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COLLISIONS OF PARTICLES

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AEC Contract No. W-7405-eng-48

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January 4, 1965

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I. INTRODUCTION

Scattering experiments provide the principal technique by which physicists attempt to understand the structure and interactions of matter on a microscopic scale. Scattering theory provides the basis for analyzing and interpreting scattering experiments.

A description of the development of scattering theory may be divided into several topics. The oldest and simplest branch of scattering theory is that of potential scattering, or scattering of two particles which interact through a local potential [1]. Potential scattering was studied extensively in the first two decades following the development of quantum mechanics in the analysis of elastic scattering of particles by atoms and of nucleon-nucleon scattering. The latter topic, in particular, led to the introduction of an elaborate theory of scattering by noncentral interactions [2]. The development of nuclear physics, with the observation of resonance reactions, indicated the need for more general descriptions of scattering. The resulting theory of resonance reactions [3] [4] has leaned only rather lightly on the details of the Schrödinger equation. Quantum field theory was developed to describe electromagnetic phenomena [5]. Of major importance in the development of scattering theory was the introduction of renormalization techniques into field theory [6].

It might be claimed that modern scattering theory began with the integral equation formulation of Lippmann and Schwinger [7] and the

introduction of S-matrix theory by Heisenberg [8] and others [9]. This work has stimulated much of the development of theoretical physics in the last decade. Of particular significance are the clarification of the study of rearrangement collisions and the development of the so-called dispersion theoretic techniques.

II. THE SCATTERING CROSS SECTION

The properties of scattering interactions are usually expressed most conveniently in terms of the scattering cross section. To define this term, we consider the following scattering experiment: A beam of particles (called beam particles) is directed on a scatterer consisting of target particles. As a result of collisions between beam and target particles, there are particles which emerge from the reactions (called reaction products) and these are detected in particle detectors. To describe this quantitatively, we suppose that the scatterer contains N_t target particles and that this is uniformly illuminated by a flux F_B (expressed as the number of beam particles per unit area per unit time arriving at the target) of beam particles. We suppose also that the scatterer is sufficiently small that the beam is negligibly attenuated in passing through it. Then, if there are δN_s scattering interactions per unit time which lead to detected particles, we define the scattering cross section $\delta\sigma$ as (10).

$$\delta\sigma = \frac{\delta N_s}{N_t F_B} \quad . \tag{1}$$

In the limit that the detectors subtend very small solid angles, as seen from the target, we define the differential scattering cross section $d\sigma$.

When a single detector, subtending a solid angle $\delta\Omega$, is used to define $\delta\sigma$, we may define the cross section per unit solid angle as

$$\frac{d\sigma}{d\Omega} = \lim_{\delta\Omega \to 0} \frac{\delta\sigma}{\delta\Omega} \quad . \tag{2}$$

The total scattering cross section σ is obtained by summing $\delta\sigma$ over all scattering events:

$$\sigma = \Sigma\, \delta\sigma \quad . \tag{3}$$

[The expression (3) does not exist for scattering by a Coulomb force.]

The scattering cross section may be expressed in terms of the square of the magnitude of a scattering amplitude (or S-matrix, or T-matrix element) and is completely described as a function of the momenta and internal states of the particles in the initial and final states. Thus, for the two particles prior to collision [11] we may take the momenta $\underset{\sim}{p}_1$ and $\underset{\sim}{p}_2$ and the internal state quantum numbers s_1 and s_2 as variables. (For example, s_1 and s_2 may describe spin orientation, isotopic spin, etc. For colliding molecules these variables will describe vibrational, rotational, and electronic states.) We may suppose there to be μ particles in the final state following the collision and specify this state by the momenta and internal variables $\underset{\sim}{k}_1, \cdots \underset{\sim}{k}_\mu, s_1' \cdots s_\mu'$. The scattering cross section may be expressed in terms of these variables. Because of symmetries, the number of variables required to describe $\delta\sigma$ may ordinarily be reduced. The most commonly encountered of these symmetries are: (1) Energy and momentum conservation; (2) Rotational invariance; (3) The Lorentz invariance of the scattering cross section $\delta\sigma$ [12]. The Lorentz invariance of $\delta\sigma$ permits one to describe the scattering in the <u>barycentric</u> co-ordinate

system--the co-ordinate system in which the total momentum of the interacting particles is zero.

This may be illustrated for the special case that there are only two particles in the initial and two in the final state and an average has been performed over all spin orientations. Then the twelve components of $\underset{\sim}{p}_1$, $\underset{\sim}{p}_2$, $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$ may be replaced by only two variables. These may be taken, for example, as the barycentric energy and angle between $\underset{\sim}{p}_1$ and $\underset{\sim}{k}_1$. Convenient variables in relativistic analyses are often chosen to be the Lorentz invariants

$$s \equiv (p_1 + p_2)^2 = (k_1 + k_2)^2$$

$$t \equiv (p_1 - k_1)^2 = (p_2 - k_2)^2 \quad , \tag{4}$$

where we have written p_1, etc. for the four component energy-momentum vector.

III. POTENTIAL SCATTERING

We briefly illustrate the discussion of Section II with the example of non-relativistic scattering by a local central potential $V(r)$. The Schrödinger equation for scattering in the barycentric co-ordinate system is

$$[\nabla_r^2 + \kappa^2 - v(r)]\psi_{\underset{\sim}{\kappa}}^{+}(\underset{\sim}{r}) = 0 \quad . \tag{5}$$

Here $\hbar\kappa$ is the momentum of particle "1" in the barycentric system and $v(r) = \frac{2M_r}{\hbar^2} V(r)$, with M_r the reduced mass of the two particles. In the limit of large separation r between the particles the wavefunction $\psi_{\underset{\sim}{\kappa}}^{+}$ has the

asymptotic form [our notation is such that we represent a unit vector in the direction of $\underset{\sim}{\kappa}$ by $\hat{\underset{\sim}{\kappa}}$]

$$\psi_{\underset{\sim}{\kappa}}^{+}(\underset{\sim}{r}) \to (2\pi)^{-3/2}\left[e^{i\underset{\sim}{\kappa}\cdot\underset{\sim}{r}} + \frac{e^{i\kappa r}}{r}\, f(\hat{\underset{\sim}{\kappa}}\cdot\hat{\underset{\sim}{r}})\right] \quad . \tag{6}$$

Here $f(\hat{\underset{\sim}{k}}\cdot\hat{\underset{\sim}{r}})$ is the scattering amplitude for scattering particle "1" from the direction $\hat{\underset{\sim}{\kappa}}$ into the direction $\hat{\underset{\sim}{r}}$. The corresponding cross section per unit solid angle is

$$\frac{d\sigma}{d\Omega} = |f(\hat{\underset{\sim}{\kappa}}\cdot\hat{\underset{\sim}{r}})|^2 \quad . \tag{7}$$

The wave function $\psi_{\underset{\sim}{\kappa}}^{+}$ may be expanded into partial waves as follows:

$$\psi_{\underset{\sim}{\kappa}}^{+}(\underset{\sim}{r}) = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi\kappa r}\, P_\ell(\hat{\underset{\sim}{\kappa}}\cdot\hat{\underset{\sim}{r}})\, i^\ell e^{i\delta_\ell}\, w_\ell(\kappa;r) \quad . \tag{8}$$

Here P_ℓ is the Legendre polynomial of order ℓ, δ_ℓ is the scattering phase shift [see Eq. (10) below], and $w_\ell(\kappa;r)$ satisfies the differential equation (13)

$$\left[\frac{d^2}{dr^2} + \kappa^2 - \frac{\ell(\ell+1)}{r^2}\, v(r)\right] w_\ell = 0 \quad . \tag{9}$$

This is to be integrated subject to the condition that w_ℓ is regular at $r = 0$. For large r, w_ℓ has the asymptotic form

$$w_\ell(\kappa;r) \to \sqrt{\frac{2}{\pi}}\, \sin\left(\kappa r - \frac{\pi\ell}{2} + \delta_\ell\right) \quad . \tag{10}$$

It is Eq. (10) which permits the determination of the phase shift δ_ℓ. The quantity

$$S_\ell(\kappa) = \exp\left[2i\,\delta_\ell\,(\kappa)\right] \tag{11}$$

is an eigenvalue of Heisenberg's S-matrix [8].

For scattering by noncentral forces the potential $V(\underset{\sim}{r},\underset{\sim}{S}_1,\underset{\sim}{S}_2)$ is a function of $\underset{\sim}{r}$ (and sometimes the orbital angular momentum operator) and the spin operators $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ of the two colliding particles (if either has no spin we consider its spin operator as being zero). Spin eigenfunctions $u(\nu_1,\nu_2)$ may be introduced as depending on the orientations ν_1 and ν_2 of the respective spins of magnitudes S_1 and S_2. Then the wave function $\psi^{+}_{\underset{\sim}{\kappa},\nu_1,\nu_2}$ is to be labeled with the initial spin orientations ν_1 and ν_2. The asymptotic form corresponding to Eq. (6) is

$$\psi^{+}_{\underset{\sim}{\kappa},\nu_1,\nu_2} \to (2\pi)^{-3/2}\Big[e^{i\underset{\sim}{\kappa}\cdot\underset{\sim}{r}}u(\nu_1,\nu_2) + \frac{e^{i\kappa r}}{r}\sum_{\nu_1',\nu_2'}\langle\nu_1',\nu_2'|f(\hat{\underset{\sim}{\kappa}},\hat{\underset{\sim}{r}})|\nu_1,\nu_2\rangle u(\nu_1',\nu_2')\Big]. \tag{12}$$

Here $\langle\nu_1',\nu_2'|f(\hat{\underset{\sim}{\kappa}},\hat{\underset{\sim}{r}})|\nu_1\nu_2\rangle$ is the scattering amplitude for scattering to a final spin orientation ν_1',ν_2'. The cross section per unit solid angle is in this case

$$\frac{d\sigma}{d\Omega} = |\langle\nu_1',\nu_2'|f(\hat{\underset{\sim}{\kappa}},\hat{\underset{\sim}{r}})|\nu_1,\nu_2\rangle|^2 . \tag{13}$$

For an unpolarized initial state, corresponding to a uniform mixture of the $(2S_1 + 1)(2S_2 + 1)$ spin states, the cross section for scattering particle "1" into the direction $\hat{\underset{\sim}{r}}$ with any spin orientation is

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{1}{(2S_1 + 1)(2S_2 + 1)}\sum_{\nu_1',\nu_2'}\sum_{\nu_1,\nu_2}|\langle\nu_1',\nu_2'|f|\nu_1,\nu_2\rangle|^2 , \tag{14}$$

where the sums extend over all spin orientations.

Following scattering by noncentral forces the particles will in general have preferred spin orientations, or be polarized. When, for example, particle "1" has spin one-half with a spin operator $\underset{\sim}{\sigma}_1$, we define its polarization vector $\underset{\sim}{P}(\nu_1,\nu_2)$ by the equation

$$\underset{\sim}{P}(\nu_1,\nu_2) = \left\{ \sum_{\nu_1'',\nu_1',\nu_2'} [\langle \nu_1'',\nu_2'|f|\nu_1,\nu_2\rangle]^* \langle \nu_1''|\underset{\sim}{\sigma}_1|\nu_1'\rangle \times \langle \nu_1', \nu_2'|f|\nu_1,\nu_2\rangle \right\} \times \left\{ \sum_{\nu_1',\nu_2'} |\langle \nu_1',\nu_2'|f|\nu_1,\nu_2\rangle|^2 \right\}^{-1} . \qquad (15)$$

For an unpolarized initial state the polarization is

$$\underset{\sim}{\bar{P}} = \frac{1}{(2S_1+1)(2S_2+1)} \sum_{\nu_1,\nu_2} \underset{\sim}{P}(\nu_1,\nu_2) . \qquad (16)$$

The study of polarization following scattering has provided an important tool for analyzing nuclear and elementary particle reactions [14] [15]. In particular, the role of noncentral interactions in nucleon-nucleon scattering has been studied in great detail [16].

IV. FORMAL SCATTERING THEORY

To describe a general scattering reaction Lippmann and Schwinger [7], [17] introduced a scattering matrix $\mathcal{T}_{ba}$ to describe scattering from an initial state χ_a to a final state χ_b [18]. This is defined as

$$\mathcal{T}_{ba} = (\chi_b, V\psi_a^+) \quad , \tag{17}$$

where ψ_a^+ is the steady state wave function for the event and V is the scattering interaction. Since momentum is conserved for an isolated scattering, we may write

$$\mathcal{T}_{ba} = \delta(\underset{\sim}{P}_b - \underset{\sim}{P}_a)T_{ba} \quad , \tag{18}$$

where $\underset{\sim}{P}_a$ and $\underset{\sim}{P}_b$ are the total momenta of the particles in the initial and final states, respectively, and T_{ba} is defined only for states b and a corresponding to $\underset{\sim}{P}_b = \underset{\sim}{P}_a$.

The scattering cross section $\delta\sigma$ [Eq. (1)] is expressed in terms of T_{ba} as [12]

$$\delta\sigma = \frac{(2\pi)^4}{v_{rel}} \sum_b \delta(\underset{\sim}{P}_b - \underset{\sim}{P}_a)\delta(E_b - E_a)|T_{ba}|^2 \quad . \tag{19}$$

Here v_{rel} is the relative velocity of beam and target particles, E_b and E_a are the respective total energies of the particles in states b and a, and the sum on b extends over those states which lead to the reaction products striking the detectors and thus to register an event. We emphasize that the expression (19) is Lorentz invariant [12].

The Heisenberg S-maritx [8] is given by the expression

$$S_{ba} = \delta_{ba} - 2\pi i\delta(E_b - E_a)\mathcal{T}_{ba} \quad , \tag{20}$$

where δ_{ba} is a Dirac δ-function. The S-matrix is unitary, so

$$\sum_b S^\dagger_{cb} S_{ba} = \delta_{ca} \quad . \tag{21}$$

On substituting Eq. (20) into this, we obtain the equivalent expression of unitarity,

$$i[\mathcal{T}_{ca} - \mathcal{T}^{\dagger}_{ca}] = 2\pi \sum_b \mathcal{T}_{cb}{}^{\dagger}\, \delta(E_b - E_a)\mathcal{T}_{ba} \ , \qquad (22)$$

which is defined only for states c and a on the same energy shell [corresponding to $E_c = E_a$].

The fundamental problem of scattering theory is to determine the $\mathcal{T}$-matrix on the energy shell (or, equivalently, the S-matrix). The first step in doing this is to make use of general symmetry principles (such as Lorentz invariance) to limit the functional forms allowed. Following this a dynamical principle is needed. Such dynamical principles [reviewed in Chapters 5 and 10 of reference (10)] have been proposed in a great variety of forms including integral equations, variational principles, and conditions of functional analyticity.

Re-arrangement collisions (that is, collisions in which bound particles re-arrange themselves) have been studied extensively following the development of formal scattering theory. Much of this [19] was stimulated by the observance of apparent paradoxes [20]. An interesting modification of Eq. (17) intended for application to re-arrangement collisions has been given by Mittleman [21].

Another, and not entirely unrelated, class of applications of formal scattering theory is to scattering by composite systems. These include the multiple scattering and optical model descriptions [22] and elaborate theories of atomic scattering processes [23] [24]. The successful development and use of variational principles for such processes should also be noted [25].

V. FIELD THEORY

Quantum field theory was originally developed to describe electromagnetic phenomena. It was applied in a promising context during the 1930's to β-decay and to the meson theory of nuclear forces. The great optimism following the development of renormalization theory [6] faded quickly for want of adequate mathematical techniques for handling strong interations. The most successful applications to strong interactions were the semi-phenomenological calculations of Chew and others [26], [5].

An interesting and novel attempt to revive field theory has been initiated by Weinberg [27].

VI. S-MATRIX THEORY

Heisenberg suggested in 1946 [8] that a proper quantum theory of scattering would deal only with observable quantities such as the S-matrix and should not require off- the-energy-shell matrix elements of such quantities as $\mathcal{T}$ [Eq. (18)]. Considerable impetus for this point of view has been given by the development of dispersion theory, following early suggestions of Wigner and others [28]. The first attempt at a systematic formulation of a dispersion relation within the context of quantum field theory was made by Gell-Mann, Goldberger, and Thirring [29]. Further development followed applications of formal scattering theory to quantum field theory [30]. The development of the Mandelstam representation [31] provided an important step toward obtaining a "dynamical principle." A further important step was the proposal by Chew and Frautschi and Blankenbecker and Goldberger [33], who suggested

that the only singularities of the S-matrix are those required by the unitarity condition (22) and that families of particles should be associated with Regge Trajectories [34].

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