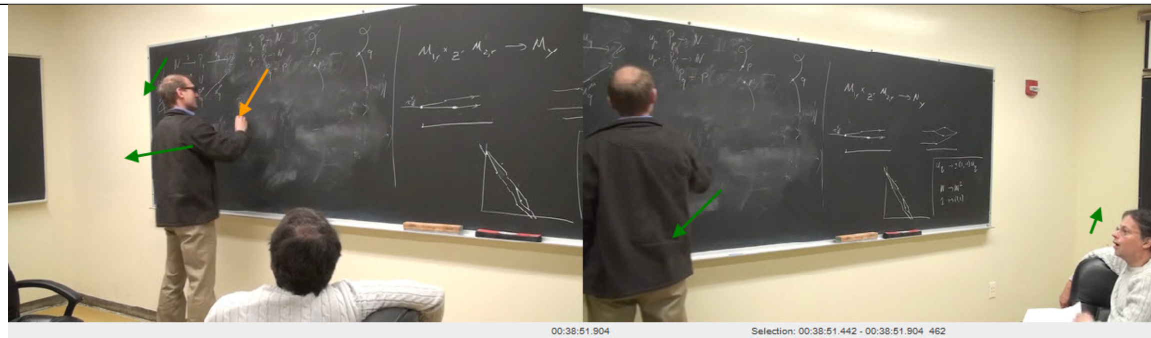
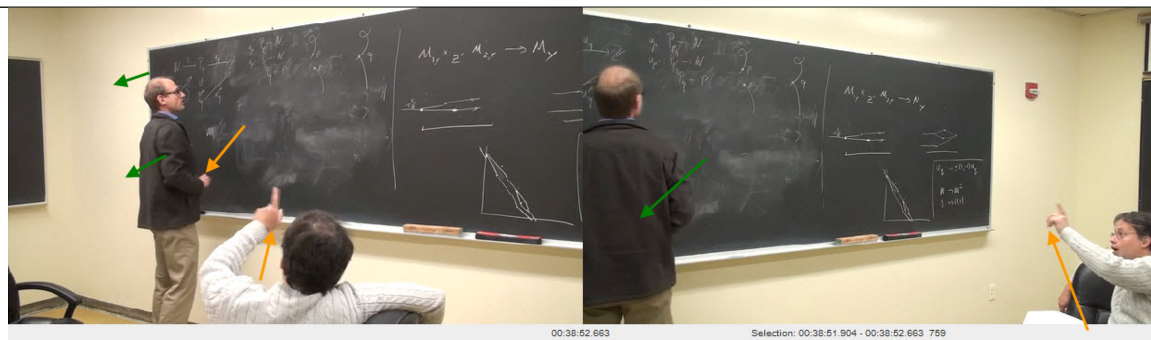
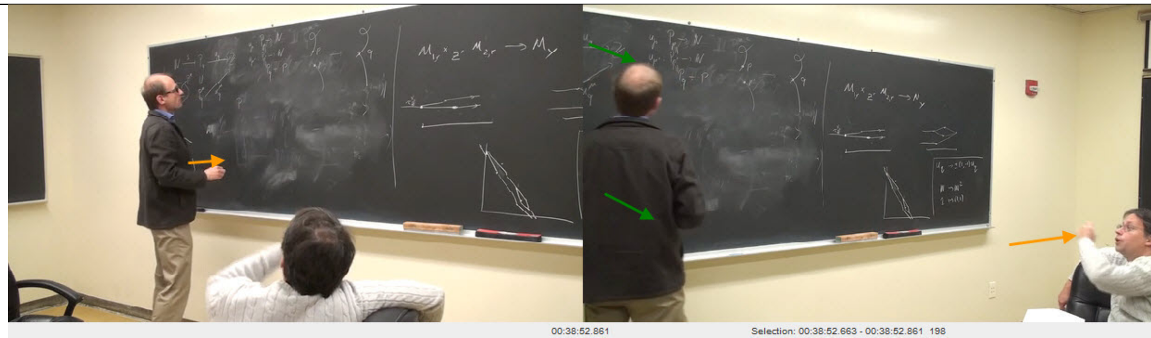
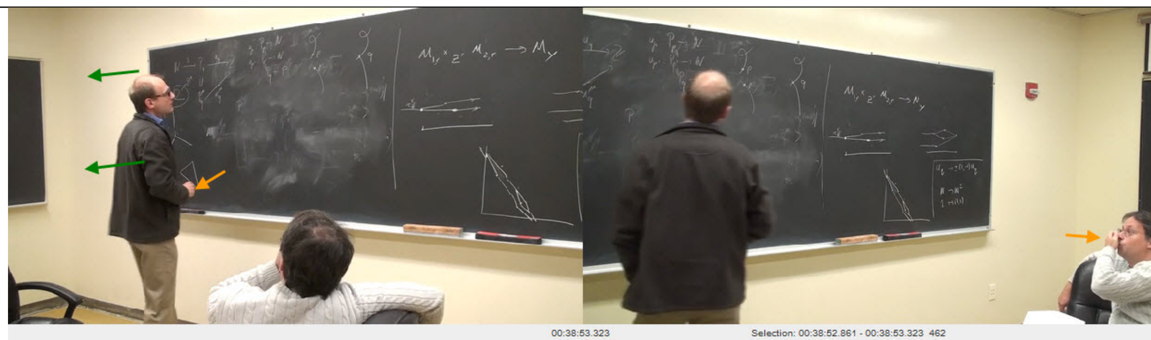
Table A.4 – inverted pyramid – *Continued from previous page*

Row	Gesture & timestamp	Transcript
69		[pause]
70		M: Oh, no no no. It's

Continued on next page

Table A.4 – inverted pyramid – *Continued from previous page*

Row	Gesture & timestamp	Transcript
71		*not*
72		P_q; so...

Continued on next page

Table A.4 – inverted pyramid – *Continued from previous page*

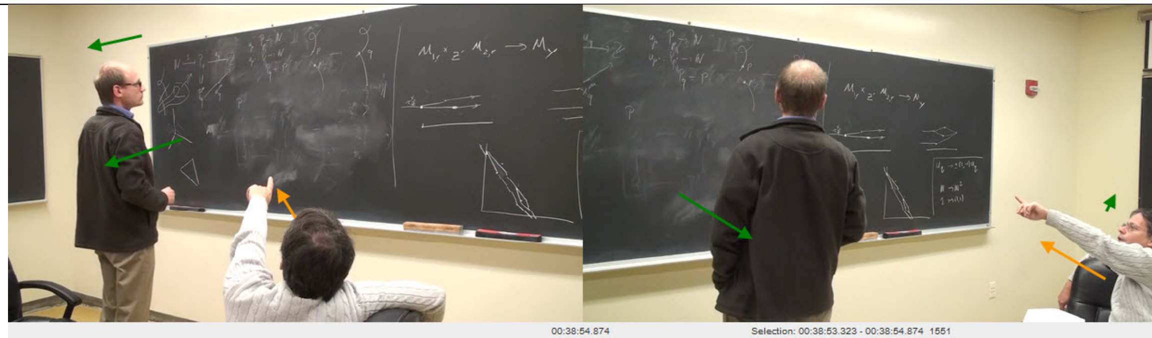
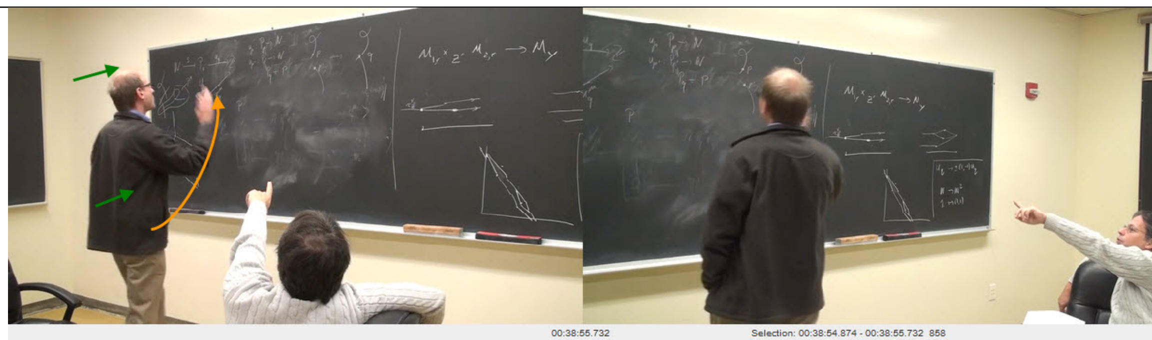
Row	Gesture & timestamp	Transcript
73		B: Um M: . . . so
74		we have to be. . . B: Oh, oh!

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Table A.4 – inverted pyramid – *Continued from previous page*

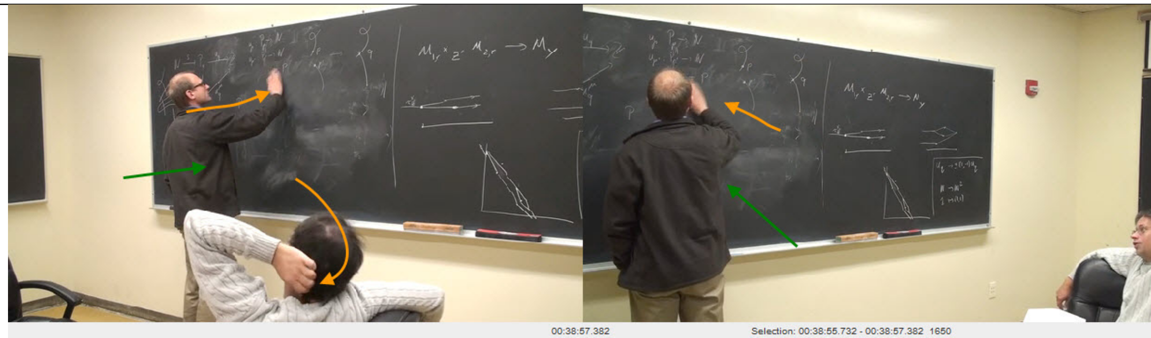
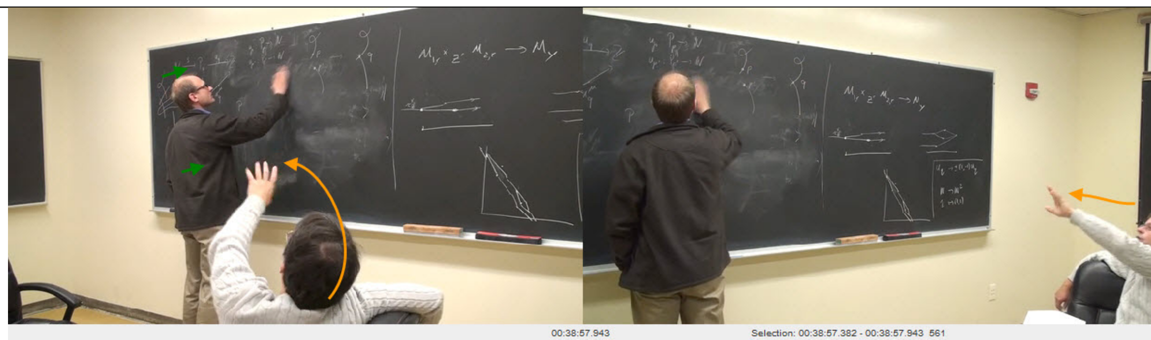
Row	Gesture & timestamp	Transcript
75		Wait, wait. M: [inaudible] B: It's *not* right.
76		M: It's completely wrong;

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Table A.4 – inverted pyramid – *Continued from previous page*

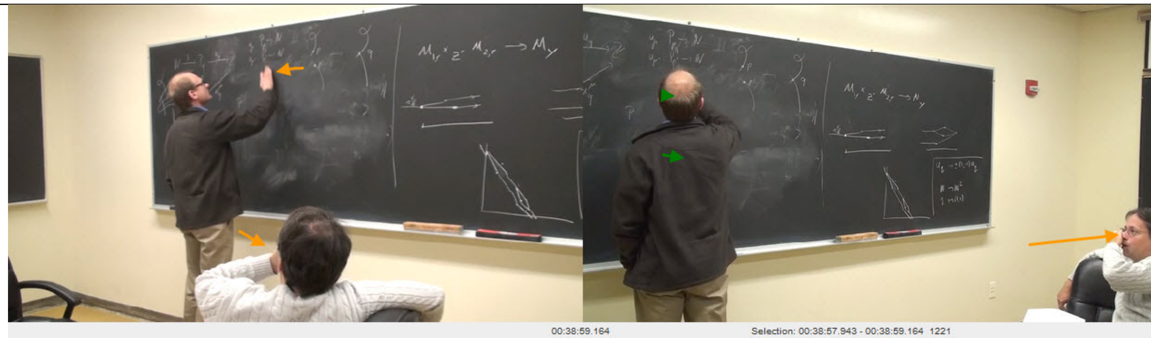
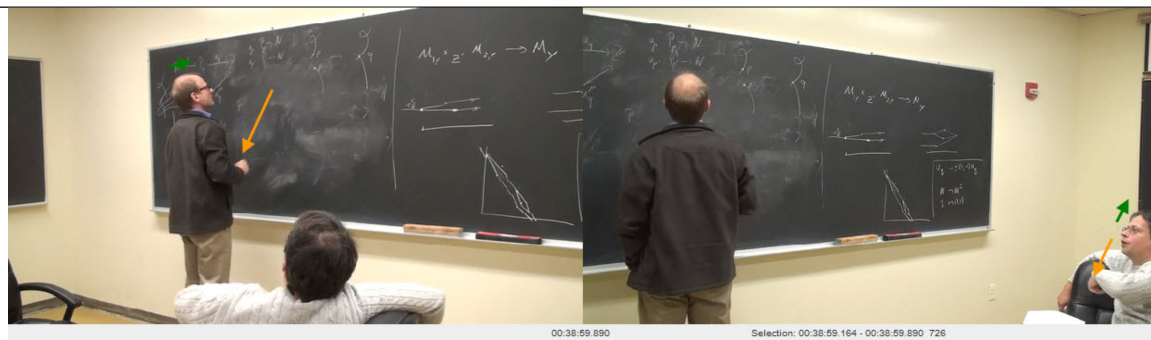
Row	Gesture & timestamp	Transcript
77		they're *not* equal because one
78		of them

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Table A.4 – inverted pyramid – *Continued from previous page*

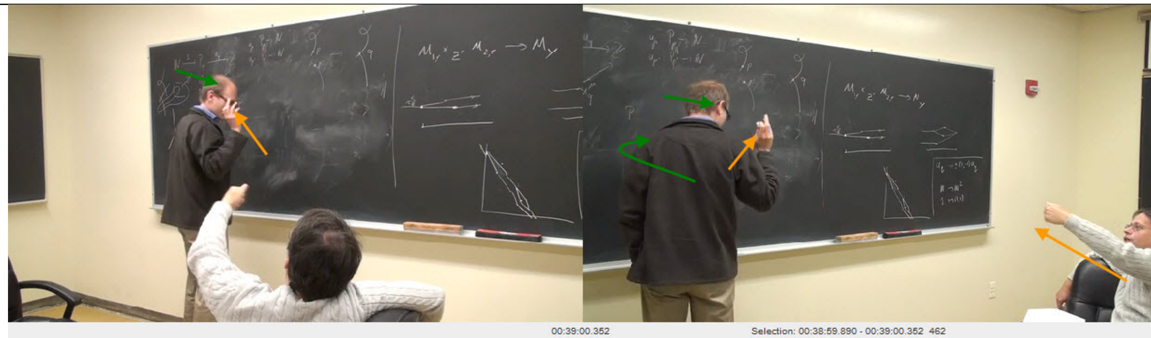
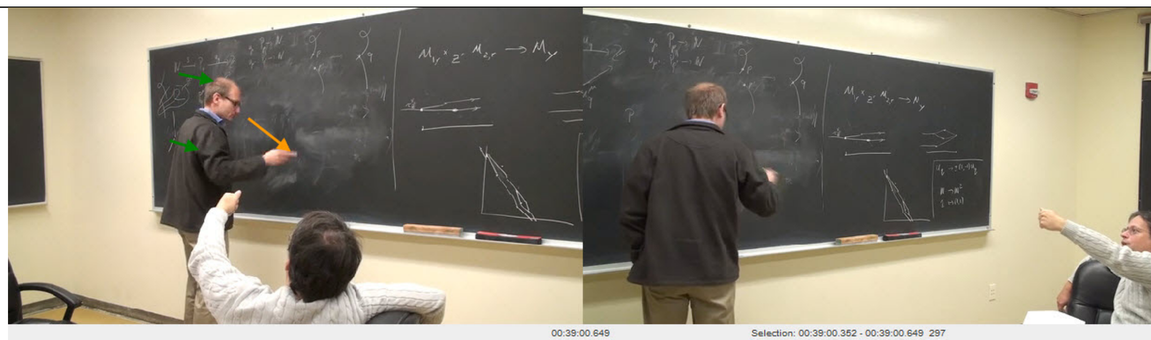
Row	Gesture & timestamp	Transcript
79		B: They are. . . M: One. . .
80		B: . . . different. . . M: . . . is. . .

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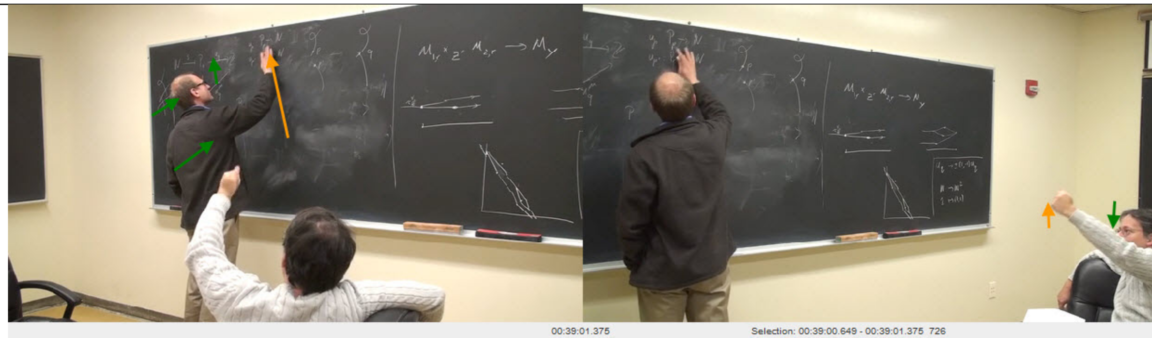
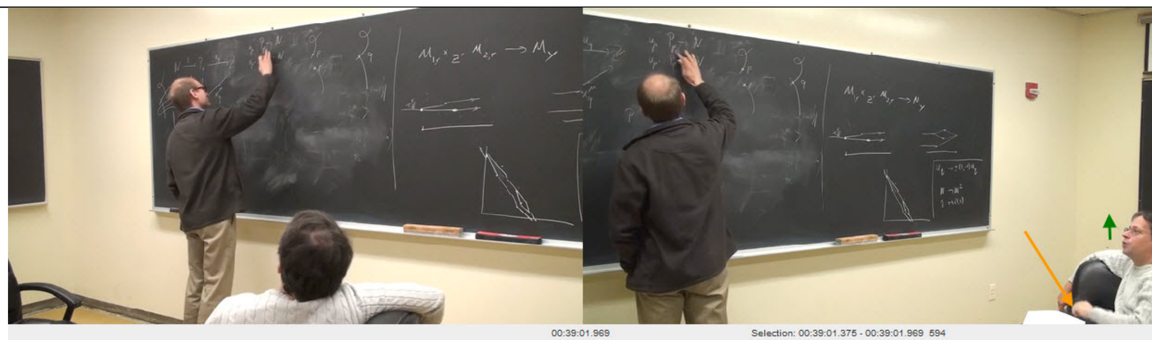
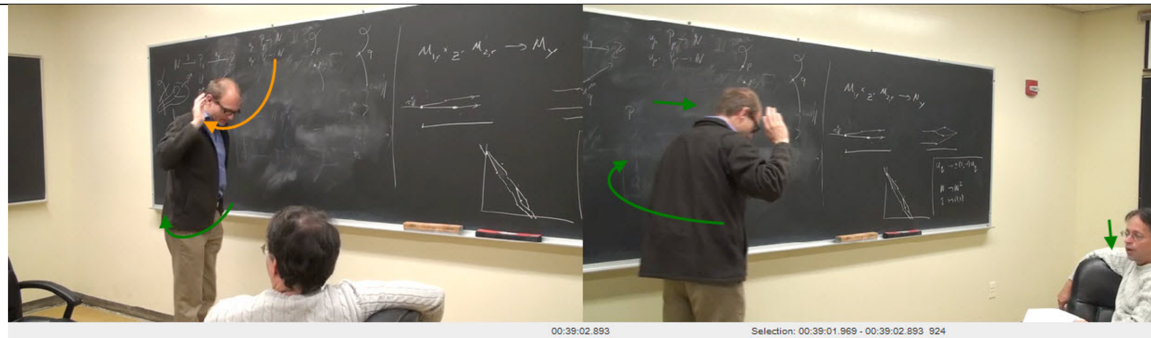
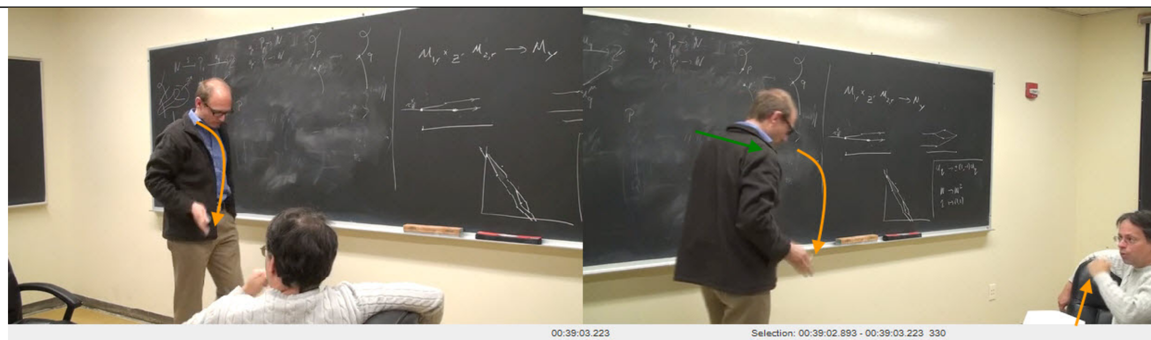
Table A.4 – inverted pyramid – *Continued from previous page*

Row	Gesture & timestamp	Transcript
81		B: . . . components. M: . . . P- P_q
82		and one is

Continued on next page

Table A.4 – inverted pyramid – *Continued from previous page*

Row	Gesture & timestamp	Transcript
83		$P_{p'}$ ["P p prime"]
84		[inaudible]. . . B: They are bo-

Continued on next page

Table A.4 – inverted pyramid – *Continued from previous page*

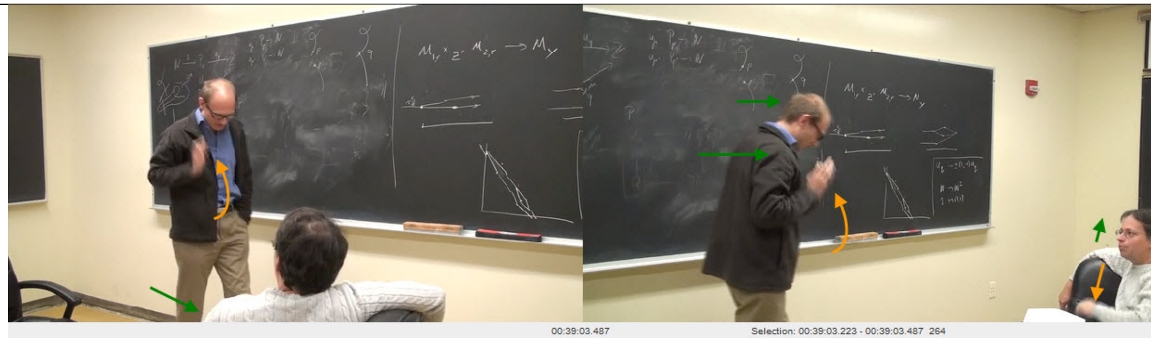
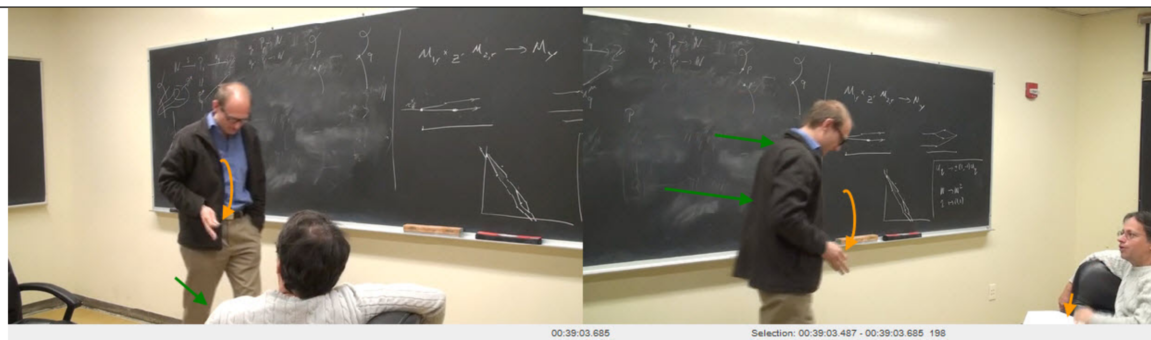
Row	Gesture & timestamp	Transcript
85		M: . . . [inaudible]. . . B: -th f-
86		M: . . . [inaudible]. . . B: -ac-

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Table A.4 – inverted pyramid – *Continued from previous page*

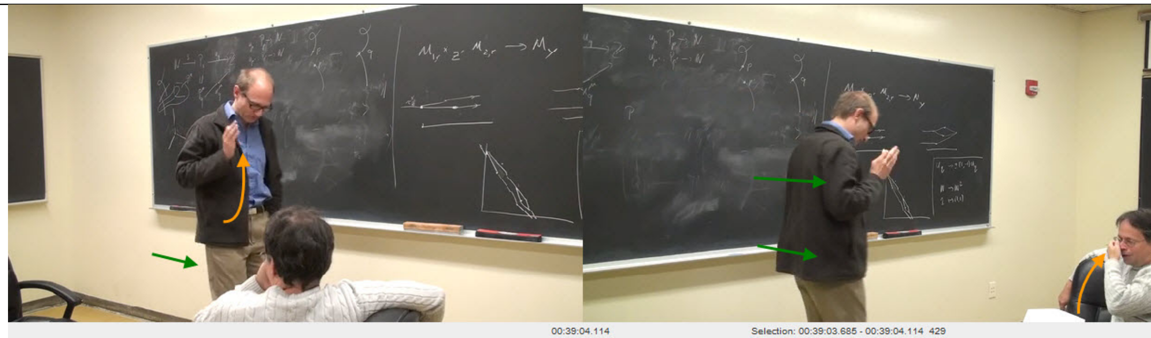
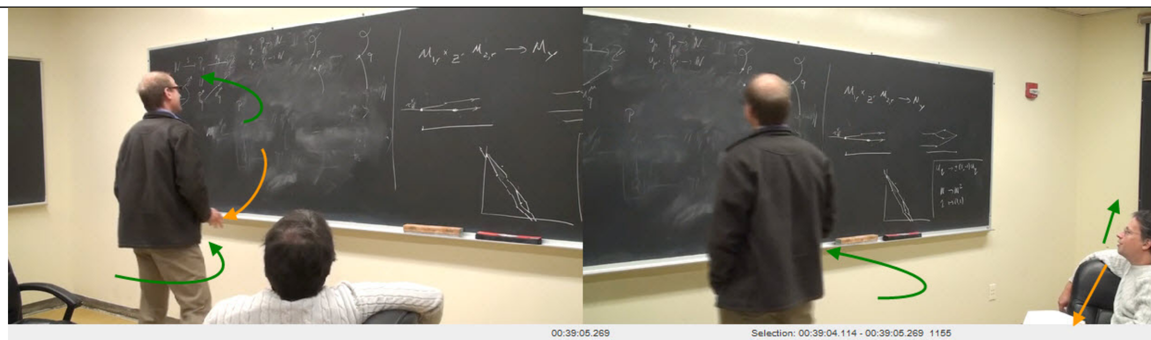
Row	Gesture & timestamp	Transcript
87		-es of
88		P_q. M: Yeah.

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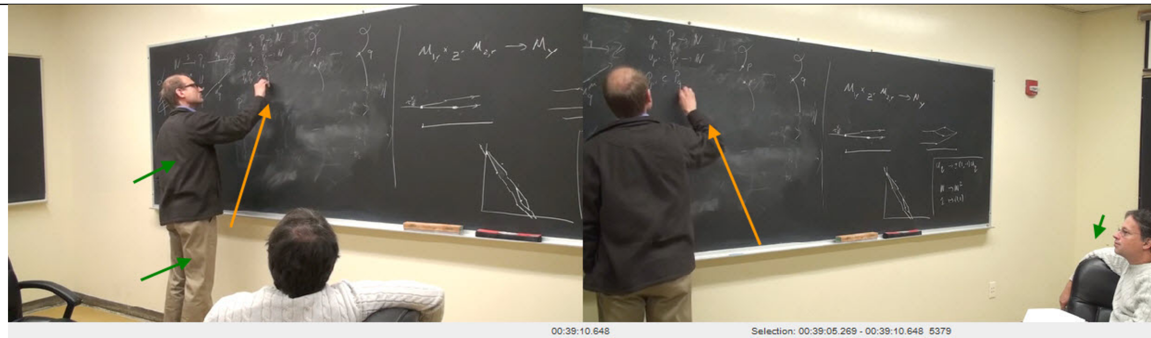
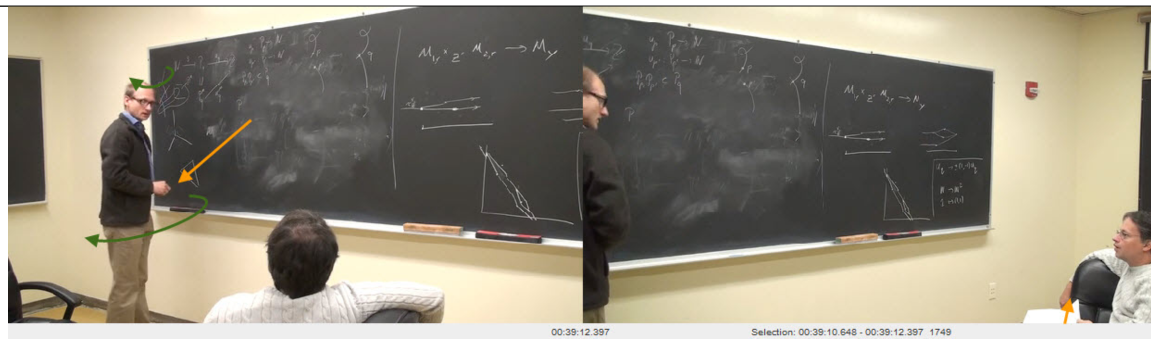
Table A.4 – inverted pyramid – *Continued from previous page*

Row	Gesture & timestamp	Transcript
89		B: [laughs] Okay. So we have P_p, $P_{p'}$ ["P p prime"] faces of
90		P_q.

Continued on next page

Table A.4 – inverted pyramid – *Continued from previous page*

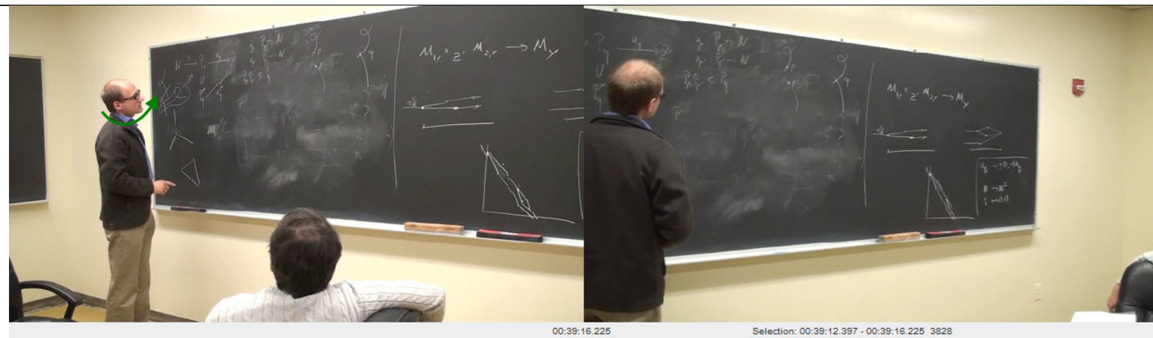
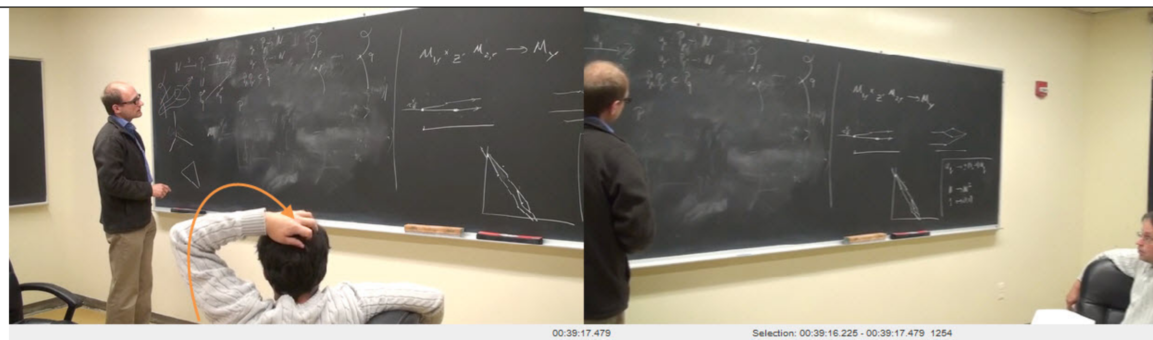
Row	Gesture & timestamp	Transcript
91		M: And then I think what we want is that, um, B: But now actually
92		we uh, we, [sigh]

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Table A.4 – inverted pyramid – *Continued from previous page*

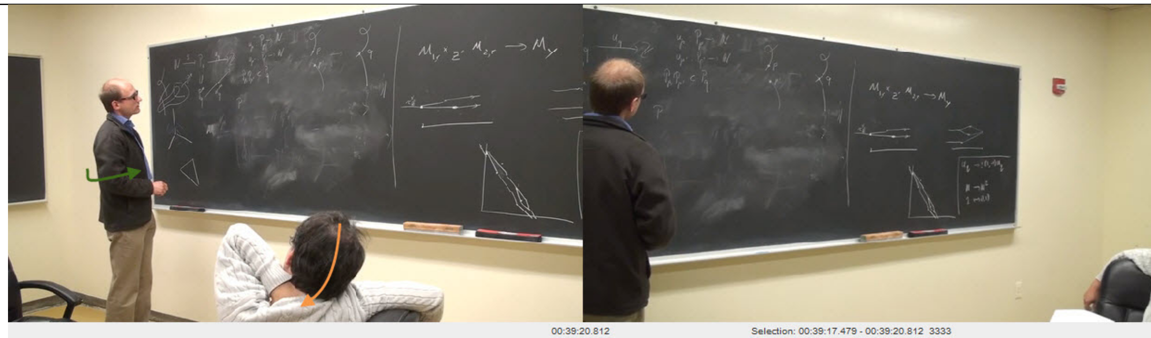
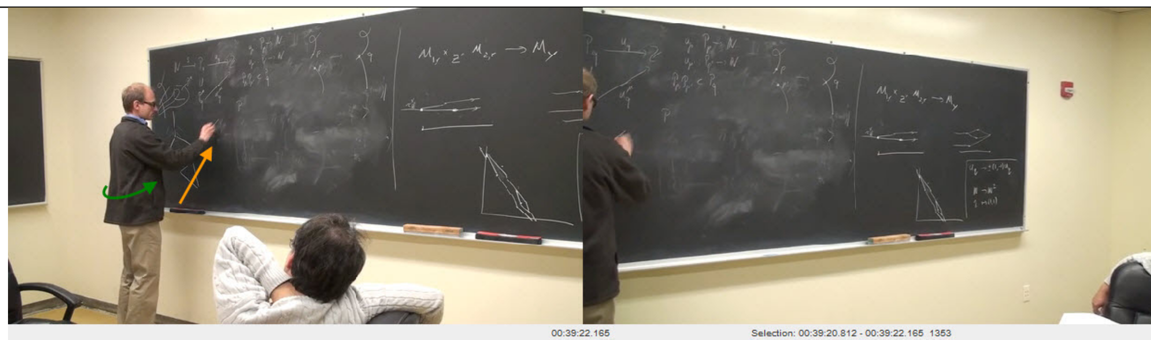
Row	Gesture & timestamp	Transcript
93		[pause] I see, oh. Oh. I see. Yeah. M: That u
94		q [indicating u_q] should be

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Table A.4 – inverted pyramid – *Continued from previous page*

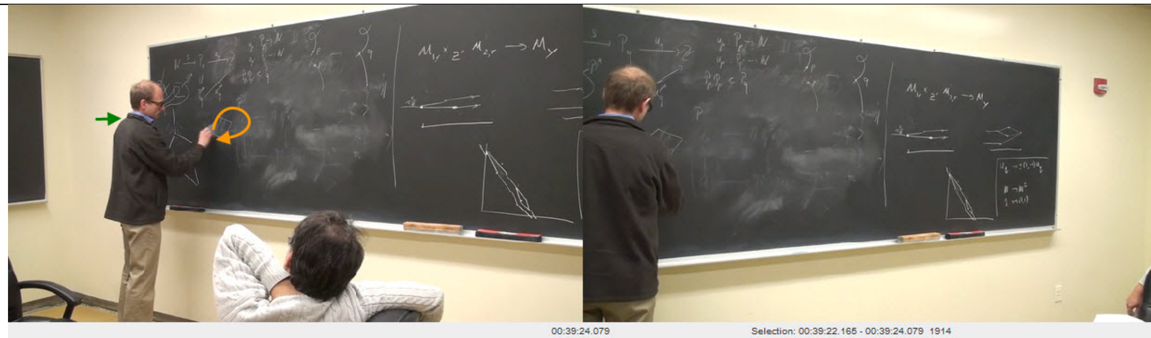
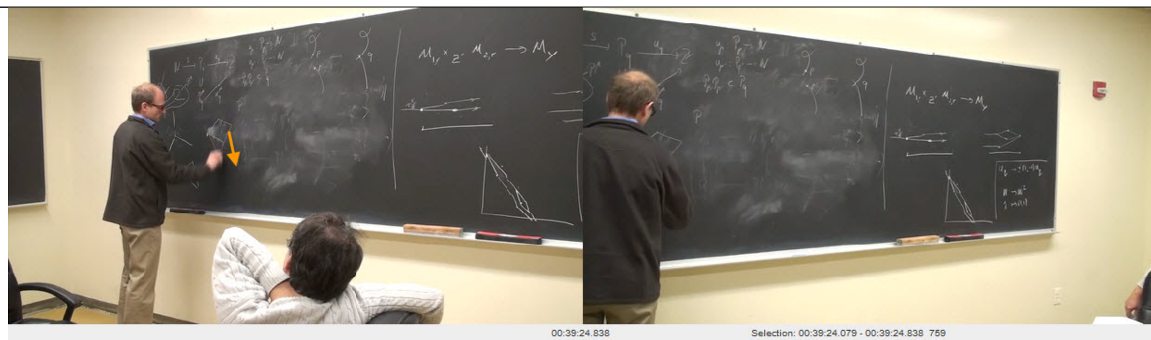
Row	Gesture & timestamp	Transcript
95		uh, [pause] something
96		that actually *lift-*

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Table A.4 – inverted pyramid – *Continued from previous page*

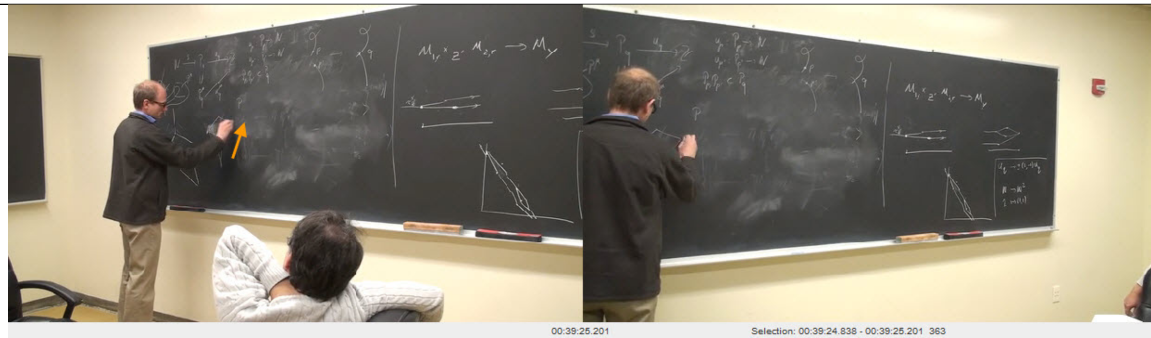
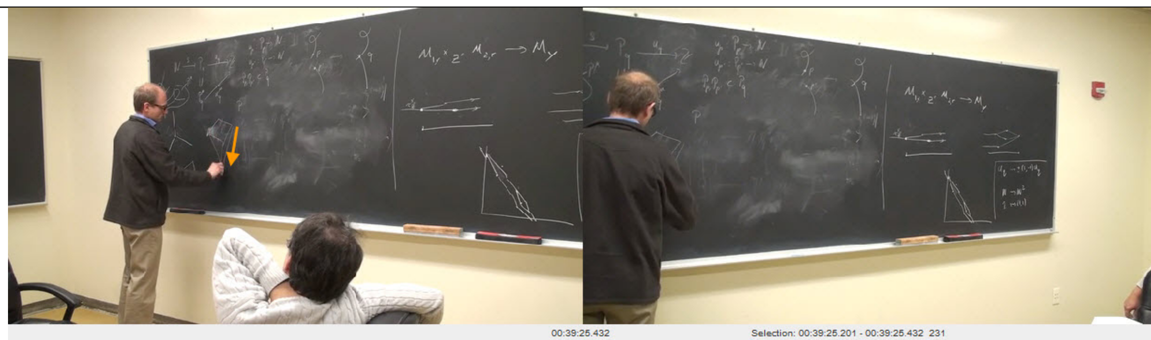
Row	Gesture & timestamp	Transcript
97		-*s*
98		[pause]

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Table A.4 – inverted pyramid – *Continued from previous page*

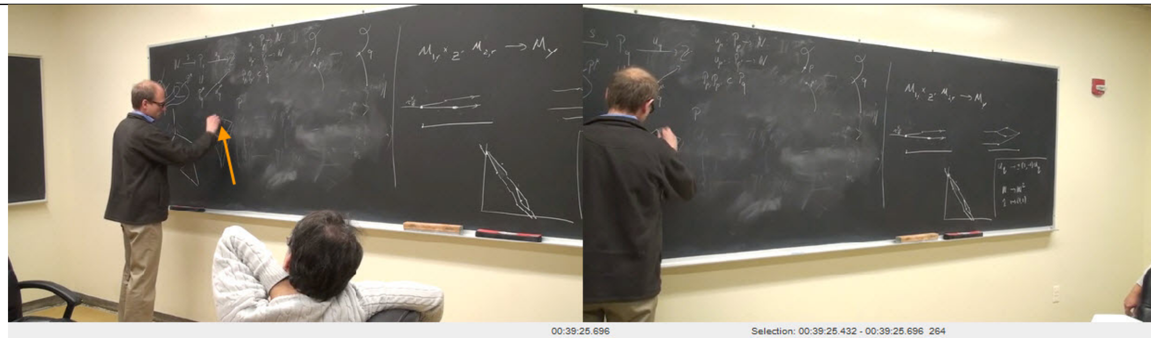
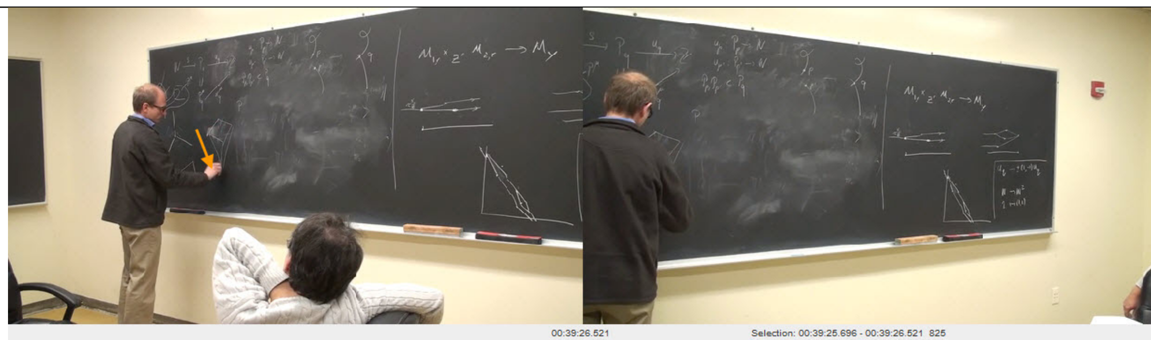
Row	Gesture & timestamp	Transcript
99		[pause]
100		[pause]

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Table A.4 – inverted pyramid – *Continued from previous page*

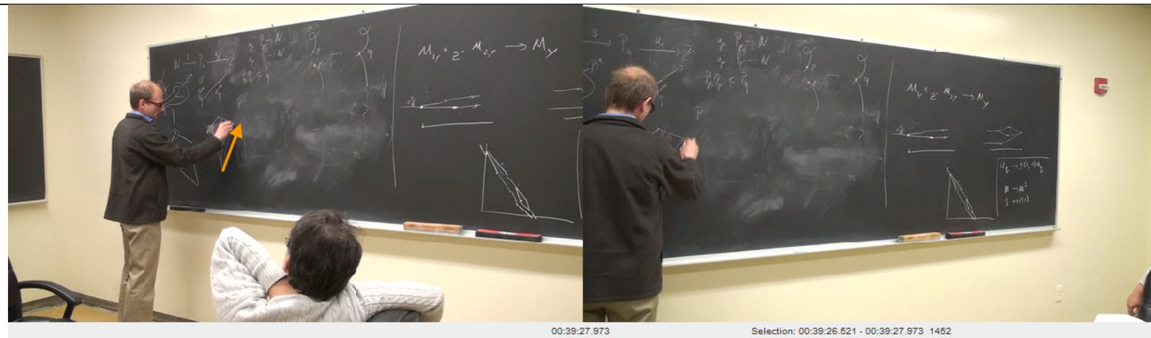
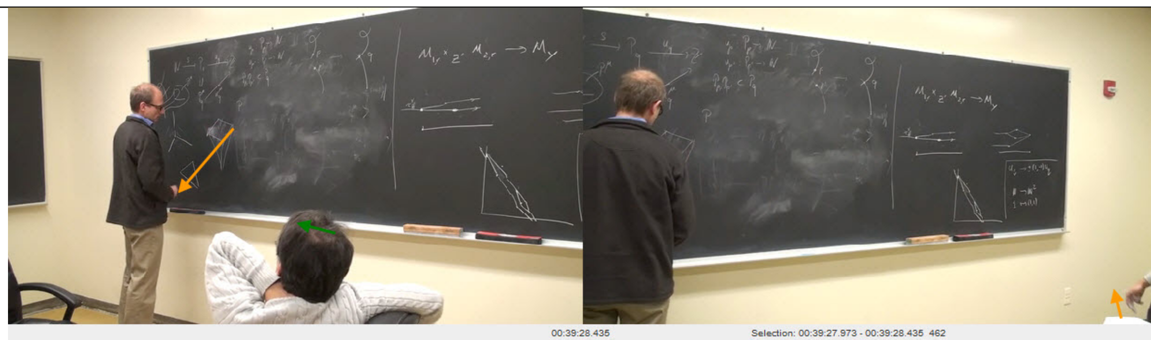
Row	Gesture & timestamp	Transcript
101		lifts th- those two
102		or rather l-

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Table A.4 – inverted pyramid – *Continued from previous page*

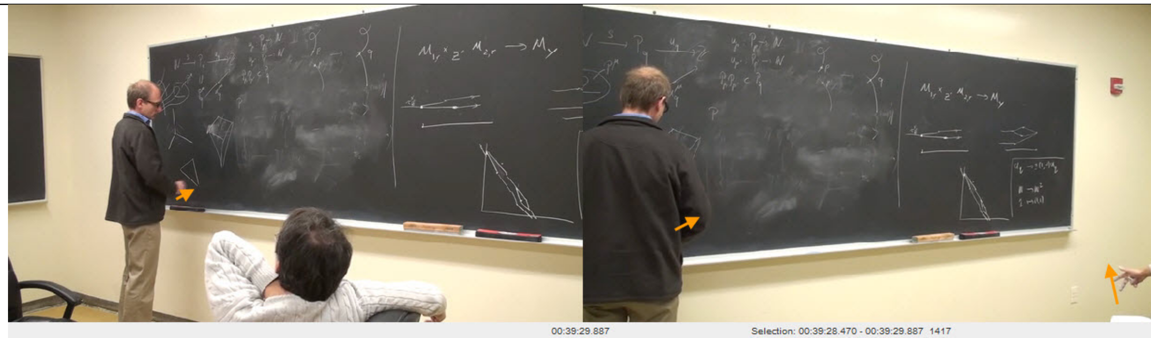
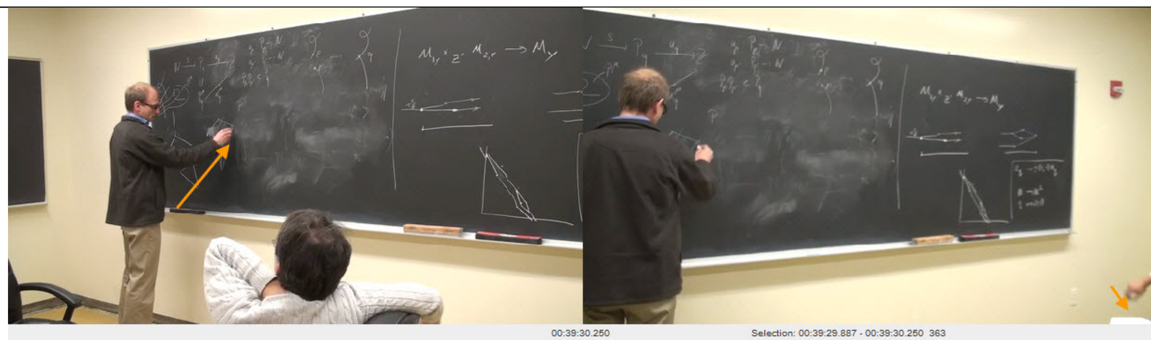
Row	Gesture & timestamp	Transcript
103		-ifts u_p and then u- mi-
104		-nus u

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Table A.4 – inverted pyramid – *Continued from previous page*

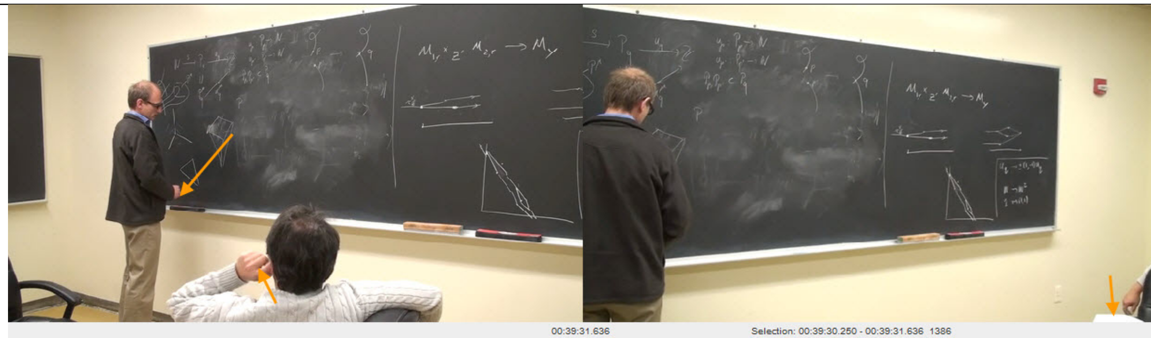
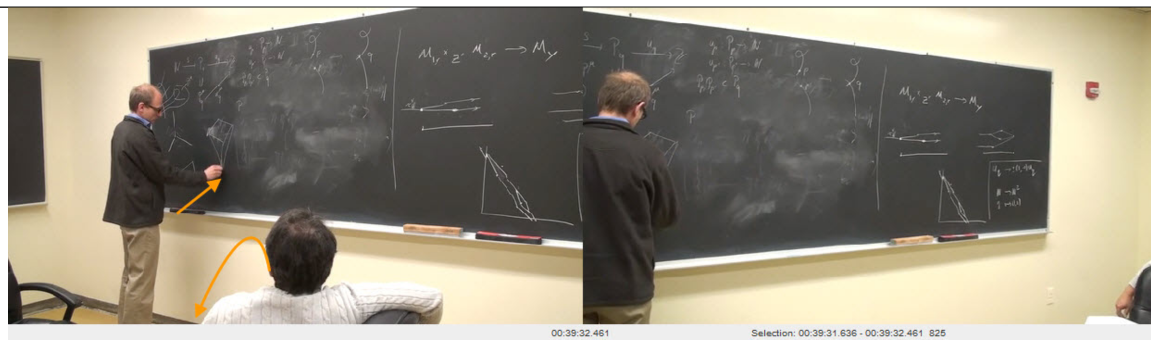
Row	Gesture & timestamp	Transcript
105		p prime [saying $u_{p'}$].
106		[pause]

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Table A.4 – inverted pyramid – *Continued from previous page*

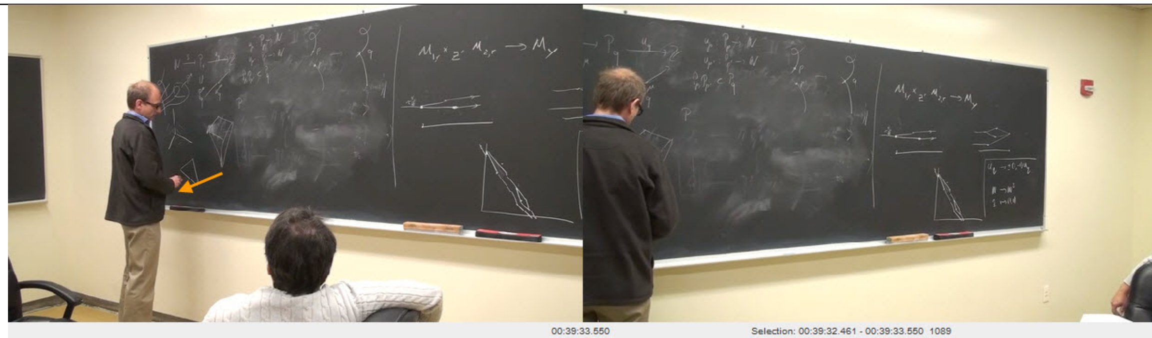
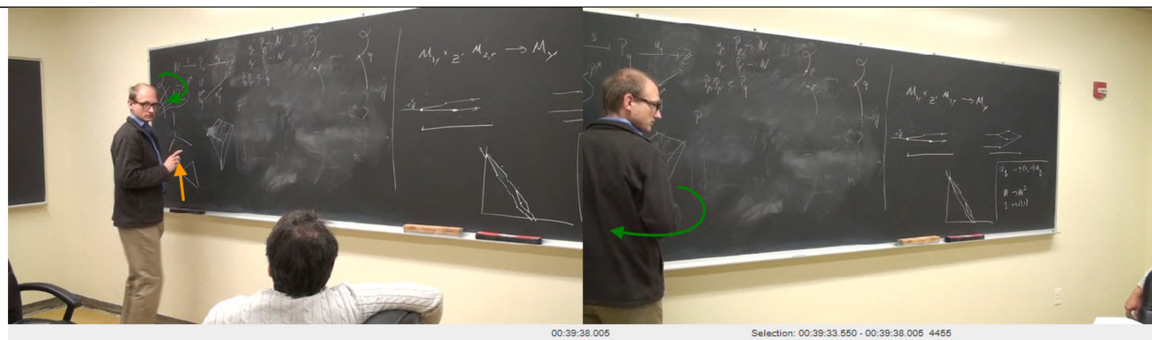
Row	Gesture & timestamp	Transcript
107		B: Yeah but
108		it's *facets*, right? Maybe that's important. It's still facets.

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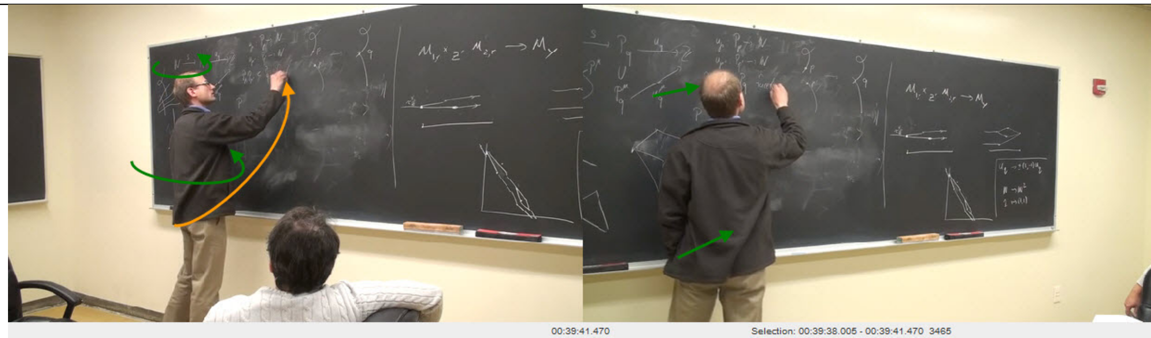
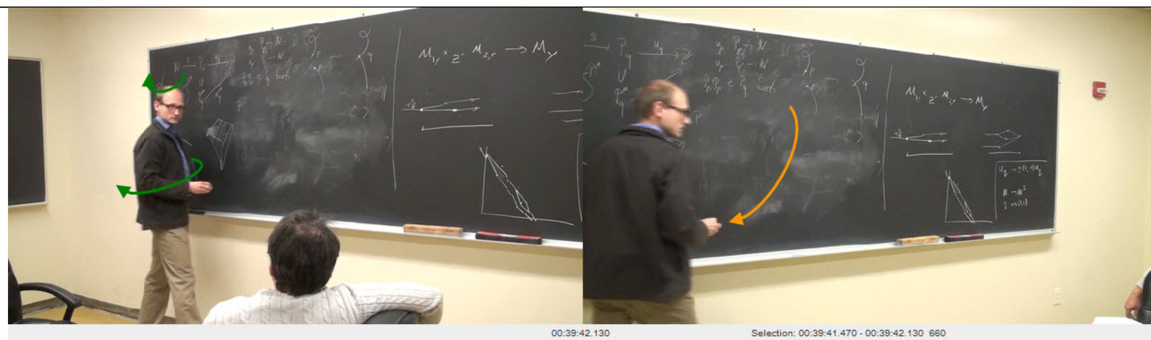
Table A.4 – inverted pyramid – *Continued from previous page*

Row	Gesture & timestamp	Transcript
109		M: Yeah. Yeah. B: Not any high codi- mension
110		things.

Continued on next page

Table A.4 – inverted pyramid – *Continued from previous page*

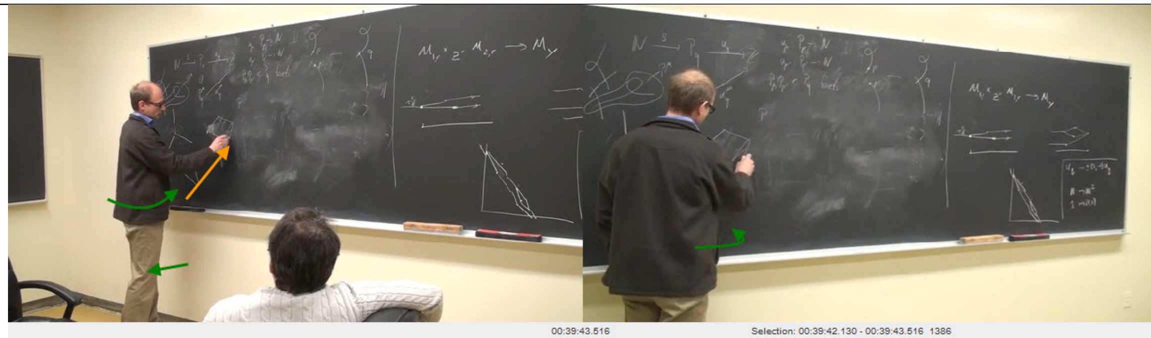
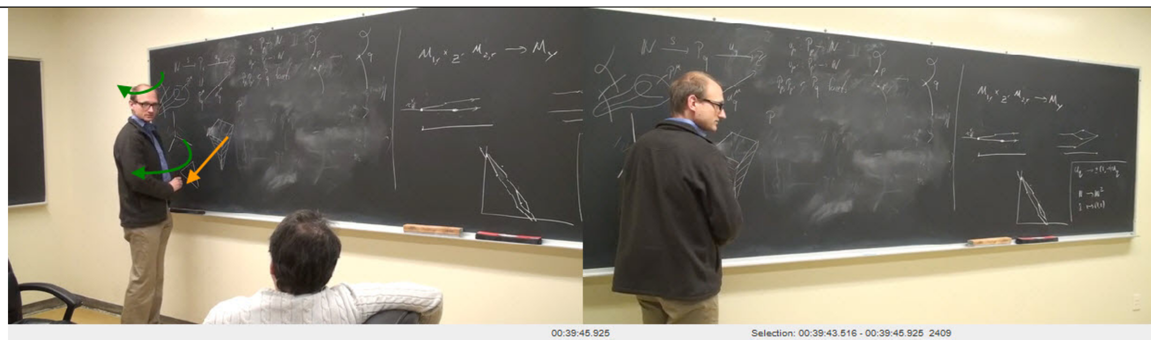
Row	Gesture & timestamp	Transcript
111		[pause] So it could be
112		for example this and that, right?

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Table A.4 – inverted pyramid – *Continued from previous page*

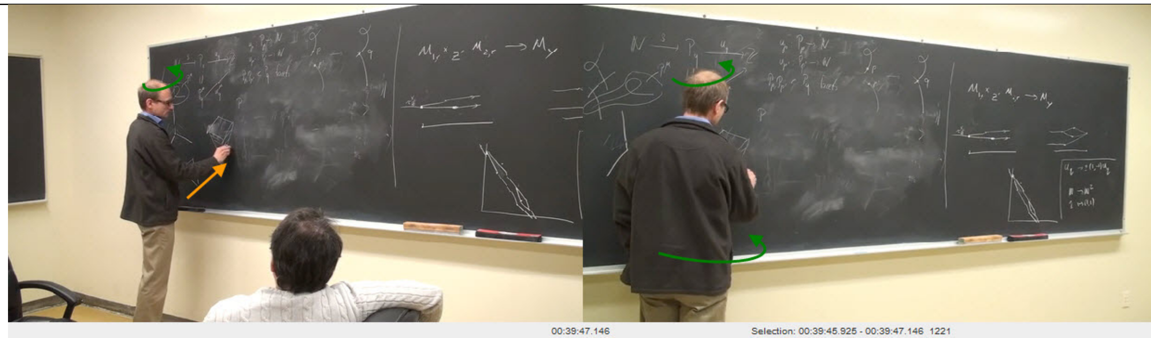
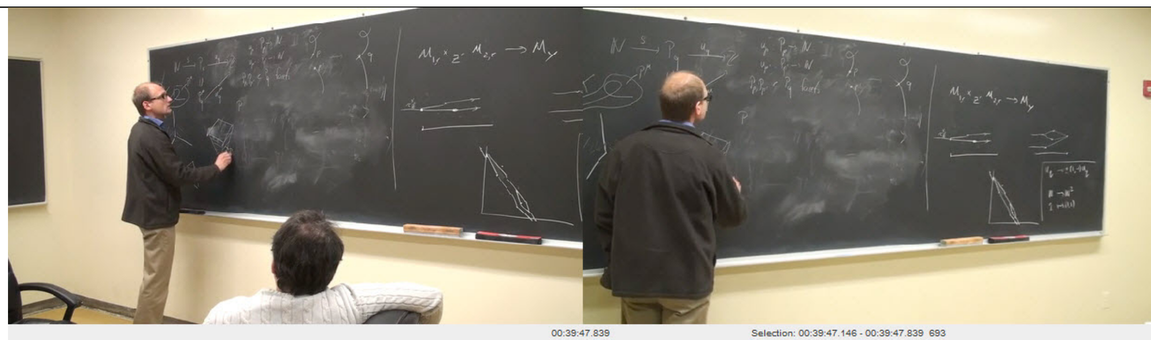
Row	Gesture & timestamp	Transcript
113		M: Mhm.
114		[pause]

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Table A.4 – inverted pyramid – *Continued from previous page*

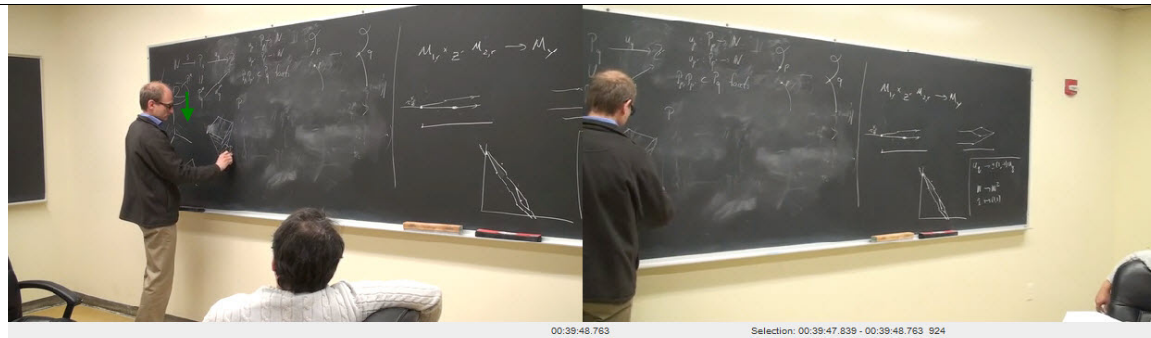
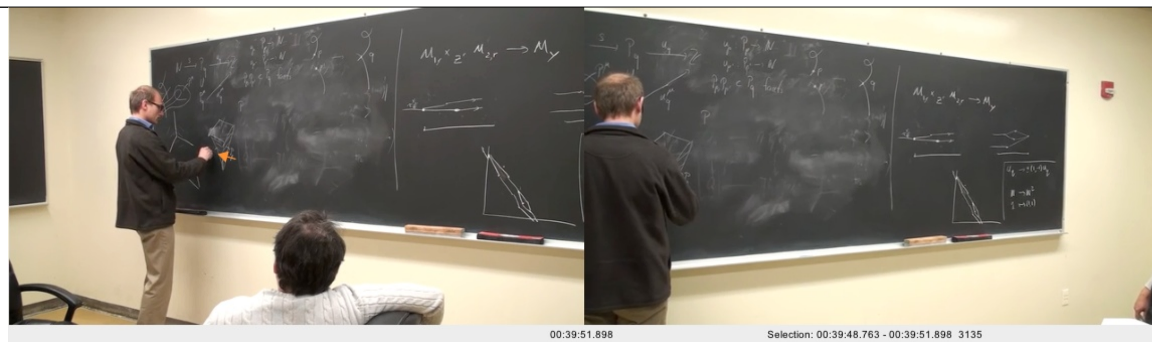
Row	Gesture & timestamp	Transcript
115		[pause]
116		[pause]

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Table A.4 – inverted pyramid – *Continued from previous page*

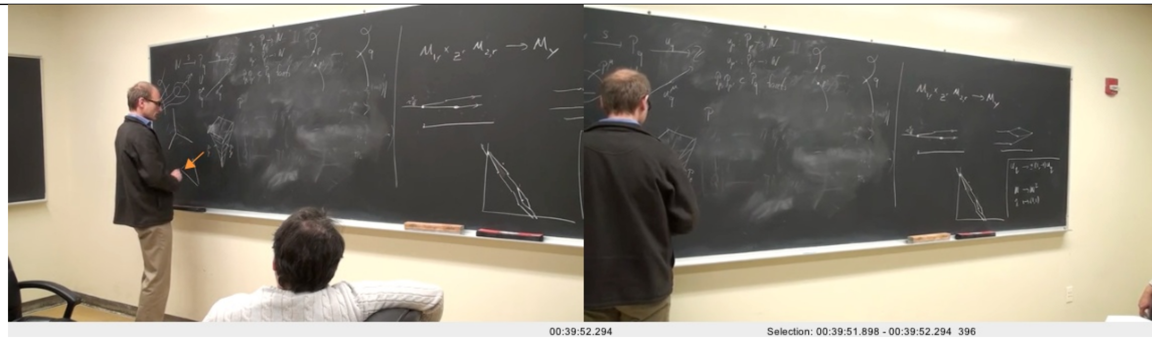
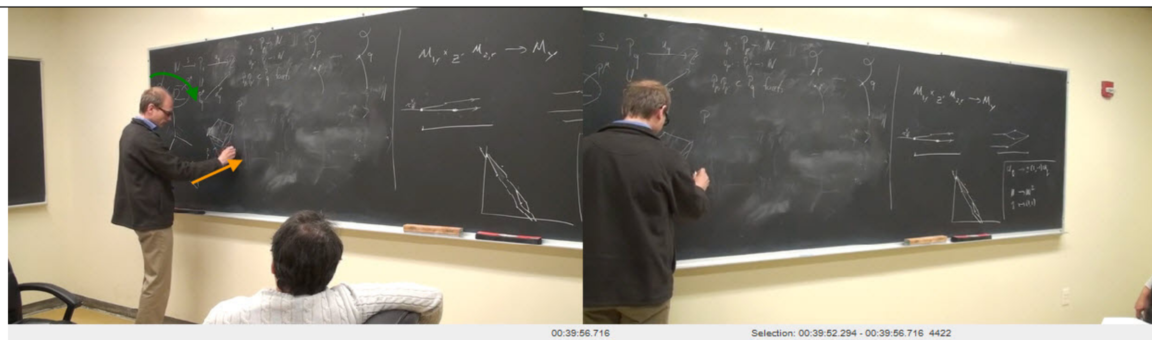
Row	Gesture & timestamp	Transcript
117		[pause]
118		B: Well, q. [pause] Eh, no.

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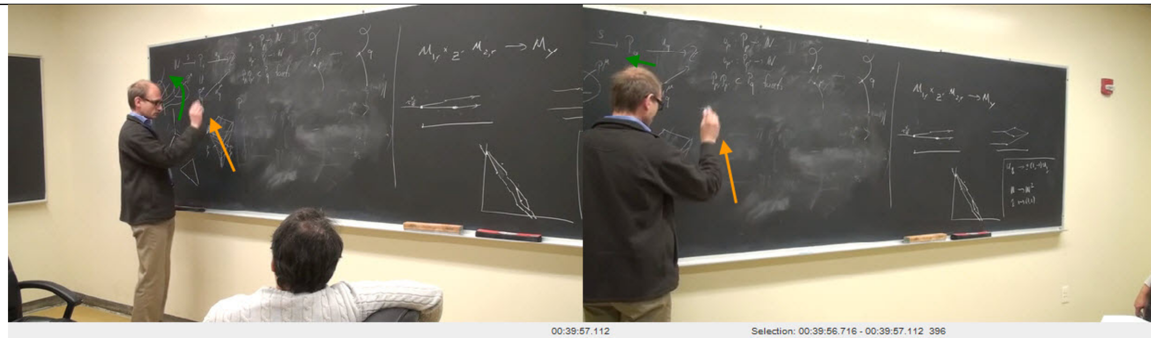
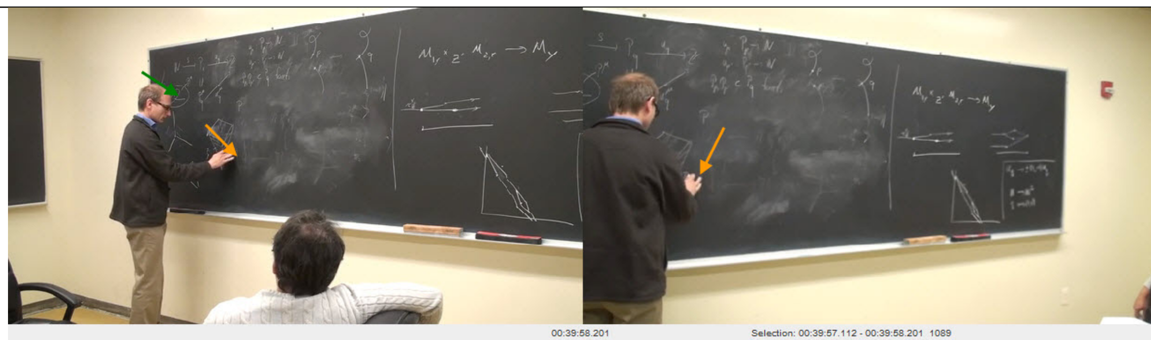
Table A.4 – inverted pyramid – *Continued from previous page*

Row	Gesture & timestamp	Transcript
119		*Not q.*
120		[pause] Now I'm

Continued on next page

Table A.4 – inverted pyramid – *Continued from previous page*

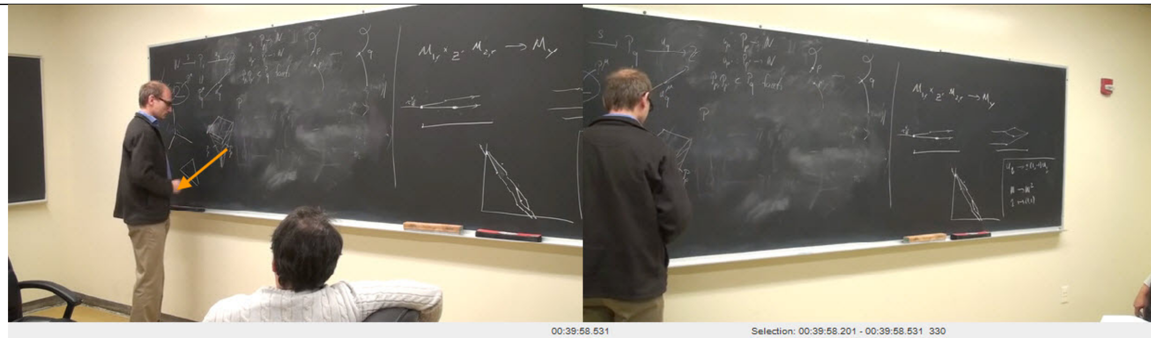
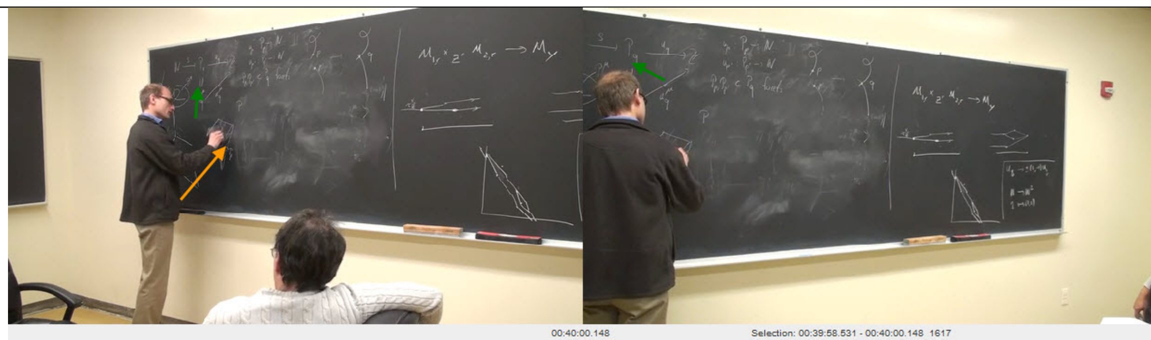
Row	Gesture & timestamp	Transcript
121		confu-
122		-sed all the time.

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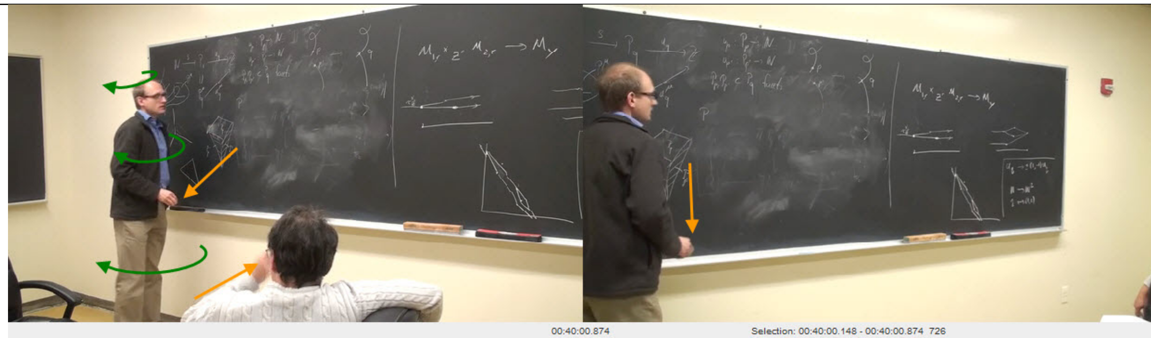
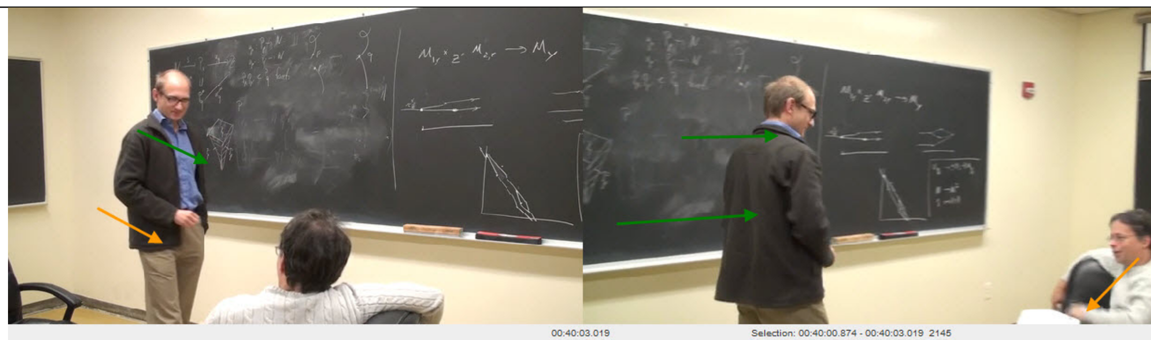
Table A.4 – inverted pyramid – *Continued from previous page*

Row	Gesture & timestamp	Transcript
123		M: That's because we're not used to
124		thinking of these things as facets.

A.5 Joseph & Bill — finger moves

Table A.5: Joseph & Bill — finger moves

Row	Gesture & timestamp	Transcript
1		[N/A]
2		[pause]

Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
3		[pause]
4		J: And then there's some, there's a, a cusp as well;

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Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
5		you can have a
6		[pause] um

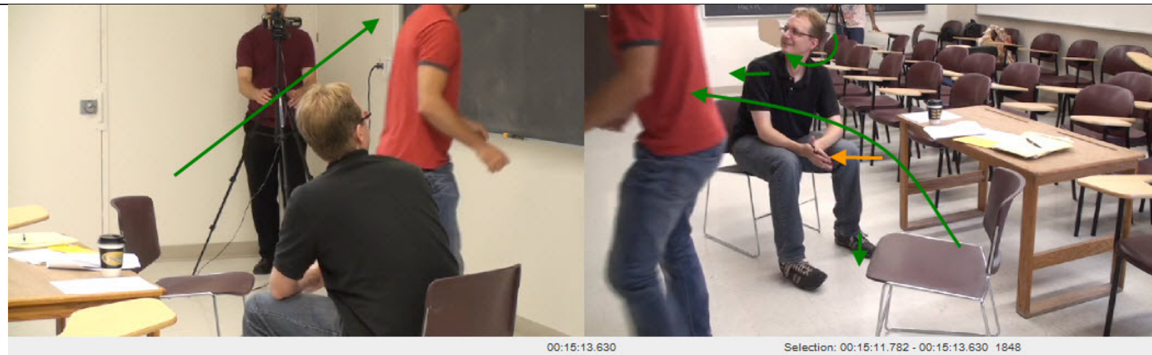
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
7		[pause]
8		[pause]

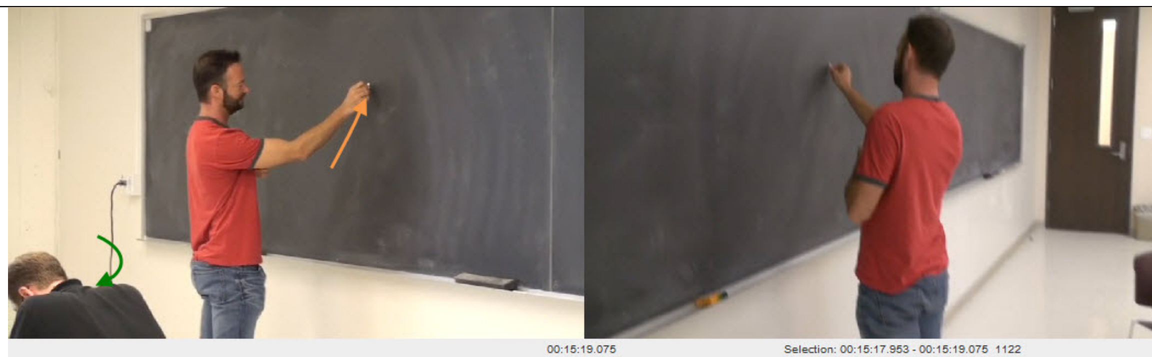
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
9		[chuckles] Check this out.
10		There a [inaudible] draw,

Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
11		it's basically rather difficult to draw. Um,
12		[pause]

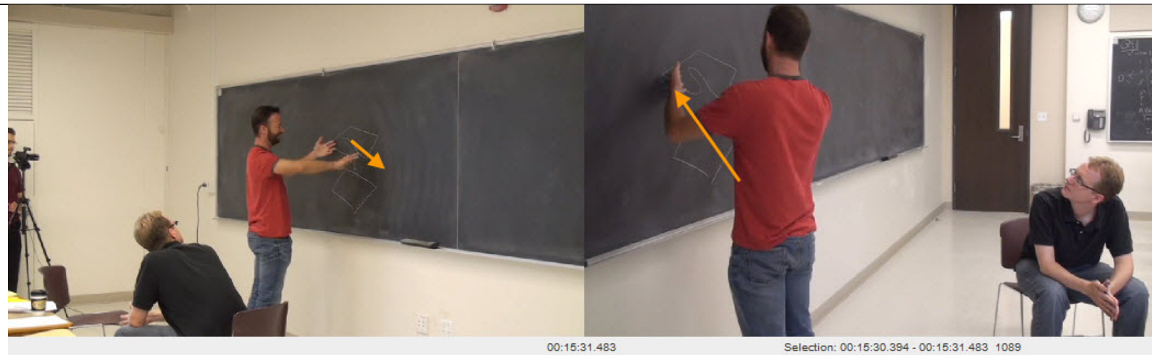
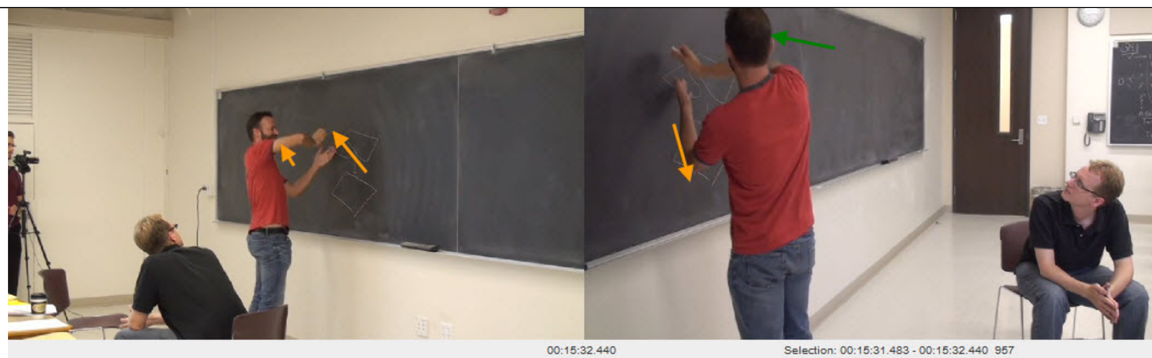
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
13		[pause]
14		[pause] If it's, so [chuckling]

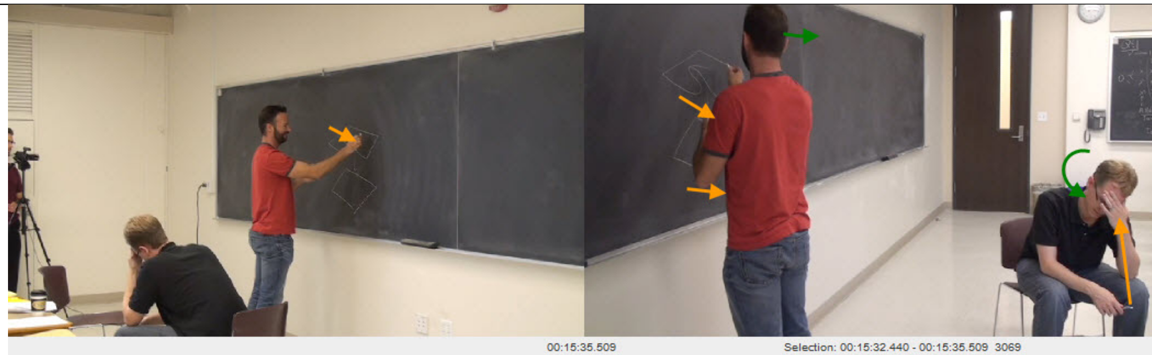
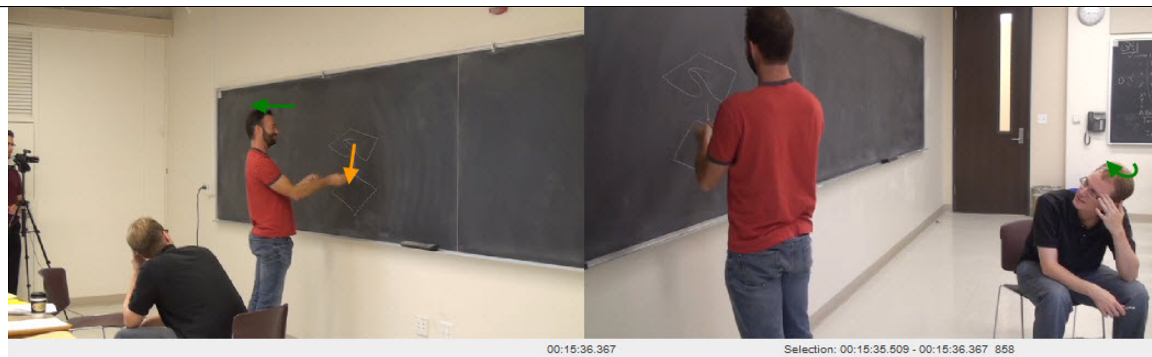
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
15		This is like a, a bla-
16		-nket that you, you

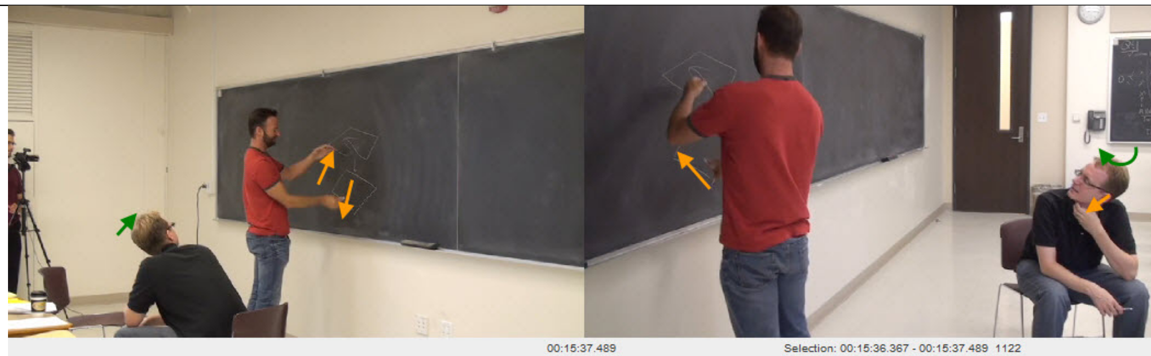
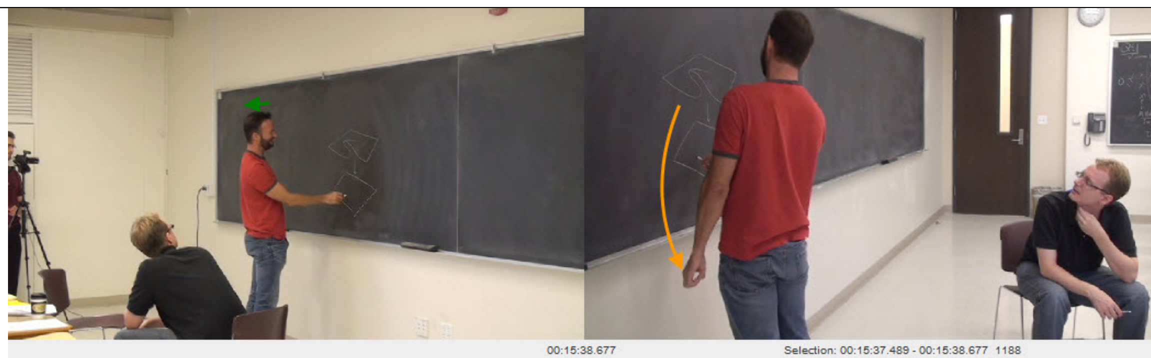
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
17		can come like this. B: Oh yeah yeah yeah.
18		Where there's three sheets

Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
19		and then there's one sheet. J: That's,
20		that's right. So, so [laughs]

Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
21		[pause] Projecting down, um [pause]
22		[inaudible]

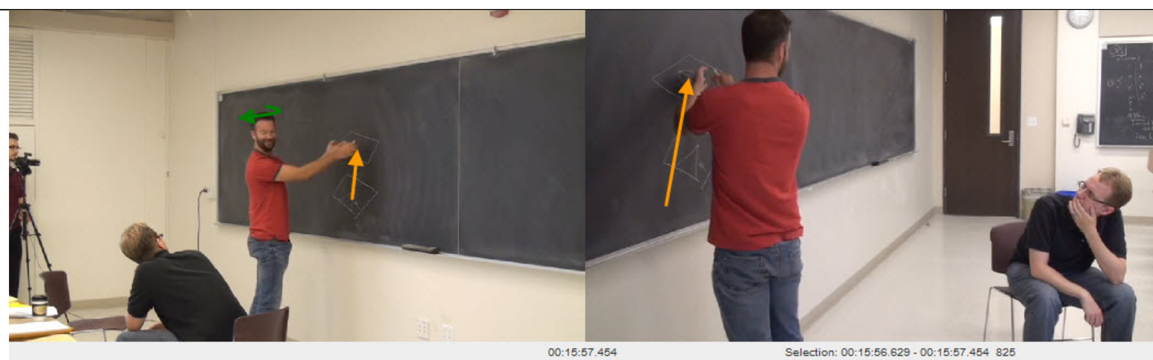
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
23		The-
24		-re's, well there's, there's, that's right, there's two answers; I mean there's there is a fold going somewhere along here. B: Mmh. J: And there's

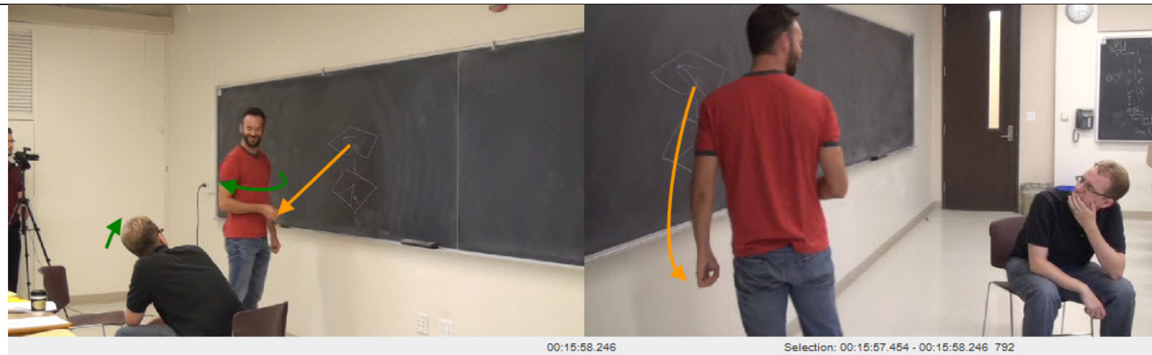
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
25		another one going somewhere along here. So you do see these two, these, these are like folds B: Right. J: in the image. But there's actually a cusp
26		where they join.

Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
27		[inaudible]
28		So, um,

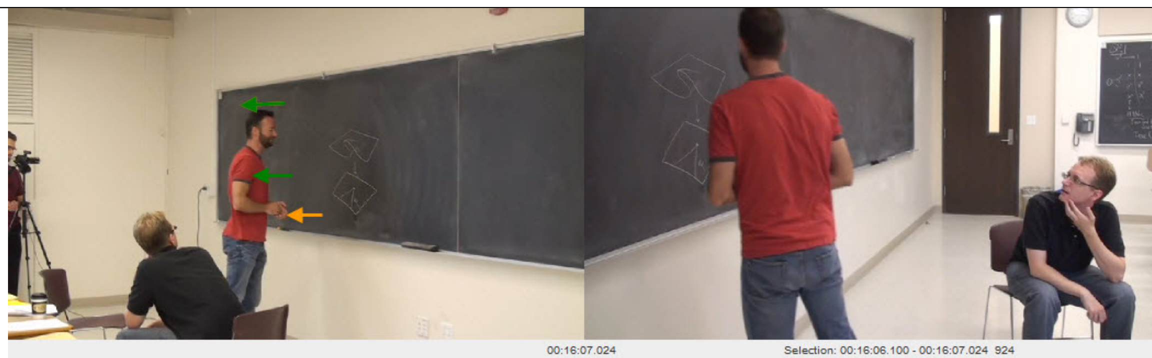
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
29		I was talking to Ben about this actually 'cause he
30		he was looking at some singularity theory and Chr-

Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
31		-istianof's crease pa- per B: Oh.
32		J: about this kind of

Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
33		stuff so
34		we actually were try-ing to

Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
35		work out what this, what this is
36		B: Oh yeah. J: And it is, in fact, these guys do actually meet

Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
37		like the
38		the equation x squared equals

Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
39		y cubed [i.e., $x^2 = y^3$] which is what
40		we normally call a cusp. So it

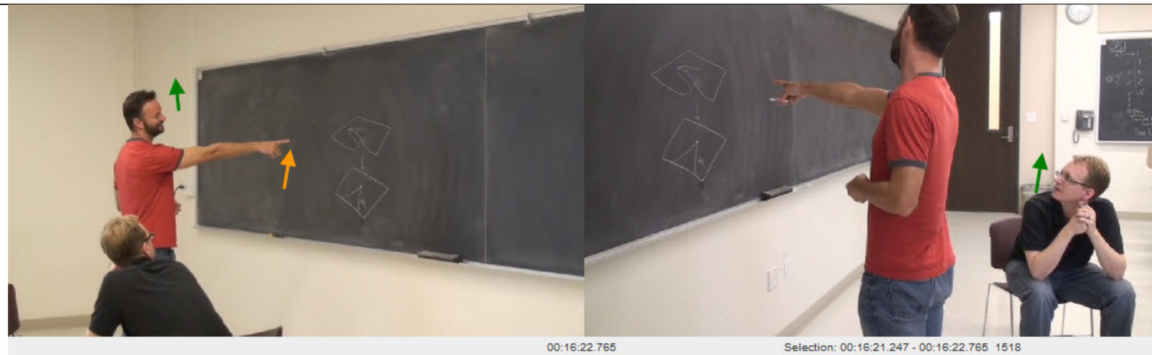
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
41		really is, it's correct. Naïve, for it-
42		from *that* pic-

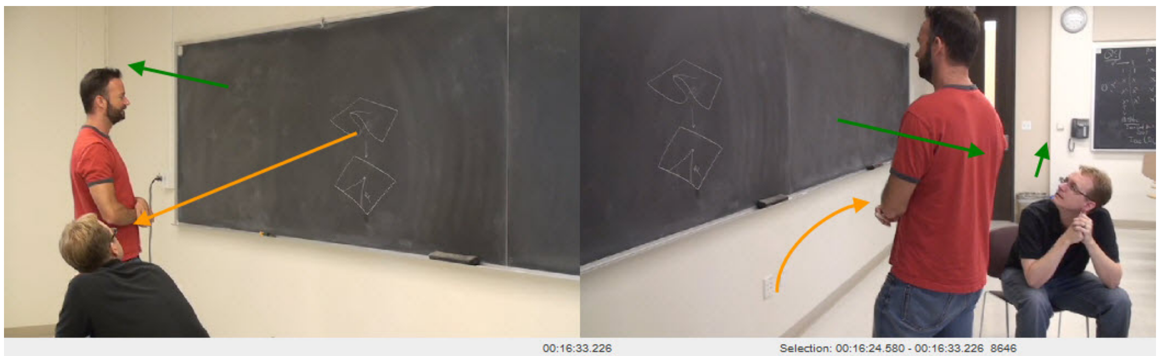
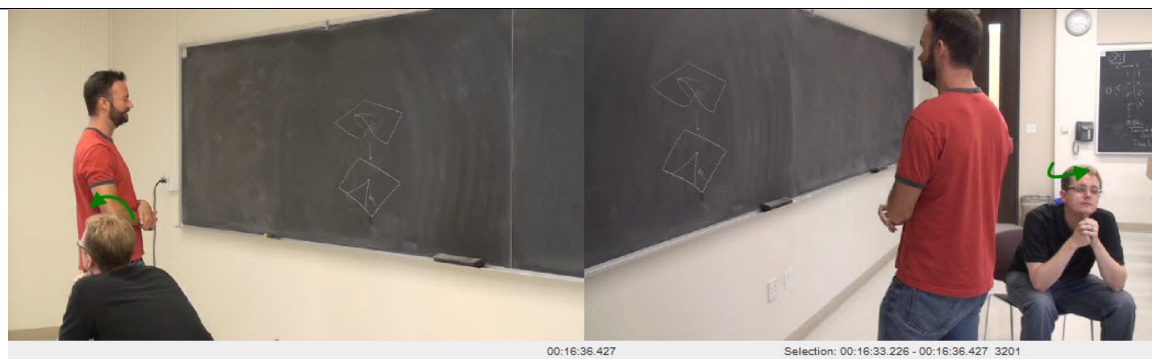
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
43		-ture, but no- not from th-
44		-is, this is critical, because n- nothing funny

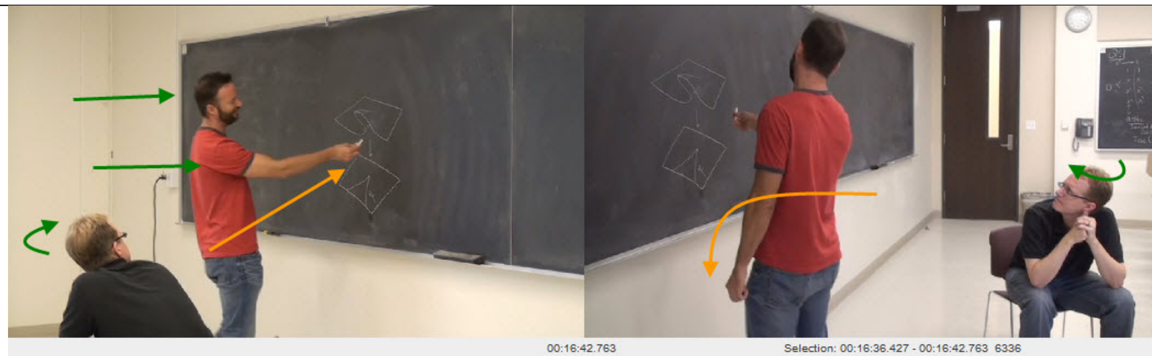
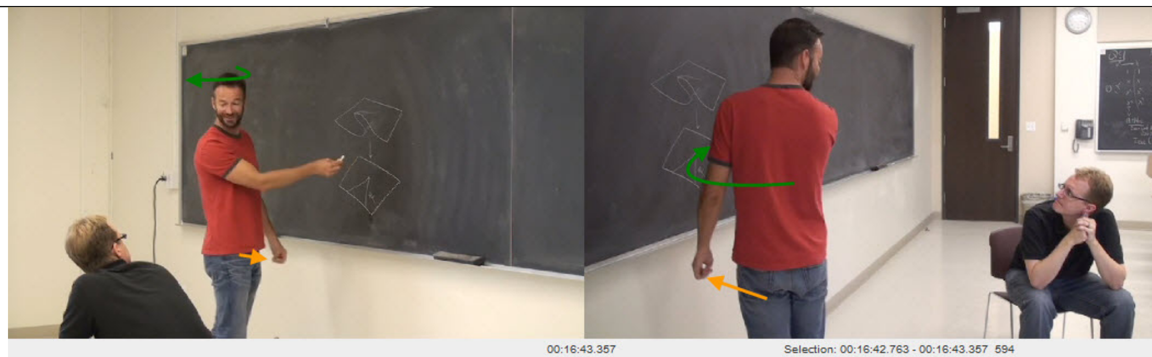
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
45		is happening there. B: Yeah. [pause] J: So anyway, so there's guys like that as well that can happen. B: Right. J: So we somehow ought to
46		find that we can actually get rid of things like this.

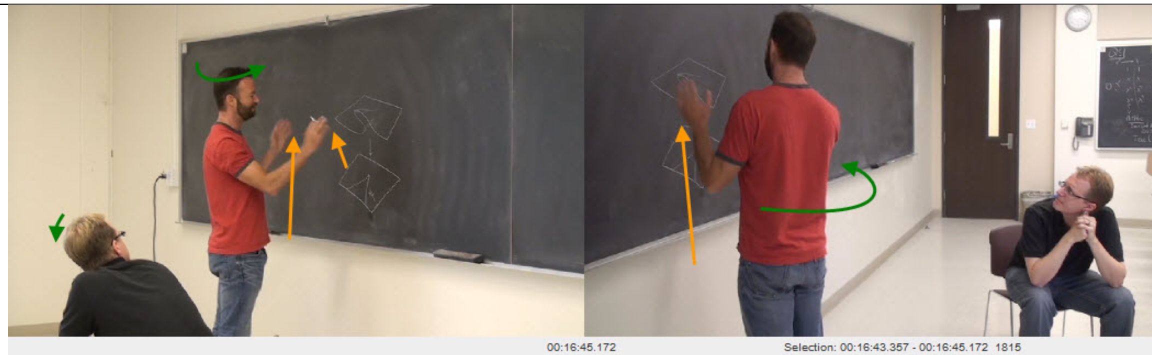
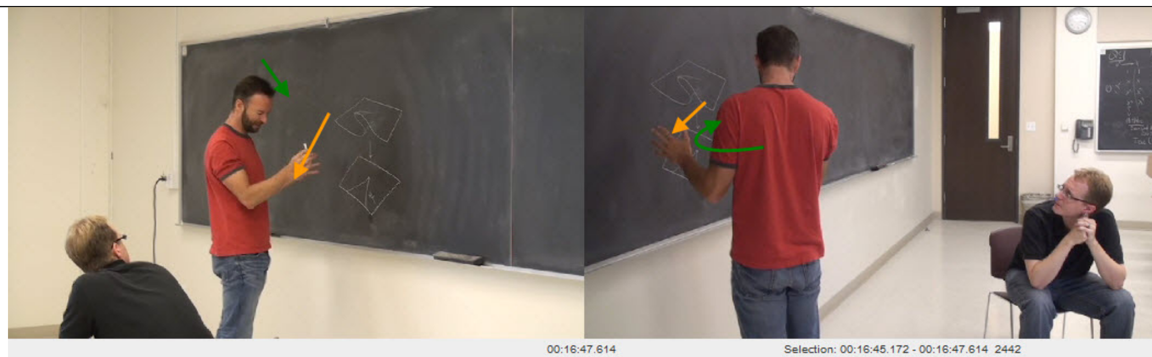
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
47		[pause] Now, I think that, I think the idea actually [laughs] it's, it's reasonably clear
48		how to do th-

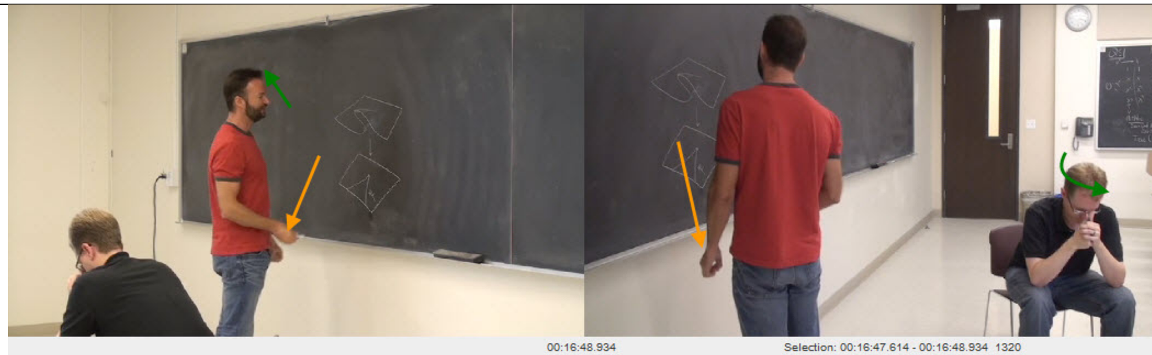
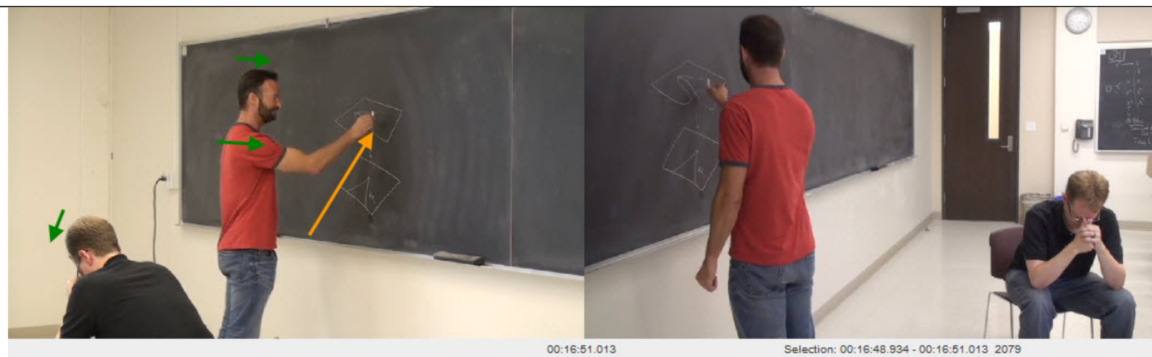
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
49		-at um, if
50		we know that the surface is a closed surface, it has some bo-

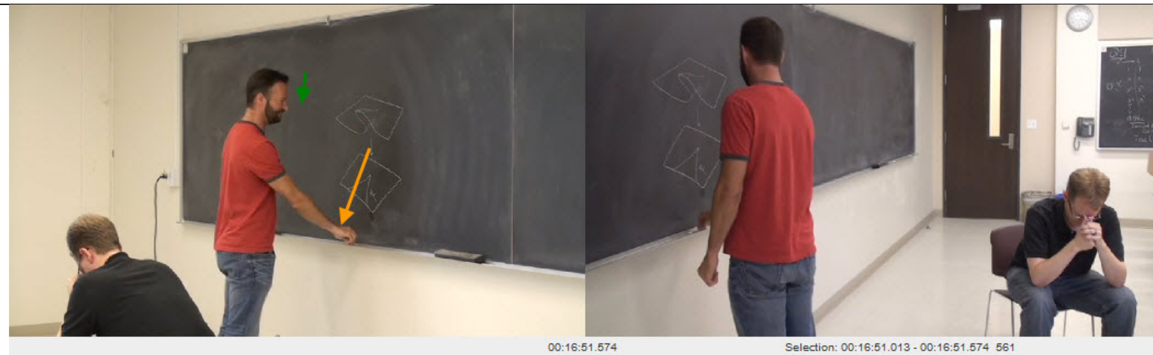
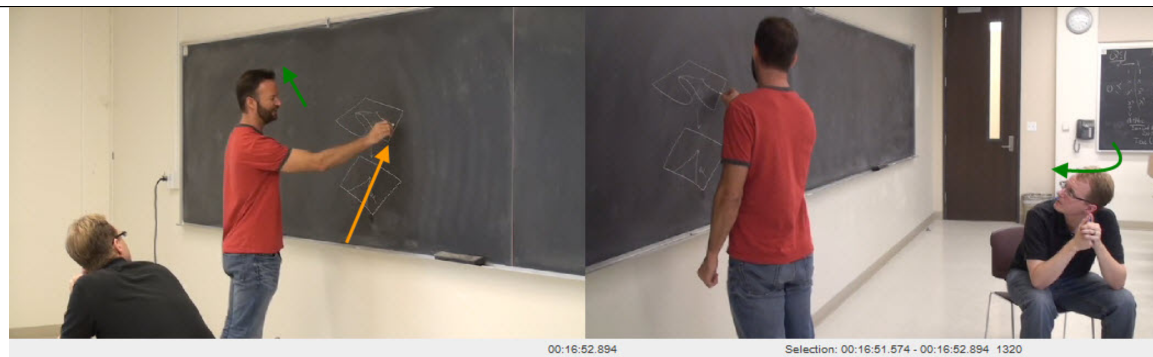
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
51		-undary somewhere.
52		Well you can, you can just find all the-

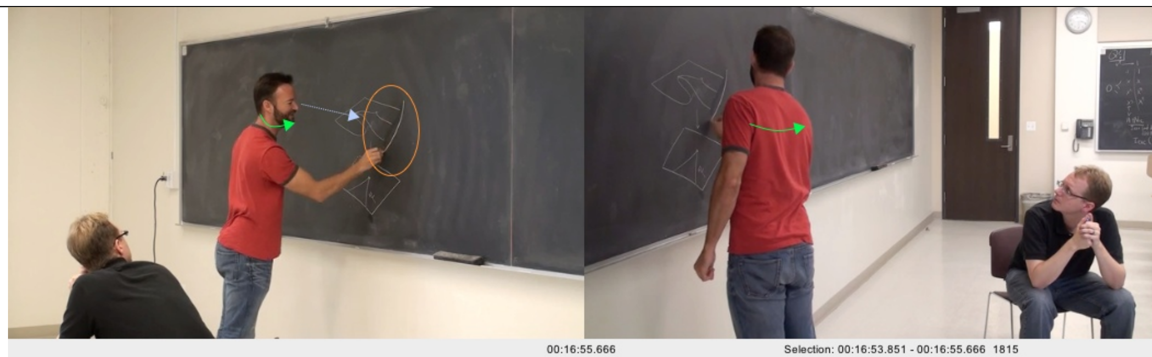
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
53		-se points
54		and draw, draw an a-

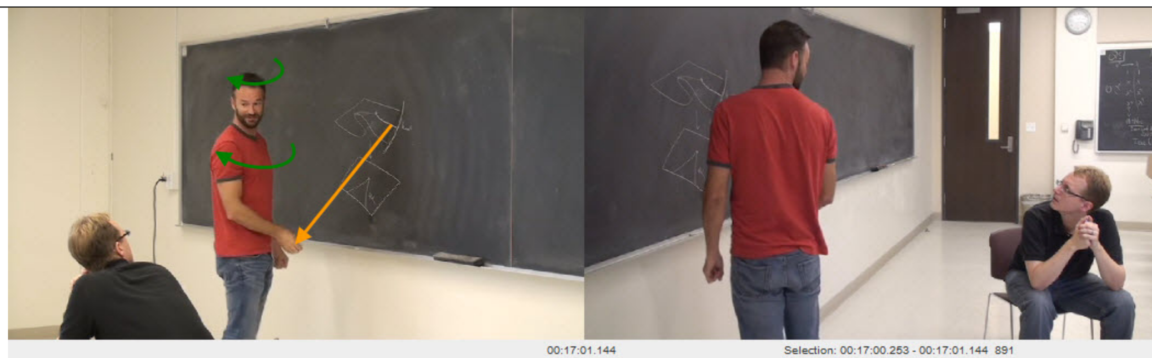
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
55		-rc that goes off to sur-
56		-face, because this is really genuinely boundary,

Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
57		this is actually the knot. B: Sure.
58		J: How about we just connect these thi-

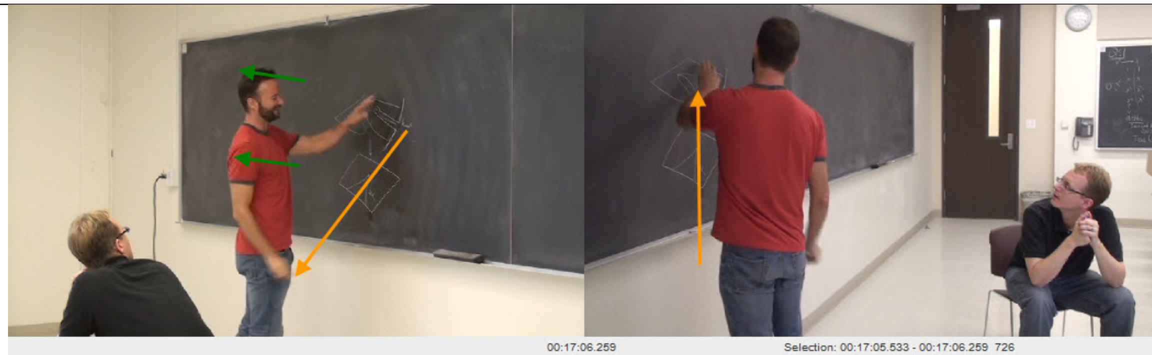
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
59		-ngs to the
60		boundary by a curve

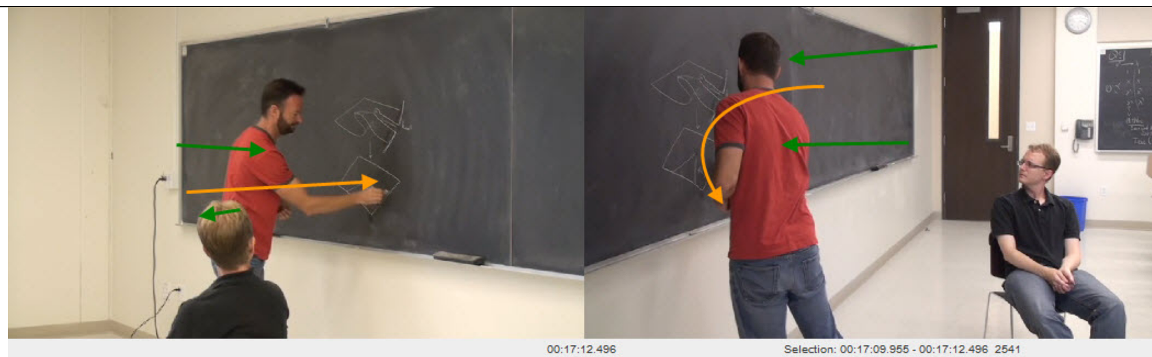
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
61		[pause] and then you do a kind of fi-
62		-nger move just pushing [laughs] just remove that part of the sur-

Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
63		-face. U-
64		-m, B: Oh yeah. J: You know, that, that

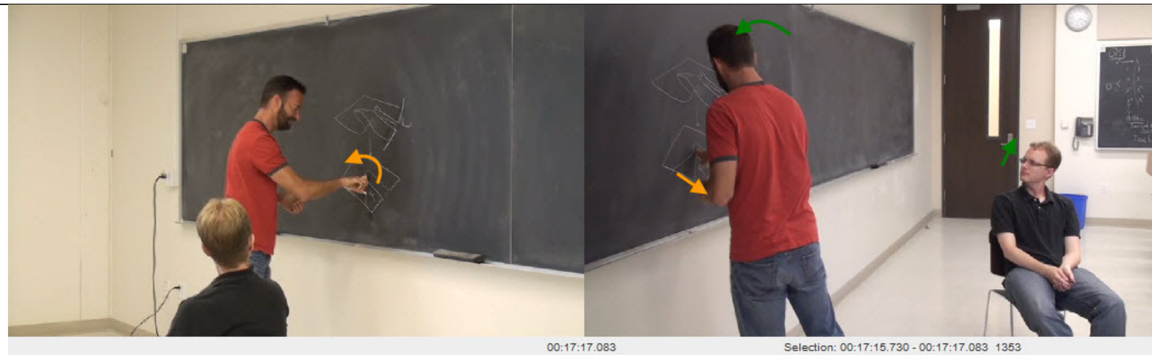
Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
65		will just get rid of them, so. B: Mmh
66		J: So you will still have, um,

Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
67		[pause] So now we come to the boundary.
68		You'll still have

Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
69		some folds there. But maybe
70		we should get rid of those next.

Continued on next page

Table A.5 – finger moves – *Continued from previous page*

Row	Gesture & timestamp	Transcript
71		B: Yeah [inaudible] J: [inaudible] or somehow. B: Yeah. J: Um.

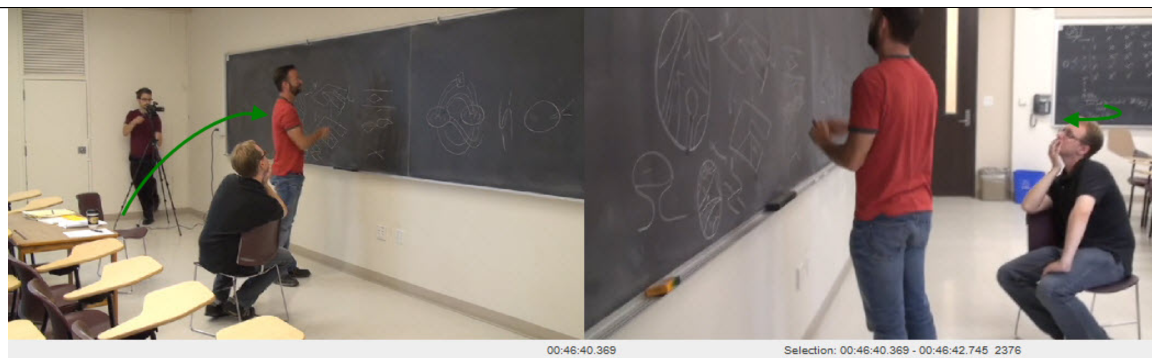
A.6 Joseph & Bill — indexical map

Table A.6: Joseph & Bill — indexical map

Row	Gesture & timestamp	Transcript
1		[N/A]
2		J: Um, I mean it's quite

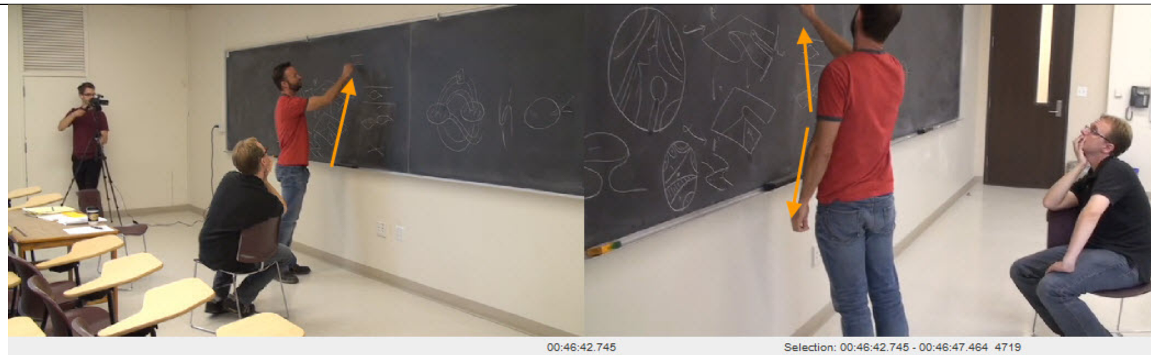
Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
3		possible that it does. I mean, how does
4		[inaudible]. So they're coming in

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
5		here. [pause] If you're go- If you move
6		to continue [inaudible]. So you go from this to [pause]

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
7		um [pause]
8		[pause] this [pause]

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
9		[pause]
10		B: [inaudible]

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
11		[inaudible continues]
12		J: Um [pause]

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
13		Something like that.
14		B: [pause] I, I think that's turned on its side.

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
15		J: Okay,
16		good.

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
17		[chuckles] Um. [long pause] B: So
18		since they're

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
19		the same, uh [long pause]
20		I mean, you can deal with that top circle, right?

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
21		[pause] Up in. . . J: [nods head in agreement]
22		[pause]

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
23		B: Clear away both of those, and then there're just
24		some sort of ways that we bootstrap. [pause] So is that, uh, set connected

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	**Gesture & timestamp**	**Transcript**
25		to that- Oh! No, that keeps them the same! [pause]
26		or maybe

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
27		[inaudible]
28		Since we invert from the two surface roots.

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
29		Right?
30		And make *this*- J: Yeah, that's right. B: Through

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
31		B: doing that kills those two. These are still two surface roots. With
32		a cycle over them. J: W- Which two are the ones described?

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
33		B: Oh. [pause] Um,
34		right? Like the… J: The, the, th- The left

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
35		and bottom?
36		B: Um, oh so I've got this

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
37		turned on its si-
38		-de. J: Okay yeah [laughs]. B: Yeah so this is the first

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
39		J: Yeah. B: And if I de-glue my square I get two of these.
40		J: Yeah. B: And they iso-morph as a whole unit.

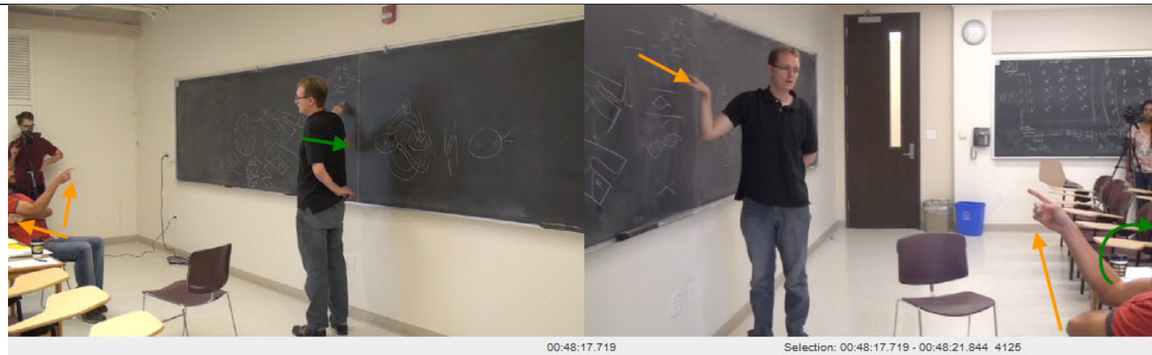
Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
41		[pause]
42		And so all left is, with this is, is preserved the

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
43		[pause] edge of the two. J: Okay, so we, so we really
44		just live in the same. . . . B: Yeah. Yeah. J: In the same complex there. Essentially.

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
45		[pause]
46		[pause]

Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
47		[pause]
48		[pause]

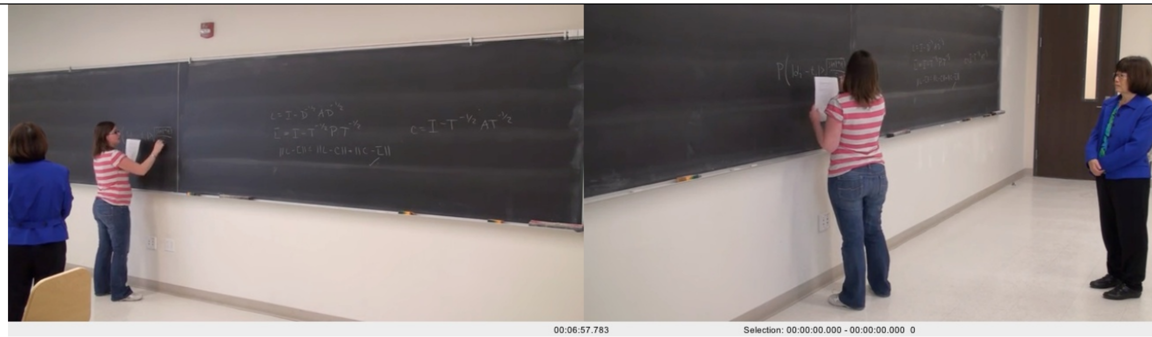
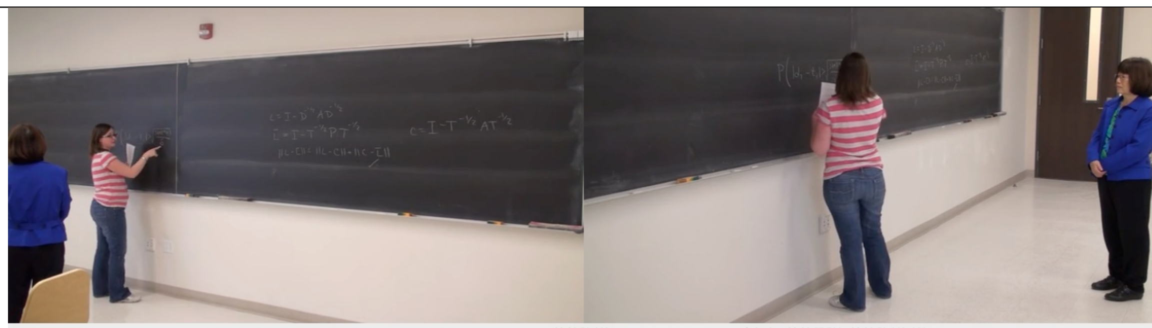
Continued on next page

Table A.6 – indexical map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
49		[pause]

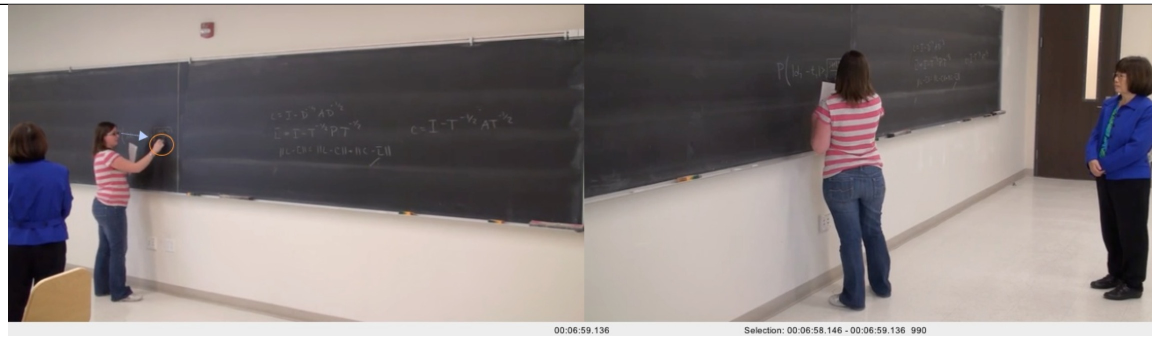
A.7 Fay & Martha — graph confining

Table A.7: Fay & Martha — graph confining

Row	Gesture & timestamp	Transcript
1		[N/A]
2		M: This is of

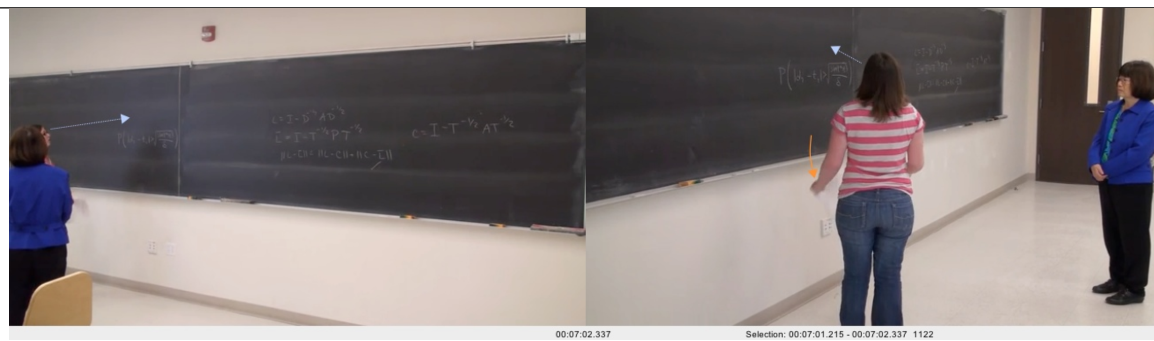
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
3		minimum degree.
4		[pause]

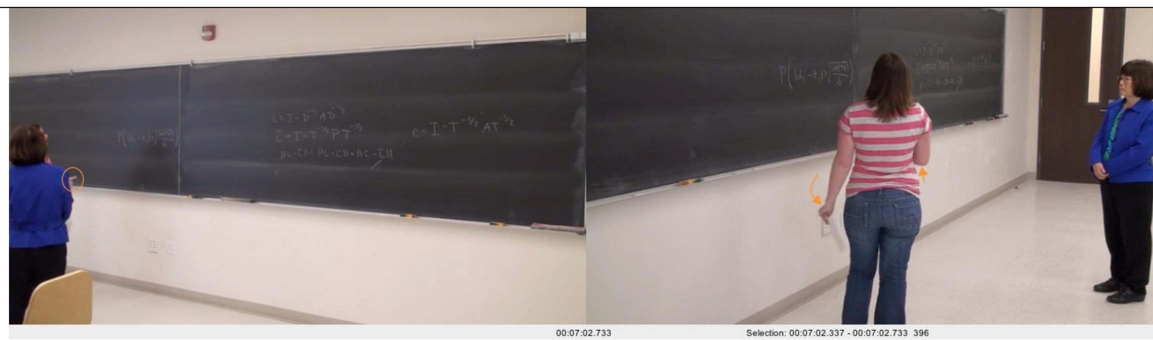
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
5		This i-
6		-s just a thing we use in com-

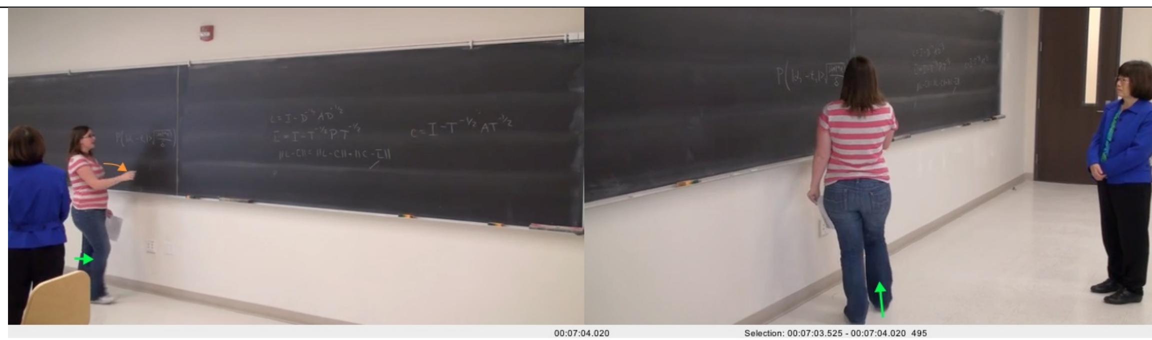
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
7		-parison.
8		Right?

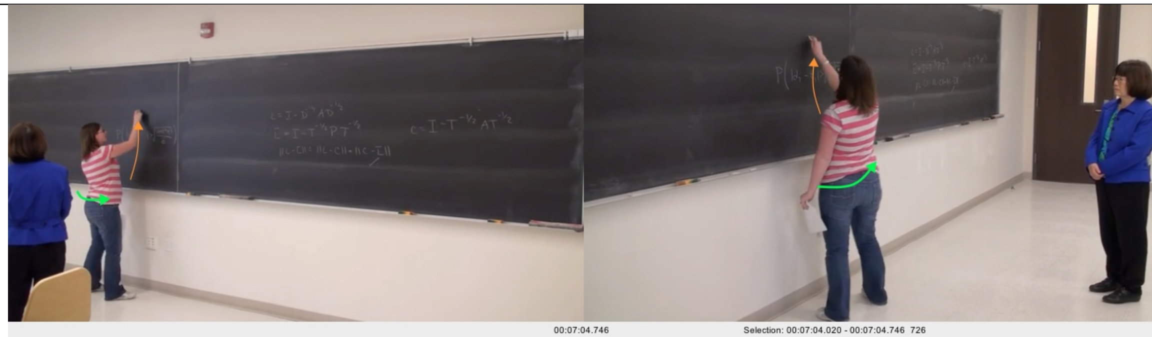
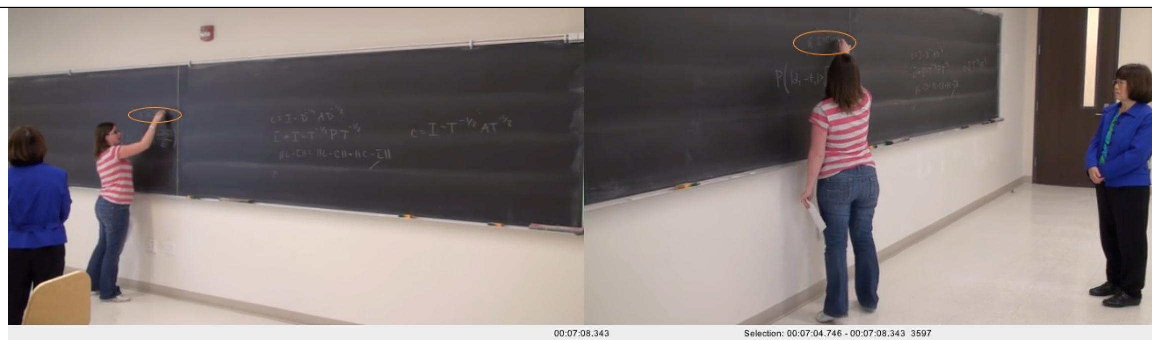
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
9		[pause]
10		And who s-

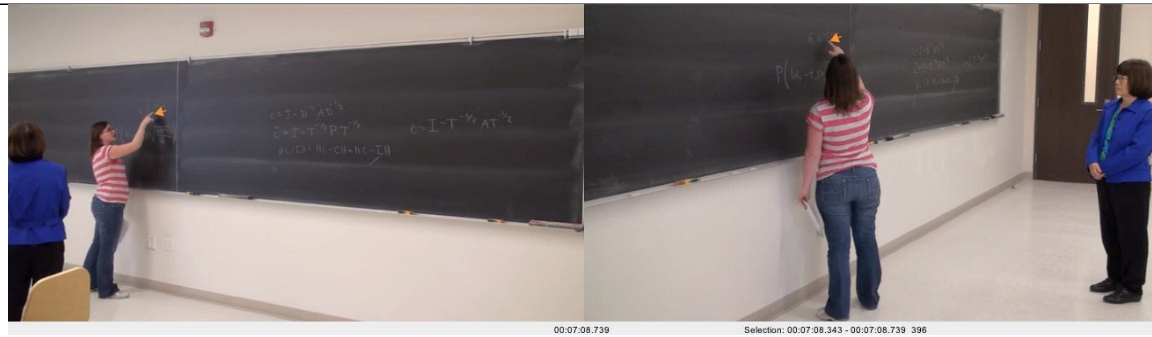
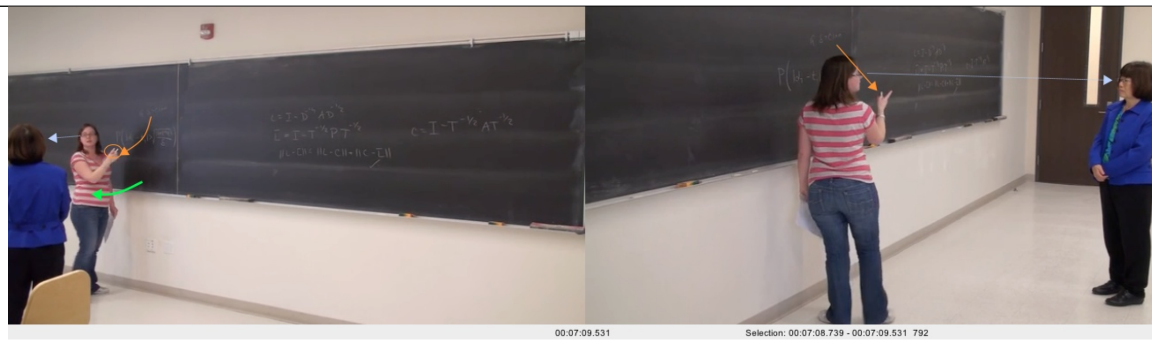
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
11		-ays that
12		if δ [“delta”] is bigger than some constant lo-

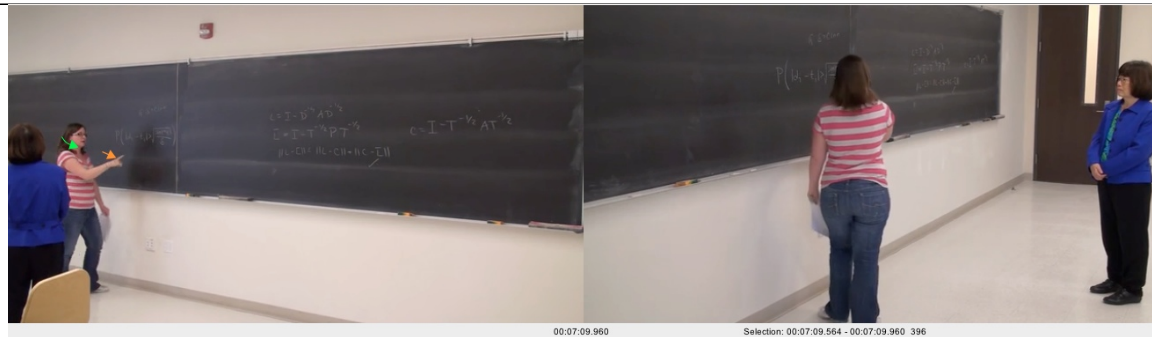
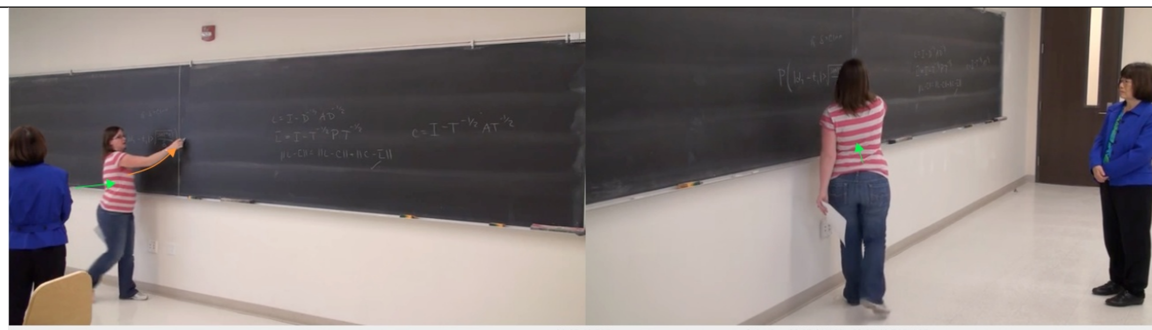
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
13		-g n [saying "$\log n$"] the-
14		-n if you make your co-

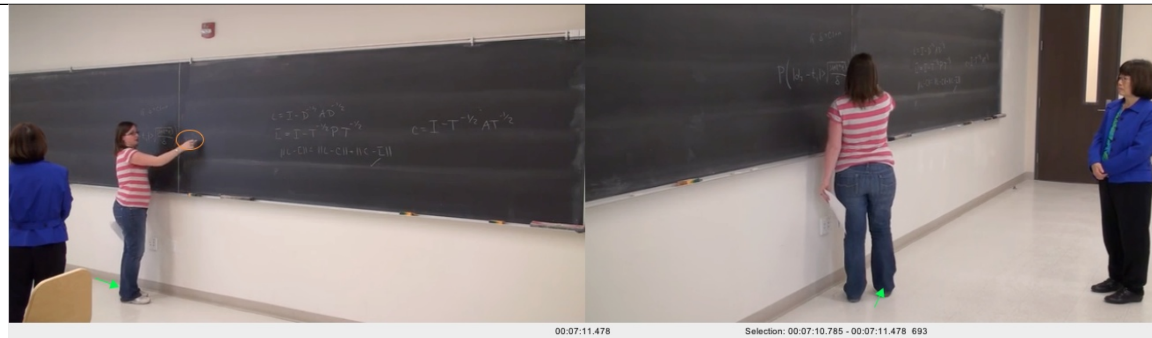
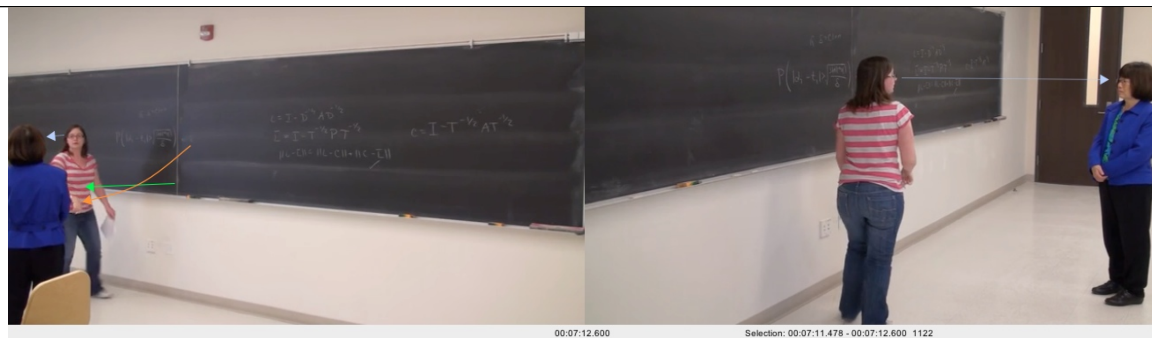
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	**Gesture & timestamp**	**Transcript**
15		-nstant l-
16		-arge enough you can make

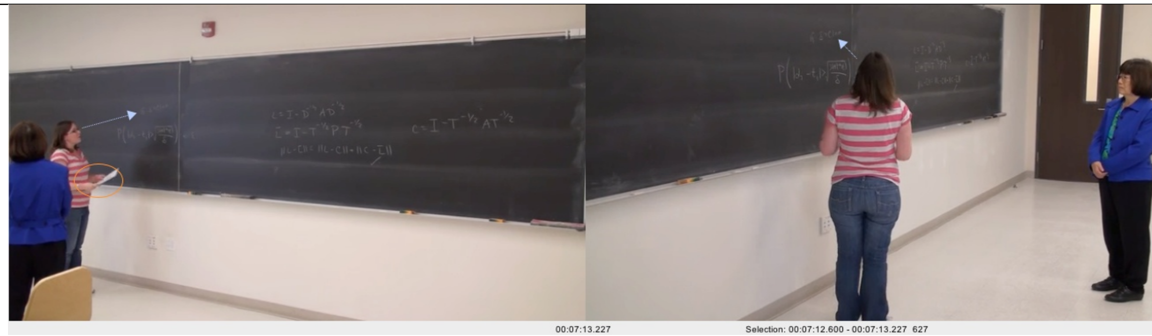
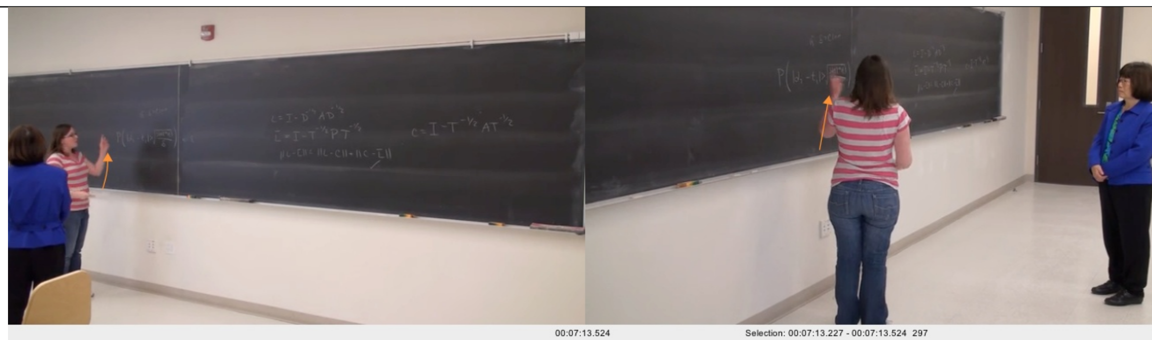
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
17		this small
18		as you want.

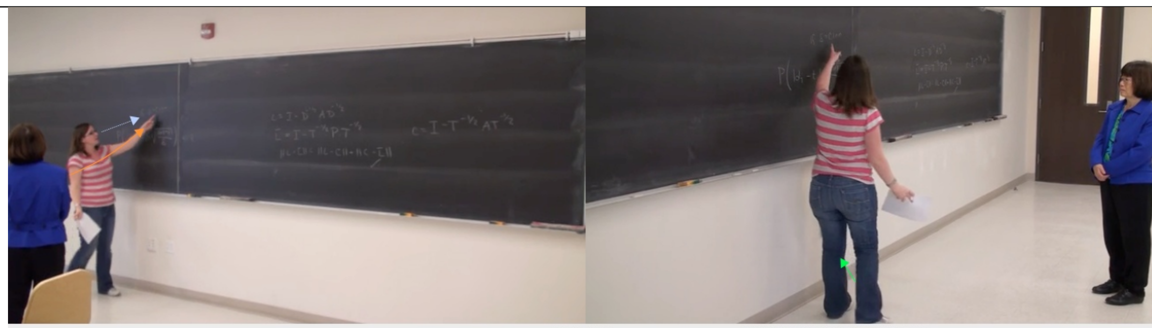
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
19		Right?
20		[pause]

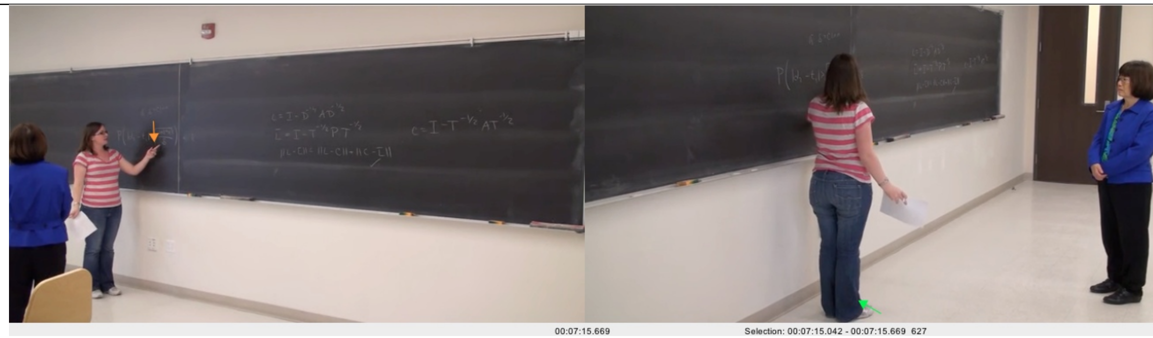
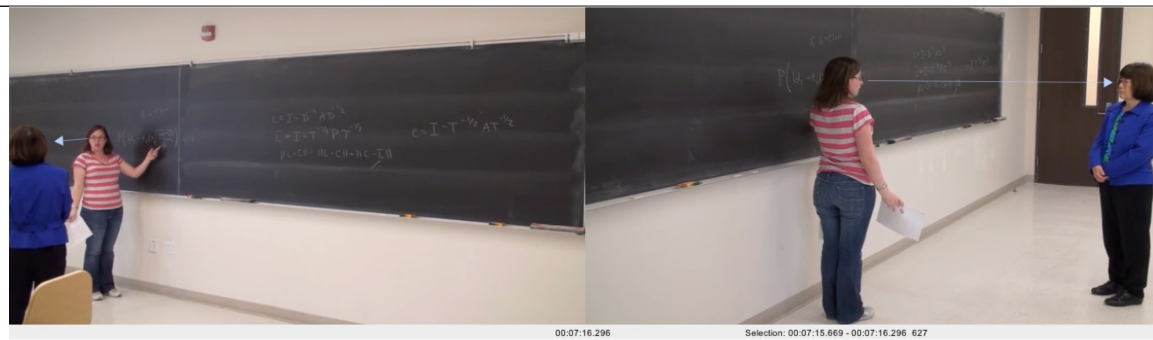
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
21		But I don't thin-
22		-k that works! If you make

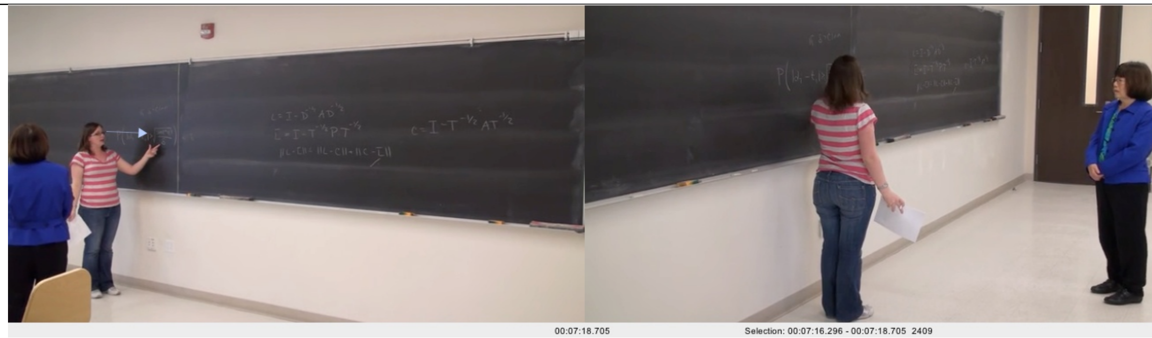
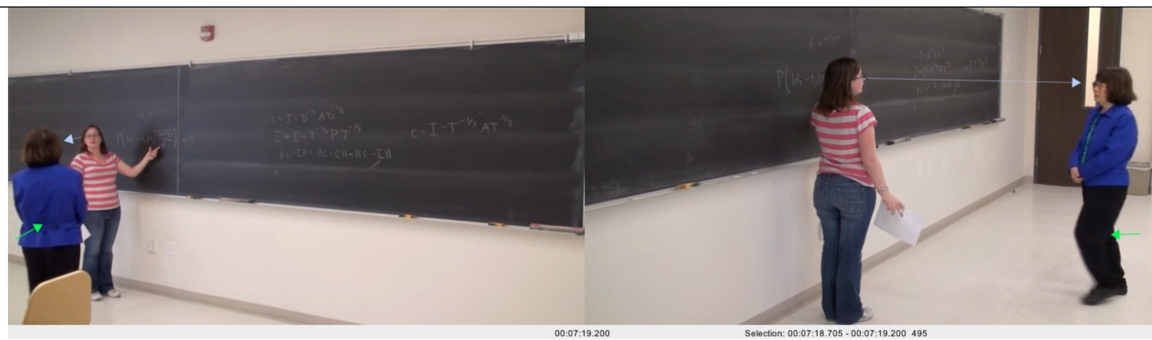
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
23		your constant
24		large-

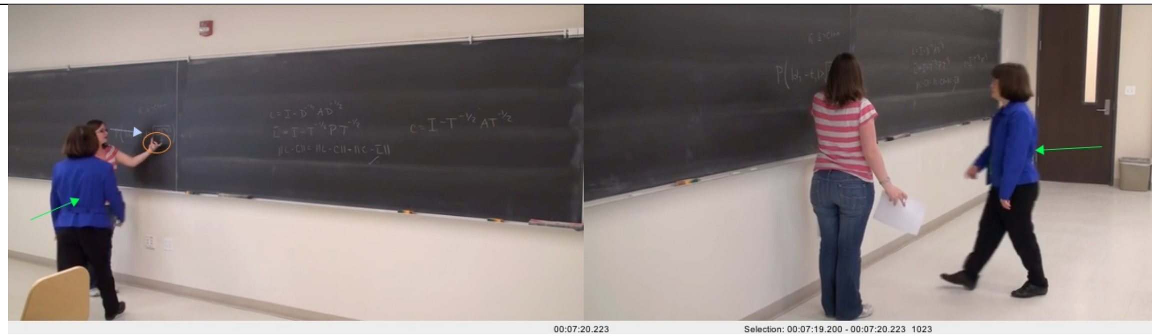
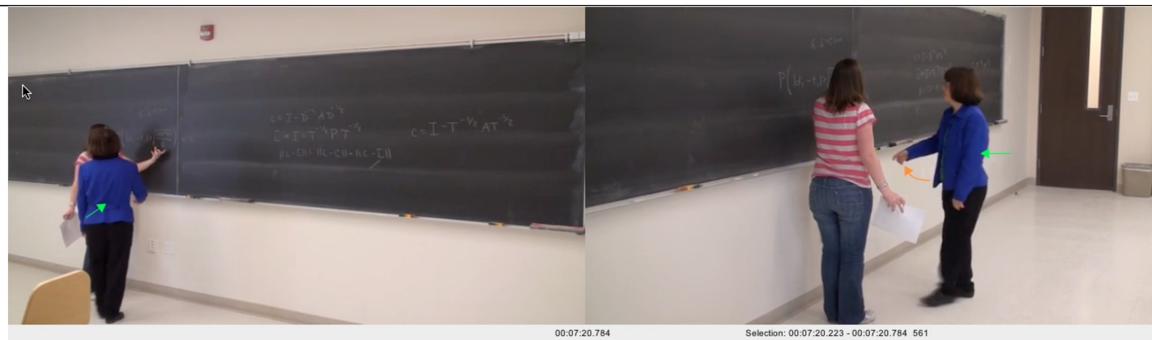
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
25		-r, [pause] the-
26		-n F: I

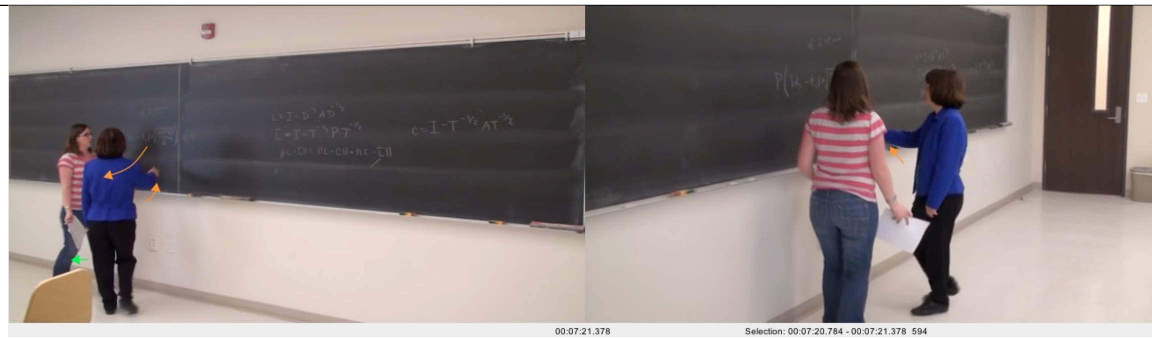
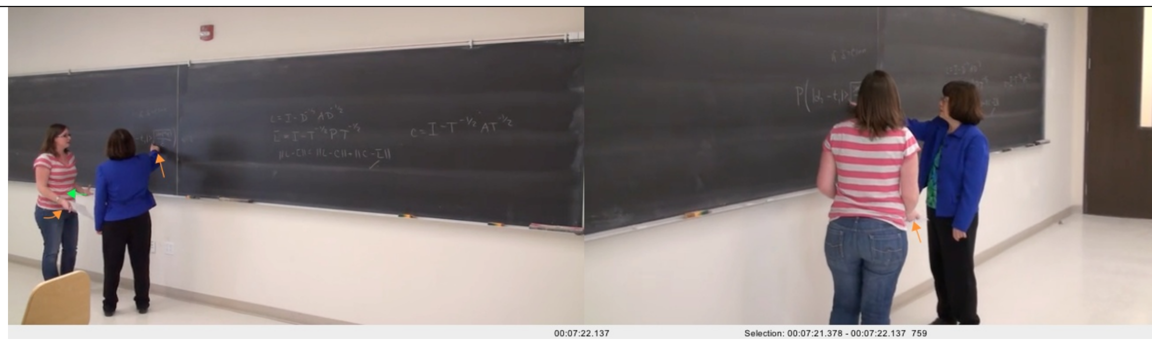
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
27		I M: You don't care- Thi-
28		-s is a small-

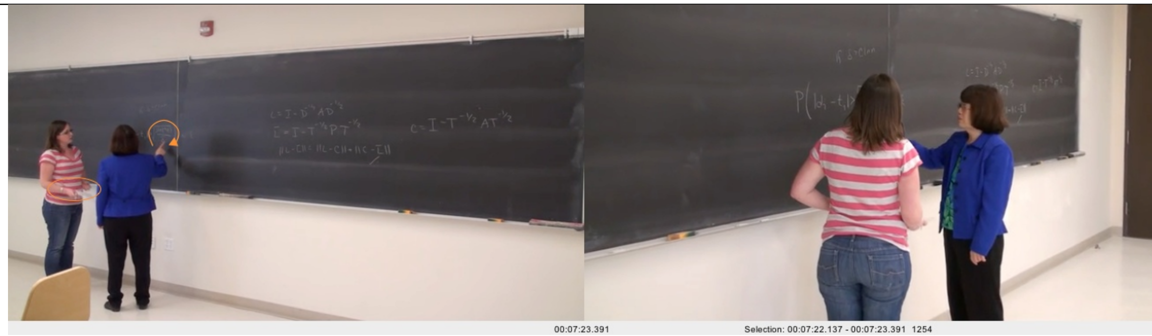
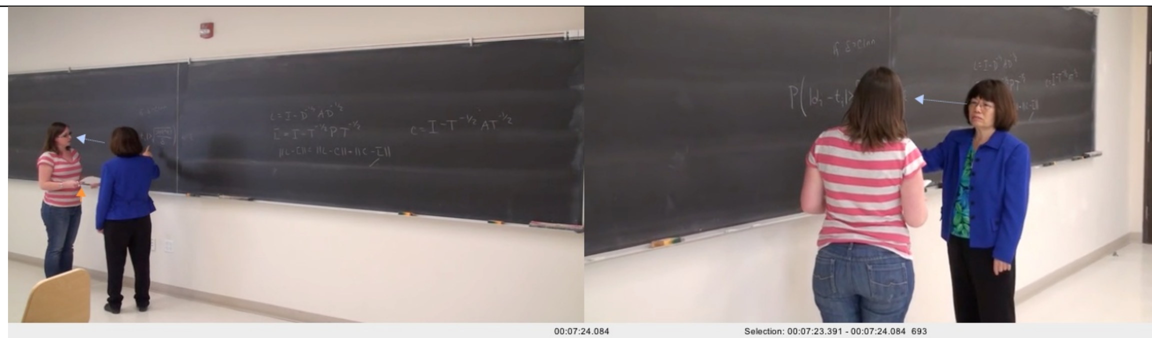
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
29		-er value!
30		F: Even- This

Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
31		is not the right
32		form.

Continued on next page

Table A.7 – graph confining – *Continued from previous page*

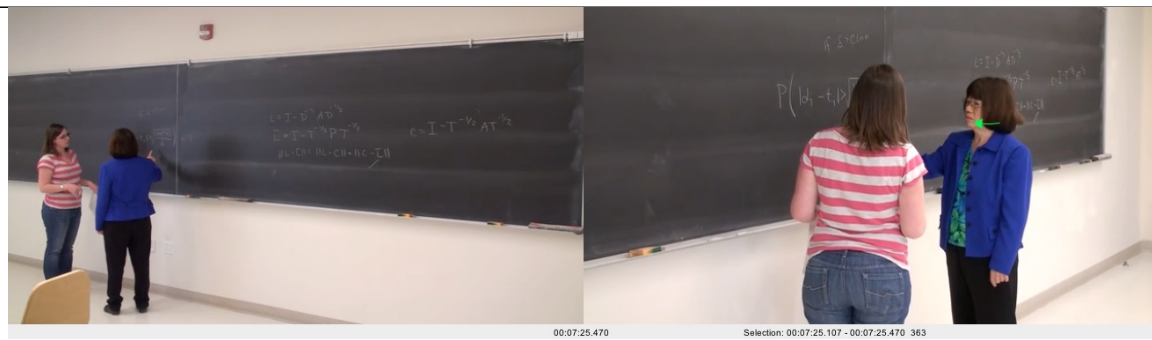
Row	Gesture & timestamp	Transcript
33		[pause] M: No?
34		F: No.

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Table A.7 – graph confining – *Continued from previous page*

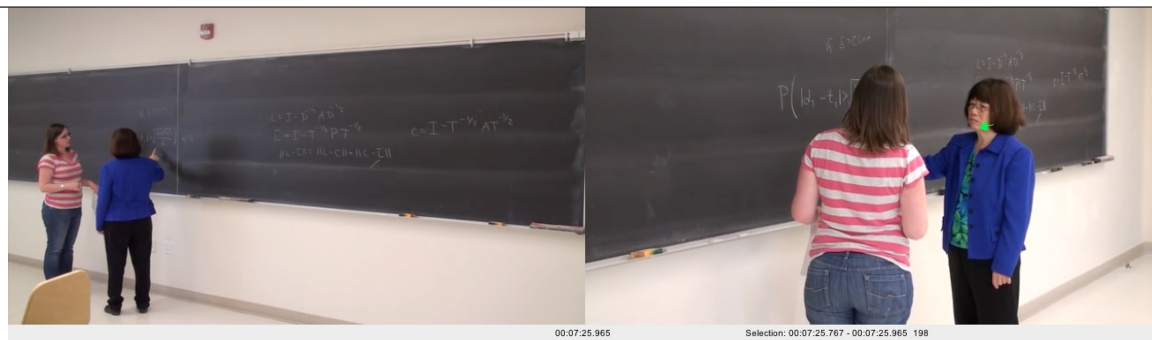
Row	Gesture & timestamp	Transcript
35		[pause]
36		[pause]

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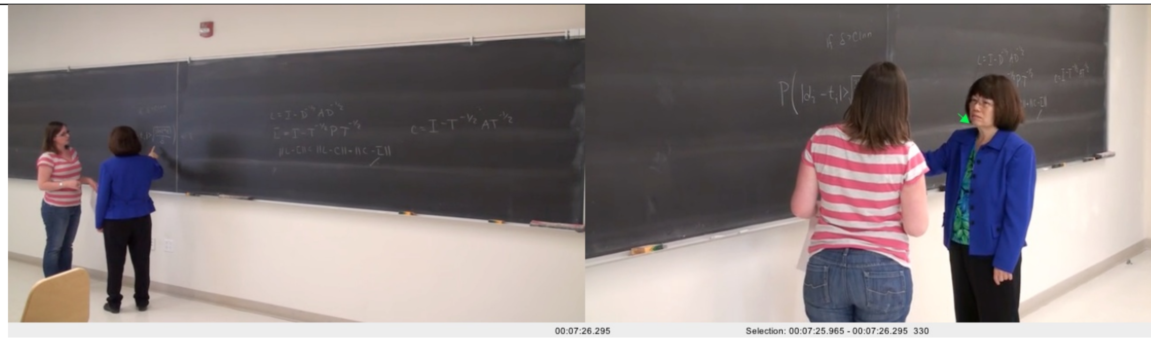
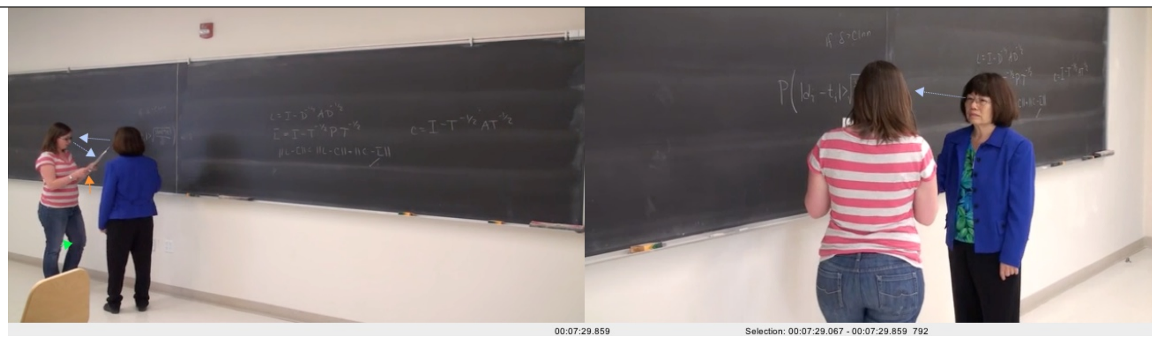
Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
37		[pause]
38		For m-

Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
39		-e, actually
40		[pause]

Continued on next page

Table A.7 – graph confining – *Continued from previous page*

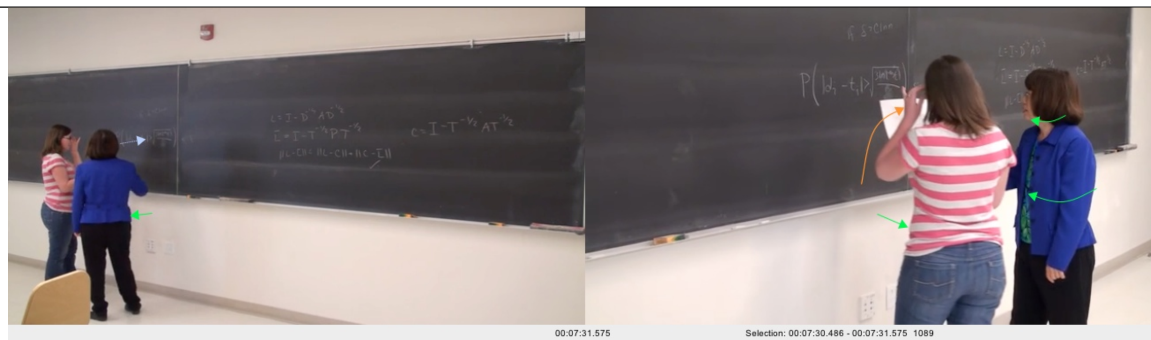
Row	Gesture & timestamp	Transcript
41		[pause]
42		M: [inaudible; sounds like "emm eye tee ell"]

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Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
43		[pause]
44		F: Because

Continued on next page

Table A.7 – graph confining – *Continued from previous page*

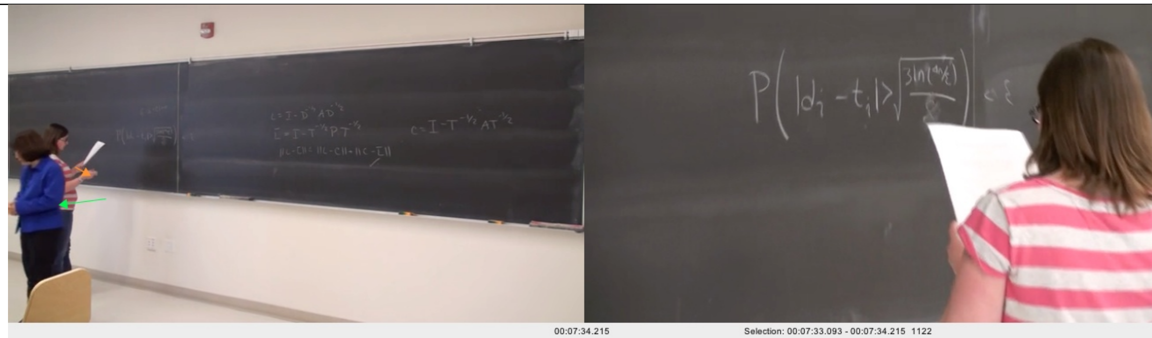
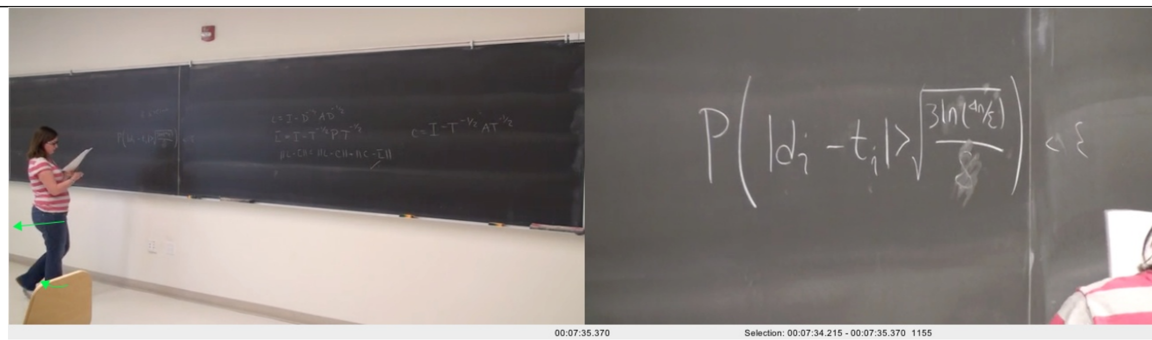
Row	Gesture & timestamp	Transcript
45		Any x [“ex”]
46		y [“why”] you h-

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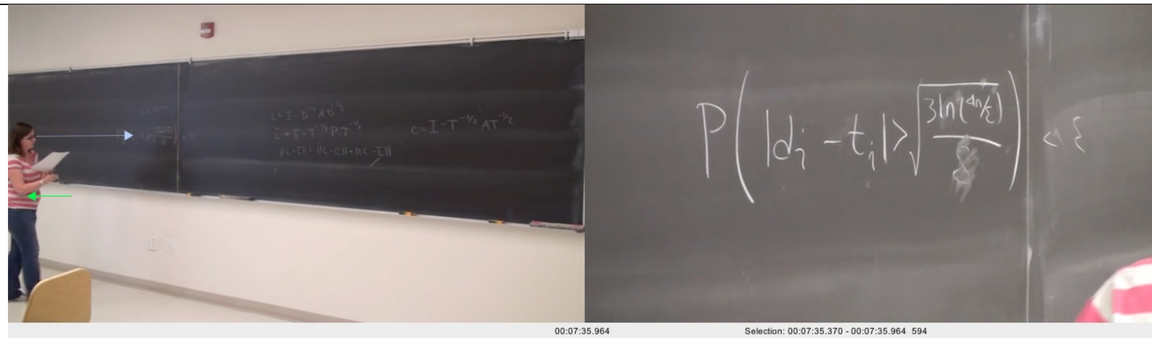
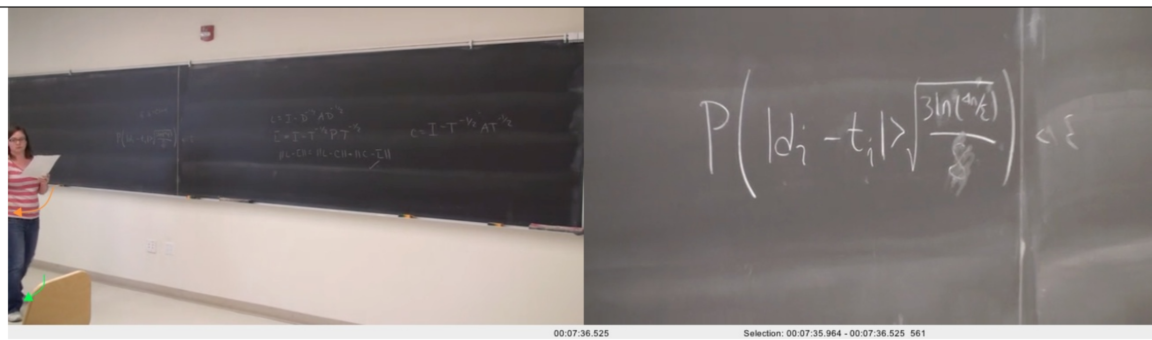
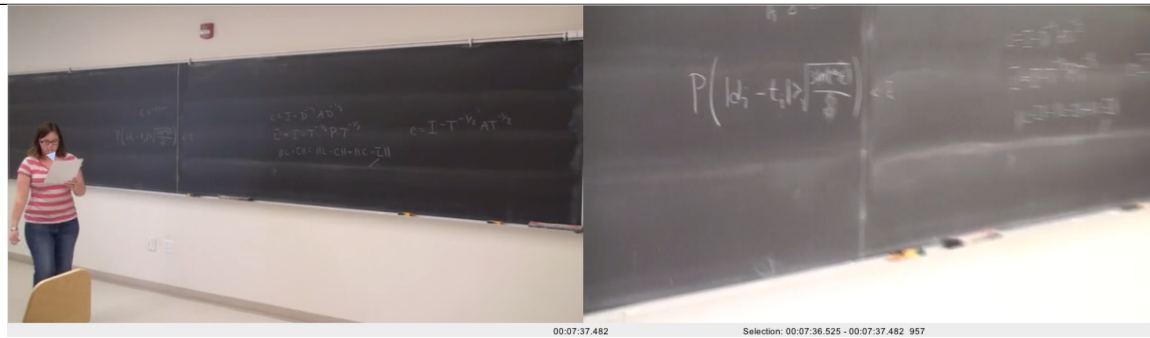
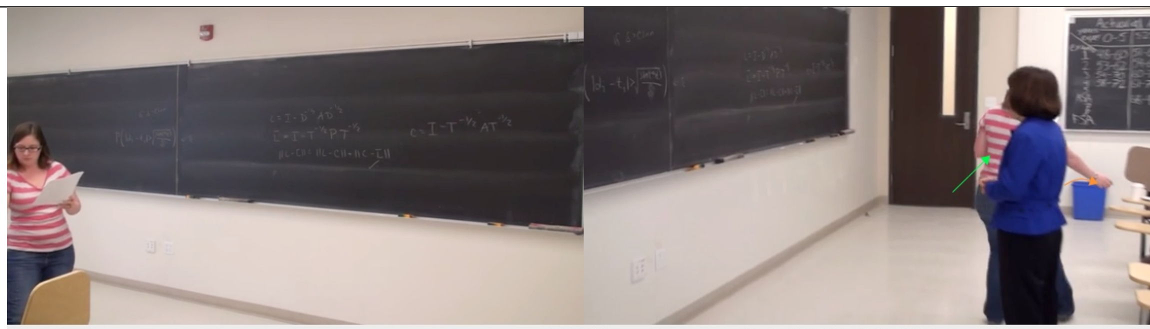
Table A.7 – graph confining – *Continued from previous page*

Row	**Gesture & timestamp**	**Transcript**
47		-ave
48		*n* ver-

Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
49		-tices.
50		M: Mhm. F: 'Kay?

Continued on next page

Table A.7 – graph confining – *Continued from previous page*

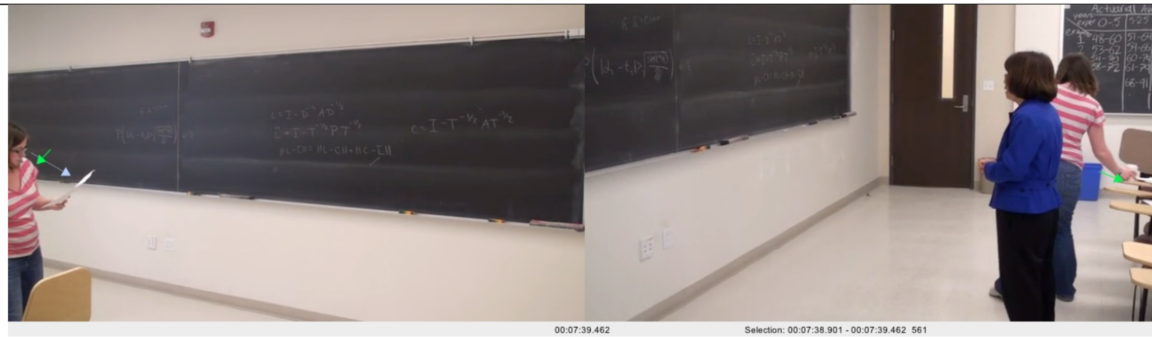
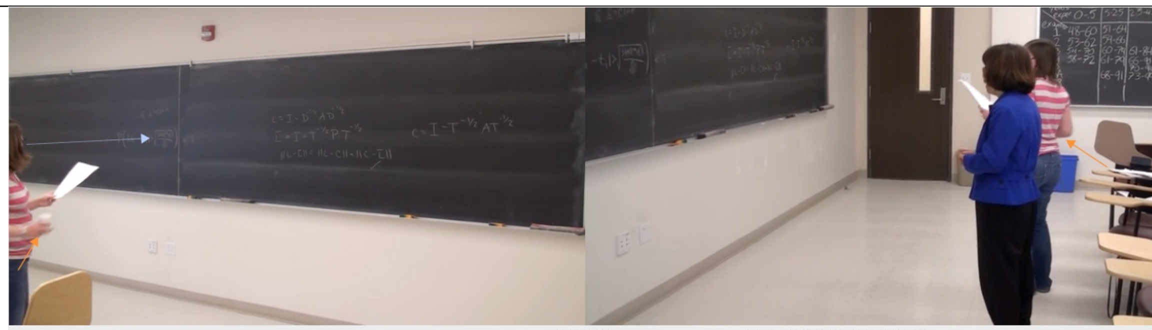
Row	Gesture & timestamp	Transcript
51		[pause]
52		You have

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Table A.7 – graph confining – *Continued from previous page*

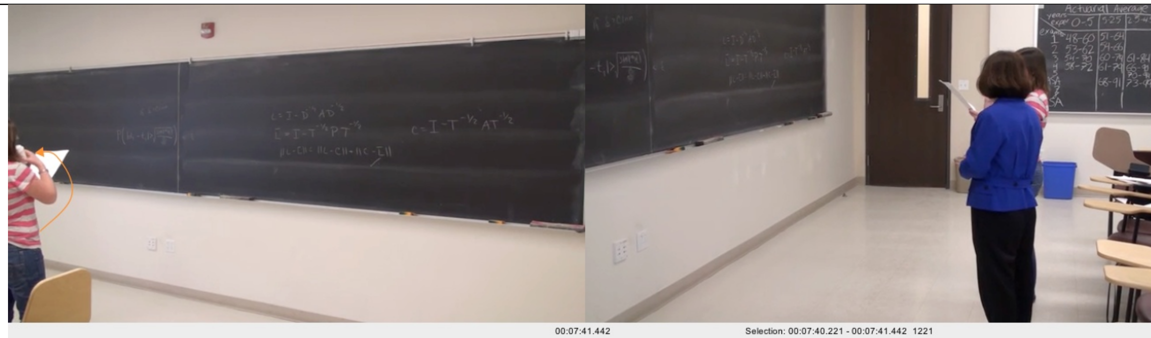
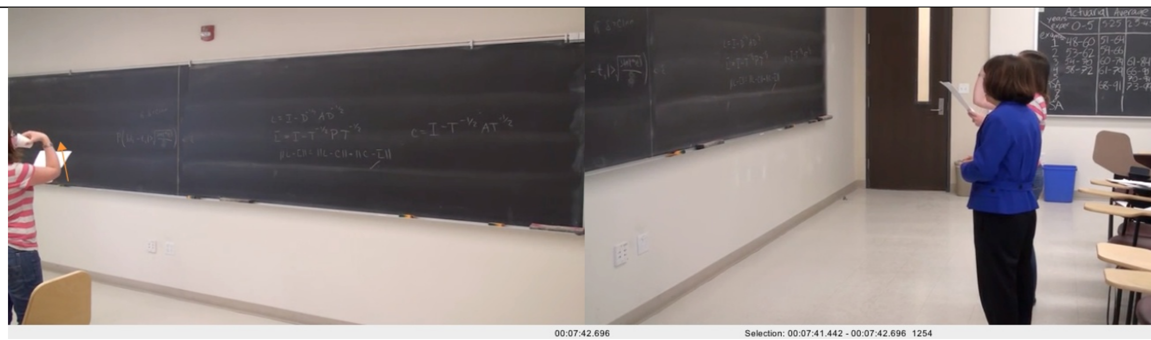
Row	Gesture & timestamp	Transcript
53		n vertices
54		[pause]

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Table A.7 – graph confining – *Continued from previous page*

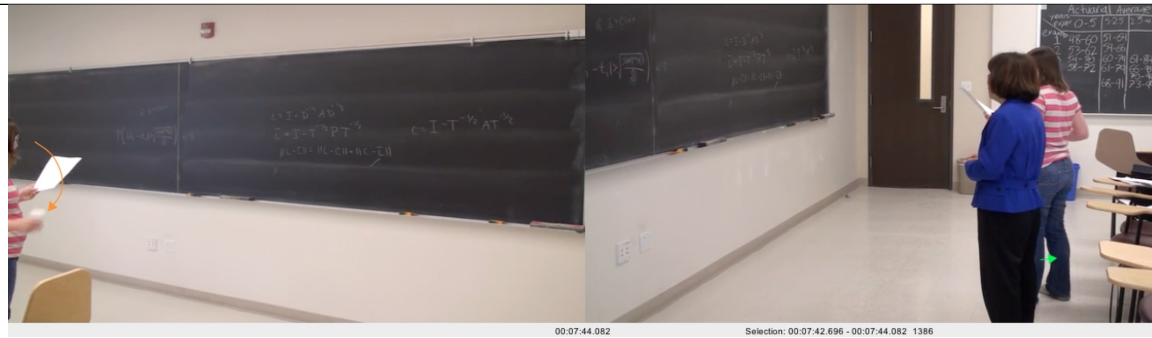
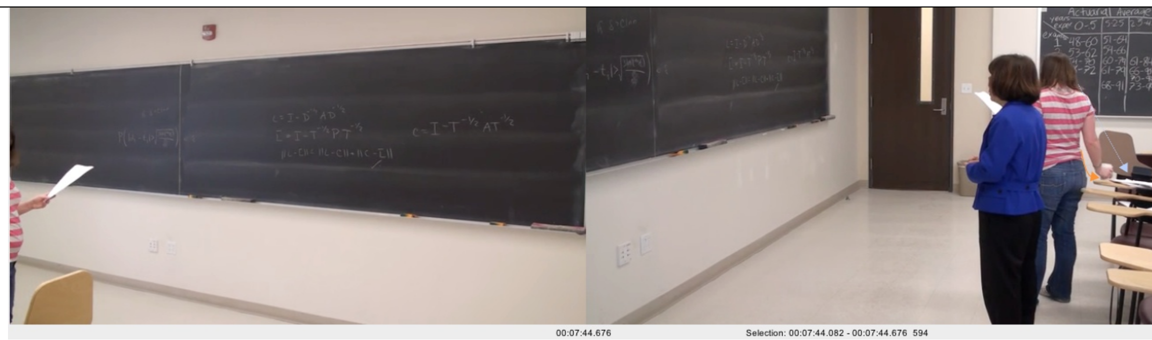
Row	Gesture & timestamp	Transcript
55		[pause]
56		So

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Table A.7 – graph confining – *Continued from previous page*

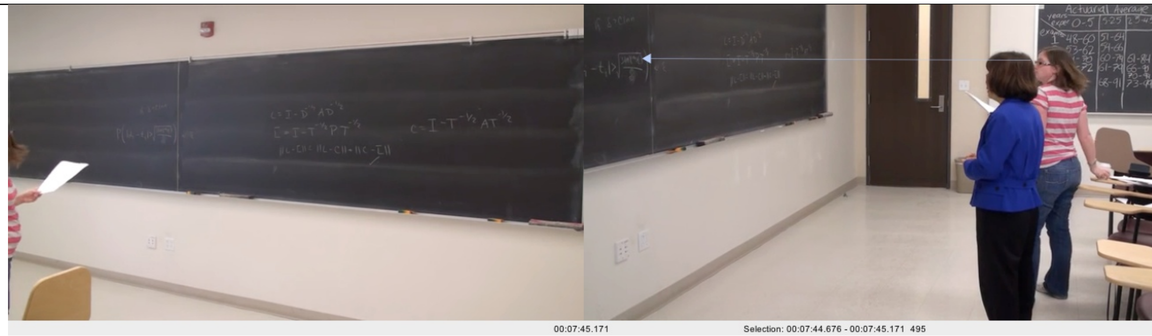
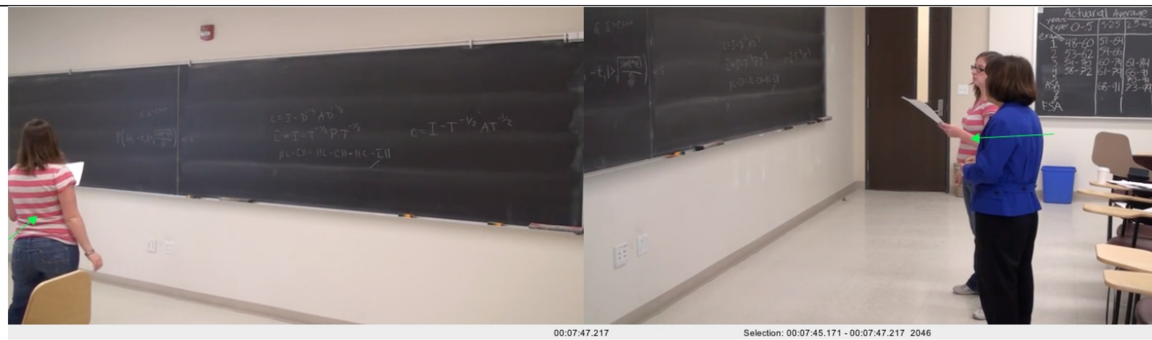
Row	Gesture & timestamp	Transcript
57		[pause]
58		[pause]

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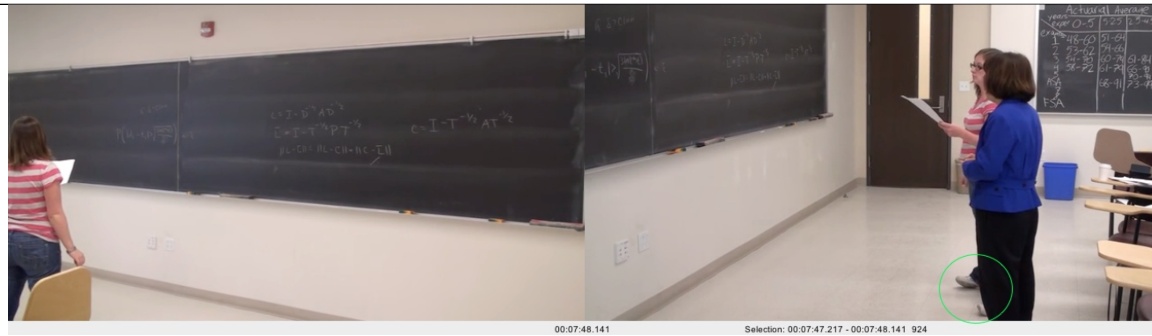
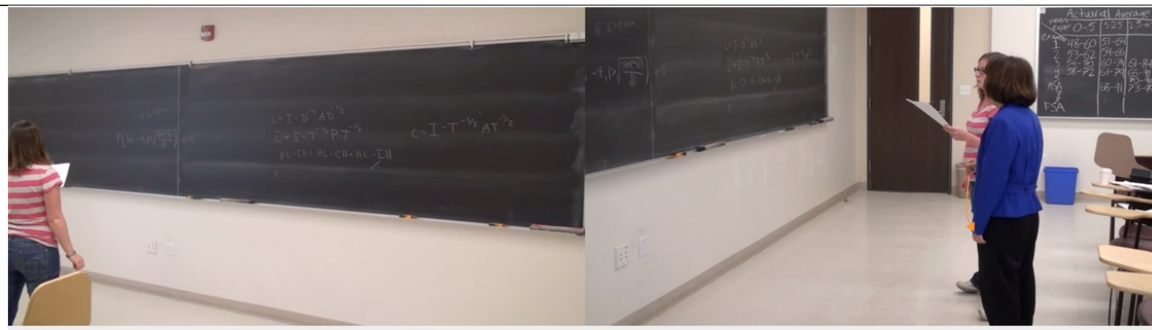
Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
59		[pause] Y-
60		eah, s-

Continued on next page

Table A.7 – graph confining – *Continued from previous page*

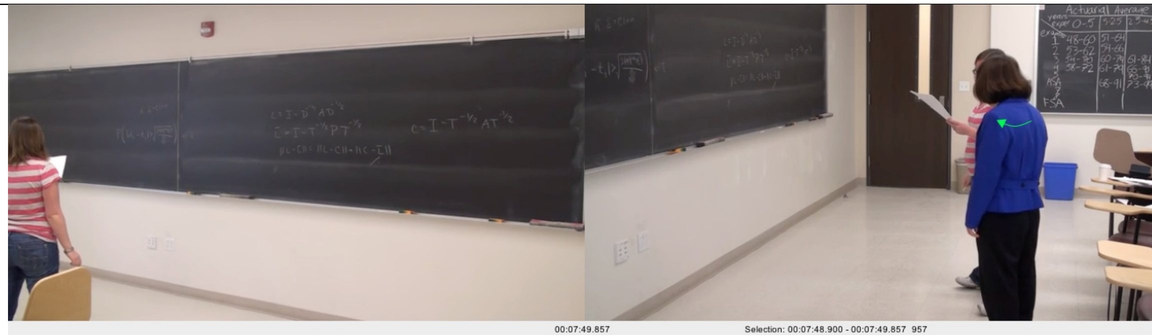
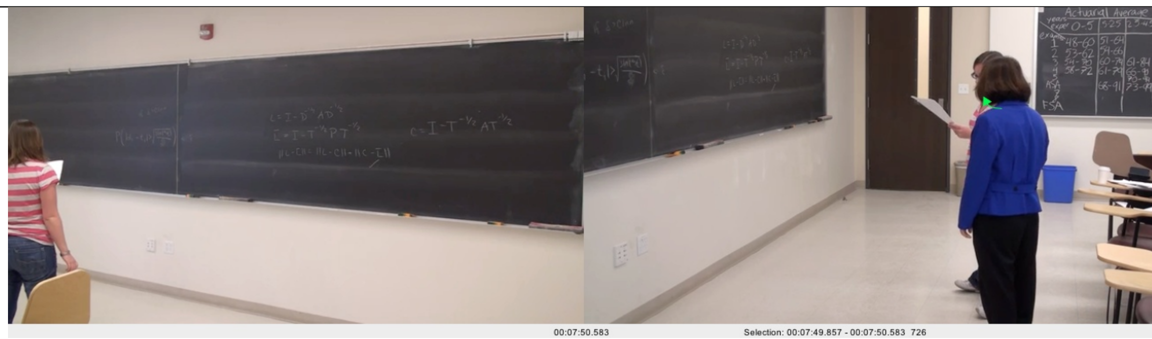
Row	Gesture & timestamp	Transcript
61		-o supp-
62		-ose, su-

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Table A.7 – graph confining – *Continued from previous page*

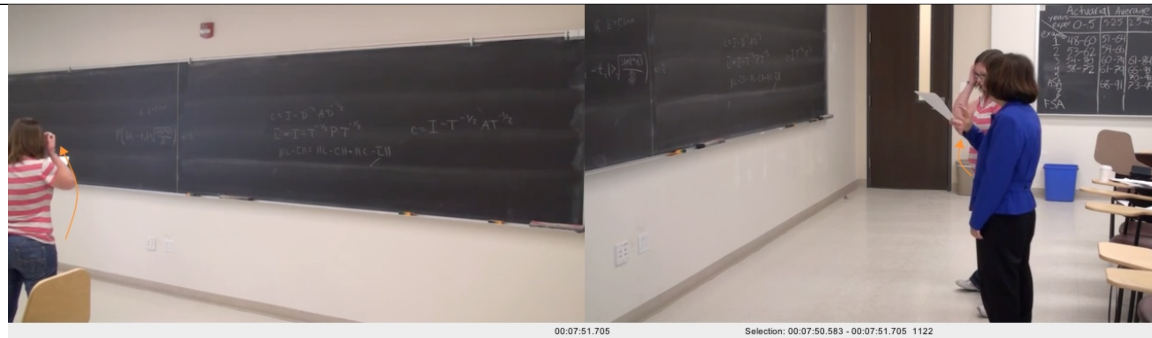
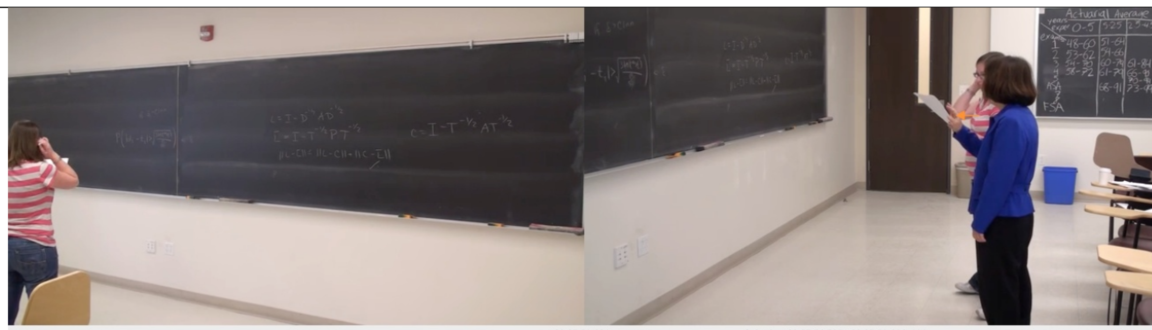
Row	Gesture & timestamp	Transcript
63		-ppose, let's forget
64		about

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Table A.7 – graph confining – *Continued from previous page*

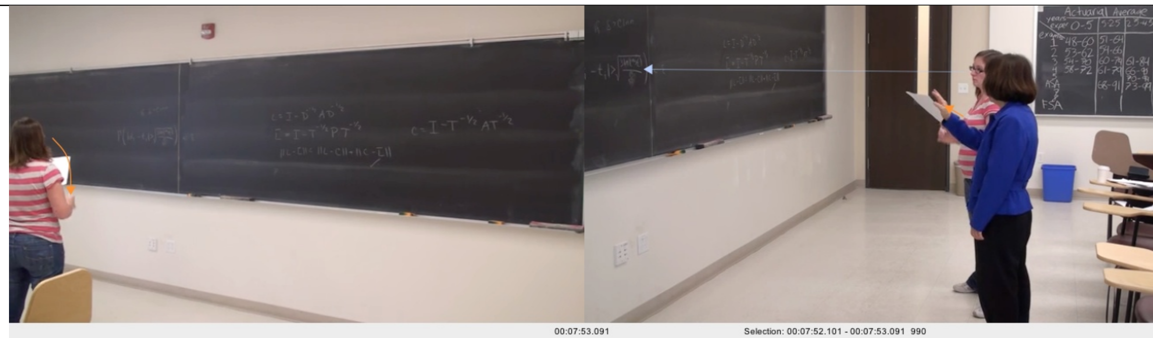
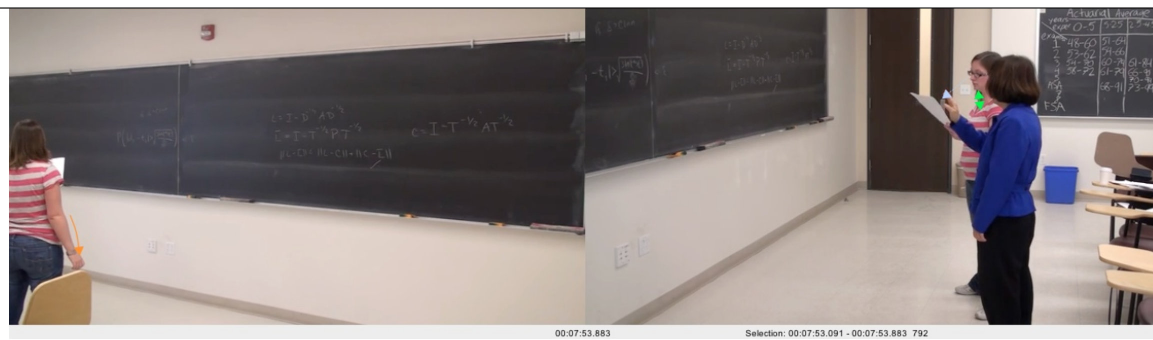
Row	Gesture & timestamp	Transcript
65		any-
66		-thing. M: Mhm.

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Table A.7 – graph confining – *Continued from previous page*

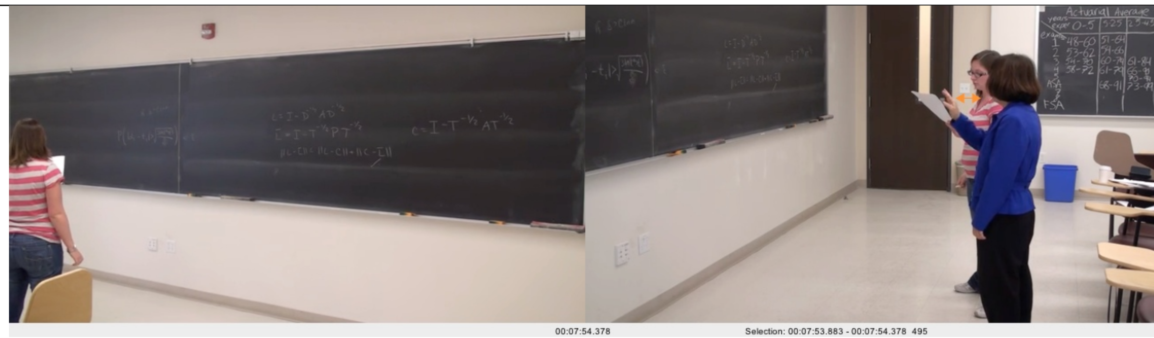
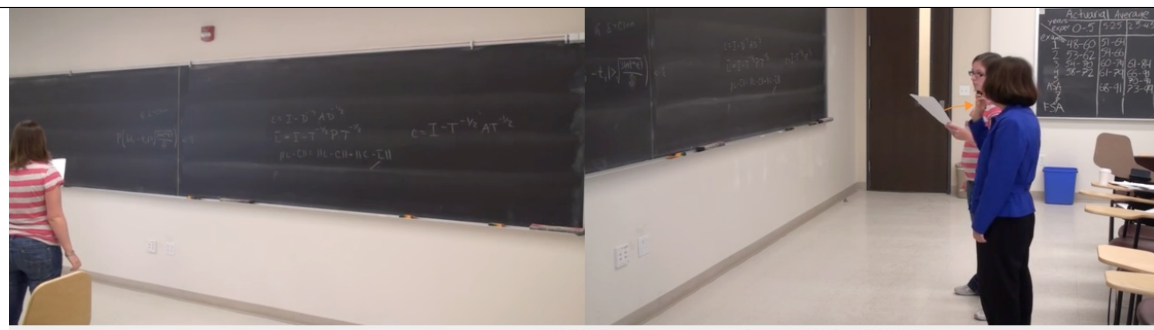
Row	Gesture & timestamp	Transcript
67		[pause]
68		F: Okay, so

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Table A.7 – graph confining – *Continued from previous page*

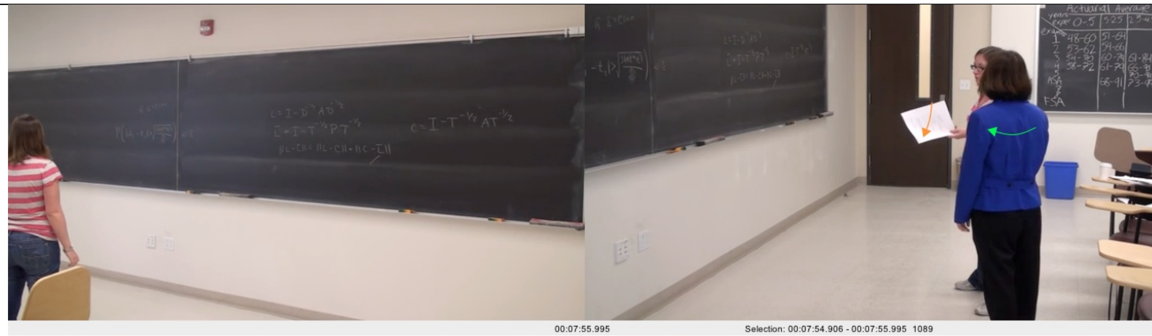
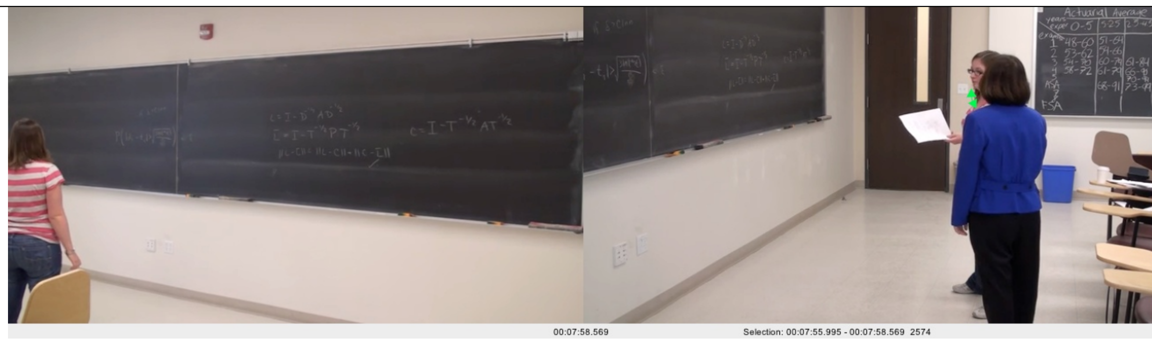
Row	Gesture & timestamp	Transcript
69		suppose all the degree
70		is more than log. [short pause] M: Mhm.

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Table A.7 – graph confining – *Continued from previous page*

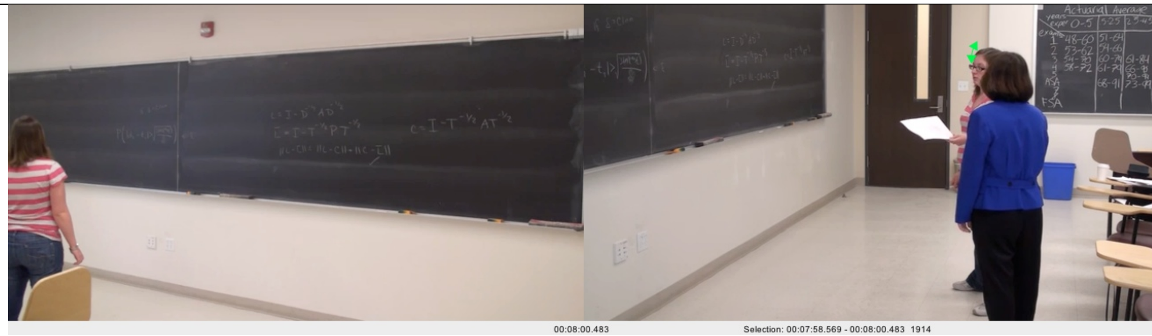
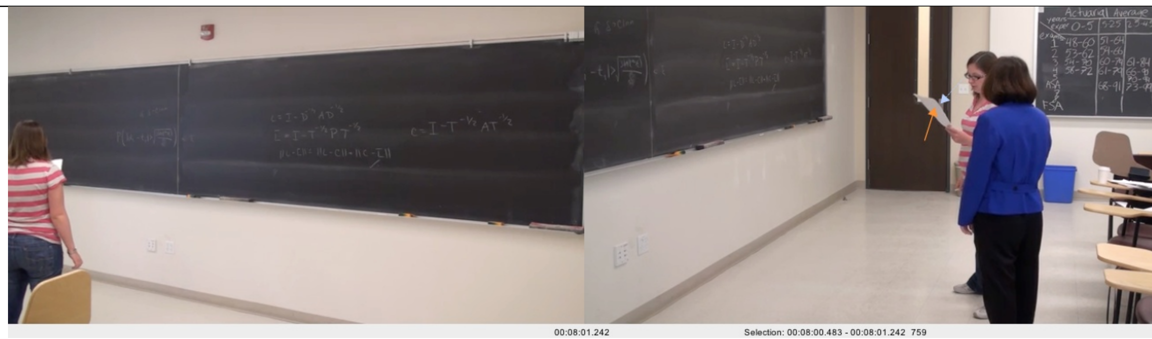
Row	Gesture & timestamp	Transcript
71		Sure. F: Suppose all the degree is more than
72		log. M: Sure.

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Table A.7 – graph confining – *Continued from previous page*

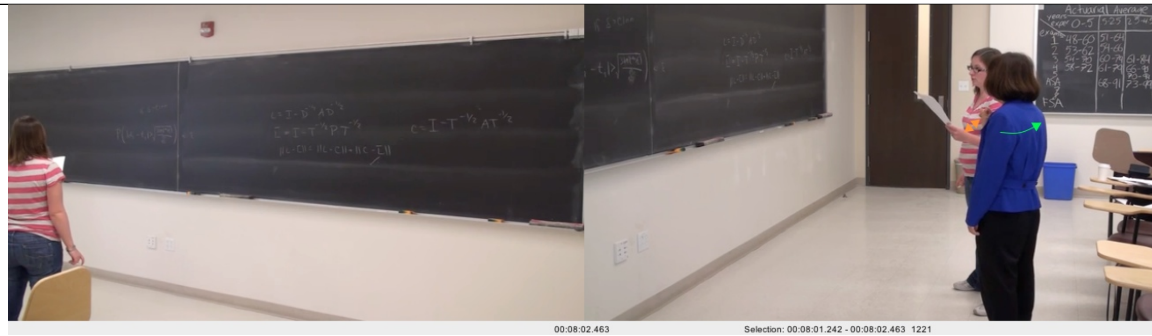
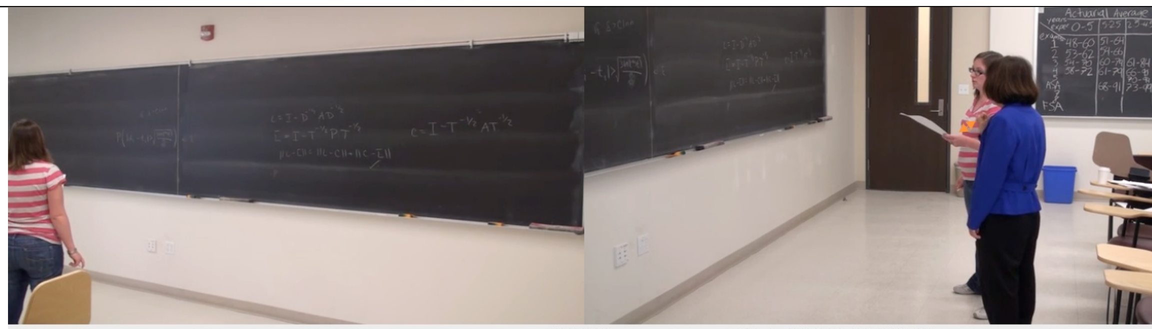
Row	Gesture & timestamp	Transcript
73		F: Then maybe we're able t-
74		-o control

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Table A.7 – graph confining – *Continued from previous page*

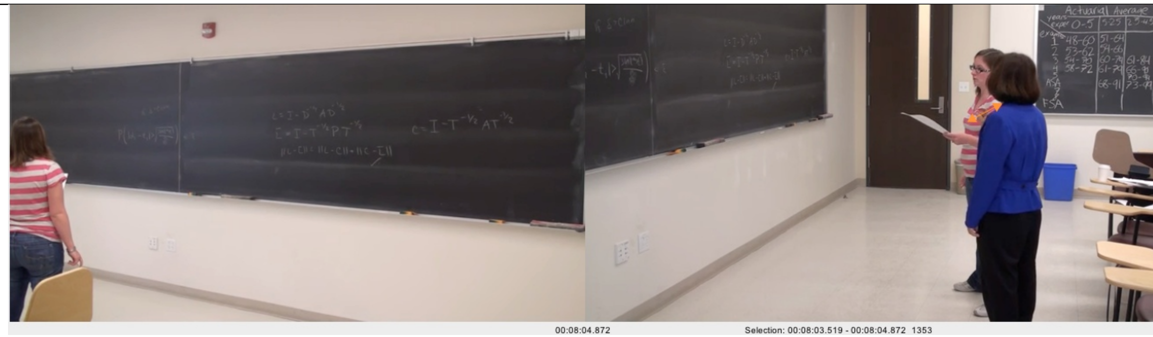
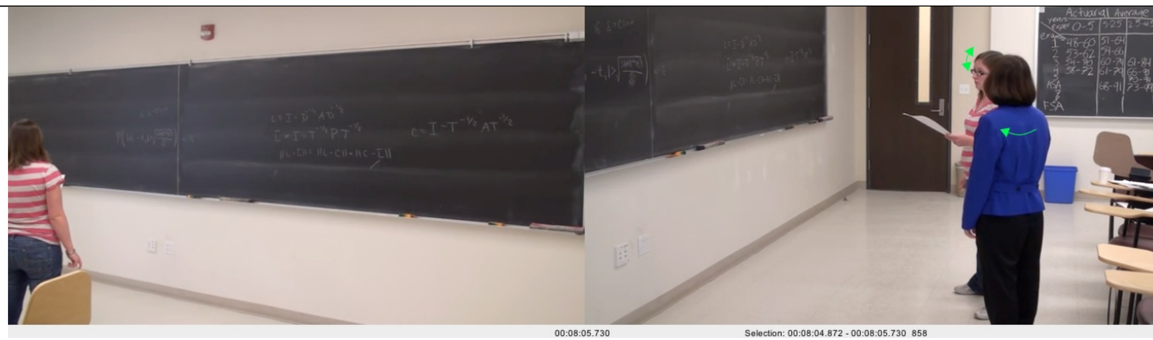
Row	Gesture & timestamp	Transcript
75		*every* degree
76		should be

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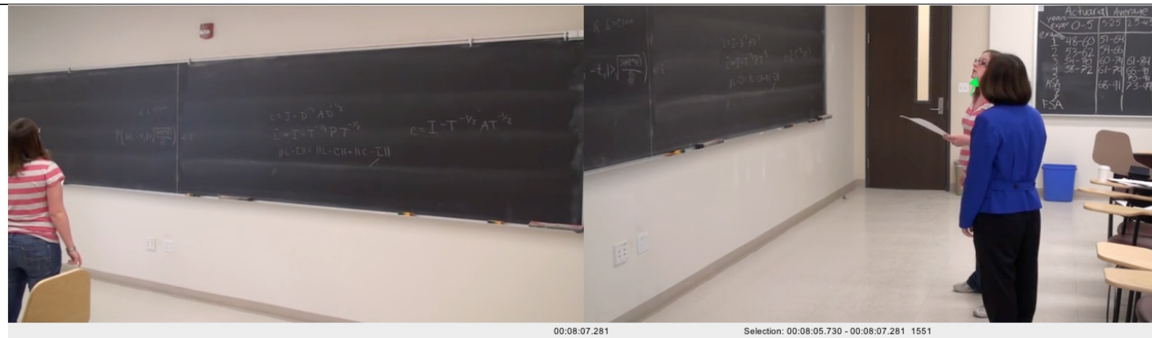
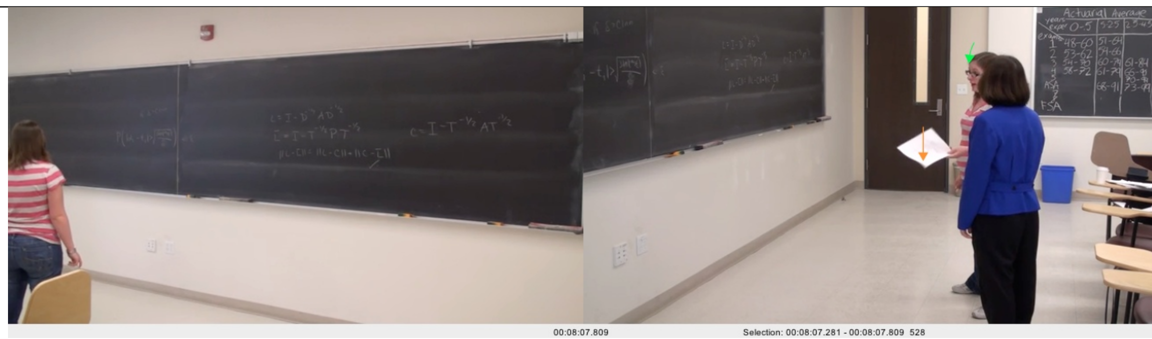
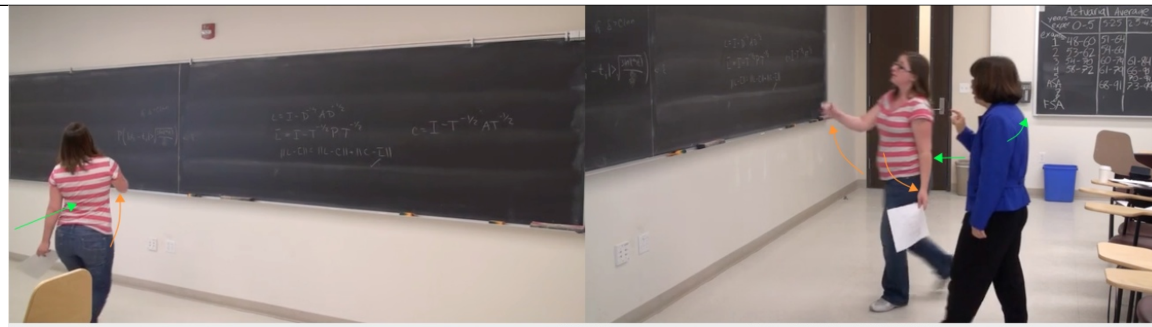
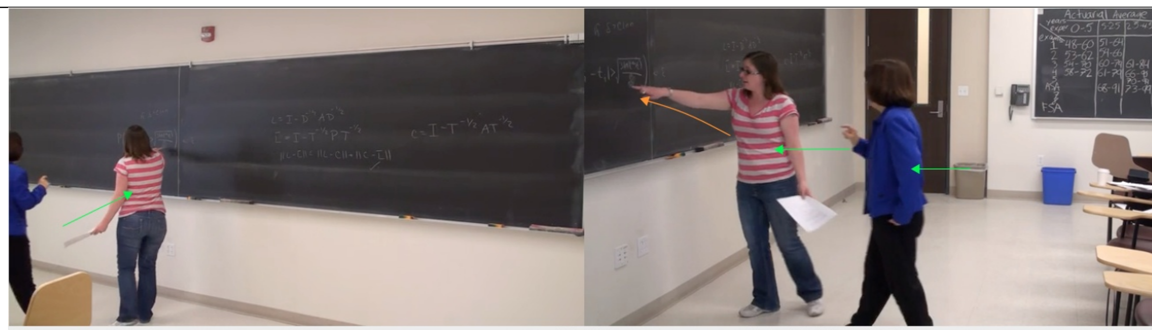
Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
77		close to its expected. . .
78		[pause] M: Ri-

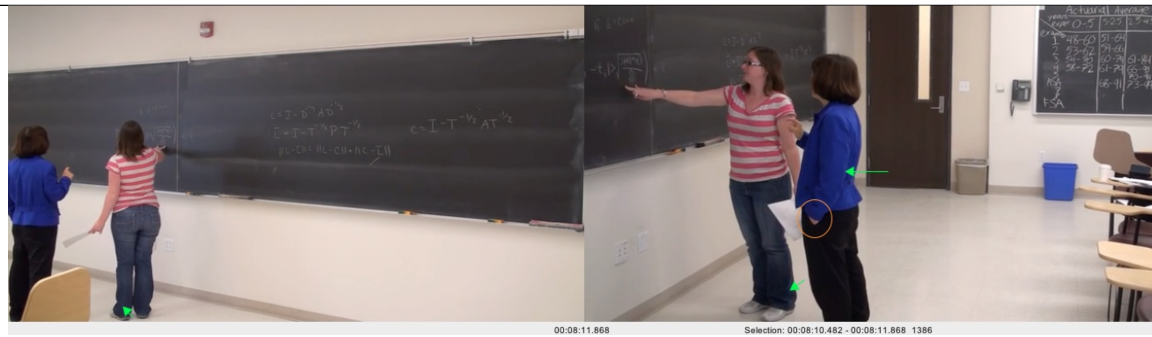
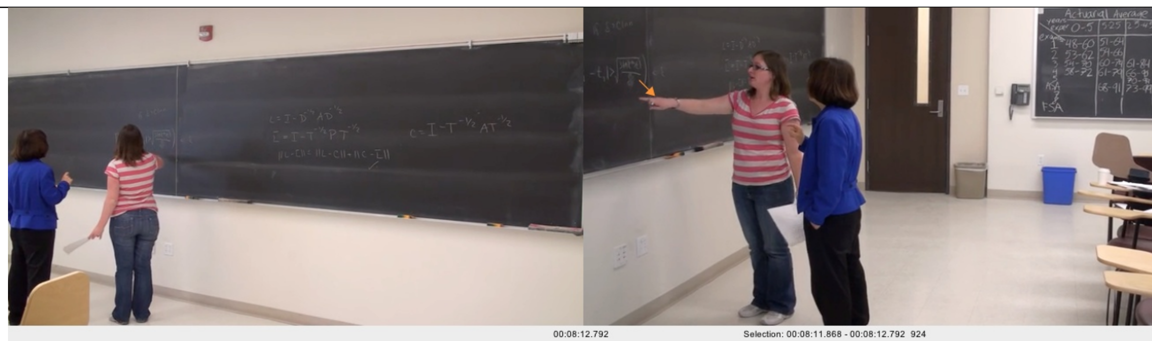
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
79	 00:08:09.327 Selection: 00:08:07.809 - 00:08:09.327 1518	-ght. Right, but what I'm really
80	 00:08:10.482 Selection: 00:08:09.327 - 00:08:10.482 1155	looking for is that,

Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
81		like an actual
82		[pause]

Continued on next page

Table A.7 – graph confining – *Continued from previous page*

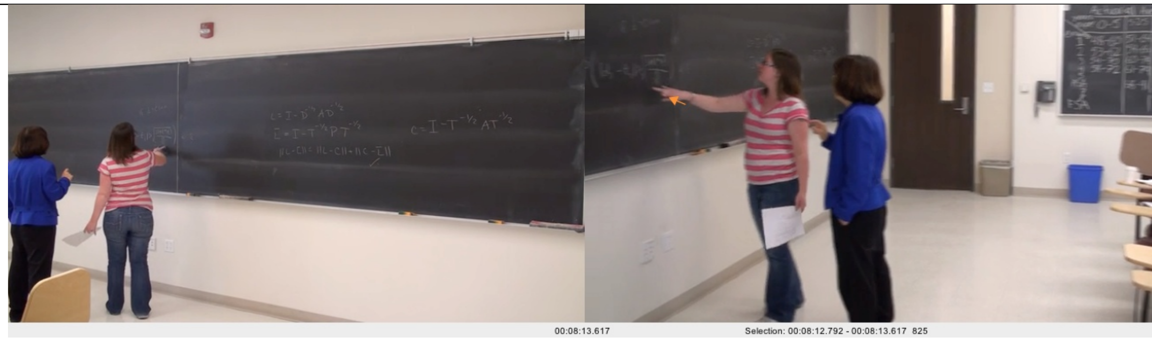
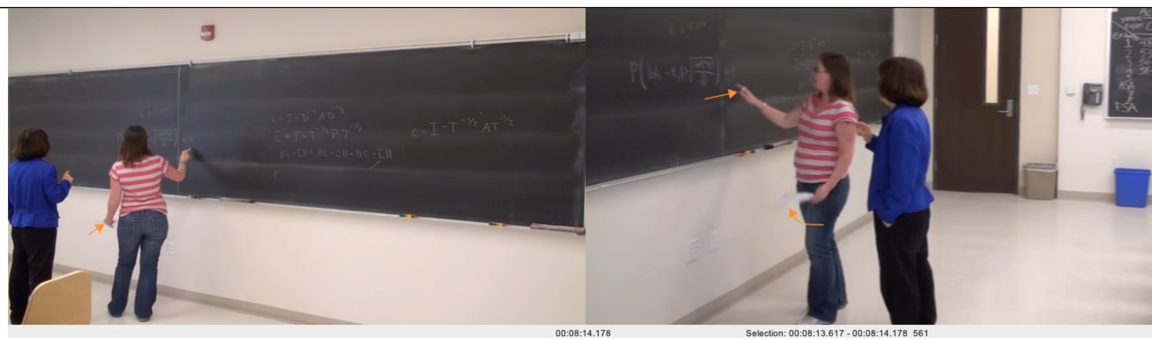
Row	Gesture & timestamp	Transcript
83		like how close
84		to its expec-

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Table A.7 – graph confining – *Continued from previous page*

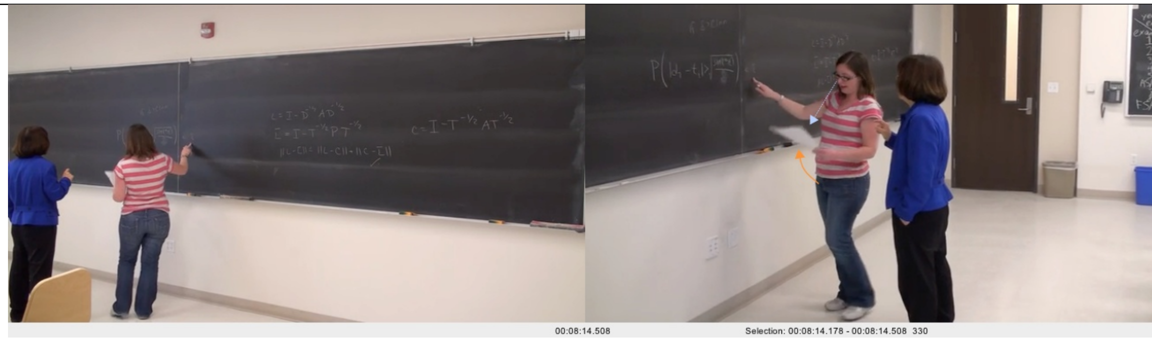
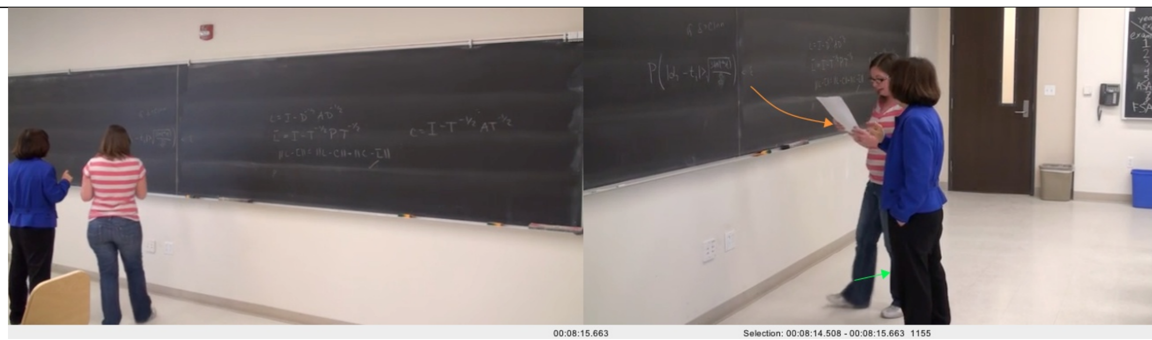
Row	Gesture & timestamp	Transcript
85		-ted
86		can I get? F: Right.

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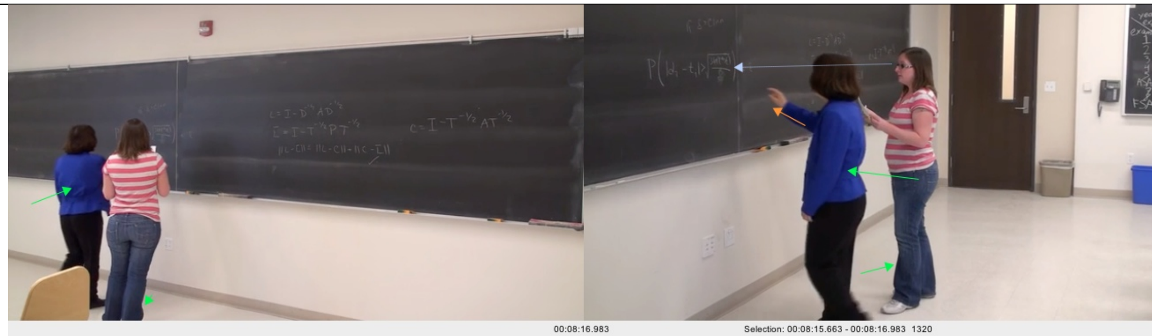
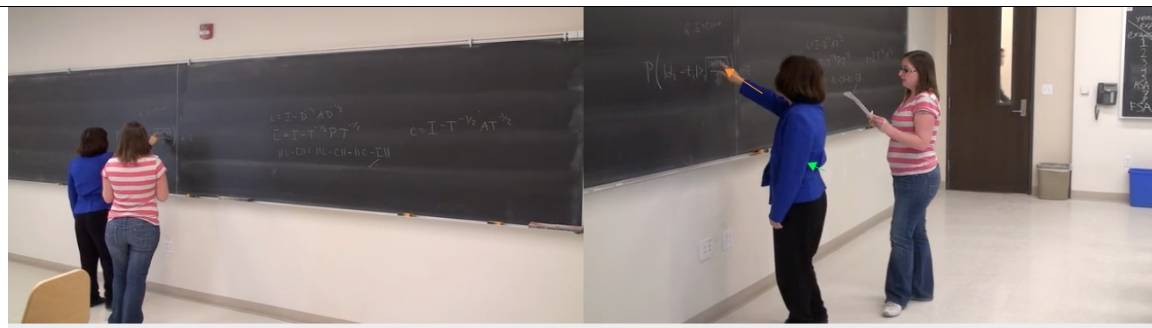
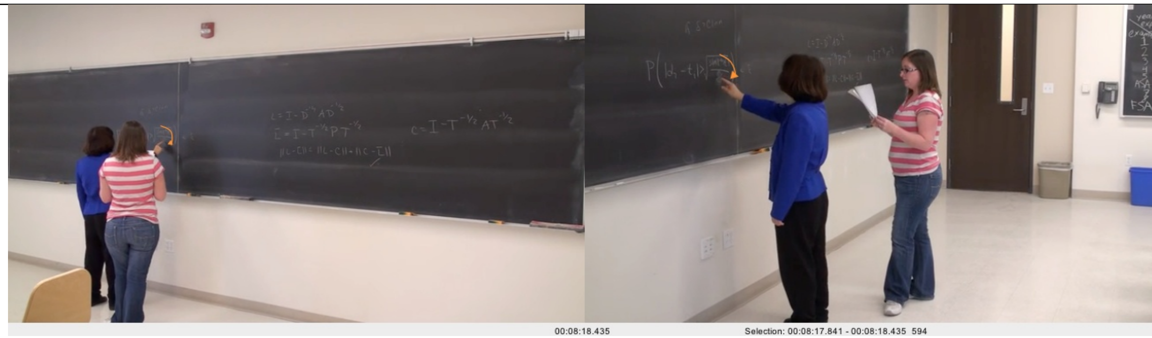
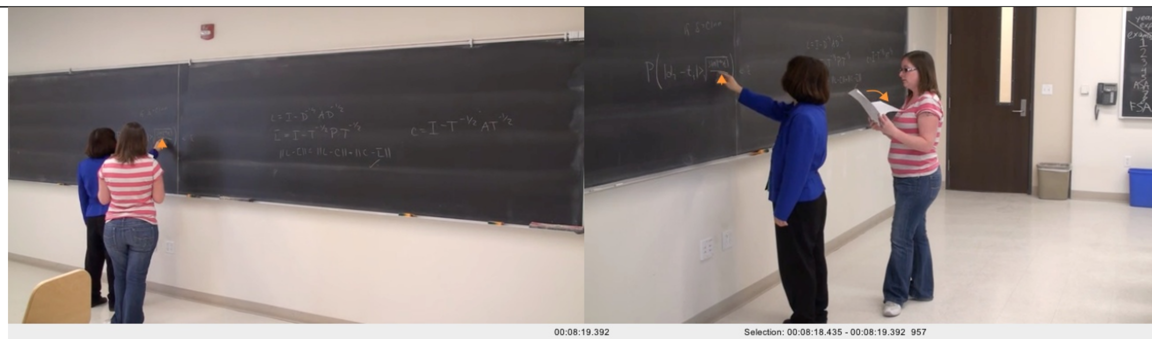
Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
87		So this
88		basically

Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
89		saying
90		you know δ [“delta”] is

Continued on next page

Table A.7 – graph confining – *Continued from previous page*

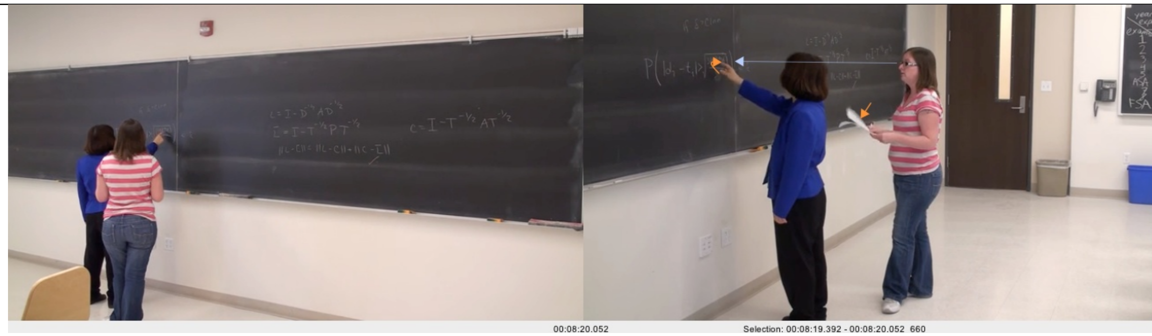
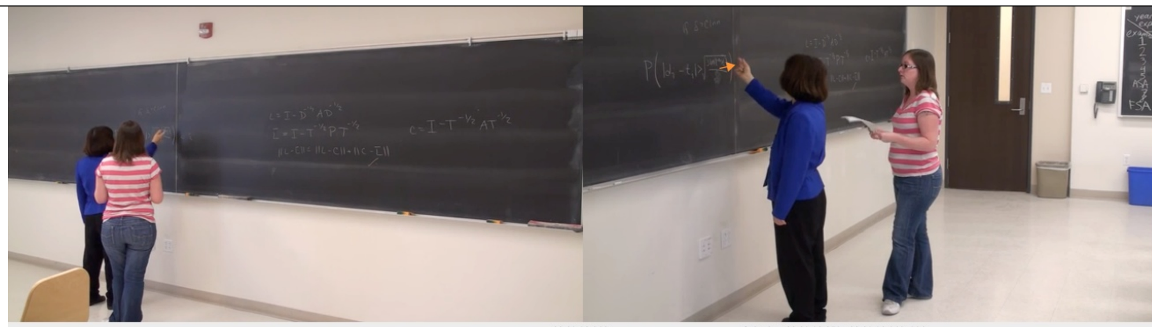
Row	Gesture & timestamp	Transcript
91		like
92		log n

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Table A.7 – graph confining – *Continued from previous page*

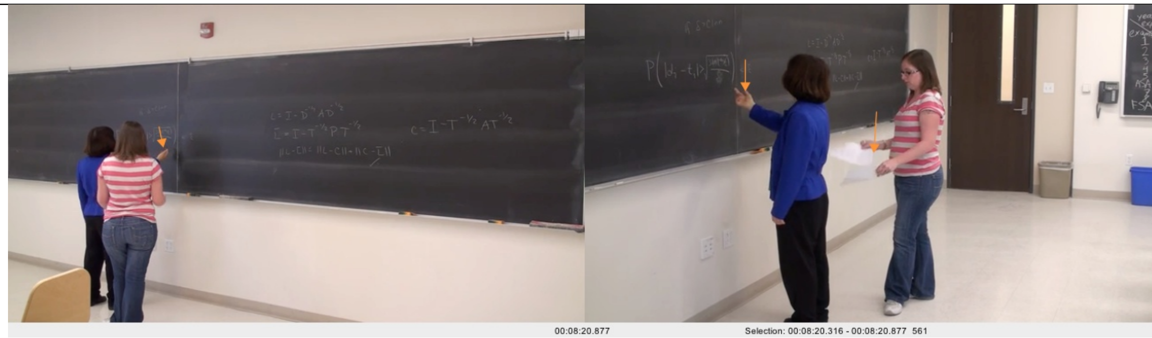
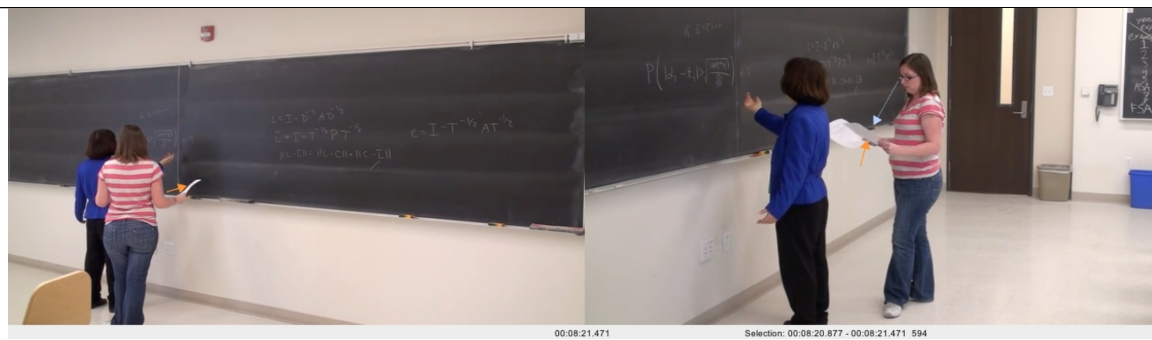
Row	**Gesture & timestamp**	**Transcript**
93		[pause]
94		Well, if I

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Table A.7 – graph confining – *Continued from previous page*

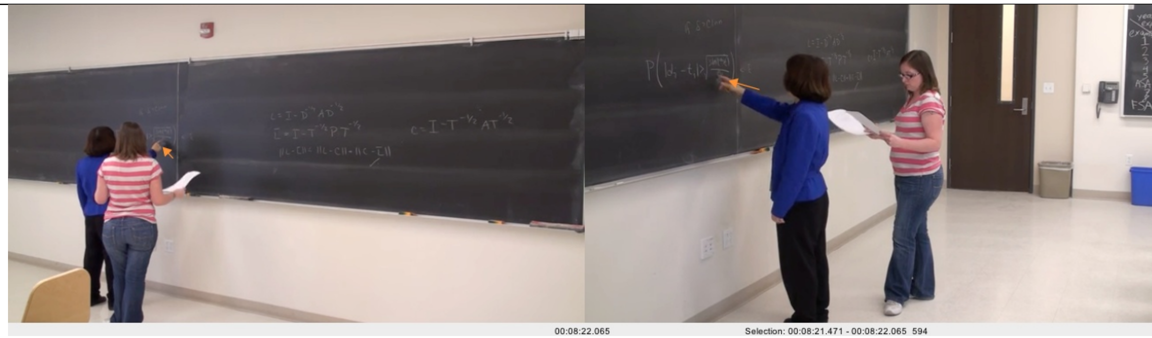
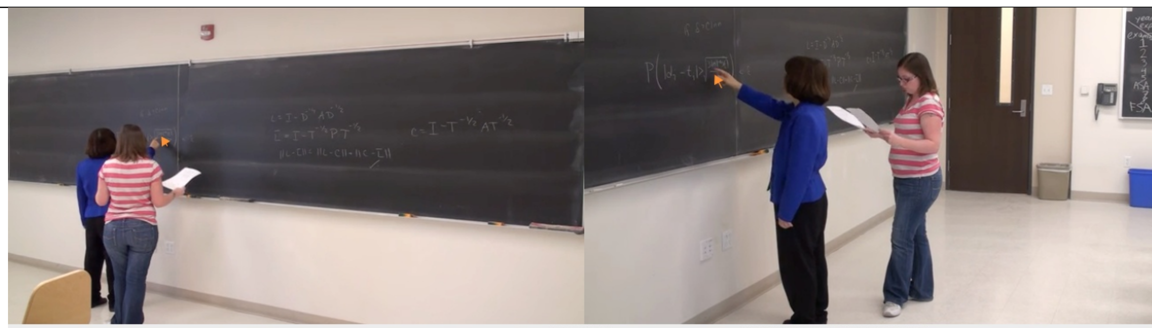
Row	Gesture & timestamp	Transcript
95		know how to
96		build up

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Table A.7 – graph confining – *Continued from previous page*

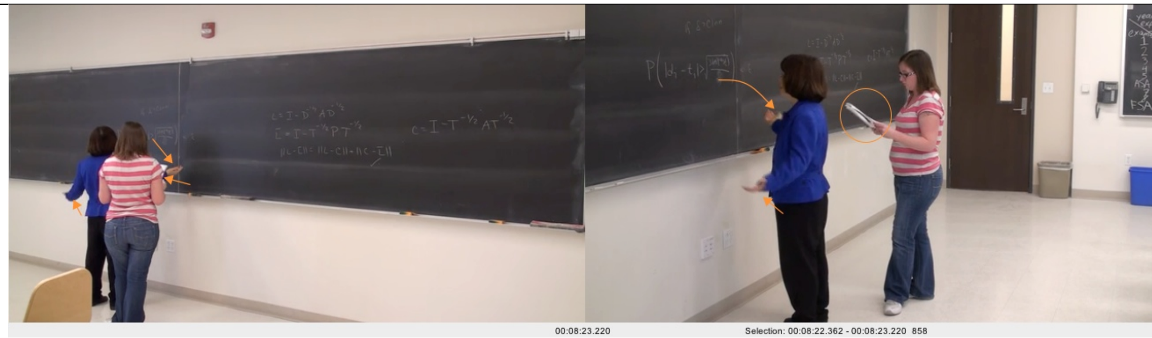
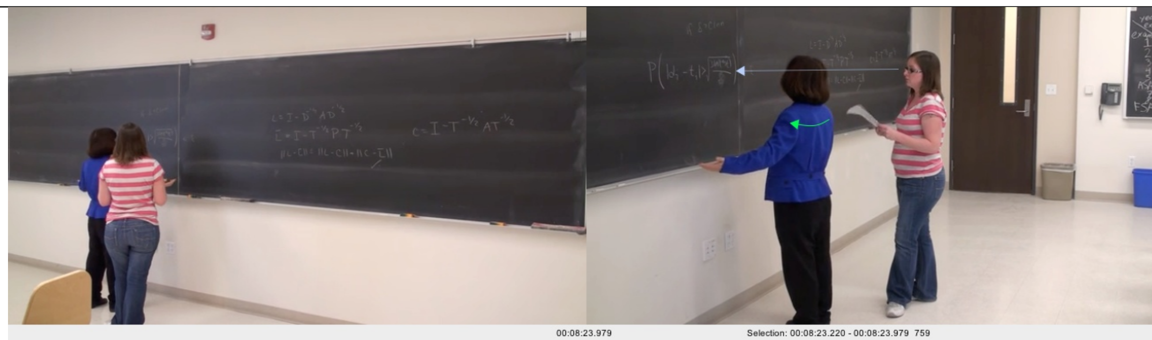
Row	Gesture & timestamp	Transcript
97		all the log n,
98		everything is

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Table A.7 – graph confining – *Continued from previous page*

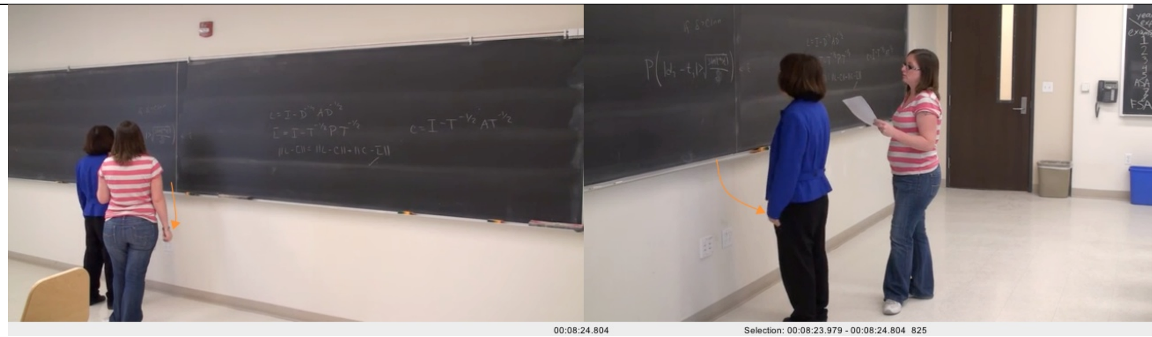
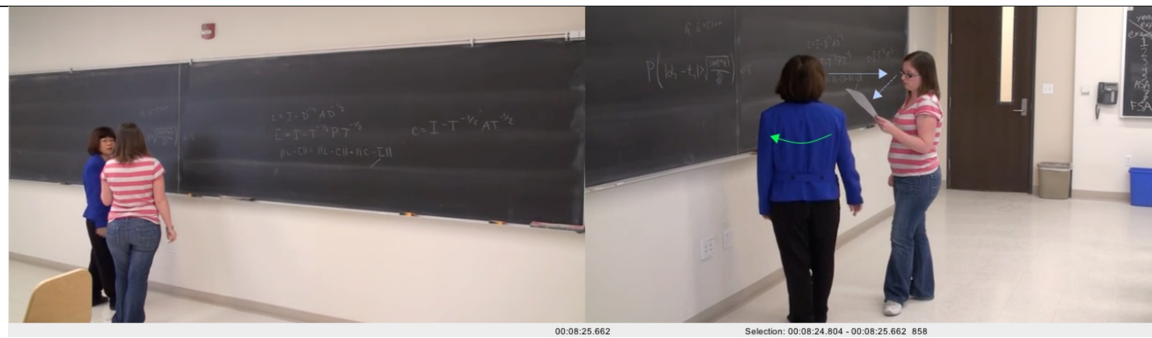
Row	Gesture & timestamp	Transcript
99		beautiful.
100		What's the pro-

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Table A.7 – graph confining – *Continued from previous page*

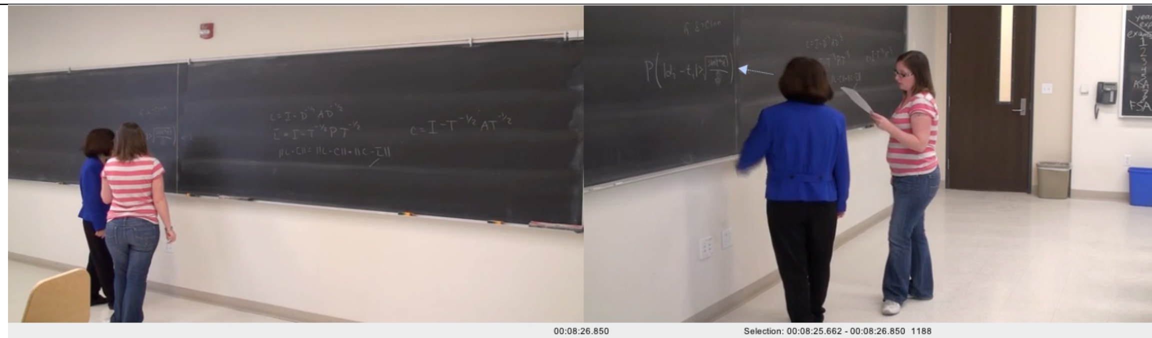
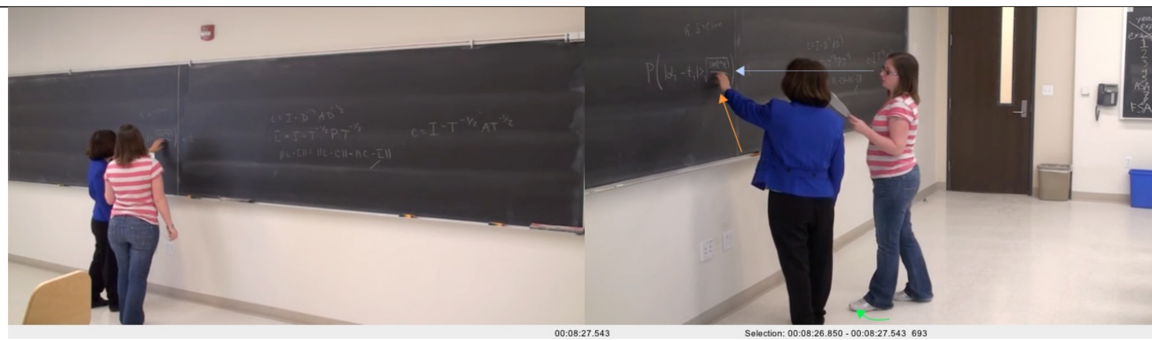
Row	Gesture & timestamp	Transcript
101		-blem? [pause] S-
102		-o this guy

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Table A.7 – graph confining – *Continued from previous page*

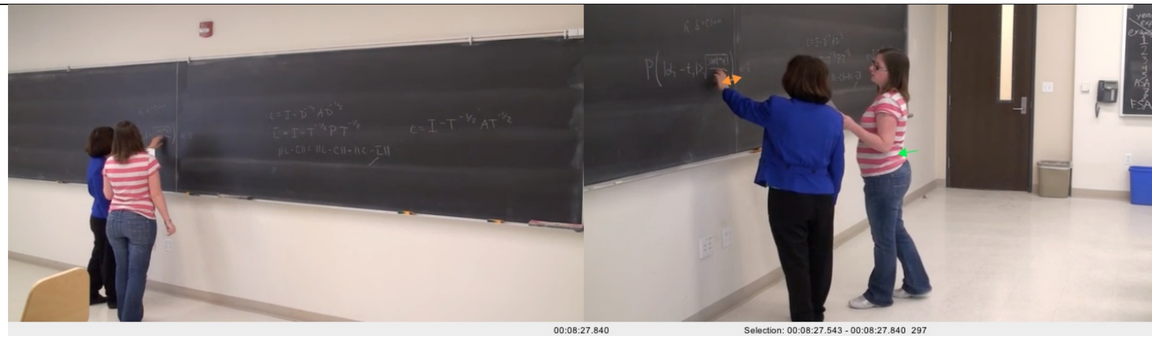
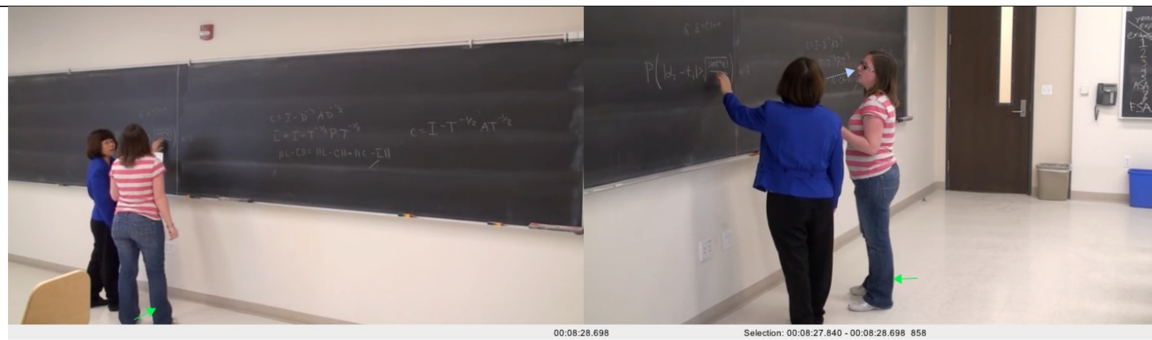
Row	Gesture & timestamp	Transcript
103		[pause]
104		should not go M: Oh.

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Table A.7 – graph confining – *Continued from previous page*

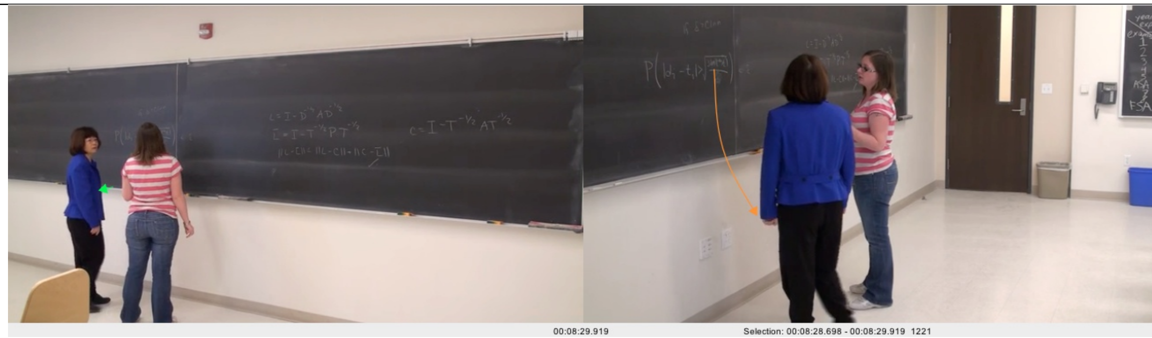
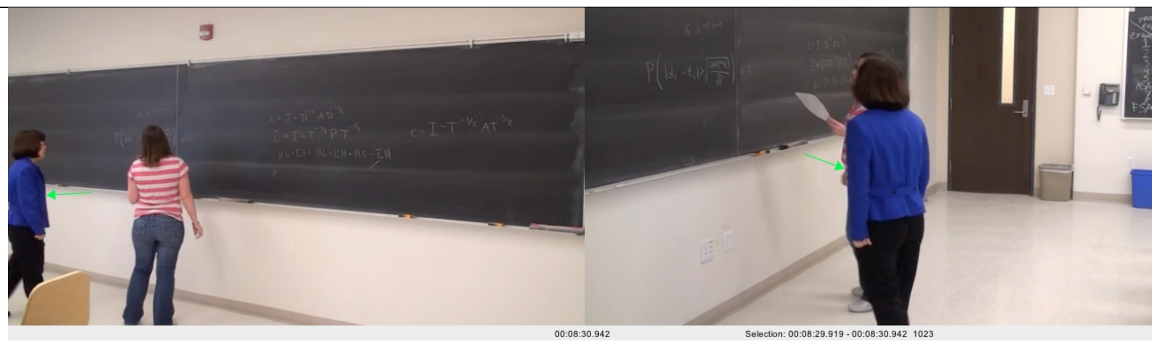
Row	Gesture & timestamp	Transcript
105		F: there. M: I see what you're saying.
106		F: Okay.

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Table A.7 – graph confining – *Continued from previous page*

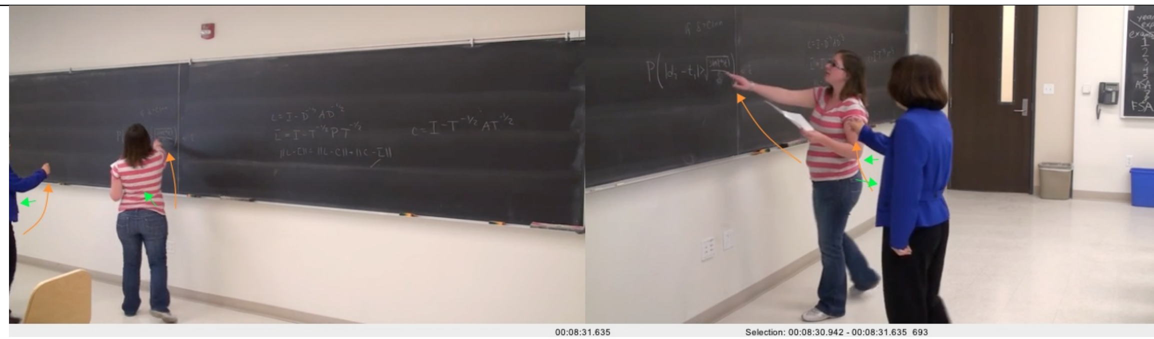
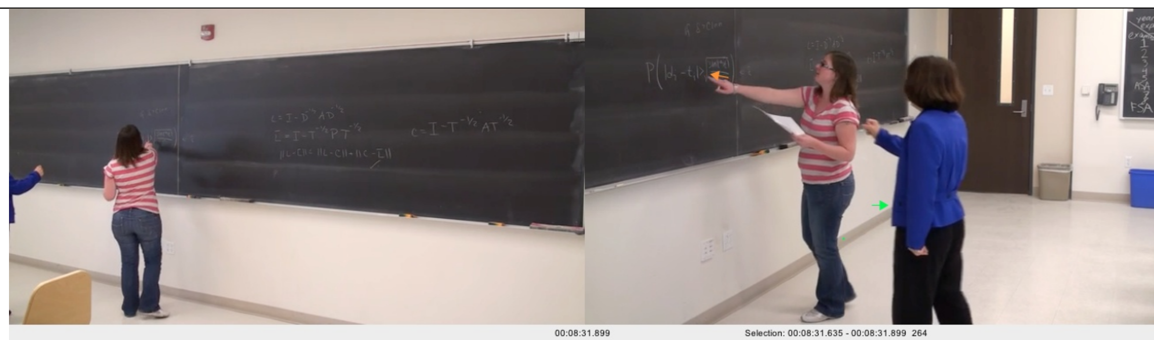
Row	Gesture & timestamp	Transcript
107		M: Yeah. She set
108		[inaudible]

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Table A.7 – graph confining – *Continued from previous page*

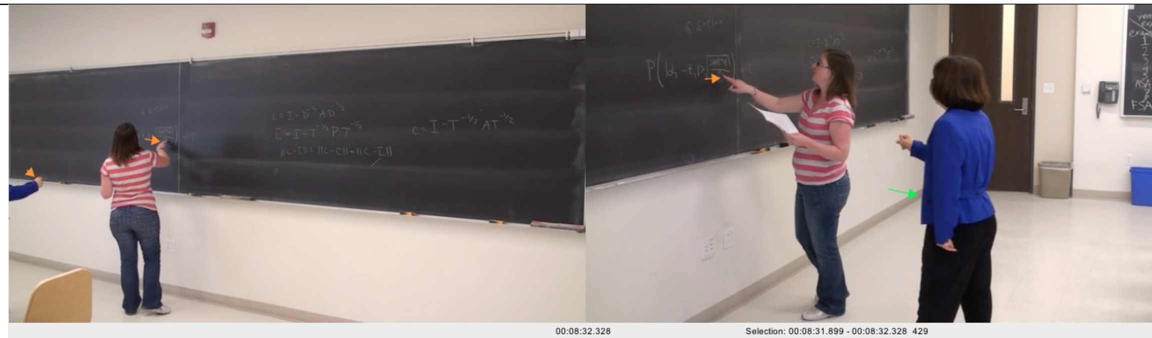
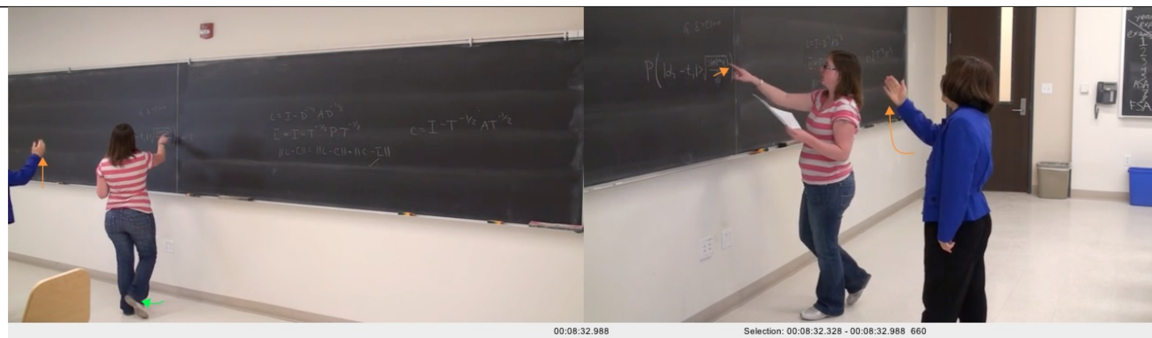
Row	Gesture & timestamp	Transcript
109		'n then
110		we just. . . F: It's no-

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Table A.7 – graph confining – *Continued from previous page*

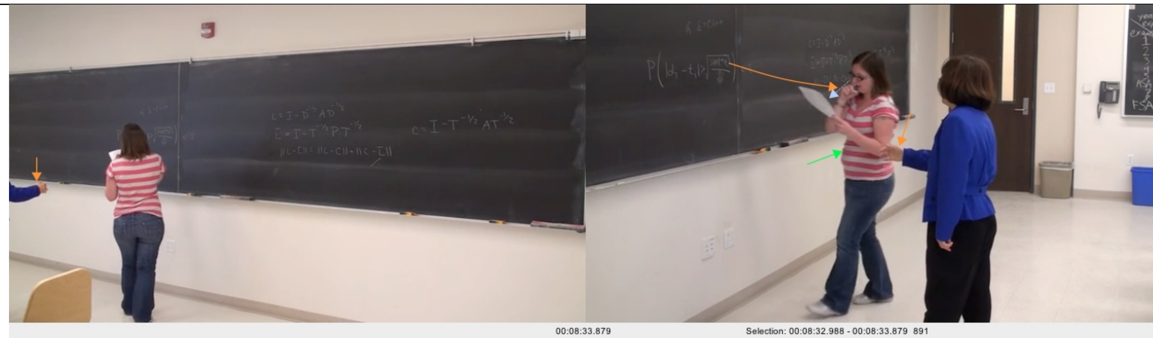
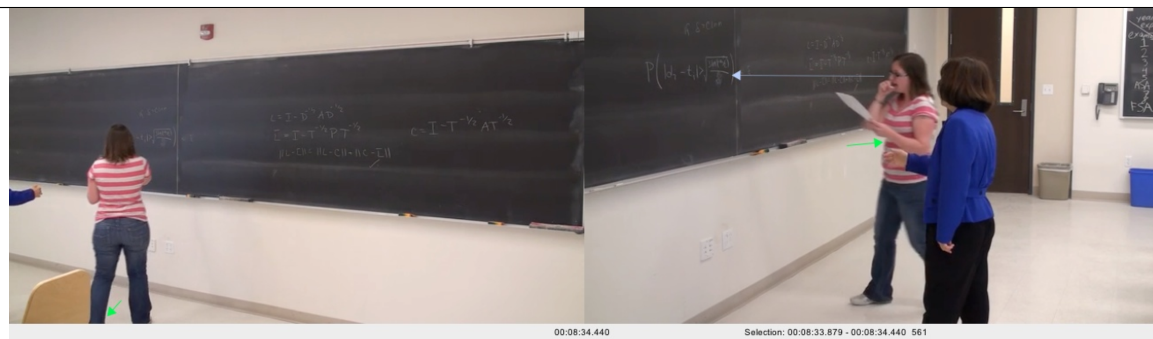
Row	Gesture & timestamp	Transcript
111		-nsense. M: Yeah.
112		F: Right?

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Table A.7 – graph confining – *Continued from previous page*

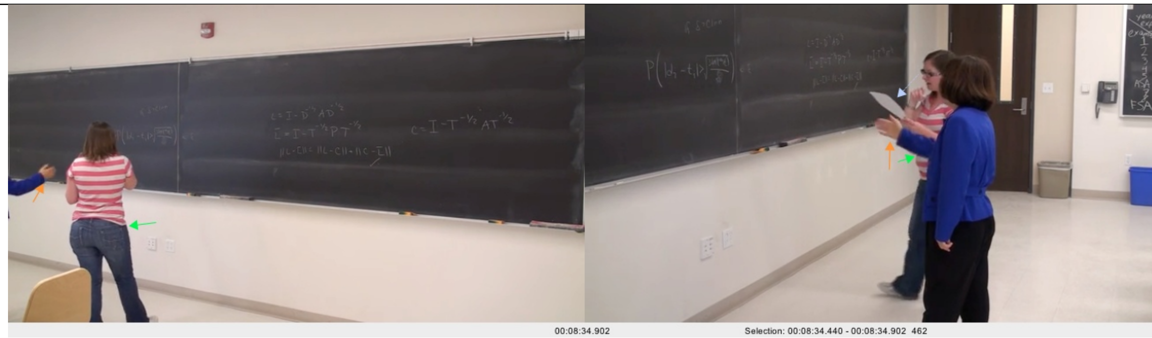
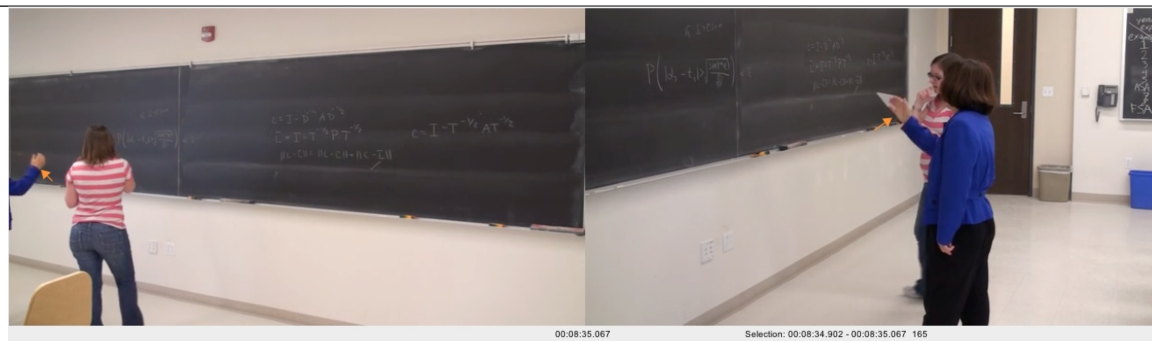
Row	Gesture & timestamp	Transcript
113		M: Yeah, I mean…
114		F: it's

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Table A.7 – graph confining – *Continued from previous page*

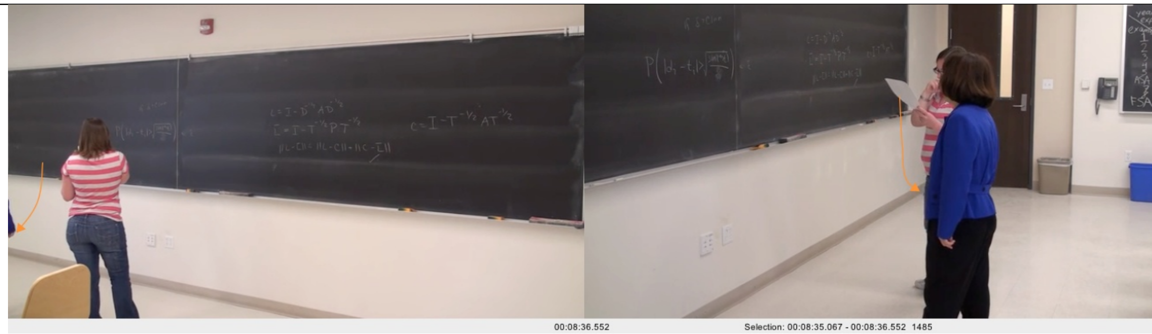
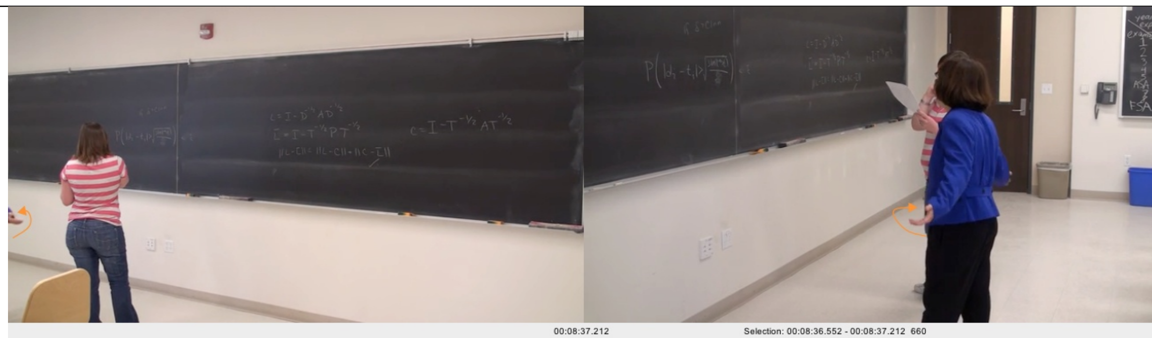
Row	Gesture & timestamp	Transcript
115		more than log,
116		I'm *happy!*

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Table A.7 – graph confining – *Continued from previous page*

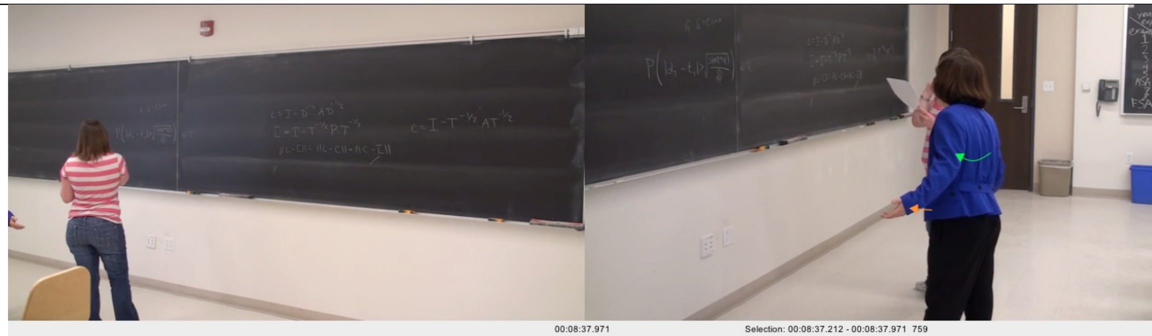
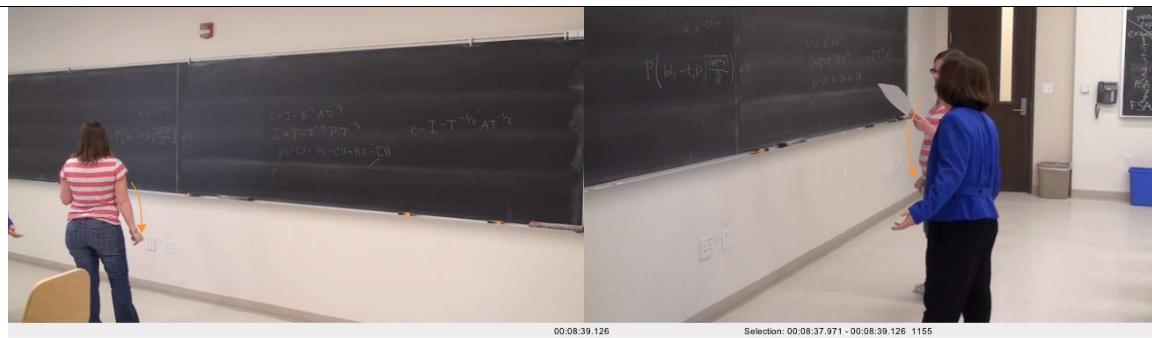
Row	Gesture & timestamp	Transcript
117		[pause]
118		M: So I. . . F: Fine!

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Table A.7 – graph confining – *Continued from previous page*

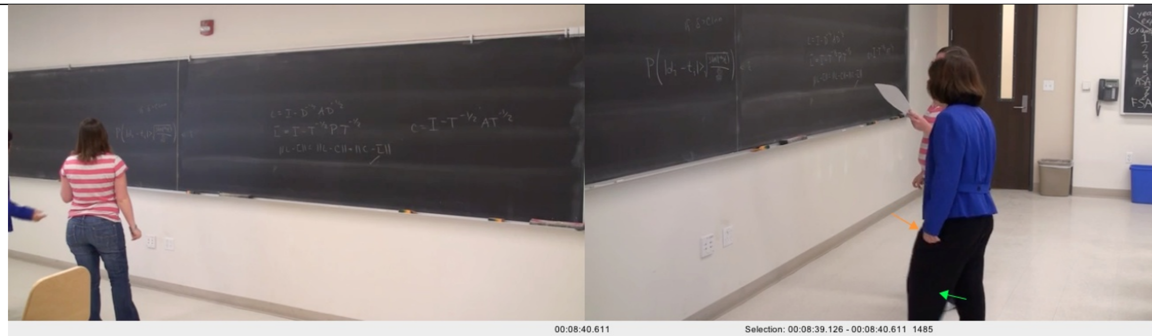
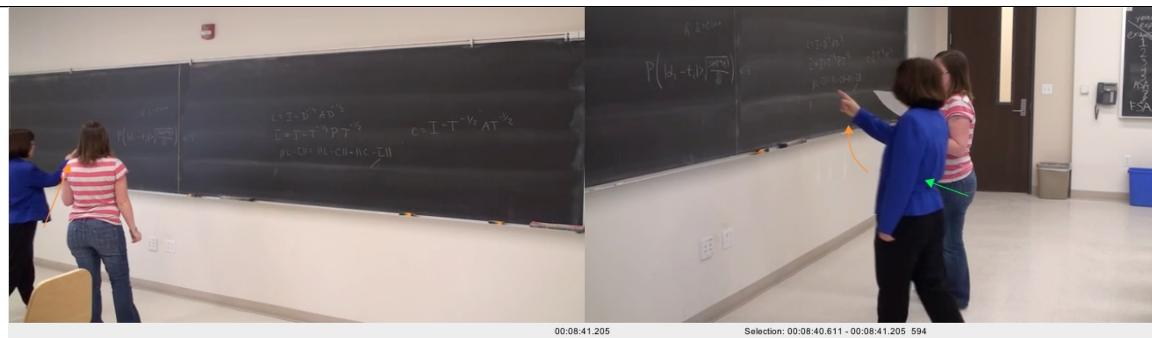
Row	Gesture & timestamp	Transcript
119		[pause] But of. . .
120		M: Right. F: . . . course you have

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Table A.7 – graph confining – *Continued from previous page*

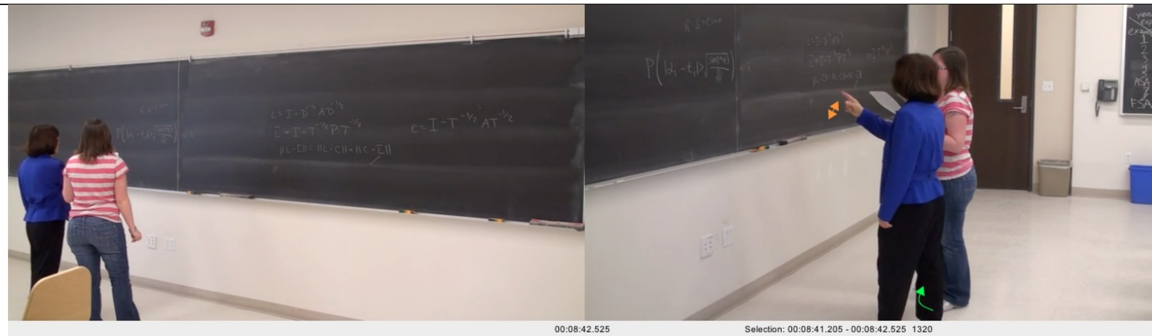
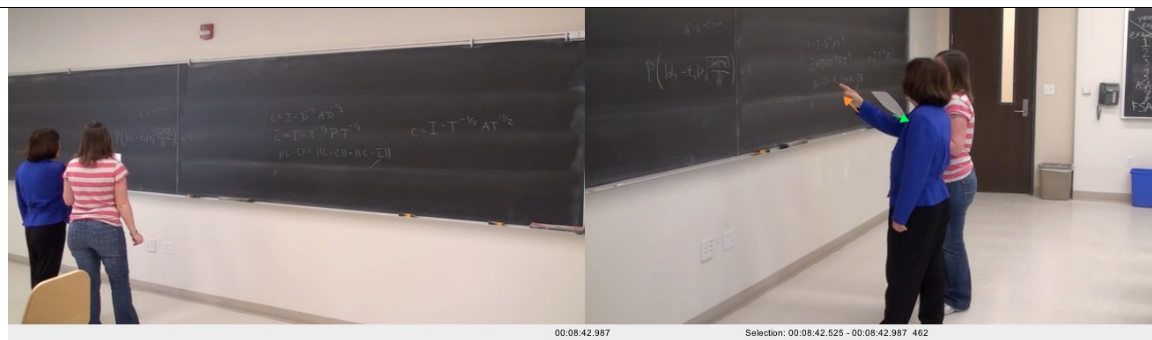
Row	Gesture & timestamp	Transcript
121		there is a square root there!
122		But st-

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Table A.7 – graph confining – *Continued from previous page*

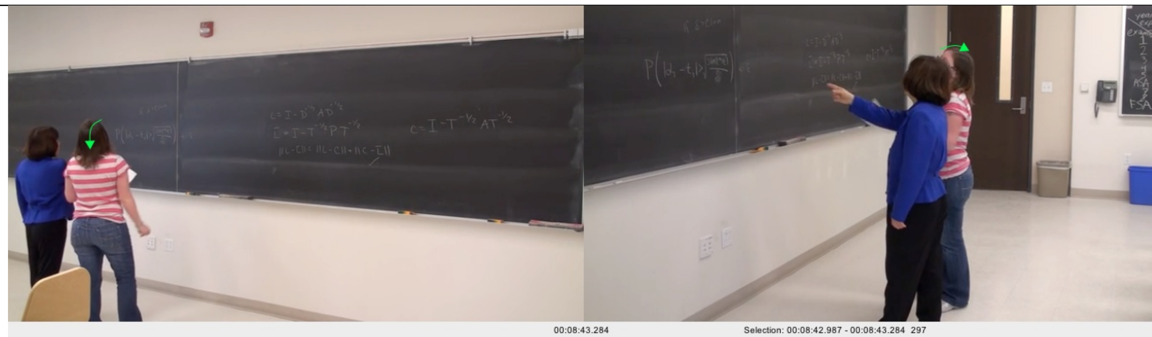
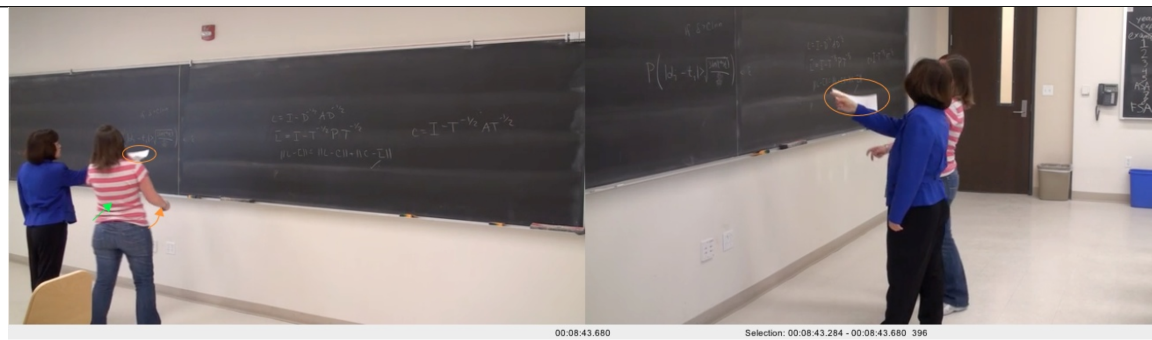
Row	Gesture & timestamp	Transcript
123		-ill sh-
124		-ouldn't. . . M: Sure. F: . . . c-

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Table A.7 – graph confining – *Continued from previous page*

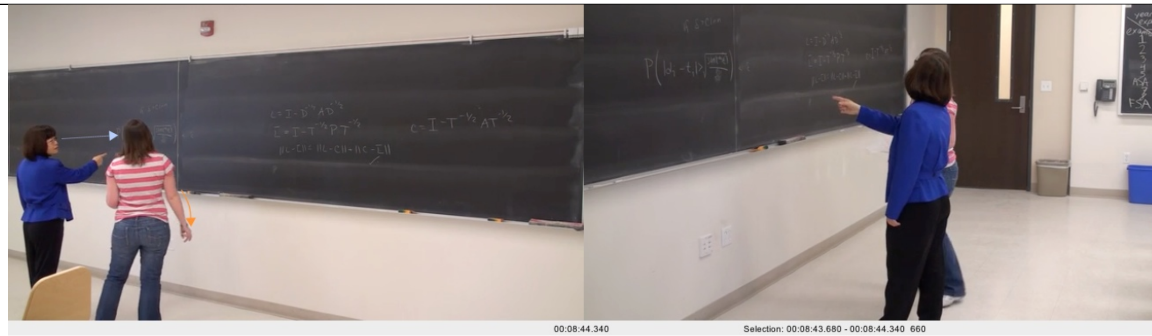
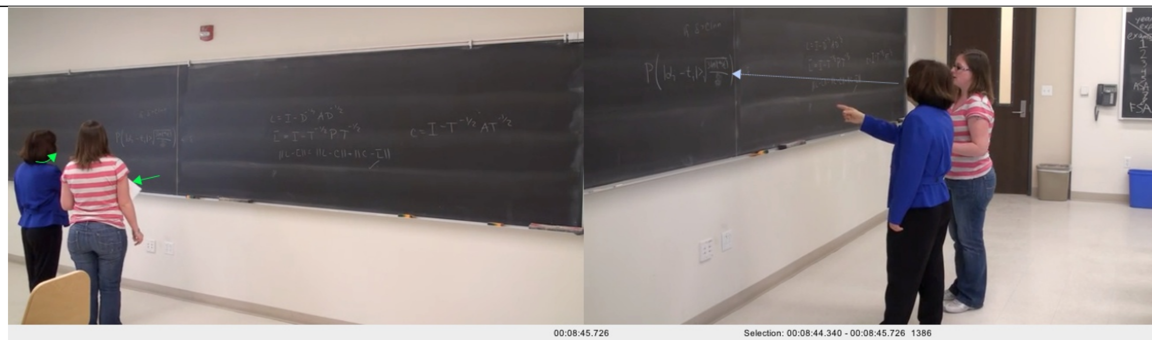
Row	Gesture & timestamp	Transcript
125		-ross any of the
126		[inaudible]. M: No, the square root shouldn't…

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Table A.7 – graph confining – *Continued from previous page*

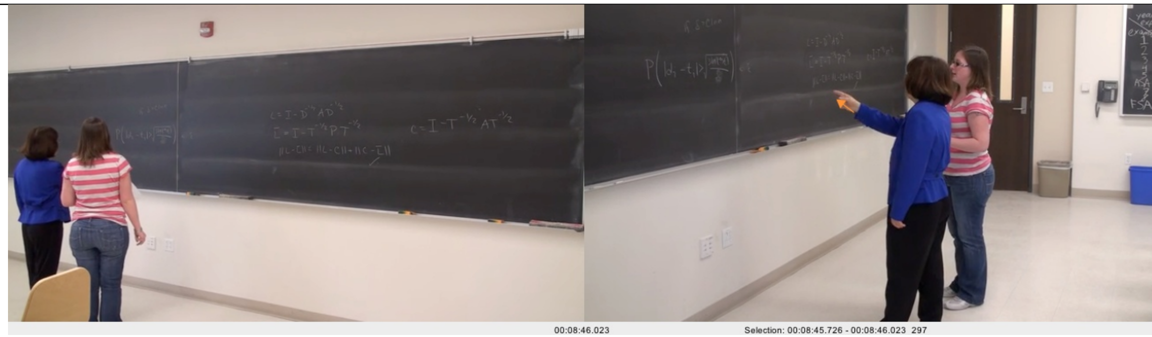
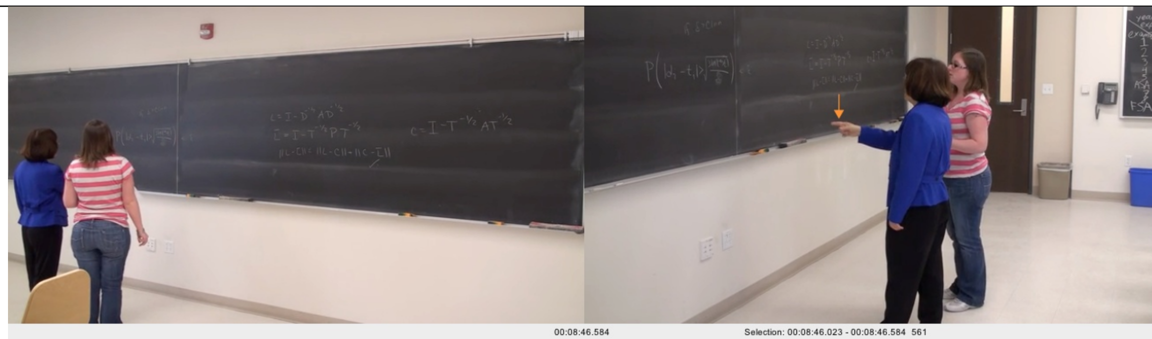
Row	Gesture & timestamp	Transcript
127		be. . . F: [inaudible] M: R-
128		-ight. F: Ridic-

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Table A.7 – graph confining – *Continued from previous page*

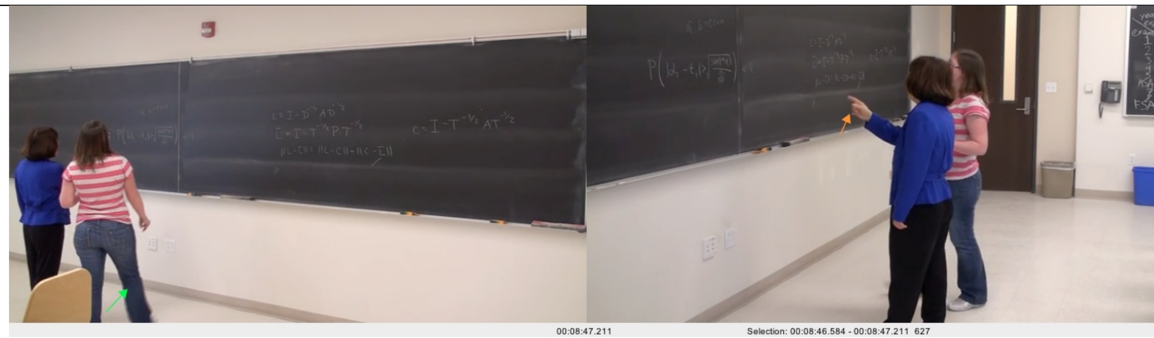
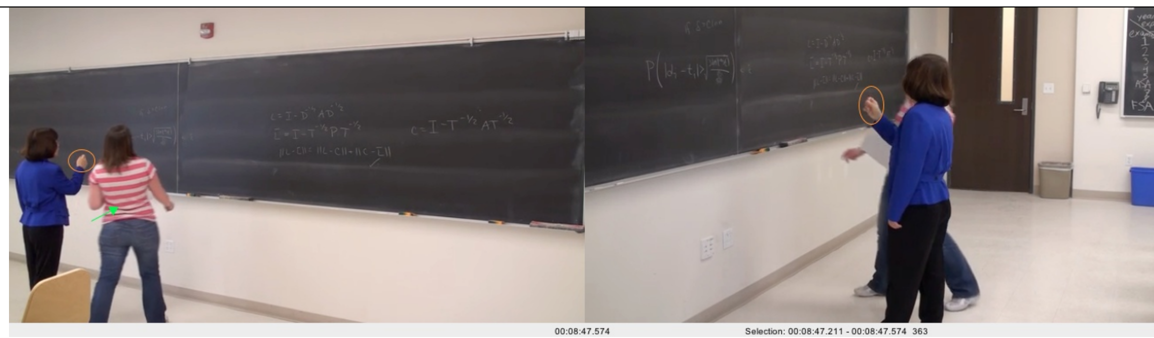
Row	Gesture & timestamp	Transcript
129		-ulous. M: So
130		[pause]

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Table A.7 – graph confining – *Continued from previous page*

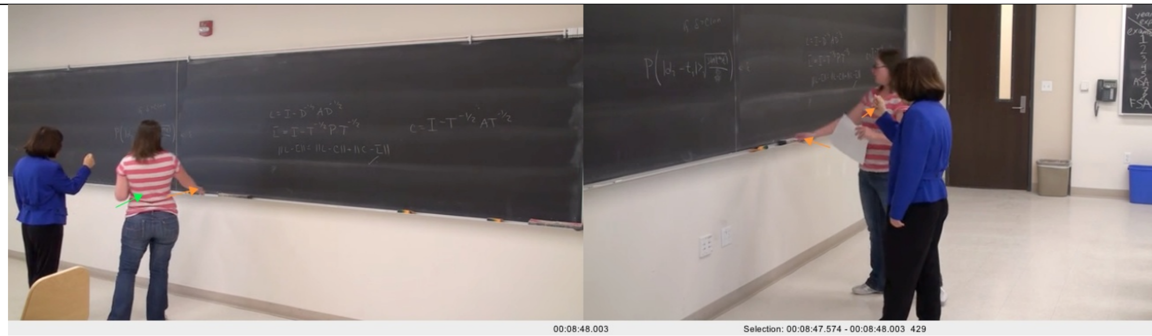
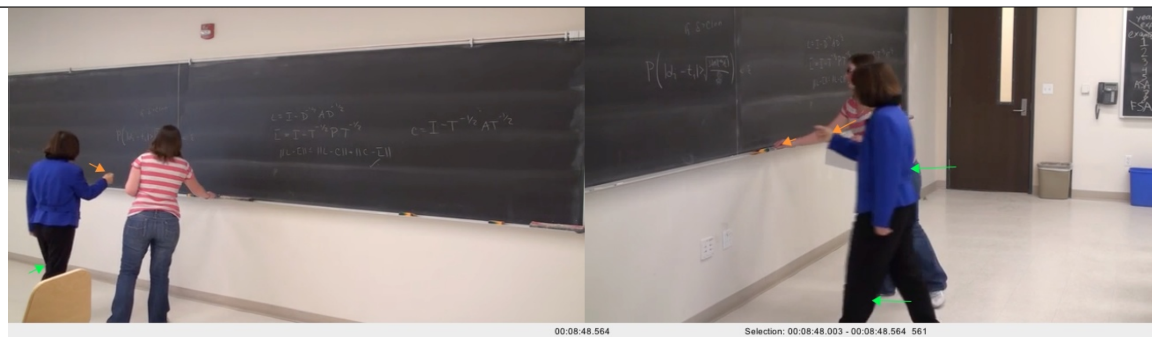
Row	Gesture & timestamp	Transcript
131		F: So
132		in fa-

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Table A.7 – graph confining – *Continued from previous page*

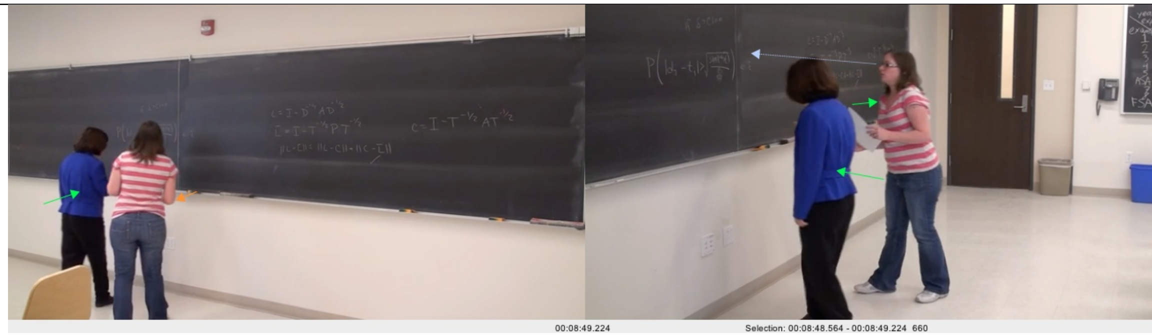
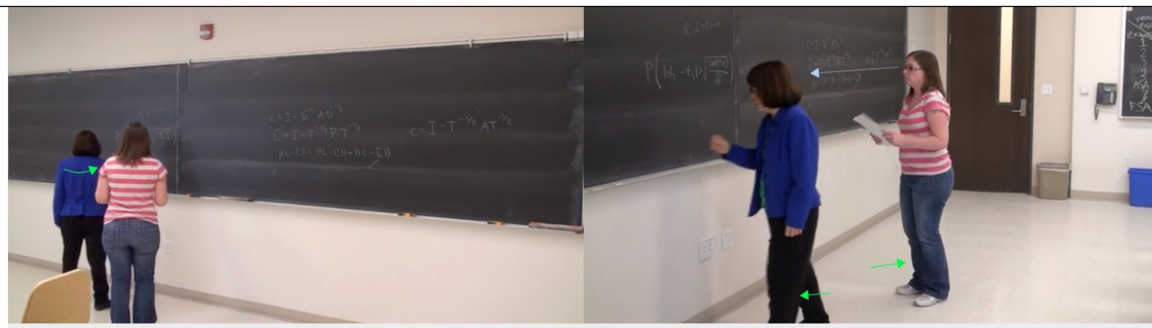
Row	Gesture & timestamp	Transcript
133		-ct the rea-
134		-son I asked

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Table A.7 – graph confining – *Continued from previous page*

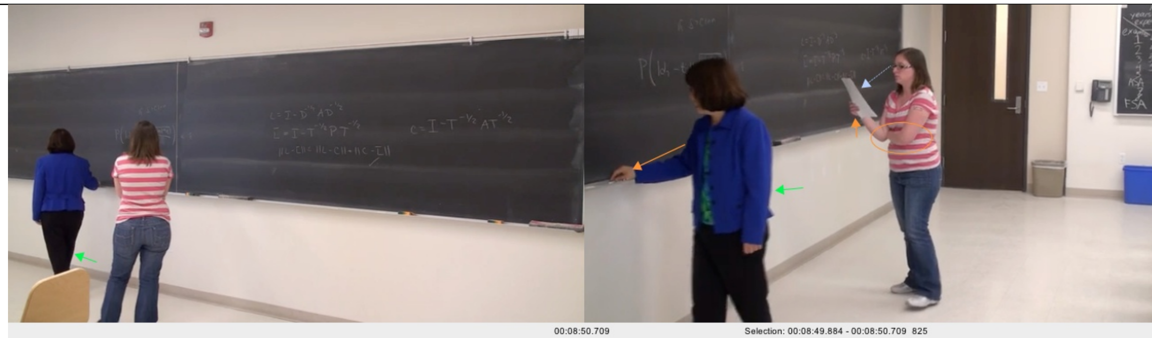
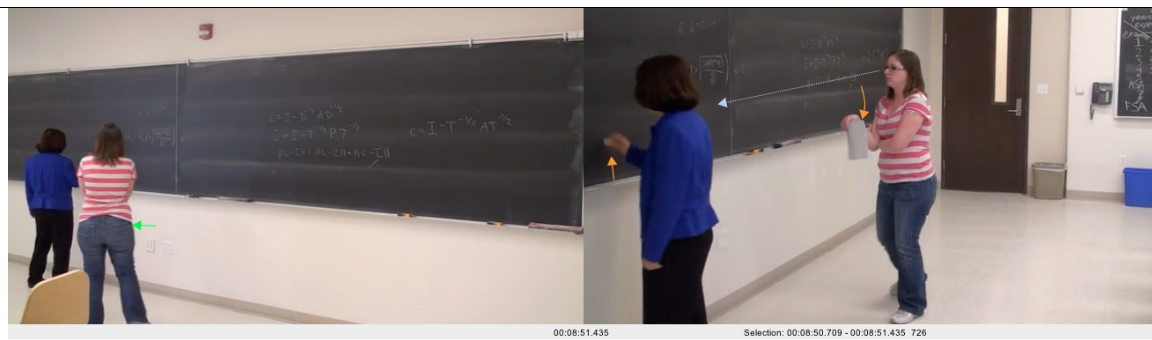
Row	Gesture & timestamp	Transcript
135		you to lo-
136		-ok at the big-

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Table A.7 – graph confining – *Continued from previous page*

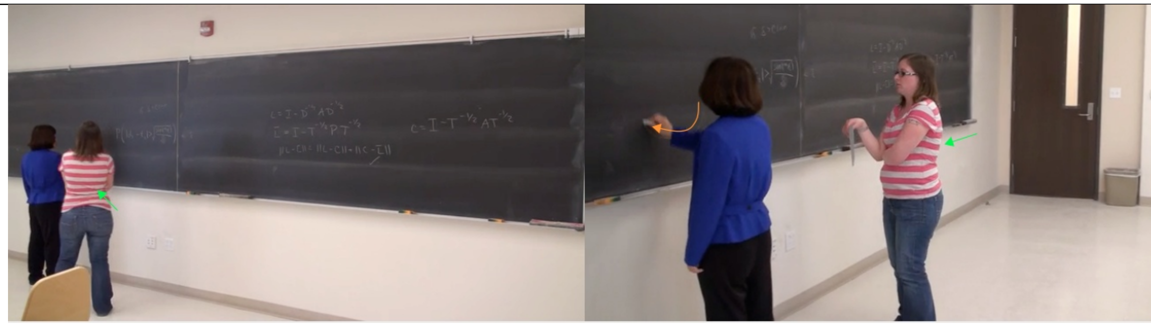
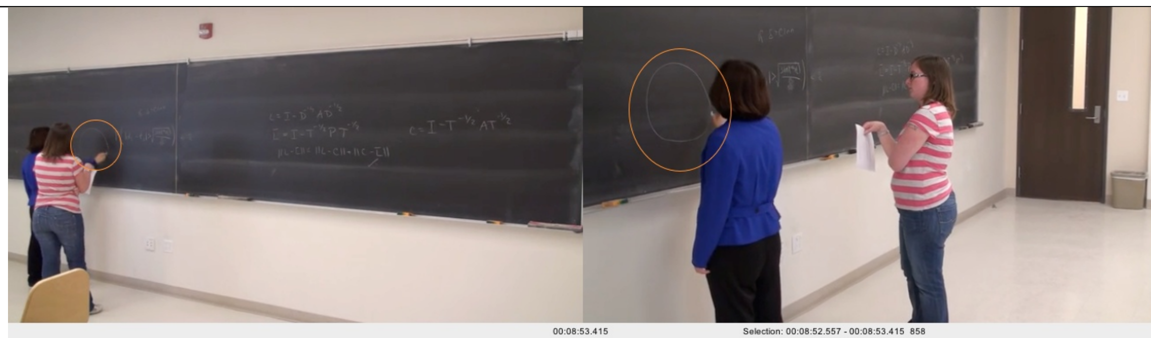
Row	Gesture & timestamp	Transcript
137		[short pause] So suppose now
138		I have a lot

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Table A.7 – graph confining – *Continued from previous page*

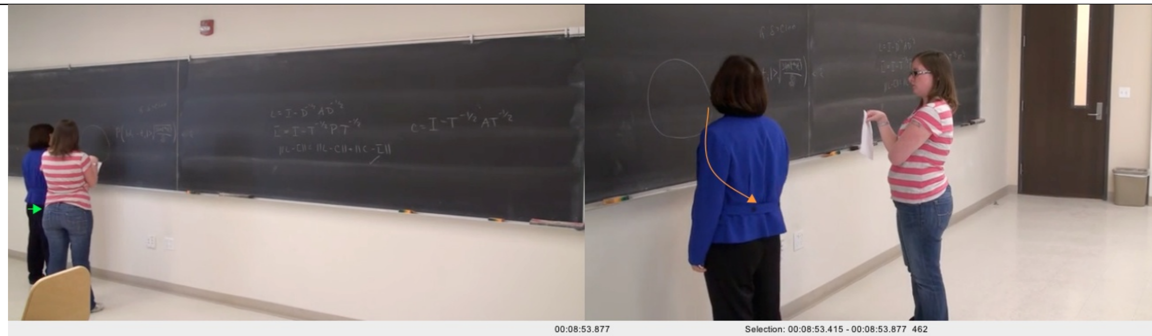
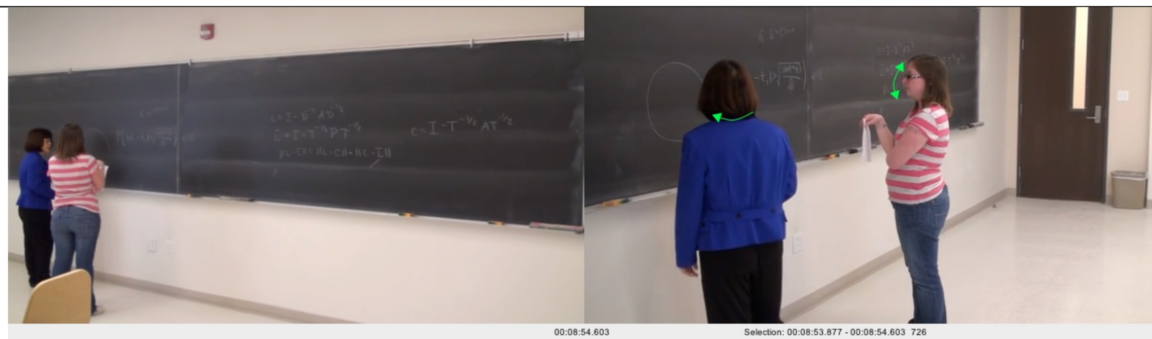
Row	Gesture & timestamp	Transcript
139		of points.
140		M: Mhm.

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Table A.7 – graph confining – *Continued from previous page*

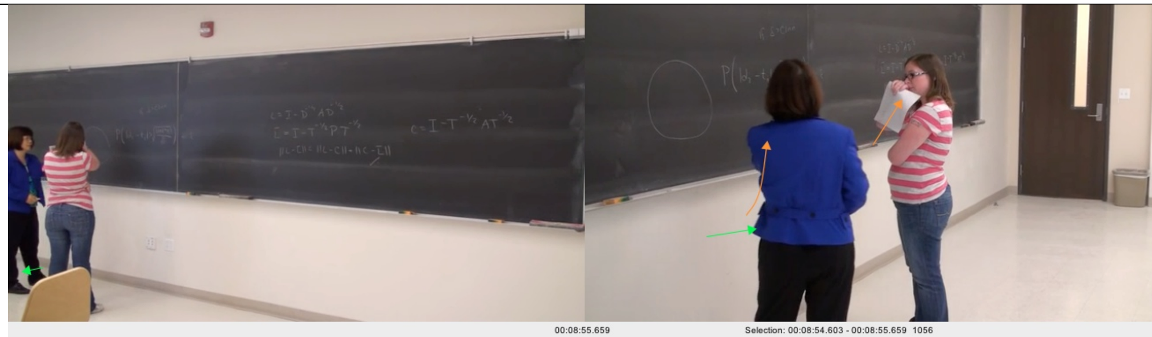
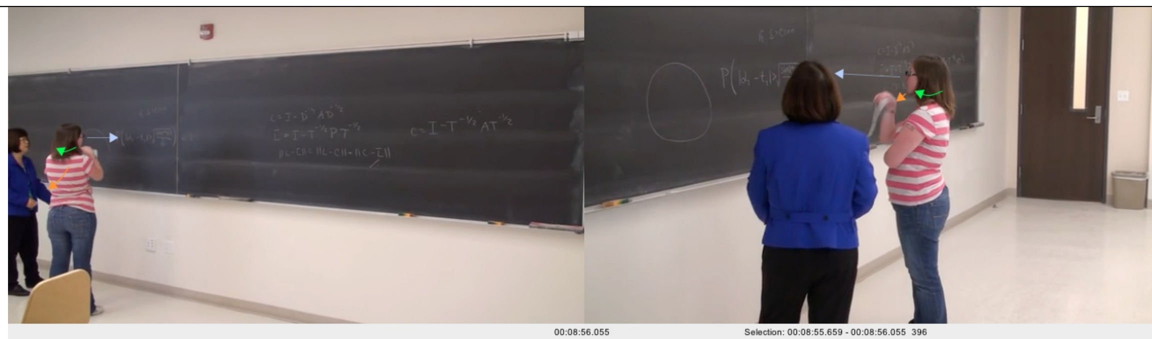
Row	Gesture & timestamp	Transcript
141		F: I don't have this,
142		every-

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Table A.7 – graph confining – *Continued from previous page*

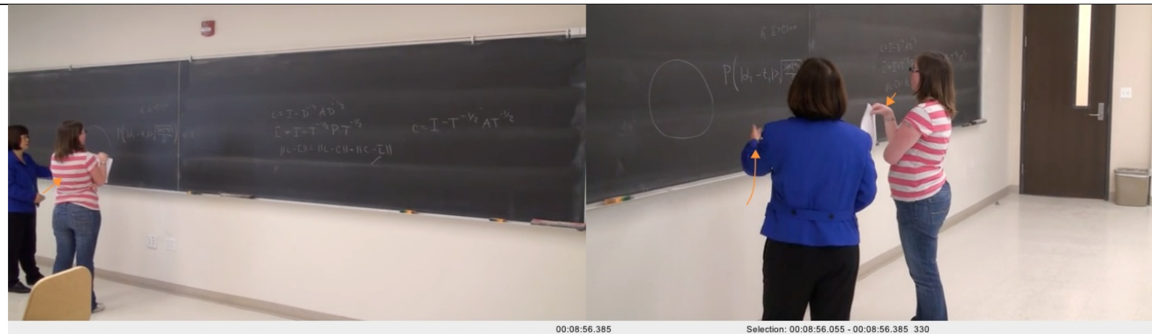
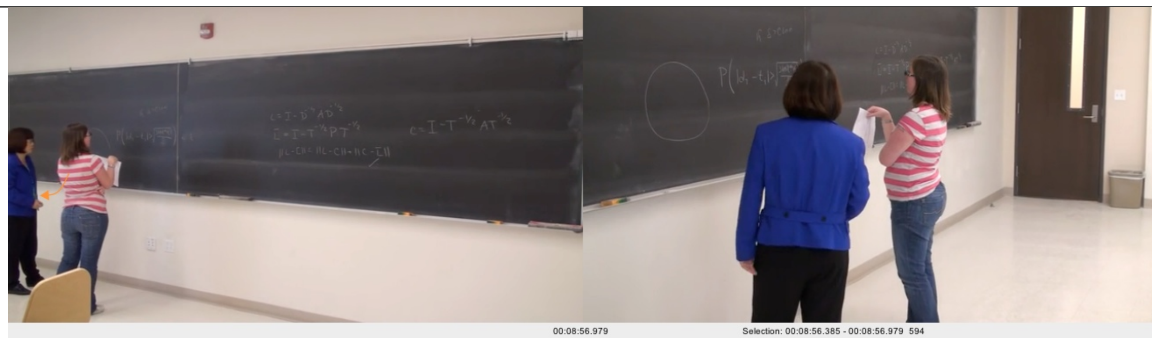
Row	Gesture & timestamp	Transcript
143		-body
144		above

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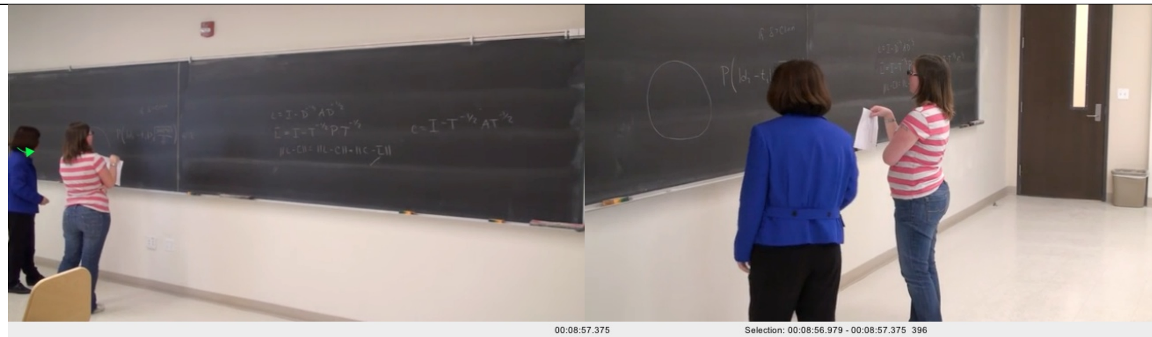
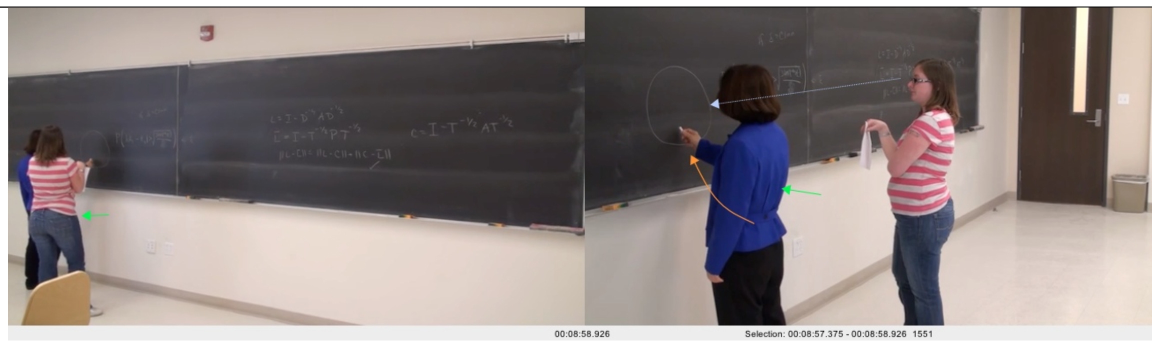
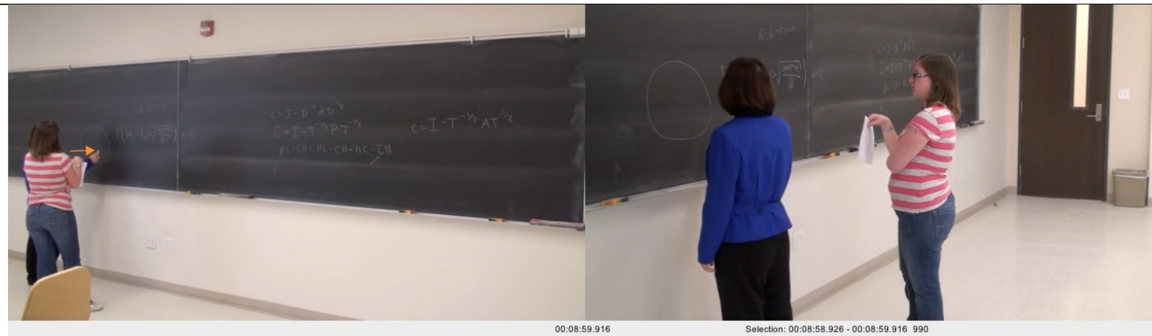
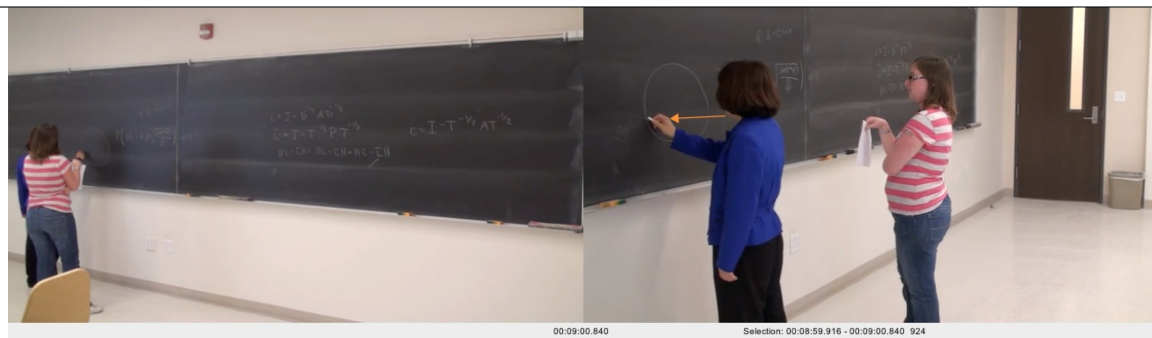
Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
145		log n.
146		[pause] S-

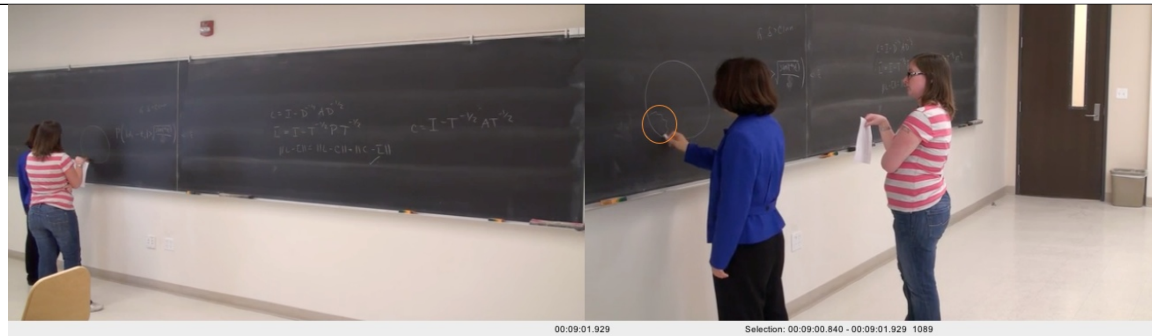
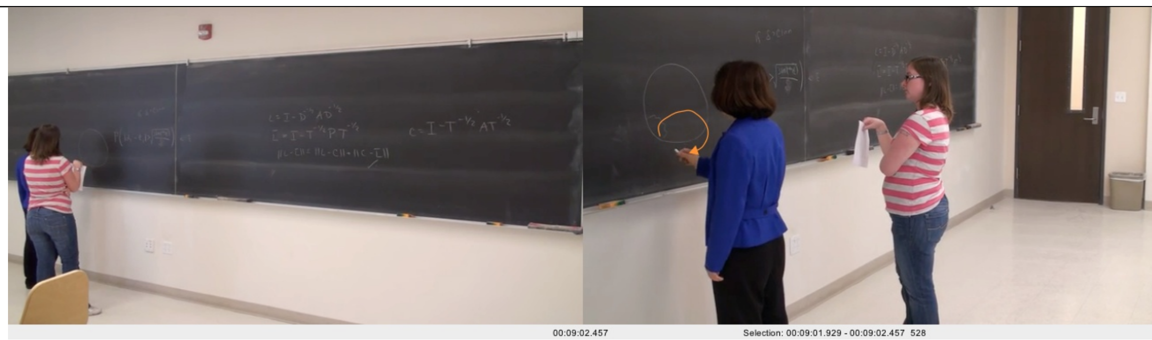
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
147		-o i-
148		So may-

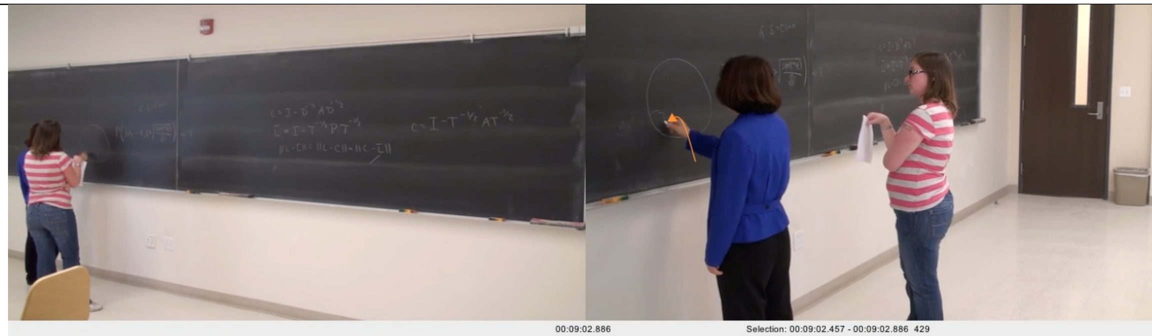
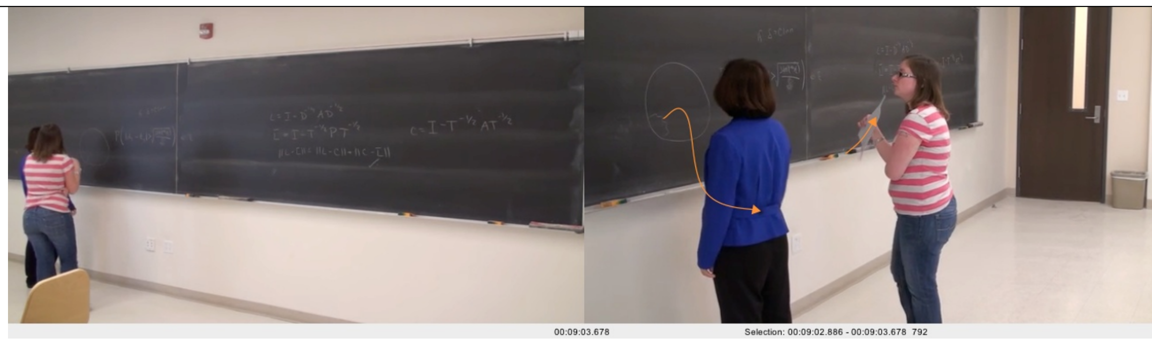
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
149		-be
150		[pause]

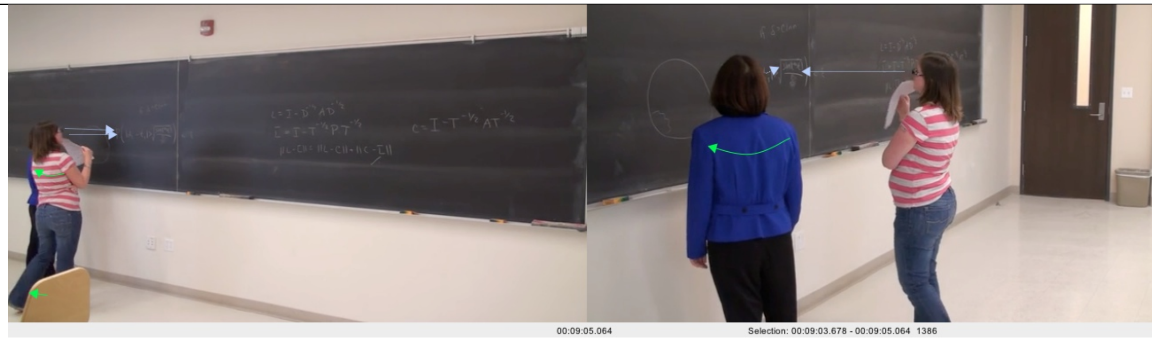
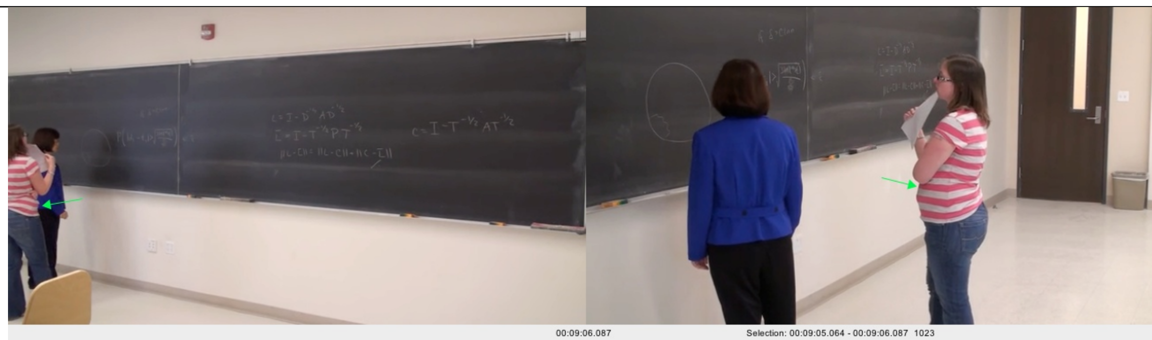
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
151		even
152		for

Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
153		points where it's big
154		expected de-

Continued on next page

Table A.7 – graph confining – *Continued from previous page*

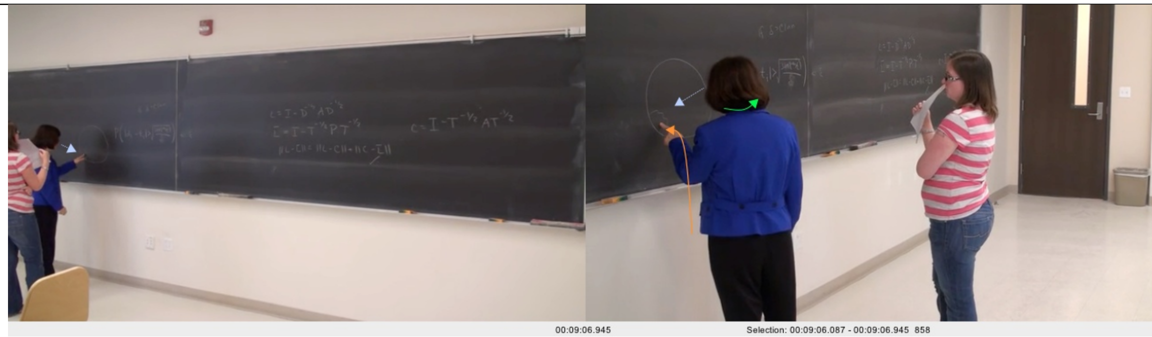
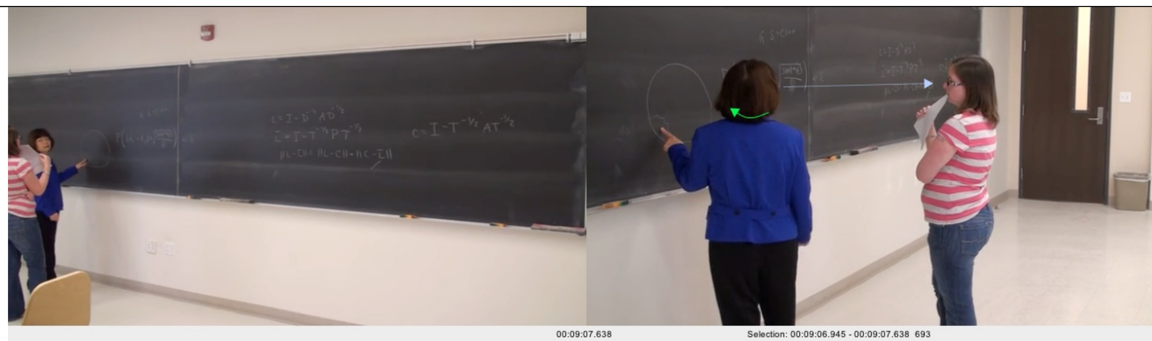
Row	Gesture & timestamp	Transcript
155		-gree, may
156		fall in

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Table A.7 – graph confining – *Continued from previous page*

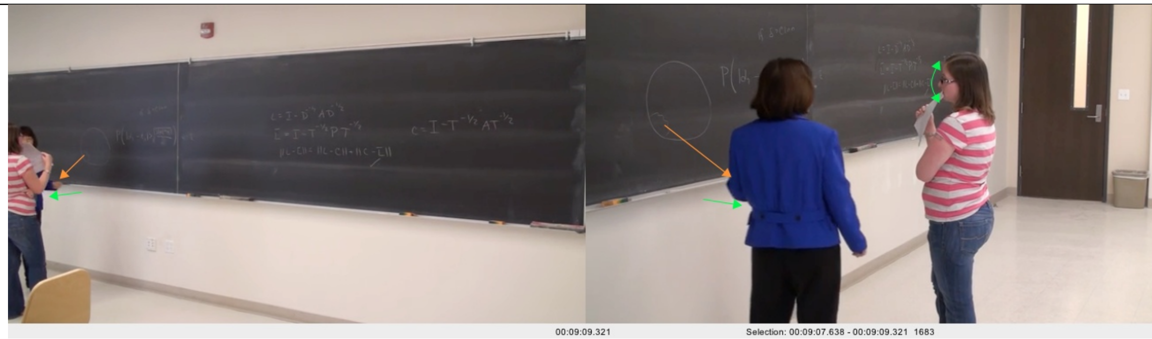
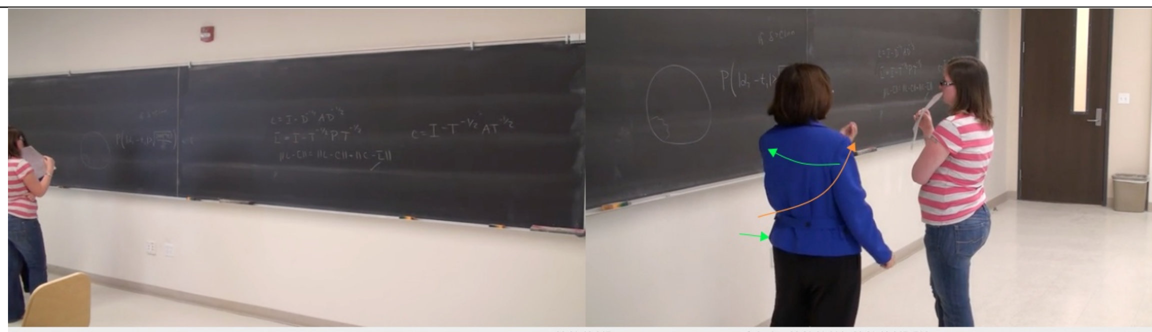
Row	Gesture & timestamp	Transcript
157		here, M: Mhm. F: may f-
158		-all into a s-

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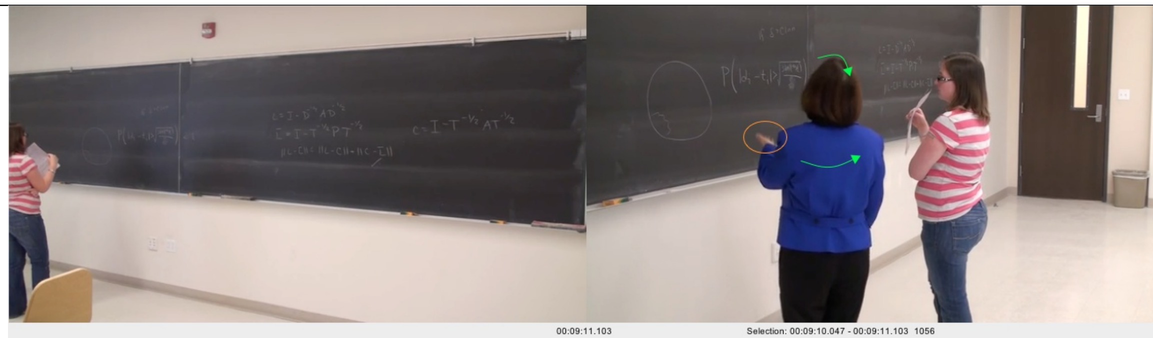
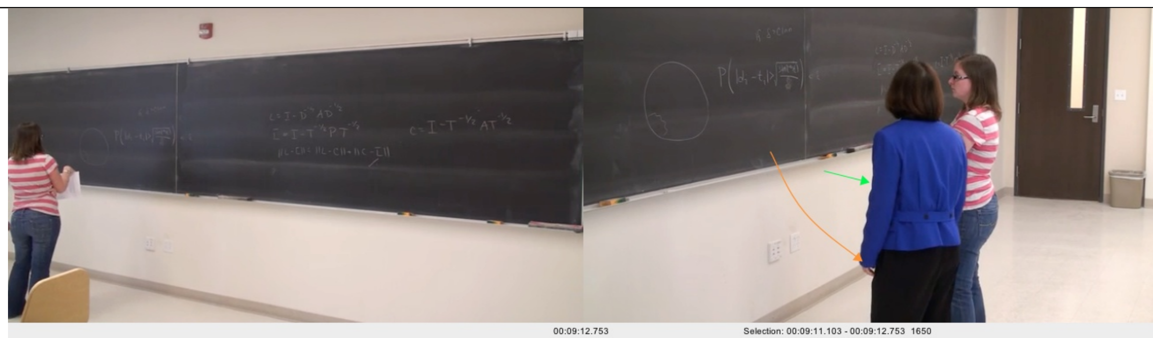
Table A.7 – graph confining – *Continued from previous page*

Row	**Gesture & timestamp**	**Transcript**
159		There's some
160		probability it has very small de-

Continued on next page

Table A.7 – graph confining – *Continued from previous page*

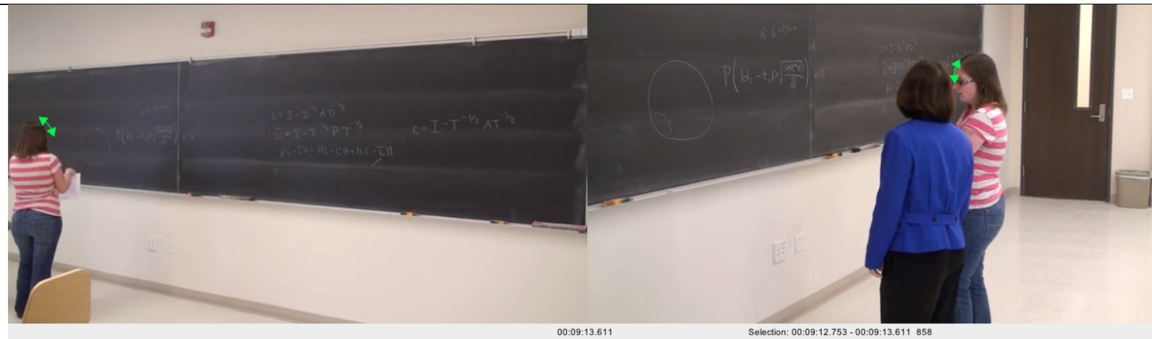
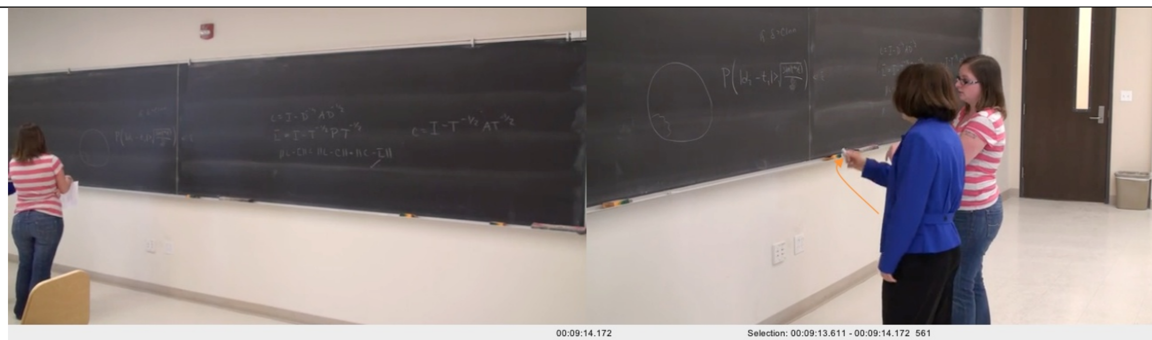
Row	Gesture & timestamp	Transcript
161		-gree. M: Ri-
162		-ght. Yeah. F: And just keep

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Table A.7 – graph confining – *Continued from previous page*

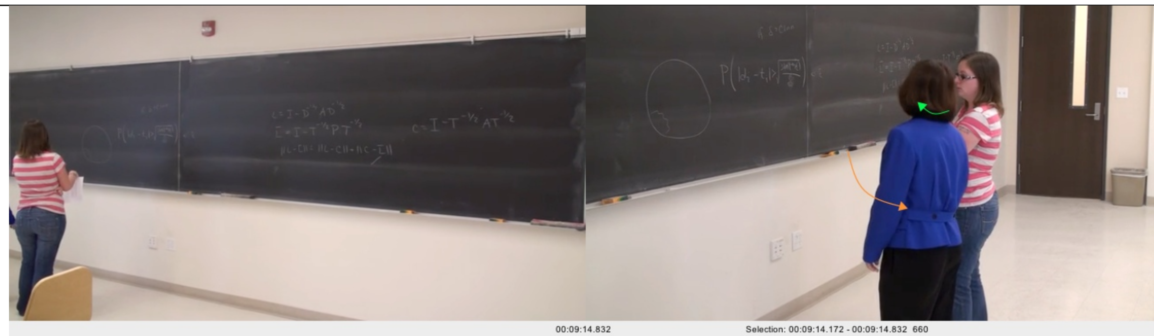
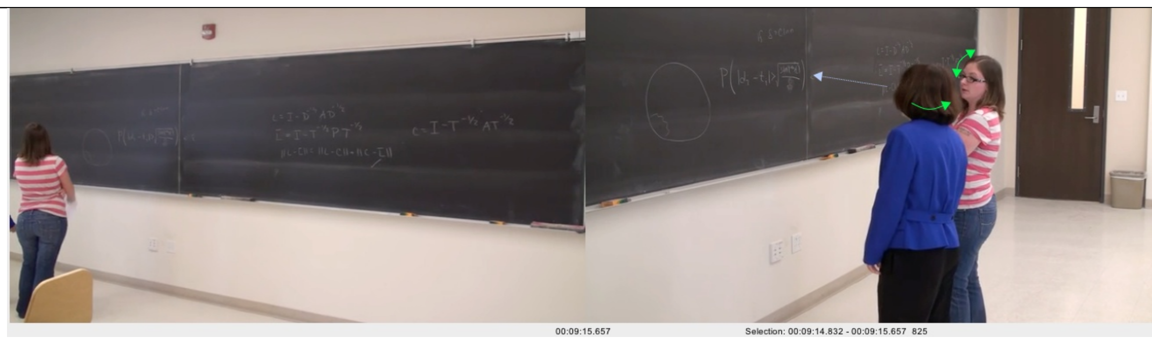
Row	Gesture & timestamp	Transcript
163		the correct. . .
164		Right? M: Yeah.

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Table A.7 – graph confining – *Continued from previous page*

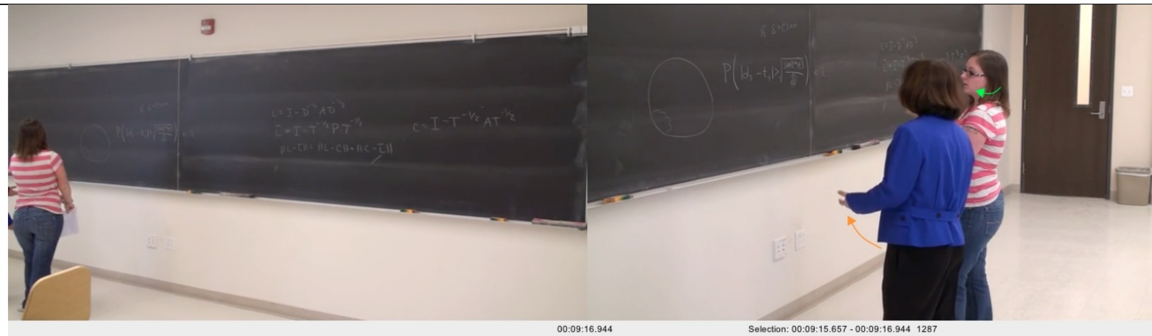
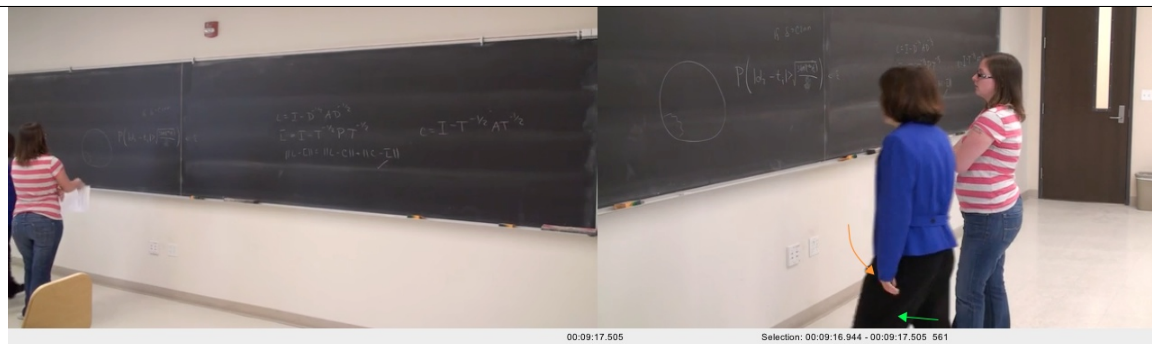
Row	Gesture & timestamp	Transcript
165		F: [pause] Then you
166		just want to

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Table A.7 – graph confining – *Continued from previous page*

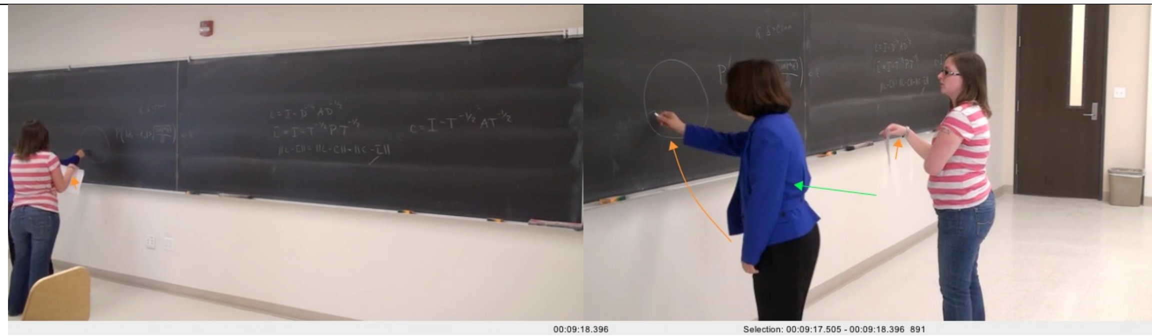
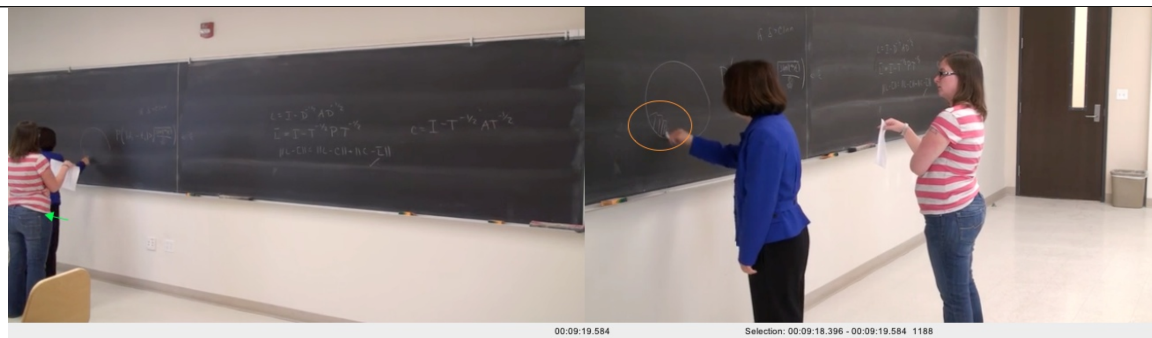
Row	Gesture & timestamp	Transcript
167		argue
168		all the things that f-

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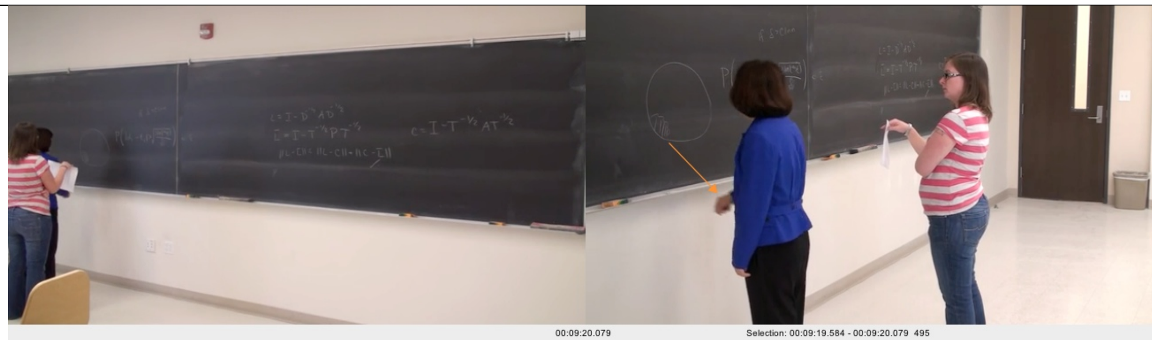
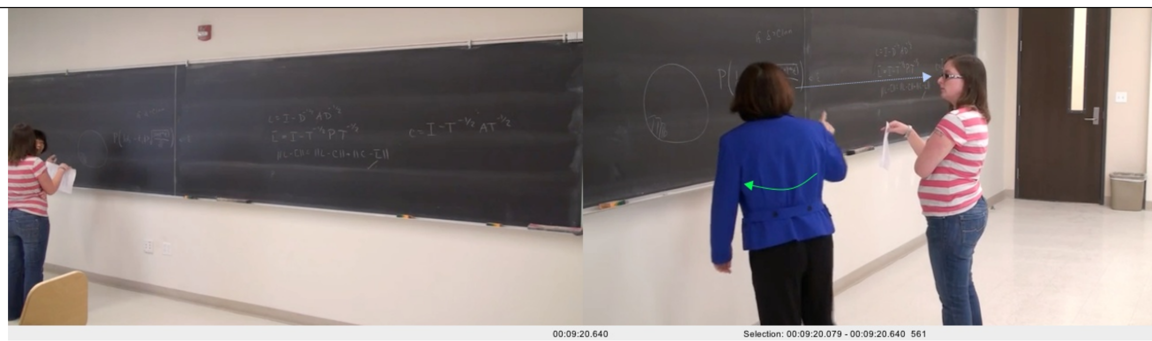
Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
169		-ail,
170		includ-

Continued on next page

Table A.7 – graph confining – *Continued from previous page*

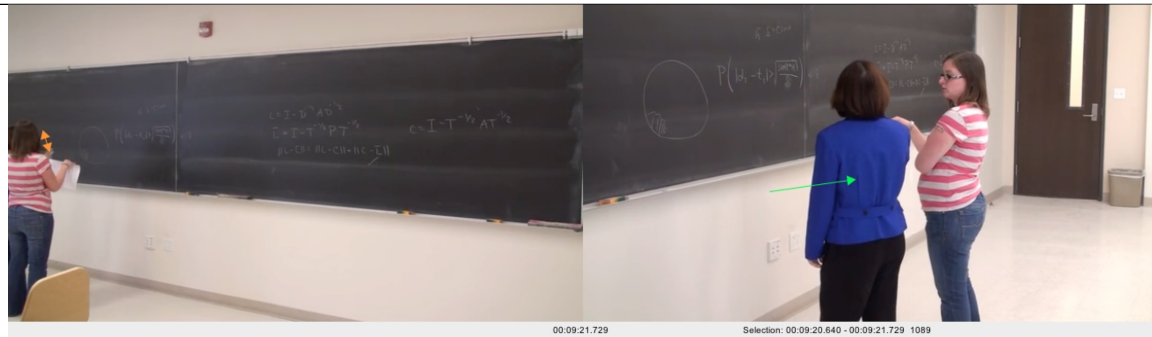
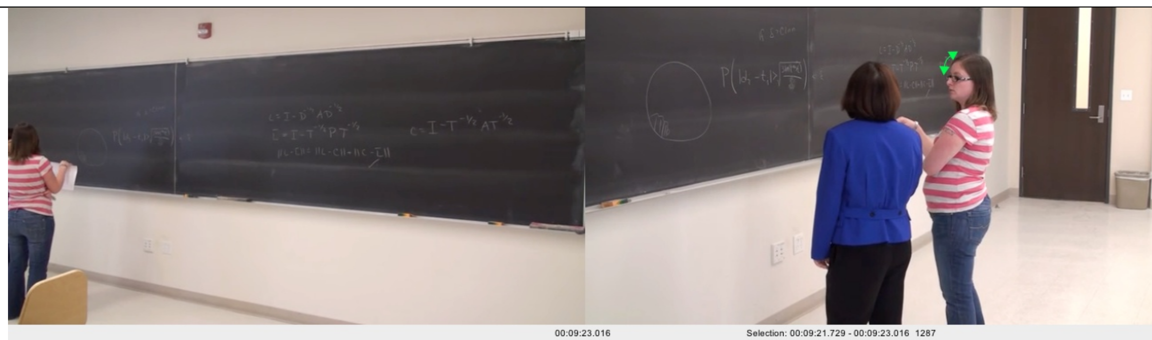
Row	Gesture & timestamp	Transcript
171		-ing those with small
172		degrees. . .

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Table A.7 – graph confining – *Continued from previous page*

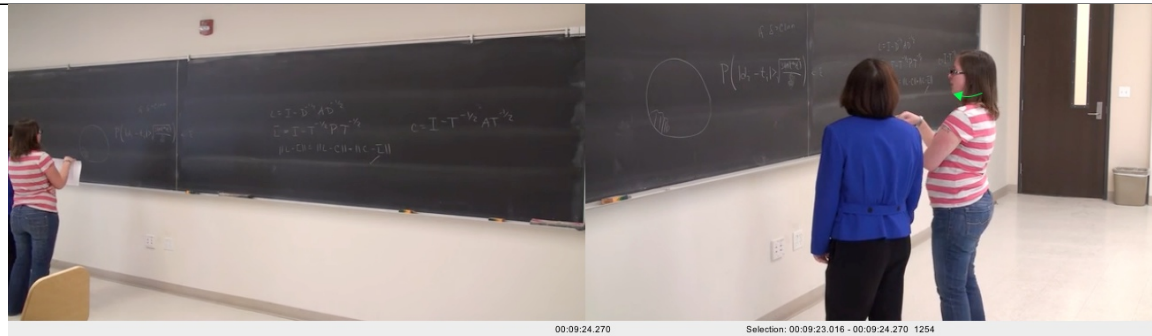
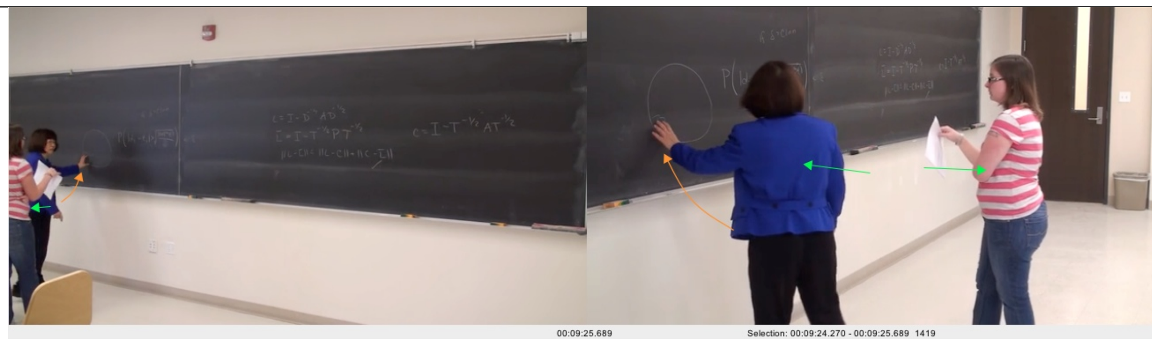
Row	Gesture & timestamp	Transcript
173		[pause] M: There are
174		not many. F: Yeah, not too many.

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Table A.7 – graph confining – *Continued from previous page*

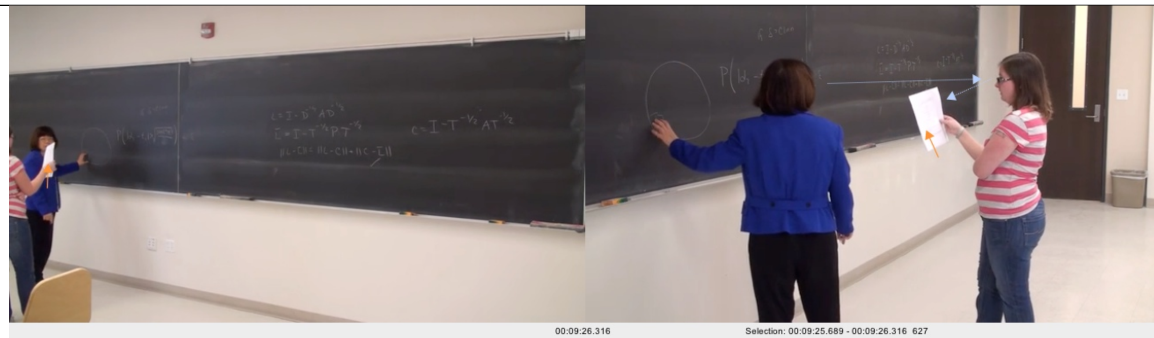
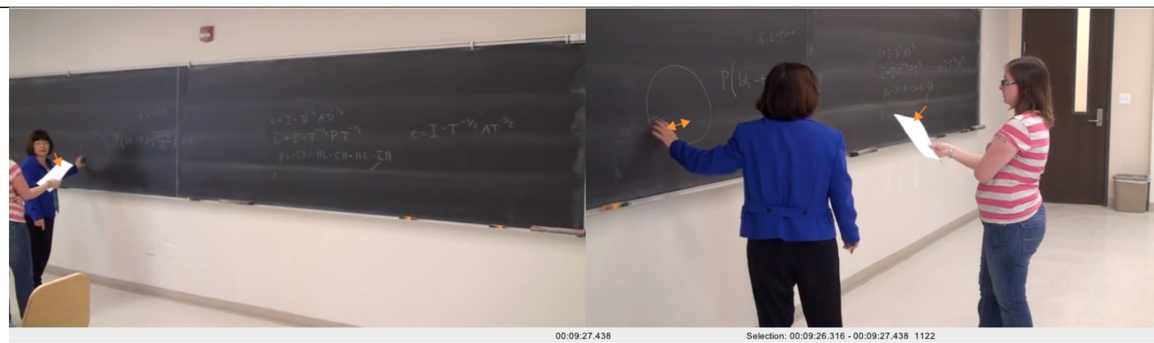
Row	**Gesture & timestamp**	**Transcript**
175		M: Yeah.
176		F: The [inaudible]

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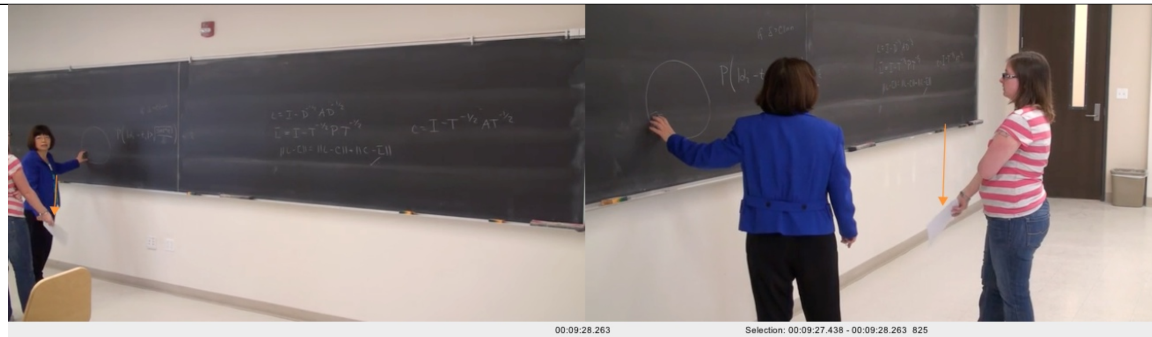
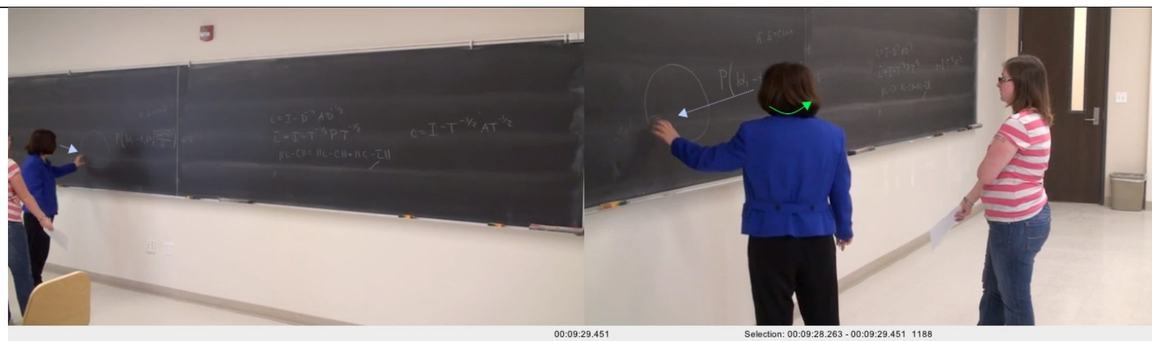
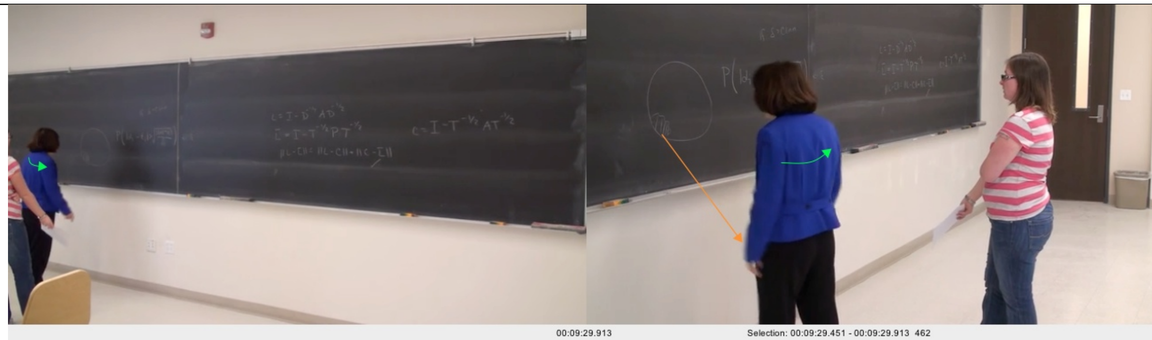
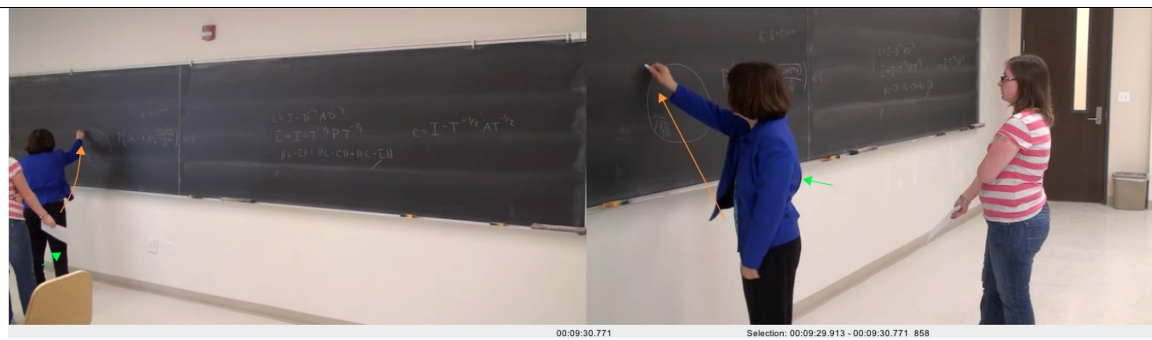
Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
177		is not so many.
178		M: Yeah. F: If- [short pause]

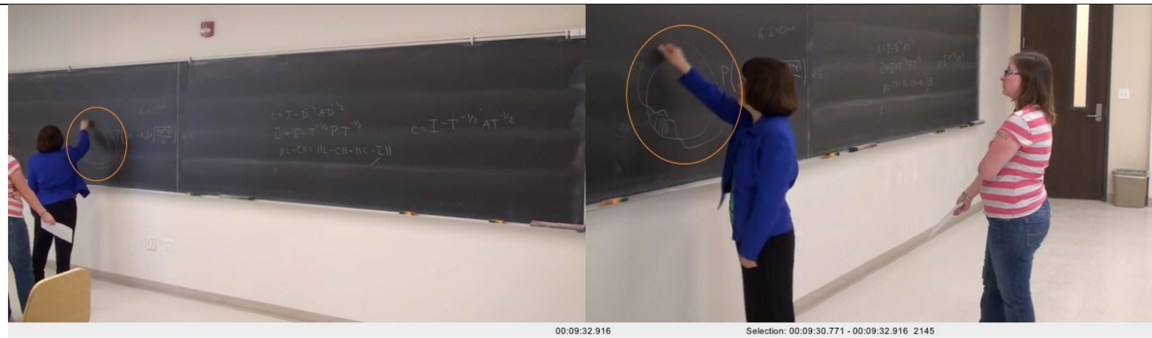
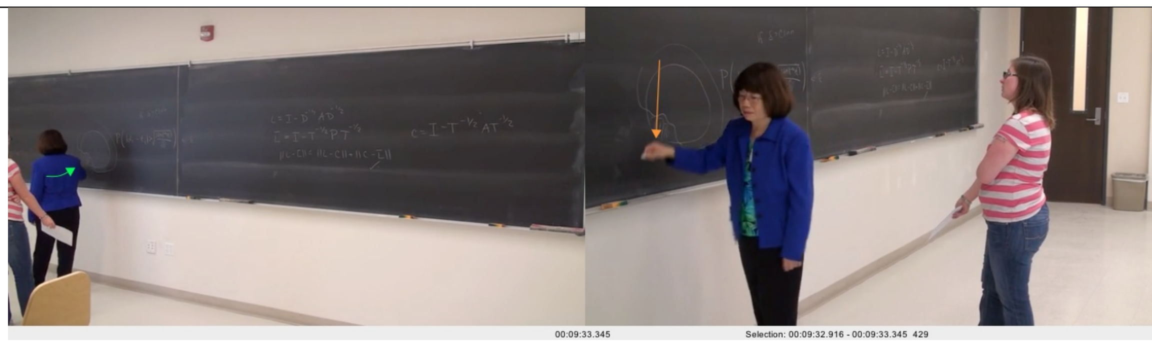
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
179		In that
180		case you you will

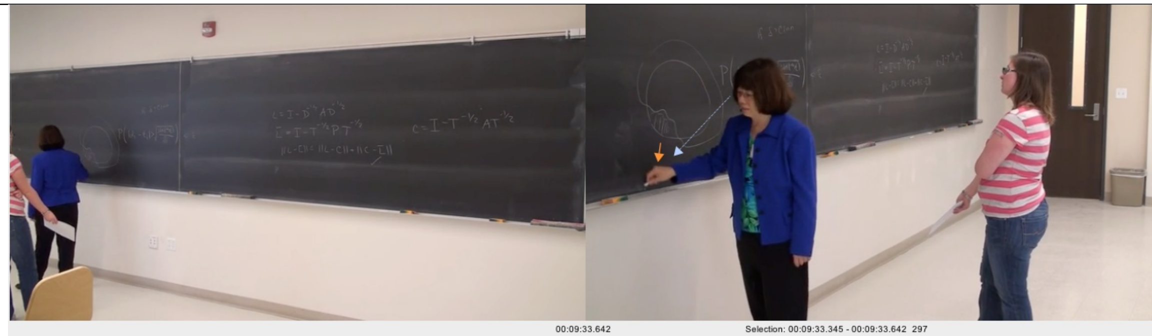
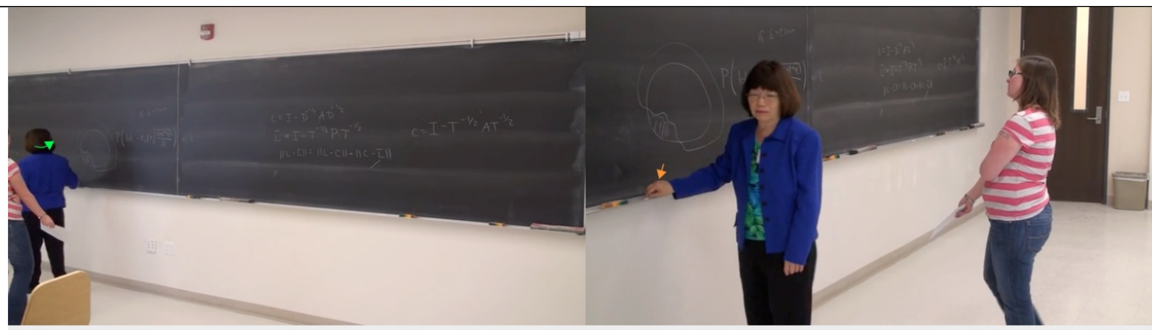
Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
181		say, “Oh. In the remaining
182		of the wor-

Continued on next page

Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
183		-ld
184		we

Continued on next page

Table A.7 – graph confining – *Continued from previous page*

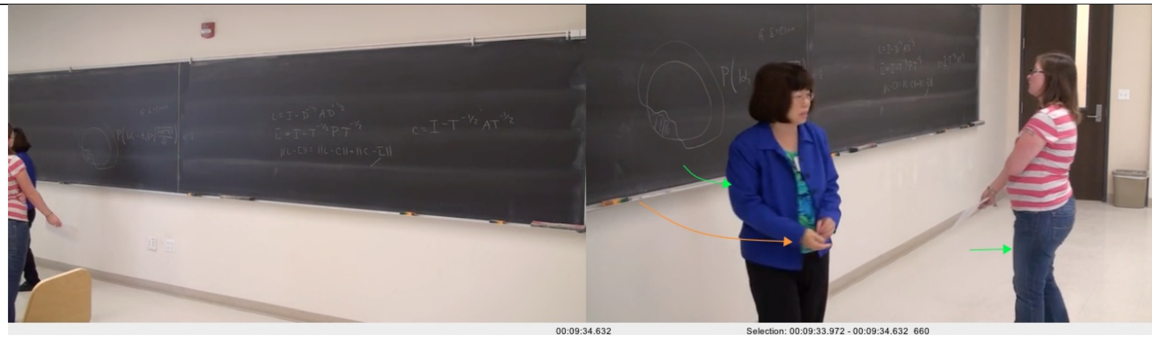
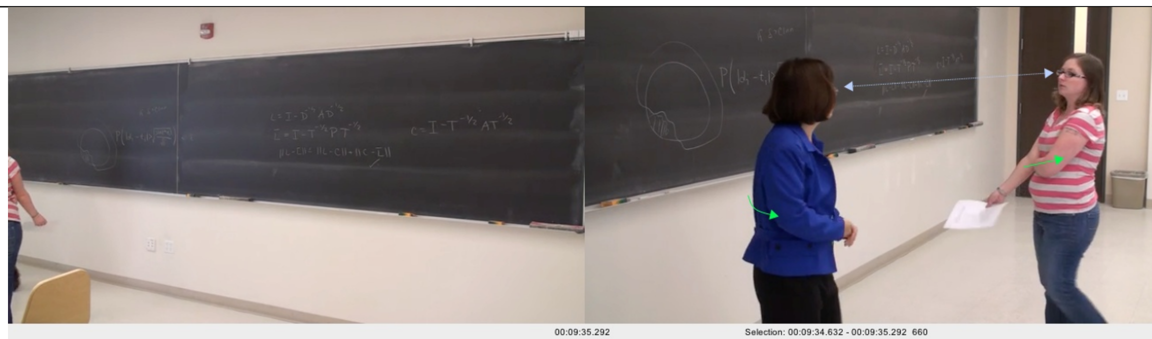
Row	Gesture & timestamp	Transcript
185		we are doing o-
186		-kay.”

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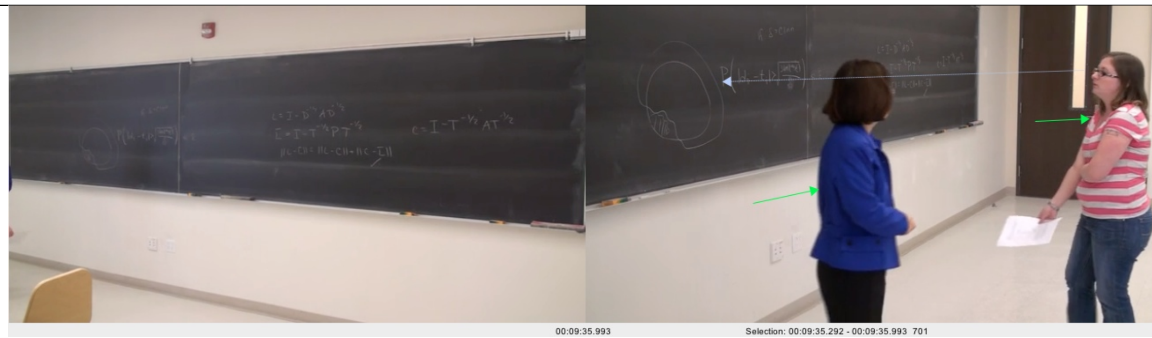
Table A.7 – graph confining – *Continued from previous page*

Row	Gesture & timestamp	Transcript
187		[pause]

A.8 Matt & Bart — adjusting a triangle

Table A.8: Matt & Bart — triangle

Row	Gesture & timestamp	Transcript
1		[N/A]
2		B: I mean, geometric-

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
3		-ally this means
4		[pause]

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
5		you're
6		talking a-

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	**Gesture & timestamp**	**Transcript**
7		-bo-
8		-ut

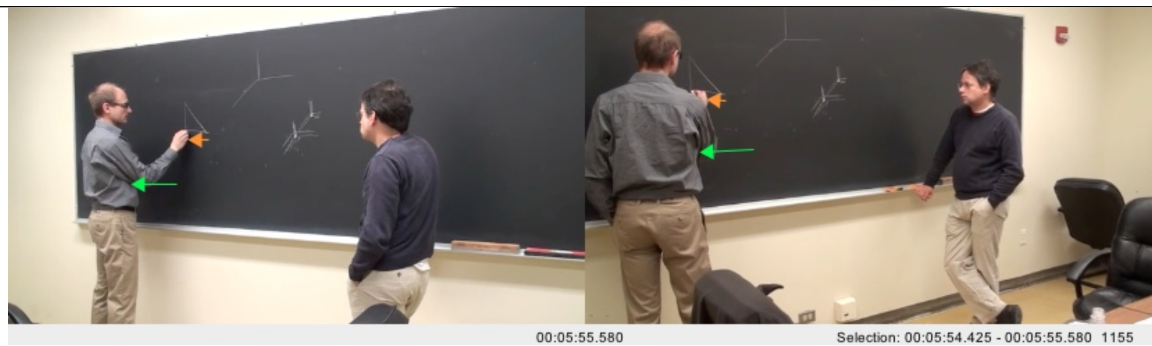
Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
9		[pause]
10		[pause]

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
11		um, [pause] a s-
12		-ituation where

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
13		[pause] where your mark p-
14		-oints maps

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
15		to the interse-
16		-ction of [pause]

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
17		the strata. These
18		two stra-

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
19		-ta, ri-
20		-ght?

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
21		M: Uh, y- [pause]
22		B: [pause] And then you have to make

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
23		a choice,
24		uh,

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
25		[pause]
26		[pause]

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
27		[pause]
28		[pause] I'm confu-

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
29		-sed.
30		[pause] I see. So, right.

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
31		Putting it *here*, actually, [pause]
32		No, I'm confused.

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
33		[pause]
34		Um, what does it mean, actually?

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
35		If this po-
36		-int moves *here*

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
37		is that even
38		possible?

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
39		[pause]
40		[pause] I mean, that,

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
41		wouldn't that mean tha-
42		-t it maps into

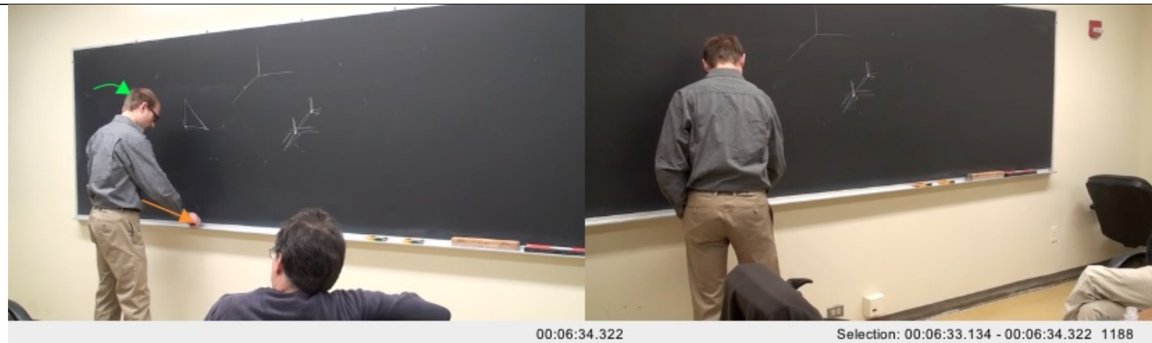
Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
43		the interior? M: Yeah, it does.
44		That's possible. We have s-

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
45		-everal components mapping into the
46		interior. [pause]

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
47		B: [pause] But with the d_is [“dee eyes”]
48		mapped to the boundary always

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
49		[pause]
50		M: We have

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
51		d_is over there. I-
52		I'm ju- I'm

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
53		not t-
54		-alking about

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
55		d_is; I'm talking about just
56		the simple strata. [pause]

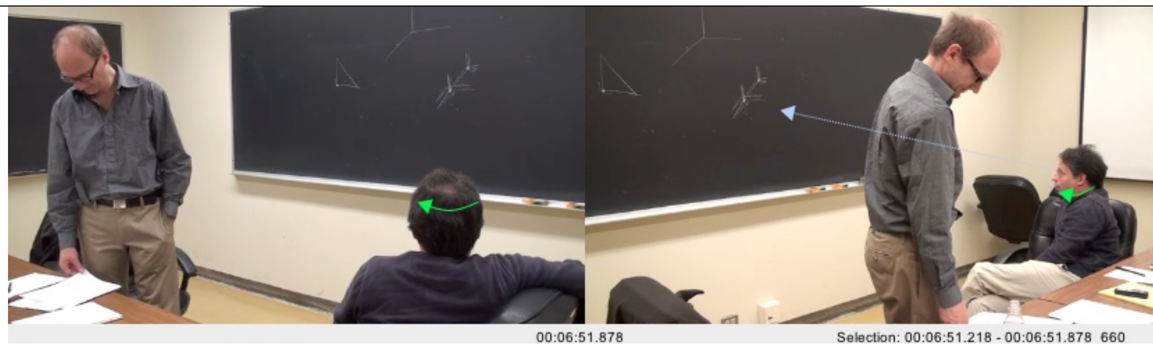
Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
57		B: [pause] I'm confused.
58		We have to say exact-

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
59		-ly what we mean by "stratum."
60		[pause]

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
61		M: I [pause]
62		Don't we know that?

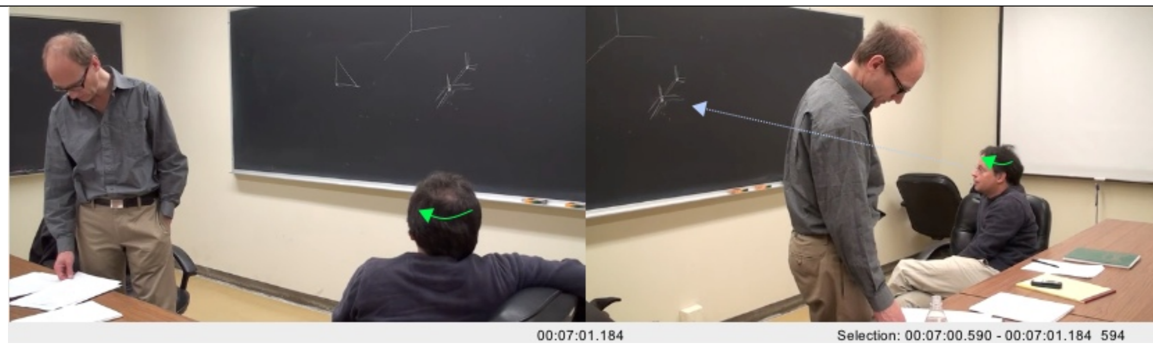
Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
63		[pause]
64		B: Well, yeah, I, eh, I, exact-

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
65		-ly; I thought we wanted the, the mark points mapped to the *d*
66		*i*s [saying "d_is"]

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
67		M: [pause] That's in the defini-
68		-tion

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
69		of the h_i.
70		[pause]

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
71		I'm
72		not talking about

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
73		the h_is; I'm ju-
74		-st talking

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
75		about the
76		one-dimensional stratum, strata in our modules.

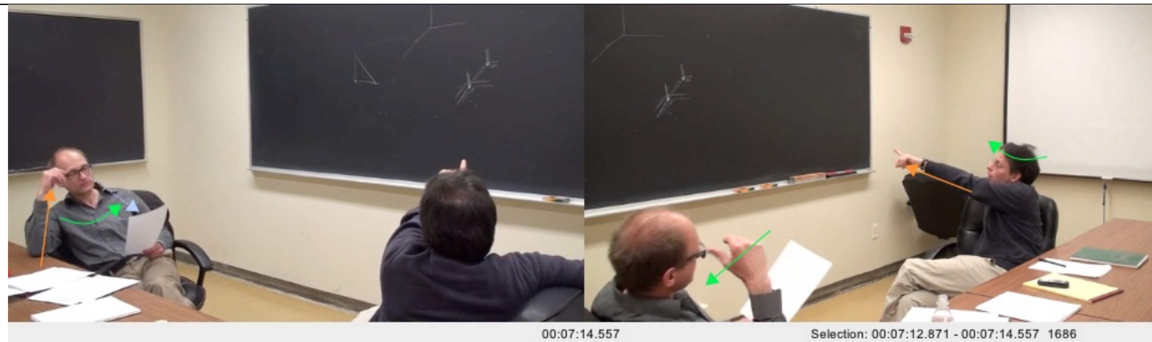
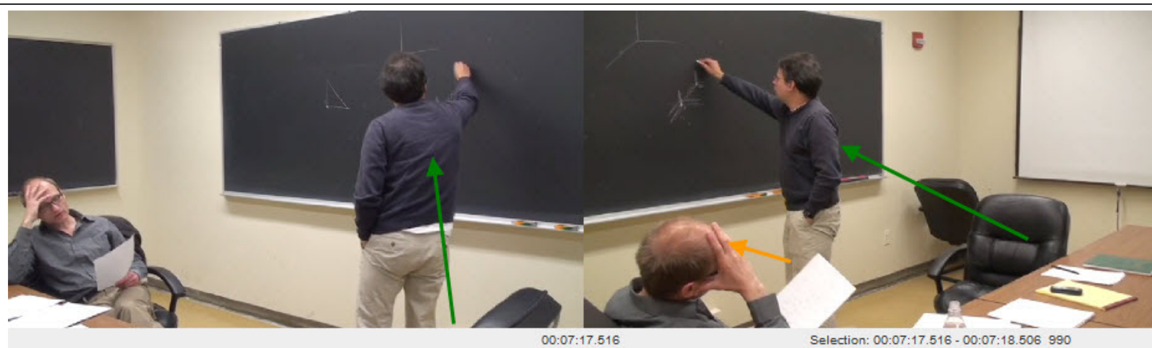
Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
77		[pause]
78		B: Let's see, the modularized space [pause]

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
79		M: And then this is what… B: Oh, I see.
80		M: …[inaudible] any… B: Yeah yeah yeah. Okay. M: …any guy over here…

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
81		B: Okay okay okay. M: is
82		B: Okay.

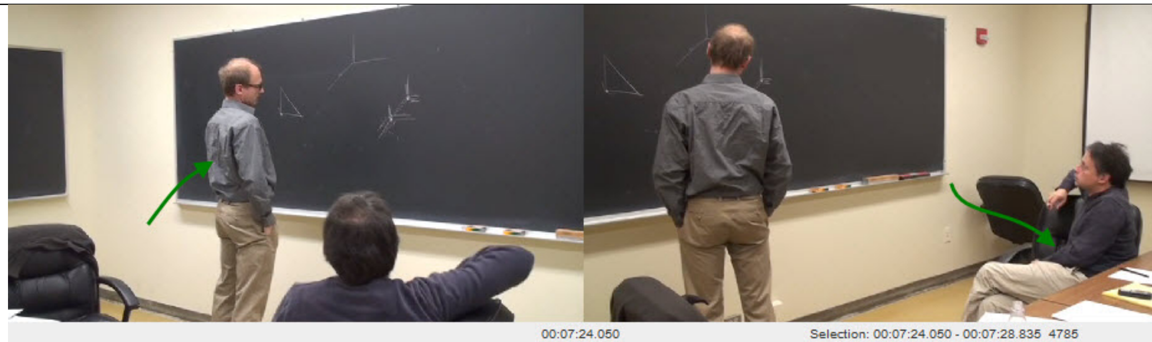
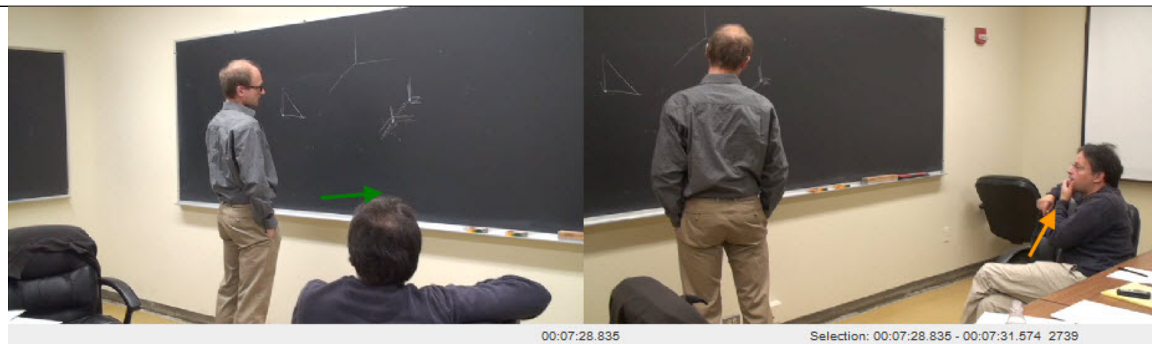
Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
83		Yeah, okay okay.
84		Because then, be-cause

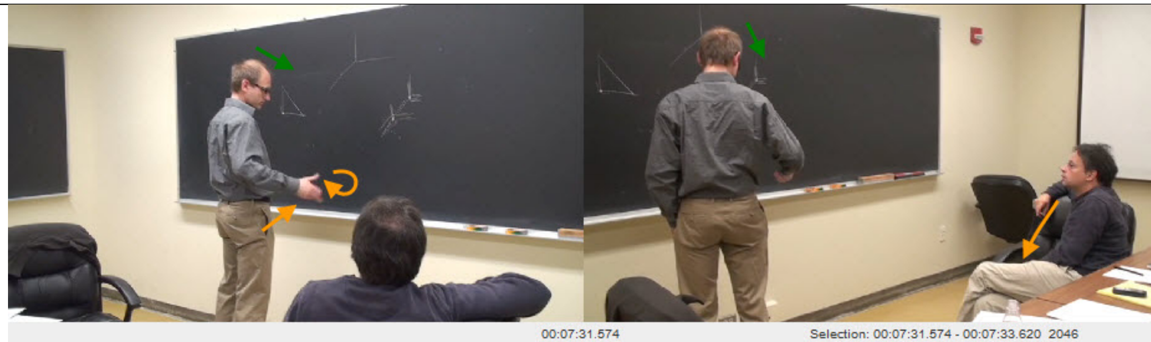
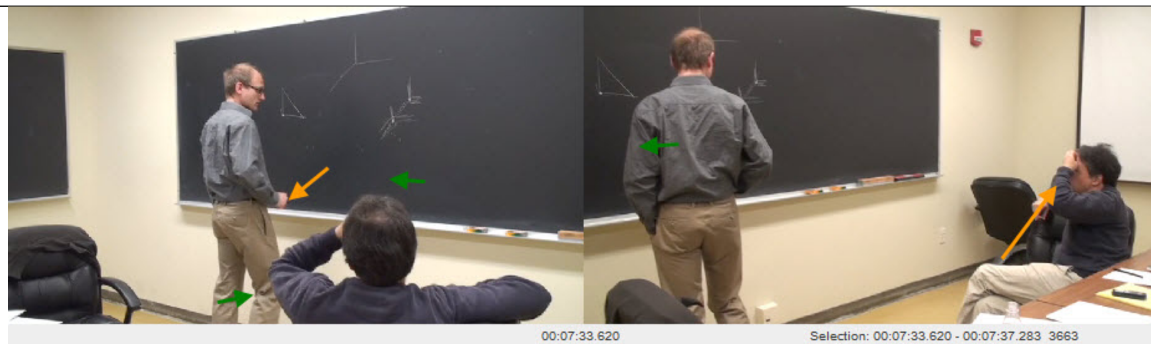
Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
85		then, but then I want [pause] but then
86		you want. . . [pause] Yeah this is usual

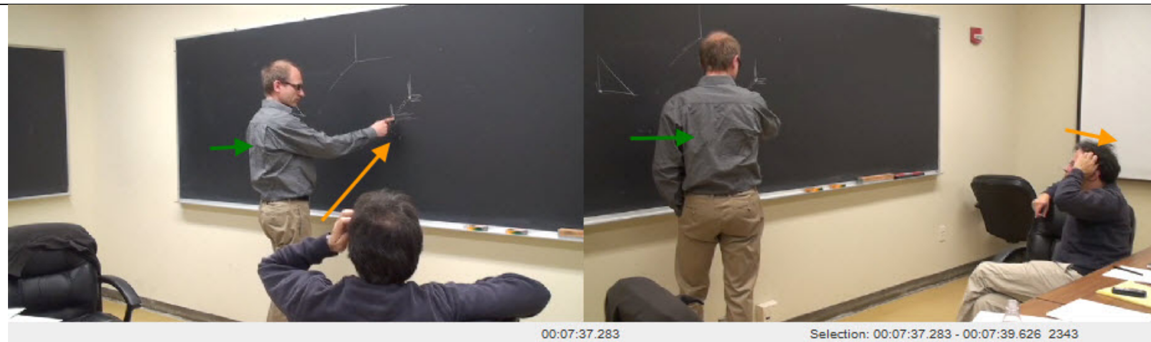
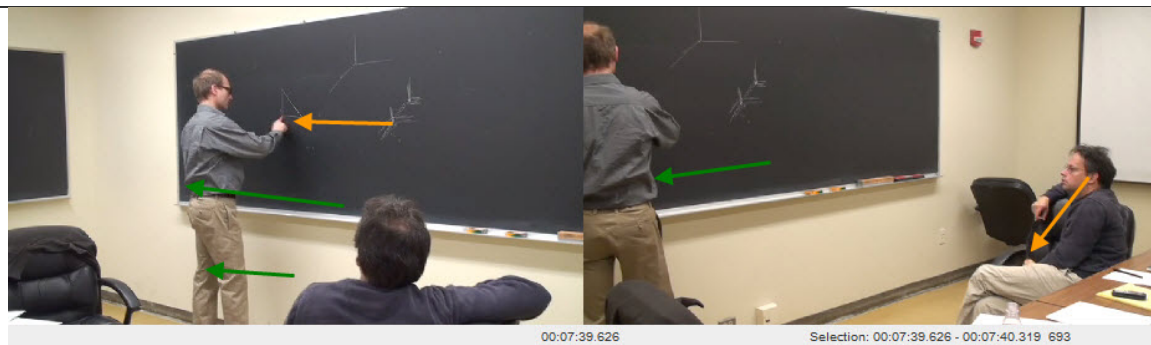
Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
87		[pause] usual kind of, uh
88		duality that

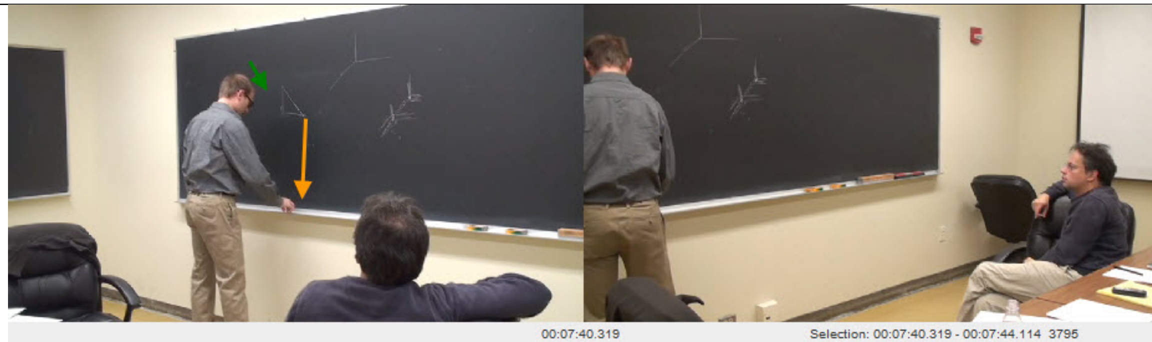
Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
89		[pause] s- Stay in a c-
90		-ell here means [pause] so this

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
91		this picture's
92		wrong. Right? [pause] We should action on

Continued on next page

Table A.8 – triangle – *Continued from previous page*

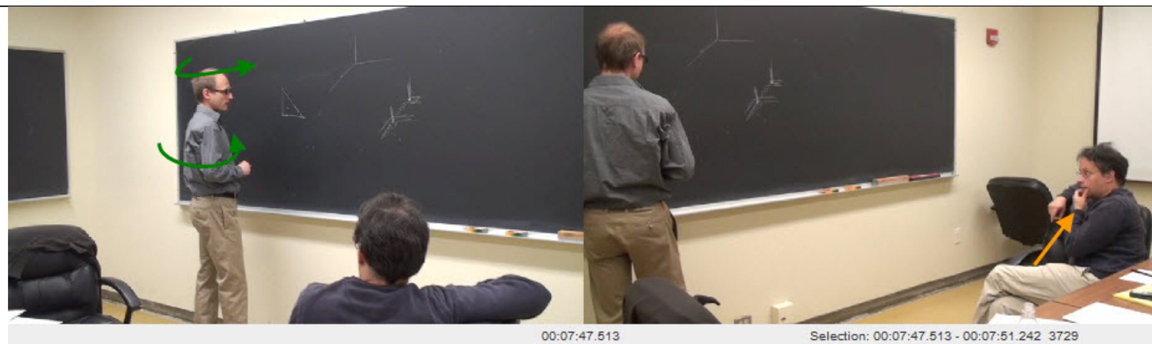
Row	Gesture & timestamp	Transcript
93		this like this. [pause] M: Yeah; right,
94		p is always going to be two.

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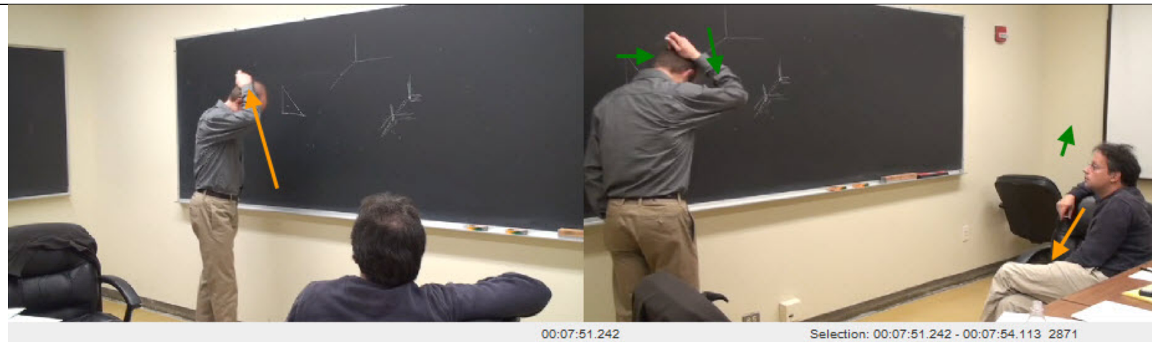
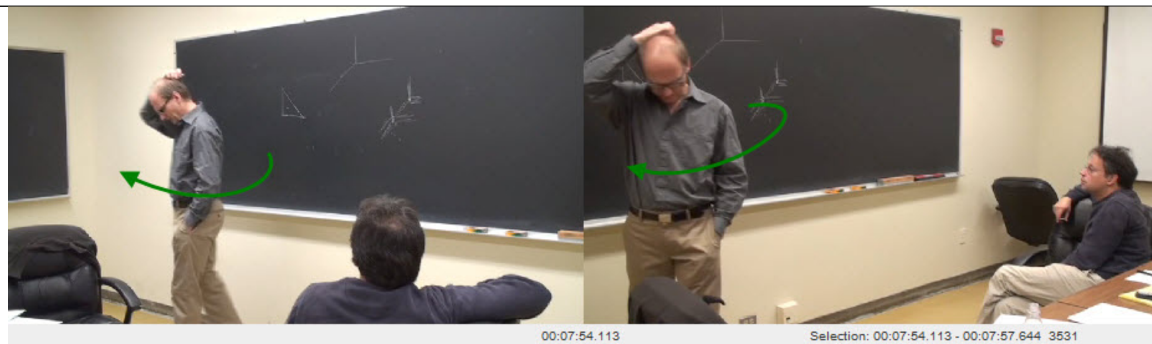
Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
95		[pause] B: Um. [pause]
96		[pause] Yeah, this is, th-

Continued on next page

Table A.8 – triangle – *Continued from previous page*

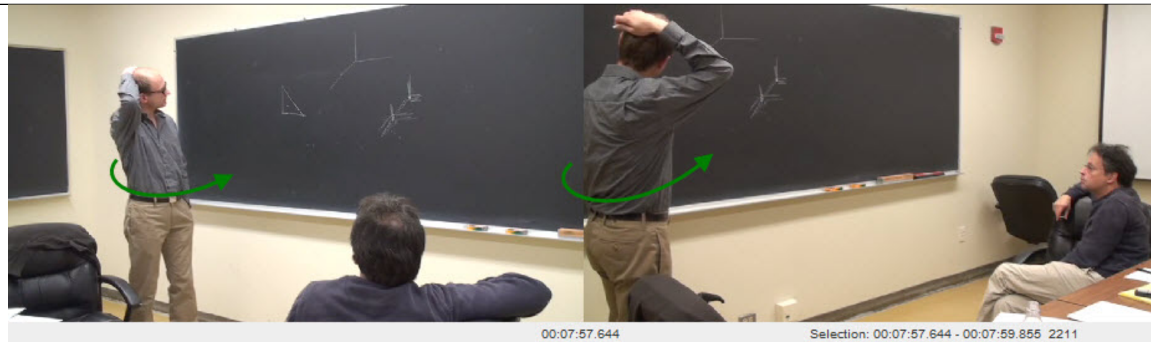
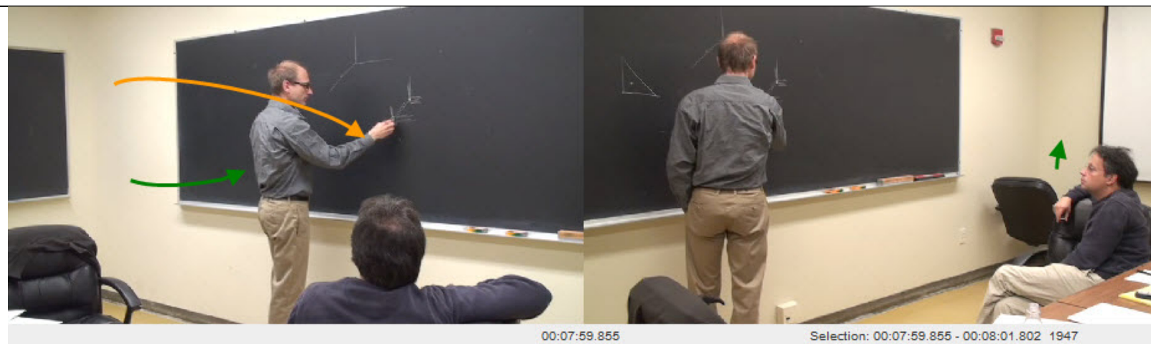
Row	Gesture & timestamp	Transcript
97		-is is thing that [pause] thing that I don't really under-
98		-stand. But here I mean that's the

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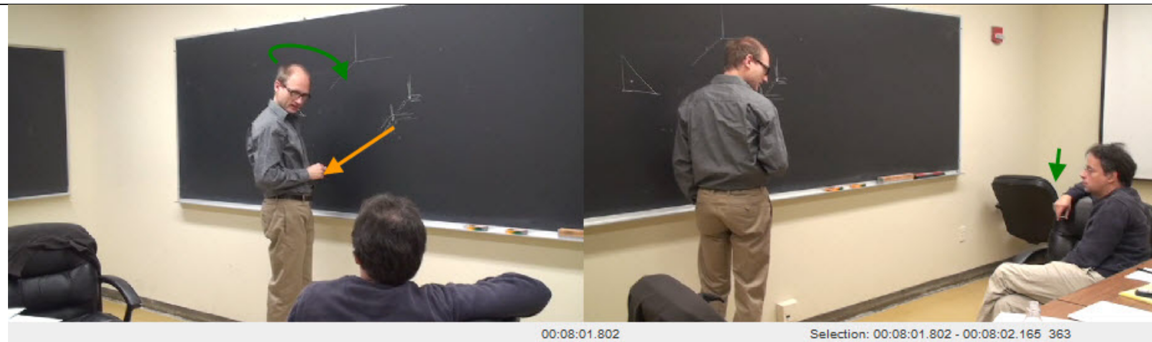
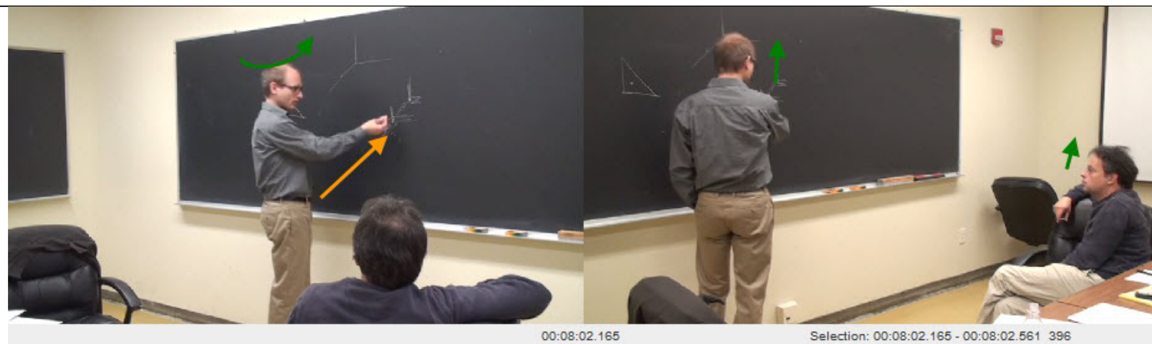
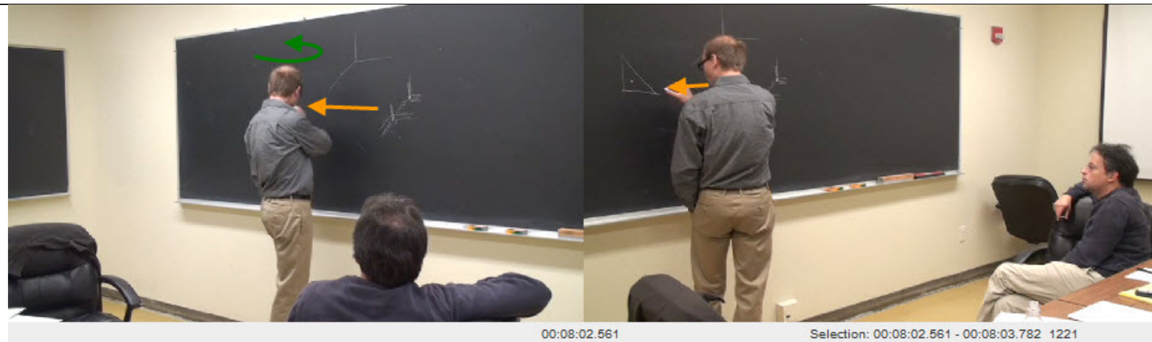
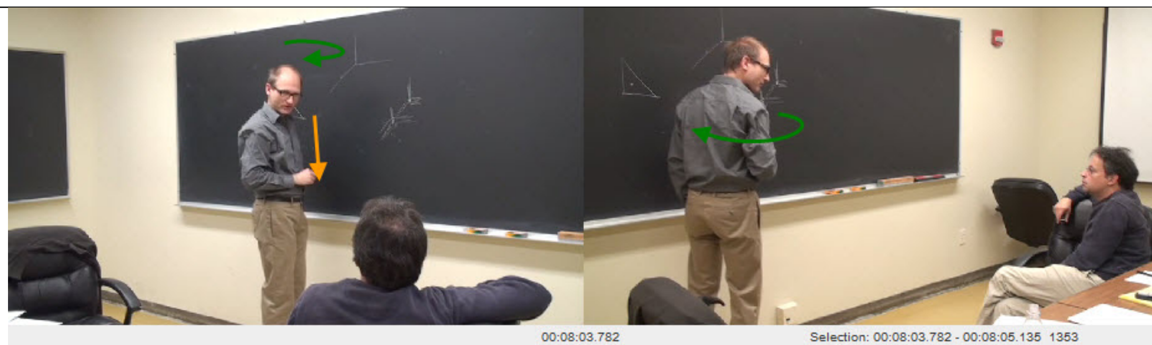
Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
99		closed condition, right? M: Mhm.
100		B: And the clo-

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
101		cl-
102		-osed moves to open on

Continued on next page

Table A.8 – triangle – *Continued from previous page*

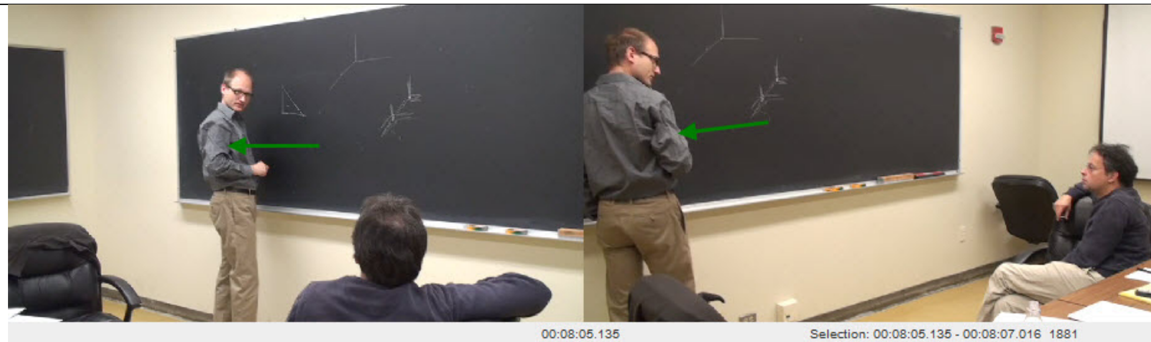
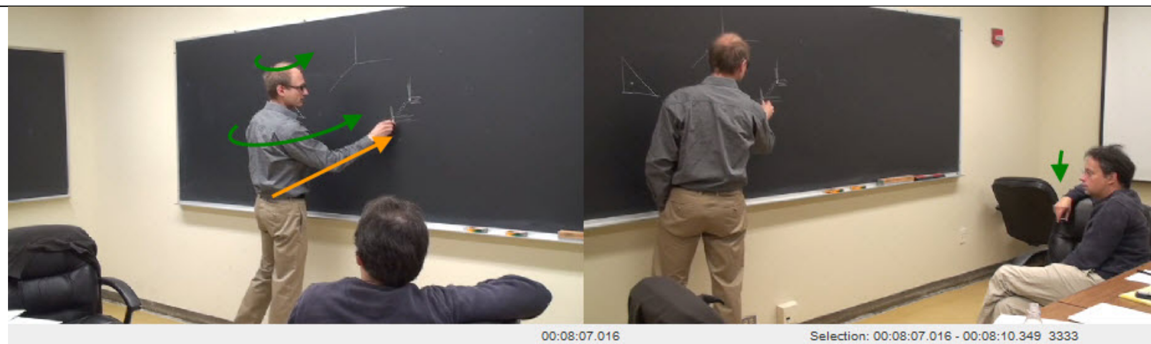
Row	Gesture & timestamp	Transcript
103		the geometric side.
104		So saying that it stays at this point

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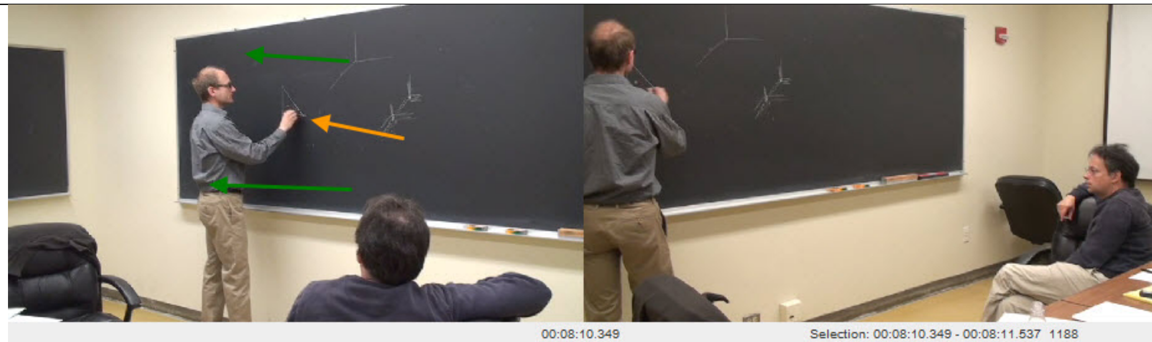
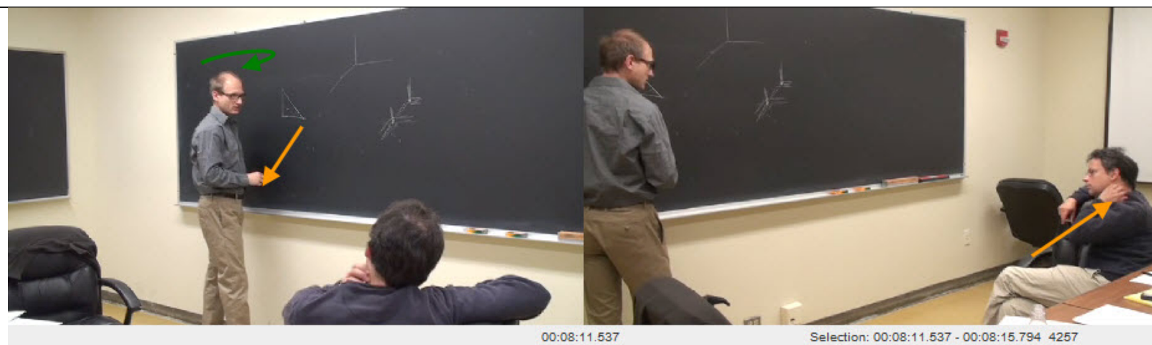
Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
105		means you move in the interior of the cell.
106		Right?

Continued on next page

Table A.8 – triangle – *Continued from previous page*

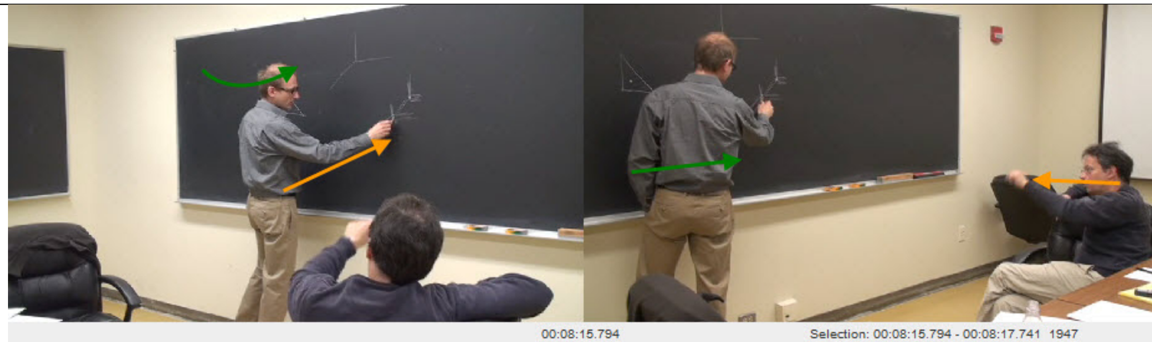
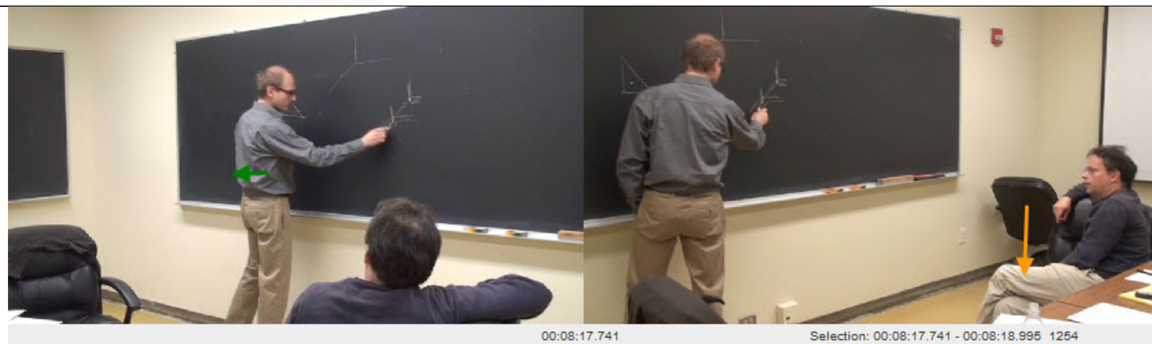
Row	Gesture & timestamp	Transcript
107		M: Mhm. B: Well if you move, uh. . . if, I mean, M: If that point
108		moves *down* B: If that point moves *down*, M: Then

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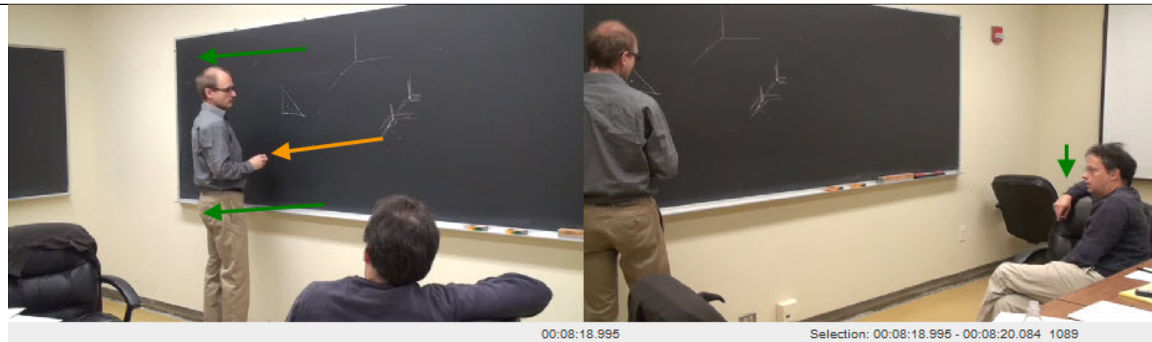
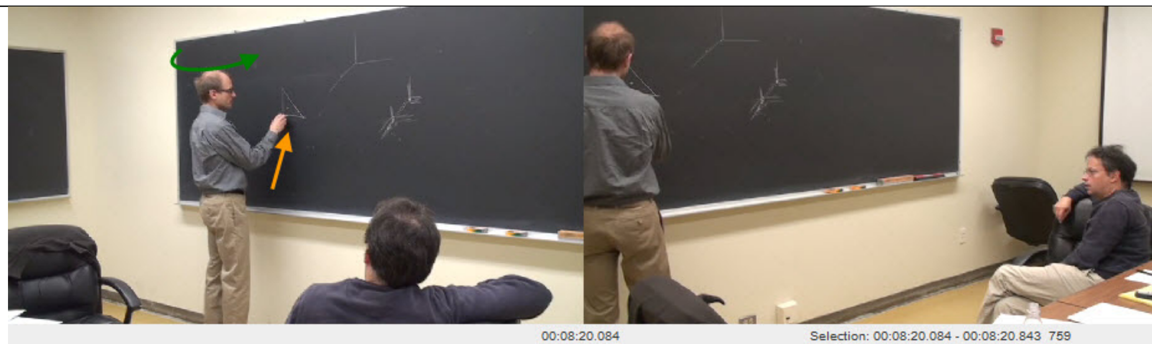
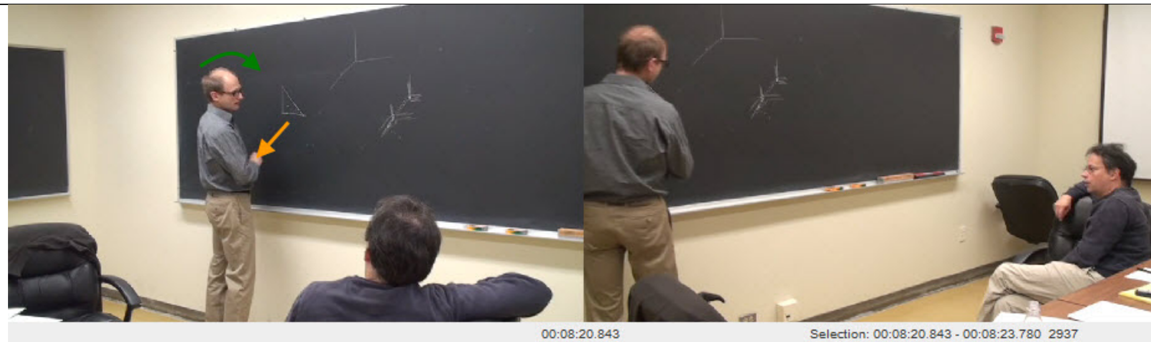
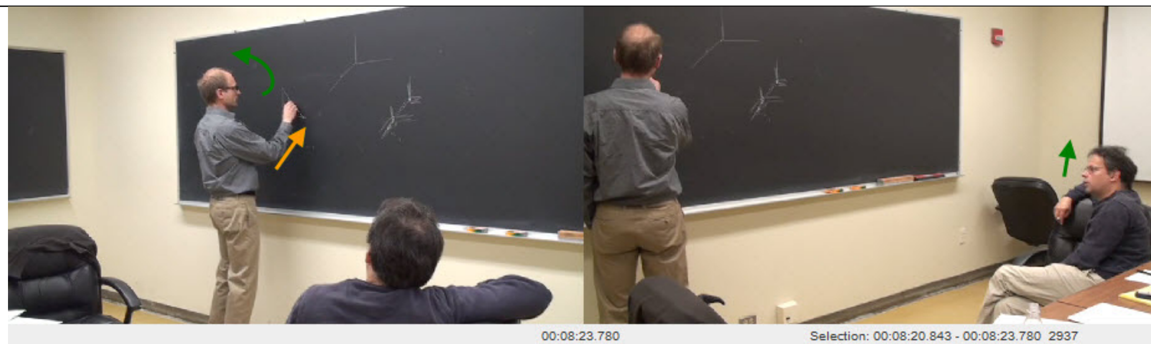
Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
109		B: that would mean
110		you actually,

Continued on next page

Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
111		uh, M: [inaudible]
112		B: uh, go to one of these, so M: [inaudible] B: I mean

Continued on next page

Table A.8 – triangle – *Continued from previous page*

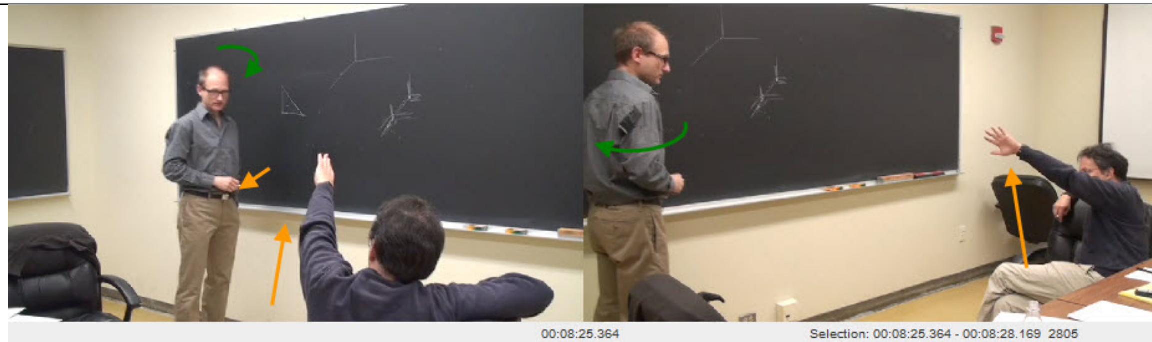
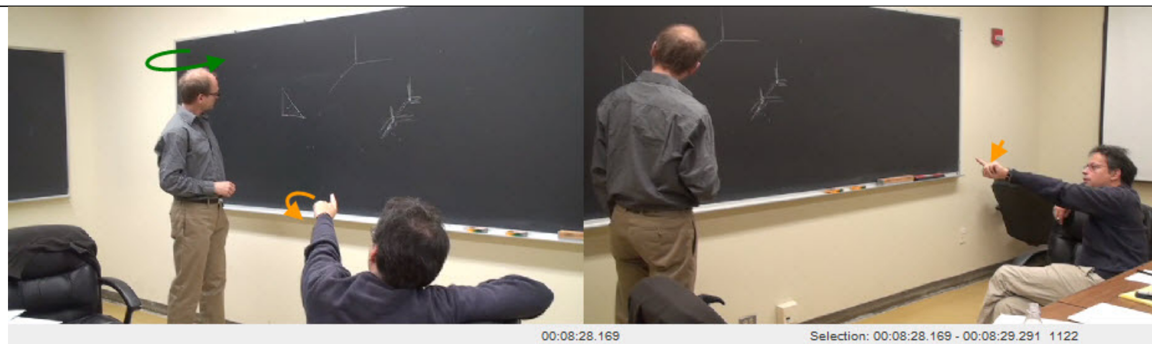
Row	Gesture & timestamp	Transcript
113		to M: The curve, B: to M: the curve
114		n- It's not the *point*; it's the *components*.

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Table A.8 – triangle – *Continued from previous page*

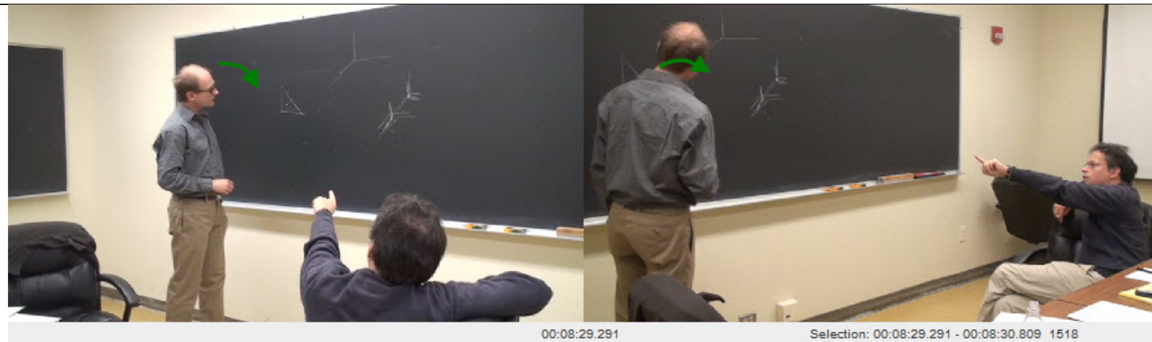
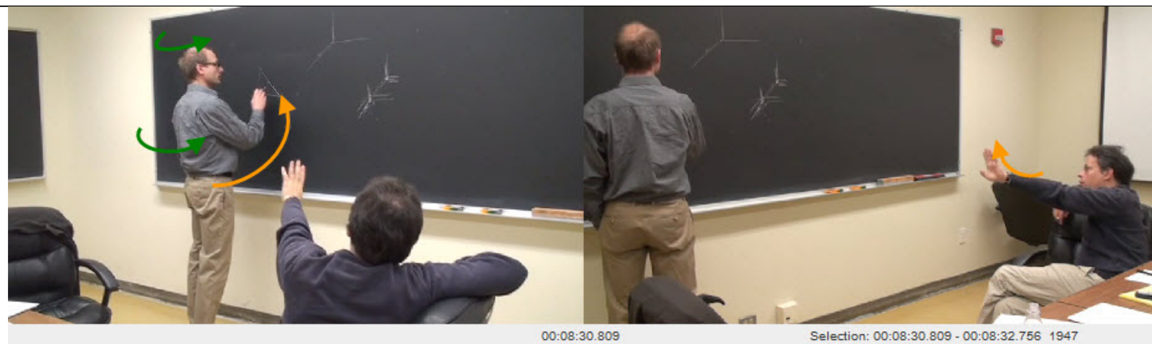
Row	Gesture & timestamp	Transcript
115		[pause]
116		B: Is it coupo- M: The, the B: uh, component

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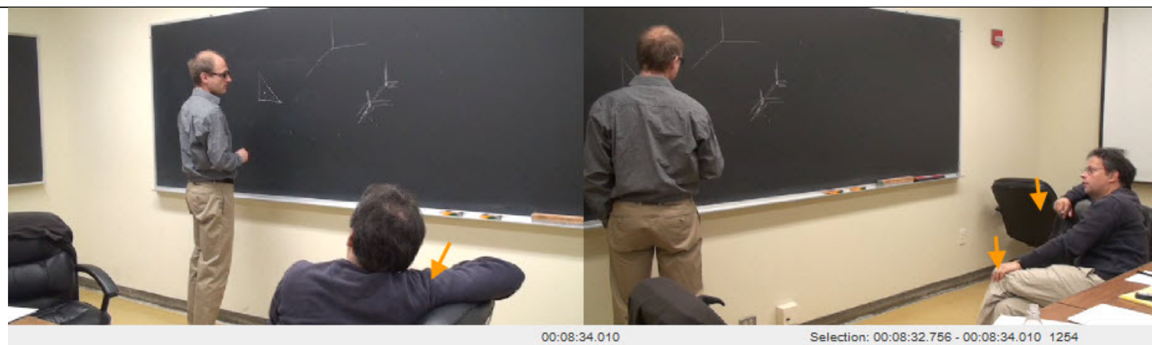
Table A.8 – triangle – *Continued from previous page*

Row	Gesture & timestamp	Transcript
117		M: point p is fixed; that's, that's in the interior.
118		[pause]

A.9 Joseph & Bill — chain map

Table A.9: Joseph & Bill — chain map

Row	Gesture & timestamp	Transcript
1	 45:55.000	[N/A]
2	 45:56.056	J: Yeah.

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
3	 45:56.848	[inaudible]
4	 45:58.201	J: So, B: Sure.

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
5	 45:58.960	J: So, uh
6	 45:59.752	[pause]

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
7	 46:00.214	This
8	 46:00.676	is- this

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
9	 46:01.501	is where we want
10	 46:02.623	Here- here's a zero.

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
11	 46:02.788	[inaudible]
12	 46:03.283	N' so

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
13	 46:04.306	[pause]
14	 46:04.867	[pause]

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
15	 46:06.121	We need to write
16	 46:07.078	down

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
17	 46:08.002	[pause]
18	 46:08.761	chain maps this

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
19	 46:09.718	way and this way.
20	 46:10.345	[pause]

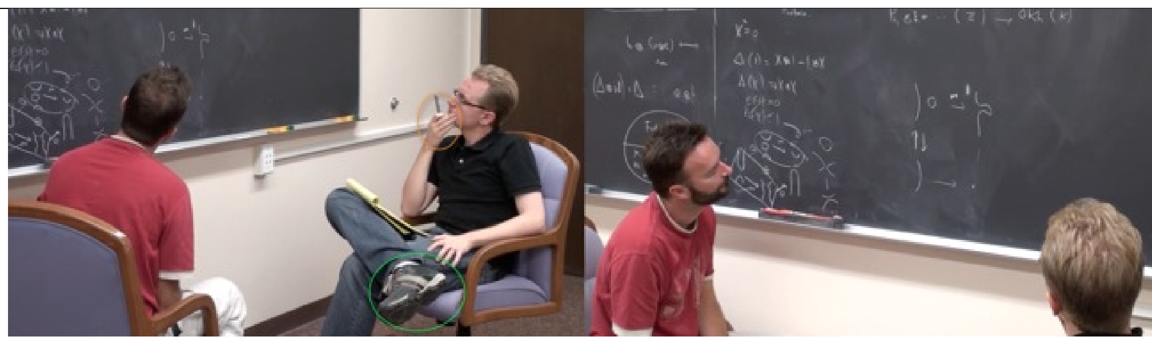
Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
21	 46:11.566	That's a multiplica-
22	 46:12.985	-tion. B: Mhm.

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
23	 46:15.064	[pause]
24	 46:16.846	[pause]

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
25	 46:24.106	[long pause] J: Um
26	 46:24.634	[pause]

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
27	 46:25.195	[pause]
28	 46:26.713	B: [inaudible murmuring]

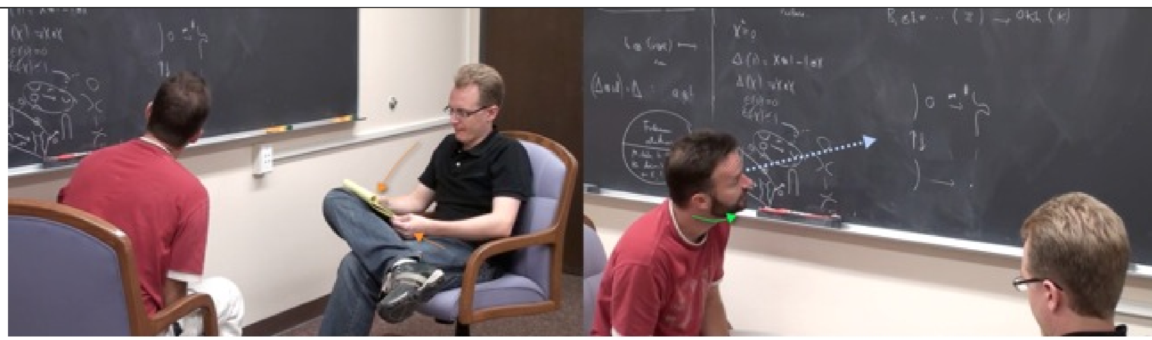
Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
29	 46:32.653	B: [continues murmuring for a moment] J: So what are the usual ones and why? I mean...
30	 46:33.181	[quick sound, seemingly not a word]

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
31	 46:34.006	What- what is the real thing-
32	 46:34.798	-ie; you have a

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
33	 46:35.953	[pause] In o-
34	 46:37.207	-ne direction you cap off

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
35	 46:38.890	with a torus don't you? B: Yes.
36	 46:39.253	[pause]

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
37	 46:41.134	J: Um [pause]
38	 46:42.652	[pause]

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
39	 46:44.038	[pause]
40	 46:44.731	[pause]

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
41	 46:46.348	[pause]
42	 46:46.942	[pause]

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
43	 46:48.097	[pause]
44	 46:48.724	[pause]

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
45	 46:49.648	[pause]
46	 64:51.364	[pause]

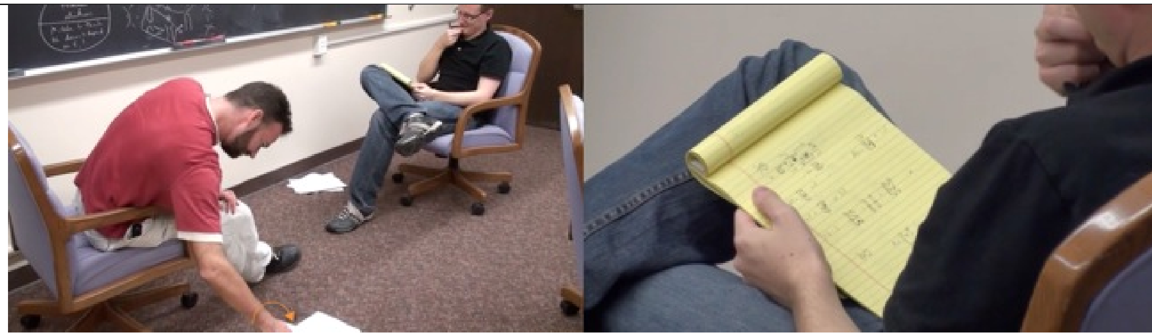
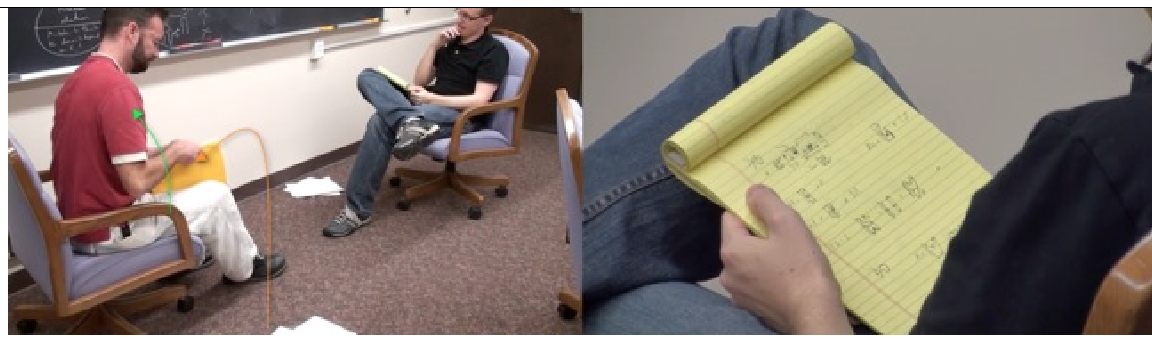
Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
47	 46:52.321	[pause]
48	 46:53.344	[pause]

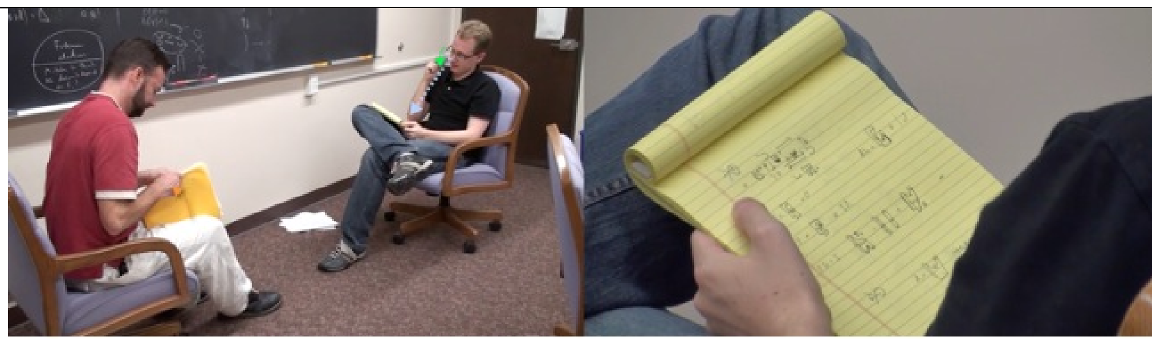
Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
49	 46:54.004	[pause]
50	 46:55.885	[pause]

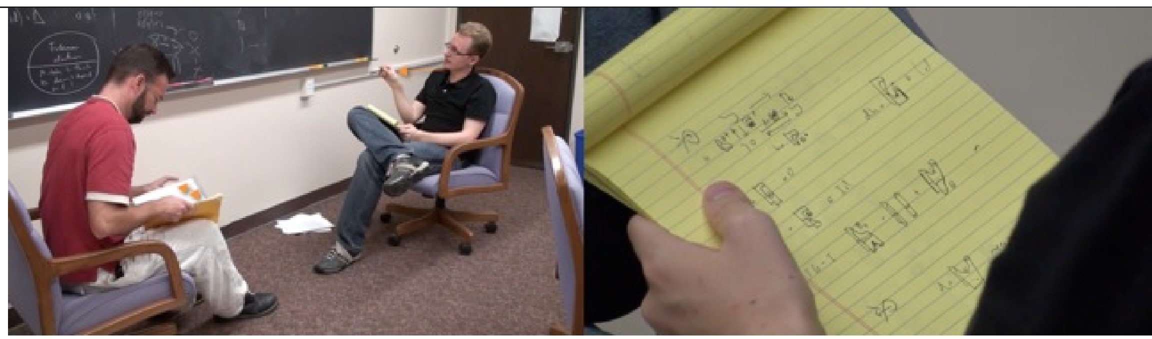
Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
51	 46:56.479	[pause]
52	 46:57.073	[pause]

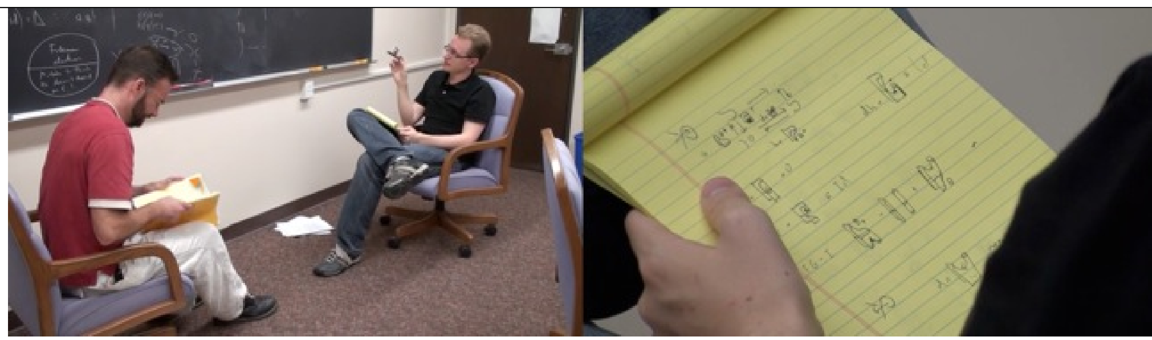
Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
53	 46:58.525	B: Yea-
54	 46:59.812	-h. The

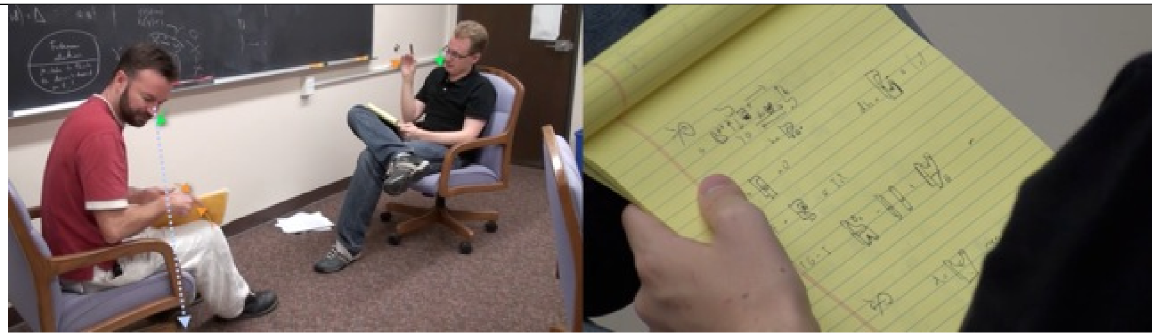
Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
55	 47:00.571	[pause]
56	 47:01.165	J: I do-

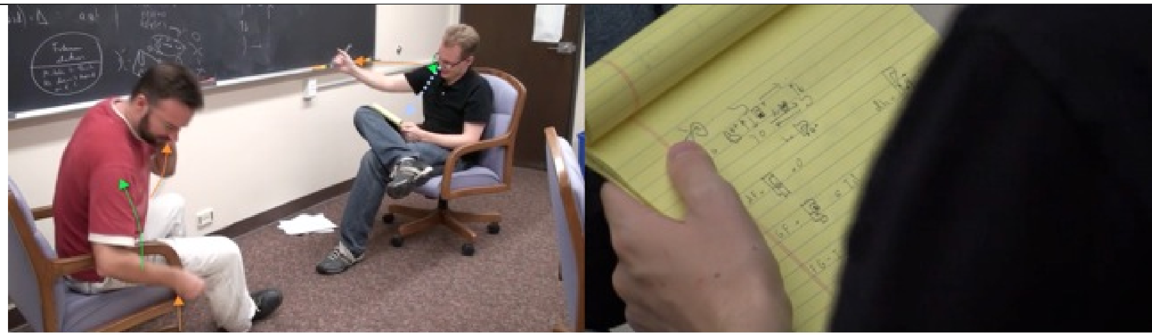
Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
57	 47:01.528	-n't have
58	 47:02.551	Draw's paper with me.

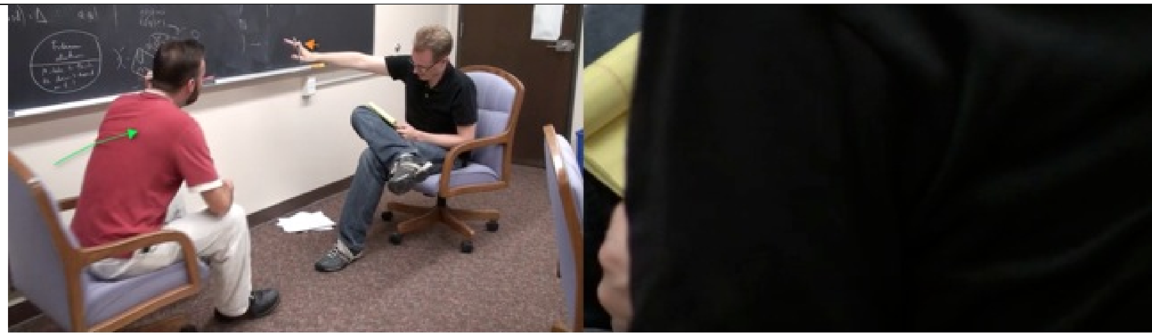
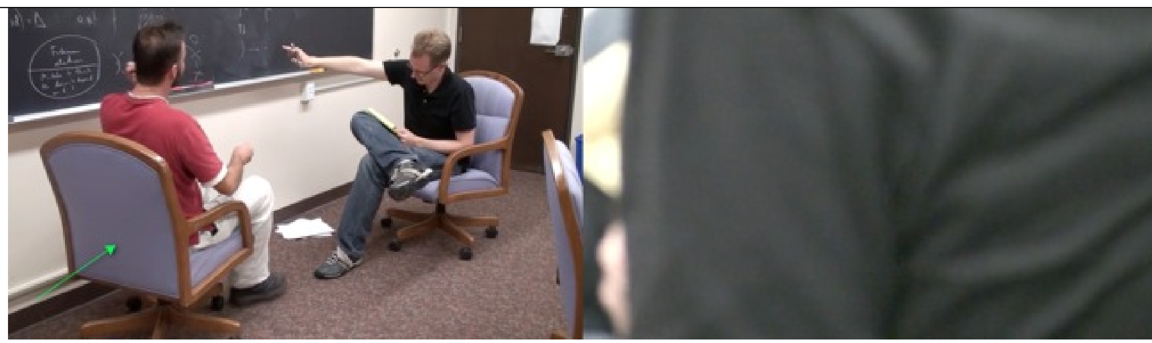
Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
59	 47:03.310	B: This J: U-
60	 47:03.739	-m. B: Th-

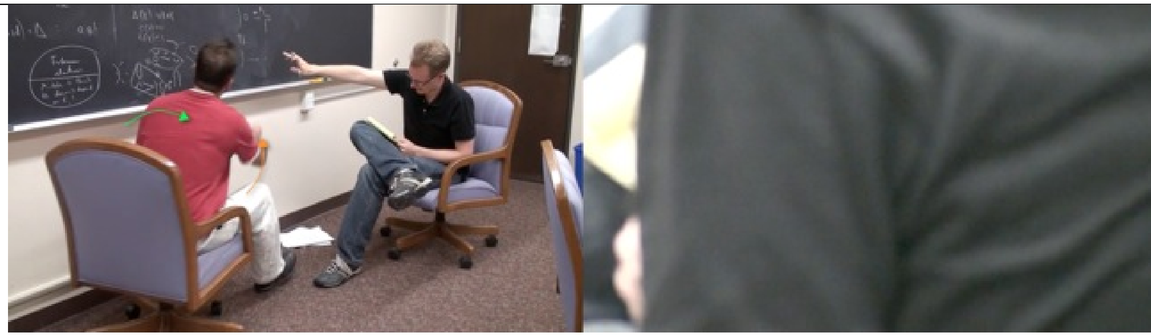
Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
61	 47:04.234	-is upper d-
62	 47:04.762	-iagram.

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
63	 47:05.224	[pause]
64	 47:05.785	[pause]

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
65	 47:06.742	[pause]
66	 47:08.194	One is a merge minus a t-

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
67	 47:08.656	-orus.
68	 47:09.547	So yo-

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
69	 47:09.646	-u g-
70	 47:09.778	-ot

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
71	 47:10.174	J: Okay.
72	 47:10.636	[pause]

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	**Gesture & timestamp**	**Transcript**
73	 47:11.428	[pause]
74	 47:12.121	Mi-

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
75	 47:14.794	-nus- Oh, okay, so that's the, that's, you know...
76	 47:15.487	B: B-

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
77	 47:15.784	-ut yeah a
78	 47:15.982	she-

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
79	 47:16.147	-et a-
80	 47:16.642	-nd a torus.

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
81	 47:17.698	[pause]
82	 47:21.130	[pause]

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
83	 47:24.859	[pause]
84	 47:25.717	[pause]

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Table A.9 – chain map – *Continued from previous page*

Row	**Gesture & timestamp**	**Transcript**
85	 47:28.258	J: Wait a minute. Uh,
86	 47:30.304	That's downwards isn't it? [laughs]

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
87	 47:31.393	How are we [laughs]
88	 47:31.888	[pause]

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Table A.9 – chain map – *Continued from previous page*

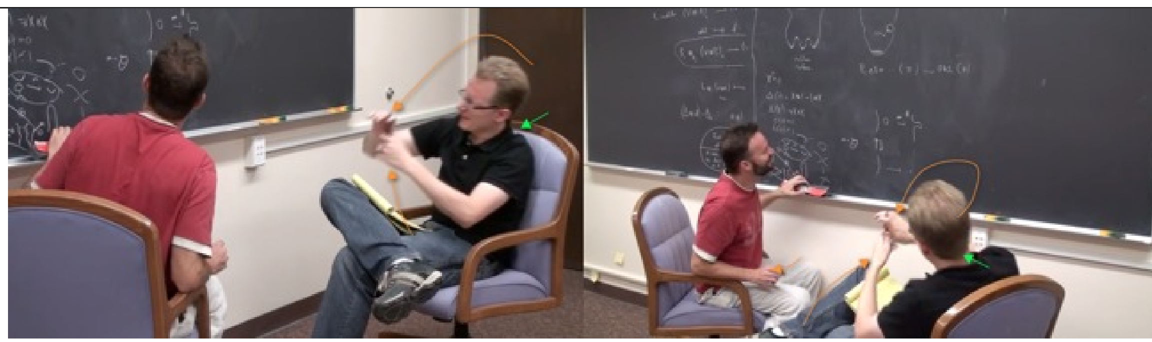
Row	Gesture & timestamp	Transcript
89	 47:32.944	[laughingly:] Which way
90	 47:33.703	are we reading B: Oh, uh,

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
91	 47:34.297	J: [chuckling] B: so,
92	 47:34.693	J: It- it B: Going

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
93	 47:35.056	this way.
94	 47:35.452	[pause]

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
95	 47:35.617	J: Go-
96	 47:36.046	-ing, going

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
97	 47:36.673	from this to th-
98	 47:37.003	-is. B: Fro-

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Table A.9 – chain map – *Continued from previous page*

Row	**Gesture & timestamp**	**Transcript**
99	 47:37.498	-m that to that,
100	 47:37.762	yeah.

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
101	 47:39.181	J: Uh, so m is
102	 47:41.260	not a merge, it's a... B: Oh, it's a split. J: ...split. B: S-

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
103	 47:42.052	-orry, J: Okay. B: sorry. [laughs]
104	 47:43.141	J: [laughs] B: Yeah it's a split.

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
105	 47:45.220	[pause] Um,
106	 47:45.748	and the-

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
107	 47:45.045	-n a sh-
108	 47:46.771	-eet and a torus. J: Yes;

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
109	 47:47.134	minu-
110	 47:47.629	-s, okay.

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
111	 47:48.586	[pause]
112	 47:50.929	[pause]

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
113	 47:51.853	J: Where-
114	 47:52.381	-as this one

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
115	 47:52.909	is just
116	 47:55.516	what, is that a merge? Is that merge mi-

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
117	 47:57.562	-nus, merge minus cap or something
118	 47:58.387	like that?

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
119	 48:05.053	B: Uh. . . J: I mean so. . . [pause] B: The other direction, uh, I got these things upside down. J: It's the same [laughs]
120	 48:05.647	[finishing chuckling]

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
121	 48:06.835	B: Heh [i.e., one-breath chuckle]
122	 48:07.660	Um

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
123	 48:08.782	[pause]
124	 48:09.607	[pause]

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
125	 48:10.531	Yeah, perhaps that J: Whuh
126	 48:11.521	B: that is the cup,

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
127	 48:13.468	the sheet cup and the sheet to-
128	 48:13.930	-rus and this

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
129	 48:14.920	is the mer-
130	 48:15.646	-ge. Maybe that's wh-

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
131	 48:16.108	-at I'm thinking.
132	 48:17.824	[pause]

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
133	 48:19.342	J: Well, B: Ah, I don't know if
134	 48:19.804	that's right.

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
135	 48:20.332	[chuckle]
136	 48:20.926	[pause]

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
137	 48:22.840	J: This, this- Well, le-
138	 48:23.533	-t's just- we- w-

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
139	 48:24.160	-e, if we u-
140	 48:24.589	-h,

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
141	 48:25.579	This is a chain map; I mea-
142	 48:26.833	-n if we go from here to here and then you mu-

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
143	 48:27.328	-ltiply
144	 48:27.559	[one-breath chuckle]

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
145	 48:28.351	So if
146	 48:28.780	we spli-

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
147	 48:29.341	-t and then multi-
148	 48:29.737	-ply we ge-

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
149	 48:30.463	-t uh
150	 48:30.628	uh

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
151	 48:30.826	uh i-
152	 48:31.453	-t's a sheet with

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
153	 48:31.750	a h-
154	 48:32.014	-andle

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
155	 48:32.476	attached
156	 48:33.598	to it. B: Yes.

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
157	 48:34.456	J: And we're subtr-
158	 48:35.182	-acting

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
159	 48:35.677	TOGETHER: A shee-
160	 48:36.106	-t with a ha-

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
161	 48:36.370	-ndle J: a-
162	 48:36.865	-ttached to it, ye-

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
163	 48:37.129	-ah.
164	 48:37.459	B: Okay. J: U-

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
165	 48:38.581	-m, so tha-
166	 48:41.782	-t seems reasonable. Um,

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
167	 48:42.508	Now of course the thing going
168	 48:43.465	in the other direction you can put ri-

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	**Gesture & timestamp**	**Transcript**
169	 48:44.026	-ght anything f-
170	 48:47.524	-for it to be a chain map. So that's not so helpful. B: Right.

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
171	 48:48.811	J: Um,
172	 48:49.042	B: But

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
173	 48:50.395	it- but it should be capped off,
174	 48:50.725	right?

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
175	 48:51.088	You ju-
176	 48:52.276	-st distinguish that extra cir-

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
177	 48:52.738	-cle
178	 48:53.794	and you look at

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
179	 48:54.586	that state,
180	 48:55.444	so. . .

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
181	 48:56.170	[pause]
182	 48:56.632	J: Tha-

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
183	 48:57.391	-t, that se-
184	 48:57.754	-ems like a r-

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
185	 48:58.843	-easonable thing.
186	 49:00.229	So, if we di-

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
187	 49:00.922	-d uh
188	 49:02.011	[pause]

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
189	 49:03.067	[pause] if
190	 49:03.397	we

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Table A.9 – chain map – *Continued from previous page*

Row	**Gesture & timestamp**	**Transcript**
191	 49:03.958	go up here
192	 49:04.420	and the we ca-

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
193	 49:05.806	-p off, we ge-
194	 49:06.961	-t uh, splitting c-

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
195	 49:07.093	-a-
196	 49:07.588	-ps with a s-

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
197	 49:08.611	identity
198	 49:10.360	minus two,

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
199	 49:11.779	so that seems like minu-
200	 49:12.241	-s the ide-

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
201	 49:13.000	-ntity.
202	 49:14.353	[pause] That's g-

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
203	 49:14.881	that's a B: Wait
204	 49:15.343	what did you do?

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
205	 49:15.871	Sorry. J: So
206	 49:16.300	if I did

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
207	 49:16.531	if I
208	 49:16.861	did let's

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
209	 49:17.257	call this eh- [saying "*f*"]
210	 49:17.620	-f and thi-

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
211	 49:17.197	-s is g-
212	 49:18.544	-ee [saying "*g*"]. B: Yeah.

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
213	 49:19.237	J: So I think I'm say-ing
214	 49:19.996	g composed

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
215	 49:22.537	with f equals minus the identity.
216	 49:22.999	[pause]

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
217	 49:27.718	B: [inaudible; sounds like "It was just a crackle on two, right?"] J: That's okay, I think.
218	 49:28.312	B: Sure.

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
219	 49:29:434	J: And f composed with g
220	 49:30.127	[pause]

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
221	 49:31.150	Um,
222	 49:33.394	[pause]

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
223	 49:35.902	[pause; Justin breathes out]
224	 49:36.892	[Justin laughs]

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
225	 49:37.750	[pause]
226	 49:39.301	[pause]

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
227	 49:40.489	J: We'll draw it; so we [one-breath chuckle]
228	 49:40.852	[finish chuckle]

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
229	 49:41.182	[pause]
230	 49:41.611	We

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
231	 49:42.304	I'm gonna draw it
232	 49:43.030	going upwards.

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
233	 49:44.383	B: Okay. J: So we cap off
234	 49:44.845	B: Yeah.

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
235	 49:47.254	J: And then we, uh,
236	 49:47.749	[pause]

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Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
237	 49:47.980	B: spl-
238	 49:48.277	-it.

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
239	 49:48.937	J: Split. [chuckle]
240	 49:49.498	[chuckle finishes]

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
241	 49:50.059	Minus
242	 49:50.851	[pause]

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
243	 49:51.313	[pause]
244	 49:52.534	[pause]

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
245	 49:53.095	[pause]
246	 49:53.326	[pause]

Continued on next page

Table A.9 – chain map – *Continued from previous page*

Row	Gesture & timestamp	Transcript
247	 49:54.613	B: [pause] Sure.
248	 49:55.500	J: Well.

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