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HA-spaces and Commutative Homology Rings

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Nicholas D. Nguyen

Committee in charge:

    Professor James P. Lin, Chair
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    Professor Justin Roberts
    Professor Daniel S. Rogalski

2013

The dissertation of Nicholas D. Nguyen is approved, and it is acceptable in quality and form for publication on microfilm:

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Chair

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2013

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Homotopy-Associative H-Spaces and their Free Loop Spaces, Submitted February 2012

ABSTRACT OF THE DISSERTATION

HA-spaces and Commutative Homology Rings

by

Nicholas D. Nguyen

Doctor of Philosophy in Mathematics

University of California San Diego, 2013

Professor James P. Lin, Chair

In the study of Lie groups, topological groups, and H-spaces, one common problem involved finding characterizations of spaces whose mod p homology is commutative, and in the case that it is not commutative, one could ask what kinds of commutators occurred in the mod p homology.

A common theme in H-space theory is trying to extend results for Lie groups to H-spaces by creating new techniques and finding new proofs. As such, our dissertation stems from recent work in characterizing Lie groups whose mod p homology is commutative using a tool called the adjoint action of the Lie group on its loop space.

Our main contribution starts with defining and studying the adjoint action, along with similar maps like the commutator, for a generalization of compact simply-connected Lie groups called finite simply-connected HA-spaces, and the homomorphism it induces on mod p cohomology (p odd). After finding a formula for the induced homomorphism in terms of the coproduct, we proceed to several applications.

Given a finite simply-connected HA-space X, we use the adjoint action, along with similar maps, and their induced homomorphisms to characterize HA-spaces whose mod p homology is commutative. This generalizes earlier work by showing that X does not need the full structure of a Lie group; it can be weakened to an HA-space structure.

For our second application, given a finite simply-connected HA-space X, we study the free loop space of X and use the adjoint action and related maps to compute products in the mod p homology of the free loop space.

Finally, given a finite simply-connected HA-space X, we demonstrate how to use the commutator on X in order to construct a new multiplication map on X which induces a commutative algebra structure in mod p homology.

1 Introduction

A major milestone in the study of topological spaces is the usage of functors from the category of (pointed) topological spaces to some category of algebraic objects like groups and modules. These algebraic objects can be used to distinguish spaces: if a functor takes two topological spaces to different algebraic objects, then the spaces are fundamentally different. We will focus on the singular cohomology over a ring R, a contravariant functor that takes a space X to the graded R-module $H^*(X; R)$, and the singular homology over a ring R, a covariant functor which takes X to the graded R-module $H_*(X; R)$. The cohomology $H^*(X; R)$ is also an algebra over R, in that objects inside it can be added, multiplied, and scaled by an element of R. Due to the many structures available in $H^*(X; R)$, it has received much focus in mathematical research. In particular, given a space, one can compute its cohomology and specify its algebra structure (how objects are added and multiplied). Conversely, given an algebra, one can ask if it arises as the cohomology of some space.

Now suppose our space has a continuous map $\mu : X \times X \to X$ such that if x_0 is the basepoint of X, then $\mu(x, x_0) = \mu(x_0, x) = x$. Examples of spaces with such a map are (from specific to general) Lie groups, topological spaces, and H-spaces; the

map μ is called a multiplication map on X, and the Lie group, topological space, or H-space is denoted (X, μ). In this case, the existence of μ imposes more structure to the cohomology and homology of X over a field $\mathbb{F}_p$ (in this dissertation, p will always denote an odd prime integer). The cohomology $H^*(X; \mathbb{F}_p) = H^*(X, \mu; \mathbb{F}_p)$ is now a Hopf algebra, with a unary operation μ^* (induced by μ) called the coproduct. Furthermore, suppose that X is a space of finite type as well: X is homotopy equivalent to a CW complex with a finite number of cells in each dimension. Then the homology $H_*(X; \mathbb{F}_p) = H_*(X, \mu; \mathbb{F}_p)$ can be given an algebra structure induced by μ. The presence of the coproduct in cohomology actually limits the possibilities for the algebra structure of $H^*(X, \mu; \mathbb{F}_p)$, and consequently, there has been much effort on computing the algebra and Hopf algebra structures of the cohomology of Lie groups, topological groups, and H-spaces.

Mathematicians have made computations of the coproduct operations in the cohomology over $\mathbb{F}_p$ of a compact connected Lie group. It started in the 1950's with Borel's work, where he found that if (G, μ) is a classical Lie group, $H^*(G, \mu; \mathbb{F}_p)$ is a primitively generated exterior algebra, the simplest kind of Hopf algebra possible [4]. From there, mathematicians like Borel, Toda, Mimura, Shimada, Araki, and Kono worked on the exceptional groups; the problem is that for certain combinations of exceptional groups (G, μ) and primes p, $H^*(G, \mu; \mathbb{F}_p)$ is not primitively generated. It was not until 1977 that Kono finished these computations with his calculation of coproducts in $H^*(E_8, \mu; \mathbb{F}_5)$ [20].

Borel also showed that if G is a compact connected Lie group, $H^*(G, \mu; \mathbb{F}_p)$

is primitively generated if and only if the homology $H_*(G, \mu; \mathbb{F}_p)$ is a (graded) commutative algebra [3]. From here, one can ask if there is a relationship between commutativity of the homology $H_*(G, \mu; \mathbb{F}_p)$ and commutativity of the group G itself. If G is abelian, then $H_*(G, \mu; \mathbb{F}_p)$ would be commutative. The converse is false: E_8 is a nonabelian Lie group, but $H_*(E_8, \mu; \mathbb{F}_7)$ is commutative. Instead, we can turn to Kono and Kozima's work on the adjoint action of G on its loop space, a map $Ad : G \times \Omega G \to \Omega G$ given pointwise by $Ad(g, l)(t) = gl(t)g^{-1}$. Notice that if G is an abelian compact connected Lie group, then $Ad(g, l) = l$ for any element g and loop l. In other words, the map Ad is equal to the projection on the second factor $p_2^{\Omega} : G \times \Omega G \to \Omega G$ given by $p_2^{\Omega}(g, l) = l$. Kono and Kozima prove that $H_*(G, \mu; \mathbb{F}_p)$ is commutative if and only if the algebra homomorphisms Ad^* and $p_2^{\Omega *}$ (from $H^*(G; \mathbb{F}_p)$ to $H^*(G \times G; \mathbb{F}_p)$) are equal [21].

As H-space theorists, we would like to know if this result holds for a larger class of spaces than the Lie groups. This question has been asked for many results about the cohomology of Lie groups in the history of H-spaces, and Kane, in his textbook on the subject, even calls Lie groups the "experimental data" of H-space theory [19]. One of the goals of this dissertation is to introduce a generalization of Lie groups that we will call HA-spaces, and show how we can generalize Ad and other maps from Kono and Kozima's work to finite simply-connected HA-spaces in a way that allows us to prove some of their results for this larger class of spaces. From there, we will use our maps for other applications: the computation of products in the homology of a type of H-space called the free loop space of an HA-space, and the construction of an

H-space whose homology is a commutative (if possibly nonassociative) algebra which is isomorphic as a vector space to the homology of a given finite simply-connected HA-space.

The remainder of the introduction will proceed as follows. We will start with some concepts and definitions from the theory of H-spaces that will be used in this dissertation; assumptions and notation will also be stated as well. We divide this background material into two sections: the algebraic material, and the topological material. Afterward, we will proceed by topic, giving a brief background of each problem and stating our results. First, we will discuss Kono and Kozima's work in more detail, along with some subsequent developments by other authors, and introduce some obstacles to generalizing their results to HA-spaces. Our results will include details on how we overcame these challenges. From there, we will discuss the problem of computing products in the homology of a free loop space, and find formulas for products in the homology over $\mathbb{F}_p$ (for any odd prime p) of a free loop space of a finite simply-connected HA-space. Afterward, we will look at the question of finding multiplication maps on a finite simply-connected topological space: given such a space X, we want to know if there is a map $\nu : X \times X \to X$ such that (X, ν) is an H-space whose homology over $\mathbb{F}_p$ (p is a fixed odd prime) is a commutative and associative algebra. In the case that our topological space is a finite simply-connected HA-space (X, μ), we find a formula for a map $\nu : X \times X \to X$ in terms of μ such that the multiplication induced by ν on the homology of X over $\mathbb{F}_p$ is commutative (but possibly nonassociative), and determine conditions on $H_*(X, \mu; \mathbb{F}_p)$ which make

$H_*(X, \nu; \mathbb{F}_p)$ a commutative and associative algebra.

1.1 Algebraic Assumptions and Definitions

Let us begin with the algebraic assumptions and definitions that will be needed. After stating our assumptions, we will introduce Hopf algebras and discuss some of their characteristics and properties. For background material on Hopf algebras, we recommend Milnor and Moore's paper, [29], and Kane's book [19].

Throughout this entire dissertation, the symbol p will always denote an odd prime. All algebras will have $\mathbb{F}_p$ as their base field. They will be assumed to be associative unless otherwise stated. Given an algebra A, we will assume A is graded, and denote the subspace of homogenous elements of degree n by A^n. For any positive integer n, we will require A^n to be a vector space (over $\mathbb{F}_p$) of finite dimension (it has a spanning set with a finite number of elements), and A^0 must equal $\mathbb{F}_p$. If we say an algebra is commutative, we mean in the graded sense. We will assume that all homomorphisms between algebras are algebra homomorphisms which preserve addition, multiplication, and scalar multiplication by the base field $\mathbb{F}_p$, and we will just write homomorphism for algebra homomorphism. The reader must be aware that we will deal with functions between algebras which might only be linear transformations which preserve addition and scalar multiplication. We will indicate when we are dealing with such a function. If a homomorphism ϕ is said to equal zero or be trivial, this means that $\phi(x) = 0$ if x has positive degree, and $\phi(x) = x$ if x has degree zero.

If A is a graded algebra, then $\bar{A}$ will denote positive degree elements of A, A^{odd} will denote the odd degree elements, and A^{even} will denote the even degree elements.

Before we continue, let us review the algebra structure on the tensor product of graded algebras.

Definition 1.1.1. Let A and B be algebras. The vector space $A \otimes B$ can be given an algebra structure as follows: let $a, c \in A$ and $b, d \in B$. Then in $A \otimes B$, we multiply the elements $a \otimes b$ and $c \otimes d$ as follows (where $|x|$ denotes the degree of an element x):

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd. \tag{1.1.1}$$

We will also abbreviate $0 \otimes 0$ as 0.

First, let us recall that the cohomology over $\mathbb{F}_p$ of a Lie group or topological group (G, μ) has a special structure:

Definition 1.1.2. An algebra A together with an algebra homomorphism $\Delta : A \to A \otimes A$ is a *Hopf algebra* $A = (A, \Delta)$ if the diagram commutes:

$$
\begin{array}{ccc}
A & \xleftarrow{\quad q_1 \quad} & A \otimes A \\
{\scriptstyle q_2} \uparrow & \searrow^{id_A} & \uparrow {\scriptstyle \Delta} \\
A \otimes A & \xleftarrow[\Delta]{\quad\quad} & A
\end{array}
$$

where

$$
q_1(a \otimes b) =
\begin{cases}
0, & |b| > 0 \\
a, & b = 1
\end{cases}
$$

and

$$q_2(a \otimes b) = \begin{cases} 0, & |a| > 0 \\ \\ b, & a = 1 \end{cases}$$

The homomorphism Δ is called the *comultiplication or coproduct* on A. According to

the diagram, for any $a \in A$,

$$\Delta(a) = a \otimes 1 + 1 \otimes a + \sum a' \otimes a'' \tag{1.1.2}$$

where $|a'|, |a''| > 0$ and $|a'| + |a''| = |a|$.

The term $\sum a' \otimes a'' = \Delta(a) - a \otimes 1 - 1 \otimes a$ is called the *reduced coproduct* and

is denoted $\overline{\Delta}(a)$.

An element $a \in A$ is *primitive* if $\overline{\Delta}(a) = 0$ (that is, $\Delta(a) = a \otimes 1 + 1 \otimes a$). The

submodule of primitive elements of A is denoted PA. Elements of degree n in PA

are denoted $P^n A$, elements of odd degree are denoted $P^{odd} A$, and elements of even

degree are denoted $P^{even} A$.

A nonzero element $a \in A$ has *height* n, $1 < n < \infty$, if $a^n = 0$, and $a^{n-1} \neq 0$.

If no such integer exists, we say a has *infinite height*.

The *module of indecomposables* is $QA = A/\bar{A}\bar{A}$, the quotient module of A by

the image of the multiplication on positive degree elements. Elements of QA are cosets

of $\bar{A}\bar{A}$. Representatives of positive degree elements of QA in A are called *generators*;

given a basis of QA, a set consisting of a representative for each and every element

of this basis of QA is called a *generating set* for A. That is, any element of A can

be written as a sum, scalar multiple, and product of elements in the generating set.

Elements of degree n in QA will be denoted $Q^n A$, elements of odd degree are denoted $Q^{odd} A$, and elements of even degree are denoted $Q^{even} A$. Elements of $\bar{A}\bar{A}$ are called *decomposables*.

If there is a choice of generating set for A which consists of primitive elements, then we say A is *primitively generated*.

If we denote the dual of A by A^*, then we say A is *coassociative* if A^* is associative, and A is *cocommutative* if A^* is commutative.

An algebra homomorphism $\varphi : (A, \Delta_A) \to (B, \Delta_B)$ between two Hopf algebras is called a *Hopf algebra homomorphism* if

$$(\varphi \otimes \varphi) \Delta_A = \Delta_B \varphi.$$

If φ is an algebra isomorphism which is a Hopf algebra homomorphism as well, we say that φ is a *Hopf algebra isomorphism* and that (A, Δ_A) and (B, Δ_B) are isomorphic as Hopf algebras.

Remark. We do not assume coassociativity or cocommutativity in our definition of a Hopf algebra.

We have the following elementary properties of PA and QA from [19]:

Theorem 1.1.3. *The vector spaces* $(QA)^*$ *and* $P(A^*)$ *are equal, and* $(PA)^* = Q(A^*)$. *Moreover, if A is a commutative (and associative) Hopf algebra (over $\mathbb{F}_p$), denote the pth power map on A by $\xi : A \to A$ (so $\xi(a) = a^p$). Then the following sequence is*

exact for each positive integer n:

$$0 \to P^n(\xi(A)) \to P^n A \to Q^n A.$$

Furthermore, any primitive element whose degree is not a multiple of $2p$ is a generator of A. That is, when n is not a multiple of $2p$, the linear transformation

$$P^n A \to Q^n A$$

is injective, and if A is also cocommutative and coassociative, the linear transformation is an isomorphism.

Remark. As we will see in Example 1.1.5, odd degree elements cannot have nontrivial pth powers, so although the exact sequence tells us that primitive elements whose degrees are not a multiple of p are generators, since odd degree elements are never nontrivial pth powers, we can be more specific and state that any primitive element whose degree is not a multiple of $2p$ is a generator.

We also have the following characterization of primitively generated Hopf algebras as well from [19]:

Theorem 1.1.4. *A Hopf algebra A is primitively generated if and only if A^* is commutative, associative, and has no pth powers (that is, if $\bar{a} \in A^*$, then $\bar{a}^p = 0$).*

Example 1.1.5. Let us list all the possible commutative Hopf algebras over $\mathbb{F}_p$ with a single algebra generator. These examples are called monogenic Hopf algebras. In

each of these cases, the generator is primitive.

1) An exterior algebra on an odd degree generator: $\wedge(x_{2n+1})$

2) A polynomial algebra on an even degree generator: $\mathbb{F}_p[x_{2n}]$

3) A truncated polynomial algebra on an even degree generator: $\mathbb{F}_p[x_{2n}]/(x_{2n}^{p^k})$

An important fact about Hopf algebras is that once these monogenic Hopf algebras are known, we can describe the algebra structure of any other Hopf algebra using the Borel Decomposition Theorem (quoted from [19]):

Theorem 1.1.6. *Let A be a commutative (and associative) Hopf algebra. Then there are monogenic Hopf algebras A_i such that A and $\underset{i}{\otimes} A_i$ are isomorphic as algebras.*

Definition 1.1.7. We will call $\underset{i}{\otimes} A_i$ in Theorem 1.1.6 the *Borel decomposition* of A.

Note that the Hopf algebra $\underset{i}{\otimes} A_i$ is primitively generated, but since the isomorphism might not preserve coproducts, A itself could have nontrivial reduced coproducts. Nevertheless, the Borel Decomposition Theorem tells us that Hopf algebras are much easier to visualize than generic algebras, so spaces whose cohomology algebras are Hopf algebras are easier to study. Thus, in the next section, we will introduce topological spaces whose cohomology algebras over $\mathbb{F}_p$ are Hopf algebras.

1.2   Topological Assumptions and Definitions

In this section, we will state any assumptions on the topological spaces we will be studying. After that, we will introduce the main objects we will be studying:

the H-spaces, and the HA-spaces. Our exposition will compare these objects to the more familiar topological groups as well. Next, we will quote an important property of the cohomology of finite simply-connected HA-spaces from Lin's paper [25] that will be referenced in our results. After that, we will introduce some examples of HA-spaces that we will focus on: loop spaces and free loop spaces. Finally, we will end this section with a discussion of the differences between topological groups and HA-spaces. For further background material on topological spaces and cohomology, we recommend the books [8], [12], [26], [33], and [38]. For further background on H-spaces, we recommend the books [19], [35], and [40].

For the rest of this document, unless specified, all spaces will have a basepoint (typically denoted with a zero subscript or superscript), and are homotopy equivalent to a CW complex which has a finite number of cells in each dimension and whose basepoint is a subcomplex. All maps and homotopies are continuous and respect the basepoint. If we say a space is *finite*, we mean it has the homotopy type of a CW complex with a finite number of cells. If we say a diagram of topological spaces and maps commutes (with no adjectives), we will mean it commutes up to homotopy. The symbol "$\simeq$" will mean "homotopic relative basepoints."

Let us define some important maps which will occur frequently in our work:

Definition 1.2.1. Given a space X with basepoint x_0, we define the *diagonal map* $\Delta_X : X \to X \times X$ and *constant (trivial) map* $k : X \to X$ by

$$\Delta_X(x) = (x, x), \text{ and } k(x) = x_0.$$

The diagonal map induces the (cup) product in cohomology,

$$H^*(X \times X; \mathbb{F}_p) \xrightarrow{\ \cong\ } H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p) \xrightarrow{\ \Delta_X^*\ } H^*(X; \mathbb{F}_p)$$

which we will also denote as Δ_X^*. That is, if $a, b \in H^*(X; \mathbb{F}_p)$,

$$\Delta_X^*(a \otimes b) = ab.$$

The identity map is denoted by $id_X : X \to X$, where $id_X(x) = x$.

The (jth) projection map $p_j : X_1 \times X_2 \times \ldots \times X_n \to X_j$ is given by

$$p_j(x_1, x_2, \ldots, x_j, \ldots, x_n) = x_j,$$

and the (jth) inclusion map $i_j : X_j \to X_1 \times X_2 \times \ldots \times X_n$ is given by

$$i_j(x_j) = (x_1^0, x_2^0, \ldots, x_j, \ldots, x_n^0),$$

where x_k^0 is the basepoint of X_k.

For any two spaces Y and Z, let $T_{Y,Z} : Y \times Z \to Z \times Y$ be defined by $T_{Y,Z}(y, z) = (z, y)$. If $Y = Z$, we will abbreviate the map as T_Y, or even just T.

Remark. Whenever we have a map $f : X \times Y \to Z$ where X, Y, Z are topological spaces, we will abuse notation in cohomology over $\mathbb{F}_p$ and call f^* to be the composi-

tion:

$$H^*(Z; \mathbb{F}_p) \xrightarrow{f^*} H^*(X \times Y; \mathbb{F}_p) \xrightarrow{\cong} H^*(X; \mathbb{F}_p) \otimes H^*(Y; \mathbb{F}_p).$$

We do the same for a map $g : X \to Y \times Z$ and denote the following composition by

g^*:

$$H^*(Y \times Z; \mathbb{F}_p) \xrightarrow{\cong} H^*(Y; \mathbb{F}_p) \otimes H^*(Z; \mathbb{F}_p) \xrightarrow{g^*} H^*(X; \mathbb{F}_p) .$$

Let us introduce the most general type of space whose cohomology over $\mathbb{F}_p$ is

a Hopf algebra:

Definition 1.2.2. Let X be a topological space with basepoint x_0. Suppose there

exists a continuous map $\mu : X \times X \to X$ such that

$$\mu(x, x_0) = \mu(x_0, x) = x.$$

That is, the following diagram commutes strictly:

$$
\begin{array}{ccc}
X & \xrightarrow{i_1} & X \times X \\
{\scriptstyle i_2}\downarrow & {\scriptstyle id_X} \searrow & \downarrow{\scriptstyle \mu} \\
X \times X & \xrightarrow{\mu} & X
\end{array}
\tag{1.2.1}
$$

Then $X = (X, \mu)$ is called an *H-space*, μ is called a *multiplication map* (or an *H-*

structure) on X, and x_0 is called a *(strict) unit or (strict) identity*. We may abbreviate

$\mu(x_1, x_2)$ using concatenation:

$$\mu(x_1, x_2) = x_1 x_2. \tag{1.2.2}$$

We can weaken the condition that X has an identity element x_0 by requiring

the diagram to commute (up to homotopy). In this case, we would say that $X =$

(X, μ) has a *homotopy identity* or *homotopy unit*. However, if X has the homotopy

type of a CW complex, then if (X, μ) has a homotopy identity, there exists a map

$\mu_0 : X \times X \to X$ such that $\mu_0(x, x_0) = \mu_0(x_0, x) = x$ and $\mu \simeq \mu_0$ [40]. Therefore,

unless specified, we will assume that our H-spaces have strict identities.

Given an H-space X with multiplication map μ, we can define the induced

homomorphism on its cohomology over the group $\mathbb{F}_p$ and use the induced homomor-

phism to define a coproduct on $H^*(X; \mathbb{F}_p)$. By definition, μ^* will be a homomor-

phism from $H^*(X; \mathbb{F}_p)$ to $H^*(X \times X; \mathbb{F}_p)$, and since the algebras $H^*(X \times X; \mathbb{F}_p)$ and

$H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p)$ are isomorphic, we have the following composition, which we

will show is a coproduct on $H^*(X; \mathbb{F}_p)$:

$$H^*(X; \mathbb{F}_p) \xrightarrow{\mu^*} H^*(X \times X; \mathbb{F}_p) \xrightarrow{\cong} H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p).$$

We will abuse notation and call this composition μ^* as well, since we may work with

more than one multiplication map on the same space X, and we will need a way to

keep track of which multiplication map induces which coproduct. Now consider the

diagram in the definition of an H-space. If we apply cohomology with coefficients in

$\mathbb{F}_p$, we get

$$H^*(X;\mathbb{F}_p) \xleftarrow{\quad i_1^* \quad} H^*(X;\mathbb{F}_p) \otimes H^*(X;\mathbb{F}_p) \qquad (1.2.3)$$

where

$$i_1^*(x \otimes y) = \begin{cases} 0, & |y| > 0 \\ \\ x, & y = 1 \end{cases}$$

and

$$i_2^*(x \otimes y) = \begin{cases} 0, & |x| > 0 \\ \\ y, & x = 1 \end{cases}$$

Hence the cohomology of an H-space $H^*(X;\mathbb{F}_p) = H^*(X,\mu;\mathbb{F}_p)$ is a Hopf algebra with coproduct μ^* induced by the H-space multiplication map (the reduced coproduct is denoted by $\bar{\mu}^*$).

Since $X = (X,\mu)$ has finite type, then in homology, we have a linear transformation

$$H_*(X;\mathbb{F}_p) \otimes H_*(X;\mathbb{F}_p) \xrightarrow{\ \cong\ } H_*(X \times X;\mathbb{F}_p) \xrightarrow{\ \mu_* \ } H_*(X;\mathbb{F}_p).$$

We will abuse notation and call this composition μ_* as well. This composition defines a product on $H_*(X;\mathbb{F}_p) = H_*(X,\mu;\mathbb{F}_p)$, making it an algebra:

$$\mu_*(\bar{x} \otimes \bar{y}) = \bar{x}\bar{y}.$$

We can obtain a Hopf algebra structure on $H_*(X; \mathbb{F}_p)$ by using the diagonal map Δ_X:

$$H_*(X; \mathbb{F}_p) \xrightarrow{\Delta_{X*}} H_*(X \times X; \mathbb{F}_p) \xrightarrow{\cong} H_*(X; \mathbb{F}_p) \otimes H_*(X; \mathbb{F}_p).$$

We will call this composition Δ_{X*} as well. It is a coproduct on $H_*(X, \mu; \mathbb{F}_p)$ because its dual map in cohomology, Δ_X^*, is a product. Note that since the following diagram commutes (strictly):

$$
\begin{array}{ccc}
X \times X & \xrightarrow{\mu} & X \\
{\scriptstyle \Delta_{X \times X}} \downarrow & & \downarrow {\scriptstyle \Delta_X} \\
X \times X \times X \times X & \xrightarrow{\mu \times \mu} & X \times X
\end{array}
$$

This implies that in homology, the induced linear transformation Δ_{X*} is an algebra homomorphism with respect to the product μ_*.

Remark. The preceding arguments also work for the case when (X, μ) only has a homotopy unit.

Notice that H-spaces require only the existence of an identity (the basepoint); nothing about associativity or existence of multiplicative inverses is mentioned. Let us introduce generalizations of these two conditions, along with a space that satisfies both conditions:

Definition 1.2.3. If the compositions $\mu(\mu \times id_X)$ and $\mu(id_X \times \mu)$ are (basepoint preserving) homotopic maps, then (X, μ) is said to be *homotopy associative*. That is,

the following diagram commutes:

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{\ id_X \times \mu\ } & X \times X \\ {\scriptstyle \mu \times id_X} \downarrow & & \downarrow {\scriptstyle \mu} \\ X \times X & \xrightarrow{\quad \mu \quad} & X \end{array} \qquad (1.2.4)$$

Elementwise, $\mu\left(\mu \times id_X\right)$ is given by

$$\mu\left(\mu \times id_X\right)(x, y, z) = (xy)z, \qquad (1.2.5)$$

and $\mu\left(id_X \times \mu\right)$ is given by

$$\mu\left(id_X \times \mu\right)(x, y, z) = x(yz). \qquad (1.2.6)$$

If the compositions are actually equal, then we call (X, μ) an *associative H-space*.

A map $i : X \to X$ is called a *(two-sided) homotopy inverse operation* for (X, μ) if the compositions $\mu(i \times id_X)\Delta_X$ and $\mu(id_X \times i)\Delta_X$ are homotopic to k. In other

words, the following diagram commutes up to homotopy:

$$X \xrightarrow{\Delta_X} X \times X \xrightarrow{id_X \times i} X \times X \qquad (1.2.7)$$

Elementwise, $\mu(i \times id_X)\Delta_X$ is given by

$$\mu(i \times id_X)\Delta_X(x) = i(x)x, \qquad (1.2.8)$$

and $\mu(id_X \times i)\Delta_X$ is given by

$$\mu(id_X \times i)\Delta_X(x) = xi(x). \qquad (1.2.9)$$

An H-space which is homotopy associative and has a two-sided homotopy inverse operation will be called an *HA-space*.

Remark. Work of James [14], Sibson [32], and Sugawara [37] show that if (X, μ) is a homotopy associative H-space, one can construct a two-sided homotopy inverse operation $i : X \to X$ which satisfies $\mu(i \times id_X)\Delta_X \simeq \mu(id_X \times i)\Delta_X \simeq k$. They also show that the map i is unique up to homotopy in that any other map $j : X \to X$ which satisfies $\mu(j \times id_X)\Delta_X \simeq \mu(id_X \times j)\Delta_X \simeq k$ is homotopic to i [14, 32, 37]. The reader must be aware that many authors like James and Sibson refer to the map i as

a "homotopy inverse," but that term is also used by modern textbooks like Hatcher's [12] in a different context with homotopy equivalences. Hence, we will refer to the map i as a "homotopy inverse operation" (in an analogy with a strict inverse operation in a group) and will often drop "two-sided."

Notice that if Diagrams 1.2.1, 1.2.4, and 1.2.7 actually commute strictly, then (X, μ) would be a topological group. Hence one can think of an HA-space as a generalization of topological groups. However, unlike groups, we cannot state the definitions for H-spaces and HA-spaces in terms of elements due to the ability to use homotopy.

If (X, μ) is an HA-space, then its homology $H_*(X, \mu; \mathbb{F}_p)$ will be an (associative) algebra, and its cohomology $H^*(X, \mu; \mathbb{F}_p)$ is a coassociative Hopf algebra. In the case that (X, μ) is a finite simply-connected HA-space, we have found that by choosing an appropriate generating set for $H^*(X, \mu; \mathbb{F}_p)$, we could simplify our calculations and find patterns that led to our results. We will use Theorem 2.1 from [25] to choose generators whose coproducts have a specified form. We will quote it here to introduce notation as well:

Theorem 1.2.4 (Lin). *Let (X, μ) be a finite simply-connected HA-space. Then every even degree nonzero element of $QH^*(X, \mu; \mathbb{F}_p)$ has a primitive representative in $H^*(X, \mu; \mathbb{F}_p)$ (a primitive generator). Let B be the sub Hopf algebra generated by the even degree primitive generators of $H^*(X; \mathbb{F}_p)$. Let*

$$R = \{r \in H^*(X, \mu; \mathbb{F}_p) : \bar{\mu}^*(r) \in B \otimes H^*(X, \mu; \mathbb{F}_p)\}.$$

The set R is a sub-coalgebra of $H^*(X, \mu; \mathbb{F}_p)$ with $R^{even} = B$, and the vector space R^{odd} is isomorphic to the vector space $QH^{odd}(X, \mu; \mathbb{F}_p)$. All odd degree nonzero elements of $QH^*(X, \mu; \mathbb{F}_p)$ have a representative in $H^*(X, \mu; \mathbb{F}_p)$ whose reduced coproduct lies in $B \otimes R^{odd}$. Hence there is a generating set for $H^*(X, \mu; \mathbb{F}_p)$ whose even degree elements are primitive and whose odd degree elements have reduced coproduct in $B \otimes R^{odd}$.

Definition 1.2.5. We will refer to the generating set of $H^*(X, \mu; \mathbb{F}_p)$ described in the previous theorem as S.

We can also generalize the idea of abelian groups with the next definition:

Definition 1.2.6. Given an H-space (X, μ), if the maps μ and $\mu T_{X,X}$ are homotopic maps, then (X, μ) is a *homotopy commutative* H-space.

If (X, μ) is an HA-space which is also homotopy commutative, then its homology $H_*(X, \mu; \mathbb{F}_p)$ will be a commutative algebra. However, it turns out that when p is odd, many compact simply-connected Lie groups which are not homotopy commutative (the groups A_n, B_n, C_n, D_n, and G_2) have homology algebras $H_*(G, \mu; \mathbb{F}_p)$ which are commutative. One central problem we will address in our results is how we can characterize such spaces and their cohomology over $\mathbb{F}_p$. Before we do so, let us give some examples of H-spaces. First, let us introduce an example that actually motivated Serre to consider a definition for H-spaces in his work on fiber spaces in [31]:

Definition 1.2.7. Let X be a simply-connected topological space with basepoint x_0. Its loop space ΩX is the space of functions $l : [0, 1] \to X$ with the property that

$l(0) = l(1) = x_0$. The space will have the compact-open topology, and its basepoint is the constant loop l_0, where $l_0(t) = x_0$ for all $t \in [0, 1]$. We can define a multiplication map $\mu_c : \Omega X \times \Omega X \to \Omega X$ given by concatenation of loops: if a and b are loops in ΩX given by $a(t)$ and $b(t)$, then for any $t \in [0, 1]$, $\mu_c(a, b)$ is a loop given by

$$\mu_c(a, b)(t) = \begin{cases} a(2t) & 0 \leq t < 1/2 \\ b(2t - 1) & 1/2 \leq t \leq 1 \end{cases}. \tag{1.2.10}$$

We also define a homotopy inverse operation $i_c : \Omega X \to \Omega X$, where for any $t \in [0, 1]$,

$$i_c(a)(t) = a(1 - t). \tag{1.2.11}$$

Then $(\Omega X, \mu_c)$ is an HA-space. Note that it has a homotopy unit rather than a strict unit.

If $X = (X, \mu)$ is an HA-space with homotopy inverse operation i, then we can define a second multiplication map $\mu_\Omega : \Omega X \times \Omega X \to \Omega X$ given by pointwise multiplication of loops. For any $t \in [0, 1]$,

$$\begin{aligned} \mu_\Omega(a, b)(t) &= \mu(a(t), b(t)) \\ &= a(t)b(t), \end{aligned} \tag{1.2.12}$$

along with another homotopy inverse operation $i_\Omega : \Omega X \to \Omega X$ given by pointwise

application of i. For any $t \in [0, 1]$,

$$i_\Omega(a)(t) = i(a(t)). \qquad (1.2.13)$$

Then $(\Omega X, \mu_\Omega)$ is an HA-space. Note that the constant loop l_0 is a strict unit. Furthermore, if (X, μ) is a topological group, then so is $(\Omega X, \mu_\Omega)$.

When $X = (X, \mu)$ is an HA-space, the maps μ_c and μ_Ω are homotopic (see Corollary 1.2.5 of [40]), so the induced coproducts μ_c^* and μ_Ω^* on $H^*(\Omega X; \mathbb{F}_p)$ are the same. Furthermore $(\Omega X, \mu_c)$ is homotopy commutative (see Corollary 1.2.5 of [40]), so $H^*(\Omega X, \mu_c; \mathbb{F}_p) = H^*(\Omega X, \mu_\Omega; \mathbb{F}_p)$ is a cocommutative Hopf algebra, and $H_*(\Omega X, \mu_c; \mathbb{F}_p) = H_*(\Omega X, \mu_\Omega; \mathbb{F}_p)$ is a commutative algebra.

The following definition introduces an important map that allows us to work with pointwise definitions of loops during calculations.

Definition 1.2.8. Let us identify S^1 as the quotient space $[0, 1]/\{0, 1\}$. Given a simply-connected topological space X and its loop space ΩX, we can define the *evaluation map* $\varepsilon : S^1 \wedge \Omega X \to X$ as follows:

$$\varepsilon(t, l) = l(t). \qquad (1.2.14)$$

The evaluation map will induce a homomorphism $\varepsilon^* : H^*(X; \mathbb{F}_p) \to H^*(S^1 \wedge \Omega X; \mathbb{F}_p)$, and since for each $n > 1$, the vector spaces $H^n(S^1 \wedge \Omega X; \mathbb{F}_p)$ and $H^{n-1}(\Omega X; \mathbb{F}_p)$ are

isomorphic, we have a composition of linear transformations in each degree $n > 1$:

$$H^n(X; \mathbb{F}_p) \xrightarrow{\varepsilon^*} H^n(S^1 \wedge \Omega X; \mathbb{F}_p) \xrightarrow{\cong} H^{n-1}(\Omega X; \mathbb{F}_p).$$

(Warning: these linear transformations will lower degree by one) We will collectively denote all of these linear transformations as the *cohomology suspension map* σ^*. Elements in the image of σ^* will be called *suspension elements.*

We also have the related map $\hat{\varepsilon} : S^1 \times \Omega X \to X$ given by $\hat{\varepsilon}(t, l) = l(t)$. It also induces the cohomology suspension map, since the following diagram commutes:

$$
\begin{array}{ccc}
S^1 \times \Omega X & \xrightarrow{\hat{\varepsilon}} & X \\
\downarrow & \nearrow_{\varepsilon} & \\
S^1 \wedge \Omega X & &
\end{array}
$$

In general, if $x \in H^n(X; \mathbb{F}_p)$, then there is a generator $s \in H^1(S^1; \mathbb{F}_p)$ such that

$$\varepsilon^*(x) = s \otimes \sigma^*(x). \tag{1.2.15}$$

As a word of warning, the reader must know that there is another linear transformation called the cohomology suspension, $\sigma : H^n(X; \mathbb{F}_p) \to H^{n-1}(\Omega X; \mathbb{F}_p)$. For example, this map appears in Proposition 2.13 of Kono and Kozima's paper [21]. The following diagram (based on material from Switzer's textbook [38]) relates the

two cohomology suspension maps for $n > 1$:

$$
\begin{array}{ccc}
H^{n-1}(\Omega X; \mathbb{F}_p) & \xleftarrow{\quad id_{\Omega X}^* \quad} & H^{n-1}(\Omega X; \mathbb{F}_p) \\
\cong \big\uparrow & {\scriptstyle \sigma^*} \nwarrow & \big\uparrow {\scriptstyle \sigma} \\
H^n(S^1 \wedge \Omega X; \mathbb{F}_p) & \xleftarrow{\quad \varepsilon^* \quad} & H^n(X; \mathbb{F}_p)
\end{array}
$$

Hence $\sigma^* = \sigma$ for any degree $n > 1$. We refer the reader to Switzer's textbook for further information about σ.

Related to loop spaces are the free loop spaces, which can receive an H-space structure from its base space as follows:

Definition 1.2.9. Let X be a simply-connected topological space. Its free loop space ΛX is the space of functions $l : [0, 1] \to X$ with the property that $l(0) = l(1)$. The space will have the compact-open topology, and its basepoint is the constant loop l_0, where $l_0(t) = x_0$ for all $t \in [0, 1]$. Notice that $\Omega X \subset \Lambda X$.

If $X = (X, \mu)$ is an HA-space, then we can define a multiplication map $\mu_\Lambda : \Lambda X \times \Lambda X \to \Lambda X$ given by pointwise multiplication of loops: for any $t \in [0, 1]$,

$$
\begin{aligned}
\mu_\Lambda(a, b)(t) &= \mu(a(t), b(t)), \\
&= a(t)b(t), \qquad\qquad (1.2.16)
\end{aligned}
$$

and a homotopy inverse operation $i_\Lambda : \Lambda X \to \Lambda X$ defined so that for any $t \in [0, 1]$,

$$
i_\Lambda(a)(t) = i(a(t)). \qquad\qquad (1.2.17)
$$

Then $(\Lambda X, \mu_\Lambda)$ is an HA-space with a strict unit. Furthermore, if (X, μ) is a topological group, then so is $(\Lambda X, \mu_\Lambda)$.

Given a simply-connected topological space X and its free loop space ΛX, we can define the *free evaluation map* $\hat{\varepsilon}_f : S^1 \times \Lambda X \to X$ as follows:

$$\hat{\varepsilon}_f(t, l) = l(t). \tag{1.2.18}$$

The loop space and free loop space are part of an important fibration sequence which we will need for many of our definitions and results:

Definition 1.2.10. Let X be any simply-connected topological space. Let $j : \Omega X \to \Lambda X$ be the inclusion

$$j(l) = l,$$

and let $\varepsilon_0 : \Lambda X \to X$ be evaluation at $t = 0$:

$$\varepsilon_0(\varphi) = \varphi(0).$$

Then we have a fibration sequence which we will call the *free loop fibration*:

$$\begin{array}{c} \Omega X \\ \downarrow{\scriptstyle j} \\ \Lambda X \\ \downarrow{\scriptstyle \varepsilon_0} \\ X \end{array}$$

Now let us discuss an important property of topological groups, loop spaces, and (strictly) associative H-spaces that distinguishes them from spaces which are merely HA-spaces.

Definition 1.2.11. Let Y be a space. If X is a space which satisfies $Y \simeq \Omega X$, then X is called a classifying space of Y.

For example, any loop space has a classifying space. Milnor has shown that given a topological group G, one can construct a space B_G such that $G \simeq \Omega B_G$, so B_G is a classifying space of G [28]. He also shows a reverse procedure: given a loop space $Y = \Omega X$, one can find a topological group G such that $Y \simeq G$. Later, Dold, Lashof, and Milgram find ways to construct a classifying space for any associative H-space: given an associative H-space X, one can find a space B_X such that $X \simeq \Omega B_X$ [7, 27]. Overall, any associative H-space is homotopy equivalent to a loop space, which in turn is homotopy equivalent to a topological group.

However, the same construction cannot be done for HA-spaces in general. As we will see in Section 2.5, HA-spaces may not be associative enough to possess classifying spaces. Thus, a challenge that occurs when generalizing a result from H-spaces with classifying spaces to H-spaces which do not is that we can no longer assume that the space is homotopy equivalent to a topological group with an associative multiplication and a strict inverse operation. Let us illustrate with some concepts from Lie group theory which have found applications in characterizing finite simply-connected topological groups whose homology over $\mathbb{F}_p$ is commutative. The following definitions are due to Kono and Kozima from [21]:

Definition 1.2.12. Let (G, μ) be a finite simply-connected topological group (for example a compact simply-connected Lie group). The *adjoint action of G on itself* $ad : G \times G \to G$ is given by

$$ad(g, h) = ghg^{-1}, \tag{1.2.19}$$

and the *commutator (map) com* $: G \times G \to G$ is given by

$$
\begin{aligned}
com(g, h) &= ghg^{-1}h^{-1} \\
&= (gh)(hg)^{-1}. \tag{1.2.20}
\end{aligned}
$$

The *adjoint action of G on its loop space* $Ad : G \times \Omega G \to \Omega G$ is defined so that given $g \in G$ and $l \in \Omega G$, for any $t \in [0, 1]$,

$$
\begin{aligned}
Ad(g, l)(t) &= gl(t)g^{-1}. \\
&= ad(g, l(t)) \tag{1.2.21}
\end{aligned}
$$

Notice that at $t = 0$,

$$
\begin{aligned}
Ad(g, l)(0) &= gl(0)g^{-1} \\
&= gg_0 g^{-1} \\
&= gg^{-1} \\
&= g_0,
\end{aligned}
$$

so $Ad(g, l)$ is indeed a loop in ΩG. There is also a map we will call $Com : G \times \Omega G \to$ ΩG. It is defined so that given $g \in G$ and $l \in \Omega G$, for any $t \in [0, 1]$,

$$
\begin{aligned}
Com(g, l)(t) &= gl(t)g^{-1}l(t)^{-1} \\
&= (gl(t))(l(t)g)^{-1} \\
&= com(g, l(t)). \quad\quad (1.2.22)
\end{aligned}
$$

Notice that at $t = 0$,

$$
\begin{aligned}
Com(g, l)(0) &= gl(0)g^{-1}l(0)^{-1} \\
&= gg_0 g^{-1} g_0 \\
&= gg^{-1} \\
&= g_0,
\end{aligned}
$$

so $Com(g, l)$ is indeed a loop in ΩG.

Notice how these definitions rely on strict associativity in order to unambiguously multiply more than two elements. Furthermore, notice how strict inverses were needed when we verified that the maps Ad and Com give outputs in ΩG.

Previously, Kono and Kozima have used these maps on Lie groups to find a characterization of compact simply-connected Lie groups whose homology over $\mathbb{F}_p$ is commutative. Let $p_2^{\Omega} : G \times \Omega G \to \Omega G$ be projection onto the second factor. Kono and Kozima prove that the induced homomorphism $Ad^* : H^*(\Omega G; \mathbb{F}_p) \to$

$H^*(G; \mathbb{F}_p) \otimes H^*(\Omega G; \mathbb{F}_p)$ is equal to the induced homomorphism of the projection onto the second factor $p_2^{\Omega*} : H^*(\Omega G; \mathbb{F}_p) \to H^*(G; \mathbb{F}_p) \otimes H^*(\Omega G; \mathbb{F}_p)$ if and only if the algebra $H_*(G, \mu; \mathbb{F}_p)$ is commutative [21]. Iwase proves this result for finite simply-connected loop spaces $(\Omega X, \mu_\Omega)$ by using the fact that loop spaces, despite being homotopy associative and possessing only a homotopy inverse operation, are homotopy equivalent to topological groups [13].

Remark. Kono, Kozima, and Iwase actually use their maps to characterize p-torsion in $H_*(G; \mathbb{Z})$. However, the absence of p-torsion in $H_*(G; \mathbb{Z})$ is equivalent to commutativity of $H_*(G, \mu; \mathbb{F}_p)$ [17].

1.3 Commutator Maps for HA-spaces

We will commence stating the main results of this dissertation. For our first set of results, we will construct a commutator map for HA-spaces (that generalizes the definition of a commutator map for topological groups) and use it to find a new characterization of finite simply-connected HA-spaces whose homology over $\mathbb{F}_p$ is a commutative algebra.

First, let us define our commutator for a simply-connected HA-space.

Definition 1.3.1. Let (X, μ) be a simply-connected HA-space. We define the com-

mutator $com : X \times X \to X$ as

$$
\begin{aligned}
com(x, y) &= \mu\left(\mu(x, y), i(\mu(y, x))\right) \\
&= (xy)i(yx). \qquad\qquad (1.3.1)
\end{aligned}
$$

Now let us look at generalizing Kono and Kozima's definition of $Com : G \times \Omega G \to \Omega G$, where $G = (G, \mu)$ is a simply-connected topological group with identity g_0. Again, notice that at $t = 0$,

$$
\begin{aligned}
Com(g, l)(0) &= gl(0)g^{-1}l(0)^{-1} \\
&= gg_0 g^{-1}g_0 \\
&= gg^{-1} \\
&= g_0,
\end{aligned}
$$

so $Com(g, l)$ is indeed a loop in ΩG. This calculation works because of the strict inverses.

Now consider a simply-connected HA-space (X, μ) with basepoint x_0 and homotopy inverse operation i. Given any $x \in X$ and $l \in \Omega X$, suppose we have a loop $\varphi \in \Lambda X$ given pointwise by

$$
\begin{aligned}
\varphi(t) &= (xl(t))i(l(t)x). \\
&= com(x, l(t)). \qquad\qquad (1.3.2)
\end{aligned}
$$

Is $\varphi \in \Omega X$? Let us plug in $t = 0$:

$$
\begin{aligned}
\varphi(0) &= (xl(0))i(l(0)x) \\
&= (xx_0)i(x_0x) \\
&= xi(x).
\end{aligned}
$$

We must be careful: i is merely a homotopy inverse operation, so it is not necessarily true that $xi(x) = x_0$ for all $x \in X$. Thus, a function which takes a pair $(x, l) \in X \times \Omega X$ to a loop given by $(xl(t))i(l(t)x)$ may give values (loops) in ΛX that might not be in ΩX.

Nevertheless, we can develop a map whose domain is $X \times \Omega X$ and whose codomain is ΩX which generalizes Kono and Kozima's definition of Com. We can use the fibration in Definition 1.2.10:

Theorem 1.3.2. *Let (X, μ) be a simply-connected HA-space. Define a map $\widehat{Com}$: $X \times \Omega X \to \Lambda X$ so that given $x \in X$ and $l \in \Omega X$, for any $t \in [0, 1]$,*

$$
\begin{aligned}
\widehat{Com}(x, l)(t) &= (xl(t))i(l(t)x) \\
&= com(x, l(t)). \tag{1.3.3}
\end{aligned}
$$

Then there exists a map $Com : X \times \Omega X \to \Omega X$ which is unique up to homotopy such that

$$
\widehat{Com} \simeq j \circ Com,
$$

and if $(X\mu)$ *is also a topological group, then this definition of Com will agree with Kono and Kozima's definition.*

Our first result using *com* and *Com* is the computation of the induced homomorphism $com^* : H^*(X, \mu; \mathbb{F}_p) \to H^*(X, \mu; \mathbb{F}_p) \otimes H^*(X, \mu; \mathbb{F}_p)$ for a finite simply-connected HA-space (X, μ).

Lemma 1.3.3. *Let* (X, μ) *be a finite simply-connected HA-space and* p *be an odd prime. If* x *is an element of* S *(see Theorem 1.2.4, then*

$$com^*(x) = \overline{\mu}^*(x) - T^*_{X,X}\overline{\mu}^*(x). \qquad (1.3.4)$$

This lemma serves as a starting point for the main results of this dissertation. From here, we can use this formula for com^* to find a formula for Com^* : $H^*(\Omega X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$ on suspension elements.

Lemma 1.3.4. *Let* (X, μ) *be a finite simply-connected HA-space and* p *be an odd prime. If* x *is an element of* S, *then*

$$Com^*(\sigma^*(x)) = (1 \otimes \sigma^*)\overline{\mu}^*(x). \qquad (1.3.5)$$

Using these formulas, we can find additional characterizations of finite simply-connected HA-spaces whose homology over $\mathbb{F}_p$ is a commutative algebra. This allows us to present a connection between the commutator *com* of an HA-space and commutativity of $H_*(X, \mu; \mathbb{F}_p)$:

Theorem 1.3.5. *Let (X, μ) be a finite simply-connected HA-space and p be an odd prime. Then*

a) the algebra $H_*(X, \mu; \mathbb{F}_p)$ is commutative iff com* is a trivial homomorphism.

b) the algebra $H_*(X, \mu; \mathbb{F}_p)$ is commutative iff Com* is a trivial homomorphism.

These results improve on previous ones by Kono, Kozima, and Iwase, who prove Theorem 1.3.5 for the special cases when (X, μ) is a finite simply-connected loop space, a topological group, or a Lie group. Their results rely on the fact that these kinds of H-spaces either have an associative multiplication and strict inverse, or are homotopy equivalent to a space with those properties (and hence possess a classifying space). We show that only homotopy associativity and existence of a homotopy inverse operation are needed to prove these results.

1.4 The Homology of Free Loop Spaces

We will use Lemmas 1.3.3 and 1.3.4 to compute coproducts in $H^*(\Lambda X; \mathbb{F}_p)$ and determine when $H^*(\Lambda X; \mathbb{F}_p)$ and $H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$ are not only isomorphic as algebras, but also as Hopf algebras.

Definition 1.4.1. Given a simply-connected HA-space (X, μ) with two-sided homotopy inverse operation i, we give the product space $X \times \Omega X$ a multiplication map $\mu_{X \times \Omega X}$, where

$$\mu_{X \times \Omega X} = (\mu \times \mu_\Omega)(id_X \times T_{\Omega X, X} \times id_{\Omega X}). \tag{1.4.1}$$

Pointwise, given $(x_1, l_1), (x_2, l_2) \in X \times \Omega X$, at $t \in [0, 1]$,

$$\mu_{X \times \Omega X}((x_1, l_1), (x_2, l_2)) = (x_1 x_2, l_1(t) l_2(t)). \qquad (1.4.2)$$

We also define a homotopy inverse operation

$$i_{X \times \Omega X} = i \times i_{\Omega}$$

This makes $(X \times \Omega X, \mu_{X \times \Omega X})$ an HA-space.

We focus on a homotopy equivalence $h_1 : \Lambda X \to X \times \Omega X$. First, we will look at the definition of h_1 for topological groups, and then discuss what happens for HA-spaces.

Definition 1.4.2. If (X, μ) is a simply-connected topological group, then we can define $h_1(\varphi) = (\varphi(0), l)$, where $l \in \Omega X$ is the loop given by

$$l(t) = \varphi(0)^{-1} \varphi(t). \qquad (1.4.3)$$

Notice that at $t = 0$,

$$
\begin{aligned}
l(0) &= \varphi(0)^{-1} \varphi(0) \\
&= x_0,
\end{aligned}
$$

so l is indeed an element of ΩX.

In the case that (X, μ) is a simply-connected topological group, we will show that the map h_1 is a homeomorphism.

In the case that (X, μ) is a simply-connected HA-space that is not known to be (homotopy equivalent to) a topological group, then like Com, we cannot write down h_1 pointwise. This is because the loop l given by

$$l(t) = \mu\left(i(\varphi(0)), \varphi(t)\right)$$

may not be an element of ΩX, because at $t = 0$,

$$l(0) = i(\varphi(0))\varphi(0),$$

which might not equal x_0 in general. Instead, we can use a map $h_1' : \Lambda X \to X \times \Lambda X$ to help us define h_1:

Theorem 1.4.3. *Let (X, μ) be a simply-connected HA-space. Define a map $h_1' :$ $\Lambda X \to X \times \Lambda X$ so that given a loop $\varphi \in \Lambda X$, for any $t \in [0, 1]$, h_1' takes a loop $\varphi \in \Lambda X$ to a pair $(\varphi(0), l)$ where $l \in \Lambda X$ is the loop given by*

$$l(t) = \mu\left(i(\varphi(0)), \varphi(t)\right). \tag{1.4.4}$$

Then there exists a map $h_1 : \Lambda X \to X \times \Omega X$ which is unique up to homotopy such that

$$h_1' \simeq (id_X \times j)h_1, \tag{1.4.5}$$

and h_1 is a homotopy equivalence. In particular, if we define $h_2 : X \times \Omega X \to \Lambda X$ by

$$h_2(x,l)(t) = xl(t), \tag{1.4.6}$$

then $h_2 h_1 \simeq id_{\Lambda X}$ and $h_1 h_2 \simeq id_{X \times \Omega X}$.

Hence $H^*(\Lambda X; \mathbb{F}_p)$ and $H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$ are isomorphic as algebras via the isomorphisms

$$h_1^* : H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p) \to H^*(\Lambda X; \mathbb{F}_p),$$

$$h_2^* : H^*(\Lambda X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p),$$

while $H_*(\Lambda X; \mathbb{F}_p)$ and $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$ are isomorphic as vector spaces via the linear transformations

$$h_{1*} : H_*(\Lambda X; \mathbb{F}_p) \to H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p),$$

$$h_{2*} : H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \to H_*(\Lambda X; \mathbb{F}_p).$$

We can ask if h_1 is an H-map; that is, we ask whether the following diagram commutes or not:

$$
\begin{array}{ccc}
\Lambda X \times \Lambda X & \xrightarrow{\ h_1 \times h_1\ } & X \times \Omega X \times X \times \Omega X \\
\Big\downarrow{\scriptstyle \mu_\Lambda} & & \Big\downarrow{\scriptstyle \mu_{X \times \Omega X}} \\
\Lambda X & \xrightarrow[\ h_1\ ]{} & X \times \Omega X
\end{array}
$$

If h_1 is an H-map, then it would induce a Hopf algebra isomorphism

$$h_1^* : H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p) \to H^*(\Lambda X; \mathbb{F}_p),$$

and an algebra isomorphism

$$h_{1*} : H_*(\Lambda X; \mathbb{F}_p) \to H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p).$$

Consequently, we can multiply elements of $H_*(\Lambda X, \mu_\Lambda; \mathbb{F}_p)$ by using h_{1*} and h_{2*} to carry out the multiplication in $H_*(X, \mu; \mathbb{F}_p) \otimes H_*(\Omega X, \mu_\Omega; \mathbb{F}_p)$.

However, it is not clear if h_1 is an H-map in general. In this case, we can use a basic tool from H-space theory, the H-deviation, to study how h_1 preserves the multiplication. The H-deviation of h_1 is a map $D_{h_1} : \Lambda X \times \Lambda X \to X \times \Omega X$, and it is defined (up to homotopy) so that D_{h_1} is nullhomotopic if and only if h_1 is an H-map.

We will find a formula for D_{h_1} in terms of Com which will allow us to compute products in $H_*(\Lambda X, \mu_\Lambda; \mathbb{F}_p)$ using three pieces of information: the product structures of $H_*(X, \mu; \mathbb{F}_p)$ and $H_*(\Omega X, \mu_\Omega; \mathbb{F}_p)$, and knowledge of

$$Com_* : H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \to H_*(\Omega X; \mathbb{F}_p).$$

Lemma 1.4.4. *Let (X, μ) be a simply-connected HA-space. Given any $x \otimes y \in$*

$H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$, we have:

$$\mu_\Lambda^*(h_1^*(x \otimes y)) = \Delta_{\Lambda X}^* \left(D_{h_1}^* \otimes [(h_1^* \otimes h_1^*)(\mu_{X \times \Omega X})^*] \right) (\mu_{X \times \Omega X}^*(x \otimes y)). \qquad (1.4.7)$$

Let $p_2^\Omega : X \times \Omega X \to \Omega X$ be projection onto the second factor, $\varepsilon_0 : \Lambda X \to X$ be defined by $\varepsilon_0(\varphi) = \varphi(0)$ and $j_2 : \Omega X \to X \times \Omega X$ be the inclusion $j_2(l) = (x_0, l)$. Then we can write D_{h_1} as the composition

$$D_{h_1} \simeq j_2 \circ Com \circ (i \times id_{\Omega X}) \circ (\varepsilon_0 \times (p_2^\Omega \circ h_1)) \circ T_{\Lambda X}. \qquad (1.4.8)$$

From here, we can use D_{h_1} to find a multiplication map

$$\tilde{\mu} : X \times \Omega X \times X \times \Omega X \to X \times \Omega X$$

such that the following diagram commutes:

$$
\begin{array}{ccc}
\Lambda X \times \Lambda X & \xrightarrow{\ h_1 \times h_1\ } & X \times \Omega X \times X \times \Omega X \\
{\scriptstyle \mu_\Lambda}\downarrow & & \downarrow{\scriptstyle \tilde{\mu}} \\
X & \xrightarrow{\ h_1\ } & X \times \Omega X
\end{array}
\qquad (1.4.9)
$$

Theorem 1.4.5. *Let* $\omega : X \times \Omega X \to \Omega X$ *be given by*

$$
\begin{aligned}
\omega(x, l) &= Com \circ (i \times id_{\Omega X})(x, l) \\
&= Com(i(x), l).
\end{aligned}
$$

Then the following choice of $\tilde{\mu} : X \times \Omega X \times X \times \Omega X \to X \times \Omega X$ will make diagram

1.4.9 commute:

$$\tilde{\mu} = (id_X \times \mu_\Omega)(\mu \times \omega \times \mu_\Omega)(id_X \times \Delta_X \times \Delta_{\Omega X} \times id_{\Omega X})(id_X \times T_{\Omega X, X} \times id_{\Omega X}),$$

$$\tilde{\mu}(x_1, l_1, x_2, l_2) = (\mu(x_1, x_2), \mu_\Omega\left(Com(i(x_2), l_1), \mu_\Omega(l_1, l_2)\right)).$$

With these results, we can we can use Com^* to determine when $H^*(\Lambda X, \mu_\Lambda; \mathbb{F}_p)$

and $H^*(X, \mu; \mathbb{F}_p) \otimes H^*(\Omega X, \mu_\Omega; \mathbb{F}_p)$ are isomorphic as Hopf algebras in the case that

(X, μ) is a finite simply-connected HA-space.

Theorem 1.4.6. *Let (X, μ) be a finite simply-connected HA-space. Then*

a) the homomorphism $D^*_{h_1}$ is trivial iff Com^* is a trivial homomorphism.

b) the algebra $H_*(X, \mu; \mathbb{F}_p)$ is commutative iff

$$h_1^* : H^*(X \times \Omega X, \mu_{X \times \Omega X}; \mathbb{F}_p) \to H^*(\Lambda X, \mu_\Lambda; \mathbb{F}_p)$$

is a Hopf algebra isomorphism.

We can use Theorem 1.4.5 and Com_* to compute products in $H_*(\Lambda X, \mu_\Lambda; \mathbb{F}_p)$.

We will study the multiplication map

$$\mu_{\Lambda *} : H_*(\Lambda X; \mathbb{F}_p) \otimes H_*(\Lambda X; \mathbb{F}_p) \to H_*(\Lambda X; \mathbb{F}_p)$$

in the homology of $(\Lambda X, \mu_\Lambda)$. Given

$$\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u} \in H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \otimes H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p),$$

we have

$$\mu_{X \times \Omega X}((\bar{x} \otimes \bar{t}), (\bar{y} \otimes \bar{u})) = \bar{x}\bar{y} \otimes \bar{t}\bar{u},$$

where $\mu_*(\bar{x} \otimes \bar{y}) = \bar{x}\bar{y}$ and $\mu_{\Omega *}(\bar{t} \otimes \bar{u}) = \bar{t}\bar{u}$. In contrast, the product of $(\bar{x} \otimes \bar{t}) \in H_*(\Lambda X; \mathbb{F}_p)$ and $(\bar{y} \otimes \bar{u}) \in H_*(\Lambda X; \mathbb{F}_p)$ can be more complicated. The following theorem gives an idea of what can happen; we will restate and prove the theorem in more generality in Chapter 4:

Theorem 1.4.7. *Let $\bar{x}, \bar{y} \in H_*(X; \mathbb{F}_p)$, $\bar{t}, \bar{u} \in H_*(\Omega X; \mathbb{F}_p)$ all have positive degree and be primitive. Then*

$$\mu_{\Lambda X *}((h_{2*} \otimes h_{2*})(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u})) = h_{2*}[\bar{x} \otimes Com_*(i_*(\bar{y}) \otimes \bar{t})\bar{u}$$

$$+ \bar{x}\bar{y} \otimes \bar{t}\bar{u}].$$

Dually, in cohomology, we can use Lemma 1.4.4, Theorem 1.4.5, and Lemma 1.3.4 to compute coproducts of elements in the image of

$$h_1^* p_2^{\Omega *} \sigma^* : H^*(X; \mathbb{F}_p) \to H^*(\Lambda X; \mathbb{F}_p).$$

Theorem 1.4.8. *Let $x \in S$. Then*

$$\mu_\Lambda^*(h_1^*(1 \otimes \sigma^*(x))) = (h_1^* \otimes h_1^*)[1 \otimes \sigma^*(x) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes \sigma^*(x)$$

$$+1 \otimes T_{\Omega X, X}^*[((i^* \otimes \sigma^*)(\bar{\mu}^*(x))] \otimes 1].$$

Kono and Kozima briefly mention an application for their work describing the difference between the group structure of $(\Lambda G, \mu_\Lambda)$ and that of $(G \times \Omega G, \mu_{G \times \Omega G})$. They define a homeomorphism of topological spaces $\Phi : \Omega G \times G \to \Lambda G$, where $\Phi(l, g)$ is the loop given by $l(t)g$. If Φ is a group homomorphism, then the following diagram would strictly commute:

$$
\begin{array}{ccc}
\Omega G \times G \times \Omega G \times G & \xrightarrow{(\mu_\Omega \times \mu)(id_{\Omega G} \times T \times id_G)} & \Omega G \times G \\
{\scriptstyle \Phi \times \Phi}\Big\downarrow & & \Big\downarrow{\scriptstyle \Phi} \\
\Lambda G \times \Lambda G & \xrightarrow[\mu_\Lambda]{} & \Lambda G
\end{array}
$$

In general, however, Φ is not known to be a group homomorphism unless (G, μ) is abelian [13]. Instead, Kono and Kozima present a modified diagram which will always commute. First, they define a map

$$\omega : \Omega G \times G \times \Omega G \times G \to \Omega G \times \Omega G \times G \times G$$

given by the composition involving the adjoint action Ad:

$$\omega = (id_{\Omega G} \times T_{G, \Omega G} \times id_G)\,(id_{\Omega G \times G} \times Ad \times id_G)\,(id_{\Omega G} \times \Delta_G \times id_{\Omega G \times G})\,;$$

for any $t \in [0, 1]$,

$$
\begin{aligned}
\omega(l_1, g_1, l_2, g_2)(t) &= (l_1, Ad(g_1, l_2), g_1, g_2)(t) \\
&= (l_1(t), g_1 l_2(t) g_1^{-1}, g_1, g_2).
\end{aligned}
$$

Then the following diagram strictly commutes whether or not Φ is a group homomorphism:

$$
\begin{array}{ccccc}
\Omega G \times G \times \Omega G \times G & \xrightarrow{\ \omega\ } & \Omega G \times \Omega G \times G \times G & \xrightarrow{\ \mu_\Omega \times \mu\ } & \Omega G \times G \\
{\scriptstyle \Phi \times \Phi} \downarrow & & & & \downarrow {\scriptstyle \Phi} \\
\Lambda G \times \Lambda G & & \xrightarrow{\hspace{4cm} \mu_\Lambda \hspace{4cm}} & & \Lambda G
\end{array}
$$

According to this diagram, if the coproduct structure of $H^*(G, \mu; \mathbb{F}_p)$, the coproduct structure of $H^*(\Omega G, \mu_\Omega; \mathbb{F}_p)$, and values of the homomorphism Ad^* are known, then we can use those pieces of information to compute coproducts in $H^*(\Lambda G, \mu_\Lambda; \mathbb{F}_p)$. However, Kono, Kozima, and Iwase do not offer any results in this direction beyond mentioning the diagram. Hamanaka applies homology to this diagram to compute products in $H_*(\Lambda G_2; \mathbb{F}_2)$ using his calculations of Ad_*, but remarks that for the other exceptional Lie groups, "it is complicated to write them out exactly" [9].

1.5   Constructing New H-spaces

Given a finite simply-connected HA-space (X, μ) and an odd prime p, we will find an explicit form for a map $\nu : X \times X \to X$ so that ν induces a new multiplication

map on $H_*(X; \mathbb{F}_p) = H_*(X, \nu; \mathbb{F}_p)$ which makes it a commutative (but possibly nonassociative) algebra. We also find conditions on the original multiplication map μ and its induced homomorphism which make $H_*(X, \nu; \mathbb{F}_p)$ commutative and associative as well.

One interesting aspect of H-space theory is that we do not restrict our spaces to just one multiplication map. The idea of finding alternate multiplication maps on H-spaces which induced given conditions on cohomology or homology comes from Harper and Zabrodsky's work [11]. Their process of constructing new multiplication maps from old ones is called altering the H-structure. Their technique can prove that the underlying topological spaces of the Lie groups (F_4, μ) and (E_6, μ) have a multiplication map ν which induces a commutative multiplication map on their homology algebras over $\mathbb{F}_3$. We adapted their methods to finite simply-connected HA-spaces (X, μ) with homotopy inverse operation i to find an explicit form for a multiplication map ν which makes $H_*(X, \nu; \mathbb{F}_p)$ a commutative (if not associative) algebra:

Theorem 1.5.1. *Let (X, μ) be a finite simply-connected HA-space and p be a fixed odd prime. Define an H-structure $\nu : X \times X \to X$ as a product*

$$\nu(x, y) = \underbrace{((com(x, y) \ldots) com(x, y))}_{\frac{p-1}{2} \ times} \mu(x, y) \qquad (1.5.1)$$

where concatenation means multiply using μ. Then (X, ν) is an H-space for which

$H_*(X, \nu; \mathbb{F}_p)$ *is a commutative algebra. Furthermore, if x is an element of S, then*

$$\nu^*(x) = \frac{1}{2} \left(\mu^*(x) + T_{X,X}^* \mu^*(x) \right). \qquad (1.5.2)$$

We can also ask when $H_*(X, \nu; \mathbb{F}_p)$ is associative. In fact, we look at certain cases when $H_*(X, \nu; \mathbb{F}_p)$ turns out to be commutative and associative, and determine a way to measure lack of associativity in terms of the original coproduct μ^*:

Theorem 1.5.2. *Let (X, μ) be a finite simply-connected HA-space. Let ν be defined as in equation 1.5.1. The Hopf algebra $H_*(X, \nu; \mathbb{F}_p)$ is associative if and only if in $H^*(X, \mu; \mathbb{F}_p)$,*

$$\frac{1}{4} \left((\mu^* \otimes 1)T^*\mu^* + (T^*\mu^* \otimes 1)\mu^* - (1 \otimes \mu^*)T^*\mu^* - (1 \otimes T^*\mu^*)\mu^* \right) = 0. \qquad (1.5.3)$$

Finally, if ν^* is coassociative, then $H^*(X, \nu; \mathbb{F}_p)$ is primitively generated as a Hopf algebra with coproduct ν^*.

1.6   Outline of the Dissertation

Our first main goal is to generalize *Com* and *Ad* beyond Lie groups and topological groups. In Chapter 2, we will review and present properties of H-spaces, HA-spaces, and their cohomology. We start off with a description of some common techniques we will use in our calculations that involve (free) loop spaces, products of spaces, homotopy associativity, and homotopy inverse operations. Next, we will col-

lect important results about the cohomology of finite H-spaces in Section 2.2. After that, we will look at HA-spaces and find properties of the homotopy inverse operation which parallel properties of a group inverse operation. From there, we discuss maps between H-spaces and introduce the H-deviation of a map, a way of measuring how maps between H-spaces preserve the multiplication on the spaces. Once we have presented this background material, then in Chapter 3, we discuss Kono, Kozima, and Iwase's results and generalize them to finite simply-connected HA-spaces. We start with some properties of the free loop fibration, and then use it to give definitions of the maps com, ad, Com, and Ad that agree with Kono and Kozima's definitions when dealing with Lie groups and topological groups. From there, we prove Lemma 1.3.3, Lemma 1.3.4, and Theorem 1.3.5. Then, in Chapter 4, we will introduce and study the map h_1 and its H-deviation D_{h_1}, along with their induced homomorphisms in cohomology. This analysis allows us to prove Lemma 1.4.4, Theorem 1.4.5 and Theorem 1.4.6. We also use Theorem 1.4.5 to compute products in $H_*(\Lambda X, \mu_\Lambda; \mathbb{F}_p)$ and obtain a generalization of Theorem 1.4.7. Finally, in Chapter 5, we give a brief introduction to alterations of H-structures: creating new multiplication maps on a space using an old one. Then we use Lemma 1.3.3 to prove Theorem 1.5.1, and afterward, we look at some examples and discuss associativity of our new multiplication map (and the product structure it induces in homology), proving Theorem 1.5.2 along the way.

2 Properties of H-spaces and HA-spaces

This chapter will cover some additional background material that we will need in this dissertation. First, we will describe the techniques we will be using in many of our calculations, especially those which involve maps whose codomain is a (free) loop space or a product of spaces. In the next section, we will go over some important properties of the cohomology of finite H-spaces and their loop spaces. We review some results from Lin's work on generators of the cohomology of finite H-spaces in [24], and collect some important properties of the cohomology suspension map defined in Section 1.2. Afterward, we will study HA-spaces and focus on maps whose codomain is an HA-space. This will allow us to give some additional properties of the homotopy inverse operation on an HA-space. Then, we will discuss maps between H-spaces and how we will deal with maps that do not preserve the multiplication. After that, we give a brief discussion of measuring associativity on an H-space using A_n-forms.

Remark. From now on, we will abbreviate the H-space (X, μ) as just X, unless we are considering multiple multiplication maps on the same space.

2.1 Loop Spaces, Products, and Homotopy

This first section presents elementary material, which we collect here for convenience. These techniques will be used in many of our more involved calculations, and as such, we feel it is important to describe them.

Throughout Chapters 3 and 4, we will be working with maps whose codomain is a loop space or a free loop space. Oftentimes, it will be convenient to work with pointwise definitions of the map (where we evaluate the loops at $t \in S^1$) during calculations. The following lemmas will justify this practice.

Lemma 2.1.1. *Let X and Y be simply-connected topological spaces and $g_1, g_2 : X \rightarrow \Lambda Y$. Let $\hat{\varepsilon}_f : S^1 \times \Lambda Y \rightarrow Y$ be the free evaluation map given by*

$$\hat{\varepsilon}_f(t, \varphi) = \varphi(t).$$

Consider the compositions $\hat{\varepsilon}_f(id_{S^1} \times g_1)$, $\hat{\varepsilon}_f(id_{S^1} \times g_2) : S^1 \times X \rightarrow Y$. Elementwise, these are given by

$$\hat{\varepsilon}_f(id_{S^1} \times g_1)(t, x) = g_1(x)(t),$$

$$\hat{\varepsilon}_f(id_{S^1} \times g_2)(t, x) = g_2(x)(t).$$

Suppose that

$$\hat{\varepsilon}_f(id_{S^1} \times g_1) \simeq \hat{\varepsilon}_f(id_{S^1} \times g_2).$$

Then

$$g_1 \simeq g_2.$$

Proof. Since

$$\hat{\varepsilon}_f(id_{S^1} \times g_1) \simeq \hat{\varepsilon}_f(id_{S^1} \times g_2),$$

there exists a map $F : S^1 \times X \times I \to Y$ such that

$$
\begin{aligned}
F(t, x, 0) &= \hat{\varepsilon}_f(id_{S^1} \times g_1)(t, x) = g_1(x)(t), \\
F(t, x, 1) &= \hat{\varepsilon}_f(id_{S^1} \times g_2)(t, x) = g_2(x)(t).
\end{aligned}
\tag{2.1.1}
$$

Define a map $G : X \times I \to \Lambda Y$ so that $G(x, u)$ is a loop given by

$$G(x, u)(t) = F(t, x, u).$$

Then according to equation 2.1.1,

$$
\begin{aligned}
G(x, 0) &= g_1(x), \\
G(x, 1) &= g_2(x),
\end{aligned}
$$

so G is a homotopy from g_1 to g_2. $\qquad\square$

A similar proof can be used to show a corresponding result for maps into a loop space:

Lemma 2.1.2. *Let X and Y be simply-connected topological spaces and $g_1, g_2 : X \to$*

ΩY. Let $\varepsilon : S^1 \wedge \Omega Y \to Y$ be the evaluation map given by

$$\varepsilon(t, \varphi) = \varphi(t).$$

Consider the compositions $\varepsilon(id_{S^1} \wedge g_1)$, $\varepsilon(id_{S^1} \wedge g_2) : S^1 \wedge X \to Y$. Elementwise,

these are given by

$$\varepsilon(id_{S^1} \wedge g_1)(t, x) \;=\; g_1(x)(t),$$

$$\varepsilon(id_{S^1} \wedge g_2)(t, x) \;=\; g_2(x)(t).$$

Suppose that

$$\varepsilon(id_{S^1} \wedge g_1) \simeq \varepsilon(id_{S^1} \wedge g_2).$$

Then

$$g_1 \simeq g_2.$$

Next, let us explain how we work with maps into a product of spaces. Given

topological spaces X, Y, and Z, and a map $f : X \to Y \times Z$, we will use projections

$p_1 : Y \times Z \to Y$ and $p_2 : Y \times Z \to Z$ to break down the map f into its components:

$$f(x) = (p_1 f(x), p_2 f(x)) .$$

We can then perform calculations on each component separately in order to determine

whether the map f is equal to or homotopic to some other map.

Lemma 2.1.3. *Let X, Y, Z be topological spaces and $f, g : X \to Y \times Z$ be maps. Let $p_1 : Y \times Z \to Y$ and $p_2 : Y \times Z \to Z$ be the projections. If*

$$p_1 f \simeq p_1 g$$

and

$$p_2 f \simeq p_2 g,$$

then

$$f \simeq g.$$

Proof. Let $H_1 : I \times X \to Y$ be a homotopy from $p_1 f$ to $p_1 g$, and let $H_2 : I \times X \to Z$ be a homotopy from $p_2 f$ to $p_2 g$. That is,

$$
\begin{aligned}
H_1(0, x) &= p_1 f(x), \\
H_1(1, x) &= p_1 g(x), \\
H_2(0, x) &= p_2 f(x), \\
H_2(1, x) &= p_2 g(x).
\end{aligned}
$$

Define $H : I \times X \to Y \times Z$ by

$$H(t, x) = (H_1(t, x), H_2(t, x)) . \tag{2.1.2}$$

Then

$$
\begin{aligned}
H(0,x) &= (H_1(0,x), H_2(0,x)) \\
&= (p_1 f(x), p_2 f(x)) \\
&= f(x),
\end{aligned}
$$

and

$$
\begin{aligned}
H(1,x) &= (H_1(1,x), H_2(1,x)) \\
&= (p_1 g(x), p_2 g(x)) \\
&= g(x),
\end{aligned}
$$

so H is a homotopy from f to g. $\qquad\square$

Remark. In the language of category theory, if we work in the category of pointed topological spaces with morphisms being (based) homotopy classes of maps, then the previous lemma states that the Cartesian product of two pointed topological spaces is a product in the category.

Let us end this section with an example on working with homotopy associativity and homotopy inverse operations for an HA-space (X, μ) with homotopy inverse operation $i : X \to X$.

Example 2.1.4. Let X be an HA-space and $f : X \times X \to X$ be defined by

$$f(x, y) = (xy)\, i(y),$$

and let $p_1 : X \times X \to X$ be projection onto the first factor:

$$p_1(x, y) = x.$$

We want to show that

$$f \simeq p_1.$$

We can expand $f(x, y)$ as

$$f(x, y) = \mu_X(\mu_X \times id_X)(id_X \times id_X \times i)(id_X \times \Delta_X)(x, y). \qquad (2.1.3)$$

We can use homotopy associativity: $\mu_X(\mu_X \times id_X) \simeq \mu_X(id_X \times \mu_X)$, to see that f is homotopic to a map which takes (x, y) to

$$\mu_X(id_X \times \mu_X)(id_X \times id_X \times i)(id_X \times \Delta_X)(x, y)$$

$$= \mu_X \left(id_X \times (\mu_X(id_X \times i)\Delta_X) \right)(x, y)$$

$$= x\, (i(y)y) \qquad (2.1.4)$$

By Definition 1.2.3,

$$\mu_X(id_X \times i)\Delta_X \simeq k,$$

so f is homotopic to a map which takes (x, y) to

$$
\begin{aligned}
\mu_X \left(id_X \times k \right) (x, y) &= \mu_X(x, x_0) \\
&= x \\
&= p_1(x, y).
\end{aligned}
$$

Therefore, $f \simeq p_1$.

As we have seen, we can use homotopy associativity to move the parentheses and obtain equation 2.1.4, and apply the homotopy inverse operation to "cancel" the product $i(y)y$ that appeared in equation 2.1.4. In full detail, we had to expand $f(x, y)$ in equation 2.1.3 in order to identify $\mu_X(\mu_X \times id_X)$ in the composition, and use homotopy associativity to make $\mu_X(id_X \times i)\Delta_X$ appear in the composition so we could replace it with k.

In practice, the process of expanding compositions with μ and i can get cumbersome, especially when more elements are being multiplied (we might have to use homotopy associativity multiple times and include numerous id_X factors). Therefore, in Chapters 3 and 4, we will refrain from writing expansions like the one in equation 2.1.3, so that calculations will look more like the next example:

Example 2.1.5. Let X be an HA-space and $f : X \times X \to X$ be defined by

$$
f(x, y) = (xy) \, i(y),
$$

and let $p_1 : X \times X \to X$ be projection onto the first factor:

$$p_1(x, y) = x.$$

We want to show that

$$f \simeq p_1.$$

We can use homotopy associativity to see that f is homotopic to a map which takes (x, y) to $x\,(i(y)y)$. Hence f is homotopic to a map which takes (x, y) to

$$x = p_1(x, y).$$

Therefore, $f \simeq p_1$.

2.2   H-spaces and their Cohomology

In Section 1.1, we saw that there are only three kinds of monogenic Hopf algebras (recall that all algebras have $\mathbb{F}_p$, p odd, as their base field), and any other Hopf algebra is isomorphic (as an algebra) to a tensor product of monogenic Hopf algebras. One can ask which Hopf algebras occur as the cohomology of some H-space.

Example 2.2.1. Let us look at exterior algebras first. Given any odd natural number n, Adams shows that there is an H-space X (which might not be a topological group), called an odd sphere localized at the odd primes, such that the cohomology of X over

$\mathbb{F}_p$ is

$$H^*(X, \mathbb{F}_p) = \wedge(x_n),$$

and X itself is a CW complex of finite type (finite number of cells in each dimension) that has the odd sphere S^n as a subcomplex [1]. Hence given an odd prime p, any exterior algebra (with any number of generators in any odd degree) over $\mathbb{F}_p$ can be realized as the cohomology of an H-space, namely the product of some odd spheres localized at the odd primes.

In contrast, when dealing with a finite H-space X, the possible degrees of even degree generators in $H^*(X; \mathbb{F}_p)$ are much more limited. Lin's Binary Theorem from [24] describes the degrees in which even degree generators can occur. Let us quote part of it:

Theorem 2.2.2 (Lin). *Let X be a finite simply-connected H-space, and let m be a positive integer. Then $Q^{2m}H^*(X; \mathbb{F}_p) = 0$ unless $m = \sum_{s=1}^{j} m_s p^s + 1$, where the m_s coefficients are either zero or one, and at most one of the m_s coefficients is zero (unless $j = 1$, in which case $m_1 = 1$).*

Lin uses the Binary Theorem to prove the following result about the cohomology of ΩX and the cohomology suspension map σ^*:

Lemma 2.2.3 (Lin). *Let X be a finite simply-connected H-space. Then*

$$H^{odd}(\Omega X; \mathbb{F}_p) = 0$$

and σ^* vanishes on even degree elements of $H^*(X; \mathbb{F}_p)$.

In general, we have the following facts about the cohomology suspension map σ^* from [39] and [6] respectively:

Theorem 2.2.4. *Let X be a simply-connected H-space.*

*1) (**Whitehead**) Decomposables in $H^*(X; \mathbb{F}_p)$ vanish on the cohomology suspension map, and the image of σ^* is contained in $PH^*(X; \mathbb{F}_p)$.*

*2) (**Clark**) Moreover, for $n > 1$, if we look at the linear transformations*

$$Q^n H^*(X; \mathbb{F}_p) \to P^{n-1} H^*(\Omega X; \mathbb{F}_p),$$

induced by σ^*, then if n is odd, then this induced linear transformation is injective on $Q^n H^*(X; \mathbb{F}_p)$, and unless $n - 1 \equiv -2 \mod 2p$, it is surjective onto $P^{n-1} H^*(X; \mathbb{F}_p)$.

In the case that X is a finite simply-connected HA-space and S is the generating set from Theorem 1.2.4, σ^* is injective on the set of odd degree elements of S.

2.3 HA-spaces and their Cohomology

Let (X, μ) be an HA-space with homotopy inverse operation i. We will introduce some properties of i, along with its induced homomorphism on cohomology. To do this, we need to look at homotopy classes of maps whose codomain is an HA-space. While it has been known that the set of (basepoint-preserving) homotopy classes of

maps whose codomain is an HA-space has a group structure (see James' paper [14] and Zabrodsky's book [40]), we would like to discuss this group structure in more detail and provide our own presentation.

For any space Y and any HA-space X, given a map $f : Y \to X$, let us denote the homotopy class of f by $[f]$. That is,

$$[f] = [g] \text{ if and only if } f \simeq g.$$

Let $\langle Y, X \rangle$ be the set of (basepoint-preserving) homotopy classes of maps from Y to X (our notation is due to Hatcher from [12]). Let us show that the HA-space structure of X can be used to define a group structure on $\langle Y, X \rangle$. First, we will work with the functions (from Y to X) themselves, and define a binary operation on functions from Y to X:

Definition 2.3.1. Let Y be any space and X be an H-space. Given $f : Y \to X$ and $g : Y \to X$, we define a function $f * g : Y \to X$ by

$$f * g = \mu \left(f \times g \right) \Delta_Y.$$

Elementwise, this means that for any $y \in Y$,

$$(f * g)(y) = f(y)g(y).$$

Let us show how the homotopy associativity of X will impose an associativity

condition on the binary operation $*$:

Proposition 2.3.2. *Given $f : Y \to X$, $g : Y \to X$, and $h : Y \to X$, the binary operation $*$ satisfies*

$$(f * g) * h \simeq f * (g * h).$$

Proof. By definition,

$$
\begin{aligned}
(f * g) * h &= \mu \left((f * g) \times h \right) \Delta_Y \\
&= \mu \left(\left(\mu \left(f \times g \right) \Delta_Y \right) \times h \right) \Delta_Y.
\end{aligned}
\tag{2.3.1}
$$

Elementwise, this map takes an element $y \in Y$ to $(f(y)g(y))h(y) \in X$, so equation 2.3.1 becomes

$$(f * g) * h = \mu \left(\mu \times id_X \right) \left(f \times g \times h \right) \left(id_Y \times \Delta_Y \right) \Delta_Y.
\tag{2.3.2}$$

Since X is homotopy associative, $\mu \left(\mu \times id_X \right) \simeq \mu \left(id_X \times \mu \right)$. Thus, we have

$$(f * g) * h \simeq \mu \left(id_X \times \mu \right) \left(f \times g \times h \right) \left(id_Y \times \Delta_Y \right) \Delta_Y.
\tag{2.3.3}$$

Elementwise, this map takes y to $f(y)(g(y)h(y))$, so the right hand side of equation

2.3.3 becomes

$$(f * g) * h \;\; \simeq \;\; \mu \left(f \times \left(\mu \left(g \times h \right) \Delta_Y \right) \right) \Delta_Y$$

$$= \;\; \mu \left(f \times \left(g * h \right) \right) \Delta_Y$$

$$= \;\; f * (g * h). \tag{2.3.4}$$

$\square$

The binary operation passes to homotopy classes in $\langle Y, X \rangle$ as follows:

$$[f] * [g] = [f * g]. \tag{2.3.5}$$

In particular, by the previous proposition,

$$([f] * [g]) * [h] \;\; = \;\; [f * g] * [h]$$

$$= \;\; [(f * g) * h]$$

$$= \;\; [f * (g * h)]$$

$$= \;\; [f] * [g * h]$$

$$= \;\; [f] * ([g] * [h]), \tag{2.3.6}$$

so $*$ is an associative binary operation on $\langle Y, X \rangle$. In fact, it gives a group structure on $\langle Y, X \rangle$:

Proposition 2.3.3. *Given an HA-space X and any space Y, $\langle Y, X \rangle$ is a group with*

binary operation $*$.

Proof. Let us show that the identity element of $\langle Y, X \rangle$ is the homotopy class of the constant map $k : Y \to X$. To do this, we look at the map $f * k$. In X,

$$(f * k)(y) = (\mu (f \times k) \Delta_Y)(y) = \mu (f(y), x_0) = f(y),$$

so $f * k = f$. Similarly, $k * f = f$, so in $\langle Y, X \rangle$,

$$[f] * [k] = [k] * [f] = [f]. \tag{2.3.7}$$

Inverse elements are defined using the homotopy inverse operation $i : X \to X$. Notice that

$$
\begin{aligned}
f * (i \circ f) &= \mu (f \times (i \circ f)) \Delta_Y \\
&= \mu (id_X \times i) (f \times f) \Delta_Y \\
&= \mu (id_X \times i) \Delta_X f,
\end{aligned}
$$

and the last map is nullhomotopic since $\mu (id_X \times i) \Delta_X$ is nullhomotopic. Similarly, $(i \circ f) * f \simeq k$, so in $\langle Y, X \rangle$,

$$[f] * [i \circ f] = [i \circ f] * [f] = [k]. \tag{2.3.8}$$

Hence we will denote $[i \circ f]$ by $[f]^{-1}$.

Equations 2.3.7 and 2.3.8, along with Proposition 2.3.2, show that $\langle Y, X \rangle$ is a group under the operation $*$ with identity $[k]$. $\qquad\square$

Here are some properties of the homotopy inverse operation i that are analogous to properties of inversion in a group, namely, that in a group G, $(g^{-1})^{-1} = g$ and $(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$. They have been used throughout the literature, but we were unable to find a reference, so for convenience, we have provided a proof using homotopy classes of maps into an HA-space. Our proof exploits the fact that $\langle Y, X \rangle$ is a group, so the properties of the group inverse operation in $\langle Y, X \rangle$ can be used to prove statements about maps being homotopic.

Lemma 2.3.4. *In X, $i \circ i \simeq id_X$ and $i \circ \mu \simeq \mu(i \times i)T_{X,X}$.*

Proof. Recall that in any group G, $(g^{-1})^{-1} = g$. Since $\langle X, X \rangle$ is a group, for any class $[f]$ (where f is a map $f : X \to X$), we have

$$([f]^{-1})^{-1} = [f].$$

In particular,

$$[i \circ i] = [i]^{-1} = ([id_X]^{-1})^{-1} = [id_X],$$

so $i \circ i \simeq id_X$.

Recall that in any group G, $(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$. Since $\langle X \times X, X \rangle$ is a group,

for any classes $[f]$ and $[g]$, we have

$$([f] * [g])^{-1} = [g]^{-1} * [f]^{-1}.$$

Let $f = p_1$ and $g = p_2$, where $p_1, p_2 : X \times X \to X$ are projection maps. Then we have

$$(p_1 * p_2)(x, y) = \mu(p_1(x, y), p_2(x, y)) = \mu(x, y), \text{ and}$$

$$
\begin{aligned}
((i \circ p_2) * (i \circ p_1))(x, y) &= \mu(i(p_2(x, y)), i(p_1(x, y))) \\
&= \mu(i(y), i(x)) \\
&= \mu(i \times i)(y, x), \\
&= (\mu(i \times i)T_{X,X})(x, y),
\end{aligned}
$$

so in $\langle X \times X, X \rangle$,

$$[i \circ \mu] = [\mu]^{-1} = ([p_1] * [p_2])^{-1} = [p_2]^{-1} * [p_1]^{-1} = [\mu(i \times i)T_{X,X}].$$

Therefore, $i \circ \mu \simeq \mu(i \times i)T_{X,X}$. □

Next, we examine i^*, the induced algebra homomorphism on $H^*(X; \mathbb{F}_p)$. In Definition 1.2.3, we see that

$$\mu(i \times id_X)\Delta_X \simeq \mu(id_X \times i)\Delta_X \simeq k. \tag{2.3.9}$$

Thus, if $x \in H^*(X; \mathbb{F}_p)$ has positive degree and $\mu^*(x) = x \otimes 1 + 1 \otimes x + \sum c \otimes d$, then we can apply cohomology to equation 2.3.9 to determine a formula for $i^*(x)$:

Proposition 2.3.5. *Suppose $x \in H^*(X; \mathbb{F}_p)$ has positive degree and $\mu^*(x) = x \otimes 1 + 1 \otimes x + \sum c \otimes d$. Then*

$$i^*(x) = -x - \sum i^*(c)d = -x - \sum c[i^*(d)]. \qquad (2.3.10)$$

Proof. When we apply cohomology to equation 2.3.9, we see that

$$
\begin{aligned}
\Delta_X^*(i^* \otimes 1)\mu^*(x) &= \Delta_X^*(i^* \otimes 1)\left(x \otimes 1 + 1 \otimes x + \sum c \otimes d\right) \\
&= \Delta_X^*\left(i^*(x) \otimes 1 + 1 \otimes x + \sum i^*(c) \otimes d\right) \\
&= i^*(x) + x + \sum i^*(c)d,
\end{aligned}
$$

$$
\begin{aligned}
\Delta_X^*(1 \otimes i^*)\mu^*(x) &= \Delta_X^*(1 \otimes i^*)\left(x \otimes 1 + 1 \otimes x + \sum c \otimes d\right) \\
&= \Delta_X^*\left(x \otimes 1 + 1 \otimes i^*(x) + \sum c \otimes i^*(d)\right) \\
&= x + i^*(x) + \sum c[i^*(d)],
\end{aligned}
$$

so

$$i^*(x) + x + \sum i^*(c)d = x + i^*(x) + \sum c[i^*(d)] = 0,$$

and therefore,

$$i^*(x) = -x - \sum i^*(c)d = -x - \sum c[i^*(d)]. \qquad (2.3.11)$$

In particular,

$$\sum i^*(c)d = \sum c[i^*(d)]. \tag{2.3.12}$$

$\square$

By dualizing to homology, we obtain the following result:

Corollary 2.3.6. *Let* $\bar{x} \in H_*(X; \mathbb{F}_p)$ *have positive degree and* $\Delta_{X*}(\bar{x}) = \bar{x} \otimes 1 + 1 \otimes \bar{x} + \sum \bar{c} \otimes \bar{d}$. *Then*

$$i_*(\bar{x}) = -\bar{x} - \sum i_*(\bar{c})\bar{d} = -\bar{x} - \sum \bar{c}[i_*(\bar{d})]. \tag{2.3.13}$$

In addition, equation 2.3.11 tells us that the map i induces multiplication by -1 on $QH^*(X; \mathbb{F}_p)$.

Example 2.3.7. If $x \in H^*(X; \mathbb{F}_p)$ is primitive, then $i^*(x) = -x$. In general, if x is a product of primitive classes $x_1, \ldots, x_n$,

$$i^*(x) = i^*(x_1, \ldots, x_n) = i^*(x_1) \ldots i^*(x_n) = (-1)^n x.$$

In general, i^* may not be multiplication by -1 on generators.

Example 2.3.8. Let $X = E_8$ and $p = 3$. For reference,

$$H^*(E_8; \mathbb{F}_3) \cong \wedge (x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47}) \otimes \mathbb{F}_p[x_8, x_{20}] / \left(x_8^3, x_{20}^3\right).$$

In particular, we have the following coproducts [10]:

$$\mu^*(x_j) = x_j \otimes 1 + 1 \otimes x_j, \ \text{where } j = 3, 7, 8, 19, 20,$$

$$\mu^*(x_{15}) = x_{15} \otimes 1 + 1 \otimes x_{15} + x_8 \otimes x_7,$$

$$\mu^*(x_{35}) = x_{35} \otimes 1 + 1 \otimes x_{35} + x_8 \otimes x_{27} - x_8^2 \otimes x_{19} + x_{20} \otimes x_{15} + x_8 x_{20} \otimes x_7.$$

Since x_i is primitive for $j = 3, 7, 8, 19, 20$, we have $i^*(x_j) = -x_j$ in those cases. For x_{15} and x_{35}, we have

$$
\begin{aligned}
i^*(x_{15}) &= -x_{15} - i^*(x_8)x_7 \\
&= -x_{15} + x_8 x_7,
\end{aligned}
$$

$$
\begin{aligned}
i^*(x_{35}) &= -x_{35} - i^*(x_8)x_{27} - i^*(-x_8^2)x_{19} - i^*(x_{20})x_{15} - i^*(x_8 x_{20})x_7 \\
&= -x_{35} + x_8 x_{27} + x_8^2 x_{19} + x_{20}x_{15} - x_8 x_{20}x_7.
\end{aligned}
$$

Given a generator x, it is possible to find another generator $\tilde{x}$ that represents the same class as x in $QH^*(X; \mathbb{F}_p)$ (again, p is odd), with $i^*(\tilde{x}) = -\tilde{x}$.

Definition 2.3.9. If $\tilde{x}$ is an algebra generator for which $i^*(\tilde{x}) = -\tilde{x}$, then $\tilde{x}$ is a -1 characteristic generator of $H^*(X; \mathbb{F}_p)$.

Proposition 2.3.10. *Suppose x is a generator of $H^*(X; \mathbb{F}_p)$ and $\mu^*(x) = x \otimes 1 + 1 \otimes$*

$x + \sum c \otimes d$. Let $\tilde{x} = x - \frac{1}{p-2} \sum i^*(c)d$. Then $i^*(\tilde{x}) = -\tilde{x}$.

Proof. Since $\mu^*(x) = x \otimes 1 + 1 \otimes x + \sum c \otimes d$, $i^*(x) = -x - \sum i^*(c)d$. Let $\tilde{x} = x - \frac{1}{p-2} \sum i^*(c)d$. Then

$$
\begin{aligned}
i^*(\tilde{x}) &= \left(\frac{p-2}{p-2}\right) i^*(\tilde{x}) \\
&= \frac{1}{p-2}[-(p-2)x - (p-2)\sum i^*(c)d - \sum i^*(i^*(c))i^*(d)] \\
&= \frac{1}{p-2}[-(p-2)x - (p-2)\sum i^*(c)d - \sum c[i^*(d)]].
\end{aligned}
$$

By equation 2.3.12, this equals

$$
\begin{aligned}
& \frac{1}{p-2}[-(p-2)x - (p-2)\sum i^*(c)d - \sum i^*(c)d] \\
&= \frac{1}{p-2}[-(p-2)x + \sum i^*(c)d] \\
&= -\tilde{x}.
\end{aligned}
$$

$\square$

Remark. Kane's book (specifically Chapter 48) contains a similar proposition [19]. However, his proof only works on maps $\Psi : X \to X$ which induce multiplication by an integer λ which multiplicatively generates $\mathbb{F}_p \backslash \{0\}$.

Although x and $\tilde{x}$ represent the same element in $QH^*(X; \mathbb{F}_p)$, their coproducts may not be equal.

Proposition 2.3.11. $\mu^*(\tilde{x}) = \frac{1}{p-2}(\mu^*(i^*(x)) - \mu^*(x))$.

Proof. By definition,

$$\tilde{x} = x - \frac{1}{p-2}\sum i^*(c)d.$$

Since $i^*(x) = -x - \sum i^*(c)d$, we can rewrite this as

$$\tilde{x} = x + \frac{1}{p-2}x + \frac{1}{p-2}i^*(x)$$

and hence

$$\tilde{x} = -\frac{1}{p-2}x + \frac{1}{p-2}i^*(x).$$

We obtain the result by taking the coproduct of both sides.

$\square$

These -1 characteristic generators will be useful in Chapter 5 when we look for different coproducts on $H^*(X; \mathbb{F}_p)$ and study whether they give primitively generated Hopf algebras.

2.4   Maps Between HA-spaces

Both Lie groups and topological groups have their own versions of homomorphisms - maps that preserve the structures on the spaces. We can define a similar notion for H-spaces as well. One interesting feature of H-space theory is that unlike group theory, we do not assume every map between H-spaces preserves the multiplication. Consequently, H-space theory has a tool for examining maps which do not preserve the multiplication, and we will need this tool in our definitions and results.

Material in this section can be found in [40].

Let us start with a generalization of group homomorphisms:

Definition 2.4.1. Let (X, μ_X) and (Y, μ_Y) be H-spaces, and $f : X \to Y$ be a map. Then f is a $\mu_X - \mu_Y$ *H-map* if $f \circ \mu_X \simeq \mu_Y \circ (f \times f)$. That is, the following diagram commutes:

$$
\begin{array}{ccc}
X \times X & \xrightarrow{\ f \times f\ } & Y \times Y \\
{\scriptstyle \mu_X}\big\downarrow & & \big\downarrow{\scriptstyle \mu_Y} \\
X & \xrightarrow[\ f\ ]{} & Y
\end{array}
$$

Frequently, we will drop "$\mu_X - \mu_Y$" and simply write "H-map."

A map that is an H-map induces a special kind of homomorphism on cohomology:

Definition 2.4.2. Let (A, μ_A^*) and (B, μ_B^*) be Hopf algebras and $\phi : A \to B$ be an algebra homomorphism. Suppose that the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\ \phi\ } & B \\
{\scriptstyle \mu_A^*}\big\downarrow & & \big\downarrow{\scriptstyle \mu_B^*} \\
A \otimes A & \xrightarrow[\ \phi \otimes \phi\ ]{} & B \otimes B
\end{array}
$$

Then we call ϕ a *Hopf algebra homomorphism.* If ϕ is also an algebra isomorphism, then we will call it a *Hopf algebra isomorphism,* and A and B are said to be *isomorphic as Hopf algebras.*

In particular, if (X, μ_X) and (Y, μ_Y) are H-spaces and $f : X \to Y$ is an H-map,

then $f^* : H^*(Y; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p)$ is a Hopf algebra homomorphism:

$$\begin{array}{ccc} H^*(Y; \mathbb{F}_p) & \xrightarrow{\quad f^* \quad} & H^*(X; \mathbb{F}_p) \\ \downarrow{\scriptstyle \mu_Y^*} & & \downarrow{\scriptstyle \mu_X^*} \\ H^*(Y; \mathbb{F}_p) \otimes H^*(Y; \mathbb{F}_p) & \xrightarrow{\quad f^* \otimes f^* \quad} & H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p) \end{array}$$

and $f_* : H_*(X; \mathbb{F}_p) \to H_*(Y; \mathbb{F}_p)$ is an algebra homomorphism:

$$\begin{array}{ccc} H_*(X; \mathbb{F}_p) \otimes H_*(X; \mathbb{F}_p) & \xrightarrow{\quad f_* \otimes f_* \quad} & H_*(Y; \mathbb{F}_p) \otimes H_*(Y; \mathbb{F}_p) \\ \downarrow{\scriptstyle \mu_{X*}} & & \downarrow{\scriptstyle \mu_{Y*}} \\ H_*(X; \mathbb{F}_p) & \xrightarrow{\quad f_* \quad} & H_*(Y; \mathbb{F}_p) \end{array}$$

Example 2.4.3. Let (X, μ) be an H-space and consider the diagonal map $\Delta_X : X \to X \times X$. If we give $X \times X$ the multiplication map $\mu_{X \times X} : X^4 \to X \times X$ given by

$$\mu_{X \times X}(x, y, z, w) = (\mu(x, z), \mu(y, w)) \,,$$

then

$$\mu_{X \times X} \left(\Delta_X \times \Delta_X \right)(x, y) = (\mu(x, y), \mu(x, y)) \,,$$

and

$$\Delta_X \mu(x, y) = (\mu(x, y), \mu(x, y)) \,,$$

so the diagonal map is an H-map. Therefore, the cup product $\Delta_X^* : H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p)$ is a Hopf algebra homomorphism. Indeed, this must occur because the coproduct map, by definition, is an algebra homomorphism.

Let us find a relationship between H-maps and homotopy inverse operations that generalizes the fact that group homomorphisms preserve multiplicative inverses.

Lemma 2.4.4. *Let (X, μ_X) and (Y, μ_Y) be HA-spaces with homotopy inverse operations i_X and i_Y respectively. Let $f : X \to Y$ be an H-map. Define a map $f^\sharp : \langle X, X \rangle \to \langle X, Y \rangle$ as follows: given $[g] \in \langle X, X \rangle$,*

$$f^\sharp([g]) = [fg]. \tag{2.4.1}$$

Then $f^\sharp$ is a group homomorphism, and

$$i_Y f \simeq f i_X. \tag{2.4.2}$$

Proof. Let us show that $f^\sharp$ is a group homomorphism. Before we proceed, let us indicate that the group operation in $\langle X, X \rangle$ will be denoted $\underset{X}{*}$, and the group operation in $\langle X, Y \rangle$ will be denoted by $\underset{Y}{*}$. Since f is an H-map, $f \circ \mu_X \simeq \mu_Y \circ (f \times f)$, so for any $g_1, g_2 : X \to X$, we have

$$
\begin{aligned}
f\mu_X(g_1 \times g_2)\Delta_X &\simeq \mu_Y(f \times f)(g_1 \times g_2)\Delta_X \\
&= \mu_Y((fg_1) \times (fg_2))\Delta_X \tag{2.4.3}
\end{aligned}
$$

(elementwise, the right hand side of equation 2.4.3 takes x to $(fg_1(x))\,(fg_2(x))$). This

means that in $\langle X, Y \rangle$,

$$
\begin{aligned}
f^{\sharp}([g_1] \underset{X}{*} [g_2]) &= [fg_1] \underset{Y}{*} [fg_2] \\
&= f^{\sharp}([g_1]) \underset{Y}{*} f^{\sharp}([g_2]),
\end{aligned}
$$

so $f^{\sharp}$ is a group homomorphism. In particular, it preserves group inverses, so for any

$[g] \in \langle X, X \rangle$,

$$
\begin{aligned}
[fi_X g] &= f^{\sharp}([i_X g] \\
&= f^{\sharp}([g]^{-1}) \\
&= \left(f^{\sharp}([g]) \right)^{-1} \\
&= [fg]^{-1} \\
&= [i_Y fg],
\end{aligned}
$$

so if we pick $g = id_X$, we see

$$
[fi_X] = [i_Y f],
$$

so $fi_X \simeq i_Y f$. $\qquad \square$

One major difference between group theory and H-space theory is that we do not assume that every map between H-spaces is an H-map. When dealing with maps which are not H-maps, we need a way to express how they fail to preserve the multiplication.

Definition 2.4.5. Let (X, μ_X) and (Y, μ_Y) be H-spaces, and $f : X \to Y$ be a map. Then the H-deviation of f (from a $\mu_X - \mu_Y$ H-map) is the map $D_f : X \times X \to Y$ which satisfies

$$\mu_Y \left(D_f \times [\mu_Y \circ (f \times f)] \right) \Delta_{X \times X} \simeq f \circ \mu_X. \tag{2.4.4}$$

In $\langle X \times X, Y \rangle$, this means

$$[D_f] * [\mu_Y \circ (f \times f)] = [f \circ \mu_X].$$

If Y is an HA-space, let i_Y denote the homotopy inverse of Y. Then D_f can be chosen as

$$D_f = \mu_Y((f \circ \mu_X) \times (i_Y \circ \mu_Y \circ (f \times f)))\Delta_{X \times X}. \tag{2.4.5}$$

Elementwise,

$$D_f(x_1, x_2) = f(x_1 x_2) i_Y \left(f(x_1) f(x_2) \right). \tag{2.4.6}$$

In particular, f is an H-map iff D_f is null-homotopic.

Example 2.4.6. Suppose we have an HA-space X, with homotopy inverse operation i. Let us examine when i is a $\mu - \mu$ H-map by studying D_i. Elementwise, we have

$$D_i(x_1, x_2) = i(x_1 x_2) i(i(x_1) i(x_2)). \tag{2.4.7}$$

Hence we can rewrite D_i as the composition

$$D_i = (i \circ \mu) * (i \circ ((i \circ p_1) * (i \circ p_2))) \qquad (2.4.8)$$

By Lemma 2.3.4, D_i is homotopic to a map which takes (x_1, x_2) to

$$i(x_1 x_2)(x_2 x_1).$$

Let us find a relationship between the map D_i and the commutativity of X as an HA-space. The following result generalizes the fact that a group is abelian if and only if its inverse operation is a homomorphism.

Theorem 2.4.7. *The homotopy inverse operation i is a $\mu - \mu$ H-map if and only if X is a homotopy commutative HA-space.*

Proof. If X is a homotopy commutative HA-space, then $\mu \simeq \mu T$, so D_i is homotopic to a map which takes (x_1, x_2) to $i(x_1 x_2)(x_1 x_2)$, and hence is null-homotopic. Thus, i is a $\mu - \mu$ H-map.

If D_i is null-homotopic, then in $\langle X \times X, X \rangle$,

$$[D_i] = [k]. \qquad (2.4.9)$$

Let $p_1, p_2 : X \times X \to X$ be projection maps. By equation 2.4.8, the left hand side of

equation 2.4.11 becomes

$$[\mu]^{-1} * ([p_1]^{-1} * [p_2]^{-1})^{-1} = [k], \qquad (2.4.10)$$

so if we (left) multiply both sides of equation 2.4.10 by $[\mu]$, we obtain

$$([p_1]^{-1} * [p_2]^{-1})^{-1} \;=\; [\mu],$$

so

$$[p_2] * [p_1] \;=\; [\mu]. \qquad (2.4.11)$$

Since $[p_2] * [p_1] = [\mu T]$ (elementwise, the maps $p_2 * p_1$ and μT take (x, y) to yx), equation 2.4.11 becomes

$$[\mu T] = [\mu],$$

so X is a homotopy commutative HA-space. $\qquad \square$

H-maps can play a role similar to group homomorphisms: if we have a familiar H-space (Y, μ_Y) and a less familiar H-space (X, μ_X), we could look for an H-map $f : X \to Y$ to help us study X and its cohomology. For example, if f is a homotopy equivalence, then $f^* : H^*(Y, \mu_Y; \mathbb{F}_p) \to H^*(X, \mu_X; \mathbb{F}_p)$ would be a Hopf algebra isomorphism. This would allow computations of coproducts in $H^*(X, \mu_X; \mathbb{F}_p)$ using the coproducts of $H^*(Y, \mu_Y; \mathbb{F}_p)$.

If f is not (known to be) an H-map, we can actually use its H-deviation to examine the difference in the coproduct structures of $H^*(X, \mu_X; \mathbb{F}_p)$ and $H^*(Y, \mu_Y; \mathbb{F}_p)$:

Lemma 2.4.8. *Let (X, μ_X) and (Y, μ_Y) be H-spaces, and $f : X \to Y$ be a homotopy-equivalence. Let $y \in H^*(Y, \mu_Y; \mathbb{F}_p)$. Then*

$$\mu_X^*(f^*(y)) = \Delta_{X \times X}^* \left(D_f^* \otimes ((f^* \otimes f^*)\mu_Y^*) \right) \mu_Y^*(y). \qquad (2.4.12)$$

In particular, the induced homomorphism D_f^* is trivial if and only if f^* is a Hopf algebra isomorphism:

$$\mu_X^*(f^*(y)) = (f^* \otimes f^*)\mu_Y^*(y). \qquad (2.4.13)$$

Proof. Let us verify equation 2.4.12 first. By definition, D_f must satisfy

$$\mu_Y \left(D_f \times [\mu_Y \circ (f \times f)] \right) \simeq f \circ \mu_X,$$

so when we apply cohomology, we see that for any $y \in H^*(Y, \mu_Y; \mathbb{F}_p)$,

$$\mu_X^*(f^*(y)) = \Delta_{X \times X}^* \left(D_f^* \otimes ((f^* \otimes f^*)\mu_Y^*) \right) \mu_Y^*(y). \qquad (2.4.14)$$

Now let us check equation 2.4.13. Write $\mu_Y^*(y) = y \otimes 1 + 1 \otimes y + \sum y' \otimes y''$.

Then if $D_f^* = 0$ on positive degree elements,

$$
\begin{aligned}
\mu_X^*(f^*(y)) &= \Delta_{X \times X}^* \left(D_f^* \otimes ((f^* \otimes f^*)\mu_Y^*) \right) \mu_Y^*(y) \\
&= \Delta_{X \times X}^* \left(D_f^* \otimes ((f^* \otimes f^*)\mu_Y^*) \right) (y \otimes 1 + 1 \otimes y \\
&\quad + \sum y' \otimes y'' \Big).
\end{aligned}
\tag{2.4.15}
$$

Since $D_f^*(y) = D_f^*(y') = 0$ and $D_f^*(1) = 1$, we can simplify equation 2.4.15 to

$$
\begin{aligned}
\mu_X^*(f^*(y)) &= \Delta_{X \times X}^* \left(D_f^* \otimes ((f^* \otimes f^*)\mu_Y^*) \right) (1 \otimes y) \\
&= (f^* \otimes f^*)\mu_Y^*(y).
\end{aligned}
$$

Hence if D_f^* is trivial, f^* is a Hopf algebra isomorphism.

On the other hand, suppose D_f^* is not a trivial homomorphism. Choose a positive degree class $y \in H^*(Y, \mu_Y; \mathbb{F}_p)$ so that $D_f^*(y) \neq 0$, and any element of smaller positive degree than y is in the kernel of D_f^*. Write $\mu_Y^*(y) = y \otimes 1 + 1 \otimes y + \sum y' \otimes y''$. Then

$$
\begin{aligned}
\mu_X^*(f^*(y)) &= \Delta_{X \times X}^* \left(D_f^* \otimes ((f^* \otimes f^*)\mu_Y^*) \right) \mu_Y^*(y) \\
&= \Delta_{X \times X}^* \left(D_f^* \otimes ((f^* \otimes f^*)\mu_Y^*) \right) (y \otimes 1 + 1 \otimes y \\
&= + \sum y' \otimes y'' \Big).
\end{aligned}
\tag{2.4.16}
$$

By our hypotheses, since each y' has degree less than that of y, $D_f^*(y') = 0$, so equation

2.4.16 gives us

$$\mu_X^*(f^*(y)) = D_f^*(y) + (f^* \otimes f^*)\mu_Y^*(y)$$

$$\neq (f^* \otimes f^*)\mu_Y^*(y).$$

Hence if D_f^* is not trivial, f^* is not a Hopf algebra isomorphism. $\square$

Another application of H-maps and H-deviations concerns H-spaces with the homotopy type of a product. Given H-spaces (X, μ_X), (Y, μ_Y), and (Z, μ_Z), and a homotopy equivalence $f : X \to Y \times Z$ with homotopy inverse $g : Y \times Z \to X$ (so $gf \simeq id_X$ and $fg \simeq id_{Y \times Z}$), let

$$\mu_{Y \times Z} : Y \times Z \times Y \times Z \to Y \times Z$$

be defined by

$$\mu_{Y \times Z} = (\mu_Y \times \mu_Z)(id_Y \times T_{Z,Y} \times id_Z).$$

We can ask if the map f is an H-map; that is, we can ask if the following diagram commutes:

$$\begin{array}{ccc} X \times X & \xrightarrow{\ f \times f\ } & Y \times Z \times Y \times Z \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle \mu_{Y \times Z}} \\ X & \xrightarrow{\quad f \quad} & Y \times Z \end{array} \qquad (2.4.17)$$

As an analogy, in group theory, we may ask if a group G is isomorphic to a direct sum or direct product of groups. In our situation, if f and g are H-maps, then diagram

2.4.17 commutes and thus the following diagram commutes:

$$
\begin{array}{ccc}
X \times X & \xleftarrow{\;\;g \times g\;\;} & Y \times Z \times Y \times Z \\
\mu_X \downarrow & & \downarrow \mu_{Y \times Z} \\
X & \xrightarrow{\quad f \quad} & Y \times Z
\end{array}
$$

However, if f and g are not H-maps, we can instead ask if the homotopy equivalences f and g induce what is called a twisted H-structure on $Y \times Z$. This twisted H-structure, if it exists, would be a map

$$ \tilde{\mu} : Y \times Z \times Y \times Z \to Y \times Z $$

which makes the diagram commute:

$$
\begin{array}{ccc}
X \times X & \xleftarrow{\;\;g \times g\;\;} & Y \times Z \times Y \times Z \\
\mu_X \downarrow & & \downarrow \tilde{\mu} \\
X & \xrightarrow{\quad f \quad} & Y \times Z
\end{array}
\tag{2.4.18}
$$

Let us explain the meaning of the term twisted H-structure. The following definition is based on one from Zabrodsky's paper [41]:

Definition 2.4.9. Let (Y, μ_Y), and (Z, μ_Z) be H-spaces, and $i_1, i_2 : Y \to Y \times Y$ be inclusions. Suppose we have a map $\omega : Y \times Y \to Z$ for which ωi_1 and ωi_2 are both nullhomotopic. Then the ω-*twisted H-structure on* $Y \times Z$ is a map $\tilde{\mu} : Y \times Z \times Y \times Z \to$

$Y \times Z$ given by the following composition:

$$\tilde{\mu} = (id_Y \times \mu_Z)(\mu_Y \times \omega \times \mu_Z)(\Delta_{Y \times Y} \times id_{Z \times Z})(id_Y \times T_{Z,Y} \times id_Z).$$

Elementwise, given $y_1, y_2 \in Y$ and $z_1, z_2 \in Z$, we have

$$\tilde{\mu}(y_1, z_1, y_2, z_2) = (\mu_Y(y_1, y_2), \mu_Z(\omega(y_1, y_2), \mu_Z(z_1, z_2))).$$

If we abbreviate $\mu_Y(y_1, y_2) = y_1 y_2$ and $\mu_Z(z_1, z_2) = z_1 \cdot z_2$, we have

$$\tilde{\mu}(y_1, z_1, y_2, z_2) = (y_1 y_2, \omega(y_1, y_2) \cdot (z_1 \cdot z_2)).$$

The map ω is called the *twisting factor*.

There are similar definitions where the domain of ω is $Y \times Z$, $Z \times Y$, or $Z \times Z$.

Example 2.4.10. In this example, let $Y = Z = F_4$ with identity element g_0. Let us find a twisted H-structure for $F_4 \times F_4$. Let $com : F_4 \times F_4 \to F_4$ be the commutator:

$$com(g, h) = ghg^{-1}h^{-1}.$$

Note that if $i_1, i_2 : F_4 \to F_4 \times F_4$ are inclusions, then

$$
\begin{aligned}
com \circ i_1(g) &= com(g, g_0) \\
&= gg_0g^{-1}g_0^{-1} \\
&= g_0,
\end{aligned}
$$

$$
\begin{aligned}
com \circ i_2(g) &= com(g_0, g) \\
&= g_0gg_0^{-1}g^{-1} \\
&= g_0.
\end{aligned}
$$

The com-twisted H-structure on $F_4 \times F_4$ is the map

$$
\tilde{\mu} : F_4 \times F_4 \times F_4 \times F_4 \to F_4 \times F_4,
$$

$$
\tilde{\mu}(g_1, h_1, g_2, h_2) = (g_1g_2, (g_1g_2g_1^{-1}g_2^{-1})h_1h_2).
$$

In the context of diagram 2.4.18, we would like to know if there is a map ω (in finding ω, we must also determine its domain among the choices in Definition 2.4.9) and an ω-twisted H-structure

$$
\tilde{\mu} : Y \times Z \times Y \times Z \to Y \times Z
$$

that makes diagram 2.4.18 commute. We will encounter this problem in Chapter 4. As we will also see in Chapter 4, functions which are not H-maps, along with their H-deviations, allow us to study new H-spaces using information from old ones, just like how one studies unfamiliar groups using homomorphisms and isomorphisms involving familiar ones.

2.5 A_n-spaces and A_n-forms

At this point, we have seen two "degrees" of associativity for an H-space (X, μ):

- (Strict) associativity: $\mu \left(\mu \times id_X \right) = \mu \left(id_X \times \mu \right)$

- Homotopy associativity: $\mu \left(\mu \times id_X \right) \simeq \mu \left(id_X \times \mu \right)$

In this section, we will introduce A_n-spaces and A_n-forms, which describe and measure how associative a multiplication map is. Although this section is optional (it will only come up in parenthetical remarks), it will allow the reader to appreciate the wide gulf between HA-spaces and topological groups. References for this material can be found in Stasheff's paper [34] (which introduced the ideas of A_n-forms) and Kane's book [19].

First, suppose we have three elements x, y, and z in the H-space (X, μ). There are two ways to multiply them using μ:

$$\mu \left(\mu \times id_X \right) (x, y, z) = (xy)z \text{ and } \mu \left(id_X \times \mu \right) (x, y, z) = x(yz).$$

If the maps $\mu\left(\mu \times id_X\right)$ and $\mu\left(id_X \times \mu\right)$ are homotopic, we can represent this using

a 1-cell called K_3:

$$K_3 = \quad \overset{(xy)z \quad x(yz)}{\underset{}{\circ\!\!-\!\!-\!\!-\!\!-\!\!\circ}}$$

The left vertex (0-cell) represents the map $\mu\left(\mu \times id_X\right)$, and the right vertex represents

$\mu\left(id_X \times \mu\right)$. The edge (1-cell) between them represents a homotopy between the

maps.

Now let us step up to multiplying four elements x, y, z, and w. There are five

ways to multiply them:

$$(xy)(zw), \ ((xy)z)w, \ (x(yz))w, \ x((yz)w), \ \text{and} \ x(y(zw)).$$

We can first consider homotopies between pairs like $(xy)(zw)$ and $((xy)z)w$, giving

a union of five 0-cells and five 1-cells, but we can impose a stronger condition and

assume all possible homotopies among all five maps exist. We can represent them

using a 2-cell called K_4:

Notice that each edge is a copy of K_3, whose vertices are maps applied to three elements formed from products of x, y, z, and w. For example, the top edge is a copy of K_3, and the vertices represent the maps $\mu\left(id_X \times \mu\right)$ and $\mu\left(\mu \times id_X\right)$ applied to the elements xy, z, and w. The 2-cell itself represents a homotopy between the homotopies (the edges).

We can continue this process of forming $(n-2)$-cells K_n to represent homotopies between maps that multiply n elements in (X, μ). For reference, here is K_2, a single vertex that represents the only way of multiplying two elements using μ:

$$K_2 = \overset{xy}{\circ}$$

Given n elements of X, $x_1, x_2, \ldots, x_n$, each map, homotopy, homotopy between homotopies, and so on represents placing pairs of parentheses in the expression "$x_1 x_2 \ldots x_n$" in order to obtain a product of elements. In particular, each map from X^n to X corresponds to placing $n - 2$ pairs of parentheses, obtaining an unambiguous order of multiplying elements in X using μ. For example, in our picture of K_4, the bottom vertex corresponds to placing two pairs of parentheses in "$xyzw$", one pair around "yz" and another pair around "yzw." A homotopy between two of these maps corresponds to placing only the $n - 3$ pairs of parentheses that the two maps have in common. For example, the top edge in our picture of K_4 corresponds to placing a pair of parentheses around "xy," and the edge in the upper right corresponds to a pair of parentheses around "xyz." Homotopies between homotopies correspond to placing

$n - 4$ pairs of parentheses, and so on.

Given K_n, a face ($n - 1$ dimensional cell) of K_n will correspond to placing only one pair of parentheses. We can formalize the idea of copies of K_j being subsets of the boundary of K_n for $j < n$ by introducing notation for faces of K_n. The face $(K_{n-s+1} \times K_s)_k$ corresponds to placing a parenthesis to the left of x_k and to the right of x_{k+s-1}: $x_1 \ldots x_{k-1}(x_k \ldots x_{k+s-1})x_{k+s} \ldots x_n$.

Hence we can construct K_n from K_j, $j < n$, as follows: K_n will be the cone on ∂K_n, where

$$\partial K_n = \bigcup_{\substack{2 \leq s \leq n-1 \\ 1 \leq k \leq n-s+1}} (K_{n-s+1} \times K_s)_k.$$

Now let us formalize the idea of homotopies (and homotopies of homotopies, and so on) and introduce A_n-forms using the cells K_n:

Definition 2.5.1. Let (X, μ) be an H-space. The space X is said to admit an A_n-form (where $2 \leq n < \infty$) if there exist maps $M_i : K_i \times X^i \to X$ for $2 \leq i \leq n$ where

$$M_2 : K_2 \times X \times X \to X \text{ is given by } M_2(k, x, y) = \mu(x, y),$$

and if $i > 2$, we have compatibility conditions that allow M_i restricted to $\partial K_i \times X^i$ to be written in terms of M_j, $j < i$: the composition

$$(K_{i-s+1} \times K_s)_k \times X^i \hookrightarrow \partial K_i \times X^i \xrightarrow{M_i} X$$

can be written as

$$(\alpha, \beta, x_1, \ldots, x_i) \mapsto M_{n-s+1}(\alpha, x_1, \ldots, x_{k-1}, M_s(\beta, x_k, \ldots, x_{k-s+1}), x_{k+s}, \ldots, x_i)).$$

A space with an A_n-form will be called an A_n-space. If the maps M_i exist for all $i \geq 2$, then X is an A_∞-space.

For example, any H-space is automatically an A_2-space by definition.

If (X, μ) is an HA-space, then let $H : I \times X^3 \to X$ be a homotopy between $\mu(id_X \times \mu)$ and $\mu(\mu \times id_X)$ (where I is the unit interval $[0, 1]$) such that

$$H(0, x, y, z) = \mu(\mu \times id_X)(x, y, z) = (xy)z, \text{ and}$$

$$H(1, x, y, z) = \mu(id_X \times \mu)(x, y, z) = x(yz)$$

Denote the point in K_2 by k_2. Since I and K_3 are homeomorphic, let $f : K_3 \to I$ be a homeomorphism taking the left vertex to 0 and the right vertex to 1. Then define

$$M_2(k_2, x, y) = \mu(x, y) \text{ and}$$

$$M_3(k, x, y, z) = H(f(k), x, y, z).$$

Then at the vertices, we have

$$M_3(f^{-1}(0), x, y, z) = H(0, x, y, z) = \mu(\mu \times id_X)(x, y, z) = (xy)z$$

and

$$M_3(f^{-1}(1), x, y, z) = H(1, x, y, z) = \mu\,(id_X \times \mu)\,(x, y, z) = x(yz).$$

Let us verify the compatibility conditions. The face $(K_2 \times K_2)_1$ corresponds to the left vertex $f^{-1}(0)$. Indeed, we have

$$M_2(k_2, M_2(k_2, x, y), z) = \mu\,(\mu \times id_X)\,(x, y, z) = (xy)z.$$

Furthermore, the face $(K_2 \times K_2)_2$ corresponds to the right vertex $f^{-1}(1)$. This time, we have

$$M_2(k_2, x, M_2(k_2, y, z)) = \mu\,(id_X \times \mu)\,(x, y, z) = x(yz).$$

Thus, an HA-space is also an A_3-space. Conversely, if (X, μ) is an H-space which is an A_3-space, we can use M_3 to define a homotopy $H : I \times X^3 \to X$ between $\mu\,(id_X \times \mu)$ and $\mu\,(\mu \times id_X)$:

$$H(r, x, y, z) = M_3(f^{-1}(r), x, y, z).$$

Then

$$H(0, x, y, z) = M_3(f^{-1}(0), x, y, z) = \mu\,(\mu \times id_X)\,(x, y, z) = (xy)z, \text{ and}$$

$$H(1, x, y, z) = M_3(f^{-1}(1), x, y, z) = \mu\,(id_X \times \mu)\,(x, y, z) = x(yz).$$

Therefore, an A_3-space is a homotopy associative H-space. From here, we can appeal to Sibson's theorem: given a homotopy associative H-space, a homotopy inverse oper-

ation $i : X \to X$ can be constructed that satisfies $\mu(i \times id_X)\Delta_X \simeq \mu(id_X \times i)\Delta_X \simeq k$ [32]. Hence any A_3-space is an HA-space.

If (X, μ) is a topological group or associative H-space, we can take advantage of strict associativity and define all the maps to simply take the product of the elements of X:

$$M_i(k, x_1, \ldots x_i) = x_1 \ldots x_i.$$

Hence any topological group and any associative H-space is an A_∞-space.

More generally, A_n-forms are designed to be homotopy invariants in the sense that if X has an A_n-form and $f : X \to Y$ is a homotopy equivalence, then we can use f to induce an A_n-form on Y. For example, any loop space ΩX is homotopy equivalent to an associative H-space (with a homotopy unit)

$$\Omega' X = \{l : [0, r] \to X : \; r \geq 0, \text{ and } l(0) = l(r) = x_0\}$$

whose multiplication map $\mu_{c'}$ is given as follows: if $a : [0, r] \to X$ and $b : [0, s] \to X$ are two loops in $\Omega' X$, then $\mu_{c'}(a, b)$ is given by the loop

$$\mu_{c'}(a, b) = \begin{cases} a(t) & 0 \leq t \leq r \\ b(t - r) & r \leq t \leq r + s \end{cases}.$$

Hence any loop space is an A_∞-space. Conversely, any A_∞-space possesses a classifying space (see Stasheff's paper [34] and Kane's book [19] for a construction). In

conclusion, we can say that any A_∞-space is homotopy equivalent to a loop space, which is homotopy-equivalent to an associative H-space, which is homotopy equivalent to a topological group.

Overall, there is a countably infinite difference in degrees of associativity between HA-spaces and topological groups: HA-spaces are only guaranteed to have A_3-forms, but topological groups have A_n-forms for any integer $n \geq 2$.

To summarize, we have:

- A_∞-spaces: topological groups, loop spaces, associative H-spaces

- A_3-spaces: HA-spaces

- A_2-spaces: H-spaces

Again, we note that A_∞-spaces possess classifying spaces, while A_3-spaces might not.

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3 The Adjoint Action and Related Maps

One outcome of Kono, Kozima, and Iwase's work is that given a finite simply-connected topological group G, the adjoint action $ad : G \times G \to G$ given by

$$ad(g, h) = ghg^{-1}$$

may not equal $p_2^G : G \times G \to G$, the projection onto the second factor, but the homomorphisms they induce in cohomology and homology over $\mathbb{F}_p$ are equal if and only if $H_*(G, \mu; \mathbb{F}_p)$ is a commutative algebra [13]. In total, they introduce the following maps:

- the commutator map $com : G \times G \to G$ given by

$$com(g, h) = ghg^{-1}h^{-1}.$$

- the adjoint action $Ad : G \times \Omega G \to \Omega G$, where $Ad(g, l)$ is the loop given pointwise

by

$$Ad(g, l)(t) = gl(t)g^{-1}.$$

- a map that we will denote as $Com : G \times \Omega G \to \Omega G$, where $Com(g, l)$ is the

  loop given pointwise by

$$Com(g, l)(t) = gl(t)g^{-1}l(t)^{-1}.$$

Overall, they use these maps to characterize commutativity of $H_*(G; \mathbb{F}_p)$. Let us

quote (and combine) Theorems 2.2 and 2.3 from Iwase's paper [13]:

Theorem 3.0.2 (Iwase). *Let G be a finite simply-connected topological group and p*

be any odd prime. Let BG and $B\Lambda G$ be the classifying spaces of G and ΛG respec-

tively. Let $j : G \to B\Lambda G$ be the inclusion, and $p_2^\Omega : G \times \Omega G \to \Omega G$ and $p_2^G : G \times G \to G$

be projections from the second factor. Then the following conditions are equivalent:

   i) The induced homomorphism $j^* : H^*(B\Lambda G; \mathbb{F}_p) \to H^*(G; \mathbb{F}_p)$ is surjective.

   ii) The Pontryagin ring $H_*(G; \mathbb{F}_p)$ is a commutative Hopf algebra. In other

words, the adjoint action of G on itself induces the trivial action

$$ad_* = p_{2*}^G : H_*(G; \mathbb{F}_p) \otimes H_*(G; \mathbb{F}_p) \to H_*(G; \mathbb{F}_p).$$

   Note: Iwase means "trivial" in that $ad_* = p_{2*}^G$, not that these maps equal zero

on positive degree elements.

   iii) The Hopf algebra $H^*(G; \mathbb{F}_p)$ is primitively generated.

iv) There is a $H^*(BG; \mathbb{F}_p)$-module isomorphism

$$H^*(B\Lambda G; \mathbb{F}_p) \cong H^*(BG; \mathbb{F}_p) \otimes H^*(G; \mathbb{F}_p).$$

v) The integral homology $H_*(G; \mathbb{Z})$ has no p-torsion.

vi) The adjoint action (of G on ΩG) induces the trivial action

$$Ad_* = p_{2*}^{\Omega} : H_*(G; \mathbb{F}_p) \otimes H_*(\Omega G; \mathbb{F}_p) \to H_*(\Omega G; \mathbb{F}_p).$$

vii) There is an isomorphism of algebras $H^*(B\Lambda G; \mathbb{F}_p) \cong H^*(BG \times G; \mathbb{F}_p).$

It was not known if these results require G to be a topological group, or if the statements can be proven to be equivalent for a larger class of spaces. If we want to generalize these results beyond topological groups, we must be careful: if we weaken the associativity condition on G (to be more precise, if we no longer require G to be an A_∞-space), we are no longer guaranteed to have classifying spaces for G and its free loop space. Consequently, we cannot generalize statements (i), (iv), and (vii) from Iwase's theorems, and we will have to find other methods of proving the equivalence of the remaining statements. A more immediate challenge is defining generalizations of the four maps: without a strict inverse, it is not immediately clear how we should define the maps and find formulas for them. Even if we allow a homotopy inverse operation, there is the question of how to define Com and Ad so that their codomain is a loop space, not a free loop space (see the discussion at the beginning of Section

1.3).

We overcome these challenges as follows. We weaken the requirements for associativity and existence of inverses to homotopy associativity and existence of a two-sided homotopy inverse operation, resulting in an HA-space. Given a simply-connected HA-space X, we will use the free loop fibration to develop our own versions of the maps ad, com, Ad, and Com that generalize the maps introduced by Kono and Kozima:

Theorem 3.0.3. *Let X be a simply-connected HA-space and p be any odd prime. There exist maps $ad : X \times X \to X$, $com : X \times X \to X$, $Ad : X \times \Omega X \to \Omega X$, and $Com : X \times \Omega X \to \Omega X$ such that if X is a topological group, these maps agree with the maps defined by Kono and Kozima.*

We will proceed to prove the following theorem, generalizing Kono, Kozima, and Iwase's results and removing the need for classifying spaces. For convenience, we will number each statement to match with the corresponding statement from Iwase's paper:

Theorem 3.0.4. *Let X be a finite simply-connected HA-space and p be any odd prime. Then the following conditions are equivalent:*

ii) The homomorphism $com^* : H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p)$ is trivial

iii) The Hopf algebra $H^*(X; \mathbb{F}_p)$ is primitively generated.

v) The integral homology $H_*(X; \mathbb{Z})$ has no p-torsion.

vi) The homomorphism $Com^* : H^*(\Omega X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$ is

trivial

Moreover, in statement (ii) of Iwase's theorem, note that he assumes the equivalence of the commutativity of $H_*(G; \mathbb{F}_p)$ and the equality of ad_* and p_{2*}^G. We will go into more detail and prove:

Theorem 3.0.5. *The homomorphism* $com^* : H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p)$ *is trivial iff* $H_*(X; \mathbb{F}_p)$ *is commutative.*

The key to the proof of our theorems will be the calculation of formulas for com^* and Com^*. Once we define the four maps, we will deduce these formulas by taking advantage of a property of the cohomology of finite simply-connected HA-spaces, Theorem 1.2.4.

Before we go further, let us point out why we work with *com* and *Com* instead of *ad* and *Ad*. A classical theorem from the topology of Lie groups is that a compact connected Lie group is abelian if and only if it is a torus [30]. According to a theorem by Araki, James, and Thomas, if a compact connected Lie group is not abelian, then it is not homotopy commutative [2]. Thus, the commutator map *com* is not nullhomotopic for these groups. Nevertheless, Borel observes that when p is odd and G is one of the classical groups A_n, B_n, C_n, or D_n, $H_*(G; \mathbb{F}_p)$ is a commutative exterior algebra [3]. Our results stem from a desire to find a relationship between the commutator *com* on a compact simply-connected Lie group G, and the commutativity of $H_*(G; \mathbb{F}_p)$.

This chapter will start with the free loop fibration from Definition 1.2.10. We will prove some important properties about the free loop fibration that will be needed for our definitions and results. Next, we will define our maps *com*, *ad*, *Com*, and *Ad* for simply-connected HA-spaces and study their induced homomorphisms in cohomology. Finally, we will use *com* and *Com* to characterize finite simply-connected HA-spaces whose homology over $\mathbb{F}_p$ is a commutative algebra.

3.1   The Free Loop Fibration

We will need to use the free loop fibration to not only prove our results, but also to define our maps in the first place. Frequently, we will face the following situation: for some space Y and a simply-connected HA-space X, we have a map $\hat{f} : Y \to \Lambda X$, and we would like to know whether:

1. If there exists a map $f : Y \to \Omega X$ such that $jf \simeq \hat{f}$:

$$
\begin{array}{ccc}
 & & \Omega X \\
 & \nearrow^{f} & \downarrow^{j} \\
Y & \xrightarrow[\hat{f}]{} & \Lambda X
\end{array}
$$

2. If the map f is unique up to homotopy: if there is another map $f_2 : Y \to \Omega X$ such that $jf_2 \simeq jf \simeq \hat{f}$, then does this imply that $f \simeq f_2$?

In this section, we will discuss these questions using a fibration sequence called the free loop fibration. We will explain why the free loop fibration is indeed a fibration,

and then prove some properties of the fibration and the maps in it.

Let us address existence of a lift by verifying that ε_0 is a fibration:

Proposition 3.1.1. *Let X be any simply-connected topological space, $j : \Omega X \to \Lambda X$*

be the inclusion

$$j(l) = l,$$

and $\varepsilon_0 : \Lambda X \to X$ be evaluation at $t = 0$:

$$\varepsilon_0(\varphi) = \varphi(0).$$

Then ε_0 is a fibration, and we have a fibration sequence which we call the free loop

fibration:

$$
\begin{array}{c}
\Omega X \\
\downarrow {\scriptstyle j} \\
\Lambda X \\
\downarrow {\scriptstyle \varepsilon_0} \\
X
\end{array}
$$

Proof. Let Y be any space and $f : Y \to \Lambda X$ be any continuous map, and suppose

that we have a homotopy $G : Y \times I \to X$ such that for any $y \in Y$,

$$G(y, 0) = \varepsilon_0(f(y)) = f(y)(0). \tag{3.1.1}$$

Furthermore, since $f(y) \in \Lambda X$, pointwise, we have

$$f(y)(0) = f(y)(1). \tag{3.1.2}$$

We will prove that ε_0 is a fibration by constructing a homotopy $F : Y \times I \to \Lambda X$ such that

$$F(y,0) = f(y), \qquad (3.1.3)$$

$$G(y,t) = \varepsilon_0(F(y,t)). \qquad (3.1.4)$$

Notice that at $t = 0$, we must have

$$\varepsilon_0(F(y,0)) = G(y,0) = f(y)(0) = f(y)(1).$$

To find F, we can instead look for a homotopy

$$F''' : (S^1 \times Y) \times I \to X$$

such that

$$F''' = \hat{\varepsilon}_f(id_{S^1} \times F).$$

In other words, we would use F''' to define F pointwise:

$$F(y,t)(s) = F'''(s,y,t).$$

Finding F''' is equivalent to finding a map

$$F'' : Y \times I \times S^1 \to X$$

where

$$F'''' = (T_{Y,S^1} \times id_I)(id_Y \times T_{I,S^1})F'',$$

which in turn is equivalent to finding a map

$$F' : Y \times I \times I \to X$$

such that

$$
\begin{aligned}
F'(y,t,s) &= F''(y,t,s) \\
&= F'''(s,y,t) \\
&= F(y,t)(s)
\end{aligned}
\tag{3.1.5}
$$

(recall that we defined S^1 to be the quotient space of $I = [0,1]$ by $\{0,1\}$). Hence F' must satisfy the following properties: since $F(y,t) \in \Lambda X$, $F(y,t)(0) = F(y,t)(1)$, and thus

$$F'(y,t,0) = F'(y,t,1),$$

while by equation 3.1.3, when $t = 0$,

$$F'(y,0,s) = f(y)(s),$$

and by equation 3.1.4, when $s = 0$,

$$F'(y, t, 0) = G(y, t).$$

We can take

$$F'(y, t, s) = \begin{cases} G(y, t - 3s) & 0 \leq s \leq \frac{t}{3} \\ f(y) \left(\frac{3s - t}{3 - 2t} \right) & \frac{t}{3} \leq s \leq 1 - \frac{t}{3} \\ G(y, t + 3s - 3) & 1 - \frac{t}{3} \leq s \leq 1 \end{cases}$$

Let us show that this map is well-defined and continuous. In particular, when $s = \frac{t}{3}$, we have

$$
\begin{aligned}
F'(y, t, \frac{t}{3}) &= G(y, t - 3(\frac{t}{3})) \\
&= G(y, 0) \\
&= f(y)(0) \\
&= f(y) \left(\frac{3(\frac{t}{3}) - t}{3 - 2t} \right),
\end{aligned}
$$

and when $t = 0$ as well, we get

$$
\begin{aligned}
f(y) \left(\frac{3(0) - 0}{3 - 2(0)} \right) &= f(y)(0) \\
&= G(y, 0) \\
&= G(y, 0 - 3(0)),
\end{aligned}
$$

and when $t = 1$, we get

$$
\begin{aligned}
f(y)\left(\frac{3(\frac{1}{3}) - 1}{3 - 2}\right) &= f(y)(0) \\
&= f(y)(1) \\
&= G(y, 0) \\
&= G(y, 1 - 3(\frac{1}{3})).
\end{aligned}
$$

When $s = 1 - \frac{t}{3}$, we have

$$
\begin{aligned}
F'(y, t, 1 - \frac{t}{3}) &= G(y, t + 3(1 - \frac{t}{3}) - 3) \\
&= G(y, 0) \\
&= f(y)(0) \\
&= f(y)(1) \\
&= f(y)\left(\frac{3 - 2t}{3 - 2t}\right) \\
&= f(y)\left(\frac{3(1 - \frac{t}{3}) - t}{3 - 2t}\right),
\end{aligned}
$$

so at $t = 0$, we get

$$
\begin{aligned}
f(y)\left(\frac{3(1) - 0}{3 - 0}\right) &= f(y)(1) \\
&= f(y)(0) \\
&= G(y, 0) \\
&= G(y, 0 + 3(1 - 0) - 3),
\end{aligned}
$$

and at $t = 1$, we get

$$
\begin{aligned}
f(y)\left(\frac{3(\frac{2}{3}) - 1}{3 - 2}\right) &= f(y)(1) \\
&= f(y)(0) \\
&= G(y, 0) \\
&= G(y, 1 + 3(1 - \frac{1}{3}) - 3).
\end{aligned}
$$

We see that F' is well-defined and continuous. Let us now check that F' preserves the basepoint $y_0 \in Y$. Depending on what s and t are, we either have

$$
F'(y_0, t, s) = G(y_0, t - 3s),
$$

$$
F'(y_0, t, s) = f(y_0)\left(\frac{3s - t}{3 - 2t}\right),
$$

or

$$
F'(y_0, t, s) = G(y_0, t + 3s - 3).
$$

Since G is a basepoint-preserving homomorphism, we must have

$$
G(y_0, t - 3s) = G(y_0, t + 3s - 3) = x_0,
$$

and since f preserves basepoints, $f(y_0) = l_0$, the constant loop at x_0, so

$$
f(y_0)\left(\frac{3s - t}{3 - 2t}\right) = x_0.
$$

Hence no matter what t and s are,

$$F'(y_0, t, s) = x_0.$$

We now verify the other requirements of F'. If $s = 0$, we have

$$
\begin{aligned}
F'(y, t, 0) &= G(y, t - 3(0)) \\
&= G(y, t),
\end{aligned}
$$

and when $s = 1$, we have

$$
\begin{aligned}
F'(y, t, 1) &= G(y, t + 3(1) - 3) \\
&= G(y, t).
\end{aligned}
$$

When $t = 0$, we have

$$
\begin{aligned}
F'(y, 0, s) &= f(y) \left(\frac{3s - 0}{3 - 2(0)} \right) \\
&= f(y)(s).
\end{aligned}
$$

Therefore, when we use F' to define $F : Y \times I \to \Lambda X$ using equation 3.1.5, we see that F is a homotopy that lifts G. Hence $\varepsilon_0 : \Lambda X \to X$ is a fibration.

Let us determine the fiber. By definition,

$$
\begin{aligned}
\varepsilon_0^{-1}(\{x_0\}) &= \{\varphi \in \Lambda X : \varepsilon_0(\varphi) = \varphi(0) = \varphi(1) = x_0\} \\
&= \Omega X,
\end{aligned}
$$

so the fiber is indeed the loop space of X. $\qquad\square$

In particular, given $\hat{f} : Y \to \Lambda X$ such that $\varepsilon_0 \hat{f} : Y \to X$ is nullhomotopic, there exists a map f such that $jf \simeq \hat{f}$:

$$
\begin{array}{ccc}
& & \Omega X \\
& \nearrow^{f} & \downarrow j \\
Y & \xrightarrow[\hat{f}]{} & \Lambda X \\
& & \downarrow \varepsilon_0 \\
& & X
\end{array}
$$

In addition to the maps in the fibration sequence, we also have the following map from X to ΛX:

Definition 3.1.2. We define $\chi : X \to \Lambda X$ as follows: given $x \in X$ and $t \in [0,1]$,

$$
\chi(x)(t) = x.
$$

Let us prove two properties of the map j. The first property will allow us to see how j interacts with the multiplication maps on ΩX and ΛX:

Lemma 3.1.3. *The map* $j : \Omega X \to \Lambda X$ *is an H-map.*

Proof. By definition, if $l_1, l_2 \in \Omega X$, at any $t \in [0,1]$,

$$
\begin{aligned}
\mu_\Lambda(l_1, l_2)(t) &= l_1(t)l_2(t) \\
&= \mu_\Omega(l_1, l_2)(t),
\end{aligned}
$$

so

$$
\mu_\Lambda(l_1, l_2) = \mu_\Omega(l_1, l_2).
$$

Hence for any $l_1, l_2 \in \Omega X$,

$$
\begin{aligned}
\mu_\Lambda(j(l_1), j(l_2)) &= \mu_\Lambda(l_1, l_2) \\
&= \mu_\Omega(l_1, l_2) \\
&= j(\mu_\Omega(l_1, l_2),
\end{aligned}
$$

so $\mu_\Lambda(j \times j) = j\mu_\Omega$. Thus, j is an H-map. $\qquad\square$

Let us determine a uniqueness condition on lifts of a map $\hat{f} : Y \to \Lambda X$. That is, suppose there exists two maps $f_1, f_2 : Y \to \Omega X$ such that $jf_1 \simeq jf_2 \simeq \hat{f}$. What conditions will guarantee that $f_1 \simeq f_2$? The following lemma will show that if X is a simply-connected HA-space, then given a homotopy $G : Y \times I \to \Lambda X$ from jf_1 to

jf_2, we can find a homotopy $F : Y \times I \to \Omega X$ from f_1 to f_2.

$$
\begin{array}{ccc}
 & \Omega X & \\
{}^{f_1, f_2}\nearrow & \big\downarrow {\scriptstyle j} & \\
Y \xrightarrow[\hat{f}]{} & \Lambda X &
\end{array}
$$

Lemma 3.1.4. *Let Y be any topological space, X be any simply-connected HA-space*

with homotopy inverse operation $i : X \to X$, and suppose we are given maps $f_1, f_2 :$

$Y \to \Omega X$, such that

$$ jf_1 \simeq jf_2. $$

Then

$$ f_1 \simeq f_2. $$

Proof. Let $G : Y \times I \to \Lambda X$ be a homotopy from jf_1 to jf_2. That is,

$$ G(y,0) = jf_1(y), \text{ and } G(y,1) = jf_2(y). $$

Let us recall a crucial property of X, and specifically its homotopy inverse operation

i: the composition

$$ \mu(id_X \times i)\Delta_X(x) = xi(x) $$

is nullhomotopic. In other words, there is a homotopy $K : X \times I \to X$ such that

$$ K(x,0) = x_0, \text{ and } K(x,1) = xi(x). \tag{3.1.6} $$

For each fixed $x \in X$, the map K defines paths in PX from x_0 to $xi(x)$ via $K(x, s)$, $s \in [0, 1]$. Since we require homotopies to preserve basepoints, for any $s \in [0, 1]$, we must have

$$K(x_0, s) = x_0.$$

We will use these paths to modify G to obtain a homotopy $F' : Y \times I \to \Lambda X$ from a map homotopic to f_1 to another map homotopic to f_2 with the property that

$$F'(x, t)(0) = F'(x, t)(1) = x_0,$$

so F' defines a map $F : Y \times I \to \Omega X$ which is a homotopy from a map homotopic to f_1 to another map homotopic to f_2. Hence $f_1 \simeq f_2$.

Let us commence modifying the homotopy G. Currently, we have

$$G(y, 0)(0) = jf_1(y)(0) = x_0,$$

$$G(y, 1)(0) = jf_2(y)(0) = x_0,$$

but if $0 < t < 1$, it may not be true that $G(y, t)(0)$ is equal to x_0. Our first modification will be to replace the loops defined by $G(y, t)$ with loops in ΛX whose starting points in X are of the form $xi(x)$. This will allow us to use the paths defined by the homotopy K. Let us define $G' : Y \times I \to \Lambda X$ as follows:

$$G' = \mu_\Lambda(id_{\Lambda X} \times [\chi \circ i \circ \varepsilon_0])\Delta_{\Lambda X} G \tag{3.1.7}$$

$$G'(y,t) = \mu_\Lambda(G(y,t), \chi(i(G(y,t)(0)))), \tag{3.1.8}$$

$$G'(y,t)(s) = \mu(G(y,t)(s), i(G(y,t)(0))). \tag{3.1.9}$$

Then at $s = 0$,

$$G'(y,t)(0) = \mu(G(y,t)(0), i(G(y,t)(0))).$$

Notice that at $t = 0$, since $G(y,0) = jf_1(y)$ is a loop that starts at x_0, $G(y,0)(0) = x_0$,

we have

$$
\begin{aligned}
G'(y,0)(s) &= \mu(G(y,0)(s), i(G(y,0)(0))) \\
&= \mu(G(y,0)(s), x_0) \\
&= G(y,0)(s) \\
&= jf_1(y)(s) \\
&= f_1(y)(s),
\end{aligned}
$$

and similarly, at $t = 1$, we get

$$
\begin{aligned}
G'(y,1)(s) &= G(y,1)(s) \\
&= jf_2(y)(s) \\
&= f_2(y)(s).
\end{aligned}
$$

Therefore, G' is actually a homotopy from jf_1 to jf_2. When $0 < t < 1$, it is still

possible that $G'(y,t)(0)$ might not equal x_0 since i is only known to be a homotopy

inverse operation. To overcome this obstacle, we use the paths from the homotopy K. Define $K' : Y \times I \times I \to X$ as the composition

$$K' = K((\varepsilon_0 G) \times id_I), \tag{3.1.10}$$

so for each $y \in Y$, $t \in I$, we have

$$K'(y, t, s) = K(G(y, t)(0), s), \tag{3.1.11}$$

and at $s = 0$,

$$
\begin{aligned}
K'(y, t, 0) &= K(G(y, t)(0), 0) \\
&= x_0,
\end{aligned}
$$

while at $s = 1$,

$$
\begin{aligned}
K'(y, t, 1) &= K(G(y, t)(0), 1) \\
&= \mu(G(y, t)(0), i(G(y, t)(0))) \\
&= G'(y, t)(0) \\
&= G'(y, t)(1),
\end{aligned}
$$

so for each $y \in Y$, $t \in I$, K' defines a path from x_0 to $\mu(G(y, t)(0), i(G(y, t)(0)))$.

Now we will define $F' : Y \times I \to \Lambda X$ pointwise:

$$F'(y,t)(s) = \begin{cases} K'(y,t,4s) & 0 \leq s \leq \frac{1}{4} \\[2ex] G'(y,t)(4s-1) & \frac{1}{4} \leq s \leq \frac{1}{2} \\[2ex] K'(y,t,2-2s) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Let us check that F' is well defined. At $s = \frac{1}{4}$, we have

$$K'(y,t,1) = G'(y,t)(0),$$

and at $s = \frac{1}{2}$, we get

$$\begin{aligned} G'(y,t)(1) &= G'(y,t)(0) \\ &= K'(y,t,1). \end{aligned}$$

Now we will verify that F' is a basepoint-preserving homotopy. Let $y_0 \in Y$ be the basepoint, and l_0 be the constant loop at x_0. Depending on what t and s are, we either have

$$F'(y_0,t)(s) = G'(y_0,t)(4s-1)$$

$$F'(y_0,t)(s) = K'(y_0,t,4s),$$

or

$$F'(y_0,t)(s) = K'(y_0,t,2-2s).$$

For any t and s, we have

$$G(y_0, t) = l_0,$$

so

$$
\begin{aligned}
G'(y_0, t) &= \mu_\Lambda(G(y_0, t), \chi(i(G(y_0, t)(0)))) \\
&= \mu_\Lambda(l_0, \chi(x_0)) \\
&= \mu_\Lambda(l_0, l_0) \\
&= l_0,
\end{aligned}
$$

which implies

$$G'(y_0, t)(s) = x_0.$$

Meanwhile,

$$
\begin{aligned}
K'(y_0, t, s) &= K(G(y_0, t)(0), s) \\
&= K(x_0, s) \\
&= x_0.
\end{aligned}
$$

Therefore, no matter what t and s are, we must have

$$F'(y_0, t)(s) = x_0,$$

so for any t,

$$F'(y_0, t) = l_0,$$

and hence F' is a basepoint-preserving homotopy. Let us check that $F'(y, t)(0) = F'(y, t)(1) = x_0$:

$$
\begin{aligned}
F'(y, t)(0) &= K'(y, t, 0) \\
&= x_0,
\end{aligned}
$$

$$
\begin{aligned}
F'(y, t)(1) &= K'(y, t, 2 - 2) \\
&= x_0.
\end{aligned}
$$

Thus, we can define a homotopy $F : Y \times I \to \Omega X$ pointwise by

$$F(y, t)(s) = F'(y, t)(s). \tag{3.1.12}$$

Let us continue working with F and examine what is happening at $t = 0$. We have

$$
\begin{aligned}
G(y, 0)(s) &= j f_1(s) \\
&= f_1(s),
\end{aligned}
$$

and for any $u \in [0,1]$,

$$
\begin{aligned}
K'(y,0,u) &= K(G(y,0)(0),u) \\
&= K(jf_1(y)(0),u) \\
&= K(x_0,u) \\
&= x_0 \\
&= l_0(u),
\end{aligned}
$$

so the loop defined by F at $t = 0$ is given pointwise by

$$
F(y,0)(s) = \begin{cases} l_0(4s) = x_0 & 0 \leq s \leq \frac{1}{4} \\ f_1(4s-1) & \frac{1}{4} \leq s \leq \frac{1}{2} \\ l_0(2-2s) = l_0(2s-2) = x_0 & \frac{1}{2} \leq s \leq 1 \end{cases}
$$

We can write down a formula for it using loop concatenation μ_c and inclusion maps $i_1, i_2 : \Omega X \to \Omega X \times \Omega X$:

$$
\begin{aligned}
F(y,0) &= \mu_c(\mu_c(l_0,f_1),l_0) \\
&= \mu_c((\mu_c i_2) \times id_{\Omega X})i_1 f_1(y),
\end{aligned}
$$

so $F(y,0)$ defines a function $f_1' : Y \to \Omega X$ with

$$
f_1' = \mu_c((\mu_c i_2) \times id_{\Omega X})i_1 f_1,
$$

which is homotopic to f_1 since $(\Omega X, \mu_c)$ is an H-space with a homotopy identity:

$$
\begin{aligned}
f_1' &= \mu_c((\mu_c i_2) \times id_{\Omega X})i_1 f_1 \\
&\simeq \mu_c(id_{\Omega X} \times id_{\Omega X})i_1 f_1 \\
&= \mu_c i_1 f_1 \\
&\simeq id_{\Omega X} f_1 \\
&= f_1.
\end{aligned}
$$

Similarly, $F(y, 1)$ defines a function $f_2' : Y \to \Omega X$ with

$$ f_2' = \mu_c((\mu_c i_2) \times id_{\Omega X})i_1 f_2 $$

which is homotopic to f_2. Therefore F represents a homotopy from f_1' to f_2', so we have

$$
\begin{aligned}
f_1 &\simeq f_1' \\
&\simeq f_2' \text{ (via the homotopy } F) \\
&\simeq f_2
\end{aligned}
$$

as functions from Y to ΩX. $\qquad \square$

In addition to the fibration and the spaces in it, we will often need to work with the products $X \times \Omega X$, $X \times \Lambda X$, and $X \times X$, along with projections from these spaces.

Let $p_1^\Omega : X \times \Omega X \to X$, $p_2^\Omega : X \times \Omega X \to \Omega X$, $p_1^\Lambda : X \times \Lambda X \to X$, $p_2^\Lambda : X \times \Lambda X \to \Lambda X$, $p_1^X : X \times X \to X$, and $p_2^X : X \times X \to X$ be projections. We can relate these projection maps in the following commutative diagrams:

$$
\begin{array}{ccc}
X \times \Omega X & \xrightarrow{\ p_1^\Omega\ } & X \\
{\scriptstyle id_X \times j}\downarrow & & \downarrow{\scriptstyle id_X} \\
X \times \Lambda X & \xrightarrow{\ p_1^\Lambda\ } & X \\
{\scriptstyle id_X \times \varepsilon_0}\downarrow & & \downarrow{\scriptstyle id_X} \\
X \times X & \xrightarrow{\ p_1^X\ } & X
\end{array}
\tag{3.1.13}
$$

$$
\begin{array}{ccc}
X \times \Omega X & \xrightarrow{\ p_2^\Omega\ } & \Omega X \\
{\scriptstyle id_X \times j}\downarrow & & \downarrow{\scriptstyle j} \\
X \times \Lambda X & \xrightarrow{\ p_2^\Lambda\ } & \Lambda X \\
{\scriptstyle id_X \times \varepsilon_0}\downarrow & & \downarrow{\scriptstyle \varepsilon_0} \\
X \times X & \xrightarrow{\ p_2^X\ } & X
\end{array}
\tag{3.1.14}
$$

3.2   Defining The Maps for HA-spaces

In this section, we will use the free loop fibration to define the adjoint actions, commutator, and related maps for HA-spaces. We will state and prove properties of our maps that generalize some of the results from Section 2 of Kono and Kozima's paper [21]; additional properties will be stated and proven in Section 3.5 at the end of this chapter.

Our first task is to define these maps for simply-connected HA-spaces. Let us start with the maps $ad : X \times X \to X$ and $com : X \times X \to X$.

Definition 3.2.1. We define $ad : X \times X \to X$ and $com : X \times X \to X$ as follows:

$$com = \mu \left(id_X \times i\right) \left(\mu \times (\mu T)\right) \Delta_{X \times X}, \tag{3.2.1}$$

$$com(x, y) = (xy)i(yx), \tag{3.2.2}$$

and

$$ad = \mu(com \times p_2^X) \Delta_{X \times X}, \tag{3.2.3}$$

$$ad(x, y) = ((xy)i(yx)) \, y. \tag{3.2.4}$$

Notice that if $X = G$ is also topological group with $i(g) = g^{-1}$, then these definitions

simplify to those of Kono and Kozima:

$$\begin{aligned} com(g, h) &= (gh)(hg)^{-1} \\ &= ghg^{-1}h^{-1}, \end{aligned}$$

and

$$\begin{aligned} ad(g, h) &= ((gh)(hg)^{-1})h \\ &= ghg^{-1}h^{-1}h \\ &= ghg^{-1}. \end{aligned}$$

To define *Com* for simply-connected HA-spaces, we must start by defining

another function whose codomain is ΛX and mimic Kono and Kozima's definition of Com.

Definition 3.2.2. Given $x \in X$ and $l \in \Omega X$, we define a function $\widehat{Com} : X \times \Omega X \to \Lambda X$ to be the loop given by

$$
\begin{aligned}
\widehat{Com}(x,l)(t) &= com(x, l(t)) \\
&= (xl(t))\, i(l(t)x).
\end{aligned}
$$

In other words, the composition $\hat{\varepsilon}_f(id_{S^1} \times \widehat{Com}) : S^1 \times X \times \Omega X \to X$ is

$$
\begin{aligned}
\hat{\varepsilon}_f(id_{S^1} \times \widehat{Com})(t,x,l) &= com(x, l(t)) \\
&= (xl(t))\, i(l(t)x).
\end{aligned}
$$

Theorem 3.2.3. *Let X be a simply-connected HA-space. There exists a map $Com :$ $X \times \Omega X \to \Omega X$ which is unique up to homotopy such that*

$$
\widehat{Com} \simeq j \circ Com,
$$

$$
\begin{array}{ccc}
 & & \Omega X \\
 & \nearrow{\scriptstyle Com} & \downarrow{\scriptstyle j} \\
X \times \Omega X & \xrightarrow[\widehat{Com}]{} & \Lambda X
\end{array}
$$

and if $X = G$ is also a topological group, then this definition of Com will agree with Kono and Kozima's definition.

Proof. We need to show that $\widehat{Com}$ lifts to ΩX in the free loop fibration. To verify this, we compute $\varepsilon_0 \widehat{Com}$ and determine if it is nullhomotopic:

$$
\begin{aligned}
\varepsilon_0 \widehat{Com}(x, l) &= \widehat{Com}(x, l)(0) \\
&= com(x, l(0)) \\
&= (xx_0)\, i(x_0 x) \\
&= xi(x),
\end{aligned}
$$

so $\varepsilon_0 \widehat{Com}$ is nullhomotopic. By the fibration sequence in Definition 1.2.10, there is a map we will call $Com : X \times \Omega X \to \Omega X$, such that

$$
j \circ Com \simeq \widehat{Com}, \tag{3.2.5}
$$

and by Lemma 3.1.4, Com is unique up to homotopy in the following sense: if $Com_2 : X \times \Omega X \to \Omega X$ is another map which satisfies

$$
j \circ Com_2 \simeq \widehat{Com},
$$

then

$$
Com_2 \simeq Com.
$$

If X is a topological group, then we may take

$$Com(x,l)(t) = xl(t)x^{-1}l(t)^{-1},$$

because for any $t \in [0,1]$,

$$
\begin{aligned}
j \circ Com(x,l)(t) &= xl(t)x^{-1}l(t)^{-1} \\
&= \widehat{Com}(x,l)(t).
\end{aligned}
$$

Hence our definition agrees with Kono and Kozima's definition in the case that X is a topological group. $\qquad\square$

Now that we have defined Com, we can perform a similar process for $Ad :$ $X \times \Omega X \to \Omega X$.

Theorem 3.2.4. *Let X be a simply-connected HA-space. Define a map $\widehat{Ad} : X \times \Omega X \to \Lambda X$ so that given $x \in X$ and $l \in \Omega X$, for any $t \in [0,1]$,*

$$\widehat{Ad}(x,l)(t) = (xl(t))i(x). \qquad\qquad (3.2.6)$$

Then there exists a map $Ad : X \times \Omega X \to \Omega X$ which is unique up to homotopy such that

$$\widehat{Ad} \simeq j \circ Ad,$$

$$\begin{array}{ccc} & & \Omega X \\ & \overset{Ad}{\nearrow} & \downarrow j \\ X \times \Omega X & \underset{\widehat{Ad}}{\longrightarrow} & \Lambda X \end{array}$$

and if $X = G$ is also a topological group, then this definition of Ad will agree with Kono and Kozima's definition.

Example 3.2.5. Given a simply-connected HA-space X with homotopy inverse operation i, let $K : X \times I \to X$ be a homotopy from k (where $k(x) = x_0$) to $\mu(id_X \times i)\Delta_X$:

$$K(x,0) = x_0, \text{ and } K(x,1) = xi(x). \tag{3.2.7}$$

We can use K to define a map $Com : X \times \Omega X \to \Omega X$ pointwise:

$$Com(x,l)(s) \;\; = \;\; \begin{cases} K(x,4s) & 0 \leq s \leq \frac{1}{4} \\ \widehat{Com}(x,l)(4s-1) & \frac{1}{4} \leq s \leq \frac{1}{2} \\ K(x,2-2s) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

For each x and l, $Com(x,l)$ is a concatenation of paths and loops. First we use the homotopy K to find a path from x_0 to $xi(x)$, the starting point of $\widehat{Com}(x,l)$. Next, we follow around the loop $\widehat{Com}(x,l)$. Finally, we return to x_0 by moving backwards along the path from x_0 to $xi(x)$ given by K.

A homotopy $F : X \times \Omega X \times I \to \Lambda X$ from $\widehat{Com}$ to $j \circ Com$ (which equals Com

pointwise) is given by

$$
F(x,l,t)(s) = \begin{cases} K(x, 4s + (1-t)) & 0 \leq s \leq \frac{t}{4} \\[2mm] \widehat{Com}(x,l)\left(\frac{4s-t}{4-3t}\right) & \frac{t}{4} \leq s \leq 1 - \frac{t}{2} \\[2mm] K(x, 2 - 2s + (1-t)) & 1 - \frac{t}{2} \leq s \leq 1 \end{cases}
$$

We can perform a similar procedure for finding an explicit choice of $Ad : X \times \Omega X \to \Omega X$.

The proof of the following lemma will demonstrate our main technique for proving statements involving Com and Ad; it will let us demonstrate the utility of defining the maps $\widehat{Com}$ and $\widehat{Ad}$.

Lemma 3.2.6. *Let X be a simply-connected HA-space, and let $i_2 : \Omega X \to X \times \Omega X$ be inclusion in the second factor. Then*

$$Ad \circ i_2 \simeq id_{\Omega X}. \tag{3.2.8}$$

Proof. According to our definition of Ad and $\widehat{Ad}$, the following diagram commutes:

$$
\begin{array}{ccc}
& & \Omega X \\
& \overset{Ad}{\nearrow} & \downarrow j \\
\Omega X \xrightarrow{\ i_2\ } & X \times \Omega X \xrightarrow[\widehat{Ad}]{} & \Lambda X
\end{array}
$$

Let us prove that

$$\widehat{Ad} \circ i_2 \simeq j \circ id_{\Omega X}.$$

Then we can obtain equation 3.2.8 by using Theorem 3.2.4 (the definitions of Ad and $\widehat{Ad}$) and Lemma 3.1.4:

$$
\begin{aligned}
j \circ Ad \circ i_2 \;\; &\simeq \;\; \widehat{Ad} \circ i_2 \\
&\simeq \;\; j \circ id_{\Omega X} \qquad\qquad (3.2.9)
\end{aligned}
$$

will imply

$$
Ad \circ i_2 \simeq id_{\Omega X}.
$$

For any $t \in [0,1]$ and $l \in \Omega X$,

$$
\begin{aligned}
\widehat{Ad} \circ i_2(l)(t) \;\; &= \;\; \widehat{Ad}(x_0, l)(t) \\
&= \;\; (x_0 l(t)) i(x_0) \\
&= \;\; l(t) x_0 \\
&= \;\; l(t) \\
&= \;\; [j(l)](t) \\
&= \;\; \left(j \circ id_{\Omega X}(l) \right)(t), \qquad\qquad (3.2.10)
\end{aligned}
$$

so

$$
\widehat{Ad} \circ i_2 = j \circ id_{\Omega X}.
$$

Therefore, equation 3.2.9 holds, so by Lemma 3.1.4,

$$Ad \circ i_2 \simeq id_{\Omega X}.$$

$\square$

Notice that working with $\widehat{Ad}$ and $\widehat{Com}$ allowed us to perform calculations, since $\widehat{Ad}$ and $\widehat{Com}$ have simple pointwise formulas while Ad and Com do not (the pointwise formula for Com in Example 3.2.5 requires us to work with the homotopy K, which does not have an explicit definition, and requires us to work with a concatenation of paths and loops on X). Future applications of this technique will be more complicated when homotopy associativity and homotopy inverse operations become involved.

Unlike Kono, Kozima, Hara, and Hamanaka, we will focus on com and Com instead of ad and Ad, but it will help to look at relationships among all four maps. Let us start with some properties involving inclusion maps:

Lemma 3.2.7. *Let X be a simply-connected HA-space and $i_1 : X \to X \times \Omega X$ and $i_2 : \Omega X \to X \times \Omega X$ be inclusions. Then the compositions $Ad \circ i_1 : X \to \Omega X$, $Com \circ i_1 : X \to \Omega X$, and $Com \circ i_2 : \Omega X \to \Omega X$ are nullhomotopic.*

Before we prove the theorem, let us discuss our presentation for the proofs of these results that involve computation on the spaces, especially when homotopy associativity and homotopy inverse operations are involved. First, we prove the result in the special case that $X = G$ is a simply-connected topological group. Here, we will use Kono and Kozima's definitions, which allow direct calculation of the maps

Com and *Ad* without needing to work in the free loop space with $\widehat{Com}$ and $\widehat{Ad}$. In a topological group, we have the advantage of an associative multiplication, and a strict inverse $i(g) = g^{-1}$. Nevertheless, we will include additional parentheses in our calculations so that they resemble the ones in the proofs for HA-spaces, in order to make the arguments in the HA-space case easier to understand.

Once we transition over to HA-spaces, our first step is to prove a similar equation involving the maps $\widehat{Com}$ and $\widehat{Ad}$. Unlike *Com* and *Ad* for HA-spaces, the maps $\widehat{Com}$ and $\widehat{Ad}$ have pointwise definitions similar to Kono and Kozima's definitions of *Com* and *Ad*, and our calculations will actually resemble the ones in the proofs for topological groups. Essentially, we will be working in the free loop space for half of the proof. For the second part of the proof, we need to go from the free loop space (with $\widehat{Com}$ and $\widehat{Ad}$) to the loop space (with *Com* and *Ad*). To do this, we use Theorems 3.2.3 and 3.2.4, along with properties of the inclusion map $j : \Omega X \to \Lambda X$, to obtain the original equation in the theorem or lemma statement.

Proof. We will focus on proving that $Ad \circ i_1$ is nullhomotopic; the proof for the other two compositions is similar.

First, suppose that $X = G$ is also a topological group. Then $i(g) = g^{-1}$ and

$Ad(g, l)(t) = (gl(t))g^{-1}$. Hence for any $t \in [0, 1]$,

$$
\begin{aligned}
Ad \circ i_1(g)(t) &= Ad(g, l_0)(t) \\
&= (gl_0(t))g^{-1} \\
&= (gg_0)g^{-1} \\
&= gg^{-1} \\
&= g_0 \\
&= l_0(t),
\end{aligned}
$$

so $Ad \circ i_1$ is equal to the constant map from G to ΩG that takes any element g to the constant loop l_0.

Now suppose X is an HA-space. We have

$$
\begin{aligned}
\widehat{Ad} \circ i_1(x)(t) &= \widehat{Ad}(x, l_0)(t) \\
&= (xl_0(t))i(x) \\
&= (xx_0)i(x) \\
&= xi(x),
\end{aligned}
$$

so the composition $\widehat{Ad} \circ i_1$ is nullhomotopic. If we let $k_0 : X \to \Omega X$ and $k_0' : X \to \Lambda X$ be trivial maps (they both take $x \in X$ to the constant loop $l_0 \in \Omega X \subset \Lambda X$), then we have

$$
\widehat{Ad} \circ i_1 \simeq k_0'. \tag{3.2.11}
$$

By definition, $j \circ Ad \simeq \widehat{Ad}$ and $j \circ k_0 = k_0'$ (since $l_0 \in \Omega X \subset \Lambda X$), so equation 3.2.11 becomes

$$j \circ Ad \circ i_1 \simeq j \circ k_0,$$

so by Lemma 3.1.4,

$$Ad \circ i_1 \simeq k_0,$$

so $Ad \circ i_1$ is nullhomotopic. $\square$

Recall that we defined ad in terms of com so that

$$ad = \mu(com \times p_2^X)\Delta_{X \times X}. \tag{3.2.12}$$

We can find a similar relationship between Com and Ad:

Theorem 3.2.8. *Let X be a simply-connected HA-space. The maps Com and Ad are related as follows:*

$$Ad \simeq \mu_\Omega(Com \times p_2^\Omega)\Delta_{X \times \Omega X}. \tag{3.2.13}$$

That is, the following diagram commutes:

$$
\begin{array}{ccc}
X \times \Omega X & \xrightarrow{\quad\quad Ad \quad\quad} & \Omega X \\
{\scriptstyle \Delta_{X \times \Omega X}} \downarrow & & \uparrow {\scriptstyle \mu_\Omega} \\
(X \times \Omega X) \times (X \times \Omega X) & \xrightarrow[Com \times p_2^\Omega]{} & \Omega X \times \Omega X
\end{array}
\tag{3.2.14}
$$

Proof. Let (G, μ) be a simply-connected topological group with a strict inverse $i(x) =$

x^{-1}. We want to prove that

$$Ad = \mu_\Omega (Com \times p_2^\Omega) \Delta_{G \times \Omega G} \tag{3.2.15}$$

(note the equals sign). For any $t \in [0, 1]$, we have

$$Ad(g, l)(t) = (gl(t))g^{-1},$$

while

$$
\begin{aligned}
\mu_\Omega (Com \times p_2^\Omega) \Delta_{G \times \Omega G}(g, l)(t) &= \mu_\Omega (Com \times p_2^\Omega)(g, l, g, l)(t) \\
&= \left((gl(t)) \, (l(t)g)^{-1} \right) l(t). \tag{3.2.16}
\end{aligned}
$$

In a topological group, $(l(t)g)^{-1} = g^{-1}l(t)^{-1}$, so equation 3.2.16 becomes

$$
\begin{aligned}
\mu_\Omega (Com \times p_2^\Omega) \Delta_{G \times \Omega G}(g, l)(t) &= \left((gl(t)) \left(g^{-1}l(t)^{-1} \right) \right) l(t) \\
&= \left((gl(t)) \, g^{-1} \right) \left(l(t)^{-1}l(t) \right) \\
&= \left((gl(t)) \, g^{-1} \right) \\
&= Ad(g, l)(t),
\end{aligned}
$$

so equation 3.2.15 holds. $\qquad\square$

In the previous proof, notice that we used the following property of strict

inverses:

$$(gh)^{-1} = h^{-1}g^{-1}.$$

Recall that Lemma 2.3.4 is the corresponding property for homotopy inverse operations. We will need it to prove Theorem 3.2.8 for HA-spaces.

Proof. Let us first prove that

$$\widehat{Ad} \simeq \mu_\Lambda(\widehat{Com} \times (p_2^\Lambda(id_X \times j)))\Delta_{X \times \Omega X}. \tag{3.2.17}$$

Pointwise, we have

$$\widehat{Ad}(x, l)(t) = (xl(t))\, i(x),$$

and

$$\mu_\Lambda(\widehat{Com} \times (p_2^\Lambda(id_X \times j)))\Delta_{X \times \Omega X}(x, l)(t) = \left(\widehat{Com}(x, l)(t)\right)[l(t)]$$
$$= ((xl(t))\, i\, (l(t)x))\, [l(t)]. \tag{3.2.18}$$

By Lemma 2.3.4, $\mu_\Lambda(\widehat{Com} \times (p_2^\Lambda(id_X \times j)))\Delta_{X \times \Omega X}$ is homotopic to a map which takes (x, l) to a loop given pointwise by

$$(x, l(t)) = ((xl(t))\, (i(x)i(l(t))))\, [l(t)]. \tag{3.2.19}$$

By homotopy associativity, $\mu_\Lambda(\widehat{Com} \times (p_2^\Lambda(id_X \times j)))\Delta_{X \times \Omega X}$ is homotopic to a map

which takes (x, l) to a loop given pointwise by

$$
\begin{aligned}
(x, l(t)) &= ((xl(t))i(x)) (i(l(t))l(t)) \\
&= Ad(x,l)(t)(i(l(t))l(t)). \quad (3.2.20)
\end{aligned}
$$

By applying the homotopy inverse operation, this map is homotopic to one that takes (x, l) to a loop given by

$$
(xl(t))i(x) = \widehat{Ad}(x,l)(t),
$$

so equation 3.2.17 holds. By definition,

$$
j \circ Ad \simeq \widehat{Ad},
$$

$$
j \circ Com \simeq \widehat{Com},
$$

and by diagram 3.1.14,

$$
jp_2^{\Omega} = p_2^{\Lambda}(id_X \times j),
$$

so equation 3.2.17 becomes

$$
\begin{aligned}
j \circ Ad &\simeq \mu_{\Lambda}((j \circ Com) \times (jp_2^{\Omega}))\Delta_{X \times \Omega X} \\
&= \mu_{\Lambda}(j \times j)(Com \times p_2^{\Omega})\Delta_{X \times \Omega X}. \quad (3.2.21)
\end{aligned}
$$

By Lemma 3.1.3, j is an H-map, so

$$\mu_\Lambda(j \times j) = j\mu_\Omega,$$

and equation 3.2.21 becomes

$$j \circ Ad \simeq j\mu_\Omega(Com \times p_2^\Omega)\Delta_{X \times \Omega X}.$$

Therefore, by Lemma 3.1.4,

$$Ad \simeq \mu_\Omega(Com \times p_2^\Omega)\Delta_{X \times \Omega X}. \tag{3.2.22}$$

$\square$

We also have the following formula involving Ad that generalizes part (i) of Proposition 2.2 of [21].

Lemma 3.2.9. *Let X be a simply-connected HA-space. Then*

$$Ad\,(id_X \times Ad) \simeq Ad\,(\mu \times id_{\Omega X}) \tag{3.2.23}$$

as maps from $X \times X \times \Omega X$ to ΩX. In other words, the following diagram commutes:

$$
\begin{array}{ccc}
X \times X \times \Omega X & \xrightarrow{\ id_X \times Ad\ } & X \times \Omega X \\
{\scriptstyle \mu \times id_{\Omega X}}\downarrow & & \downarrow{\scriptstyle Ad} \\
X \times \Omega X & \xrightarrow[\quad Ad \quad]{} & \Omega X
\end{array}
\tag{3.2.24}
$$

Let us prove this lemma for topological groups.

Proof. Let (G, μ) be a simply-connected topological group with strict inverse $i(g) = g^{-1}$. At any $t \in [0, 1]$, we have

$$
\begin{aligned}
Ad(id_G \times Ad)(g, h, l)(t) &= g\left((hl(t))h^{-1}\right)g^{-1} \\
&= ((gh)l(t))\left(h^{-1}g^{-1}\right) \\
&= ((gh)l(t))(gh)^{-1} \\
&= Ad\left(\mu \times id_{\Omega G}\right)(g, h, l)(t).
\end{aligned}
$$

Hence $Ad(id_G \times Ad) = Ad\left(\mu \times id_{\Omega G}\right)$. Notice that we actually have equality when G is a topological group. $\square$

Remark. If G is a simply-connected topological group, then the homotopy relations in Lemmas 3.2.6 and 3.2.9 become equalities, and hence $Ad : G \times \Omega G \to \Omega G$ defines a group action of G on ΩG.

In order to prove this lemma for HA-spaces, we will need to define another map as well.

Proof. Let (X, μ) be a simply-connected HA-space. Define $\widehat{Ad} : X \times \Lambda X \to \Lambda X$ so that at any $t \in [0, 1]$,

$$\widehat{Ad}(x, \varphi)(t) = (x\varphi(t))i(x). \tag{3.2.25}$$

Notice that if $l \in \Omega X \subset \Lambda X$, then

$$
\begin{aligned}
\widehat{\widehat{Ad}}(x,l)(t) &= (xl(t))i(x) \\
&= \widehat{Ad}(x,l)(t),
\end{aligned}
$$

so by definition,

$$
\widehat{\widehat{Ad}}(id_X \times j) = \widehat{Ad} \tag{3.2.26}
$$

as maps from $X \times \Omega X$ to ΛX.

First, let us show that

$$
\widehat{\widehat{Ad}}\left(id_X \times \widehat{Ad}\right) \simeq \widehat{Ad}\left(\mu \times id_{\Omega X}\right) \tag{3.2.27}
$$

as maps from $X \times X \times \Omega X$ to ΛX. Pointwise, at any $t \in [0,1]$,

$$
\widehat{\widehat{Ad}}\left(id_X \times \widehat{Ad}\right)(x,y,l)(t) = (x\left((yl(t))i(y)\right))i(x),
$$

and

$$
\widehat{Ad}\left(\mu \times id_{\Omega X}\right)(x,y,l)(t) = ((xy)l(t))i(xy).
$$

By Lemma 2.3.4, we see that $\widehat{Ad}\left(\mu \times id_{\Omega X}\right)$ is homotopic to a map which takes (x,y,l)

to

$$
((xy)l(t))\left(i(y)i(x)\right).
$$

By using homotopy associativity, we can conclude that this map is homotopic to $\widehat{\widehat{Ad}}\left(id_X \times \widehat{Ad}\right)$.

To finish the proof of the lemma, recall that $j \circ Ad \simeq \widehat{Ad}$. Then on the left side of equation 3.2.27, we have

$$
\begin{aligned}
\widehat{\widehat{Ad}}\left(id_X \times \widehat{Ad}\right) &\simeq \widehat{\widehat{Ad}}\left(id_X \times (j \circ Ad)\right) \\
&= \widehat{\widehat{Ad}}(id_X \times j)\left(id_X \times Ad\right). \tag{3.2.28}
\end{aligned}
$$

By equation 3.2.26, we can simplify the previous equation and obtain

$$
\begin{aligned}
\widehat{\widehat{Ad}}\left(id_X \times \widehat{Ad}\right) &\simeq \widehat{Ad}\left(id_X \times Ad\right) \\
&\simeq j \circ Ad(id_X \times Ad). \tag{3.2.29}
\end{aligned}
$$

Meanwhile, on the right side of equation 3.2.27, we have

$$
\widehat{Ad}\left(\mu \times id_{\Omega X}\right) \simeq j \circ Ad\left(\mu \times id_{\Omega X}\right). \tag{3.2.30}
$$

Therefore, by combining equations 3.2.27, 3.2.29, and 3.2.30, we obtain

$$
j \circ Ad(id_X \times Ad) \simeq j \circ Ad\left(\mu \times id_{\Omega X}\right), \tag{3.2.31}
$$

so by applying Lemma 3.1.4, we can conclude that $Ad\left(id_X \times Ad\right)$ and $Ad\left(\mu \times id_{\Omega X}\right)$ are homotopic maps. $\qquad\square$

Let us find a relationship between *com* and *Com* using the evaluation maps.

Lemma 3.2.10. *Let X be a simply-connected HA-space. Then*

$$com(id_X \times \hat{\varepsilon})(T_{S^1,X} \times id_{\Omega X}) \simeq \hat{\varepsilon}(id_{S^1} \times Com)$$

as maps from $S^1 \times X \times \Omega X$ to X.

Proof. Given $(t, l) \in S^1 \times \Omega X$, by definition,

$$\hat{\varepsilon}_f(id_{S^1} \times j)(t, l) = l(t)$$

$$= \hat{\varepsilon}(t, l),$$

so the following diagram commutes:

Pointwise, we have

$$\hat{\varepsilon}_f(id_{S^1} \times \widehat{Com})(t, x, l) = com(x, l(t))$$

$$= com(id_X \times \hat{\varepsilon})(x, t, l)$$

$$= com(id_X \times \hat{\varepsilon})(T_{S^1,X} \times id_{\Omega X})(t, x, l),$$

so therefore,

$$com(id_X \times \hat{\varepsilon})(T_{S^1,X} \times id_{\Omega X}) \simeq \hat{\varepsilon}(id_{S^1} \times Com).$$

$\square$

Additional properties of *ad*, *com*, *Ad*, and *Com* (generalizing those found in Kono and Kozima's paper) that were not used in our results will be presented and proven in Section 3.5. For convenience, let us summarize the properties we proved so far for a simply-connected HA-space X:

1. Let $i_2 : \Omega X \to X \times \Omega X$ be inclusion in the second factor. Then

$$Ad \circ i_2 \simeq id_{\Omega X}. \tag{3.2.32}$$

2. Let $i_1 : X \to X \times \Omega X$ and $i_2 : \Omega X \to X \times \Omega X$ be inclusions. Then the compositions $Ad \circ i_1 : X \to \Omega X$, $Com \circ i_1 : X \to \Omega X$, and $Com \circ i_2 : \Omega X \to \Omega X$ are nullhomotopic.

3. The maps Com and Ad are related as follows:

$$Ad \simeq \mu_\Omega (Com \times p_2^\Omega) \Delta_{X \times \Omega X}. \tag{3.2.33}$$

4. As maps from $X \times X \times \Omega X$ to ΩX, we have

$$Ad\,(id_X \times Ad) \simeq Ad\,(\mu \times id_{\Omega X}) \tag{3.2.34}$$

5. As maps from $S^1 \times X \times \Omega X$ to X,

$$com(id_X \times \hat{\varepsilon})(T_{S^1,X} \times id_{\Omega X}) \simeq \hat{\varepsilon}(id_{S^1} \times Com).$$

3.3   Induced Homomorphisms

Now let us study the cohomology and homology of a *finite* simply-connected HA-space using our maps. We will find formulas (Theorem 3.3.3 and Theorem 3.3.5) for the induced homomorphisms of our maps. To do this, we will take the results from the previous section on relationships between the maps and apply cohomology and homology to them.

Let us begin by listing the induced homomorphisms in cohomology and their domains and codomains. We have:

$$com^* : H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p),$$

$$ad^* : H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p),$$

$$Com^* : H^*(\Omega X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p),$$

$$Ad^* : H^*(\Omega X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p).$$

In homology, we have the following induced linear transformations:

$$com_* : H_*(X; \mathbb{F}_p) \otimes H_*(X; \mathbb{F}_p) \to H_*(X; \mathbb{F}_p),$$

$$ad_* : H_*(X; \mathbb{F}_p) \otimes H_*(X; \mathbb{F}_p) \to H_*(X; \mathbb{F}_p),$$

$$Com_* : H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \to H_*(\Omega X; \mathbb{F}_p),$$

$$Ad_* : H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \to H_*(\Omega X; \mathbb{F}_p).$$

Let us apply cohomology and homology to Lemmas 3.2.6, 3.2.7, 3.2.9, 3.2.10, along with Theorem 3.2.8 from the previous section to obtain properties of, and relationships between, these induced homomorphisms and linear transformations. Recall that for any $x \in H^*(X; \mathbb{F}_p)$, $t \in H^*(\Omega X; \mathbb{F}_p)$,

$$i_1^*(x \otimes t) = \begin{cases} 0, & |t| > 0 \\ x, & t = 1 \end{cases}$$

$$i_2^*(x \otimes t) = \begin{cases} 0, & |x| > 0 \\ t, & x = 1 \end{cases}$$

and for any $\bar{x} \in H_*(X; \mathbb{F}_p)$, $\bar{t} \in H_*(\Omega X; \mathbb{F}_p)$,

$$i_{1*}(\bar{x}) = \bar{x} \otimes 1 \in H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p),$$

$$i_{2*}(\bar{t}) = 1 \otimes \bar{t} \in H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p).$$

Proposition 3.3.1. *We have the following properties of* com^*, ad^*, Com^*, *and* Ad^*:

1. *As homomorphisms from* $H^*(\Omega X; \mathbb{F}_p)$ *to* $H^*(\Omega X; \mathbb{F}_p)$,

$$i_2^* \circ Ad^* = id_{\Omega X}^*,$$

so for any $t \in H^*(\Omega X; \mathbb{F}_p)$,

$$i_2^* \circ Ad^*(t) = t.$$

Hence $Ad^*(t) = 1 \otimes t + \ldots$

2. *As homomorphisms from* $H^*(\Omega X; \mathbb{F}_p)$ *to* $H^*(X; \mathbb{F}_p)$,

$$i_1^* \circ Ad^* = i_1^* \circ Com^* = 0,$$

and as a homomorphism from $H^*(\Omega X; \mathbb{F}_p)$ *to* $H^*(\Omega X; \mathbb{F}_p)$,

$$i_2^* \circ Com^* = 0.$$

In particular, for any $t \in H^*(\Omega X; \mathbb{F}_p)$,

$$Com^*(t) \in \bar{H}^*(\Omega X; \mathbb{F}_p) \otimes \bar{H}^*(\Omega X; \mathbb{F}_p).$$

3. As homomorphisms from $H^*(\Omega X; \mathbb{F}_p)$ to $H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$,

$$Ad^* = \Delta^*_{X \times \Omega X}(Com^* \otimes p_2^{\Omega *})\mu^*_\Omega.$$

4. As homomorphisms from $H^*(\Omega X; \mathbb{F}_p)$ to $H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$,

$$(1 \otimes Ad^*)Ad^* = (\mu^* \otimes 1)Ad^*.$$

5. As homomorphisms from $H^*(X; \mathbb{F}_p)$ to $H^*(S^1; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$,

$$(T^*_{S^1, X} \otimes 1)(1 \otimes \hat{\varepsilon}^*)com^* = (1 \otimes Com^*)\hat{\varepsilon}^*.$$

In cohomology, there is a generator $s \in H^1(S^1; \mathbb{F}_p)$ such that if x is a element of $H^*(X; \mathbb{F}_p)$, then

$$\hat{\varepsilon}^*(x) = s \otimes \sigma^*(x),$$

$$\left(T^*_{S^1, X} \otimes 1\right)(1 \otimes \hat{\varepsilon}^*)(com^*(x)) = s \otimes Com^*(\sigma^*(x)). \qquad (3.3.1)$$

If we write

$$com^*(x) = \sum a \otimes b \in H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p),$$

then since $T^*_{S^1, X}(a \otimes s) = (-1)^{|a||s|}(s \otimes a) = (-1)^{|a|}(s \otimes a)$,

$$Com^*(\sigma^*(x)) = (-1)^{|a|} \sum a \otimes \sigma^*(b). \qquad (3.3.2)$$

We have the following properties of com_*, ad_*, Com_*, and Ad_*:

6. As linear transformations from $H_*(\Omega X; \mathbb{F}_p)$ to $H_*(\Omega X; \mathbb{F}_p)$,

$$Ad_* i_{2*} = id_{\Omega X*},$$

so for any $\bar{t} \in H_*(\Omega X; \mathbb{F}_p)$,

$$Ad_* i_{2*}(\bar{t}) = Ad_*(1 \otimes \bar{t}) = \bar{t}.$$

7. As linear transformations from $H_*(X; \mathbb{F}_p)$ to $H_*(\Omega X; \mathbb{F}_p)$,

$$Ad_* i_{1*} = Com_* i_{1*} = 0,$$

so for any $\bar{x} \in H_*(X; \mathbb{F}_p)$,

$$Ad_*(\bar{x} \otimes 1) = Com_*(\bar{x} \otimes 1) = 0.$$

As a linear transformation from $H_*(\Omega X; \mathbb{F}_p)$ to $H_*(\Omega X; \mathbb{F}_p)$,

$$Com_* i_{2*} = 0,$$

so for any $\bar{t} \in H_*(\Omega X; \mathbb{F}_p)$,

$$Com_* i_{2*}(\bar{t}) = Com_*(1 \otimes \bar{t}) = 0.$$

8. *As linear transformations from* $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$ *to* $H_*(\Omega X; \mathbb{F}_p)$,

$$Ad_* = \mu_{\Omega*}(Com_* \otimes p_{2*}^{\Omega})\Delta_{X \times \Omega X*}.$$

9. *As linear transformations from* $H_*(X; \mathbb{F}_p) \otimes H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$ *to*

$H_*(\Omega X; \mathbb{F}_p)$,

$$Ad_*(1 \otimes Ad_*) = Ad_*(\mu_* \otimes 1).$$

Hence given $\bar{x}, \bar{y} \in H_*(X; \mathbb{F}_p)$, and $\bar{t} \in H_*(\Omega X; \mathbb{F}_p)$,

$$Ad_*(\bar{x} \otimes Ad_*(\bar{y} \otimes \bar{t})) = Ad_*(\bar{x}\bar{y} \otimes \bar{t}).$$

10. *As linear transformations from* $H_*(S^1; \mathbb{F}_p) \otimes H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$ *to*

$H_*(X; \mathbb{F}_p)$,

$$com_*(1 \otimes \hat{\varepsilon}_*)(T_{S^1, X*} \otimes 1) = \hat{\varepsilon}_*(1 \otimes Com_*).$$

Remark. Parts 6 and 9 of Proposition 3.3.1 tell us that Ad_* gives $H_*(\Omega X; \mathbb{F}_p)$ the structure of a module over $H_*(X; \mathbb{F}_p)$.

By our definition of *com*, we can show that the induced homomorphism of *com*

in $H^*(X; \mathbb{F}_p)$ is a homomorphism $com^* : H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p)$ which can be written as a composition:

$$com^* = \Delta^*_{X \times X}(\mu^* \otimes (T^*\mu^*))(1 \otimes i^*)\mu^*. \tag{3.3.3}$$

Example 3.3.2. Suppose $x \in H^*(X; \mathbb{F}_p)$ is primitive. Then $\mu^*(x) = x \otimes 1 + 1 \otimes x$, and $i^*(x) = -x$. Hence $com^*(x) = 0$.

Let us find an elementwise formula for com^* in terms of the coproduct μ^*. To do this, we need to determine a choice of generators for $H^*(X; \mathbb{F}_p)$ that will facilitate calculation. In particular, since com^* is a composition involving μ^*, we would like to pick generators based on their coproducts. If X is a finite simply-connected HA-space, the set S defined after Theorem 1.2.4 will suffice: if an odd generator x has reduced coproduct $\sum x' \otimes x''$, we want each x' to be an element of B. Recall that B is primitively generated, so by Theorem 1.1.4, B is cocommutative: if $b \in B$, then

$$\mu^*(b) = T^*_X \mu^*(b). \tag{3.3.4}$$

With this, we can find a formula for $com^*(x)$ when $x \in S$:

Theorem 3.3.3. *Let X be a finite simply-connected HA-space. Let $x \in S$. Then*

$$com^*(x) = \bar{\mu}^*(x) - T^*_X \bar{\mu}^*(x). \tag{3.3.5}$$

Proof. If x is a primitive generator, then $com^*(x) = 0$. Thus, we will compute

$com^*(x)$ where x is a generator of $H^*(X; \mathbb{F}_p)$ that is not primitive. Write $\mu^*(x) = x \otimes 1 + 1 \otimes x + \sum c \otimes d$, where each $c \in B$. Next, we apply $(1 \otimes i^*)$. By equation 2.3.11, we obtain

$$x \otimes 1 + 1 \otimes i^*(x) + \sum c \otimes i^*(d) = x \otimes 1 - 1 \otimes x - \sum 1 \otimes c[i^*(d)] + \sum c \otimes i^*(d). \quad (3.3.6)$$

Now we need to apply $\Delta^*_{X \times X}(\mu^* \otimes (T^* \mu^*))$. Notice that the summand $\sum c \otimes i^*(d)$ is in $B \otimes H^*(X)$, so since

$$\mu^*(c) = T^*_X \mu^*(c), \quad (3.3.7)$$

for these terms, we have

$$\Delta^*_{X \times X}(\mu^* \otimes (T^*_X \mu^*)) = \Delta^*_{X \times X}((T^*_X \mu^*) \otimes (T^*_X \mu^*)) = (T^*_X \mu^*)\Delta^*_X. \quad (3.3.8)$$

Since $\mu^*(1) = T^*_X \mu^*(1) = 1 \otimes 1$, the above equation also works on the terms $-1 \otimes x - \sum 1 \otimes c[i^*(d)]$. Thus, when we apply $\Delta^*_{X \times X}(\mu^* \otimes (T^*_X \mu^*))$ to the right hand side of equation 3.3.6, we get

$$\Delta^*_{X \times X}(\mu^* \otimes (T^*_X \mu^*))(x \otimes 1) + (T^*_X \mu^*)\Delta^*_X(-1 \otimes x - \sum 1 \otimes c[i^*(d)] + \sum c \otimes i^*(d))$$

$$= \mu^*(x) - T^*_X \mu^*(x) - (T^*_X \mu^*)(\sum(-c[i^*(d)] + c[i^*(d)]))$$

$$= \mu^*(x) - T^*_X \mu^*(x). \quad (3.3.9)$$

Since the summand $x \otimes 1 + 1 \otimes x$ in $\mu^*(x)$ is invariant under T^*_X (due to the zero

degree factors), we can replace the coproduct with the reduced coproduct $\bar{\mu}^*(x)$:

$$com^*(x) = \bar{\mu}^*(x) - T_X^* \bar{\mu}^*(x). \tag{3.3.10}$$

$\square$

Recall that we defined $ad = \mu(com \times p_2^X)\Delta_{X \times X}$. Let us use this to compute $ad^*(x)$

when $x \in S$:

Lemma 3.3.4. *Let X be a finite simply-connected HA-space. Let $x \in S$. Then*

$$ad^*(x) = 1 \otimes x + com^*(x). \tag{3.3.11}$$

Proof. We have

$$ad^* = \Delta_{X \times X}^*(com^* \otimes p_2^{X*})\mu^*.$$

If $\mu^*(x) = x \otimes 1 + 1 \otimes x + \sum c \otimes d$, with each $c \in B$, then

$$(com^* \otimes p_2^{X*})\mu^*(x)$$

$$= com^*(x) \otimes p_2^{X*}(1) + 1 \otimes 1 \otimes p_2^{X*}(x) + \sum com^*(c) \otimes p_2^{X*}(x)$$

$$= com^*(x) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes x + \sum com^*(c) \otimes 1 \otimes d,$$

so

$$ad^*(x) = com^*(x) + 1 \otimes x + \sum com^*(c)(1 \otimes d). \tag{3.3.12}$$

Note that if we expand $c = \sum c_1 \ldots c_n$ as a sum of products of primitive generators c_j in B, then we have $com^*(c) = \sum \prod_{j=1}^{n} com^*(c_j) = 0$ since com^* vanishes on primitive elements. Hence equation 3.3.12 becomes

$$
\begin{aligned}
ad^*(x) &= com^*(x) + 1 \otimes x + \sum com^*(c)(1 \otimes d) \\
&= 1 \otimes x + com^*(x).
\end{aligned}
$$

In particular, $ad^*(x) = 1 \otimes x = p_2^{X*}(x)$ if and only if $com^*(x) = 0$. $\square$

Now that we have a formula for com^* on S, we can find a formula for

$$
Com^* : H^*(\Omega X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)
$$

on suspension elements in $H^*(\Omega X; \mathbb{F}_p)$:

Lemma 3.3.5. *Let X be a finite simply-connected HA-space. Let $x \in S$. Then*

$$
Com^*(\sigma^*(x)) = (1 \otimes \sigma^*)(\bar{\mu}^*(x)). \tag{3.3.13}
$$

Proof. By Lemma 3.2.10, we have

$$
com(id_X \times \hat{\varepsilon})(T_{S^1,X} \times id_{\Omega X}) \simeq \hat{\varepsilon}(id_{S^1} \times Com).
$$

This implies that in cohomology, there is a generator $s \in H^1(S^1; \mathbb{F}_p)$ such that if x is

a element of $H^*(X; \mathbb{F}_p)$, then

$$\hat{\varepsilon}^*(x) = s \otimes \sigma^*(x),$$

$$\left(T^*_{S^1,X} \otimes 1\right)(1 \otimes \hat{\varepsilon}^*)(com^*(x)) = (1 \otimes Com^*)(\hat{\varepsilon}^*(x)). \qquad (3.3.14)$$

When we try to get a formula in $H^*(\Omega X; \mathbb{F}_p)$ with $\sigma^*(x)$ and without $s \in H^1(S^1; \mathbb{F}_p)$, we need to be mindful of $T^*_{S^1,X}$, which causes a sign change on terms in $com^*(x)$ whose left factors have odd degree: if $a \in H^*(X; \mathbb{F}_p)$ has odd degree, then since $|s| = 1$,

$$
\begin{aligned}
T^*_{S^1,X}(a \otimes s) &= (-1)^{|a||s|} s \otimes a \\
&= (-1)^{|a|} s \otimes a.
\end{aligned}
$$

Consequently, if we write

$$com^*(x) = \sum a \otimes b \in H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p),$$

then

$$(-1)^{|a|} \sum a \otimes \sigma^*(b) = Com^*(\sigma^*(x)). \qquad (3.3.15)$$

Hence if no term in the expansion of $com^*(x)$ has the form $a \otimes b$ where a has odd

degree and b is not in the kernel of σ^*, then equation 3.3.15 simplifies to

$$(1 \otimes \sigma^*)com^*(x) = Com^*(\sigma^*(x)). \qquad (3.3.16)$$

Let $x \in S$. We can show that no term in the expansion of $com^*(x)$ has the form $a \otimes b$ where a has odd degree and b is not in the kernel of σ^*. If x has even degree, x is primitive, so $com^*(x) = 0$, and there are no terms in $com^*(x)$ with the form $a \otimes b$ where a has odd degree and b is not in the kernel of σ^*. If x has odd degree, then if x is primitive, $com^*(x) = 0$. If x is not primitive, we can write its reduced coproduct as

$$\bar{\mu}^*(x) = \sum b \otimes r,$$

where each b has even degree and each r has odd degree. Hence by Theorem 3.3.3,

$$com^*(x) = \sum b \otimes r - \sum r \otimes b.$$

The terms in the expansion of $com^*(x)$ with an odd degree element in their left factor are all of the form $r \otimes b$, but since each b has even degree, each b is in the kernel of σ^* by Lemma 2.2.3, so in the end, equation 3.3.16 holds for any element of S.

Now let us prove the lemma. If x is an even degree generator in S, then x is primitive, so $\bar{\mu}^*(x) = 0$. Furthermore, since x has even degree, by Lemma 2.2.3,

$\sigma^*(x) = 0$. Hence by Theorem 3.3.3,

$$
\begin{aligned}
Com^*(\sigma^*(x)) &= Com^*(0) \\
&= 0 \\
&= (1 \otimes \sigma^*)(0) \\
&= (1 \otimes \sigma^*)(\bar{\mu}^*(x)) \quad\quad (3.3.17)
\end{aligned}
$$

Now suppose x is an odd degree generator in S. We have that

$$
\begin{aligned}
Com^*(\sigma^*(x)) &= (1 \otimes \sigma^*)com^*(x). \\
&= (1 \otimes \sigma^*)[\bar{\mu}^*(x) - T_X^*\bar{\mu}^*(x)]. \quad\quad (3.3.18)
\end{aligned}
$$

Since $\bar{\mu}^*(x) \in B \otimes H^*(X; \mathbb{F}_p)$, $T_X^*\bar{\mu}^*(x) \in H^*(X; \mathbb{F}_p) \otimes B$. By definition, every element of B is either an even degree generator in S, or a product of even degree generators. Thus, by Lemma 2.2.3, when we apply $(1 \otimes \sigma^*)$ to $[\bar{\mu}^*(x) - T_X^*\bar{\mu}^*(x)]$, the terms from $-T_X^*\bar{\mu}^*(x)$ will vanish. Therefore,

$$
Com^*(\sigma^*(x)) = (1 \otimes \sigma^*) \circ [\bar{\mu}^*(x)]. \quad\quad (3.3.19)
$$

$\square$

By Theorem 3.2.8, $Ad \simeq \mu_\Omega(Com \times p_2^\Omega)\Delta_{X \times \Omega X}$, so since $\sigma^*(x)$ is primitive, by Lemma

3.3.5, we have:

$$Ad^*(\sigma^*(x)) = \Delta^*_{X \times \Omega X}(Com^* \otimes p_2^{\Omega*})\mu^*_\Omega(\sigma^*(x))$$

$$= \Delta^*_{X \times \Omega X}(Com^* \otimes p_2^{\Omega*})\left(\sigma^*(x) \otimes 1 + 1 \otimes \sigma^*(x)\right)$$

$$= \Delta^*_{X \times \Omega X}\left(Com^*(\sigma^*(x)) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes \sigma^*(x)\right)$$

$$= Com^*(\sigma^*(x)) + 1 \otimes \sigma^*(x)$$

$$= 1 \otimes \sigma^*(x) + (1 \otimes \sigma^*)\bar{\mu}^*(x). \tag{3.3.20}$$

In general, we have this relationship between the triviality of Ad^* and Com^*:

Lemma 3.3.6. *Let X be a finite simply-connected HA-space. In $H^*(\Omega X; \mathbb{F}_p)$, $Ad^* = p_2^{\Omega*}$ iff $Com^* = 0$.*

Proof. By part 3 of Proposition 3.3.1,

$$Ad^* = \Delta^*_{X \times \Omega X}(Com^* \otimes p_2^{\Omega*})\mu^*_\Omega,$$

so in cohomology, for any $y \in H^*(\Omega X; \mathbb{F}_p)$, if we write $\mu^*_\Omega(y) = y \otimes 1 + 1 \otimes y + \sum y' \otimes y''$, then we have

$$Ad^*(y) = Com^*(y) + 1 \otimes y + \sum(Com^*(y'))(1 \otimes y''). \tag{3.3.21}$$

Suppose that $Com^* = 0$. Then by equation 3.3.21,

$$
\begin{aligned}
Ad^*(y) &= Com^*(y) + 1 \otimes y + \sum (Com^*(y'))(1 \otimes y'') \\
&= 1 \otimes y \\
&= p_2^{\Omega*}(y).
\end{aligned}
\tag{3.3.22}
$$

Now suppose that $Ad^* = p_2^{\Omega*}$. We will show that $Com^* = 0$ by induction on degree. First, choose y so that y has positive degree and there are no elements of $\bar{H}^*(\Omega X; \mathbb{F}_p)$ with degree less than y. Then y is primitive, and hence by equation 3.3.21,

$$Ad^*(y) = Com^*(y) + 1 \otimes y,$$

and by our hypothesis,

$$Ad^*(y) = p_2^{\Omega*}(y) = 1 \otimes y,$$

so

$$Com^*(y) + 1 \otimes y = 1 \otimes y,$$

and hence $Com^*(y) = 0$.

For our induction, suppose $Ad^* = p_2^{\Omega*}$, and $Com^*(z) = 0$ for all $z \in \bar{H}(\Omega X; \mathbb{F}_p)$ with degree less than n. Consider any class y with degree n. We have

$$Ad^*(y) = Com^*(y) + 1 \otimes y + \sum (Com^*(y'))(1 \otimes y''),$$

and by our hypothesis,

$$Ad^*(y) = p_2^{\Omega*}(y) = 1 \otimes y.$$

Since each y' has positive degree less than n, $Com^*(y') = 0$, so

$$Com^*(y) + 1 \otimes y + \sum (Com^*(y'))(1 \otimes y'') = Com^*(y) + 1 \otimes y,$$

so

$$Com^*(y) + 1 \otimes y = 1 \otimes y,$$

and thus $Com^*(y) = 0$. Hence $Com^* = 0$ by induction on degree. $\qquad\square$

The computation of Com^* and Ad^* on elements of $H^*(\Omega X; \mathbb{F}_p)$ which are not in the image of the suspension (or in the subalgebra generated by suspension elements) is still an open problem when the integral cohomology of X has p-torsion. See [10] for some examples using the exceptional Lie groups. Nevertheless, we can use these results to find new ways of characterizing finite simply-connected HA-spaces whose integral homology has p-torsion.

3.4 Homology Algebras and HA-space Commutator Maps

Given a finite simply-connected HA-space, we will find a connection between the commutator map *com* and commutativity of $H_*(X; \mathbb{F}_p)$ and then prove Theorem

3.0.4.

As stated in the introduction to this chapter, many of the classical compact simply-connected Lie groups have commutative homology algebras (with coefficients in $\mathbb{F}_p$, p odd). The key to finding a relationship between homotopy commutativity of the group (or HA-space) and commutativity of its homology is not to look at *com* itself, but rather at the induced homomorphism *com**.

Throughout this section, X will be a finite simply-connected HA-space.

Theorem 3.4.1. *Let X be a finite simply-connected HA-space. The homomorphism com** : $H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p)$ *is trivial iff* $H_*(X; \mathbb{F}_p)$ *is commutative.*

Proof. If $H_*(X; \mathbb{F}_p)$ is commutative, then $H^*(X; \mathbb{F}_p)$ is cocommutative, so if x is any element of S,

$$\overline{\mu}^*(x) = T_X^* \overline{\mu}^*(x).$$

By Theorem 3.3.3,

$$com^*(x) = \overline{\mu}^*(x) - T_X^* \overline{\mu}^*(x) = 0,$$

so *com** vanishes on the generating set, and hence on $H^*(X; \mathbb{F}_p)$.

If $H_*(X; \mathbb{F}_p)$ is not commutative, then $H^*(X; \mathbb{F}_p)$ is not primitively generated [16]. Thus, in S, there is an odd element x whose reduced coproduct is $\sum c \otimes d \neq 0$, where each c has even degree and each d has odd degree. Then by Theorem 3.3.3,

$$com^*(x) = \sum c \otimes d - \sum d \otimes c.$$

No term of the form $c \otimes d$ could equal any term of the form $d \otimes c$ for degree reasons: the left factor of $c \otimes d$ has even degree, while the left factor of $d \otimes c$ has odd degree. Hence $com^*(x) \neq 0$. $\qquad\qquad\square$

Now let us turn to Theorem 3.0.4. Zabrodsky has shown that $H^*(X; \mathbb{F}_p)$ is primitively generated if and only if $H_*(X; \mathbb{Z})$ has no p-torsion in Theorem D of [41], and Kane has shown that $H^*(X; \mathbb{F}_p)$ is primitively generated if and only if $H_*(X; \mathbb{F}_p)$ is commutative in Theorem 1.1 of [16], so by Theorem 3.4.1, we have the equivalence of statements (ii), (iii), and (v) in Theorem 3.0.4. Let us finish things by showing that statements (iii) and (vi) are equivalent:

Theorem 3.4.2. *Let X be a finite simply-connected HA-space. The Hopf algebra $H^*(X; \mathbb{F}_p)$ is primitively generated if and only if the homomorphism $Com^* : H^*(\Omega X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$ is trivial.*

Proof. Suppose that $H^*(X; \mathbb{F}_p)$ is primitively generated. Zabrodsky shows that since X is an HA-space, then $H^*(X; \mathbb{F}_p)$ being primitively generated implies $H^*(X; \mathbb{F}_p)$ must be a free algebra [41]. Since $H^*(X; \mathbb{F}_p)$ is finite dimensional, it must be an exterior algebra on odd degree generators. Consequently, by the Leray-Samelson theorem (quoted from [19]), since $H_*(X; \mathbb{F}_p)$ is associative (due to the fact that X is an HA-space), it must also be an exterior algebra on odd degree generators.

Let us use part 9 of Proposition 3.3.1 to show that if $\bar{x}$ is a positive degree element of $H_*(X; \mathbb{F}_p)$, and $\bar{u} \in H_*(\Omega X; \mathbb{F}_p)$, then $Ad_*(\bar{x} \otimes \bar{u}) = 0$. By Lemma 2.2.3, $H_*(\Omega X; \mathbb{F}_p)$ has no odd degree elements, so any element $\bar{u} \in H_*(\Omega X; \mathbb{F}_p)$ has

even degree. Thus, if $\bar{x} \in H_*(X; \mathbb{F}_p)$ has odd degree, then $Ad_*(\bar{x} \otimes \bar{u})$ has odd degree, so it must be zero. Otherwise, if $\bar{x}$ has even degree, then since $H_*(X; \mathbb{F}_p)$ is an exterior algebra on odd degree elements, $\bar{x}$ must be a product of two elements $\bar{x} = \bar{y}\bar{z} = \mu_*(\bar{y} \otimes \bar{z})$ where $\bar{z}$ has odd degree (note that $\bar{y}$ and $\bar{z}$ might be decomposables as well). Then

$$
\begin{aligned}
Ad_*(\bar{x} \otimes \bar{u}) &= Ad_*(\mu_* \otimes 1)(\bar{y} \otimes \bar{z} \otimes \bar{u}) \\
&= Ad_*(1 \otimes Ad_*)(\bar{y} \otimes \bar{z} \otimes \bar{u}) \\
&= Ad_*(\bar{y} \otimes Ad_*(\bar{z} \otimes \bar{u})),
\end{aligned}
$$

and since $Ad_*(\bar{z} \otimes \bar{u})$ would have odd degree, it must be zero, so $Ad_*(\bar{x} \otimes \bar{u}) = 0$ for any $\bar{x} \in H_*(X; \mathbb{F}_p)$ with positive degree. On the other hand, by part 6 of Proposition 3.3.1, $Ad_*(1 \otimes \bar{u}) = \bar{u}$. In comparison, we have

$$
p_{2*}^{\Omega}(\bar{x} \otimes \bar{u}) = \begin{cases} 0 & |\bar{x}| > 0 \\ \\ \bar{u} & \bar{x} = 1 \end{cases}
$$

so $Ad_* = p_{2*}^{\Omega}$. Dually, in cohomology, we have $Ad^* = p_2^{\Omega*}$, so by Lemma 3.3.6, Com^* is trivial on $H^*(\Omega X; \mathbb{F}_p)$.

Now suppose $H^*(\Omega X; \mathbb{F}_p)$ is not primitively generated. Then there exists a generator $z \in S$, with

$$
\bar{\mu}^*(z) = \sum b \otimes r \neq 0,
$$

and since any even degree element of S is primitive, z must have odd degree, so each b has even degree, and each r is a generator of $H^*(\Omega X; \mathbb{F}_p)$. Since z, and each r, are odd degree generators, by Theorem 2.2.4, $\sigma^*(z) \neq 0$, and $\sigma^*(r) \neq 0$ for each r. Hence

$$
\begin{aligned}
Com^*(\sigma^*(x)) &= (1 \otimes \sigma^*) \circ [\bar{\mu}^*(x)] \\
&= \sum b \otimes \sigma^*(r) \\
&\neq 0.
\end{aligned}
$$

Hence Com^* is not trivial on $H^*(\Omega X; \mathbb{F}_p)$. $\qquad\square$

With this result, we have shown that Kono, Kozima, and Iwase's results do not require a space to be homotopy equivalent to a topological group. That is, we do not need the entire structure of a finite simply-connected topological group: a finite simply-connected HA-space will be enough.

3.5 Other Properties of the Maps

Let us prove some other properties of the maps ad, com, Ad, and Com, generalizing the results from Kono and Kozima's paper [21] (specifically from Section 2 of their paper). These properties were not used in our results from this chapter, but will be needed in a few examples in Chapter 4.

Remark. Part 3 of Proposition 2.2 from Kono and Kozima's paper [21] states that Ad commutes with diagonal maps, so we will not prove it ourselves. The material

involving maximal tori of Lie groups is not applicable to HA-spaces, so they will not be generalized.

Proposition 3.5.1. *Let X be a simply-connected HA-space. The maps Com and Ad are related as follows:*

$$Com \simeq \mu_\Omega(Ad \times i_\Omega)(id_X \times \Delta_{\Omega X}). \qquad (3.5.1)$$

That is, the following diagram commutes:

$$
\begin{array}{ccc}
X \times \Omega X & \xrightarrow{\quad Com \quad} & \Omega X \\
{\scriptstyle id_X \times \Delta_{\Omega X}}\big\downarrow & & \big\uparrow{\scriptstyle \mu_\Omega} \\
X \times \Omega X \times \Omega X & \xrightarrow[\quad Ad \times i_\Omega \quad]{} & \Omega X \times \Omega X
\end{array}
\qquad (3.5.2)
$$

Proof. First, suppose $X = G$ is also a topological group, so $Ad(g,l)(t) = (gl(t))g^{-1}$, $Com(g,l)(t) = (gl(t))(l(t)g)^{-1}$, and $i_\Omega(l)(t) = l(t)^{-1}$. Then

$$
\begin{aligned}
\mu_\Omega(Ad \times i_\Omega)(id_G \times \Delta_{\Omega G})(g,l)(t) &= \mu_\Omega(Ad \times i_\Omega)(g,l,l)(t) \\
&= \left((gl(t))g^{-1}\right) l(t)^{-1} \\
&= (gl(t))\left(g^{-1}l(t)^{-1}\right) \\
&= (gl(t))(l(t)g)^{-1} \\
&= Com(g,l)(t).
\end{aligned}
$$

Now suppose X is only known to be an HA-space. Let us first prove that

$$\mu_\Lambda(id_{\Lambda X} \times j)(\widehat{Ad} \times i_\Omega)(id_X \times \Delta_{\Omega X}) \simeq \widehat{Com} \qquad (3.5.3)$$

as maps from $X \times \Omega X$ to ΛX. We have

$$\mu_\Lambda(id_{\Lambda X} \times j)(\widehat{Ad} \times i_\Omega)(id_X \times \Delta_{\Omega X})(x, l)(t) = ((xl(t))i(x))\, i(l(t)).$$

By homotopy associativity, $\mu_\Omega(Ad \times i_\Omega)(id_X \times \Delta_{\Omega X})$ is homotopic to a map which takes (x, l) to a loop given by

$$(xl(t))(i(x)i(l(t))).$$

By Lemma 2.3.4, this map is homotopic to one that takes (x, l) to a loop given by

$$(xl(t))i(l(t)x) = \widehat{Com}(x, l)(t).$$

Hence

$$\mu_\Lambda(id_{\Lambda X} \times j)(\widehat{Ad} \times i_\Omega)(id_X \times \Delta_{\Omega X}) \simeq \widehat{Com}.$$

By definition, $j \circ Ad \simeq \widehat{Ad}$ and $j \circ Com \simeq \widehat{Com}$, so

$$\mu_\Lambda(j \times j)(Ad \times i_\Omega)(id_X \times \Delta_{\Omega X}) \simeq j \circ Com.$$

Since j is an H-map, $\mu_\Lambda(j \times j) \simeq j\mu_\Omega$, so

$$j\mu_\Omega(Ad \times i_\Omega)(id_X \times \Delta_{\Omega X}) \simeq j \circ Com,$$

so by Lemma 3.1.4,

$$\mu_\Omega(Ad \times i_\Omega)(id_X \times \Delta_{\Omega X}) \simeq Com.$$

$\square$

Proposition 3.5.2. *Let X be a simply-connected HA-space. Let $D : X \times \Omega X \times \Omega X \to X \times \Omega X \times X \times \Omega X$ be defined as*

$$
\begin{aligned}
D(x, l_1, l_2) &= (x, l_1, x, l_2) \\
&= (id_X \times T_{\Omega X, X} \times id_{\Omega X})(\Delta_X \times id_{\Omega X \times \Omega X})(x, l_1, l_2).
\end{aligned}
$$

Then

$$\mu_\Omega(Ad \times Ad)D \simeq Ad(id_X \times \mu_{\Omega X}).$$

That is, the following diagram commutes:

$$
\begin{array}{ccccc}
X \times \Omega X \times \Omega X & \xrightarrow{D} & X \times \Omega X \times X \times \Omega X & \xrightarrow{Ad \times Ad} & \Omega X \times \Omega X \\
{\scriptstyle id_X \times \mu_\Omega}\downarrow & & & & \downarrow{\scriptstyle \mu_\Omega} \\
X \times \Omega X & \xrightarrow{Ad} & & & \Omega X
\end{array}
$$

Proof. Suppose $X = G$ is also a topological group. Then

$$
\begin{aligned}
\mu_\Omega (Ad \times Ad) D(g, l_1, l_2)(t) &= \mu_\Omega (Ad \times Ad)(g, l_1, g, l_2)(t) \\
&= \left((g l_1(t)) g^{-1} \right) \left((g l_2(t)) g^{-1} \right) \\
&= \left((g l_1(t))(g^{-1} g) \right) \left(l_2(t) g^{-1} \right) \\
&= \left((g l_1(t)) \right) \left(l_2(t) g^{-1} \right) \\
&= (g(l_1(t) l_2(t))) g^{-1} \\
&= Ad(g, \mu_\Omega(l_1, l_2))(t) \\
&= Ad(id_G \times \mu_\Omega)(g, l_1, l_2)(t).
\end{aligned}
$$

Now suppose X is only known to be an HA-space. We will first prove that

$$
\mu_\Lambda(\widehat{Ad} \times \widehat{Ad}) D \simeq \widehat{Ad}(id_X \times \mu_{\Omega X}) \tag{3.5.4}
$$

as maps from $X \times \Omega X \times \Omega X$ to ΛX. We have

$$
\mu_\Lambda(\widehat{Ad} \times \widehat{Ad}) D(x, l_1, l_2)(t) = ((x l_1(t)) i(x)) ((x l_2(t)) i(x)).
$$

By homotopy associativity, $\mu_\Lambda(\widehat{Ad} \times \widehat{Ad}) D$ is homotopic to a map which takes (x, l_1, l_2) to a loop given by

$$
((x l_1(t))(i(x) x)) (l_2(t) i(x)).
$$

By definition of the homotopy inverse operation, this map is homotopic to one that

takes (x, l_1, l_2) to a loop given by

$$((xl_1(t)))\,(l_2(t)i(x))\,.$$

By homotopy associativity, this map is homotopic to one that takes (x, l_1, l_2) to a loop given by

$$(x(l_1(t)l_2(t)))i(x) = \widehat{Ad}(id_X \times \mu_\Omega)(x, l_1, l_2)(t).$$

Hence

$$\mu_\Lambda(\widehat{Ad} \times \widehat{Ad})D \simeq \widehat{Ad}(id_X \times \mu_{\Omega X}).$$

By definition, $j \circ Ad \simeq \widehat{Ad}$, so

$$\mu_\Lambda(j \times j)(Ad \times Ad)D \simeq j \circ Ad(id_X \times \mu_{\Omega X}).$$

Since j is an H-map, $\mu_\Lambda(j \times j) \simeq j\mu_\Omega$, so

$$j\mu_\Omega(Ad \times Ad)D \simeq j \circ Ad(id_X \times \mu_{\Omega X}),$$

so by Lemma 3.1.4,

$$\mu_\Omega(Ad \times Ad)D \simeq Ad(id_X \times \mu_{\Omega X}).$$

$\square$

Proposition 3.5.3. *Let X be a simply-connected HA-space. Let $D : X \times \Omega X \times \Omega X \to$*

$X \times \Omega X \times X \times \Omega X$ *be defined as*

$$
\begin{aligned}
D(x, l_1, l_2) &= (x, l_1, x, l_2) \\
&= (id_X \times T_{X,\Omega X} \times id_{\Omega X})(\Delta_X \times id_{\Omega X \times \Omega X})(x, l_1, l_2).
\end{aligned}
$$

Then

$$
\mu_\Omega(Com \times Com)D \simeq Com(id_X \times \mu_{\Omega X}).
$$

That is, the following diagram commutes:

Proof. Unlike the other properties proven in this section, we will immediately work

with HA-spaces, and we will not need the free loop space. We will exploit the fact

that ΩX is a homotopy commutative HA-space:

$$
\mu_\Omega T_{\Omega X} \simeq \mu_\Omega.
$$

By Proposition 3.5.1, $\mu_\Omega(Com \times Com)D$ is homotopic to

$$
\mu_\Omega((\mu_\Omega(Ad \times i_\Omega)(id_X \times \Delta_{\Omega X})) \times (\mu_\Omega(Ad \times i_\Omega)(id_X \times \Delta_{\Omega X})))D.
$$

Elementwise, this map takes (x, l_1, l_2) to an element of ΩX given (in terms of Ad) by

$$(Ad(x, l_1)i_\Omega(l_1)) \, (Ad(x, l_2)i_\Omega(l_2))$$

(here, concatenation indicates multiplying using μ_Ω). By homotopy commutativity and homotopy associativity (of ΩX), this map is homotopic to one that takes (x, l_1, l_2) to

$$(Ad(x, l_1)Ad(x, l_2)) \, (i_\Omega(l_2)i_\Omega(l_1)) \, ,$$

and by Lemma 2.3.4, this map is homotopic to one that takes (x, l_1, l_2) to

$$(Ad(x, l_1)Ad(x, l_2)) \, i_\Omega \, (l_1 l_2) \, . \tag{3.5.5}$$

Now, elementwise, we have

$$\mu_\Omega(Ad \times Ad)D(x, l_1, l_2) = Ad(x, l_1)Ad(x, l_2),$$

and by Proposition 3.5.2, this is homotopic to $Ad(id_X \times \mu_{\Omega X})$, which is given elementwise by

$$Ad(id_X \times \mu_{\Omega X})(x, l_1, l_2) = Ad(x, l_1 l_2).$$

Thus, the map described in equation 3.5.5 is homotopic to one that takes (x, l_1, l_2) to

$$(Ad(x, l_1 l_2)) \, i_\Omega \, (l_1 l_2) \, .$$

By Proposition 3.5.1, this map is homotopic to one that takes (x, l_1, l_2) to

$$Com(x, l_1 l_2) = Com(id_X \times \mu_{\Omega X})(x, l_1, l_2).$$

Therefore,

$$\mu_\Omega(Com \times Com)D \simeq Com(id_X \times \mu_{\Omega X}).$$

$\square$

Let us apply cohomology and homology to the previous propositions to find properties of the induced homomorphisms and induced linear transformations:

Proposition 3.5.4. *We have the following properties of com^*, ad^*, Com^*, and Ad^*:*

1. *In $H^*(\Omega X; \mathbb{F}_p)$, $Com^* = (1 \otimes \Delta^*_{\Omega X})(Ad^* \otimes i^*_\Omega)\mu^*_\Omega$.*

2. *In $H^*(\Omega X; \mathbb{F}_p)$,*

$$D^*(Ad^* \otimes Ad^*)\mu_{\Omega *} = (1 \otimes \mu^*_{\Omega X})Ad^*.$$

3. *In $H^*(\Omega X; \mathbb{F}_p)$,*

$$D^*(Com^* \otimes Com^*)\mu_{\Omega *} = (1 \otimes \mu^*_{\Omega X})Com^*.$$

We have the following properties of com_*, ad_*, Com_*, and Ad_*:

4. *In $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$, $Com_* = \mu_{\Omega *}(Ad_* \otimes i_{\Omega *})(1 \otimes \Delta_{\Omega X *})$.*

5. In $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$,

$$\mu_{\Omega *}(Ad_* \otimes Ad_*)D_* = Ad_*(1 \otimes \mu_{\Omega X*})$$

6. In $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$,

$$\mu_{\Omega *}(Com_* \otimes Com_*)D_* = Com_*(1 \otimes \mu_{\Omega X*})$$

Let us end with a relationship between Ad, Com, and H-maps.

Lemma 3.5.5. *Let (X, μ_X) and (Y, μ_Y) be simply-connected HA-spaces. Let $f : X \to Y$ be an H-map. Then the following diagrams commute:*

$$
\begin{array}{ccc}
X \times \Omega X & \xrightarrow{\ Ad_X\ } & \Omega X \\
{\scriptstyle f \times \Omega f}\downarrow & & \downarrow{\scriptstyle \Omega f} \\
Y \times \Omega Y & \xrightarrow[\ Ad_Y\ ]{} & \Omega Y
\end{array}
$$

$$
\begin{array}{ccc}
X \times \Omega X & \xrightarrow{\ Com_X\ } & \Omega X \\
{\scriptstyle f \times \Omega f}\downarrow & & \downarrow{\scriptstyle \Omega f} \\
Y \times \Omega Y & \xrightarrow[\ Com_Y\ ]{} & \Omega Y
\end{array}
$$

(we will use subscripts X and Y to distinguish between the respective maps for each space). For reference, $\Omega f : \Omega X \to \Omega Y$ is pointwise application of f; if $l \in \Omega X$,

$$(\Omega f)(l)(t) = f(l(t)),$$

and similarly, $\Lambda f : \Lambda X \to \Lambda Y$ is pointwise application of f; if $\varphi \in \Lambda X$,

$$(\Lambda f)(\varphi)(t) = f(\varphi(t)).$$

Proof. Let us first prove this lemma for the special case that (G, μ_G) and (H, μ_H) are simply-connected topological groups, and $f : G \to H$ is a group homomorphism. For any $g \in G$ and $l \in \Omega G$, if $t \in [0, 1]$,

$$(\Omega f) Ad_G(g, l)(t) = f((gl(t))g^{-1}) \in H. \tag{3.5.6}$$

Since f is a group homomorphism, 3.5.6 becomes

$$
\begin{aligned}
(\Omega f) Ad_G(g, l)(t) &= (f(g)f(l(t)))f(g)^{-1} \\
&= Ad_H(f(g), (\Omega f)(l))(t) \\
&= Ad_H(f \times \Omega f)(g, l)(t).
\end{aligned}
$$

The proof for the second diagram is similar.

Now suppose that (X, μ_X) and (Y, μ_Y) are simply-connected HA-spaces with homotopy inverse operations given by i_X and i_Y respectively, and $f : X \to Y$ is an

H-map. We will first prove that the following diagram with free loop spaces commutes:

$$
\begin{array}{ccc}
X \times \Omega X & \xrightarrow{\widehat{Ad}_X} & \Lambda X \\
{\scriptstyle f \times \Omega f}\downarrow & & \downarrow{\scriptstyle \Lambda f} \\
Y \times \Omega Y & \xrightarrow[\widehat{Ad}_Y]{} & \Lambda Y
\end{array}
\tag{3.5.7}
$$

For reference, here are the maps involved:

$$
\begin{array}{ccc}
 & & \Omega X \\
 & \nearrow{\scriptstyle Ad_X} & \downarrow{\scriptstyle j_X} \\
X \times \Omega X & \xrightarrow[\widehat{Ad}_X]{} & \Lambda X
\end{array}
$$

$$
\begin{array}{ccc}
 & & \Omega Y \\
 & \nearrow{\scriptstyle Ad_Y} & \downarrow{\scriptstyle j_Y} \\
Y \times \Omega Y & \xrightarrow[\widehat{Ad}_Y]{} & \Lambda Y
\end{array}
$$

For any $x \in X$ and $l \in \Omega X$, if $t \in [0,1]$,

$$
(\Lambda f)\widehat{Ad}_X(x,l)(t) = f((xl(t))i(x)) \in Y.
$$

Since f is an H-map, $(\Lambda f)\widehat{Ad}_X$ is homotopic to a map which takes (x,l) to a loop given by

$$
(f(x)f(l(t)))\, f(i_X(x)).
$$

By Lemma 2.4.4, $fi_X \simeq i_Y f$, so $(\Lambda f)\widehat{Ad}_X$ is homotopic to a map which takes (x,l)

to a loop given by

$$(f(x)f(l(t)))\, i_Y\,(f(x)) = \widehat{Ad}_Y(f(x), (\Omega f)(l))(t)$$

$$= \widehat{Ad}_Y(f \times \Omega f)(x, l)(t).$$

Therefore, diagram 3.5.7 commutes.

By definition, $j_X Ad_X \simeq \widehat{Ad}_X$ and $j_Y Ad_Y \simeq \widehat{Ad}_Y$. Meanwhile, if $l \in \Omega X \subset \Lambda X$, then $(\Omega f)(l) \in \Omega Y \subset \Lambda Y$, so

$$j_Y(\Omega f)(l) = (\Omega f)(l)$$

$$= (\Lambda f)(j_X(l)),$$

so $j_Y(\Omega f) = (\Lambda f)j_X$. Since we have proven that

$$\Lambda f \circ \widehat{Ad}_X \simeq \widehat{Ad}_Y(f \times \Omega f),$$

then since $j_X Ad_X \simeq \widehat{Ad}_X$ and $j_Y Ad_Y \simeq \widehat{Ad}_Y$, we must have

$$\Lambda f \circ j_X Ad_X \simeq j_Y Ad_Y(f \times \Omega f),$$

so since $j_Y(\Omega f) = (\Lambda f)j_X$,

$$j_Y \Omega f \circ Ad_X \simeq j_Y Ad_Y(f \times \Omega f),$$

and therefore, by Lemma 3.1.4,

$$\Omega f \circ Ad_X \simeq Ad_Y(f \times \Omega f),$$

so the diagram is commutative:

$$\begin{array}{ccc}
X \times \Omega X & \xrightarrow{\;Ad_X\;} & \Omega X \\
{\scriptstyle f \times \Omega f}\downarrow & & \downarrow{\scriptstyle \Omega f} \\
Y \times \Omega Y & \xrightarrow[\;Ad_Y\;]{} & \Omega Y
\end{array}$$

The proof for the second diagram with Com_X and Com_Y is similar. $\square$

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4 The Free Loop Space of an HA-space

One of Kono and Kozima's motivations for introducing the adjoint action came from their study of the free loop space $(\Lambda G, \mu_\Lambda)$ of a Lie group (G, μ). One can ask how the group structure on $(\Lambda G, \mu_\Lambda)$ differs from the product group structure on $(G \times \Omega G, \mu_{G \times \Omega G})$ where

$$\mu_{G \times \Omega G} = (\mu \times \mu_\Omega)(id_G \times T_{\Omega G, G} \times id_{\Omega G}).$$

Kono and Kozima show that there is a homeomorphism of topological spaces $\Phi : \Omega G \times G \to \Lambda G$ where $\Phi(l, g)$ is the loop given by $l(t)g$. If this homeomorphism is also an H-map, then the following diagram would commute:

$$
\begin{array}{ccc}
\Omega G \times G \times \Omega G \times G & \xrightarrow{\;(\mu_\Omega \times \mu)(id_{\Omega G} \times T \times id_G)\;} & \Omega G \times G \\
{\scriptstyle \Phi \times \Phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\
\Lambda G \times \Lambda G & \xrightarrow{\qquad\qquad \mu_\Lambda \qquad\qquad} & \Lambda G
\end{array}
$$

Kono and Kozima present a modified diagram that changes the maps in the top row.

They define a map

$$\omega : \Omega G \times G \times \Omega G \times G \to \Omega G \times \Omega G \times G \times G$$

given by the composition

$$\omega = \left(id_{\Omega G} \times T_{G, \Omega G} \times id_G \right) \left(id_{\Omega G \times G} \times Ad \times id_G \right) \left(id_{\Omega G} \times \Delta_G \times id_{\Omega G \times G} \right) ;$$

for any $t \in [0, 1]$,

$$
\begin{aligned}
\omega(l_1, g_1, l_2, g_2)(t) &= (l_1, Ad(g_1, l_2), g_1, g_2)(t) \\
&= (l_1(t), g_1 l_2(t) g_1^{-1}, g_1, g_2).
\end{aligned}
$$

Then the following diagram commutes whether or not Φ is a group homomorphism:

$$
\begin{array}{ccccc}
\Omega G \times G \times \Omega G \times G & \xrightarrow{\ \omega\ } & \Omega G \times \Omega G \times G \times G & \xrightarrow{\ \mu_\Omega \times \mu\ } & \Omega G \times G \\
{\scriptstyle \Phi \times \Phi} \downarrow & & & & \downarrow {\scriptstyle \Phi} \\
\Lambda G \times \Lambda G & & \xrightarrow{\hspace{6cm} \mu_\Lambda \hspace{6cm}} & & \Lambda G
\end{array} \qquad (4.0.1)
$$

An application of this diagram would be the computation of products in the algebra

$H_*(\Lambda G, \mu_\Lambda; \mathbb{F}_p)$ using three ingredients: the product structures of $H_*(G, \mu; \mathbb{F}_p)$ and

$H_*(\Omega G, \mu_\Omega; \mathbb{F}_p)$, and knowledge of the induced linear transformation Ad_*. However,

the furthest that anyone has gone in the past is Hamanaka, who computed the map

$Ad_* : H_*(G; \mathbb{F}_2) \otimes H_*(\Omega G; \mathbb{F}_2) \to H_*(\Omega G; \mathbb{F}_2)$ for the cases when G is an exceptional Lie group other than E_8 [9]. Hamanaka proceeded to apply homology to the previous diagram to provide a formula for $\mu_{\Lambda*}$ in terms of the maps μ_*, $\mu_{\Omega*}$, and Ad_*, but is only able to provide an example for $G = G_2$, remarking that the other cases were too complicated to write down.

In this chapter, we present another approach to computing products in the algebra $H_*(\Lambda X, \mu_\Lambda; \mathbb{F}_p)$, where X is a finite simply-connected HA-space . Our method stems from the H-deviation of another homotopy equivalence, $h_1 : \Lambda X \to X \times \Omega X$. In the case that X is a topological group, we can write down h_1:

$$h_1(\varphi) = (\varphi(0), l)$$

where $l \in \Omega X$ is a loop given pointwise by

$$l(t) = \varphi(0)^{-1}\varphi(t).$$

However, if X is only known to be an HA-space, we will need the free loop fibration to define h_1.

This chapter starts with an introduction to the free loop space ΛX and its relation to the product space $X \times \Omega X$. We will focus on the homotopy equivalence $h_1 : \Lambda X \to X \times \Omega X$ and its H-deviation. Our most important finding is a formula for the H-deviation of h_1 which involves the map $Com : X \times \Omega X \to \Omega X$ (from Chapter 3). We then use the H-deviation of h_1 to find a twisted H-structure on $X \times \Omega X$, obtaining

a diagram similar to Kono and Kozima's diagram 4.0.1 for HA-spaces. From here, we can use the induced homomorphisms (in cohomology) of h_1 and Com, along with the twisted H-structure, to find another characterization of finite simply-connected HA-spaces whose homology over $\mathbb{F}_p$ is commutative, and compute products in $H_*(\Lambda X; \mathbb{F}_p)$ using μ_*, $\mu_{\Omega*}$, and Com_*.

Throughout this chapter, we will let $p_1^\Omega : X \times \Omega X \to X$, $p_2^\Omega : X \times \Omega X \to \Omega X$, $p_1^\Lambda : X \times \Lambda X \to X$, $p_2^\Lambda : X \times \Lambda X \to \Lambda X$ be projections. Recall that the following diagrams commute strictly:

$$
\begin{array}{ccc}
X \times \Omega X & \xrightarrow{\ p_1^\Omega\ } & X \\
{\scriptstyle id_X \times j}\downarrow & & \downarrow{\scriptstyle id_X} \\
X \times \Lambda X & \xrightarrow[\ p_1^\Lambda\ ]{} & X
\end{array}
\qquad (4.0.2)
$$

$$
\begin{array}{ccc}
X \times \Omega X & \xrightarrow{\ p_2^\Omega\ } & \Omega X \\
{\scriptstyle id_X \times j}\downarrow & & \downarrow{\scriptstyle j} \\
X \times \Lambda X & \xrightarrow[\ p_2^\Lambda\ ]{} & \Lambda X
\end{array}
\qquad (4.0.3)
$$

4.1 The Homotopy Equivalence and its H-deviation

We have noticed a few mentions of the homotopy equivalence of ΛX and $X \times \Omega X$ with various conditions on what kind of space X needs to be throughout the literature, but we were unable to find a proof. In particular, given a simply-connected HA-space X, we would like a map $h_1 : \Lambda X \to X \times \Omega X$ which is a homotopy equivalence. We will define h_1 in this section, and prove that it is a homotopy equivalence.

First, let us introduce and define h_1 for the special case when $X = G$ is a

simply-connected topological group and prove that it is a homeomorphism in this case using elementwise calculations. This will prepare the reader for the more general case when X is a simply-connected HA-space when additional tools such as the free loop fibration are needed due to an inability to write down h_1 pointwise.

Definition 4.1.1. Let G be a simply-connected topological group. We define a map $h_1 : \Lambda G \to G \times \Omega G$ as follows. Given a loop $\varphi \in \Lambda G$, at $t \in [0, 1]$, let

$$h_1(\varphi)(t) = (\varphi(0), \varphi(0)^{-1}\varphi(t)).$$

Notice that at $t = 0$, the second component is

$$\varphi(0)^{-1}\varphi(0) = x_0,$$

so the loop given by $\varphi(0)^{-1}\varphi(t)$ is indeed an element of ΩG. We also define a map $h_2 : G \times \Omega G \to \Lambda G$ so that given $(g, l) \in G \times \Omega G$, at $t \in [0, 1]$,

$$h_2(g, l)(t) = gl(t).$$

Theorem 4.1.2. *Let X be a simply-connected topological group. Then $h_2 h_1 = id_{\Lambda G}$ and $h_1 h_2 = id_{G \times \Omega G}$, so h_1 and h_2 are homeomorphisms.*

Proof. Let us compute $h_2 h_1$ and $h_1 h_2$ elementwise. At $t \in [0, 1]$, for any $\varphi \in \Lambda G$,

$$
\begin{aligned}
h_2 h_1(\varphi)(t) &= h_2(\varphi(0), \varphi(0)^{-1}\varphi(t)) \\
&= \varphi(0)\left(\varphi(0)^{-1}\varphi(t)\right) \\
&= \left(\varphi(0)\varphi(0)^{-1}\right)\varphi(t) \\
&= \varphi(t),
\end{aligned}
$$

so elementwise, $h_2 h_1 = id_{\Lambda G}$. Meanwhile, for any $g \in G$ and $l \in \Omega G$,

$$
\begin{aligned}
h_1 h_2(g, l)(t) &= h_1(gl(t)) \\
&= \left(gl(0), (gl(0))^{-1}(gl(t))\right) \\
&= \left(gg_0, \left(g_0^{-1}g^{-1}\right)(gl(t))\right) \\
&= \left(g, \left(g^{-1}\right)(gl(t))\right) \\
&= \left(g, \left(g^{-1}g\right)l(t)\right) \\
&= (g, l(t)),
\end{aligned}
$$

so elementwise, $h_1 h_2 = id_{G \times \Omega G}$. $\qquad \square$

Now let us transition to the case when X is a simply-connected HA-space. We cannot simply copy the definition of h_1 for topological groups, because a loop l given by

$$
l(t) = i(\varphi(0))\varphi(t)
$$

has its basepoint at $i(\varphi(0))\varphi(0)$, which may not equal x_0, so this loop might not be an element of ΩX. Just like in Chapter 3 when defining Com and Ad, we will need the free loop fibration.

Recall that we have a fibration sequence

$$
\begin{array}{c}
\Omega X \\
\downarrow{\scriptstyle j} \\
\Lambda X \\
\downarrow{\scriptstyle \varepsilon_0} \\
X
\end{array}
$$

where $j : \Omega X \to \Lambda X$ is the inclusion

$$j(l) = l,$$

and $\varepsilon_0 : \Lambda X \to X$ is evaluation at $t = 0$:

$$\varepsilon_0(\varphi) = \varphi(0).$$

Recall that the map j is an H-map, and it has unique lifts (see Section 3.1). We will need these facts when defining the homotopy equivalences.

Definition 4.1.3. Given an HA-space X, its loop space ΩX, and its free loop space ΛX, let us define a map $h_1' : \Lambda X \to X \times \Lambda X$ so that at $t \in [0, 1]$, $h_1'(\varphi)$ is given by

$$h_1'(\varphi)(t) = (\varphi(0), i(\varphi(0))\varphi(t)).$$

Notice that

$$p_1^\Lambda h_1'(\varphi)(t) \;=\; \varphi(0)$$

$$\qquad\qquad =\; \varepsilon_0(\varphi). \qquad\qquad (4.1.1)$$

When we apply $\varepsilon_0 p_2^\Lambda$, we get

$$\varepsilon_0 p_2^\Lambda h_1'(\varphi) \;=\; p_2^\Lambda h_1'(\varphi)(0)$$

$$\qquad\qquad =\; i(\varphi(0))\varphi(0), \qquad\qquad (4.1.2)$$

so the composition $\varepsilon_0 p_2^\Lambda h_1' : \Lambda X \to X$ is nullhomotopic. Hence there is a map $h_1'' : \Lambda X \to \Omega X$ such that the diagram commutes:

$$(4.1.3)$$

From here, we can define $h_1 : \Lambda X \to X \times \Omega X$ as

$$h_1 = (\varepsilon_0 \times h_1'') \Delta_{\Lambda X}. \qquad\qquad (4.1.4)$$

Then by equation 4.1.1 and diagram 4.1.3, we have the following commutative dia-

gram:

$$
\begin{array}{c}
\Lambda X \xrightarrow[h'_1]{} X \times \Lambda X
\end{array}
\qquad (4.1.5)
$$

(diagram showing $\Lambda X \xrightarrow{h_1} X \times \Omega X$, with $id_X \times j$ downward to $X \times \Lambda X$, and h'_1 from ΛX to $X \times \Lambda X$)

Unlike the case when X is a topological group, we cannot write down a pointwise

definition of h_1.

Now let us define a map $h'_2 : X \times \Lambda X \to \Lambda X$ as follows: $h'_2(x, \varphi)$ is a loop

given by

$$
h'_2(x, \varphi)(t) = \mu(x, \varphi(t)) = x\varphi(t).
$$

We define $h_2 : X \times \Omega X \to \Lambda X$ to be the restriction of h'_2 to $X \times \Omega X$. Then the

following diagram commutes:

$$
\qquad (4.1.6)
$$

(diagram showing two triangles: $\Lambda X \xrightarrow{h_1} X \times \Omega X$, $id_X \times j$ to $X \times \Lambda X$, h_2 to ΛX, with h'_1 and h'_2; then $\Lambda X \xrightarrow{h_1} X \times \Omega X$, $id_X \times j$ to $X \times \Lambda X$, with h'_1)

Let us show that h_1 is a homotopy equivalence:

Proposition 4.1.4. *The spaces ΛX and $X \times \Omega X$ are homotopy equivalent as topo-*

logical spaces.

Proof. We will prove that the compositions $h_2 h_1 : \Lambda X \to \Lambda X$ and $h_1 h_2 : X \times \Omega X \to$

$X \times \Omega X$ are homotopic to the identity maps $id_{\Lambda X}$ and $id_{X \times \Omega X}$ respectively.

To prove that $h_2 h_2 \simeq id_{\Lambda X}$, let us look at the composition $h'_2 h'_1 : \Lambda X \to \Lambda X$.

Elementwise, it is a loop given by

$$h'_2 h'_1(\varphi)(t) = \varphi(0) \left(i(\varphi(0)) \varphi(t) \right). \tag{4.1.7}$$

By homotopy associativity, this map is homotopic to one that takes φ to a loop given by

$$\left(\varphi(0) i(\varphi(0)) \right) \varphi(t).$$

By using the homotopy inverse operation, this map is homotopic to one that takes φ to a loop given by $\varphi(t)$, so

$$h'_2 h'_1 \simeq id_{\Lambda X}.$$

Hence by diagram 4.1.6,

$$h_2 h_1 \simeq id_{\Lambda X}. \tag{4.1.8}$$

To prove that $h_1 h_2 \simeq id_{X \times \Omega X}$, let us look at $h'_1 h_2 : X \times \Omega X \to X \times \Lambda X$. Pointwise, it is given by

$$
\begin{aligned}
h'_1 h_2(x, l)(t) &= (xl(0), i(xl(0)) [xl(t)]) \\
&= (x, i(x)[xl(t)]).
\end{aligned}
$$

Let us use projection maps to study each component of $h'_1 h_2(x, l)(t)$. In particular,

we have

$$p_1^\Lambda h_1' h_2(x, l) \;=\; x$$

$$=\; p_1^\Lambda (id_X \times j)(id_{X \times \Omega X})(x, l), \qquad (4.1.9)$$

while at any $t \in [0, 1]$,

$$p_2^\Lambda h_1' h_2(x, l)(t) = i(x)[xl(t)], \qquad (4.1.10)$$

so by homotopy associativity, $p_2^\Lambda h_1' h_2$ is homotopic to a map which takes (x, l) to the loop given by

$$(i(x)x)\, l(t),$$

and this map, in turn, is homotopic to one that takes (x, l) to the loop given by

$$l(t) \;=\; j(l)(t)$$

$$=\; p_2^\Lambda (id_X \times j)(id_{X \times \Omega X})(x, l)(t). \qquad (4.1.11)$$

Consequently, by equations 4.1.10 and 4.1.11,

$$p_2^\Lambda h_1' h_2 \simeq p_2^\Lambda (id_X \times j) id_{X \times \Omega X} \qquad (4.1.12)$$

and therefore, by equations 4.1.9 and 4.1.12,

$$h_1' h_2 \simeq (id_X \times j) id_{X \times \Omega X}. \qquad (4.1.13)$$

Thus, by diagram 4.1.6, $(id_X \times j)h_1' \simeq h_1$, so we can rewrite the previous equation as

$$(id_X \times j)h_1h_2 \simeq (id_X \times j)id_{X \times \Omega X}. \qquad (4.1.14)$$

We need to show that $h_1h_2 \simeq id_{X \times \Omega X}$. We will proceed by proving that

$$p_1^\Omega h_1 h_2 \simeq p_1^\Omega id_{X \times \Omega X}$$

and

$$p_2^\Omega h_1 h_2 \simeq p_2^\Omega id_{X \times \Omega X}.$$

When we apply p_1^Λ on both sides of equation 4.1.14, we get

$$p_1^\Lambda (id_X \times j)h_1h_2 \simeq p_1^\Lambda (id_X \times j)id_{X \times \Omega X}. \qquad (4.1.15)$$

and by diagram 4.0.2,

$$p_1^\Lambda (id_X \times j) \simeq id_X p_1^\Omega,$$

so we obtain

$$id_X p_1^\Omega h_1 h_2 \simeq id_X p_1^\Omega id_{X \times \Omega X},$$

and hence

$$p_1^\Omega h_1 h_2 \simeq p_1^\Omega id_{X \times \Omega X}. \qquad (4.1.16)$$

When we apply p_2^Λ on both sides of equation 4.1.14 and use diagram 4.0.3, we obtain

$$jp_2^\Omega h_1 h_2 \simeq jp_2^\Omega id_{X \times \Omega X},$$

so by Lemma 3.1.4,

$$p_2^\Omega h_1 h_2 \simeq p_2^\Omega id_{X \times \Omega X}. \tag{4.1.17}$$

Therefore, by combining equations 4.1.16 and 4.1.17, we obtain

$$h_1 h_2 \simeq id_{X \times \Omega X}. \tag{4.1.18}$$

Hence, we can conclude that h_1 is a homotopy equivalence. $\qquad\square$

In particular, the induced homomorphisms

$$h_1^* : H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p) \to H^*(\Lambda X; \mathbb{F}_p)$$

and

$$h_2^* : H^*(\Lambda X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$$

are algebra isomorphisms. Hence any element of $H^*(\Lambda X; \mathbb{F}_p)$ has the form

$$h_1^*(x \otimes y)$$

where $x \in H^*(X; \mathbb{F}_p)$ and $y \in H^*(\Omega X; \mathbb{F}_p)$.

An immediate application of the homotopy equivalence h_1 is that we can use it to help us compute the homomorphism on cohomology induced by ε_0. By equation 4.1.4, the following diagram commutes:

$$\begin{array}{c} \xymatrix{ \Lambda X \ar@/^/[rr]^{h_1} \ar[dr]_{\varepsilon_0} & & X \times \Omega X \ar@/^/[ll]^{h_2} \ar[dl]^{p_1^{\Omega}} \\ & X & } \end{array} \qquad (4.1.19)$$

Hence if $x \in H^*(X; \mathbb{F}_p)$, then

$$\varepsilon_0^*(x) = h_1^* p_1^{\Omega*}(x) = h_1^*(x \otimes 1). \qquad (4.1.20)$$

Now that we have introduced the homotopy equivalence h_1, we can proceed to discuss when it is an H-map, and examine its H-deviation, a map $D_{h_1} : \Lambda X \times \Lambda X \to X \times \Omega X$. We will find a way to write D_{h_1} as a composition of maps, including Com.

One may ask how the homotopy equivalences interact with the HA-space structures. Recall that we give $X \times \Omega X$ a product HA-space structure with

$$\mu_{X \times \Omega X} = (\mu \times \mu_\Omega)(id_X \times T_{\Omega X, X} \times id_{\Omega X}) : X \times \Omega X \times X \times \Omega X \to X \times \Omega X. \quad (4.1.21)$$

Pointwise, given $(x_1, l_1), (x_2, l_2) \in X \times \Omega X$, at $t \in [0, 1]$,

$$\mu_{X \times \Omega X}((x_1, l_1), (x_2, l_2))(t) = (x_1 x_2, l_1(t) l_2(t)). \qquad (4.1.22)$$

Thus, what we would like to know is if h_1 and h_2 are H-maps. If they are H-maps, then in particular, the induced algebra isomorphism in cohomology

$$h_1^* : H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p) \to H^*(\Lambda X; \mathbb{F}_p)$$

would be a Hopf algebra isomorphism, and the induced *linear transformation* in homology

$$h_{2*} : H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \to H_*(\Lambda X; \mathbb{F}_p)$$

is a vector space isomorphism which is also an algebra isomorphism. Thus knowing the coproducts in $H^*(X; \mathbb{F}_p)$ and $H^*(\Omega X; \mathbb{F}_p)$ would allow easy computation of coproducts in $H^*(\Lambda X; \mathbb{F}_p)$: we compute the coproducts in $H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$ and apply h_1^* to get coproducts in $H^*(\Lambda X; \mathbb{F}_p)$. Dually, we can find the product of two elements in $H_*(\Lambda X; \mathbb{F}_p)$ by multiplying their preimages under h_{2*} in $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$ and then applying h_{2*}.

However, h_1 might not be an H-map. Instead, we will determine how it fails to preserve the multiplication by using its H-deviation:

$$D_{h_1} : \Lambda X \times \Lambda X \to X \times \Omega X,$$

$$D_{h_1} = \mu_{X \times \Omega X} \left((h_1 \circ \mu_\Lambda) \times \left((i \times i_\Omega) \circ (\mu_{X \times \Omega X}) \circ (h_1 \times h_1) \right) \right) \Delta_{\Lambda X \times \Lambda X}.$$

In order to study D_{h_1}, we will also need the H-deviation of h_1', which is given as

follows. First, we need the multiplication map on $X \times \Lambda X$:

$$\mu_{X \times \Lambda X} = (\mu \times \mu_\Lambda)(id_X \times T_{\Lambda X, X} \times id_{\Lambda X}) : X \times \Lambda X \times X \times \Lambda X \to X \times \Lambda X.$$

Pointwise, given $(x_1, \varphi_1), (x_2, \varphi_2) \in X \times \Lambda X$, at $t \in [0, 1]$,

$$\mu_{X \times \Lambda X}((x_1, \varphi_1), (x_2, \varphi_2))(t) = (x_1 x_2, \varphi_1(t)\varphi_2(t)). \qquad (4.1.23)$$

From there, we define the H-deviation of h'_1 as a map

$$D_{h'_1} : \Lambda X \times \Lambda X \to X \times \Lambda X,$$

$$D_{h'_1} = \mu_{X \times \Lambda X} \left((h'_1 \circ \mu_\Lambda) \times ((i \times i_\Lambda) \circ (\mu_{X \times \Lambda X}) \circ (h'_1 \times h'_1)) \right) \Delta_{\Lambda X \times \Lambda X}. \qquad (4.1.24)$$

We can write out $D_{h'_1}$ pointwise. Given $\varphi_1, \varphi_2 \in \Lambda X$, $D_{h'_1}(\varphi_1, \varphi_2) \in X \times \Lambda X$ is given as follows: its first component is

$$(\varphi_1(0)\varphi_2(0)) \, i \, (\varphi_1(0)\varphi_2(0)) \, ,$$

and its second component is a loop (in ΛX) given by

$$(i \, (\varphi_1(0)\varphi_2(0)) \, (\varphi_1(t)\varphi_2(t))) \, i \, ((i \, (\varphi_1(0)) \, \varphi_1(t)) \, (i \, (\varphi_2(0)) \, \varphi_2(t))) \, .$$

Recall that according to diagram 4.1.5,

$$(id_X \times j) \, h_1 \simeq h_1'. \tag{4.1.25}$$

We can use this to find a formula involving D_{h_1} and $D_{h_1'}$. Note that $id_X \times j :$
$X \times \Omega X \to X \times \Lambda X$ is an H-map (the proof is similar to that of Lemma 3.1.3). From
there, by applying Proposition 1.4.2 from Zabrodsky's book $[40]$, we see that

$$(id_X \times j) \, D_{h_1} \simeq D_{h_1'}, \tag{4.1.26}$$

Let us combine equation 4.1.26 with diagrams 4.0.2 and 4.0.3; the resulting diagrams
in the following proposition will be used in the proof of Lemma 4.1.6.

Proposition 4.1.5. *The following diagrams commute:*

$$\tag{4.1.27}$$

$$\tag{4.1.28}$$

We will show how the H-deviation of h_1 allows us to examine coproducts in $H^*(\Lambda X; \mathbb{F}_p)$. The first part of Lemma 4.1.6 is a restatement of Lemma 2.4.8 with h_1^*, but the second part of the lemma is specific to D_{h_1}, so the proof following the lemma will focus on proving equation 4.1.30.

Lemma 4.1.6. *Given any $x \otimes y \in H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$, we have:*

$$\mu_\Lambda^*(h_1^*(x \otimes y)) = \Delta_{\Lambda X \times \Lambda X}^* \left[\left(D_{h_1}^* \otimes [(h_1^* \otimes h_1^*)(\mu_{X \times \Omega X})^*] \right) (\mu_{X \times \Omega X})^*(x \otimes y) \right]. \quad (4.1.29)$$

In addition, we can write the map D_{h_1} as the composition

$$D_{h_1} \simeq j_2 \circ Com \circ (i \times id_{\Omega X}) \circ (\varepsilon_0 \times (p_2^\Omega \circ h_1)) \circ T_{\Lambda X} \quad (4.1.30)$$

where $j_2 : \Omega X \to X \times \Omega X$ is defined by $j_2(l) = (x_0, l)$.

This means that the coproduct structure in $H^*(\Lambda X; \mathbb{F}_p)$ can be determined from the coproduct structure of $H^*(X; \mathbb{F}_p)$ and $H^*(\Omega X; \mathbb{F}_p)$, along with knowledge of the map Com^*.

We will prove equation 4.1.30 for topological groups first, and then prove it for HA-spaces. We will use the definition of h_1 from Definition 4.1.1.

Proof. Let $X = G$ be a finite simply-connected topological group. We will prove that

$$D_{h_1} = j_2 \circ Com \circ (i \times id_{\Omega G}) \circ (\varepsilon_0 \times (p_2^\Omega \circ h_1)) \circ T_{\Lambda G}$$

(notice the equality sign). We can write

$$D_{h_1} = \mu_{G \times \Omega G} \left((h_1 \circ \mu_\Lambda) \times \left((i \times i_\Omega) \circ (\mu_{G \times \Omega G}) \circ (h_1 \times h_1) \right) \right) \Delta_{\Lambda G \times \Lambda G}$$

pointwise: for any $\varphi_1, \varphi_2 \in \Lambda G$, $D_{h_1}(\varphi_1, \varphi_2) \in G \times \Omega G$ is given as follows: its first component is

$$(\varphi_1(0)\varphi_2(0)) \left(\varphi_1(0)\varphi_2(0) \right)^{-1}$$

$$= g_0, \tag{4.1.31}$$

and its second component is a loop in ΩG given by

$$\left((\varphi_1(0)\varphi_2(0))^{-1} (\varphi_1(t)\varphi_2(t)) \right) \left(\left(\varphi_1(0)^{-1}\varphi_1(t) \right) \left(\varphi_2(0)^{-1}\varphi_2(t) \right) \right)^{-1}$$

$$= \left(\left(\varphi_2(0)^{-1}\varphi_1(0)^{-1} \right) \varphi_1(t) \right) \left(\varphi_2(t) \left(\left(\varphi_2(0)^{-1}\varphi_2(t) \right)^{-1} \left(\varphi_1(0)^{-1}\varphi_1(t) \right)^{-1} \right) \right)$$

$$= \left(\varphi_2(0)^{-1} \left(\varphi_1(0)^{-1}\varphi_1(t) \right) \right) \left(\varphi_2(t) \left(\left(\varphi_2(t)^{-1}\varphi_2(0) \right) \left(\varphi_1(0)^{-1}\varphi_1(t) \right)^{-1} \right) \right)$$

$$= \left(\varphi_2(0)^{-1} \left(\varphi_1(0)^{-1}\varphi_1(t) \right) \right) \left(\left(\varphi_2(t)\varphi_2(t)^{-1} \right) \left(\varphi_2(0) \left(\varphi_1(0)^{-1}\varphi_1(t) \right)^{-1} \right) \right)$$

$$= \left(\varphi_2(0)^{-1} \left(\varphi_1(0)^{-1}\varphi_1(t) \right) \right) \left(\varphi_2(0) \left(\varphi_1(0)^{-1}\varphi_1(t) \right)^{-1} \right)$$

$$= \left(\varphi_2(0)^{-1} \left(\varphi_1(0)^{-1}\varphi_1(t) \right) \right) \left(\left(\varphi_1(0)^{-1}\varphi_1(t) \right) \varphi_2(0) \right)^{-1}$$

$$= Com(\varphi_2(0)^{-1}, \varphi_1(0)^{-1}\varphi_1(t)). \tag{4.1.32}$$

By definition,

$$i\left(\varepsilon_0(\varphi_2) \right) = \varphi_2(0)^{-1}$$

and

$$\varphi_1(0)^{-1}\varphi_1(t) = p_2^{\Omega} h_1(\varphi_1)(t),$$

so $D_{h_1}(\varphi_1, \varphi_2)$ is given by

$$
\begin{aligned}
D_{h_1}(\varphi_1, \varphi_2) &= (g_0, Com(i(\varepsilon_0(\varphi_2)), p_2 h_1(\varphi_1)) \\
&= j_2 \circ Com \circ (i \times id_{\Omega G}) \circ (\varepsilon_0 \times (p_2^{\Omega} \circ h_1)) \circ T_{\Lambda G}(\varphi_1, \varphi_2).
\end{aligned}
$$

Hence

$$D_{h_1} = j_2 \circ Com \circ (i \times id_{\Omega G}) \circ (\varepsilon_0 \times (p_2^{\Omega} \circ h_1)) \circ T_{\Lambda G}.$$

$\square$

When we prove Lemma 4.1.6 and equation 4.1.30 for finite simply-connected HA-spaces, we will need to work with h_1' and $D_{h_1'}$, since these maps have pointwise formulas that we can use for calculations. We will also need to define another map $\widehat{Com} : X \times \Lambda X \to \Lambda X$ similar to $\widehat{Ad}$ from Lemma 3.2.9.

Proof. Let us start by writing out $D_{h_1'}$ again. This time, we will need to define a map $\widehat{Com} : X \times \Lambda X \to \Lambda X$, where for any $t \in [0, 1]$,

$$
\begin{aligned}
\widehat{Com}(x, \varphi)(t) &= com(x, \varphi(t)) \\
&= (x\varphi(t))i(\varphi(t)x). \qquad (4.1.33)
\end{aligned}
$$

Notice that

$$\widehat{\widehat{Com}}(id_X \times j) = \widehat{Com}. \tag{4.1.34}$$

Let us first prove that $p_1^\Lambda D_{h_1'}$ is nullhomotopic. Note that

$$p_1^\Lambda D_{h_1'}(\varphi_1, \varphi_2) = (\varphi_1(0)\varphi_2(0)) \, i \, (\varphi_1(0)\varphi_2(0)) \,,$$

so $p_1^\Lambda D_{h_1'}$ is nullhomotopic. Now let us prove that

$$p_2^\Lambda D_{h_1'} \simeq \widehat{\widehat{Com}}(i \times id_{\Lambda X})(\varepsilon_0 \times (p_2^\Lambda h_1'))T_{\Lambda X}.$$

The verification of this equation will imitate the calculations in equations 4.1.31 and 4.1.32 from the previous proof. By definition, $p_2^\Lambda D_{h_1'}(\varphi_1, \varphi_2) \in \Lambda X$ is a loop in ΛX given by

$$(i \, (\varphi_1(0)\varphi_2(0)) \, (\varphi_1(t)\varphi_2(t))) \, i \, ((i \, (\varphi_1(0)) \, \varphi_1(t)) \, (i \, (\varphi_2(0)) \, \varphi_2(t))) \,.$$

By using homotopy associativity, along with Lemma 2.3.4 (see equation 4.1.32 for a similar calculation), we can see that $p_2^\Lambda D_{h_1'}$ is homotopic to each of the following

maps (defined by their image of (φ_1, φ_2)):

$$(i\,(\varphi_1(0)\varphi_2(0))\,(\varphi_1(t)\varphi_2(t)))\,i\,((i(\varphi_1(0))\varphi_1(t))\,(i(\varphi_2(0))\varphi_2(t)))$$

$$((i(\varphi_2(0))i(\varphi_1(0)))\,\varphi_1(t))\,(\varphi_2(t)\,(i\,(i(\varphi_2(0))\varphi_2(t))\,i\,(i(\varphi_1(0))\varphi_1(t))))$$

$$(i(\varphi_2(0))\,(i(\varphi_1(0))\varphi_1(t)))\,(\varphi_2(t)\,((i(\varphi_2(t))\varphi_2(0))\,i\,(i(\varphi_1(0))\varphi_1(t))))$$

$$(i(\varphi_2(0))\,(i(\varphi_1(0))\varphi_1(t)))\,((\varphi_2(t)i(\varphi_2(t)))\,(\varphi_2(0)i\,(i(\varphi_1(0))\varphi_1(t))))$$

$$(i(\varphi_2(0))\,(i(\varphi_1(0))\varphi_1(t)))\,(\varphi_2(0)i\,(i(\varphi_1(0))\varphi_1(t)))$$

$$(i(\varphi_2(0))\,(i(\varphi_1(0))\varphi_1(t)))\,i\,((i(\varphi_1(0))\varphi_1(t))\,\varphi_2(0))$$

and therefore, $p_2^\Lambda D_{h_1'}$ is homotopic to a map $D_2 : \Lambda X \times \Lambda X \to X \times \Omega X$ which takes (φ_1, φ_2) to an element of $X \times \Lambda X$ given pointwise by

$$D_2(\varphi_1, \varphi_2)(t) = [i\,(\varphi_2(0))\,(i(\varphi_1(0))\varphi_1(t))]\,i\,[(i\,(\varphi_1(0))\,\varphi_1(t))\,i\,(\varphi_2(0))]\,.$$

Now, recall that

$$p_2^\Lambda h_1'(\varphi_1) = i(\varphi_1(0))\varphi_1(t) \in \Lambda X,$$

so

$$
\begin{aligned}
D_2(\varphi_1, \varphi_2)(t) &= [i\,(\varphi_2(0))\,(i(\varphi_1(0))\varphi_1(t))]\,i\,[(i\,(\varphi_1(0))\,\varphi_1(t))\,i\,(\varphi_2(0))] \\
&= \widehat{Com}\left(i(\varphi_2(0)), p_2^\Lambda h_1'(\varphi_1)\right).
\end{aligned}
\tag{4.1.35}
$$

Elementwise, we have

$$
\begin{aligned}
\big(i(\varphi_2(0)), p_2^\Lambda h_1'(\varphi_1) \big) &= (i \times id_{\Lambda X})(\varphi_2(0), p_2^\Lambda h_1'(\varphi_1)) \\
&= (i \times id_{\Lambda X})(\varepsilon_0 \times (p_2^\Lambda h_1'))(\varphi_2, \varphi_1) \\
&= (i \times id_{\Lambda X})(\varepsilon_0 \times (p_2^\Lambda h_1'))T_{\Lambda X}(\varphi_1, \varphi_2),
\end{aligned}
$$

so we can expand equation 4.1.35 as

$$
\widehat{\widehat{Com}}\big(i(\varphi_2(0)), p_2^\Lambda h_1'(\varphi_1) \big) = \widehat{\widehat{Com}}(i \times id_{\Lambda X})(\varepsilon_0 \times (p_2^\Lambda h_1'))T_{\Lambda X}(\varphi_1, \varphi_2). \qquad (4.1.36)
$$

Hence

$$
p_2^\Lambda D_{h_1'} \simeq p_2^\Lambda D_2 = \widehat{\widehat{Com}}(i \times id_{\Lambda X})(\varepsilon_0 \times (p_2^\Lambda h_1'))T_{\Lambda X}. \qquad (4.1.37)
$$

Now let us prove that

$$
\begin{aligned}
j p_2^\Omega D_{h_1} &\simeq p_2^\Lambda D_{h_1'} \\
&\simeq \widehat{\widehat{Com}}(i \times id_{\Lambda X})(\varepsilon_0 \times (p_2^\Lambda h_1'))T_{\Lambda X} \\
&\simeq j \circ Com(i \times id_{\Omega X})(\varepsilon_0 \times (p_2^\Omega h_1))T_{\Lambda X}.
\end{aligned}
$$

According to diagram 4.1.28,

$$
j p_2^\Omega D_{h_1} \simeq p_2^\Lambda D_{h_1'}. \qquad (4.1.38)
$$

Meanwhile, by combining diagrams 4.1.5 and 4.0.3,

$$p_2^\Lambda h_1' \simeq j \circ p_2^\Omega h_1,$$

so we see that the right side of equation 4.1.37 becomes

$$\widehat{\widehat{Com}}(i \times id_{\Lambda X})(\varepsilon_0 \times (p_2^\Lambda h_1'))T_{\Lambda X} \simeq \widehat{\widehat{Com}}(i \times id_{\Lambda X})(\varepsilon_0 \times (j \circ p_2^\Omega h_1))T_{\Lambda X}. \quad (4.1.39)$$

We need to more j to the left end of the composition. Given $\varphi_1, \varphi_2 \in \Lambda X$, the map on the right can be expanded elementwise:

$$\widehat{\widehat{Com}}(i \times id_{\Lambda X})(\varepsilon_0 \times (j \circ p_2^\Omega h_1))T_{\Lambda X}(\varphi_1, \varphi_2)$$

$$= \widehat{\widehat{Com}}(i(\varphi_2(0)), j \circ p_2^\Omega h_1(\varphi_1))$$

$$= \widehat{\widehat{Com}}(id_X \times j)(i \times id_{\Omega X})(\varphi_2(0), p_2^\Omega h_1(\varphi_1))$$

$$= \widehat{\widehat{Com}}(id_X \times j)(i \times id_{\Omega X})(\varepsilon_0 \times (p_2^\Omega h_1))T_{\Lambda X}(\varphi_1, \varphi_2),$$

and by equation 4.1.34, $\widehat{\widehat{Com}}(id_X \times j) = \widehat{Com}$, so equation 4.1.39 becomes

$$\widehat{\widehat{Com}}(i \times id_{\Lambda X})(\varepsilon_0 \times (p_2^\Lambda h_1'))T_{\Lambda X} \simeq \widehat{Com}(i \times id_{\Omega X})(\varepsilon_0 \times (p_2^\Omega h_1))T_{\Lambda X}. \quad (4.1.40)$$

Since $j \circ Com \simeq \widehat{Com}$, we can simplify equation 4.1.40 to

$$\widehat{\widehat{Com}}(i \times id_{\Lambda X})(\varepsilon_0 \times (p_2^{\Lambda} h_1'))T_{\Lambda X}$$

$$\simeq \widehat{Com}(i \times id_{\Omega X})(\varepsilon_0 \times (p_2^{\Omega} h_1))T_{\Lambda X}$$

$$\simeq j \circ Com(i \times id_{\Omega X})(\varepsilon_0 \times (p_2^{\Omega} h_1))T_{\Lambda X}. \tag{4.1.41}$$

Thus, when we combine equations 4.1.38 and 4.1.41, we obtain

$$j p_2^{\Omega} D_{h_1} \simeq j \circ Com(i \times id_{\Omega X})(\varepsilon_0 \times (p_2^{\Omega} h_1))T_{\Lambda X},$$

so by Lemma 3.1.4,

$$p_2^{\Omega} D_{h_1} \simeq Com(i \times id_{\Omega X})(\varepsilon_0 \times (p_2^{\Omega} h_1))T_{\Lambda X}. \tag{4.1.42}$$

To conclude the proof, recall that $p_1^{\Lambda} D_{h_1'}$ is nullhomotopic, so by diagram 4.1.27, $p_1^{\Omega} D_{h_1}$ is nullhomotopic as well. Since

$$D_{h_1}(\varphi_1, \varphi_2) = (p_1^{\Omega} D_{h_1}(\varphi_1, \varphi_2), p_2^{\Omega} D_{h_1}(\varphi_1, \varphi_2)),$$

$$\left(j_2 \circ p_2^{\Omega}\right) \circ D_{h_1}(\varphi_1, \varphi_2) = (x_0, p_2^{\Omega} D_{h_1}(\varphi_1, \varphi_2)),$$

and $p_1^{\Omega} D_{h_1}$ is nullhomotopic, we have $D_{h_1} \simeq \left(j_2 \circ p_2^{\Omega}\right) \circ D_{h_1}$. Therefore, by equation 4.1.42,

$$D_{h_1} \simeq j_2 \circ Com(i \times id_{\Omega X})(\varepsilon_0 \times (p_2^{\Omega} h_1))T_{\Lambda X}.$$

$\square$

We can use this lemma for some applications. First, we can find a twisted H-structure on $X \times \Omega X$ using h_1 and D_{h_1}. Second, we can look at multiplication in the homology algebra $H_*(\Lambda X; \mathbb{F}_p)$ using information from the product and coproduct structures of $H_*(X; \mathbb{F}_p)$ and $H_*(\Omega X; \mathbb{F}_p)$, along with knowledge of Com_*. After that, we can look for a relationship between h_1^* being a Hopf algebra isomorphism and commutativity of the homology over $\mathbb{F}_p$ of the original space X itself.

4.2 Twisted H-structures and ΛX

Let X be a simply-connected HA-space. This section will restate the results of Section 4.2 in the context of twisted H-structures. We will show how to obtain a twisted H-structure on $X \times \Omega X$ using the H-deviation of the homotopy equivalence $h_1 : \Lambda X \to X \times \Omega X$.

Although ΛX and $X \times \Omega X$ have the same homotopy type, it may not be true that the homotopy equivalence h_1 is an H-map. Instead, we can use the H-deviation and Lemma 4.1.6 to find a map $\omega : X \times \Omega X \to \Omega X$ and an ω-twisted multiplication structure $\tilde{\mu}$ on $X \times \Omega X$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\Lambda X \times \Lambda X & \xrightarrow{\ h_1 \times h_1\ } & X \times \Omega X \times X \times \Omega X \\
{\scriptstyle \mu_\Lambda} \downarrow & & \downarrow {\scriptstyle \tilde{\mu}} \\
\Lambda X & \xrightarrow[\ h_1\ ]{} & X \times \Omega X
\end{array}
\qquad (4.2.1)
$$

or equivalently,

$$\begin{array}{ccc}
\Lambda X \times \Lambda X & \xleftarrow{\quad h_2 \times h_2 \quad} & X \times \Omega X \times X \times \Omega X \\
\mu_\Lambda \downarrow & & \downarrow \tilde{\mu} \\
\Lambda X & \xleftarrow{\quad h_2 \quad} & X \times \Omega X
\end{array} \qquad (4.2.2)$$

Theorem 4.2.1. *Let X be a simply-connected HA-space. Let $\omega : X \times \Omega X \to \Omega X$ be given by*

$$\begin{aligned}
\omega(x, l) &= Com \circ (i \times id_{\Omega X})(x, l) \\
&= Com(i(x), l).
\end{aligned}$$

Then the following choice of $\tilde{\mu} : X \times \Omega X \times X \times \Omega X \to X \times \Omega X$ will make diagram 4.2.1 commute:

$$\tilde{\mu} = (id_X \times \mu_\Omega)(\mu \times \omega \times \mu_\Omega)(id_X \times \Delta_X \times \Delta_{\Omega X} \times id_{\Omega X})(id_X \times T_{\Omega X, X} \times id_{\Omega X}),$$

$$\tilde{\mu}(x_1, l_1, x_2, l_2) = (\mu(x_1, x_2), \mu_\Omega (Com(i(x_2), l_1), \mu_\Omega(l_1, l_2))) .$$

Note that this theorem implies that h_1 is a $\mu_\Lambda - \tilde{\mu}$ H-map.

Proof. We will prove that diagram 4.2.1 commutes as follows. Recall that

$$(\mu_{X \times \Omega X}) \circ (D_{h_1} \times [(\mu_{X \times \Omega X}) \circ (h_1 \times h_1)]) \circ \Delta_{\Lambda X \times \Lambda X} \simeq h_1 \circ \mu_\Lambda,$$

so it suffices to show that

$$(\mu_{X \times \Omega X}) \circ (D_{h_1} \times [(\mu_{X \times \Omega X}) \circ (h_1 \times h_1)]) \circ \Delta_{\Lambda X \times \Lambda X} \simeq \tilde{\mu}(h_1 \times h_1).$$

Throughout this proof, given $x_1, x_2 \in X$ and $l_1, l_2 \in \Omega X$, we will write

$$\mu(x_1, x_2) = x_1 x_2,$$

$$\mu_\Omega(l_1, l_2) = l_1 l_2.$$

Given $\varphi_j \in \Lambda X$, we will write

$$h_1(\varphi_j) = (y_j, \alpha_j)$$

where $y_j \in X$, $\alpha_j \in \Omega X$.

First, let $i_1 : X \to X \times \Omega X$ and $i_2 : \Omega X \to X \times \Omega X$ be inclusions. Let us verify that the compositions ωi_1 and ωi_2 are nullhomotopic. For any $x \in X$ and $l \in \Omega X$,

$$
\begin{aligned}
\omega i_1(x) &= Com(i \times id_{\Omega X})(x, l_0) \\
&= Com(i(x), l_0) \\
&= Com \circ i_1 \circ i(x),
\end{aligned}
$$

and by Lemma 3.2.7, $Com \circ i_1$ is nullhomotopic, so therefore ωi_1 is nullhomotopic.

Similarly, we have

$$\begin{aligned}
\omega i_2(l) &= Com(i \times id_{\Omega X})(x_0, l) \\
&= Com(i(x_0), l) \\
&= Com(x_0, l) \\
&= Com \circ i_2(l),
\end{aligned}$$

and by Lemma 3.2.7, $Com \circ i_2$ is nullhomotopic, so therefore ωi_2 is nullhomotopic.

Let us compute $\tilde{\mu}(h_1 \times h_1)$. For any $\varphi_1, \varphi_2 \in \Lambda X$,

$$\begin{aligned}
\tilde{\mu}(h_1 \times h_1)(\varphi_1, \varphi_2) &= \tilde{\mu}(h_1(\varphi_1), h_1(\varphi_2)) \\
&= \tilde{\mu}(y_1, \alpha_1, y_2, \alpha_2) \\
&= (id_X \times \mu_\Omega)(\mu \times \omega \times \mu_\Omega)(y_1, y_2, y_2, \alpha_1, \alpha_1, \alpha_2) \\
&= (id_X \times \mu_\Omega)(y_1 y_2, Com(i(y_2), \alpha_1), \alpha_1 \alpha_2) \\
&= (y_1 y_2, \mu_\Omega(Com(i(y_2), \alpha_1), (\alpha_1 \alpha_2))). \quad\quad (4.2.3)
\end{aligned}$$

Now let us write out

$$(\mu_{X \times \Omega X}) \circ (D_{h_1} \times [(\mu_{X \times \Omega X}) \circ (h_1 \times h_1)]) \circ \Delta_{\Lambda X \times \Lambda X}$$

elementwise. For any $\varphi_1, \varphi_2 \in \Lambda X$,

$$(\mu_{X \times \Omega X})(D_{h_1} \times [(\mu_{X \times \Omega X})(h_1 \times h_1)])\Delta_{\Lambda X \times \Lambda X}(\varphi_1, \varphi_2)$$

$$= (\mu_{X \times \Omega X})(D_{h_1}(\varphi_1, \varphi_2), (\mu_{X \times \Omega X})(y_1, \alpha_1, y_2, \alpha_2))$$

$$= (\mu_{X \times \Omega X})(D_{h_1}(\varphi_1, \varphi_2), (y_1 y_2, \alpha_1 \alpha_2))$$

$$= (\mu(p_1^\Omega D_{h_1}(\varphi_1, \varphi_2), p_1^\Omega(y_1 y_2, \alpha_1 \alpha_2)), \mu_\Omega(p_2^\Omega D_{h_1}(\varphi_1, \varphi_2), p_2^\Omega(y_1 y_2, \alpha_1 \alpha_2)))$$

$$= (\mu(p_1^\Omega D_{h_1}(\varphi_1, \varphi_2), y_1 y_2), \mu_\Omega(p_2^\Omega D_{h_1}(\varphi_1, \varphi_2), \alpha_1 \alpha_2)).$$

By Lemma 4.1.6, $p_1^\Omega D_{h_1}$ is nullhomotopic, while

$$p_2^\Omega D_{h_1} \simeq Com(i \times id_{\Omega X})(\varepsilon_0 \times (p_2^\Omega h_1))T_{\Lambda X}.$$

Let us write out the map on the right side elementwise. Recall that

$$p_1^\Omega h_1(\varphi_j) = \varepsilon_0(\varphi_j),$$

so according to the notation of this proof,

$$\varepsilon_0(\varphi_j) = y_j,$$

$$h_1(\varphi_j) = \alpha_j.$$

Hence

$$Com(i \times id_{\Omega X})(\varepsilon_0 \times (p_2^{\Omega} h_1)) T_{\Lambda X}(\varphi_1, \varphi_2)$$

$$= Com(i(y_2), \alpha_1).$$

Thus, $(\mu_{X \times \Omega X})(D_{h_1} \times [(\mu_{X \times \Omega X})(h_1 \times h_1)]) \Delta_{\Lambda X \times \Lambda X}$, and hence $h_1 \circ \mu_{\Lambda}$, is homotopic to a map which takes (φ_1, φ_2) to

$$(y_1 y_2, \mu_{\Omega}(Com(i(y_2), \alpha_1), (\alpha_1 \alpha_2))) = \tilde{\mu}(h_1 \times h_1)(\varphi_1, \varphi_2).$$

Therefore, $h_1 \circ \mu_{\Lambda} \simeq \tilde{\mu}(h_1 \times h_1)$. $\qquad \square$

In the next section, we will use this theorem to compute $\mu_{\Lambda *}$ in $H_*(\Lambda X; \mathbb{F}_p)$ using μ_*, $\mu_{\Omega *}$, and Com_*.

4.3   Multiplication in the Homology of the Free Loop Space

Let X be a simply-connected HA-space. Since h_1 and h_2 are homotopy equivalences, their induced *linear transformations* in homology,

$$h_{1*} : H_*(\Lambda X; \mathbb{F}_p) \to H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$$

and

$$h_{2*} : H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \to H_*(\Lambda X; \mathbb{F}_p)$$

are vector space isomorphisms (for which $h_{1*}h_{2*}$ and $h_{2*}h_{1*}$ are identity linear transformations), but not necessarily algebra isomorphisms. We will use Theorem 4.2.1 to give a formula for multiplying two elements in $H_*(\Lambda X; \mathbb{F}_p)$ in terms of the multiplication structures on $H_*(X; \mathbb{F}_p)$ and $H_*(\Omega X; \mathbb{F}_p)$, along with the linear transformation $Com_* : H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \to H_*(\Omega X; \mathbb{F}_p)$.

Since h_{2*} is a vector space isomorphism, any element of $H_*(\Lambda X; \mathbb{F}_p)$ has the form

$$h_{2*}(\bar{x} \otimes \bar{t})$$

where $\bar{x} \in H_*(X; \mathbb{F}_p)$ and $\bar{t} \in H_*(\Omega X; \mathbb{F}_p)$.

Remark. We must stress that h_{2*} and h_{1*} may not be algebra isomorphisms. For example, in $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$, if $\bar{x} \in H_*(X; \mathbb{F}_p)$ and $\bar{t} \in H_*(\Omega X; \mathbb{F}_p)$ ($\bar{t}$ has even degree),

$$
\begin{aligned}
\bar{x} \otimes \bar{t} &= \mu_{X \times \Omega X *}(\bar{x} \otimes 1 \otimes 1 \otimes \bar{t}) \\
&= (\bar{x} \otimes 1)(1 \otimes \bar{t}) \\
&= (1 \otimes \bar{t})(\bar{x} \otimes 1) \\
&= \mu_{X \times \Omega X *}(1 \otimes \bar{t} \otimes \bar{x} \otimes 1)
\end{aligned}
$$

and we will see that in $H_*(\Lambda X; \mathbb{F}_p)$, if $Com_* : H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p) \to H_*(\Omega X; \mathbb{F}_p)$

is not trivial, then there exists choices of $\bar{x} \in H_*(X; \mathbb{F}_p)$ and $\bar{t} \in H_*(\Omega X; \mathbb{F}_p)$ such

that

$$h_{2*}(\bar{x} \otimes \bar{t}) \neq \mu_{\Lambda*}(h_{2*}(1 \otimes \bar{t}) \otimes h_{2*}(\bar{x} \otimes 1)).$$

To avoid confusion, we will never suppress $\mu_{\Lambda*}$ using concatenation, but for conve-

nience, for any $\bar{x}, \bar{y} \in H_*(X; \mathbb{F}_p)$, $\bar{t}, \bar{u} \in H_*(\Omega X; \mathbb{F}_p)$, we will write:

$$\mu_*(\bar{x} \otimes \bar{y}) = \bar{x}\bar{y},$$

$$\mu_{\Omega*}(\bar{t} \otimes \bar{u}) = \bar{t}\bar{u},$$

$$
\begin{aligned}
\mu_{X \times \Omega X*}(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u}) &= (\bar{x} \otimes \bar{t})(\bar{y} \otimes \bar{u}) \\
&= \bar{x}\bar{y} \otimes \bar{t}\bar{u}.
\end{aligned}
$$

Let us use Theorem 4.2.1 to examine the multiplication of $H_*(\Lambda X; \mathbb{F}_p)$:

Theorem 4.3.1. *Let X be a simply-connected HA-space. Let $\bar{x}, \bar{y} \in H_*(X; \mathbb{F}_p)$,*

$\bar{t}, \bar{u} \in H_*(\Omega X; \mathbb{F}_p)$, and the coproducts of $\bar{y}$ and $\bar{t}$ are as follows:

$$\Delta_{X*}(\bar{y}) = \bar{y} \otimes 1 + 1 \otimes \bar{y} + \sum_{\beta} \bar{y}'_\beta \otimes \bar{y}''_\beta,$$

$$\Delta_{\Omega X*}(\bar{t}) = \bar{t} \otimes 1 + 1 \otimes \bar{t} + \sum_{\gamma} \bar{t}'_\gamma \otimes \bar{t}''_\gamma.$$

Then if $\bar{y}$ and $\bar{t}$ have positive degree,

$$\mu_{\Lambda*}(h_{2*}(\bar{x} \otimes \bar{t}) \otimes h_{2*}(\bar{y} \otimes \bar{u}))$$

$$= \quad h_{2*}[\bar{x} \otimes Com_*(i_*(\bar{y}) \otimes \bar{t})\bar{u}$$

$$+ \sum_{\beta} \bar{x}\bar{y}''_{\beta} \otimes Com_*(i_*(\bar{y}'_{\beta}) \otimes \bar{t})\bar{u}$$

$$+ \sum_{\gamma} \bar{x} \otimes Com_*(i_*(\bar{y}) \otimes \bar{t}'_{\gamma})\bar{t}''_{\gamma}\bar{u} \qquad (4.3.1)$$

$$+ \sum_{\beta,\gamma} \bar{x}\bar{y}''_{\beta} \otimes Com_*(i_*(\bar{y}'_{\beta}) \otimes \bar{t}'_{\gamma})\bar{t}''_{\gamma}\bar{u}$$

$$+ \bar{x}\bar{y} \otimes \bar{t}\bar{u}].$$

If either $\bar{y} = 1$ or $\bar{t} = 1$,

$$\mu_{\Lambda*}(h_{2*}(\bar{x} \otimes \bar{t}) \otimes h_{2*}(\bar{y} \otimes \bar{u})) = h_{2*}(\bar{x}\bar{y} \otimes \bar{t}\bar{u}). \qquad (4.3.2)$$

Proof. When we apply homology to diagram 4.2.2, we see that if $\bar{x}, \bar{y} \in H_*(X; \mathbb{F}_p)$, $\bar{t}, \bar{u} \in H_*(\Omega X; \mathbb{F}_p)$, then

$$\mu_{\Lambda*}(h_{2*}(\bar{x} \otimes \bar{t}) \otimes h_{2*}(\bar{y} \otimes \bar{u}))$$

$$= \quad h_{2*}\tilde{\mu}_*(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u}), \qquad (4.3.3)$$

where

$$\tilde{\mu}_* = (1 \otimes \mu_{\Omega*})(\mu_* \otimes \omega_* \otimes \mu_{\Omega*})(1 \otimes \Delta_{X*} \otimes \Delta_{\Omega X*} \otimes 1)(1 \otimes T_{\Omega X, X*} \otimes 1),$$

$$\omega_* = (Com_*)(i_* \otimes 1).$$

By applying h_{1*} on equation 4.3.3, we get

$$h_{1*}\mu_{\Lambda*}(h_{2*}(\bar{x} \otimes \bar{t}) \otimes h_{2*}(\bar{y} \otimes \bar{u}))$$

$$= h_{1*}h_{2*}\tilde{\mu}_*(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u})$$

$$= \tilde{\mu}_*(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u}), \tag{4.3.4}$$

so to prove the theorem, it suffices to compute $\tilde{\mu}_*(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u})$ and then apply h_{2*}.

At some point, we will need to evaluate $(\Delta_{X*} \otimes \Delta_{\Omega X*})(\bar{y} \otimes \bar{t})$. Because of this, we will

break the argument into four cases, depending on whether $\bar{y} = 1$ or not, and whether

$\bar{t} = 1$ or not.

First, suppose that both $\bar{y}$ and $\bar{t}$ have positive degree. Before we compute $\tilde{\mu}_*$,

let us write down $(\Delta_{X*} \otimes \Delta_{\Omega X*})(\bar{y} \otimes \bar{t})$:

$$(\Delta_{X*} \otimes \Delta_{\Omega X*})(\bar{y} \otimes \bar{t})$$

$$= \bar{y} \otimes 1 \otimes \bar{t} \otimes 1 + \bar{y} \otimes 1 \otimes 1 \otimes \bar{t} + \sum_{\gamma} \bar{y} \otimes 1 \otimes \bar{t}'_\gamma \otimes \bar{t}''_\gamma$$

$$+ 1 \otimes \bar{y} \otimes \bar{t} \otimes 1 + 1 \otimes \bar{y} \otimes 1 \otimes \bar{t} + \sum_{\gamma} 1 \otimes \bar{y} \otimes \bar{t}'_\gamma \otimes \bar{t}''_\gamma$$

$$+ \sum_{\beta} \bar{y}'_\beta \otimes \bar{y}''_\beta \otimes \bar{t} \otimes 1 + \sum_{\beta} \bar{y}'_\beta \otimes \bar{y}''_\beta \otimes 1 \otimes \bar{t}$$

$$+ \sum_{\beta,\gamma} \bar{y}'_\beta \otimes \bar{y}''_\beta \otimes \bar{t}'_\gamma \otimes \bar{t}''_\gamma. \tag{4.3.5}$$

First, we apply $(1 \otimes T_{\Omega X, X*} \otimes 1)$ to $\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u}$. Since $H_*(\Omega X; \mathbb{F}_p)$ has no nonzero

elements of odd degree, $\bar{t}$ has even degree, so

$$(1 \otimes T_{\Omega X, X*} \otimes 1)(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u}) = \bar{x} \otimes \bar{y} \otimes \bar{t} \otimes \bar{u}.$$

Next, we apply $(1 \otimes \Delta_{X*} \otimes \Delta_{\Omega X*} \otimes 1)$. By equation 4.3.5, we get

$$(1 \otimes \Delta_{X*} \otimes \Delta_{\Omega X*} \otimes 1)(1 \otimes T_{\Omega X, X*} \otimes 1)(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u})$$

$$= \bar{x} \otimes \bar{y} \otimes 1 \otimes \bar{t} \otimes 1 \otimes \bar{u} + \bar{x} \otimes \bar{y} \otimes 1 \otimes 1 \otimes \bar{t} \otimes \bar{u} + \sum_\gamma \bar{x} \otimes \bar{y} \otimes 1 \otimes \bar{t}'_\gamma \otimes \bar{t}''_\gamma \otimes \bar{u}$$

$$+ \bar{x} \otimes 1 \otimes \bar{y} \otimes \bar{t} \otimes 1 \otimes \bar{u} + \bar{x} \otimes 1 \otimes \bar{y} \otimes 1 \otimes \bar{t} \otimes \bar{u} + \sum_\gamma \bar{x} \otimes 1 \otimes \bar{y} \otimes \bar{t}'_\gamma \otimes \bar{t}''_\gamma \otimes \bar{u}$$

$$+ \sum_\beta \bar{x} \otimes \bar{y}'_\beta \otimes \bar{y}''_\beta \otimes \bar{t} \otimes 1 \otimes \bar{u} + \sum_\beta \bar{x} \otimes \bar{y}'_\beta \otimes \bar{y}''_\beta \otimes 1 \otimes \bar{t} \otimes \bar{u}$$

$$+ \sum_{\beta,\gamma} \bar{x} \otimes \bar{y}'_\beta \otimes \bar{y}''_\beta \otimes \bar{t}'_\gamma \otimes \bar{t}''_\gamma \otimes \bar{u}. \tag{4.3.6}$$

Now we apply $(\mu_* \otimes \omega_* \otimes \mu_{\Omega*})$. By part 7 of Proposition 3.3.1, we see that for any

$\bar{z} \in H_*(X; \mathbb{F}_p)$, $\bar{v} \in H_*(\Omega X; \mathbb{F}_p)$,

$$Com_*(\bar{z} \otimes 1) = Com_*(1 \otimes \bar{v}) = 0,$$

and since Com_* is a graded linear transformation,

$$Com_*(1 \otimes 1) = 1,$$

so we get

$$(\mu_* \otimes \omega_* \otimes \mu_{\Omega*})(1 \otimes \Delta_{X*} \otimes \Delta_{\Omega X*} \otimes 1)(1 \otimes T_{\Omega X, X*} \otimes 1)(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u})$$

$$= \bar{x}\bar{y} \otimes 1 \otimes \bar{t}\bar{u}$$

$$+ \bar{x} \otimes Com_*(i_*(\bar{y}) \otimes \bar{t}) \otimes \bar{u} + \sum_\gamma \bar{x} \otimes Com_*(i_*(\bar{y}) \otimes \bar{t}'_\gamma) \otimes \bar{t}''_\gamma \bar{u}$$

$$+ \sum_\beta \bar{x}\bar{y}'_\beta \otimes Com_*(i_*(\bar{y}''_\beta) \otimes \bar{t}) \otimes \bar{u}$$

$$+ \sum_{\beta,\gamma} \bar{x}\bar{y}'_\beta \otimes Com_*(i_*(\bar{y}''_\beta) \otimes \bar{t}'_\gamma) \otimes \bar{t}''_\gamma \bar{u}. \qquad (4.3.7)$$

Finally, we apply $(1 \otimes \mu_{\Omega*})$, which gives us $\tilde{\mu}_*(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u})$:

$$\tilde{\mu}_*(\bar{x} \otimes \bar{t} \otimes \bar{y} \otimes \bar{u})$$

$$= \bar{x}\bar{y} \otimes \bar{t}\bar{u}$$

$$+ \bar{x} \otimes Com_*(i_*(\bar{y}) \otimes \bar{t})\bar{u} + \sum_\gamma \bar{x} \otimes Com_*(i_*(\bar{y}) \otimes \bar{t}'_\gamma)\bar{t}''_\gamma \bar{u}$$

$$+ \sum_\beta \bar{x}\bar{y}'_\beta \otimes Com_*(i_*(\bar{y}''_\beta) \otimes \bar{t})\bar{u}$$

$$+ \sum_{\beta,\gamma} \bar{x}\bar{y}'_\beta \otimes Com_*(i_*(\bar{y}''_\beta) \otimes \bar{t}'_\gamma)\bar{t}''_\gamma \bar{u}. \qquad (4.3.8)$$

We obtain the formula in the theorem by applying h_{2*}. This proves the main part of the theorem. Now we proceed to the special cases where one of $\bar{y}$ or $\bar{t}$ has degree zero.

Suppose $\bar{y} = 1$ and $\bar{t}$ still has positive degree. Then

$$(\Delta_{X*} \otimes \Delta_{\Omega X*})(1 \otimes \bar{t}) = 1 \otimes 1 \otimes \bar{t} \otimes 1 + 1 \otimes 1 \otimes 1 \otimes \bar{t} + \sum_{\gamma} 1 \otimes 1 \otimes \bar{t}'_\gamma \otimes \bar{t}''_\gamma,$$

so

$$\tilde{\mu}_*(\bar{x} \otimes \bar{t} \otimes 1 \otimes \bar{u})$$

$$= (1 \otimes \mu_{\Omega*})(\bar{x} \otimes Com_*(1 \otimes \bar{t}) \otimes \bar{u} + \bar{x} \otimes Com_*(1 \otimes 1) \otimes \bar{t}\bar{u}$$

$$+ \sum_{\gamma} \bar{x} \otimes Com_*(1 \otimes \bar{t}'_\gamma) \otimes \bar{t}''_\gamma \bar{u})$$

$$= \bar{x} \otimes \bar{t}\bar{u}.$$

Now suppose $\bar{t} = 1$ and $\bar{y}$ has positive degree. Then

$$(\Delta_{X*} \otimes \Delta_{\Omega X*})(\bar{y} \otimes 1) = \bar{y} \otimes 1 \otimes 1 \otimes 1 + 1 \otimes \bar{y} \otimes 1 \otimes 1 + \sum_{\beta} \bar{y}'_\beta \otimes \bar{y}''_\beta \otimes 1 \otimes 1,$$

so

$$\tilde{\mu}_*(\bar{x} \otimes 1 \otimes \bar{y} \otimes \bar{u})$$

$$= (1 \otimes \mu_{\Omega*})(\bar{x}\bar{y} \otimes Com_*(1 \otimes 1) \otimes \bar{u} + \bar{x} \otimes Com_*(i_*(\bar{y}) \otimes 1) \otimes \bar{u}$$

$$+ \sum_{\beta} \bar{x}\bar{y}'_\beta \otimes Com_*(i_*(\bar{y}''_\beta) \otimes 1) \otimes \bar{u})$$

$$= \bar{x}\bar{y} \otimes \bar{u}.$$

Finally, if $\bar{y} = 1$ and $\bar{t} = 1$, then

$$(\Delta_{X*} \otimes \Delta_{\Omega X*})(1 \otimes 1) = 1 \otimes 1 \otimes 1 \otimes 1,$$

so

$$\tilde{\mu}_*(\bar{x} \otimes 1 \otimes 1 \otimes \bar{u})$$

$$= (1 \otimes \mu_{\Omega*})(\bar{x} \otimes Com_*(1 \otimes 1) \otimes \bar{u})$$

$$= \bar{x} \otimes \bar{u}.$$

We obtain the formulas in the theorem by applying h_{2*}. $\qquad\square$

We have the following corollary about the linear transformations h_{1*} and h_{2*} and how they affect generators and decomposables.

Corollary 4.3.2. *The linear transformation h_{2*} sends decomposables of $H_*(X;\mathbb{F}_p)\otimes H_*(\Omega X;\mathbb{F}_p)$ to decomposables of $H_*(\Lambda X;\mathbb{F}_p)$. In addition, if $\{\bar{c}_1,\ldots,\bar{c}_r\}$ is a generating set for $H_*(X;\mathbb{F}_p) \otimes H_*(\Omega X;\mathbb{F}_p)$, then a possible generating set for $H_*(\Lambda X;\mathbb{F}_p)$ is a subset of $\{h_{2*}(\bar{c}_1),\ldots,h_{2*}(\bar{c}_r)\}$.*

Proof. A decomposable in $H_*(X;\mathbb{F}_p) \otimes H_*(\Omega X;\mathbb{F}_p)$ will be a sum of one of the following kinds of elements:

1) An element of the form $\bar{x}\bar{y}\otimes 1 = (\bar{x}\otimes 1)(\bar{y}\otimes 1)$ where $\bar{x},\bar{y} \in H_*(X;\mathbb{F}_p)$ have positive degree.

2) An element of the form $1 \otimes \bar{t}\bar{u} = (1 \otimes \bar{t})(1 \otimes \bar{u})$ where $\bar{t}, \bar{u} \in H_*(\Omega X; \mathbb{F}_p)$ have positive degree.

3) An element of the form $\bar{x} \otimes \bar{t} = (\bar{x} \otimes 1)(1 \otimes \bar{t})$ where $\bar{x} \in H_*(X; \mathbb{F}_p)$ and $\bar{t} \in H_*(\Omega X; \mathbb{F}_p)$ have positive degree.

By Theorem 4.3.1 (specifically equation 4.3.2), we have

$$h_{2*}(\bar{x}\bar{y} \otimes 1) = \mu_{\Lambda*}(h_{2*}(\bar{x} \otimes 1) \otimes h_{2*}(\bar{y} \otimes 1)),$$

$$h_{2*}(1 \otimes \bar{t}\bar{u}) = \mu_{\Lambda*}(h_{2*}(1 \otimes \bar{t}) \otimes h_{2*}(1 \otimes \bar{u})),$$

$$h_{2*}(\bar{x} \otimes \bar{t}) = \mu_{\Lambda*}(h_{2*}(\bar{x} \otimes 1) \otimes h_{2*}(1 \otimes \bar{t})),$$

so $h_{2*}(\bar{x}\bar{y} \otimes 1)$, $h_{2*}(1 \otimes \bar{t}\bar{u})$, and $h_{2*}(\bar{x} \otimes \bar{t})$ are decomposables as well. Hence h_{2*} sends decomposables to decomposables.

For the rest of this proof, let us abbreviate $C = H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$. Since h_{2*} sends decomposables to decomposables, we have an induced linear transformation on the module of indecomposables,

$$h_{2*}^Q : Q(H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)) \rightarrow QH_*(\Lambda X; \mathbb{F}_p),$$

$$h_{2*}^Q(\bar{c} + \bar{C}\bar{C}) = h_{2*}(\bar{c}) + \bar{H}_*(\Lambda X; \mathbb{F}_p)\bar{H}_*(\Lambda X; \mathbb{F}_p).$$

Note that h_{2*}^Q is surjective: given

$$\bar{z} + \bar{H}_*(\Lambda X; \mathbb{F}_p)\bar{H}_*(\Lambda X; \mathbb{F}_p) \in QH_*(\Lambda X; \mathbb{F}_p),$$

we have

$$\begin{aligned} h_{2*}^Q\left(h_{1*}(\bar{z}) + \bar{C}\bar{C}\right) &= h_{2*}h_{1*}(\bar{z}) + \bar{H}_*(\Lambda X; \mathbb{F}_p)\bar{H}_*(\Lambda X; \mathbb{F}_p) \\ &= \bar{z} + \bar{H}_*(\Lambda X; \mathbb{F}_p)\bar{H}_*(\Lambda X; \mathbb{F}_p). \end{aligned}$$

However, h_{2*} may send generators to decomposables, so h_{2*}^Q may not be injective. Consequently, if

$$\mathcal{B} = \left\{\bar{c}_1 + \bar{C}\bar{C}, \ldots, \bar{c}_r + \bar{C}\bar{C}\right\}$$

is a basis of $Q(H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p))$, then the set

$$h_{2*}^Q(\mathcal{B}) = \left\{h_{2*}(\bar{c}_1) + \bar{H}_*(\Lambda X; \mathbb{F}_p)\bar{H}_*(\Lambda X; \mathbb{F}_p), \ldots, h_{2*}(\bar{c}_r) + \bar{H}_*(\Lambda X; \mathbb{F}_p)\bar{H}_*(\Lambda X; \mathbb{F}_p)\right\}$$

will span $QH_*(\Lambda X; \mathbb{F}_p)$, but it may not be linearly independent. Then a subset of $h_{2*}^Q(\mathcal{B})$ will be a basis for $QH_*(\Lambda X; \mathbb{F}_p)$. Therefore, if $\{\bar{c}_1, \ldots, \bar{c}_r\}$ is a generating set for $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$, then a subset of $\{h_{2*}(\bar{c}_1), \ldots, h_{2*}(\bar{c}_r)\}$ will be a generating set for $H_*(\Lambda X; \mathbb{F}_p)$. $\square$

We also have the following corollary about powers in $H_*(\Lambda X; \mathbb{F}_p)$.

Corollary 4.3.3. *Let* $\bar{x} \in H_*(X; \mathbb{F}_p)$, $\bar{t} \in H_*(\Omega X; \mathbb{F}_p)$ *and* $n \geq 1$. *Then*

$$[h_{2*}(\bar{x} \otimes 1)]^n = h_{2*}(\bar{x}^n \otimes 1), \tag{4.3.9}$$

where $[h_{2*}(\bar{x} \otimes 1)]^n$ *means to take the power using* $\mu_{\Lambda *}$, *and*

$$[h_{2*}(1 \otimes \bar{t})]^n = h_{2*}(1 \otimes \bar{t}^n). \tag{4.3.10}$$

Proof. Let us use induction on n. Suppose

$$[h_{2*}(\bar{x} \otimes 1)]^{n-1} = h_{2*}(\bar{x}^{n-1} \otimes 1).$$

Then by equation 4.3.2,

$$
\begin{aligned}
h_{2*}(\bar{x}^n \otimes 1) &= \mu_{\Lambda *}(h_{2*}(\bar{x}^{n-1} \otimes 1) \otimes h_{2*}(\bar{x} \otimes 1)) \\
&= \mu_{\Lambda *}([h_{2*}(\bar{x} \otimes 1)]^{n-1} \otimes h_{2*}(\bar{x} \otimes 1)) \\
&= [h_{2*}(\bar{x} \otimes 1)]^n.
\end{aligned}
$$

Hence equation 4.3.9 holds by induction. The proof of equation 4.3.10 is similar. $\square$

Example 4.3.4. Let us illustrate Theorem 4.3.1. Let $\bar{x}, \bar{y} \in H_*(X; \mathbb{F}_p)$, $\bar{t}, \bar{u} \in H_*(\Omega X; \mathbb{F}_p)$. We will use the same notation as in the theorem for their coproducts.

According to Theorem 4.3.1, if $\bar{t} = \bar{u} = 1$, we have

$$\mu_{\Lambda *}(h_{2*}(\bar{x} \otimes 1) \otimes h_{2*}(\bar{y} \otimes 1)) = h_{2*}(\bar{x}\bar{y} \otimes 1), \qquad (4.3.11)$$

and if instead $\bar{x} = \bar{y} = 1$, we have

$$\mu_{\Lambda *}(h_{2*}(1 \otimes \bar{t}) \otimes h_{2*}(1 \otimes \bar{u})) = h_{2*}(1 \otimes \bar{t}\bar{u}), \qquad (4.3.12)$$

and if $\bar{y} = 1$ and $\bar{t} = 1$, we have already seen that

$$\mu_{\Lambda *}(h_{2*}(\bar{x} \otimes 1) \otimes h_{2*}(1 \otimes \bar{u})) = h_{2*}(\bar{x} \otimes \bar{u}), \qquad (4.3.13)$$

so in these cases, h_{2*} acts like an algebra homomorphism on $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$.

Now, let us look at $\mu_{\Lambda *}(h_{2*}(1 \otimes \bar{t}) \otimes h_{2*}(\bar{y} \otimes 1))$ with the additional assumption that $\bar{y}$ and $\bar{t}$ are primitive. By Corollary 2.3.6, $i_*(\bar{y}) = -\bar{y}$, so by Theorem 4.3.1, we have

$$
\begin{aligned}
\mu_{\Lambda *}(h_{2*}(1 \otimes \bar{t}) \otimes h_{2*}(\bar{y} \otimes 1)) &= h_{2*}(1 \otimes Com_*(i_*(\bar{y}) \otimes \bar{t}) + \bar{y} \otimes \bar{t}) \\
&= h_{2*}(-1 \otimes Com_*(\bar{y} \otimes \bar{t}) + \bar{y} \otimes \bar{t}). \quad (4.3.14)
\end{aligned}
$$

Note that if $Com_*(\bar{y} \otimes \bar{t})$ vanishes (for example, if $\bar{y}$ has odd degree), then

$$
\begin{aligned}
\mu_{\Lambda*}(h_{2*}(1 \otimes \bar{t}) \otimes h_{2*}(\bar{y} \otimes 1)) &= h_{2*}(1 \otimes Com_*(i_*(\bar{y}) \otimes \bar{t}) + \bar{y} \otimes \bar{t}) \\
&= h_{2*}(-1 \otimes Com_*(\bar{y} \otimes \bar{t}) + \bar{y} \otimes \bar{t}) \\
&= h_{2*}(\bar{y} \otimes \bar{t}).
\end{aligned}
\tag{4.3.15}
$$

Let us see what happens if $Com_*(\bar{y} \otimes \bar{t})$ does not vanish in a concrete example in $H_*(\Lambda F_4; \mathbb{F}_3)$. We will use Theorems 1 and 2 from Hara and Hamanaka's paper [10] for data about coproducts in $H_*(\Omega F_4; \mathbb{F}_3)$ and computations of Ad_*. We will use $\bar{x}$ to denote elements in $H_*(F_4; \mathbb{F}_3)$ (instead of y) and $\bar{t}$ to denote elements in $H_*(\Omega F_4; \mathbb{F}_3)$ (instead of t). Note that Hara and Hamanaka use ad (with a lower case) to denote the map Kono and Kozima call Ad. For reference, a generating set of $H_*(F_4; \mathbb{F}_3) \otimes H_*(\Omega F_4; \mathbb{F}_3)$ would be

$$
\left\{ \bar{x}_3 \otimes 1, \bar{x}_7 \otimes 1, \bar{x}_8 \otimes 1, 1 \otimes \bar{t}_2, 1 \otimes \bar{t}_6, 1 \otimes \bar{t}_{10}, 1 \otimes \bar{t}_{14}, 1 \otimes \bar{t}_{22} \right\}.
$$

According to Hara and Hamanaka, in $H_*(\Omega F_4; \mathbb{F}_3)$, we have the following calculation of $Ad_* : H_*(F_4; \mathbb{F}_3) \otimes H_*(\Omega F_4; \mathbb{F}_3) \to H_*(\Omega F_4; \mathbb{F}_3)$:

$$
Ad_*(\bar{x}_8 \otimes \bar{t}_2) = \bar{t}_{10}.
$$

According to part 4 of Proposition 3.5.4, if $\bar{t} \in H_*(\Omega F_4; \mathbb{F}_3)$ is primitive ($\Delta_{\Omega X*}(\bar{t}) =$

$\bar{t} \otimes 1 + 1 \otimes \bar{t}$), then for any $\bar{x} \in H_*(F_4; \mathbb{F}_3)$,

$$
\begin{aligned}
Com_*(\bar{x} \otimes \bar{t}) &= \mu_{\Omega*}(Ad_* \otimes i_{\Omega*})(\bar{x} \otimes \Delta_{\Omega X*}(\bar{t})) \\
&= \mu_{\Omega*}(Ad_* \otimes i_{\Omega*})(\bar{x} \otimes \bar{t} \otimes 1 + \bar{x} \otimes 1 \otimes \bar{t}) \\
&= \mu_{\Omega*}(Ad_*(\bar{x} \otimes \bar{t}) \otimes 1 + Ad_*(\bar{x} \otimes 1) \otimes \bar{t}) \\
&= Ad_*(\bar{x} \otimes \bar{t}) \quad\quad\quad (4.3.16)
\end{aligned}
$$

since $Ad_*(\bar{x} \otimes 1) = 0$ (by part 7 of Proposition 3.3.1). Thus, in $H_*(\Omega F_4; \mathbb{F}_3)$, we have

$$
Com_*(\bar{x}_8 \otimes \bar{t}_2) = \bar{t}_{10}.
$$

Therefore, in $H_*(\Lambda F_4; \mathbb{F}_3)$, by equation 4.3.13,

$$
\mu_{\Lambda*}(h_{2*}(\bar{x}_8 \otimes 1) \otimes h_{2*}(1 \otimes \bar{t}_2)) = h_{2*}(\bar{x}_8 \otimes \bar{t}_2),
$$

while by equation 4.3.14,

$$
\begin{aligned}
\mu_{\Lambda*}(h_{2*}(1 \otimes \bar{t}_2) \otimes h_{2*}(\bar{x}_8 \otimes 1)) &= h_{2*}(1 \otimes Com_*(i_*(\bar{x}_8) \otimes \bar{t}_2) + \bar{x}_8 \otimes \bar{t}_2) \\
&= h_{2*}(-1 \otimes Com_*(\bar{x}_8 \otimes \bar{t}_2) + \bar{x}_8 \otimes \bar{t}_2) \\
&= -h_{2*}(1 \otimes \bar{t}_{10}) + h_{2*}(\bar{x}_8 \otimes \bar{t}_2)) \\
&\neq \mu_{\Lambda X*}(h_{2*}(\bar{x}_8 \otimes 1) \otimes h_{2*}(1 \otimes \bar{t}_2)).
\end{aligned}
$$

Hence there is a nontrivial commutator $[h_{2*}(\bar{x}_8 \otimes 1), h_{2*}(1 \otimes \bar{t}_2)]$ in $H_*(\Lambda F_4; \mathbb{F}_3)$,

$$[h_{2*}(\bar{x}_8 \otimes 1), h_{2*}(1 \otimes \bar{t}_2)] = h_{2*}(1 \otimes \bar{t}_{10})$$

$$= h_{2*}(1 \otimes Ad_*(\bar{x}_8 \otimes \bar{t}_2)). \qquad (4.3.17)$$

Notice that this means $h_{2*}(1 \otimes \bar{t}_{10}) \in H_*(\Lambda F_4; \mathbb{F}_3)$ is a decomposable; recall

that $1 \otimes \bar{t}_{10} \in H_*(F_4; \mathbb{F}_3) \otimes H_*(\Omega F_4; \mathbb{F}_3)$ is a generator.

Let us continue our calculation of products and commutators in $H_*(\Lambda F_4; \mathbb{F}_3)$.

If $j = 3$ or 7, and k is either $2, 6, 10, 14,$ or 22, then for any $\bar{u} \in H_*(\Omega F_4; \mathbb{F}_3)$ (so $\bar{u}$ has

even degree),

$$Com_*(\bar{x}_j \otimes \bar{u}) = 0,$$

so by Theorem 4.3.1,

$$\mu_{\Lambda*}(h_{2*}(1 \otimes \bar{t}_k) \otimes h_{2*}(\bar{x}_j \otimes 1)) = h_{2*}(\bar{x}_j \otimes \bar{t}_k)$$

$$= \mu_{\Lambda*}(h_{2*}(\bar{x}_j \otimes 1) \otimes h_{2*}(1 \otimes \bar{t}_k)),$$

and consequently,

$$[h_{2*}(\bar{x}_j \otimes 1), h_{2*}(1 \otimes \bar{t}_k)] = 0.$$

Next, we look at $\mu_{\Lambda*}(h_{2*}(1 \otimes \bar{t}_6) \otimes h_{2*}(\bar{x}_8 \otimes 1))$. The element $\bar{t}_6 \in H_*(\Omega F_4; \mathbb{F}_3)$ is not

primitive; its coproduct (again using data from Hara and Hamanaka's paper [10]) is

$$\Delta_{\Omega X*}(\bar{t}_6) = \bar{t}_6 \otimes 1 + 1 \otimes \bar{t}_6 - \bar{t}_2^2 \otimes \bar{t}_2 - \bar{t}_2 \otimes \bar{t}_2^2.$$

In order to use $Ad_*(\bar{x}_8 \otimes \bar{t}_6)$ to compute $Com_*(\bar{x}_8 \otimes \bar{t}_6)$ (with part 5 of Proposition 3.5.4), we will also need the following: $i_{\Omega*}(\bar{t}_2^2)$, $i_{\Omega*}(\bar{t}_6)$, $Ad_*(\bar{x}_8 \otimes \bar{t}_2)$, and $Ad_*(\bar{x}_8 \otimes \bar{t}_2^2)$. Let us compute $i_{\Omega*}(\bar{t}_6)$. Since $i_{\Omega*}(\bar{t}_2) = -\bar{t}_2$, by applying homology to Lemma 2.3.4,

$$
\begin{aligned}
i_{\Omega*}(\bar{t}_2^2) &= i_{\Omega*}(\mu_{\Omega*}(\bar{t}_2 \otimes \bar{t}_2)) \\
&= \mu_{\Omega*}(i_{\Omega*} \otimes i_{\Omega*})(T_{\Omega X*}(\bar{t}_2 \otimes \bar{t}_2)) \\
&= \mu_{\Omega*}(-\bar{t}_2 \otimes (-\bar{t}_2)) \\
&= \bar{t}_2^2,
\end{aligned}
$$

so by Corollary 2.3.6,

$$
\begin{aligned}
i_{\Omega*}(\bar{t}_6) &= -\bar{t}_6 + i_{\Omega*}(\bar{t}_2^2)\bar{t}_2 + i_{\Omega*}(\bar{t}_2)\bar{t}_2^2 \\
&= -\bar{t}_6 + \bar{t}_2^3 + (-\bar{t}_2)\bar{t}_2^2 \\
&= -\bar{t}_6.
\end{aligned}
$$

We already know that $Com_*(\bar{x}_8 \otimes \bar{t}_2) = \bar{t}_{10} = Ad_*(\bar{x}_8 \otimes \bar{t}_2)$ and $Ad_*(\bar{x}_8 \otimes \bar{t}_6) = \bar{t}_{14} - \bar{t}_{10}\bar{t}_2^2$. Before we compute $Com_*(\bar{x}_8 \otimes \bar{t}_6)$, we also need $Ad_*(\bar{x}_8 \otimes \bar{t}_2^2)$, which can be calculated

using part 5 of Proposition 3.5.4:

$$
\begin{aligned}
Ad_*(\bar{x}_8 \otimes \bar{t}_2^2) \;=\;& \mu_\Omega \left(Ad_*(\bar{x}_8 \otimes \bar{t}_2), Ad_*(1 \otimes \bar{t}_2) \right) \\[2mm]
& +\mu_\Omega \left(Ad_*(1 \otimes \bar{t}_2), Ad_*(\bar{x}_8 \otimes \bar{t}_2) \right) \\[2mm]
=\;& \bar{t}_{10}\bar{t}_2 + \bar{t}_2\bar{t}_{10} \\[2mm]
=\;& -\bar{t}_{10}\bar{t}_2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
& Com_*(\bar{x}_8 \otimes \bar{t}_6) \\[2mm]
=\;& \mu_{\Omega*}(Ad_* \otimes i_{\Omega*})(\bar{x}_8 \otimes \Delta_{\Omega X *}(\bar{t}_6)) \\[2mm]
=\;& \mu_{\Omega*}(Ad_* \otimes i_{\Omega*})(\bar{x}_8 \otimes \bar{t}_6 \otimes 1 + \bar{x}_8 \otimes 1 \otimes \bar{t}_6 - \bar{x}_8 \otimes \bar{t}_2^2 \otimes \bar{t}_2 - \bar{x}_8 \otimes \bar{t}_2 \otimes \bar{t}_2^2) \\[2mm]
=\;& \mu_{\Omega*}(Ad_*(\bar{x}_8 \otimes \bar{t}_6) \otimes 1 - Ad_*(\bar{x}_8 \otimes 1) \otimes \bar{t}_6 \\[2mm]
& +Ad_*(\bar{x}_8 \otimes \bar{t}_2^2) \otimes \bar{t}_2 - Ad_*(\bar{x}_8 \otimes \bar{t}_2) \otimes \bar{t}_2^2) \\[2mm]
=\;& \mu_{\Omega*}((\bar{t}_{14} - \bar{t}_{10}\bar{t}_2^2) \otimes 1 - 0 - \bar{t}_{10}\bar{t}_2 \otimes \bar{t}_2 - \bar{t}_{10} \otimes \bar{t}_2^2) \\[2mm]
=\;& \bar{t}_{14} - \bar{t}_{10}\bar{t}_2^2 - \bar{t}_{10}\bar{t}_2^2 - \bar{t}_{10}\bar{t}_2^2 \\[2mm]
=\;& \bar{t}_{14}.
\end{aligned}
$$

Therefore, by Theorem 4.3.1,

$$\mu_{\Lambda*}(h_{2*}(1 \otimes \bar{t}_6) \otimes h_{2*}(\bar{x}_8 \otimes 1))$$

$$= h_{2*}[1 \otimes Com_*(i_*(\bar{x}_8) \otimes \bar{t}_6) - 1 \otimes Com_*(i_*(\bar{x}_8) \otimes \bar{t}_2^2)\bar{t}_2$$

$$-1 \otimes Com_*(i_*(\bar{x}_8) \otimes \bar{t}_2)\bar{t}_2^2 + \bar{x}_8 \otimes \bar{t}_6]. \qquad (4.3.18)$$

By part 6 of Proposition 3.5.4,

$$Com_*(\bar{x}_8 \otimes \bar{t}_2^2) = \mu_\Omega (Com_*(\bar{x}_8 \otimes \bar{t}_2), Com_*(1 \otimes \bar{t}_2))$$

$$+\mu_\Omega (Com_*(1 \otimes \bar{t}_2), Com_*(\bar{x}_8 \otimes \bar{t}_2))$$

$$= 0.$$

Hence equation 4.3.18 becomes

$$\mu_{\Lambda*}(h_{2*}(1 \otimes \bar{t}_6) \otimes h_{2*}(\bar{x}_8 \otimes 1)) = h_{2*}[-1 \otimes \bar{t}_{14} + 1 \otimes \bar{t}_{10}\bar{t}_2^2 + \bar{x}_8 \otimes \bar{t}_6].$$

Thus, we have another commutator:

$$[h_{2*}(\bar{x}_8 \otimes 1), h_{2*}(1 \otimes \bar{t}_6)] = h_{2*}(1 \otimes \bar{t}_{14} - 1 \otimes \bar{t}_{10}\bar{t}_2^2)$$

$$= h_{2*}(1 \otimes Ad_*(\bar{x}_8 \otimes \bar{t}_6)). \qquad (4.3.19)$$

Since the remaining generators of $H_*(\Omega F_4; \mathbb{F}_3)$, namely $\bar{t}_{10}$, $\bar{t}_{14}$, and $\bar{t}_{22}$, are primitive, by calculations similar to the one for $\mu_{\Lambda F_4*}(1 \otimes \bar{t}_2 \otimes \bar{x}_8 \otimes 1)$ in Example

4.3.4, we have

$$Com_*(\bar{x}_8 \otimes \bar{t}_{10}) = Ad_*(\bar{x}_8 \otimes \bar{t}_{10})$$

$$= 1 \otimes \kappa \bar{t}_6^3$$

where $\kappa = \pm 1 \in \mathbb{F}_3$,

$$Com_*(\bar{x}_8 \otimes \bar{t}_{14}) = Ad_*(\bar{x}_8 \otimes \bar{t}_{14})$$

$$= 1 \otimes \bar{t}_{22},$$

$$Com_*(\bar{x}_8 \otimes \bar{t}_{22}) = Ad_*(\bar{x}_8 \otimes \bar{t}_{22})$$

$$= -1 \otimes \bar{t}_{10}^3,$$

so

$$\mu_{\Lambda*}(h_{2*}(1 \otimes \bar{t}_{10}) \otimes h_{2*}(\bar{x}_8 \otimes 1)) = h_{2*}(1 \otimes Com_*(i_*(\bar{x}_8) \otimes \bar{t}_{10}) + \bar{x}_8 \otimes \bar{t}_{10})$$

$$= -h_{2*}(1 \otimes \kappa \bar{t}_6^3) + h_{2*}(\bar{x}_8 \otimes \bar{t}_{10}),$$

$$\mu_{\Lambda*}(h_{2*}(1 \otimes \bar{t}_{14}) \otimes h_{2*}(\bar{x}_8 \otimes 1)) = h_{2*}(1 \otimes Com_*(i_*(\bar{x}_8) \otimes \bar{t}_{14}) + \bar{x}_8 \otimes \bar{t}_{14})$$

$$= -h_{2*}(1 \otimes \bar{t}_{22}) + h_{2*}(\bar{x}_8 \otimes \bar{t}_{14}),$$

$$\mu_{\Lambda*}(h_{2*}(1 \otimes \bar{t}_{22}) \otimes h_{2*}(\bar{x}_8 \otimes 1)) \quad = \quad h_{2*}(1 \otimes Com_*(i_*(\bar{x}_8) \otimes \bar{t}_{22}) + \bar{x}_8 \otimes \bar{t}_{22})$$

$$= \quad h_{2*}(1 \otimes \bar{t}_{10}^3) + h_{2*}(\bar{x}_8 \otimes \bar{t}_{22}).$$

At this point, we have seen generators of $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$ whose image under h_{2*} is a decomposable in $H_*(\Lambda X; \mathbb{F}_p)$. By Corollary 4.3.2, there is a subset of

$$\{h_{2*}(\bar{x}_3 \otimes 1), h_{2*}(\bar{x}_7 \otimes 1), h_{2*}(\bar{x}_8 \otimes 1), h_{2*}(1 \otimes \bar{t}_2),$$

$$h_{2*}(1 \otimes \bar{t}_6), h_{2*}(1 \otimes \bar{t}_{10}), h_{2*}(1 \otimes \bar{t}_{14}), h_{2*}(1 \otimes \bar{t}_{22})\}$$

which is generating set for $H_*(\Lambda F_4; \mathbb{F}_3)$. Based on our calculations, we could take

$$\mathcal{B} = \{h_{2*}(\bar{x}_3 \otimes 1), h_{2*}(\bar{x}_7 \otimes 1), h_{2*}(\bar{x}_8 \otimes 1), h_{2*}(1 \otimes \bar{t}_2), h_{2*}(1 \otimes \bar{t}_6)\}.$$

We can use equations 4.3.11, 4.3.12, and 4.3.15 to show that these five elements of $\mathcal{B}$ are indeed generators; there is no way to write any of these five elements as (sums of) products of each other. Since the heights of $\bar{x}_3, \bar{x}_7, \bar{x}_8 \in H_*(F_4; \mathbb{F}_3)$ are $2, 2, 3$ respectively, $\bar{t}_2 \in H_*(\Omega F_4; \mathbb{F}_3)$ has height 3, and $\bar{t}_6 \in H_*(\Omega F_4; \mathbb{F}_3)$ has infinite height, by Corollary 4.3.3, the heights of the elements of $\mathcal{B}$ are $2, 2, 3, 3$, and infinity respectively. Furthermore, we have these commutators involving $\bar{x}_8 \otimes 1$ and $1 \otimes \bar{t}_j$ for $j = 2, 6, 10, 14, 22$ in $H_*(\Lambda X; \mathbb{F}_p)$:

$$[h_{2*}(\bar{x}_8 \otimes 1), h_{2*}(1 \otimes \bar{t}_2)] = h_{2*}(1 \otimes \bar{t}_{10}),$$

$$[h_{2*}(\bar{x}_8 \otimes 1), h_{2*}(1 \otimes \bar{t}_6)] = h_{2*}(1 \otimes \bar{t}_{14} - 1 \otimes \bar{t}_{10}\bar{t}_2^2),$$

$$[h_{2*}(\bar{x}_8 \otimes 1), h_{2*}(1 \otimes \bar{t}_{10})] = \pm h_{2*}(1 \otimes \bar{t}_6^3),$$

$$[h_{2*}(\bar{x}_8 \otimes 1), h_{2*}(1 \otimes \bar{t}_{14})] = h_{2*}(1 \otimes \bar{t}_{22}),$$

$$[h_{2*}(\bar{x}_8 \otimes 1), h_{2*}(1 \otimes \bar{t}_{22})] = -h_{2*}(1 \otimes \bar{t}_{10}^3).$$

The generating set $\mathcal{B}$, along with information on heights and commutators, give us a complete picture of the algebra $H_*(\Lambda F_4; \mathbb{F}_3)$. We see that h_{2*} takes $1 \otimes \bar{t}_{10}$, $1 \otimes \bar{t}_{14}$, and $1 \otimes \bar{t}_{22}$ (which are generators of $H_*(F_4; \mathbb{F}_3) \otimes H_*(\Omega F_4; \mathbb{F}_3)$) to decomposables in $H_*(\Lambda F_4; \mathbb{F}_3)$. We note that $\bar{t}_{10}$, $\bar{t}_{14}$, and $\bar{t}_{22}$ are in the image of Com_*. In addition, for $j = 2, 6, 10, 14, 22$,

$$[h_{2*}(\bar{x}_8 \otimes 1), h_{2*}(1 \otimes \bar{t}_j)] = h_{2*}(1 \otimes Ad_*(\bar{x}_8 \otimes \bar{t}_j)).$$

We are currently exploring further connections between commutators in $H_*(\Lambda F_4; \mathbb{F}_3)$ and the linear transformation Ad_*.

Let us end this section by briefly looking at what happens in cohomology. We can use Theorem 4.2.1 and Lemma 3.3.5 to compute coproducts of elements in the image of

$$h_1^* p_2^{\Omega *} \sigma^* : H^*(X; \mathbb{F}_p) \to H^*(\Lambda X; \mathbb{F}_p).$$

Theorem 4.3.5. *Let X be a finite simply-connected HA-space. In $H^*(X; \mathbb{F}_p)$, let $x \in S$ (see Theorem 1.2.4 and the discussion afterward for the definitions of S).*

Then

$$\mu_\Lambda^*(h_1^*(1 \otimes \sigma^*(x))) \;\; = \;\; h_1^* \otimes h_1^*[1 \otimes \sigma^*(x) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes \sigma^*(x)$$

$$+ 1 \otimes \left(T_{\Omega X, X}^*(i^* \otimes \sigma^*)(\bar\mu^*(x))\right) \otimes 1].$$

Proof. When we apply cohomology to diagram 4.2.1, we see that

$$\mu_\Lambda^* h_1^* = (h_1^* \otimes h_1^*)\tilde\mu^*,$$

where

$$\tilde\mu^* = (1 \otimes T_{\Omega X, X}^* \otimes 1)(1 \otimes \Delta_X^* \otimes \Delta_{\Omega X}^* \otimes 1)(\mu^* \otimes \omega^* \otimes \mu_\Omega^*)(1 \otimes \mu_\Omega^*),$$

$$\omega^* = (i^* \otimes 1)Com^*.$$

Recall that $\sigma^*(x) \in PH^*(\Omega X; \mathbb{F}_p)$, so

$$\mu_\Omega^*(\sigma^*(x)) = \sigma^*(x) \otimes 1 + 1 \otimes \sigma^*(x),$$

and by Lemma 3.3.5,

$$Com^*(\sigma^*(x)) = (1 \otimes \sigma^*)(\bar\mu^*(x)),$$

so

$$
\begin{aligned}
\omega^*(\sigma^*(x)) &= (i^* \otimes 1)Com^*(\sigma^*(x)) \\[2mm]
&= (i^* \otimes 1)(1 \otimes \sigma^*)(\bar{\mu}^*(x)) \\[2mm]
&= (i^* \otimes \sigma^*)(\bar{\mu}^*(x)).
\end{aligned} \tag{4.3.20}
$$

Let us compute $\tilde{\mu}^*(1 \otimes \sigma^*(x))$. We start with $(1 \otimes \mu_\Omega^*)$:

$$
\begin{aligned}
&(1 \otimes \mu_\Omega^*)(1 \otimes \sigma^*(x)) \\[2mm]
&= 1 \otimes \sigma^*(x) \otimes 1 + 1 \otimes 1 \otimes \sigma^*(x).
\end{aligned}
$$

Next, we apply $(\mu^* \otimes \omega^* \otimes \mu_\Omega^*)$ and use equation 4.3.20:

$$
\begin{aligned}
&(\mu^* \otimes \omega^* \otimes \mu_\Omega^*)(1 \otimes \mu_\Omega^*)(1 \otimes \sigma^*(x)) \\[2mm]
&= \mu^*(1) \otimes \omega^*(\sigma^*(x)) \otimes \mu_\Omega^*(1) + \mu^*(1) \otimes \omega^*(1) \otimes \mu_\Omega^*(\sigma^*(x)) \\[2mm]
&= 1 \otimes 1 \otimes (i^* \otimes \sigma^*)(\bar{\mu}^*(x)) \otimes 1 \otimes 1 \\[2mm]
&\quad +1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma^*(x) \otimes 1 + 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma^*(x).
\end{aligned}
$$

To apply $(1 \otimes \Delta_X^* \otimes \Delta_{\Omega X}^* \otimes 1)$, we multiply the second and third factors, and multiply

the fourth and fifth factors.

$$(1 \otimes \Delta_X^* \otimes \Delta_{\Omega X}^* \otimes 1)(\mu^* \otimes \omega^* \otimes \mu_\Omega^*)(1 \otimes \mu_\Omega^*)(1 \otimes \sigma^*(x))$$

$$= 1 \otimes (i^* \otimes \sigma^*)(\bar{\mu}^*(x)) \otimes 1 + 1 \otimes 1 \otimes \sigma^*(x) \otimes 1 + 1 \otimes 1 \otimes 1 \otimes \sigma^*(x).$$

Finally, we apply $(1 \otimes T_{\Omega X, X}^* \otimes 1)$:

$$(1 \otimes T_{\Omega X, X}^* \otimes 1)(1 \otimes \Delta_X^* \otimes \Delta_{\Omega X}^* \otimes 1)(\mu^* \otimes \omega^* \otimes \mu_\Omega^*)(1 \otimes \mu_\Omega^*)(1 \otimes \sigma^*(x))$$

$$= \tilde{\mu}^*(1 \otimes \sigma^*(x))$$

$$= (1 \otimes T_{\Omega X, X}^* \otimes 1)(1 \otimes (i^* \otimes \sigma^*)(\bar{\mu}^*(x)) \otimes 1)$$

$$+ 1 \otimes \sigma^*(x) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes \sigma^*(x).$$

Therefore, when we apply $(h_1^* \otimes h_1^*)$, we obtain

$$\mu_\Lambda^*(h_1^*(1 \otimes \sigma^*(x))) = (h_1^* \otimes h_1^*)\tilde{\mu}^*(1 \otimes \sigma^*(x))$$

$$= (h_1^* \otimes h_1^*)[1 \otimes \sigma^*(x) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes \sigma^*(x)$$

$$+ 1 \otimes T_{\Omega X, X}^*[((i^* \otimes \sigma^*)(\bar{\mu}^*(x))] \otimes 1].$$

$\square$

Overall, it will be worthwhile to continue learning about the maps Ad, Com, and their induced homomorphisms and linear transformations in order to examine coproducts in $H^*(\Lambda X; \mathbb{F}_p)$, and multiplication and commutators in $H_*(\Lambda X; \mathbb{F}_p)$. When

Hara and Hamanaka made their calculations of Ad_* in [10], they needed to use the fact that for the Lie groups $G = F_4, E_6, E_7, E_8$, for any positive even integer k,

$$\dim Q^k H^*(G; \mathbb{F}_3) \le 1.$$

Future efforts will need to find ways of bypassing this requirement.

4.4 The Cohomology of the Free Loop Space and h_1

For this section, we will restrict ourselves to finite simply-connected spaces. Recall Iwase's observation that the homeomorphism $\Phi : \Omega G \times G \to \Lambda G$ from the introduction to this chapter will be a group homomorphism if G is abelian. In that case, there would be an isomorphism of Hopf algebras $\Phi^* : H^*(\Lambda G; \mathbb{F}_p) \to H^*(\Omega G; \mathbb{F}_p) \otimes H^*(G; \mathbb{F}_p)$. In this section, we will show that a weaker condition than (homotopy) commutativity will suffice for showing that our homotopy equivalence h_1 will induce a Hopf algebra isomorphism, and this condition applies to finite simply-connected HA-spaces as well.

Given a finite simply-connected HA-space X, we can characterize commutativity of $H_*(X; \mathbb{F}_p)$ using h_1^* by applying Lemma 4.1.6 and Theorem 3.4.2.

Theorem 4.4.1. *The algebra $H_*(X; \mathbb{F}_p)$ is commutative iff the induced homomorphism h_1^* is a Hopf algebra isomorphism.*

Proof. By Theorem 3.4.2, commutativity of $H_*(X; \mathbb{F}_p)$ is equivalent to Com^* being

trivial.

If $Com^* = 0$, then by Lemma 4.1.6,

$$D_{h_1}^* = T_{\Lambda X}^* \circ (\varepsilon_0^* \otimes (h_1^* \circ p_2^{\Omega *})) \circ (i^* \otimes 1) \circ Com^* \circ j_2^* = 0,$$

so $D_{h_1}^* = 0$, and hence by equation 4.1.29,

$$\mu_\Lambda^* \circ h_1^*(x \otimes y) = (h_1^* \otimes h_1^*) \circ (\mu_{X \times \Omega X})^*(x \otimes y).$$

Hence h_1^* is not only an algebra isomorphism, it also preserves coproducts, so it is a

Hopf algebra isomorphism. This implies that $H^*(\Lambda X; \mathbb{F}_p)$ (with coproduct μ_Λ^*) and

$H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$ (with coproduct $\mu_{X \times \Omega X}^*$) are isomorphic as Hopf algebras.

Now suppose that h_1^* is a Hopf algebra isomorphism. Then by Lemma 2.4.8,

$D_{h_1}^* = 0$, so by Lemma 4.1.6,

$$D_{h_1}^* = T_{\Lambda X}^* \circ (\varepsilon_0^* \otimes (h_1^* \circ p_2^{\Omega *})) \circ (i^* \otimes 1) \circ Com^* \circ j_2^* = 0.$$

We will prove that if $D_{h_1}^* = 0$, then $Com^* = 0$. Now, note that i^*, ε_0^*, $p_2^{\Omega *}$, h_1^*, and

$T_{\Lambda X}^*$ are all injective homomorphisms:

1) By Lemma 2.3.4, $i^* \circ i^* = id_X^*$, so i^* is a bijection.

2) By equation 4.1.20, $\varepsilon_0^*(x) = x \otimes 1$.

3) Recall that $p_2^{\Omega *}(y) = 1 \otimes y$.

4) The maps h_1 and $T_{\Lambda X}$ are homotopy equivalences, so their induced homo-

morphisms are algebra isomorphisms.

Although j_2^* is not injective on $H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$, we have $j_2^*(1 \otimes y) = y$, so j_2^* is injective on $\{1\} \otimes H^*(\Omega X; \mathbb{F}_p)$ and surjective onto $H^*(\Omega X; \mathbb{F}_p)$. Therefore, in order for $D_{h_1}^* = 0$, we must have $Com^* = 0$. □

Thus, if the algebra $H_*(X; \mathbb{F}_p)$ is commutative, then h_1^* is a Hopf algebra isomorphism. Consequently, if the coproduct structures of $H^*(X; \mathbb{F}_p)$ and $H^*(\Omega X; \mathbb{F}_p)$ are known, then the coproduct structure of $H^*(X\Lambda; \mathbb{F}_p)$, and dually the multiplication in $H_*(\Lambda X; \mathbb{F}_p)$ are easily determined in this case. As we have seen, if $H_*(X; \mathbb{F}_p)$ is not commutative and Com_* is not trivial, then the multiplication in $H_*(\Lambda X; \mathbb{F}_p)$ can look very different from the tensor product multiplication structure on $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$.

As a corollary, we have the following connection between commutativity of $H_*(X; \mathbb{F}_p)$ and commutativity of $H_*(\Lambda X; \mathbb{F}_p)$:

Corollary 4.4.2. *The algebra $H_*(X; \mathbb{F}_p)$ is commutative iff the algebra $H_*(\Lambda X; \mathbb{F}_p)$ is commutative.*

Proof. If the algebra $H_*(X; \mathbb{F}_p)$ is commutative, then by Theorem 4.4.1, h_1^* is a Hopf algebra isomorphism, and equivalently h_{1*} is an algebra isomorphism. Thus, $H_*(X; \mathbb{F}_p) \otimes H_*(\Omega X; \mathbb{F}_p)$ and $H_*(\Lambda X; \mathbb{F}_p)$ are isomorphic as algebras. Since $H_*(X; \mathbb{F}_p)$ and $H_*(\Omega X; \mathbb{F}_p)$ are commutative (see Definition 1.2.7), this implies $H_*(\Lambda X; \mathbb{F}_p)$ is commutative as well.

If the algebra $H_*(X; \mathbb{F}_p)$ is not commutative, then $H^*(X; \mathbb{F}_p)$ is not primitively

generated, so there is a generator $z \in S$ with $\bar{\mu}^*(z) \neq 0$. Since even degree elements in

S are primitive by definition, this generator z must have odd degree (and $|z| > 1$ since

we assume X is simply-connected), so by Theorem 2.2.4, $\sigma^*(z) \neq 0$. By Theorem

4.3.5, the coproduct of $h_1^*(1 \otimes \sigma^*(z))$ in $H^*(\Lambda X; \mathbb{F}_p)$ is

$$
\begin{aligned}
\mu_\Lambda^*(h_1^*(1 \otimes \sigma^*(z))) \;=\; & (h_1^* \otimes h_1^*)[1 \otimes \sigma^*(z) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes \sigma^*(z) \\
& + 1 \otimes T_{\Omega X, X}^*[((i^* \otimes \sigma^*)(\bar{\mu}^*(z))] \otimes 1].
\end{aligned}
$$

Let us write out the reduced coproduct. First, we write

$$
\bar{\mu}^*(z) = \sum z' \otimes z'',
$$

where each z'' is an odd degree generator (again $|z''| > 1$), so $\sigma^*(z'') \neq 0$. Then

$$
\begin{aligned}
\bar{\mu}_\Lambda^*(h_1^*(1 \otimes \sigma^*(z))) \;=\; & (h_1^* \otimes h_1^*)[1 \otimes T_{\Omega X, X}^*[((i^* \otimes \sigma^*)(\bar{\mu}^*(z))] \otimes 1] \\
\;=\; & (h_1^* \otimes h_1^*)[\sum 1 \otimes \sigma^*(z'') \otimes i^*(z') \otimes 1].
\end{aligned}
$$

To compute $T_{\Lambda X}^* \bar{\mu}_\Lambda^*(h_1^*(1 \otimes \sigma^*(z)))$, we note that the following diagram commutes

strictly:

$$T_{X \times \Omega X}(h_1 \times h_1)(\varphi_1, \varphi_2) = T_{X \times \Omega X}(h_1(\varphi_1), h_1(\varphi_2))$$

$$= (h_1(\varphi_2), h_1(\varphi_1))$$

$$= (h_1 \times h_1)(\varphi_2, \varphi_1)$$

$$= (h_1 \times h_1)T_{\Lambda X}(\varphi_2, \varphi_1).$$

Hence

$$T^*_{\Lambda X}\bar{\mu}^*_\Lambda(h^*_1(1 \otimes \sigma^*(z))) = T^*_{\Lambda X}(h^*_1 \otimes h^*_1)[\sum 1 \otimes \sigma^*(z'') \otimes i^*(z') \otimes 1]$$

$$= (h^*_1 \otimes h^*_1)T^*_{X \times \Omega X}[\sum 1 \otimes \sigma^*(z'') \otimes i^*(z') \otimes 1]$$

$$= (h^*_1 \otimes h^*_1)[\sum i^*(z') \otimes 1 \otimes 1 \otimes \sigma^*(z'')].$$

In $H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$, we have

$$\sum 1 \otimes \sigma^*(z'') \otimes i^*(z') \otimes 1 \neq \sum i^*(z') \otimes 1 \otimes 1 \otimes \sigma^*(z'')$$

since the second factor of every term on the left has positive degree, while the second factor of every term on the right has degree zero. Since $(h^*_1 \otimes h^*_1)$ is an algebra isomorphism, it is injective, so

$$(h^*_1 \otimes h^*_1)[\sum 1 \otimes \sigma^*(z'') \otimes i^*(z') \otimes 1] \neq (h^*_1 \otimes h^*_1)[\sum i^*(z') \otimes 1 \otimes 1 \otimes \sigma^*(z'')],$$

and therefore

$$\bar{\mu}^*_\Lambda(h^*_1(1 \otimes \sigma^*(z))) \neq T^*_{\Lambda X}\bar{\mu}^*_\Lambda(h^*_1(1 \otimes \sigma^*(z))),$$

so $H^*(\Lambda X; \mathbb{F}_p)$ is not cocommutative, and hence $H_*(\Lambda X; \mathbb{F}_p)$ is not commutative. $\quad\square$

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5 Altering H-space Structures

Given an H-space X with multiplication map μ, we may ask if there is another multiplication map on X which induces a different coproduct structure on its cohomology with desired properties. For example, suppose X is homotopy associative with multiplication map μ, and for some odd prime p, $H_*(X, \mu; \mathbb{F}_p)$ is not commutative. One may ask if there is a multiplication map which makes $H_*(X, \nu; \mathbb{F}_p)$ commutative, and furthermore, if the coproduct ν^* makes the cohomology algebra $H^*(X, \nu; \mathbb{F}_p)$ primitively generated.

For example, if G is one of the classical compact simply-connected Lie groups A_n, B_n, C_n, and D_n, or the exceptional group G_2, $H_*(G; \mathbb{F}_3)$ is commutative, while the opposite is true if G is one of the following groups: F_4, E_6, E_7, or E_8. The main difference between these two sets of Lie groups is that the homomorphism com^* does not vanish on the cohomology over $\mathbb{F}_3$ of F_4, E_6, E_7, and E_8. Thus, one can ask if there is an alternative multiplication map on F_4, E_6, E_7, and E_8 which induces a commutative homology algebra over $\mathbb{F}_3$, and if com or com^* will play a role.

The idea of modifying the multiplication map of an H-space to produce desired properties in cohomology originated from Harper and Zabrodsky's work [11]. They

used the fact that an H-space (X, μ) has a map $\psi_\lambda : (X, \mu) \to (X, \mu)$ (where λ is an integer) which is analogous to raising an element to the λth power in a group (because of this, Harper and Zabrodsky call ψ_λ a power map for X). In particular, ψ_λ induces multiplication by λ on $QH^*(X; \mathbb{F}_p)$. Harper and Zabrodsky use the H-deviation of ψ_λ from a $\mu - \mu$ H-map to construct new multiplication maps on X and produce primitively generated cohomology algebras under specific conditions.

While Harper and Zabrodsky focused on power maps where λ multiplicatively generates the group $\mathbb{F}_p \backslash \{0\}$, we instead looked at a specific power map for $\lambda = -1$: the homotopy inverse operation of an HA-space. Consequently, the multiplication maps we obtained (in terms of the H-deviation of the homotopy inverse operation) could actually be rewritten in terms of *com*. Hence we present our formulas for new multiplication maps in terms of *com*, and utilize our results from Chapter 3 to determine properties of the cohomology algebra using the new coproduct.

Since we will be dealing with multiple multiplication maps on the same space (different H-spaces with the same underlying topological space), from now on, we will write (X, μ) to indicate an H-space X with multiplication map μ. Consequently, we will write the cohomology of (X, μ) over $\mathbb{F}_p$ with coproduct induced by μ as $H^*(X, \mu; \mathbb{F}_p)$; the homology with product induced by μ will be denoted $H_*(X, \mu; \mathbb{F}_p)$. As a warning, the homology might no longer be an associative algebra.

5.1 Cocommutative Hopf Algebra Structures

Let us begin by defining what it means to alter an H-structure. Basically, we start with one multiplication map on $X = (X, \mu)$, and use the binary operation from Section 2.3 to combine it with another map to produce a new multiplication map. Once we have introduced alterations of H-structures, we introduce iterated multiplication maps on (X, μ) in order to discuss multiplying more than two elements at a time, along with their induced homomorphisms, the iterated coproducts in $H^*(X, \mu; \mathbb{F}_p)$. Finally, given a finite simply-connected HA-space (X, μ), we will show how to construct a multiplication map $\nu : X \times X \to X$ using iterated multiplication maps such that $H_*(X, \nu; \mathbb{F}_p)$ is commutative. Throughout this chapter, the symbol "$*$" will always be defined using the original multiplication map μ, and concatenation of elements will mean multiply using μ. In addition, $T : X \times X \to X \times X$ will be given by

$$T(x, y) = (y, x).$$

Definition 5.1.1. Given an H-space (X, μ) an *altered multiplication map (or altered H-structure)* on X is obtained by choosing a map $w : X \times X \to X$ which must factor through the smash product. That is, if $i_1, i_2 : X \to X \times X$ are inclusions and $k : X \to X$ is the constant map $k(x) = x_0$, then

$$wi_1 \simeq wi_2 \simeq k. \tag{5.1.1}$$

From here, we define a map

$$\nu : X \times X \to X$$

by

$$\nu = w * \mu = \mu \left(w \times \mu \right) \Delta_X. \qquad (5.1.2)$$

Elementwise, we have

$$\nu(x, y) = w(x, y)(xy).$$

We need to check if (X, ν) has a homotopy identity and verify:

$$\nu i_1 \simeq \nu i_2 \simeq id_X.$$

To do this, we use the fact that (X, μ) is an H-space, so by Definition 1.2.2,

$$\mu i_1 = \mu i_2 = id_X. \qquad (5.1.3)$$

By using equation 5.1.2, we have:

$$
\begin{aligned}
\nu i_1 &= \mu \left(w \times \mu \right) \Delta_X i_1 \\
&= \mu \left(w \times \mu \right) (i_1 \times i_1) \Delta_X \\
&= \mu \left((w i_1) \times (\mu i_1) \right) \Delta_X. \qquad (5.1.4)
\end{aligned}
$$

By equations 5.1.1 and 5.1.3, we have

$$\nu i_1 \quad \simeq \quad \mu(k \times id_X)\Delta_X$$

$$= \quad \mu i_2$$

$$= \quad id_X.$$

Similarly, we can show that

$$\nu i_2 \simeq id_X,$$

so (X, ν) is an H-space (with a homotopy unit).

The process of creating ν is also called *altering* μ.

A justification for this method of producing new multiplication maps on X comes from Theorem 1.4.3 from [40]. In fact, if μ and ν are two multiplication maps for a space X, and (X, μ) is also homotopy associative and has homotopy inverse operation i, then we can set $w = \nu * (i \circ \mu)$ so that $\nu = (\nu * (i \circ \mu)) * \mu \simeq w * \mu$. In other words, if we start with a homotopy associative multiplication map μ, we can obtain any other multiplication map (up to homotopy) by altering μ.

Before we continue, we need to determine an order of operations for using our binary operation on three or more maps.

Definition 5.1.2. Given an H-space (X, μ), we define the *iterated multiplication maps* inductively:

$$\mu^2 = \mu(\mu \times id_X) : X^{\times 3} \to X, \tag{5.1.5}$$

and for $n > 1$,

$$\mu^n = \mu^{n-1}(\mu \times id_X^{\times(n-1)}) : X^{\times(n+1)} \to X. \tag{5.1.6}$$

Essentially, when multiplying more than two elements, we multiply from the left. For example,

$$\mu^n(x_1, x_2, \ldots, x_{n+1}) = (\ldots((x_1 x_2)x_3)\ldots x_n)x_{n+1}.$$

We will call

$$\mu^{n*} : H^*(X, \mu; \mathbb{F}_p) \to H^*(X, \mu; \mathbb{F}_p)^{\otimes(n+1)}$$

the *nth (iterated) coproduct* on $H^*(X, \mu; \mathbb{F}_p)$. We have

$$\mu^{n*}(x) = (\mu^* \otimes 1^{\otimes(n-1)})\mu^{(n-1)*}(x). \tag{5.1.7}$$

Given any space Y, we define the *iterated diagonal maps* inductively:

$$\Delta_Y^2 = (\Delta_Y \times id_Y)\Delta_Y : Y \to Y^{\times 3}, \tag{5.1.8}$$

and for $n > 1$,

$$\Delta_Y^n = (\Delta_Y \times id_Y^{\times(n-1)})\Delta_Y^{n-1} : Y \to Y^{\times(n+1)}. \tag{5.1.9}$$

$$\Delta_Y^n(y) = (y, y, \ldots, y).$$

We successively apply Δ_Y on the left each time. In $H^*(Y; \mathbb{F}_p)$, we have the *nth*

iterated cup product

$$\Delta_Y^{n*} : H^*(Y; \mathbb{F}_p)^{\otimes n+1} \to H^*(Y; \mathbb{F}_p),$$

and

$$
\begin{aligned}
\Delta_Y^{n*}(y_1 \otimes y_2 \otimes \ldots \otimes y_{n+1}) &= \Delta_Y^{(n-1)*}(\Delta_Y^*(y_1 \otimes y_2) \otimes \ldots \otimes y_{n+1}) \\
&= \Delta_Y^{(n-1)*}(y_1 y_2 \otimes \ldots \otimes y_{n+1}) \\
&= y_1 y_2 \ldots y_{n+1}.
\end{aligned}
\tag{5.1.10}
$$

Hence Δ_Y^{n*} takes $y_1 \otimes y_2 \otimes \ldots \otimes y_{n+1}$ to the product $y_1 y_2 \ldots y_{n+1}$.

Suppose $f_1, \ldots f_n$ are n maps from Y to X ($n > 1$). Then the map

$$f_1 * f_2 * \ldots * f_n : Y \to X$$

is defined by the composition

$$f_1 * f_2 * \ldots * f_n = \mu^{n-1} (f_1 \times \ldots \times f_n) \Delta_Y^{n-1}. \tag{5.1.11}$$

In addition, we abbreviate

$$f * \ldots * f \qquad (n \text{ times}) \tag{5.1.12}$$

as $n \cdot f$.

Let us determine what $\mu^{n*}(x)$, the nth coproduct of $x \in H^*(X, \mu; \mathbb{F}_p)$, could look like. That is, we compute it up to a sum of subalgebras.

Lemma 5.1.3. *Let $x \in S$, where S is the generating set from Theorem 1.2.4. Then $\mu^{n*}(x)$ has the form*

$$\sum_{i=0}^{n} 1^{\otimes n-i} \otimes x \otimes 1^{\otimes i} + \sum_{k=0}^{n-1} \left(\sum_{j=0}^{n-1-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-1-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right).$$
$$(5.1.13)$$

Proof. Since the iterated coproducts are defined inductively, we will prove the lemma using induction.

In order to perform induction, we need some facts about the sub Hopf algebra B. Let $b \in \bar{B}$. Since B is a Hopf algebra, we have $\mu^*(b) \in B \otimes B$. We can be more specific and note that

$$\mu^*(b) = b \otimes 1 + \sum b' \otimes b'' + 1 \otimes b,$$

where b', $b'' \in B$ have positive degree (so they are in $\bar{B}$), and thus $\mu^*(b)$ will have the form

$$\bar{B} \otimes B + 1 \otimes \bar{B}. \qquad (5.1.14)$$

We will need this observation in an inductive proof of formula 5.1.13.

Now let x be an element of the generating set S. Then $\mu^*(x)$ has the form

$$x \otimes 1 + 1 \otimes x + B \otimes H^*(X, \mu; \mathbb{F}_p).$$

Since the terms in the reduced coproduct must have factors which have positive degree, we can be more specific and write

$$x \otimes 1 + 1 \otimes x + \bar{B} \otimes H^*(X, \mu; \mathbb{F}_p). \tag{5.1.15}$$

Suppose that the $(n-1)$th coproduct of x, $\mu^{(n-1)*}(x)$, has the form

$$\sum_{i=0}^{n-1} 1^{\otimes n-1-i} \otimes x \otimes 1^{\otimes i}$$
$$+ \sum_{k=0}^{n-2} \left(\sum_{j=0}^{n-2-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-2-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right). \tag{5.1.16}$$

By definition, the nth coproduct of x is

$$\mu^{n*}(x) = (\mu^* \otimes 1^{\otimes(n-1)})\mu^{(n-1)*}(x).$$

Let us apply $\mu^* \otimes 1^{\otimes(n-1)}$ on each term of equation 5.1.16 individually. By equation 5.1.15, we have

$$(\mu^* \otimes 1^{\otimes(n-1)}) \left(\sum_{i=0}^{n-1} 1^{\otimes n-1-i} \otimes x \otimes 1^{\otimes i} \right)$$
$$= \sum_{i=0}^{n-2} 1 \otimes 1^{\otimes n-1-i} \otimes x \otimes 1^{\otimes i} + (x \otimes 1) \otimes 1^{\otimes n-1} + (1 \otimes x) \otimes 1^{\otimes n-1}$$
$$+ (\bar{B} \otimes H^*(X, \mu; \mathbb{F}_p)) \otimes 1^{\otimes n-1}$$
$$= \sum_{i=0}^{n-2} 1^{\otimes n-i} \otimes x \otimes 1^{\otimes i} + x \otimes 1 \otimes 1^{\otimes n-1} + 1 \otimes x \otimes 1^{\otimes n-1}$$
$$+ \bar{B} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes n-1}. \tag{5.1.17}$$

The last term in equation 5.1.17 will give us the $k = n - 1$ term in equation 5.1.13.

When we examine the sum

$$\sum_{i=0}^{n-2} 1^{\otimes n-i} \otimes x \otimes 1^{\otimes i} + x \otimes 1 \otimes 1^{\otimes n-1} + 1 \otimes x \otimes 1^{\otimes n-1},$$

we see that it equals

$$\sum_{i=0}^{n} 1^{\otimes n-i} \otimes x \otimes 1^{\otimes i}, \tag{5.1.18}$$

since the $i = n - 1$ term is $1 \otimes x \otimes 1^{\otimes n-1}$ and the $i = n$ term is $x \otimes 1 \otimes 1^{\otimes n-1}$.

Consequently, we have

$$\left(\mu^* \otimes 1^{\otimes(n-1)}\right) \left(\sum_{i=0}^{n-1} 1^{\otimes n-1-i} \otimes x \otimes 1^{\otimes i}\right)$$

$$= \sum_{i=0}^{n} 1^{\otimes n-i} \otimes x \otimes 1^{\otimes i} + \bar{B} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes n-1}. \tag{5.1.19}$$

For the other term in 5.1.16, we can equation 5.1.14 when applying $\mu^* \otimes 1^{\otimes(n-1)}$. It will help to break up the inner sum into two parts: the $j = 0$ term (whose leftmost factor is $\bar{B}$), and the remaining terms $j = 1, \ldots, n - 2 - k$ (which have 1 as their

leftmost factor). We have

$$
(\mu^* \otimes 1^{\otimes(n-1)}) \left(\sum_{k=0}^{n-2} \left(\sum_{j=0}^{n-2-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-2-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right) \right)
$$

$$
= (\mu^* \otimes 1^{\otimes(n-1)}) \left(\sum_{k=0}^{n-2} \left(\bar{B} \otimes B^{\otimes n-2-k-0} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right) \right)
$$

$$
+ (\mu^* \otimes 1^{\otimes(n-1)}) \left(\sum_{k=0}^{n-2} \left(\sum_{j=1}^{n-2-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-2-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right) \right)
$$

$$
= \sum_{k=0}^{n-2} \Big((\bar{B} \otimes B) \otimes B^{\otimes n-2-k} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k}
$$

$$
+ (1 \otimes \bar{B}) \otimes B^{\otimes n-2-k} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \Big)
$$

$$
+ \sum_{k=0}^{n-2} \left(\sum_{j=1}^{n-2-k} 1 \otimes 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-2-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right). \tag{5.1.20}
$$

This will give us the $k = 0, \ldots, n-2$ terms in equation 5.1.13. To see this, we will need to change the indexing of the inner sum in the last term:

$$
\sum_{j=1}^{n-2-k} 1 \otimes 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-2-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k}
$$

$$
= \sum_{j=1}^{n-2-k} 1^{\otimes j+1} \otimes \bar{B} \otimes B^{\otimes n-1-k-(j+1)} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k}
$$

will go from $j = 2$ to $j = n - 1 - k$:

$$
\sum_{j=2}^{n-1-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-1-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k}.
$$

The terms in the second to last line of equation 5.1.13,

$$\sum_{k=0}^{n-2} \left(\bar{B} \otimes B \otimes B^{\otimes n-2-k} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right.$$

$$\left. +1 \otimes \bar{B} \otimes B^{\otimes n-2-k} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right)$$

$$= \sum_{k=0}^{n-2} \left(\bar{B} \otimes B^{\otimes n-1-k} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right.$$

$$\left. +1 \otimes \bar{B} \otimes B^{\otimes n-1-k-1} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right)$$

will become the $j = 0$ and $j = 1$ terms respectively in the sum

$$\sum_{k=0}^{n-2} \left(\sum_{j=0}^{n-1-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-1-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right). \qquad (5.1.21)$$

Consequently, we have

$$(\mu^* \otimes 1^{\otimes(n-1)}) \left(\sum_{k=0}^{n-2} \left(\sum_{j=0}^{n-2-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-2-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right) \right)$$

$$= \sum_{k=0}^{n-2} \left(\sum_{j=0}^{n-1-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-1-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right) \qquad (5.1.22)$$

Thus, by equations 5.1.19 and 5.1.22, the nth coproduct $\mu^{n*}(x)$ will have the form

$$\sum_{i=0}^{n} 1^{\otimes n-i} \otimes x \otimes 1^{\otimes i} + \bar{B} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes n-1}$$

$$+ \sum_{k=0}^{n-2} \left(\sum_{j=0}^{n-1-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-1-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right), \qquad (5.1.23)$$

and since

$$\bar{B} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes n-1}$$

$$+ \sum_{k=0}^{n-2} \left(\sum_{j=0}^{n-1-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-1-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right)$$

$$= \sum_{k=0}^{n-1} \left(\sum_{j=0}^{n-1-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-1-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right),$$

we can simplify equation 5.1.23 to

$$\sum_{i=0}^{n} 1^{\otimes n-i} \otimes x \otimes 1^{\otimes i} + \sum_{k=0}^{n-1} \left(\sum_{j=0}^{n-1-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes n-1-k-j} \otimes H^*(X, \mu; \mathbb{F}_p) \otimes 1^{\otimes k} \right).$$

Therefore, by induction, we have formula 5.1.13. $\qquad\square$

The following result shows that any finite simply-connected HA-space can be given a new multiplication map that will result in an H-space whose cohomology over $\mathbb{F}_p$ (with coproduct induced by the new multiplication map) is cocommutative.

Theorem 5.1.4. *Let (X, μ) be a finite simply-connected HA-space with homotopy inverse i. Let $\nu = \left(\frac{p-1}{2} \cdot com \right) * \mu$ Then (X, ν) is an H-space for which $H_*(X, \nu; \mathbb{F}_p)$ is a commutative (non-associative) algebra, and there is a generating set for $H^*(X, \nu; \mathbb{F}_p)$ such that if x is an element of this set,*

$$\nu^*(x) = \frac{1}{2} \left(\mu^*(x) + T^* \mu^*(x) \right). \tag{5.1.24}$$

Proof. We have

$$
\begin{aligned}
\nu &= \left(\frac{p-1}{2} \cdot com \right) * \mu \\
&= \mu^{[(p-1)/2]}(com^{\times(p-1)/2} \times \mu)\Delta_{X\times X}^{[(p-1)/2]},
\end{aligned}
$$

so when we apply cohomology, we obtain

$$
\nu^* = \Delta_{X\times X}^{[(p-1)/2]*}((com^*)^{\otimes(p-1)/2} \otimes \mu^*)\mu^{[(p-1)/2]*}.
$$

Let $x \in S$. By Lemma 5.1.3, $\mu^{[(p-1)/2]*}(x)$ has the form

$$
\begin{aligned}
&\sum_{i=0}^{(p-1)/2} 1^{\otimes[(p-1)/2]-i} \otimes x \otimes 1^{\otimes i} \\
&+ \sum_{k=0}^{[(p-1)/2]-1} \left(\sum_{j=0}^{[(p-1)/2]-1-k} 1^{\otimes j} \otimes \bar{B} \otimes B^{\otimes[(p-1)/2]-1-k-j} \right. \\
&\left. \otimes H^*(X,\mu;\mathbb{F}_p) \otimes 1^{\otimes k} \right).
\end{aligned} \tag{5.1.25}
$$

Next, we apply $(com^*)^{\otimes(p-1)/2} \otimes \mu^*$. Since B is generated by primitive even degree classes, by Example 3.3.2, com^* vanishes on $\bar{B}$ (so the terms in the second and third lines of equation 5.1.25 vanish on $(com^*)^{\otimes(p-1)/2} \otimes \mu^*$). Hence, when we apply

$(com^*)^{\otimes(p-1)/2} \otimes \mu^*$ to the expression in 5.1.25, we obtain

$$
((com^*)^{\otimes(p-1)/2} \otimes \mu^*)\mu^{[(p-1)/2]*}(x)
$$

$$
= \sum_{i=0}^{(p-1)/2-1} (1 \otimes 1)^{\otimes[(p-1)/2]-i} \otimes com^*(x) \otimes (1 \otimes 1)^{\otimes i}
$$

$$
+(1 \otimes 1)^{\otimes(p-1)/2} \otimes \mu^*(x) \tag{5.1.26}
$$

Finally, we apply an iterated cup product $\Delta_{X \times X}^{[(p-1)/2]*}$ on equation 5.1.26. For each term,

we will obtain a product of elements (most of them will be $(1 \otimes 1)$) in $H^*(X, \mu; \mathbb{F}_p) \otimes$

$H^*(X, \mu; \mathbb{F}_p)$. For example,

$$
\Delta_{X \times X}^{[(p-1)/2]*}((1 \otimes 1)^{\otimes(p-1)/2} \otimes \mu^*(x))
$$

$$
= (1 \otimes 1) \ldots (1 \otimes 1)\mu^*(x)
$$

$$
= \mu^*(x).
$$

Hence when we apply $\Delta_{X \times X}^{[(p-1)/2]*}$ on equation 5.1.26, we obtain

$$
\nu^*(x) = \left(\sum_{i=0}^{(p-1)/2-1} com^*(x) \right) + \mu^*(x).
$$

$$
= \frac{p-1}{2}com^*(x) + \mu^*(x). \tag{5.1.27}
$$

By Theorem 3.3.3, if $x \in S$, $com^*(x) = \mu^*(x) - T^*\mu^*(x)$, so

$$
\begin{aligned}
\nu^*(x) &= \frac{p-1}{2} \left(\mu^*(x) - T^*\mu^*(x) \right) + \mu^*(x) \\
&= \frac{1}{2} \left(\mu^*(x) + T^*\mu^*(x) \right).
\end{aligned}
$$

Thus, when $x \in S$,

$$
\nu^*(x) = \frac{1}{2} \left(\mu^* + T^*\mu^* \right)(x),
$$

and in particular,

$$
T^*\nu^*(x) = \frac{1}{2} \left(T^*\mu^* + \mu^* \right)(x) = \nu^*(x),
$$

and since S is a generating set, we can conclude that $H^*(X, \nu; \mathbb{F}_p)$ is cocommutative.

$\square$

Example 5.1.5. Consider the exceptional Lie group (F_4, μ) with its usual Lie group multiplication map μ. Recall that

$$
H^*(F_4, \mu; \mathbb{F}_3) \cong \wedge(x_3, x_7, x_{11}, x_{15}) \otimes \mathbb{F}_3[x_8]/(x_8^3).
$$

According to the previous theorem, if we want its mod 3 cohomology to be a cocommutative Hopf algebra we should replace μ by $\nu = com * \mu$. That is,

$$
\nu(g, h) = ghg^{-1}h^{-1}gh. \tag{5.1.28}
$$

Let us compute ν^* on the generating set S for $H^*(F_4, \mu; \mathbb{F}_3)$ (which is isomorphic to

$H^*(F_4, \nu; \mathbb{F}_3)$ as an algebra, but not as a Hopf algebra). The elements x_3, x_7, and x_8 are already primitive (their coproducts are symmetric), so $\mu^* = \nu^*$ on these elements. Instead, let us focus on x_{11} and x_{15}.

We have x_{11}, whose coproduct is $\mu^*(x_{11}) = x_{11} \otimes 1 + x_8 \otimes x_3 + 1 \otimes x_{11}$. If we compute $\nu^*(x_{11})$ without using the previous theorem, the calculations would look like this:

$$
\begin{aligned}
\nu^*(x_{11}) &= com^*(x_{11}) + com^*(x_8)\mu^*(x_3) + \mu^*(x_{11}) \\
&= (x_8 \otimes x_3 - x_3 \otimes x_8) + 0 + (x_{11} \otimes 1 + x_8 \otimes x_3 + 1 \otimes x_{11}) \\
&= x_{11} \otimes 1 - x_8 \otimes x_3 - x_3 \otimes x_8 + 1 \otimes x_{11}.
\end{aligned}
$$

Let us use the theorem on $\nu^*(x_{15})$:

$$
\nu^*(x_{15}) = x_{15} \otimes 1 - x_8 \otimes x_7 - x_7 \otimes x_8 + 1 \otimes x_{15}.
$$

Thus, $\nu^* = T^*\nu^*$ on this generating set for $H^*(F_4, \nu; \mathbb{F}_3)$. Hence $H^*(F_4, \nu; \mathbb{F}_3)$ is cocommutative.

Now let us look at (E_8, μ). This time, we want a multiplication map ν so that $H^*(E_8, \nu; \mathbb{F}_5)$ is cocommutative. Recall that

$$
H^*(E_8, \mu; \mathbb{F}_5) \cong \wedge(x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}) \otimes \mathbb{F}_5[x_{12}]/(x_{12}^5).
$$

According to the theorem, we should take

$$\nu(g, h) = ghg^{-1}h^{-1}ghg^{-1}h^{-1}gh. \qquad (5.1.29)$$

For example, in the generating set S, let us look at three non-primitive elements: x_{15}, x_{27}, and x_{39}. Their original coproducts are as follows:

$$\bar{\mu}^*(x_{15}) = x_{12} \otimes x_3,$$

$$\bar{\mu}^*(x_{27}) = x_{12} \otimes x_{15} + 3x_{12}^2 \otimes x_3,$$

$$\bar{\mu}^*(x_{39}) = x_{12} \otimes x_{27} + 3x_{12}^2 \otimes x_{15} + x_{12}^3 \otimes x_3,$$

and the new coproducts are:

$$\bar{\nu}^*(x_{15}) = 3x_{12} \otimes x_3 + 3x_3 \otimes x_{12},$$

$$\bar{\nu}^*(x_{27}) = 3x_{12} \otimes x_{15} + 4x_{12}^2 \otimes x_3 + 3x_{15} \otimes x_{12} + 4x_3 \otimes x_{12}^2,$$

$$\begin{aligned} \bar{\nu}^*(x_{39}) &= 3x_{12} \otimes x_{27} + 4x_{12}^2 \otimes x_{15} + 3x_{12}^3 \otimes x_3 \\ &\quad + 3x_{27} \otimes x_{12} + 4x_{15} \otimes x_{12}^3 + 3x_3 \otimes x_{12}^3. \end{aligned}$$

Although $H_*(X, \nu; \mathbb{F}_p)$ is now commutative, remember that $H_*(X, \mu; \mathbb{F}_p)$ was associative (by virtue of (X, μ) being an HA-space). In the next section, we explore

the associativity of $H_*(X, \nu; \mathbb{F}_p)$ and look at examples where $H_*(X, \nu; \mathbb{F}_p)$ is both commutative and associative.

5.2 Associativity and Primitive Generators

This section deals with questions of associativity, both for the new H-spaces as well as the homology algebras. We start with a method of checking when $H_*(X, \nu; \mathbb{F}_p)$ is not only commutative, but also associative, using a formula called a coassociator consisting of the original coproduct. We then give an example of computing coassociators. After that, we look at (X, ν) and give sufficient conditions for (X, ν) to not be an HA-space. Finally, we look at examples of (X, ν) for which $H_*(X, \nu; \mathbb{F}_p)$ is commutative, but not associative.

Theorem 5.2.1. *Let (X, μ) be a finite simply-connected HA-space with homotopy inverse i. Let $\nu = \left(\frac{p-1}{2} \cdot com\right) * \mu$. The Hopf algebra $H_*(X, \nu; \mathbb{F}_p)$ is associative if and only if in $H^*(X, \mu; \mathbb{F}_p)$,*

$$\frac{1}{4} \left((\mu^* \otimes 1)T^*\mu^* + (T^*\mu^* \otimes 1)\mu^* - (1 \otimes \mu^*)T^*\mu^* - (1 \otimes T^*\mu^*)\mu^*\right) = 0. \qquad (5.2.1)$$

Finally, if $H_*(X, \nu; \mathbb{F}_p)$ is associative, then $H^*(X, \nu; \mathbb{F}_p)$ is primitively generated as a Hopf algebra with coproduct ν^*.

Proof. To check associativity of $H_*(X, \nu; \mathbb{F}_p)$, we can dualize and check coassociativity of $H^*(X, \nu; \mathbb{F}_p)$. This means that we need to compute the difference of the second

coproducts for ν^* and check if it equals zero:

$$(\nu^* \otimes 1)\nu^* - (1 \otimes \nu^*)\nu^*.$$

Taking the second coproduct on the left gives us

$$
\begin{aligned}
(\nu^* \otimes 1)\nu^* &= \frac{1}{2}\left((\mu^* + T^*\mu^*) \otimes 1\right) \circ \frac{1}{2}\left(\mu^* + T^*\mu^*\right) \\
&= \frac{1}{4}\left((\mu^* \otimes 1)\mu^* + (\mu^* \otimes 1)T^*\mu^* + (T^*\mu^* \otimes 1)\mu^* + (T^*\mu^* \otimes 1)T^*\mu^*\right),
\end{aligned}
$$

while taking the second coproduct on the right gives us

$$
\begin{aligned}
(1 \otimes \nu^*)\nu^* &= \frac{1}{2}\left(1 \otimes (\mu^* + T^*\mu^*)\right) \circ \frac{1}{2}\left(\mu^* + T^*\mu^*\right) \\
&= \frac{1}{4}\left((1 \otimes \mu^*)\mu^* + (1 \otimes \mu^*)T^*\mu^* + (1 \otimes T^*\mu^*)\mu^* + (1 \otimes T^*\mu^*)T^*\mu^*\right).
\end{aligned}
$$

Since μ is homotopy associative, on (X, μ), we have

$$\mu(\mu \times id_X) \simeq \mu(id_X \times \mu).$$

Elementwise, we have

$$\mu(\mu \times id_X)(x, y, z) = (\mu T)((\mu T) \times id_X)(z, y, x)$$

and

$$\mu(id_X \times \mu)(x, y, z) = (\mu T)(id_X \times (\mu T))(z, y, x),$$

so

$$(\mu T)((\mu T) \times id_X) \simeq (\mu T)(id_X \times (\mu T)).$$

Therefore, in cohomology, μ^* and $T^*\mu^*$ must satisfy

$$(\mu^* \otimes 1)\mu^* = (1 \otimes \mu^*)\mu^*$$

and

$$(T^*\mu^* \otimes 1)T^*\mu^* = (1 \otimes T^*\mu^*)T^*\mu^*.$$

Therefore,

$$
\begin{aligned}
&(\nu^* \otimes 1)\nu^* - (1 \otimes \nu^*)\nu^* \\
&= \ \frac{1}{4}\left((\mu^* \otimes 1)T^*\mu^* + (T^*\mu^* \otimes 1)\mu^* - (1 \otimes \mu^*)T^*\mu^* - (1 \otimes T^*\mu^*)\mu^*\right),
\end{aligned}
$$

so $H^*(X, \nu; \mathbb{F}_p)$ is coassociative if and only if

$$
\begin{aligned}
&(\nu^* \otimes 1)\nu^* - (1 \otimes \nu^*)\nu^* \\
&= \ \frac{1}{4}\left((\mu^* \otimes 1)T^*\mu^* + (T^*\mu^* \otimes 1)\mu^* - (1 \otimes \mu^*)T^*\mu^* - (1 \otimes T^*\mu^*)\mu^*\right) = 0.
\end{aligned}
$$

Finally, since X is a finite space, a generating set for $H^*(X, \nu; \mathbb{F}_p)$ will have a finite number of even degree elements. Hence there are no pth powers in the homology ring $H_*(X, \nu; \mathbb{F}_p)$ [24]. Thus, if $H_*(X, \nu; \mathbb{F}_p)$ is associative, then $H_*(X, \nu; \mathbb{F}_p)$ is a commutative and associative algebra with no pth powers, so by Theorem 1.1.4, $H^*(X, \nu; \mathbb{F}_p)$ is primitively generated as a Hopf algebra. $\qquad\square$

Remark. The expression in 5.2.1 is called the coassociator (of the coproduct ν).

Let us compute some coassociators of generators in some of the exceptional Lie groups.

Example 5.2.2. Let us return to (F_4, ν). The coassociator from Theorem 5.2.1 vanishes on the generating set S, so $H_*(F_4, \nu; \mathbb{F}_3)$ is not only commutative, but also associative. Since $H_*(F_4, \nu; \mathbb{F}_3)$ has no 3rd powers, we can conclude that $H^*(F_4, \nu; \mathbb{F}_3)$ is primitively generated. Can we find a generating set of primitive elements for $H^*(F_4, \nu; \mathbb{F}_3)$? Our calculations have shown that the -1 characteristic generators (see Proposition 2.3.10) corresponding to the ones from S will work:

$$\tilde{x}_3 = x_3, \ \tilde{x}_7 = x_7, \ \tilde{x}_8 = x_8,$$

$$\tilde{x}_{11} = x_{11} + x_8 x_3, \ \tilde{x}_{15} = x_{15} + x_8 x_7.$$

For example,

$$
\begin{aligned}
\nu^*(\tilde{x}_{11}) &= \nu^*(x_{11}) + \nu^*(x_8 x_3) \\[6pt]
&= x_{11} \otimes 1 - x_8 \otimes x_3 - x_3 \otimes x_8 + 1 \otimes x_{11} \\[6pt]
&+ x_8 x_3 \otimes 1 + x_8 \otimes x_3 + x_3 \otimes x_8 + 1 \otimes x_8 x_3 \\[6pt]
&= x_{11} \otimes 1 + x_8 x_3 \otimes 1 + 1 \otimes x_{11} + 1 \otimes x_8 x_3 \\[6pt]
&= \tilde{x}_{11} \otimes 1 + 1 \otimes \tilde{x}_{11}. \tag{5.2.2}
\end{aligned}
$$

Now, since $H_*(F_4, \nu; \mathbb{F}_3)$ is associative, we can ask if (F_4, ν) is homotopy as-sociative, or even associative. However, if it turned out to be an HA-space, then since $H^*(F_4, \nu; \mathbb{F}_3)$ is primitively generated, Theorem 3.2 from [41] would require $H^*(F_4, \nu; \mathbb{F}_3)$ to be a free algebra. Since $H^*(F_4, \nu; \mathbb{F}_3)$ is finite-dimensional, this would mean that $H^*(F_4, \nu; \mathbb{F}_3)$ must be an exterior algebra, which is false. Therefore, (F_4, ν) is an H-space that is not an HA-space.

In general, given a finite simply-connected HA-space (X, μ) with altered mul-tiplication map $\nu = \left(\frac{p-1}{2} \cdot com\right) * \mu$, let us show when (X, ν) is not an HA-space:

Theorem 5.2.3. *Given a finite simply-connected HA-space (X, μ) with altered mul-tiplication map $\nu = \left(\frac{p-1}{2} \cdot com\right) * \mu$, suppose that $H_*(X, \mu; \mathbb{F}_p)$ is not commutative. Then the H-space (X, ν) is not an HA-space.*

Proof. There are two cases to consider: whether $H_*(X, \nu; \mathbb{F}_p)$ is associative or not.

If $H_*(X, \nu; \mathbb{F}_p)$ is not associative, then (X, ν) cannot be (homotopy equivalent

to) an HA-space, since the homology of an HA-space must be an associative algebra.

If $H_*(X, \nu; \mathbb{F}_p)$ is associative, then by Theorem 5.2.1, $H^*(X, \nu; \mathbb{F}_p)$ is primitively generated. Since (X, μ) is a finite simply-connected HA-space and the algebra $H_*(X, \mu; \mathbb{F}_p)$ is not commutative, the integral homology $H_*(X; \mathbb{Z})$ has p-torsion [17]. Hence $H^*(X; \mathbb{F}_p)$ is not an exterior algebra [5]. Then $H^*(X, \nu; \mathbb{F}_p)$ (as an algebra) must have even degree generators, and since $H^*(X, \nu; \mathbb{F}_p)$ has finite dimension, these even degree generators must produce truncated polynomial algebras, so $H^*(X\nu; \mathbb{F}_p)$ is not a free algebra.

Let us finish with a proof by contradiction. If (X, ν) is an HA-space, then since $H^*(X, \nu; \mathbb{F}_p)$ is primitively generated, by Theorem 3.2 from [41], $H^*(X\nu; \mathbb{F}_p)$ is a free algebra, contradicting our earlier observation that $H^*(X\nu; \mathbb{F}_p)$ is not a free algebra. Hence the H-space (X, ν) is not an HA-space. $\qquad\square$

Now we turn to some examples of nontrivial coassociators.

Example 5.2.4. Let us look at the Lie group (E_8, μ). We will start with $H^*(E_8, \mu; \mathbb{F}_3)$ and then look at $H^*(E_8, \mu; \mathbb{F}_5)$. For reference,

$$H^*(E_8; \mathbb{F}_3) \cong \wedge(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47}) \otimes \mathbb{F}_p[x_8, x_{20}] / \left(x_8^3, x_{20}^3\right).$$

A generating set for $H^*(E_8, \mu; \mathbb{F}_3)$ will have an element x_{35} whose coproduct is

$$\mu^*(x_{35}) = x_{35} \otimes 1 + 1 \otimes x_{35} + x_8 \otimes x_{27} - x_8^2 \otimes x_{19} + x_{20} \otimes x_{15} + x_8 x_{20} \otimes x_7.$$

Let

$$\nu(g, h) = ghg^{-1}h^{-1}gh. \tag{5.2.3}$$

Then in $H^*(E_8, \nu; \mathbb{F}_3)$, we have

$$\nu^*(x_{35}) = x_{35} \otimes 1 + 1 \otimes x_{35} - x_8 \otimes x_{27} + x_8^2 \otimes x_{19} - x_{20} \otimes x_{15} - x_8 x_{20} \otimes x_7$$

$$-x_{27} \otimes x_8 + x_{19} \otimes x_8^2 - x_{15} \otimes x_{20} - x_7 \otimes x_8 x_{20},$$

and the coassociator on x_{35} is

$$x_8 \otimes x_8 \otimes x_{19} + x_8 \otimes x_{20} \otimes x_7 + x_{20} \otimes x_8 \otimes x_7$$

$$-x_{19} \otimes x_8 \otimes x_8 - x_7 \otimes x_8 \otimes x_{20} - x_7 \otimes x_{20} \otimes x_8. \tag{5.2.4}$$

For reference,

$$x_8 \otimes x_8 \otimes x_{19} \text{ came from computing } (T^*\mu^* \otimes 1) \text{ on } -x_8^2 \otimes x_{19},$$

$$x_8 \otimes x_{20} \otimes x_7 + x_{20} \otimes x_8 \otimes x_7 \text{ came from } (T^*\mu^* \otimes 1) \text{ on } x_8 x_{20} \otimes x_7,$$

$$-x_{19} \otimes x_8 \otimes x_8 \text{ came from } -(1 \otimes \mu^*) \text{ on } x_{19} \otimes x_8^2,$$

$$-x_7 \otimes x_8 \otimes x_{20} - x_7 \otimes x_{20} \otimes x_8 \text{ came from } -(1 \otimes \mu^*) \text{ on } x_7 \otimes x_8 x_{20},$$

so the main causes for the nonzero coassociator are the terms $-x_8^2 \otimes x_{19} + x_8 x_{20} \otimes x_7$ in $\overline{\mu}^*(x_{35})$. Notice that their left factors are decomposables.

In fact, due to the presence of two even degree elements in $QH^*(E_8, \mu; \mathbb{F}_3)$ linked by a Steenrod operation $\mathcal{P}^3$ (namely the elements represented by x_8 and x_{20} in S), it is impossible to find a multiplication map on E_8 which induces a product structure on $H_*(E_8; \mathbb{F}_3)$ that is both commutative and associative [18].

Meanwhile, in $H^*(E_8, \nu; \mathbb{F}_5)$, where

$$\nu(g,h) = ghg^{-1}h^{-1}ghg^{-1}h^{-1}gh, \tag{5.2.5}$$

we have $\frac{1}{4} = 4 = -1$ in $\mathbb{F}_5$, so the coassociator of x_{27} is

$$4x_{12} \otimes x_{12} \otimes x_3 + x_3 \otimes x_{12} \otimes x_{12}. \tag{5.2.6}$$

These came from $(T^*\mu^* \otimes 1)(3x_{12}^2 \otimes x_3)$ and $-(1 \otimes \mu^*)(3x_3 \otimes x_{12}^2)$ respectively. The coassociator for x_{39} is

$$4x_{12} \otimes x_{12} \otimes x_{15} + 2x_{12}^2 \otimes x_{12} \otimes x_3 + 2x_{12} \otimes x_{12}^2 \otimes x_3$$

$$+x_{15} \otimes x_{12} \otimes x_{12} + 3x_3 \otimes x_{12}^2 \otimes x_{12} + 3x_3 \otimes x_{12} \otimes x_{12}^2$$

$$= -(T^*\mu^* \otimes 1)(3x_{12}^2 \otimes x_{15}) - (T^*\mu^* \otimes 1)(x_{12}^3 \otimes x_3)$$

$$+(1 \otimes \mu^*)(3x_{15} \otimes x_{12}^2) + (1 \otimes \mu^*)(x_3 \otimes x_{12}^3). \tag{5.2.7}$$

Again, the terms in the original reduced coproducts with decomposable factors cause nonzero terms in the coassociator. Unlike $QH^*(E_8, \mu; \mathbb{F}_3)$, $QH^*(E_8, \mu; \mathbb{F}_5)$ only has one element with even degree. There may be a possibility of an entirely different

multiplication map on E_8 which induces an algebra structure on $H_*(E_8; \mathbb{F}_5)$ that is both commutative and associative.

Example 5.2.5. Our final example is (E_7, μ). Recall that

$$H^*(E_7, \mu; \mathbb{F}_3) \cong \wedge(x_3, x_7, x_{11}, x_{15}, x_{19}, x_{27}, x_{35}) \otimes \mathbb{F}_3[x_8]/(x_8^3).$$

In $H^*(E_7, \mu; \mathbb{F}_3)$, we have a generator x_{35} with

$$\mu^*(x_{35}) = x_{35} \otimes 1 + x_8 \otimes x_{27} - x_8^2 \otimes x_{19} + 1 \otimes x_{35},$$

If we let

$$\nu(g, h) = ghg^{-1}h^{-1}gh, \tag{5.2.8}$$

then the coassociator of x_{35} in $H^*(E_7, \nu; \mathbb{F}_3)$ is

$$x_8 \otimes x_8 \otimes x_{19} - x_{19} \otimes x_8 \otimes x_8$$

$$= (T^*\mu^* \otimes 1)\left(-x_8^2 \otimes x_{19}\right) - (1 \otimes \mu^*)\left(x_{19} \otimes x_8^2\right). \tag{5.2.9}$$

Notice that the nonzero terms in the coassociator of x_{35} came from applying left or right coproducts on terms with decomposable factors in the original reduced coproduct of x_{35}.

Overall, if

$$\nu(g, h) = ghg^{-1}h^{-1}gh,$$

then $H^*(F_4, \nu; \mathbb{F}_3)$ and $H^*(E_6, \nu; \mathbb{F}_3)$ are primitively generated, while $H^*(E_7, \nu; \mathbb{F}_3)$ and $H^*(E_8, \nu; \mathbb{F}_3)$ are cocommutative, but not coassociative. One property that both $H^*(F_4, \nu; \mathbb{F}_3)$ and $H^*(E_6, \nu; \mathbb{F}_3)$ have in common is that the generators can be chosen (using Theorem 1.2.4) so that each of their original reduced coproducts consists only of sums of tensor products of primitive generators. On the other hand, a generating set for the algebras $H^*(F_4, \nu; \mathbb{F}_3)$, $H^*(E_6, \nu; \mathbb{F}_3)$ or $H^*(E_7, \nu; \mathbb{F}_3)$ would have only one even degree generator, while a generating set for $H^*(E_8, \nu; \mathbb{F}_3)$ would require two. It is an open question as to whether there is a multiplication map on E_7 that makes its mod 3 homology commutative and associative.

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