Title
Representation theory of SL(2,p) and its subgroups

Author
Kirby, Gary David

Publication Date
2013

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA
UNIVERSITY OF CALIFORNIA SANTA CRUZ

REPRESENTATION THEORY OF $\mathbf{SL}_2(\mathbb{F}_P)$ AND ITS SUBGROUPS

A thesis submitted in partial satisfaction
of the requirements for the degree of

MASTER OF ARTS

in

MATHEMATICS

by

Gary David Kirby

June 2013

The Thesis of Gary David Kirby
is approved:

Professor Martin H. Weissman, Chair

Professor Robert Boltje

Professor Samit Dasgupta

Tyrus Miller
Vice Provost and Dean of Graduate Studies

Table of Contents

List of Figures

Abstract

Representation Theory of $\mathrm{SL}_2(\mathbb{F}_p)$ and its Subgroups

by

Gary David Kirby

The aim of this thesis is to describe the principal series representations of $\mathrm{SL}_2(\mathbb{F}_p)$, together with the character tables of some of its subgroups. We describe all irreducible characters and conjugacy classes of a Borel subgroup $B \leq \mathrm{SL}_2(\mathbb{F}_p)$, the standard torus $T \leq \mathrm{SL}_2(\mathbb{F}_p)$, and the unipotent subgroup $U \trianglelefteq B$. We go on to completely describe the principal series of $\mathrm{SL}_2(\mathbb{F}_p)$, those representations induced from characters of B.

To my family and the memory of

Dorthy Guerin

Acknowledgments

First and foremost I would like to thank my masters thesis advisor, Marty Weissman for assisting me through this process. This thesis would not have been possible if it weren't for Marty's great ability to describe the topics within this paper in a simplistic manner throughout the term that we worked together. I'd also like to give a special thanks to the remainder of the mathematics faculty at UCSC. I would not be where I am today if it weren't for the excellent teaching and amount of rigor demanded in and outside the classroom throughout my career at UCSC.

Last but not least, I'd like to give recognition to my fellow math graduates. You all were my constant support system and I would not have made it though this arduous journey without you. In particular, I wish to acknowledge Gabriel Martins and Philipp Perepelitsky for the countless hours they spent away from their own work sharing ideas and explaining concepts about Representation theory. This thesis would have been near impossible without the generosity and knowledge you both shared with me. With this, I thank you and hope to return the favor one day.

Introduction

We describe the principal series representations of $\mathrm{SL}_2(\mathbb{F}_p)$. The irreducible representations of $\mathrm{SL}_2(\mathbb{F}_p)$ were independently classified in 1907 by H. Jordan [4], who calculated the ordinary character table of $\mathrm{SL}_2(\mathbb{F}_q)$. Jordan's methods were based off of Frobenius's work in the article *Über Gruppencharakatere* [2], where Frobenius determined the group characters of $\mathrm{SL}_2(\mathbb{F}_p)$.

Chapter 1 reviews the fundamental tools required for the theory of representations and characters. This includes basic definitions and elementary results on representations and characters. In Chapter 2, we introduce the character tables of cyclic groups, $\mathbb{F}_p$ and $\mathbb{F}_p^{\times}$, which will play a major role in determining the representations of the Borel subgroup (B) and principal series of $\mathrm{SL}_2(\mathbb{F}_p)$. In Chapter 3, we will determine the conjugacy classes of B. Using this, by inflating and inducing representations from subgroups of B, we deduce its character table. The main results of this thesis are contained in Chapter 4. We consider a one-dimensional character of B and induce to $\mathrm{SL}_2(\mathbb{F}_p)$. Using Frobenius reciprocity, we will describe the principal series representations, dependent on the one-dimensional character, and highlight the main result at the end of Chapter 4.

Chapter 1

Background

Before we begin, let us define several important definitions and state some fundamental results about representations and characters. For more detail and proofs see Isaacs [3] or Serre [7].

1.1 Definition A *representation* of a finite group G is a pair (π, V) where V is a finite-dimensional complex vector space and π is a group homomorphism $\pi : G \to \mathrm{GL}(V)$. If W is a subspace of V which is stable under G then W is a *subrepresentation* of V. A representation V is called *irreducible* if there is no proper nonzero subrepresentation W of V.

1.2 Definition If (π, V) is a representation of G, its *character* χ is the complex-valued function on the group defined by

$$\chi(g) = \mathrm{Tr}(\pi(g)).$$

In particular, we have

$$\chi(hgh^{-1}) = \chi(g),$$

so χ is constant on each conjugacy class of G. Such a function is called a *class function.* The class functions form a complex vector space $CF(G, \mathbb{C})$ with the usual addition and scalar multiplication. An *irreducible character* is the character of an irreducible representation. Let $\mathrm{Irr}(G)$ be the set of irreducible characters.

1.3 Remark If χ is the character of a representation (π, V) of dimension n, we have $\chi(1) = n$. In other words,

$$\dim(V) = \mathrm{Tr}(\pi(1)).$$

1.4 Definition Let χ be the character of a representation (π, V). We define the degree of χ to be equal to the dimension of V, i.e.

$$\deg(\chi) = \dim(V) = \chi(1).$$

1.5 Definition On the complex vector space $CF(G, \mathbb{C})$ of class functions from G to $\mathbb{C}$, we introduce the *Schur inner product*

$$\langle f_1, f_2 \rangle_G := \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

Characters of a representation are class functions. So if χ is the character of the representation (π, V), for a fixed group G, one has two orthogonality relations which are enormously useful in the study of characters.

1.6 Theorem (First Orthogonality Relation) *The irreducible characters of G form a set of orthonormal vectors in $CF(G, \mathbb{C})$ with respect to the Schur inner product. In other words, let $\chi_i, \chi_j \in \mathrm{Irr}(G)$; then*

$$\langle \chi_i, \chi_j \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \chi_j(g^{-1}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Note that $\chi(g^{-1}) = \overline{\chi(g)}$.

1.7 Theorem (Second Orthogonality Relation) *Let $g, h \in G$. Then*

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(g) \overline{\chi(h)} = 0$$

if g is not conjugate to h in G. Otherwise, the sum is equal to the size of the centralizer of g in G.

1.8 Corollary *Fix a finite group G. The size of $\mathrm{Irr}(G)$ is equal to the number of conjugacy classes of G.*

1.9 Proposition *Fix a finite group G and let $\chi_1, ..., \chi_c \in Irr(G)$ where c is the number of conjugacy classes. If $n_i = \deg(\chi_i)$ then*

$$\sum_{i=1}^{c} n_i^2 = |G|.$$

Chapter 2

Character tables of $\mathbb{F}_p$ and $\mathbb{F}_p^\times$

Let p be an odd prime for the remainder of this paper unless otherwise stated. Let $\mathbb{F}_p$ be the field with p elements. Then $(\mathbb{F}_p, +)$ is a cyclic group of order p. Every homomorphism in $\mathrm{Hom}(\mathbb{F}_p, \mathbb{C}^\times)$ is uniquely determined by its value on 1, a power of $\zeta := e^{2\pi i/p}$. Thus we obtain p one-dimensional representations of $\mathbb{F}_p$, giving the character table described in Figure 2.1 with $\alpha_n(i) = \zeta^{in}$.

Figure 2.1: Character table of $\mathbb{F}_p$

$(\mathbb{F}_p, +)$	0	1	2	$\cdots$	i	$\cdots$	$p-1$
α_0	1	1	1	$\cdots$	1	$\cdots$	1
α_1	1	ζ	ζ^2	$\cdots$	ζ^i	$\cdots$	ζ^{p-1}
α_2	1	ζ^2	ζ^4	$\cdots$	ζ^{2i}	$\cdots$	$\zeta^{2(p-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
α_n	1	ζ^n	ζ^{2n}	$\cdots$	ζ^{in}	$\cdots$	$\zeta^{n(p-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
α_{p-1}	1	ζ^{p-1}	$\zeta^{2(p-1)}$	$\cdots$	$\zeta^{i(p-1)}$	$\cdots$	$\zeta^{(p-1)(p-1)}$

Since $\mathbb{F}_p^\times$ is an abelian group every element is its own conjugacy class. On account of $\mathbb{F}_p$ a field of p elements, $\mathbb{F}_p^\times$ is a cyclic group of order $p-1$. Let a be a generator of $\mathbb{F}_p^\times$. We obtain the character table of $\mathbb{F}_p^\times$ with $\xi := e^{2\pi i/(p-1)}$, described in Figure 2.2, where $\varphi_j(a^i) = \xi^{ij}$.

Figure 2.2: Character table of $\mathbb{F}_p^\times$

$(\mathbb{F}_p^\times, \cdot)$	1	a	a^2	$\cdots$	a^i	$\cdots$	a^{p-2}
φ_0	1	1	1	$\cdots$	1	$\cdots$	1
φ_1	1	ξ	ξ^2	$\cdots$	ξ^i	$\cdots$	ξ^{p-2}
φ_2	1	ξ^2	ξ^4	$\cdots$	ξ^{2i}	$\cdots$	$\xi^{2(p-2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
φ_j	1	ξ^j	ξ^{2j}	$\cdots$	ξ^{ij}	$\cdots$	$\xi^{j(p-2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
φ_{p-2}	1	ξ^{p-2}	$\xi^{2(p-2)}$	$\cdots$	$\xi^{i(p-2)}$	$\cdots$	$\xi^{(p-2)(p-2)}$

Chapter 3

Representations of the Borel Subgroup

Let $G = \mathrm{SL}_2(\mathbb{F}_p)$ and consider the Borel subgroup,

$$B = \left\{ \begin{pmatrix} u & x \\ 0 & u^{-1} \end{pmatrix} : u \in \mathbb{F}_p^\times, x \in \mathbb{F}_p \right\}.$$

It can be shown that $B = TU = T \ltimes U$, with U the normal subgroup of upper triangular unipotent matrices and T the collection of diagonal matrices in B. Let $Z = \{\pm I_2\}$, the center of B.

Note that we have an isomorphism $\mathbb{F}_p^\times \to T$, given by $u \mapsto \mathrm{diag}(u, u^{-1})$ for any $u \in \mathbb{F}_p^\times$. This implies that the character tables of $\mathbb{F}_p^\times$ and T are the same. Similarly, we have an isomorphism from $\mathbb{F}_p \to U$, given by

$$x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

So $\mathbb{F}_p$ and U have identical character tables.

The following proposition describes the centralizers of specific elements in B.

3.1 Proposition *Let $h \in B$.*

(i) If $h \in Z$, then $Z_B(h) = B$,

(ii) if $h \in T$ and $h \notin Z$, then $Z_B(h) = T$,

(iii) if $h \in ZU$ and $h \notin Z$, then $Z_B(h) = ZU$.

Proof Part (i) is clear since Z is the center of B. For (ii) let

$$g = \begin{pmatrix} c & y \\ 0 & c^{-1} \end{pmatrix}$$

be an element in B. If

$$h = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \in T$$

with $u \neq \pm 1$, then conjugating h by g yields

$$ghg^{-1} = \begin{pmatrix} c & y \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} c^{-1} & -y \\ 0 & c \end{pmatrix} = \begin{pmatrix} u & cy(u^{-1} - u) \\ 0 & u^{-1} \end{pmatrix}. \quad (3.1)$$

So $h = ghg^{-1}$ if and only if $y = 0$, since c is nonzero and $u \neq u^{-1}$. Hence $g \in Z_B(h)$ if and only if $g \in T$. Now let

$$g = \begin{pmatrix} c & y \\ 0 & c^{-1} \end{pmatrix},$$

as before and let

$$h = \begin{pmatrix} \varepsilon & x \\ 0 & \varepsilon \end{pmatrix}$$

be an element of ZU, with x nonzero and $\varepsilon = \pm 1$. Conjugating h by g we have,

$$ghg^{-1} = \begin{pmatrix} c & y \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} \varepsilon & x \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} c^{-1} & -y \\ 0 & c \end{pmatrix} = \begin{pmatrix} \varepsilon & xc^2 \\ 0 & \varepsilon \end{pmatrix}. \quad (3.2)$$

So $h = ghg^{-1}$ if and only if $c = \pm 1$. Thus $g \in Z_B(h)$ if and only if

$$g = \begin{pmatrix} \pm 1 & y \\ 0 & \pm 1 \end{pmatrix} \in ZU.$$

□

Using the above proposition, the orbit stabilizer equation will help us determine the conjugacy classes of B. If $h \in T$ and $h \notin Z$, then $Z_B(h) = T$ which implies the conjugacy class of h has size p. By equation (3.1), every element of $T \setminus \{\pm I_2\}$ is in a different conjugacy class, therefore there are $p-3$ conjugacy classes of size p coming from non-central elements of T.

If $h \in ZU$ and $h \notin Z$, then $Z_B(h) = ZU$, which implies the conjugacy class of h has size $\frac{p-1}{2}$. It suffices to see when the matrices

$$\begin{pmatrix} \varepsilon & x \\ 0 & \varepsilon \end{pmatrix}, \begin{pmatrix} \varepsilon' & y \\ 0 & \varepsilon' \end{pmatrix}$$

with $x, y \in \mathbb{F}_p^\times, \varepsilon = \pm 1$, and $\varepsilon' = \pm 1$ are conjugate, to determine the number of such classes. Equation (3.2) implies they are conjugate if and only if $\varepsilon = \varepsilon'$ and y/x is a square modulo p. This implies that there are four conjugacy classes of size $\frac{p-1}{2}$. Since

$$2 + p(p-3) + 4\left(\frac{p-1}{2}\right) = p(p-1) = |B|,$$

we have exhausted the conjugacy classes of B. Figure 3.1 below describes these $p+3$ conjugacy classes.

Figure 3.1: Conjugacy classes of B

Representative	εI_2 $\varepsilon \in \{\pm 1\}$	$\begin{pmatrix} a^i & 0 \\ 0 & a^{-i} \end{pmatrix}$ $a^i \in \mathbb{F}_p^\times \setminus \{\pm I_2\}$	$\begin{pmatrix} \varepsilon & 1 \\ 0 & \varepsilon \end{pmatrix}$	$\begin{pmatrix} \varepsilon & a \\ 0 & \varepsilon \end{pmatrix}$
Centralizer	B	T	ZU	ZU
Cardinality	1	p	$(p-1)/2$	$(p-1)/2$
Number of classes	2	$p-3$	2	2
Total Contribution	2	$p^2 - 3p$	$p-1$	$p-1$

To determine the character table of B, we may induce and inflate representations from the subgroups T and U.

3.2 Definition Let $N \trianglelefteq G$ and let $\nu : G \to G/N$, given by $g \mapsto gN$ denote the canonical epimorphism. For any representation $\rho : G/N \to \mathrm{GL}(V)$ we write

$$\mathrm{Inf}_{G/N}^{G}(\rho) := \rho \circ \nu : G \to \mathrm{GL}(V).$$

This representation of G is called the *inflation* of the representation ρ. If φ is the character of representation ρ, we write $\inf_{G/N}^{G}(\varphi)$ for the character of $\mathrm{Inf}_{G/N}^{G}(\varphi)$.

To begin let us determine the derived subgroup B' of B to find all one dimensional representations of B. By definition, the derived subgroup is the smallest normal subgroup such that the quotient is abelian. We know U is a normal subgroup of B and B/U is isomorphic to T, which is abelian. So the derived subgroup B' must be a subgroup of U. Moreover, B' is the trivial subgroup or all of U since U is a cyclic group of order p; however, B is not abelian, so it must be that B' is all of U. Therefore, every one-dimensional character of B is an inflation of a character from B/U. This implies that there are $p-1$ one-dimensional characters.

Let $\chi_j = \inf_T^B(\varphi_j)$ be the character of $\mathrm{Inf}_T^B(\varphi_j)$ for all $0 \le j \le p-2$. Therefore, we obtain a portion of the character table of B :

B	εI_2 $\varepsilon \in \{\pm 1\}$	$\begin{pmatrix} a^i & 0 \\ 0 & a^{-i} \end{pmatrix}$ $a^i = \mathbb{F}_p^\times \setminus \{\pm 1\}$	$\begin{pmatrix} \varepsilon & 1 \\ 0 & \varepsilon \end{pmatrix}$	$\begin{pmatrix} \varepsilon & a \\ 0 & \varepsilon \end{pmatrix}$
χ_0	1	1	1	1
χ_j	ε^j	ξ^{ji}	ε^j	ε^j

3.3 Definition Let H be a subgroup of G and let $\pi : G \to \mathrm{GL}(V)$ be a representation of G. The *restriction of* π *to* H, denoted $\mathrm{Res}_H^G(\pi)$, is defined by

$$[\mathrm{Res}_H^G(\pi)](h) = \pi(h)$$

for all h in H. If χ is the character of π, write $\mathrm{res}_H^G(\chi)$ for the character of $\mathrm{Res}_H^G(\chi)$.

3.4 Definition Let G be a finite group, H a subgroup of G, and let $\sigma : H \to \mathbb{C}^\times$ be a one-dimensional character of H. The *induced representation*, $\mathrm{Ind}_H^G(\sigma)$, has vector space

$$V = \mathrm{Ind}_H^G(\sigma) = \{f : G \to \mathbb{C} : f(hg) = \sigma(h)f(g), \forall g \in G, h \in H\}$$

and representation $\rho : G \to \mathrm{GL}(V)$, given by the G action $[\rho(g)f](x) = f(xg)$ for all $x, g \in G$. The dimension of the induced representation is given by the index of G and H, i.e.

$$\dim(\mathrm{Ind}_H^G(\sigma)) = [G : H].$$

The character of $\mathrm{Ind}_H^G(\sigma)$ is given by (see [6])

$$[\mathrm{ind}_H^G(\sigma)](g) = \sum_{\substack{x\in H\backslash G \\ g\in x^{-1}Hx}} \sigma(xgx^{-1}).$$

In particular, $[\mathrm{ind}_H^G(\sigma)](g) = 0$ unless g is conjugate to an element in H, and if $g = 1$ then $[\mathrm{ind}_H^G(\sigma)](1) = \dim(V)$.

To determine if characters are irreducible, we will use the following theorem without proof, see Isaacs [3] or Serre [7] for details.

3.5 Theorem (Frobenius Reciprocity) *If W is a representation of H and U a representation of G, then*

$$\langle ind_H^G(\psi), \chi\rangle_G = \langle \psi, res_H^G(\chi)\rangle_H.$$

Consider the normal subgroup ZU of B and once again let α be a nontrivial irreducible character of U. Define $\alpha^{\pm} : ZU \to \mathbb{C}^{\times}$, given by $\alpha^{+}(\pm I_2 u) = \alpha(u)$ and $\alpha^{-}(\pm I_2 u) = \pm\alpha(u)$ for $u \in U$. Both α^{+} and α^{-} define one-dimensional characters of ZU. Then consider $\mathrm{Ind}_{ZU}^B(\alpha^{+})$ and $\mathrm{Ind}_{ZU}^B(\alpha^{-})$, both of dimension $\frac{p-1}{2}$. Since α is a nontrivial character of U, we know $\alpha = \alpha_n$ for some $1 \le n \le p-1$.

3.6 Proposition *$Ind_{ZU}^B(\alpha_n^{\pm})$ is an irreducible representation of dimension $\frac{p-1}{2}$.*

Proof Consider $\chi = \mathrm{ind}_{ZU}^B(\alpha_n^{\pm})$, the function given by

$$\chi(b) = \sum_{\substack{x\in ZU\backslash B \\ xbx^{-1}\in ZU}} \alpha_n^{\pm}(xbx^{-1}).$$

Since $[B : ZU] = \frac{p-1}{2}$,

$$\dim(\mathrm{Ind}_{ZU}^B(\alpha_n^{\pm 1})) = \frac{p-1}{2}.$$

Consider

$$\left\{ \begin{pmatrix} a^i & 0 \\ 0 & a^{-i} \end{pmatrix} : 1 \le i \le \frac{p-3}{2} \right\},$$

a set of coset representatives for $ZU \setminus B$. Then if $zu = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$,

$$\begin{aligned}
\chi(zu) &= \sum_{x \in ZU \setminus B} \alpha_n^{\pm}(xzux^{-1}) \\
&= \sum_{i=1}^{\frac{p-3}{2}} \alpha_n^{\pm}\left(\begin{pmatrix} a^i & 0 \\ 0 & a^{-i} \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-i} & 0 \\ 0 & a^i \end{pmatrix} \right) \\
&= \sum_{i=1}^{\frac{p-3}{2}} \alpha_n^{\pm} \begin{pmatrix} \varepsilon & ya^{2i} \\ 0 & \varepsilon \end{pmatrix} \\
&= \sum_{i=1}^{\frac{p-3}{2}} \alpha_{na^{2i}}^{\pm} \begin{pmatrix} \varepsilon & y \\ 0 & \varepsilon \end{pmatrix}.
\end{aligned}$$

Thus, $\mathrm{res}_{ZU}^B(\mathrm{ind}_{ZU}^B(\alpha_n^{\pm})) = \sum_{s \in \mathbb{F}_p^{\times 2}} \alpha_{sn}^{\pm}$ and the first orthogonality relation gives us

$$\left\langle \alpha_n^{\pm}, \sum_{s \in \mathbb{F}_p^{\times 2}} \alpha_{sn}^{\pm} \right\rangle_{ZU} = \langle \alpha_n^{\pm}, \alpha_n^{\pm} \rangle_{ZU} = 1.$$

Therefore, $\mathrm{Ind}_{ZU}^B(\alpha_n^{\pm})$ is irreducible. Furthermore,

$$\mathrm{Res}_{ZU}^B(\mathrm{Ind}_{ZU}^B(\alpha_n^{\pm})) = \bigoplus_{s \in \mathbb{F}_p^{\times 2}} \alpha_{ns}^{\pm}.$$

∎

Define $\mu_n^{\pm} = \mathrm{ind}_{ZU}^B(\alpha_n^{\pm})$. This gives us four distinct irreducible characters of dimension $\frac{p-1}{2}$: $\mu_1^+, \mu_1^-, \mu_a^+, \mu_a^-$. Recall a is a generator of $\mathbb{F}_p^{\times}$. We summarize the character values of B in Figure 3.2, where a representative of each class is given.

The numbers τ and τ' are defined as follows:

$$\tau = \sum_{s \in \mathbb{F}_p^{\times 2}} \alpha_s^+ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \sum_{s \in \mathbb{F}_p^{\times 2}} \zeta^s.$$

Similarly, define

$$\tau' = \sum_{s \in \mathbb{F}_p^{\times 2}} \alpha_{as}^+ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \sum_{s \in \mathbb{F}_p^{\times 2}} \zeta^{as}.$$

Figure 3.2: Character Table of B

B	εI_2 $\varepsilon \in \{\pm 1\}$	$\begin{pmatrix} a^i & 0 \\ 0 & a^{-i} \end{pmatrix}$ $a^i \in \mathbb{F}_p^\times \setminus \{\pm 1\}$	$\begin{pmatrix} \varepsilon & 1 \\ 0 & \varepsilon \end{pmatrix}$	$\begin{pmatrix} \varepsilon & a \\ 0 & \varepsilon \end{pmatrix}$
Cardinality	1	p	$(p-1)/2$	$(p-1)/2$
Num. of classes	2	$p-3$	2	2
χ_0	1	1	1	1
χ_j	ε^j	ξ^{ji}	ε^j	ε^j
μ_1^+	$\frac{p-1}{2}$	0	τ	τ'
μ_1^-	$\varepsilon\frac{p-1}{2}$	0	$\varepsilon\tau$	$\varepsilon\tau'$
μ_a^+	$\frac{p-1}{2}$	0	τ'	τ
μ_a^-	$\varepsilon\frac{p-1}{2}$	0	$\varepsilon\tau'$	$\varepsilon\tau$

Since

$$\sum_{s\in\mathbb{F}_p^{\times 2}} \zeta^s + \sum_{s\in\mathbb{F}_p^{\times 2}} \zeta^{as} + 1 = 0,$$

we obtain

$$\tau' = -1 - \tau.$$

Chapter 4

Principal Series of $SL_2(\mathbb{F}_p)$

Recall $G = \mathrm{SL}_2(\mathbb{F}_p)$. Fix a character χ of B, inflated from T, so that

$$\chi\begin{pmatrix} a^i & x \\ 0 & a^{-i} \end{pmatrix} = \xi^{ij},$$

for some j. Consider the representation

$$V = \mathrm{Ind}_B^G(\chi) = \{f : G \to \mathbb{C} : f(bg) = \chi(b)f(g), \forall b \in B, g \in G\}.$$

Let ρ be the representation of G on V given in Definition 3.4. Since χ is a one-dimensional character of B,

$$\dim(V) = [G : B] = p + 1.$$

Is (ρ, V) an irreducible representation? We know that $\mathrm{ind}_B^G(\chi)$ is an irreducible character if

$$\langle \mathrm{ind}_B^G(\chi), \mathrm{ind}_B^G(\chi)\rangle_G = \langle \chi, \mathrm{res}_B^G(\mathrm{ind}_B^G(\chi))\rangle_B = 1,$$

by Frobenius Reciprocity. This implies that $\mathrm{ind}_B^G(\chi)$ is irreducible only if χ shows up in $\mathrm{res}_B^G(\mathrm{ind}_B^G(\chi))$ only once.

The Bruhat Decomposition states that, if

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then

$$G = B \sqcup BwB.$$

A nice exposition of Bruhat Decomposition for G can be seen in Bonnafé [1]. This implies that we may view $V = \mathrm{Ind}_B^G(\chi)$ through two of its proper subspaces. Define

$$V_1 = \{f : B \to \mathbb{C} : f(bg) = \chi(b)f(g), \forall b, g \in B\}$$

and

$$V_w = \{f : BwB \to \mathbb{C} : f(bg) = \chi(b)f(g), \forall b \in B, g \in BwB\}.$$

Then V_1 and V_w can be naturally identified with subspaces of V. Namely, for $f \in V_1$, we can identify it with $\tilde{f} \in V$, defined as

$$\tilde{f}(g) := \begin{cases} f(g) & \text{if } g \in B \\ 0 & \text{if } g \in BwB \end{cases}$$

for all $g \in G$, and similarly for V_w. Through this identification we get $V = V_1 \oplus V_w$ since $G = B \sqcup BwB$. We can see that the dimension of V_1 is one by the isomorphism $V_1 \mapsto \mathbb{C}$, given by $f \mapsto f(1)$. Thus, the dimension of V_w is p since

$$\dim(V) = \dim(V_1) + \dim(V_w).$$

4.1 Proposition *The subspaces V_1 and V_w of V are stable under $\rho(B)$.*

Proof The space V_1 is the collection of f in V such that $f(x) = 0$ for all $x \in BwB$ and V_w is the collection of f in V such that $f(x) = 0$ for all $x \in B$. Since $[\rho(b)f](x) = f(xb)$ and the sets B and BwB are stable under right multiplication by B, the result follows. □

This implies that the restriction of $\mathrm{Ind}_B^G(\chi)$ back to B is the direct sum of V_1 and V_w, i.e.

$$\mathrm{Res}_B^G(\mathrm{Ind}_B^G(\chi)) = V_1 \oplus V_w.$$

Let ψ_1 be the character of V_1 and ψ_w the character of V_w as representations of B. Then by Proposition 4.1 we have

$$\mathrm{res}_B^G(\mathrm{ind}_B^G(\chi)) = \psi_1 + \psi_w.$$

Let us define

$$V_w^U = \{f \in V_w : f(gu) = f(g), \forall u \in U, g \in BwB\},$$

the U invariant subspace of V_w. Let us analyze what it explicitly means for f to be in V_w^U. If $f \in V_w^U$, then $f : BwB \to \mathbb{C}$ satisfies $f(bg) = \chi(b)f(g)$ for all $b \in B$ and $g \in BwB$. Additionally, $f(gu) = f(g)$ for all $u \in U$ and $g \in BwB$, as defined above.

4.2 Theorem *The dimension of V_w^U is one.*

Proof *Claim 1: $BwB = BwU$.* It is clear that $BwU \subseteq BwB$ since U is contained in B. Conversely, let $g \in BwB$; thus $g = bwb'$ for some $b, b' \in B$. Since $B = TU$, we may let $b' = tu$ for some $t \in T$ and $u \in U$. Then

$$\begin{aligned} bwb' &= bwtu \\ &= bwtw^{-1}wu \\ &= bt^{-1}wu \\ &= b''wu \end{aligned}$$

which is an element of BwU with $b'' = bt^{-1}$.

So if we look at the map from $V_w^U \to \mathbb{C}$ given by $f \mapsto f(w)$ we see that the dimension of V_w^U is at most one.

Claim 2: The map $\theta : B \times U \to BwU$ given by $(b, u) \mapsto bwu$ is a bijection of finite sets. Let bwu be an element of BwU. Then there exists $b \in B$ and $u \in U$ such that $\theta(b, u) = bwu$, implying θ is a surjection. Moreover the cardinality of $B \times U$ is easily seen to be $p^3 - p^2$ and

$$|BwU| = |BwB| = |G| - |B| = p^3 - p^2.$$

Therefore θ is a surjection of finite sets with the same cardinality and thus a bijection.

Now consider the map $\tilde{f} : B \times U \to \mathbb{C}$, given by $\tilde{f}(b, u) = \chi(b)$ and define $f = \tilde{f} \circ \theta^{-1}$. Then

$$f(bwu) = f(\theta(b, u)) = \tilde{f}(b, u) = \chi(b).$$

Then $f \in V_w^U$ and f is nonzero, which implies that the dimension of V_w^U is one. □

4.3 Definition Suppose $G = H \ltimes N$ and let (ϕ, W) be a representation of G. The *deflation* of (ϕ, W) is the representation of H with vector space

$$W^N = \{w \in W : \phi(n)w = w, \forall n \in N\}$$

with H action given by $[\mathrm{Def}_H^G(\phi(h))w] = \phi(h)w$, for all $w \in W^N$. Notice this is a well defined action since for any $w \in W^N$ we have $\phi(h)w \in W^N$. To see this, note that if $n \in N$, then

$$\begin{aligned}\phi(n)\phi(h)w &= \phi(nh)w \\ &= \phi(hh^{-1}nh)w \\ &= \phi(h)\phi(h^{-1}nh)w \\ &= \phi(h)w\end{aligned}$$

since $h^{-1}nh \in N$ and $w \in W^N$.

4.4 Lemma *Let (ϕ, W) be a representation of $G = H \ltimes N$ and ψ the character of W. Define W^N as in Definition 4.3. Then*

$$[\mathrm{def}_H^G(\psi)](h) = \frac{1}{|N|}\sum_{n\in N}\psi(hn). \tag{4.1}$$

for all $h \in H$.

Proof Define the projection map $\mathrm{pr}^N : W \to W$, given by

$$\mathrm{pr}^N(w) = \frac{1}{|N|}\sum_{n\in N}\phi(n)w.$$

We have that $[\mathrm{def}_H^G(\psi)](h)$ is the trace of $\phi(h)$ on W^N for all $h \in H$. Notice that

$$\mathrm{Tr}(\phi(h)|_{W^N}) = \mathrm{Tr}(\mathrm{pr}^N \circ \phi(h)).$$

Hence,

$$\begin{aligned}
[\mathrm{def}_H^G(\psi)](h) &= \mathrm{Tr}(\mathrm{pr}^N \circ \phi(h)) \\
&= \mathrm{Tr}(\phi(h) \circ \mathrm{pr}^N) \\
&= \mathrm{Tr}\left(\phi(h) \cdot \frac{1}{|N|} \sum_{n \in N} \phi(n)\right) \\
&= \frac{1}{|N|} \sum_{n \in N} \mathrm{Tr}(\phi(hn)) \\
&= \frac{1}{|N|} \sum_{n \in N} \psi(hn).
\end{aligned}$$

∎

4.5 Theorem *Let ψ be a character of B and φ be a character of T. Then*

$$\langle \psi, \mathrm{inf}_T^B(\varphi) \rangle_B = \langle \mathrm{def}_T^B(\psi), \varphi \rangle_T.$$

Proof We calculate,

$$\begin{aligned}
\langle \psi, \mathrm{inf}_T^B(\varphi) \rangle_B &= \frac{1}{|B|} \sum_{b \in B} \psi(b) \overline{[\mathrm{inf}_T^B(\varphi)](b)} \\
&= \frac{1}{|T|} \frac{1}{|U|} \sum_{t \in T} \sum_{u \in U} \psi(tu) \overline{[\mathrm{inf}_T^B(\varphi)](tu)} \\
&= \frac{1}{|T|} \sum_{t \in T} [\mathrm{def}_T^B(\psi)](t) \overline{\varphi(t)} \\
&= \langle \mathrm{def}_T^B(\psi), \varphi \rangle_T.
\end{aligned}$$

The third equality follows from equation (4.1) and the fact that $[\mathrm{inf}_T^B(\varphi)](tu) = \varphi(t)$ for all $b = tu$ in B. ∎

4.6 Theorem *Let χ be a one-dimensional character of B inflated from the character φ of T. Then*

$$\langle ind_B^G(\chi), ind_B^G(\chi) \rangle_G = 1 + \langle \varphi, \varphi^{-1} \rangle_T.$$

Proof Recall that V_1 is a representation of B with character ψ_1. Let ρ_1 be the representation associated to V_1 and notice $\rho_1(b)f = \chi(b)f$. Since the dimension of V_1 is one,

$$\psi_1(b) = \mathrm{Tr}(\rho_1(b)) = \chi(b),$$

for all $b \in B$. Thus $\chi = \psi_1$. Hence, using Frobeinus reciprocity, we see

$$\begin{aligned}
\langle \mathrm{ind}_B^G(\chi), \mathrm{ind}_B^G(\chi)\rangle_G &= \langle \chi, \mathrm{res}_B^G(\mathrm{ind}_B^G(\chi))\rangle_B \\
&= \langle \chi, \psi_1 + \psi_w\rangle_B \\
&= \langle \chi, \chi\rangle_B + \langle \chi, \psi_w\rangle_B \\
&= 1 + \langle \mathrm{inf}_T^B(\varphi), \psi_w\rangle_B \\
&= 1 + \langle \varphi, \mathrm{def}_T^B(\psi_w)\rangle_T.
\end{aligned}$$

Note that $\psi_w^U = \mathrm{def}_T^B(\psi_w)$ since ψ_w^U is the character of V_w^U. It suffices to compute $[\rho(t)f](w)$. Recall $f \mapsto f(w)$ is a complex linear isomorphism between V_w^U and $\mathbb{C}$. Hence,

$$\begin{aligned}
[\rho(t)f](w) &= f(wt) \\
&= f(wtw^{-1}w) \\
&= f(t^{-1}w) \\
&= \varphi(t)^{-1}f(w).
\end{aligned}$$

So $\rho(t)f = \varphi^{-1}(t)f$ for all $f \in V_w^U$. Thus, $\psi_w^U = \varphi^{-1}$ and hence,

$$\langle \varphi, \psi_w^U\rangle_T = \langle \varphi, \varphi^{-1}\rangle_T$$

giving us

$$\langle \mathrm{ind}_B^G(\chi), \mathrm{ind}_B^G(\chi)\rangle_G = 1 + \langle \varphi, \varphi^{-1}\rangle_T.$$

∎

4.7 Corollary *Let φ be defined as above.*

(i) If $\varphi \neq \varphi^{-1}$, then $Ind_B^G(\chi)$ is irreducible.

(ii) If $\varphi = \varphi^{-1}$, then $Ind_B^G(\chi)$ splits into two nonequivalent representations.

Proof Using Theorem 4.6, if $\varphi \neq \varphi^{-1}$ then $\langle \text{ind}_B^G(\chi), \text{ind}_B^G(\chi)\rangle_G = 1$. This implies that $\text{Ind}_B^G(\chi)$ is an irreducible representation and thus proves (i). Similarly, if $\varphi = \varphi^{-1}$ then $\langle \text{ind}_B^G(\chi), \text{ind}_B^G(\chi)\rangle_G = 2$. Since $\text{ind}_B^G(\chi)$ is an $\mathbb{N}$-linear combination of irreducibles (which must satisfy that the squares of their multiplicities sum to 2) it must be $\text{ind}_B^G(\chi)$ is the sum of two nonequivalent irreducible characters. Therefore, $\text{Ind}_B^G(\chi)$ must split into two nonequivalent representations, proving (ii).

□

4.8 Theorem *Let φ_1 and φ_2 be characters of T and $\chi_i = inf_T^B(\varphi_i)$ for $i = 1, 2$. If $\varphi_1 \neq \varphi_2^{\pm 1}$ then $Ind_B^G(\chi_1)$ is not isomorphic to $Ind_B^G(\chi_2^{\pm 1})$.*

Proof If $\text{Ind}_B^G(\chi_1)$ is isomorphic to $\text{Ind}_B^G(\chi_2^{\pm 1})$ then we may consider

$$\text{Hom}_G(\text{Ind}_B^G(\chi_1), \text{Ind}_B^G(\chi_2)).$$

The Schur inner product of the corresponding characters then must satisfy

$$\langle \text{ind}_B^G(\chi_1), \text{ind}_B^G(\chi_2)\rangle_G \neq 0.$$

Hence,

$$\begin{aligned}
\langle \text{ind}_B^G(\chi_1), \text{ind}_B^G(\chi_2)\rangle_G &= \langle \chi_1, \text{res}_B^G(\text{ind}_B^G(\chi_2))\rangle_B \\
&= \langle \text{inf}_T^B(\varphi_1), \psi_1 + \psi_w\rangle_B \\
&= \langle \text{inf}_T^B(\varphi_1), \psi_1\rangle_B + \langle \text{inf}_T^B(\varphi_1), \psi_w\rangle_B \\
&= \langle \varphi_1, \text{def}_B^G(\psi_1)\rangle_T + \langle \varphi_1, \text{def}_B^G(\psi_w)\rangle_T \\
&= \langle \varphi_1, \varphi_2\rangle_T + \langle \varphi_1, \varphi_2^{-1}\rangle_T.
\end{aligned}$$

This implies that $\langle \varphi_1, \varphi_2\rangle_T \neq 0$ or $\langle \varphi_1, \varphi_2^{-1}\rangle_T \neq 0$. Therefore $\varphi_1 = \varphi_2^{\pm 1}$. □

Only two irreducible characters of T satisfy the properties imposed in Corollary 4.7 (ii), namely when $\varphi = \varphi_0$, the trivial character, or $\varphi = \varphi_{\frac{p-1}{2}}$, the Legendre character. If φ is the trivial character on T then $\chi = \text{inf}_B^G(\varphi)$ is the trivial character on B. Then $\text{Ind}_B^G(\chi)$ contains the constant functions, which is a copy of the trivial representation of G. Since the dimension of $\mathrm{Ind}_B^G(\chi)$ is $p+1$, its complement must be an irreducible representation of dimension p; this representation is called the Steinberg representation (see Reeder [6]). Therefore

$$\mathrm{Ind}_B^G(\chi) = \mathbb{C} \oplus \mathrm{St},$$

where St denotes the Steinberg representation.

If $\varphi = \varphi_{\frac{p-1}{2}}$, denote $\varphi = \mathrm{Leg}$ for the Legendre character where

$$\varphi \begin{pmatrix} a^i & 0 \\ 0 & a^{-i} \end{pmatrix} = \mathrm{Leg}(a^i) = \begin{cases} 1 & \text{if } i \text{ is even} \\ -1 & \text{if } i \text{ is odd.} \end{cases}$$

4.9 Definition Let H be a subgroup G. For any element $g \in G$ we set ${}^gH := gHg^{-1}$. Let (ϕ, W) be a complex representation of H. Consider the isomorphism

$$c_{g^{-1}} : {}^gH \to H,$$

given by $x \mapsto g^{-1}xg$. We set ${}^g\phi = \phi \circ c_{g^{-1}}$, this defines a representation of ${}^gH$. In other words, for any $x \in {}^gH$ and $w \in W$ we have

$${}^g\phi(x)w = \phi(g^{-1}xg)w.$$

Let ψ be a character of (ϕ, W). We define ${}^g\psi$ as the character associated to $({}^g\phi, W)$. It is given by

$${}^g\psi(x) = \psi(g^{-1}xg)$$

for $x \in {}^gH$.

To determine the dimensions of the two nonequivalent representations of $\mathrm{Ind}_B^G(\chi)$, we state the following theorem (see Isaacs [3] for proof), which we will use in Corollary 4.11 below.

4.10 Theorem (Mackey decomposition formula) *Let U and H be subgroups of G and let (ϕ, W) be a complex representation of H. Then one has an isomorphism*

$$Res_U^G(Ind_H^G(\phi)) \cong \bigoplus_{g \in U \backslash G/H} Ind_{U \cap {}^gH}^U(Res_{U \cap {}^gH}^{{}^gH}({}^g\phi)).$$

4.11 Corollary *Let φ be the Legendre character. Then $\mathrm{Ind}_B^G(\chi)$ splits into two nonequivalent representations, each of dimension $\frac{p+1}{2}$.*

Proof We have already shown in Corollary 4.7 (ii), $\mathrm{Ind}_B^G(\chi)$ splits into two nonequivalent representations. In order to calculate the dimensions of these representations, we will show their corresponding characters are conjugate under the action of $\tilde{G} = \mathrm{GL}_2(\mathbb{F}_p)$.

Denote by $\tilde{B}$ the Borel subgroup of $\tilde{G}$, whose elements are the upper triangular matrices, and $\tilde{T}$ the subgroup of diagonal matrices. Similarly to B and G, we have

$$\tilde{B} = \tilde{T} \ltimes U \text{ and } \tilde{G} = \tilde{B} \sqcup \tilde{B}w\tilde{B}. \tag{4.2}$$

One may reference Bonnafé [1] for more details on (4.2). Define

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Since A and T are normal subgroups of $\tilde{T}$ with trivial intersection and $|\tilde{T}|$ is equal to $|AT|$, namely $(p-1)^2$, we have

$$\tilde{T} = A \times T.$$

Define $\tilde{\varphi}$, a character of $\tilde{T}$ where

$$\tilde{\varphi}\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \varphi \begin{pmatrix} v^{-1} & 0 \\ 0 & v \end{pmatrix} = \mathrm{Leg}(v).$$

By the Mackey decomposition, we obtain

$$\begin{aligned} \mathrm{res}_G^{\tilde{G}}(\mathrm{ind}_{\tilde{B}}^{\tilde{G}}(\mathrm{inf}_{\tilde{T}}^{\tilde{B}}(\tilde{\varphi}))) &= \mathrm{ind}_{G\cap\tilde{B}}^{G}(\mathrm{res}_{G\cap\tilde{B}}^{\tilde{B}}(\mathrm{inf}_{\tilde{T}}^{\tilde{B}}(\tilde{\varphi}))) \\ &= \mathrm{ind}_B^G(\mathrm{res}_B^{\tilde{B}}(\mathrm{inf}_{\tilde{T}}^{\tilde{B}}(\tilde{\varphi}))), \end{aligned}$$

since $G \cap \tilde{B} = B$. Furthermore, since $\tilde{B} = \tilde{T}U = ATU$ we have

$$\begin{aligned} \mathrm{inf}_{\tilde{T}}^{\tilde{B}}(\tilde{b}) &= \tilde{\varphi}(at) \\ &= \varphi(t) \\ &= \mathrm{inf}_T^B(\varphi)(tu) \end{aligned}$$

for any $\tilde{b} = atu \in \tilde{B}$. This implies that for any $b = tu \in B$,

$$\begin{aligned}\mathrm{res}_B^{\tilde{B}}(\inf_{\tilde{T}}^{\tilde{B}}(\tilde{\varphi}))(b) &= \tilde{\varphi}(t)\\ &= \varphi(t)\\ &= \inf_T^B(\varphi)(tu).\end{aligned}$$

Therefore

$$\mathrm{ind}_B^G(\mathrm{res}_B^{\tilde{B}}(\inf_{\tilde{T}}^{\tilde{B}}(\tilde{\varphi}))) = \mathrm{ind}_B^G(\inf_T^B(\varphi)).$$

For simplicity denote $\tilde{R}(\tilde{\varphi}) = \mathrm{ind}_{\tilde{B}}^{\tilde{G}}(\inf_{\tilde{T}}^{\tilde{B}}(\tilde{\varphi}))$ as the induced character from $\tilde{B}$. I claim ${}^w\tilde{\varphi} \neq \tilde{\varphi}$. To see this, let

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in \tilde{T}.$$

Conjugating by w, we obtain

$$w \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} w^{-1} = \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}.$$

Then,

$$\tilde{\varphi} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \mathrm{Leg}(v)$$

and

$${}^w\tilde{\varphi} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \tilde{\varphi} \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix} = \mathrm{Leg}(u),$$

implying ${}^w\tilde{\varphi} \neq \tilde{\varphi}$.

Since ${}^w\tilde{\varphi} \neq \tilde{\varphi}$, we see by a similar argument to the proof of Theorem 4.6 that

$$\langle \tilde{R}(\tilde{\varphi}), \tilde{R}(\tilde{\varphi}) \rangle_{\tilde{G}} = 1.$$

Hence $\tilde{R}(\tilde{\varphi})$ is irreducible. By Clifford Theory, since G is a normal subgroup of $\tilde{G}$, $\mathrm{res}_G^{\tilde{G}}(\tilde{R}(\tilde{\varphi})) = \mathrm{ind}_B^G(\inf_T^B(\varphi))$ is equal to the sum of two irreducible characters in G that are conjugate under the action given by $\tilde{G}$, see [3, Theorem 6.2]. Therefore, these two irreducible characters have the same dimension in G. Since the dimension of $\mathrm{Ind}_B^G(\chi)$ is $p + 1$, these irreducible characters must have dimension

$\frac{p+1}{2}$. Ergo if φ is the Legendre character, $\mathrm{Ind}_B^G(\chi)$ is equivalent to the direct sum of two inequivalent irreducible representations, each of dimension $\frac{p+1}{2}$. ∎

We summarize all constructed principal series representations of $\mathrm{SL}_2(\mathbb{F}_p)$ in the following theorem (cf. [5, Theorem 2.1]).

4.12 Theorem *Let φ_1, φ_2 be two irreducible characters of T and $\chi_i = \mathrm{inf}_T^B(\varphi_i)$, for $i = 1, 2$.*

(i) Suppose that φ_i^2 is not the trivial character for $i = 1, 2$. Then $\mathrm{Ind}_B^G(\chi_i)$ is an irreducible representation of dimension $p+1$. Furthermore, $\mathrm{Ind}_B^G(\chi_1)$ and $\mathrm{Ind}_B^G(\chi_2)$ are equivalent if and only if $\varphi_1 = \varphi_2^{\pm 1}$;

(ii) If φ_1 is the trivial character, then $\mathrm{Ind}_B^G(\chi_1)$ is the direct sum of the trivial representation and Steinberg representation, where St is an irreducible representation of dimension p;

(iii) Let $\varphi_1 = Leg$, the Legendre character. Then $\mathrm{Ind}_B^G(\chi_1)$ is the direct sum of π^+ and π^-, both irreducible of dimension $\frac{p+1}{2}$.

These are called the principal series representations of G.

We have accounted for

$$2 + \frac{p-3}{2} + 2 = \frac{p+5}{2}$$

irreducible representations of G. There are $p+4$ conjugacy classes in G (see [1]), so there are $\frac{p+3}{2}$ many irreducible representations remaining. The sum of the squares of the dimensions of the principal series is

$$1 + p^2 + \left(\frac{p-3}{2}\right)(p+1)^2 + 2\left(\frac{p+1}{2}\right)^2 = \frac{(p-1)p(p+3)}{2}.$$

Since $|G| = (p-1)p(p+1)$ we see that the sum of squares of the dimensions of remaining irreducible representations is

$$\frac{(p-1)^2 p}{2}.$$

The remaining representations are referred to as the cuspidal representations of G (see [5, Theorem 2.3]) which are beyond the scope of this paper.

Bibliography

[1] Cédric Bonnafé. *Representations of* $\mathrm{SL}_2(\mathbb{F}_q)$, volume 13 of *Algebra and Applications*. Springer-Verlag London Ltd., London, 2011.

[2] F.G. Frobenius. *Über Gruppencharaktere*. Preussische Akademie der Wissenschaften Berlin: Sitzungsberichte der Preußischen Akademie der Wissenschaften zu Berlin. Reichsdr., 1896.

[3] I. Martin Isaacs. *Character theory of finite groups*. AMS Chelsea Publishing, Providence, RI, 2006. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423].

[4] Herbert E. Jordan. Group-characters of various types of linear groups. *American Journal of Mathematics*, 29.

[5] John D. Lafferty and Daniel Rockmore. Fast fourier analysis for sl 2 over a finite field and related numerical experiments. *Experimental Mathematics*, 1:115–139, 1992.

[6] Mark Reeder. Characters of $\mathrm{SL}_2(p)$. Technical report, Boston College, December 2008.

[7] Jean-Pierre Serre. *Linear representations of finite groups*. Springer-Verlag, New York, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.