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KINEMATICAL-SINGULARITY FREE INVARIANT AMPLITUDES FOR TWO PARTICLE SCATTERING PROCESSES. I. GENERAL KINEMATICAL CONSIDERATIONS*

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ABSTRACT

A study is made of the kinematical properties in four-momenta space of scattering amplitudes involving two incoming and two outgoing massive particles with arbitrary spins. For reasons of simplicity in considering analytic properties, the M function, or Spinor Amplitude, formalism is employed, a choice that has no practical effect on the results. Complex covariance at points where the scattering functions are holomorphic is employed to find points where there are kinematical restrictions on them. The effect of discrete symmetries is then accounted for. It is shown that a decomposition into sums involving invariant amplitudes free of any kinematical singularities or zeroes and kinematical factors, or "standard covariants," is possible, provided the latter have certain precisely specified properties derived from the kinematical behavior of the Spinor Amplitudes. Our approach is contrasted to that using center-of-mass helicity amplitudes. Actual discussion of the methods by which standard covariants may be constructed is left for a second paper.

INTRODUCTION

In considering a scattering process involving any number of particles with spin, one is faced with two main problems. The first involves the treatment of the dynamical properties of the system, which arise from the interactions that the particles undergo, and the second involves the kinematical properties, which are imposed by the requirement of Lorentz invariance. One would like to treat the two properties separately, decomposing scattering amplitudes into sums involving purely kinematical factors and functions of the independent invariants formed from the four-momenta, such that the latter contain only dynamical singularities and zeroes. Then one would be able to write dispersion relations in terms of invariants,[1,2] and make dynamical approximations for the scattering functions.

Initially, one must regard the scattering amplitudes as functions of the four-momenta of the particles involved. The choice of amplitudes to be used is far from unique, since, if one wishes to regard the individual particle states as unitary representations of the inhomogeneous Lorentz group, one has the freedom, among other possible choices, of using the so-called canonical basis,[3,4] the helicity basis,[5] or the spinor basis.[3,6] However, it is simplest to work with the spinor amplitudes,[3,6] also called M functions,[7-10] since it is only these which can be holomorphic except for dynamical singularities.[7-10]

Even when there are no spins and, consequently, no kinematical factors involved, it is not trivial to express the analytic properties in terms of invariants formed from the four-momenta. Hepp[11] and

Williams[12] showed that a multisheeted invariant function of any number of four-vectors is expressible as a function of invariants formed from the four-vectors on a dense subdomain that is "saturated" with respect to the mapping between vectors and invariants. The result of Hepp and Williams is a generalization of a theorem of Hall and Wightman[13] that is not applicable to scattering functions on the mass shell.

For scattering amplitudes with arbitrary spins, the separation of the kinematical and dynamical properties on "saturated" domains was carried out explicitly by Williams[12] for the case of four massive particles, while Hepp[11] showed how to treat the case of any number of particles. Unfortunately, in order to simplify the problem as much as possible, these authors coupled all spinor indices with the same transformation properties, using Clebsch-Gordan coefficients. These couplings are not invariant under discrete symmetry transformations, and complicate any attempt to make practical use of the results. Hepp did prove for four-particle amplitudes that it is, in principle, possible to find combinations of the coupled-spinor amplitudes that have definite signature under the various discrete symmetry operations and that are free of kinematical singularities. When only two of the four particles have spins, Fox[14] was able to actually find such combinations.

Most other treatments have relied almost exclusively on perturbation theory, the philosophy of which was summarized by Hearn,[15] to justify their results. The best known example is the familiar A and B amplitudes for pion-nucleon scattering.[16] Recently Scadron and Jones[17] have given some of the relations that are needed to apply this philosophy to scattering amplitudes involving two incoming and two outgoing particles with arbitrary spins. A few authors have tried to justify their results on the basis of the Hall-Wightman Theorem, an example being the invariant amplitudes for nucleon-nucleon scattering.[18]

Because of the results of Hepp,[11] one is justified in using the criteria of perturbation theory[15] to get invariant amplitudes free of kinematical singularities, even when using theories that do not acknowledge the validity of the perturbation expansion of the scattering amplitude.[19] However, as anyone who has tried will readily acknowledge, in practice the approach involves tiresome and seemingly endless algebraic manipulations that offer no physical insight. The main purpose of this paper is to present an alternative means of obtaining kinematical singularity-free invariant amplitudes for the case of two incoming and two outgoing particles, using some simple analytic and Lorentz covariance properties that scattering amplitudes are believed to satisfy as functions of their four-momenta on the mass shell.[7-10]

The criteria that we develop to justify the absence of kinematical singularities in a particular expansion of the scattering function for two incoming and two outgoing particles are actually a generalization of those used by Williams[12] in connection with his invariant amplitudes. However, in contrast to Williams, we avoid the coupling of spinor indices of the M functions, and, consequently, we find it possible to account for the restrictions that the discrete symmetries impose on the invariant amplitudes.

Section 2 introduces the S-matrix elements in canonical form and the motives for preferring the M functions, or spinor amplitudes. Following this, we review the analytic properties of the "connected parts" of the M functions, referred to as M^c functions.

In Sec. 3, after considering some consequences of Hall and Wightman's treatment of complex four-vectors,[13] we explain their relevance to Stapp's theorem,[8,20] which states, among other things, that the M^c functions are covariant under complex Lorentz transformations at all points where they are regular. We define "saturated" domains in the space of the complex four momenta and use Stapp's theorem to find those points on such domains where there are kinematical restrictions on the M^c functions--that is, points at which Lorentz covariance imposes a linear relationship between their values.

Section 4 is concerned with the effect that the various discrete symmetries can have. One result is that, if we divide the M^c functions into parts having positive and negative signature under spatial inversion, whether or not it is actually a symmetry, the number of independent components with a given signature is the same at all points on a "saturated" domain where there are three linearly independent four-momenta. This is true even though, as found by Hall and Wightman,[13] some of these points have the same invariants formed from the four-momenta as other points where there are only two linearly independent momenta.

In Sec. 5 we discuss the theorem of Hepp and Williams mentioned earlier. Following this we show, for those cases in which only four particles are involved, how our earlier discussion of the kinematical and discrete symmetry restrictions at various points is relevant to the problem of decomposing the scattering functions on "saturated" domains into sums of invariant amplitudes describing only the dynamics and of kinematical factors or "standard covariants." This approach is compared to recent attempts to describe the analytic properties in terms of center-of-mass (c.m.) helicity amplitudes.[21] The methods and practical problems involved in actually constructing the standard covariants are left for the following paper.[22]

In reading this paper, we urge the reader to give particular attention to the discussion in Secs. 3 and 4 of special points in the space of complex four-vectors, since this discussion proves essential in defining the properties of the standard covariants in Sec. 5. One can, however, omit Part C of Sec. 4, where restriction due to symmetries other than parity are considered, without interrupting the smooth flow of the paper, and it would probably be advisable to do so at the first reading. Of the four appendices, the first three are concerned with the Lorentz Group and Spinors and play an important part in establishing our notation. The final appendix gives the proof of a lemma in Sec. 5 concerning the standard covariants.

We avoid any discussion of massless particles because, when one or two of the particles in a process involving two incoming and two outgoing particles are massless, the problem of finding standard covariants for cases in which discrete symmetry operations are relevant has been explicitly solved by Zwanziger.[23] Compared with the case in which all four particles are massive, this is not difficult to do.

2. BASIC PROPERTIES OF TWO PARTICLE SCATTERING AMPLITUDES

A. The S Matrix

Let us consider a scattering experiment in which particles 1 and 2 are incoming and particles 3 and 4 are outgoing. We specify the results by the following information:[24]

(a) $t_a = \{t_1, t_2\}$ and $t_b = \{t_3, t_4\}$, where $t_i = \{m_i, j_i, q_i\}$ specifies the "type" of the particle. Here m_i is the mass of the particle, j_i is its spin, and q_i stands for all additive quantum numbers independent of the Lorentz Group, i.e., baryon number, electric charge, and strangeness. We ignore the effects of isotopic spin.

(b) $k_a = \{k_1, k_2\}$ and $k_b = \{k_3, k_4\}$, where $k_i = (k_i^0, \underset{\sim}{k}_i)$ specifies the (real) four-momentum of the particle. The energy is positive and the mass shell constraint $k_i \cdot k_i = (m_i)^2$ is satisfied. It will prove convenient to introduce the notation $K_i = \{k_i, t_i\}$ and $\overline{K}_i = \{k_i, \overline{t}_i\}$, with obvious meanings for K_a and K_b.

(c) $\alpha_a = \{\alpha_1, \alpha_2\}$ and $\alpha_b = \{\alpha_3, \alpha_4\}$, where α_i is the spin quantum number of the <u>i</u>th particle. Its value is one of the numbers $-j_i, -j_i + 1, \cdots, j_i - 1, j_i$.

The spin measurements for a given particle refer to that particle's rest system. In such a frame, one specifies a unit three-vector along which the values, α_i, of the spin quantum numbers are measured. In the "canonical" convention,[3,4] which we adopt here, this rest-frame vector is chosen along the 3 axis. It is a simple matter to relate this choice to others; for example, the helicity convention,[5] where the rest-frame vector with respect to which the spin components are specified is along the direction of motion of the particle.[25]

The relationship between measurements on the initial and final particles is expressed by the S-matrix elements $S(K_b; K_a)$, $\alpha_b; \alpha_a$, which contain a four-momentum conservation delta function that expresses invariance under space-time translations.[26] To each of the incoming and outgoing particles one may assign a momentum-space wavefunction $\psi_{\alpha_i}{}^{i}(k_i)$ with the norm

$$\sum_{\alpha_i} \int \frac{d\underset{\sim}{k}_i}{k_i^0} \psi_{\alpha_i}{}^{i*}(k_i)\, \psi_{\alpha_i}{}^{i}(k_i) = 1 \; ; \quad i=1,\cdots,4 \quad . \tag{1}$$

Then, the probability that the outgoing particles are in the state specified by their assigned wavefunctions if the incoming particles have the given wavefunctions is $|S(\psi_b,\psi_a)|^2$, where

$$S(\psi_b,\psi_a) = \prod_{i=1}^{4} \int \frac{d\underset{\sim}{k}_i}{k_i^0} \sum_{\alpha_i} \psi_{\alpha_3}{}^{3*}(k_3)\, \psi_{\alpha_4}{}^{4*}(k_4)\, S(K_b; K_a)_{\alpha_b;\, \alpha_a} \times \psi_{\alpha_1}{}^{1}(k_1)\, \psi_{\alpha_2}{}^{2}(k_2) \; . \tag{2}$$

The invariance of probabilities under a simultaneous proper orthochronous Lorentz transformation of both incoming and outgoing states leads to the relation[3,6,9,27]

$$S_{\alpha_b;\,\dot{\alpha}_a}(\Lambda K_b;\,\Lambda K_a) = D^{(j_b)}\big(A(k_b)\big)_{\alpha_b}^{\alpha_b'}\, D^{(j_a)}\big(A^*(k_a)\big)_{\dot{\alpha}_a}^{\dot{\alpha}_a'}\, S_{\alpha_b';\,\dot{\alpha}_a'}(K_b;\,K_a)\,. \qquad (3)$$

The notation is such that $D^{(j_b)}\big(A(k_b)\big)_{\alpha_b}^{\alpha_b'} = \prod_{i=3}^{4} D^{(j_i)}\big(A(k_i)\big)_{\alpha_i}^{\alpha_i'}$ and $D^{(j_a)}\big(A^*(k_a)\big)_{\dot{\alpha}_a}^{\dot{\alpha}_a'} = \prod_{i=1}^{2} D^{(j_i)}\big(A^*(k_i)\big)_{\dot{\alpha}_i}^{\dot{\alpha}_i'}$. According to the spinor calculus convention introduced in Appendix A, each of the raised indices on the transformation matrices is summed with the corresponding S-matrix index, and dots are introduced over incoming spin indices to indicate that they transform like the complex conjugate of outgoing ones. Here $\Lambda = \Lambda(A,A^*)$, where, as explained in the Appendix, A is a matrix in $SL(2,C)$, and $A(k_i)$, defined by (A.33), is the matrix in $SU(2)$ corresponding to the "Wigner rotation" in (A.32). We have adopted the convention $\Lambda K_i = \{\Lambda k_i, t_i\}$.

B. The M Functions

The fact that the spin transformation matrices in (3) depend on the four-momenta of the particles leads to "kinematical" singularities in the four-momenta when one analytically continues such an expression out of the physical region of the original process. This, in turn, leads to somewhat complicated crossing relations between the amplitudes that are most natural for describing certain physical processes that are all actually described by a single set of functions of the four-momenta. To simplify the various properties just mentioned, it is convenient to introduce spinor amplitudes, or M functions.[3,6-10]

We may define M functions with lower undotted spinor indices assigned to outgoing particles and lower dotted indices to incoming particles by the relation

$$M_{\alpha_b;\,\dot{\alpha}_a}(K_b;\,K_a) \;=\; D^{(j_b)}\big(\mathcal{B}(k_b)\big)_{\alpha_b}^{\;\alpha'_b}\; D^{(j_a)}\big(\mathcal{B}^*(k_a)\big)_{\dot{\alpha}_a}^{\;\dot{\alpha}'_a}\; S_{\alpha'_b;\,\dot{\alpha}'_a}(K_b;\,K_a)\;, \qquad (4)$$

where $\mathcal{B}(k_i)$, defined in (A.24), is the Hermitian matrix in SL(2,C) corresponding to the Hermitian "boost" $L(k_i)$ in (A.21) that carries $\bar{k}_i = (m_i,\,0)$ into k_i. Then, using $\Lambda = \Lambda(A,\,A^*)$, the following simple covariance property is easily verified from (3), (4), and (A.33):

$$M_{\alpha_b;\,\alpha_a}(\Lambda K_b;\,\Lambda K_a) \;=\; D^{(j_b)}(A)_{\alpha_b}^{\;\alpha'_b}\; D^{(j_a)}(A^*)_{\dot{\alpha}_a}^{\;\dot{\alpha}'_a}\; M_{\alpha'_b;\,\dot{\alpha}'_a}(K_b;\,K_a)\;. \qquad (5)$$

Instead, we could have defined M functions with all lower undotted indices. It may be verified from (5) and

$$M_{\alpha_b;\,\alpha_a}(K_b;\,K_a) \;=\; D^{(j_a)}(\sigma\cdot k_a C^{-1}/m_a)_{\alpha_a}^{\;\dot{\alpha}'_a}\; M_{\alpha_b;\,\dot{\alpha}'_a}(K_b;\,K_a)\;, \qquad (6)$$

with the aid of (B.6), that

$$M_{\alpha_b;\,\alpha_a}(\Lambda K_b;\,\Lambda K_a) \;=\; D^{(j_b)}(A)_{\alpha_b}^{\;\alpha'_b}\; D^{(j_a)}(A)_{\alpha_a}^{\;\alpha'_a}\; M_{\alpha'_b;\,\alpha'_a}(K_b;\,K_a)\;. \qquad (7)$$

As a matter of fact, one is free to choose individual spinor indices to have any desired transformation properties, since the type of spinor index chosen has no effect on the physics or on the analytic properties. The matrix that transforms a particular spinor index from one type to another is a holomorphic function of that particle's four-momentum.

In the usual way, we may express the M functions for a given process as a sum of a "no-scattering" part and a "connected part," M^0, in which all the particles interact.[28] It is only the latter part in which we are interested, and we write it so as to explicitly show the four-momentum conservation delta function:

$$M^0(K_b; K_a) = i\,\delta^4(k_1 + k_2 - k_3 - k_4)\, M^c(K_b; K_a) , \tag{8}$$

where the spinor indices have been suppressed.

Equation (8) defines M^c functions that are free of delta functions and it is these which are regarded as analytic functions in complex four-momenta space in S-matrix theory.[7-10] To discuss these analytic properties, one has to take into consideration the fact that, of the 16 components of the four-momenta, the mass shell constraints and four-momentum conservation allow only eight to be functionally independent. There does not exist any set of eight independent parameters such that the 16 components of the $(k_i)^\nu$ are globally holomorphic functions of these coordinates. However, there always exist sets of "local" coordinates, and "analyticity on the mass shell" may be well-defined in terms of these coordinates.[8,29]

As usual, we use the term holomorphic to designate the property of being analytic and single-valued. A domain is a connected open set, and, for our purposes, it is sufficient to regard the "domain of holomorphy" of a function as a union of sheets, a sheet being the maximum domain on which the mapping from the local coordinates to the function is single-valued.[30] Note that the definition of the domain of holomorphy implies that the sheets may overlap; in fact, the location of sheets is somewhat arbitrary, and any point on the domain of holomorphy lies on the interior of some sheet. We do not include poles and branch points on any sheet, even though it is customary in physics to speak of a pole as lying on a particular sheet when the point at which it occurs is the limit of points on that sheet and not of points on some other sheet.

For physical reasons one defines a "physical sheet"[9] whose boundary includes the physical regions of the various processes related by crossing.[2,6,9,10] Although the singularities of the M^c functions are believed to be determined by unitarity, their exact locations and nature do not affect the discussion in this paper.

3. PROPERTIES IN COMPLEX FOUR-MOMENTA SPACE

A. Stapp's Theorem

There is no need in this section to distinguish between incoming and outgoing particles.[24] We may also avoid specifying spinor index types by writing the covariance relation, of which (5) and (7) are particular examples, in the form

$$M^c\big(\Lambda(A,A^*)K\big) = \Lambda(A,A^*)\, M^c(K)\,. \tag{9}$$

Any function satisfying this property in some region of four-momenta space is said to be $L_+^\uparrow$-covariant in that region.

Following Stapp,[8,20] let us define the domain of regularity of the set of M^c functions with the same incoming and outgoing particles to be the intersections of their domains of holomorphy. A theorem of Stapp[8,20] says that, because the M^c functions satisfy (9) in some physical region, they are $\mathcal{L}_+$-covariant everywhere on their domain of regularity. By $\mathcal{L}_+$-covariance one means the generalization of (9) to complex Lorentz transformations. A function that is $\mathcal{L}_+$-covariant on a domain D satisfies

$$M^c\big(\Lambda(A,B)K\big) = \Lambda(A,B)\, M^c(K) \tag{10}$$

whenever both k and $\Lambda(A,B)$ k $\in$ D. As explained in Appendix C, A and B $\in$ SL(2,C) and $\Lambda(A,B) \in \mathcal{L}_+$. The notation on the right-hand side of (10) indicates that the matrix $D^{(j_i)}(A)$ acts from the left on the spinor index belonging to the <u>i</u>th particle if it is lower undotted, while the matrix $D^{(j_i)}(B)$ acts on it if it is lower dotted.

The $\mathcal{L}_+$ orbit, $\mathcal{L}_+ k$, of any point $k = \{k_1, \cdots, k_4\}$ in the space of the complex four-momenta is the set of all points $\Lambda k = \{\Lambda k_1, \cdots, \Lambda k_4\}$ obtained by letting $\Lambda \in \mathcal{L}_+$ take on all possible values. Stapp's Theorem for the M^c functions under consideration then says:[8,20]

S1. The domain of regularity of the M^c functions is a union of sheets. (Recall that branch points and poles are not included on the sheets, which may overlap.) If a sheet contains a point k, it contains every point on the same $\mathcal{L}_+$ orbit.

S2. The M^c functions are $\mathcal{L}_+$-covariant on each sheet; that is, they satisfy (10) there.

S3. The sheets may be chosen such that all points in a real domain corresponding to physical points lie on a single sheet.

The above statements are actually valid for delta function-free M^c functions describing the interaction of any number of particles.[31]

B. Properties of $\mathcal{L}_+$ Orbits

At any point $k = \{k_1, \cdots, k_4\}$ in the space of the complex four-momenta we may define the scalar invariants $k_i \cdot k_j$. For the case of an arbitrary number of particles, one should also consider the pseudoscalar invariants formed by contracting the completely antisymmetric tensor $\epsilon^{\mu\nu\lambda\rho}$ with the four-momenta. The invariants taken together are then referred to as $\mathcal{L}_+$ invariants, since they are invariant under any $\Lambda \in \mathcal{L}_+$, while the scalars alone are referred to as $\mathcal{L}$ invariants, since they are invariant under any $\Lambda \in \mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_-$, where

$\mathcal{L}_-$ is the set of improper Lorentz transformations. In the case under consideration the pseudoscalars vanish identically, because four-momentum conservation allows at most three of the momenta to be linearly independent at any point. Consequently, it makes no difference whether we refer to two distinct points as having the same $\mathcal{L}_+$ invariants or the same $\mathcal{L}$ invariants, and we will use the former terminology.

Even though all points on the same $\mathcal{L}_+$ orbit have the same $\mathcal{L}_+$ invariants, one cannot always specify orbits by the values of their invariants, since two distinct points with the same $\mathcal{L}_+$ invariants do not necessarily lie on the same $\mathcal{L}_+$ orbit, as will be seen below. Following Hall and Wightman,[13] let us consider any real or complex point $k = \{k_1, \cdots, k_n, k_{n+1}, \cdots, k_4\}$, where the vectors are ordered such that the first $n(k)$ are linearly independent at the point under consideration. Because of four-momentum conservation, $n(k) \leq 3$, and, because the mass shell condition prevents the four-momenta from vanishing identically, $n(k) \geq 1$.

Let us define the Gram determinant

$$G(k_1, k_2, k_3) = \det \begin{pmatrix} k_1 \cdot k_1 & k_1 \cdot k_2 & k_1 \cdot k_3 \\ k_2 \cdot k_1 & k_2 \cdot k_2 & k_2 \cdot k_3 \\ k_3 \cdot k_1 & k_3 \cdot k_2 & k_3 \cdot k_3 \end{pmatrix}, \tag{11}$$

and let $r(k)$ be the rank of this determinant at the point k. Hall and Wightman[13] gave the following relationship between the rank of the

Gram determinant at any point and the number of linearly independent four-vectors at that point:

r(k)	n(k)
3	3
2	2 or 3
1	1 or 2

(12)

The possibility of having $n > r$ for $r \leq 2$ is, as will be seen below, a consequence of the fact that one can have complex light-like vectors in the space orthogonal to the first r vectors when $r \leq 2$. From the considerations of Hall and Wightman regarding the properties of complex four-vectors we can make the following remarks, keeping (12) in mind:

H1. (a) At any $r = n = 3, 2,$ or 1 point, one can write

$$k_i = \sum_{j=1}^{r} \mathcal{a}_{ij} k_j, \qquad \text{for } i=r+1,\cdots,3 , \tag{13}$$

where the $\mathcal{a}_{ij}$'s are finite scalar coefficients. (Recall that the first $r = n$ vectors are linearly independent at the given point and that k_4 is globally determined by four-momentum conservation.)

(b) If k and k' are any two $r = n$ points with the same $\mathcal{L}_+$ invariants, they lie on the same $\mathcal{L}_+$ orbit.

H2. (a) There exist $r = 2$, $n = 3$ points with the same $\mathcal{L}_+$ invariants as any given $r = n = 2$ point. For example, consider the point k determined by (13) with $r = 2$. In the spatial direction orthogonal to the two linearly independent vectors k_1 and k_2 one may define real unit space-like vectors $\hat{e}^1$ and $\hat{e}^2$ such that $\hat{e}^1 \cdot \hat{e}^2 = 0$. Then define

$$\omega_\pm = \hat{e}^1 \pm i\, \hat{e}^2 . \tag{14}$$

It follows that $\omega_\pm \cdot k_1 = \omega_\pm \cdot k_2 = \omega_\pm \cdot \omega_\pm = 0$. Now consider two points $k^{(+)}$ and $k^{(-)}$ such that $k^{(\pm)} = \{k_1, k_2, k_3^{(\pm)}, k_4^{(\pm)}\}$, where

$$k_3^{(\pm)} = k_3 + C\omega_\pm = \sum_{i=1}^{2} \mathcal{a}_{3i} k_i + C\omega_\pm . \tag{15}$$

Here k_1, k_2, and k_3 are the same as for the $r = n = 2$ point in (13), while $C \neq 0$ is an arbitrary real or complex number. The points $k^{(+)}$ and $k^{(-)}$ are two distinct $n = 3$ points with the same $\mathcal{L}_+$ invariants as the $r = n = 2$ point obtained by putting $C = 0$ in (15). They are related by an improper Lorentz transformation that changes ω_+ into ω_-, while leaving k_1 and k_2 the same.

(b) The points $k^{(+)}$ and $k^{(-)}$ determined by (15) and the $r = n = 2$ point obtained by putting $C = 0$ in that equation all lie on different $\mathcal{L}_+$ orbits. Any other $r = 2$, $n = 3$ point with the same $\mathcal{L}_+$ invariants lies on either the $\mathcal{L}_+$ orbit of $k^{(+)}$ or the $\mathcal{L}_+$ orbit of $k^{(-)}$. Any point on one of these two orbits is related to any point on the other by means of an improper Lorentz transformation. As a consequence of H1.(b), we may state that, for any set of values of the $\mathcal{L}_+$ invariants for which the rank of the Gram determinant is 2, there exist three different $\mathcal{L}_+$ orbits.[35]

(c) Consider the limit $C \to 0$ in (15), which yields an $r = n = 2$ point with the same $\mathscr{L}_+$ invariants. This means that any neighborhood of an $r = n = 2$ point contains points of every $r = 2$, $n = 3$ orbit with the same $\mathscr{L}_+$ invariants.

H3. (a) Similar remarks enable one to construct an infinite number of $r = 1$, $n = 2$ orbits having the same invariants as any $r = n = 1$ orbit. In this case, however, any two $r = 1$, $n = 2$ points related by an improper Lorentz transformation lie on the same $\mathscr{L}_+$ orbit.

(b) The occurrence of $r = 1$ points is possible only in M^c functions for which the sum of some of the masses equals the sum of the others. This follows from the conservation of energy and the fact that every $r = n = 1$ orbit contains a point of the form $k_i = (\pm m_i, 0)$ for all $k_i \in k$.

H4. Let us define the little group $\mathscr{G}_+(k)$ at a point k to be the set of proper Lorentz transformations that leave k invariant; i.e., $\Lambda \in \mathscr{G}_+(k) \to \det \Lambda = 1$ and $\Lambda k = k$. At any point k with $n = 3$, the only matrix in $\mathscr{G}_+(k)$ is the unit matrix. However, if $n \leq 2$, $\mathscr{G}_+(k)$ is an infinite set.

We now introduce some terminology first adopted by Hepp.[11,12,20] The I_+-saturated kernel, $\mathscr{U}_a^{(I_+)}$, of any sheet $\mathscr{U}_a$ on the domain of regularity of the M^c functions for a given process is the largest subdomain of $\mathscr{U}_a$ such that, if it contains a point, it contains all points with the same $\mathscr{L}_+$ invariants. The I_+-saturated kernel of the domain of regularity of the M^c functions is the union of the I_+-saturated kernels of all sheets.

Every $r = n$ point on the domain of regularity lies on the I_+-saturated kernel. This remark is a consequence, first of all, of H1.(b) and H2.(c), which show that if an $r = n$ point k lies on a given sheet $\mathcal{U}_a$, then every neighborhood of k on the sheet contains points of every $\mathcal{L}_+$ orbit with the same invariants, and then of S1, which guarantees that every point with the same $\mathcal{L}_+$ invariants lies on $\mathcal{U}_a$. However, an $r = 2$, $n = 3$ point on the domain of regularity does not necessarily lie on the I_+-saturated kernel, since it is possible to have a singularity at an $r = n = 2$ point which is a limit point of the $r = 2$, $n = 3$ orbit. Physical points on the domain of regularity are always on the I_+-saturated kernel, since such points are real and the construction in (14) and (15) shows that $r \neq n$ points are always complex.

C. Kinematical Restrictions

Each of the M^c-function components is a different function, and the number of such independent functions is the same as the number of values that the spinor indices take on. This is given by

$$N = \prod_{i=1}^{4} (2j_i + 1) . \qquad (16)$$

The result in (16) is just the number of independent scattering experiments at a fixed physical value of the four-momenta, at least on a dense subset of the physical points. Of course, for the four particle M^c functions under consideration, discrete symmetries can lead to a relation between the results of various experiments, so that the number that are actually independently determined is less than the number in (16). Such restrictions will be completely ignored in this section and will be the concern of the next one.

It may be possible to find sets of measure zero in the complex four-momenta space on which one can determine, from physical considerations and the analytic properties, a linear relationship, with constant coefficients, between the values of the M^c-function components at any point on such a set.[36] Such a restriction can be of dynamical or kinematical origin, but it is only the latter type which will be of concern to us. We will show that at any point on the I_+-saturated kernel where the rank of the Gram determinant is less than 3, Lorentz covariance alone requires the number of M^c-function components whose values can be independently assigned to be less than the number in (16); that is, there are kinematical restrictions at such points. For a given point k with $n \leq 2$ on a given sheet $\mathcal{U}_a$ this statement follows immediately from remark H4 and the $\mathcal{L}_+$ covariance relation (10), the latter of which becomes, when $\Lambda(A_k,B_k) \in \mathcal{G}_+(k)$,

$$M^c(K) = \Lambda(A_k,B_k)\, M^c(K) . \tag{17}$$

Equation (17) is a linear relation among the values of the various M^c-function components at the point under consideration. Although the little group at any $r = 2$, $n = 3$ point contains only the unit matrix, according to remark H4, we will also find a restriction at any such point lying on the I_+-saturated kernel of some sheet, due to the fact that the $\mathcal{L}_+$ orbit on which it lies contains $r = n = 2$

limit points on the same sheet. It is trivial to extend our considerations to $r = 1$ points and they will be omitted from our discussion.

It is sufficient to restrict our considerations to a single point of any given $r = n = 2$ orbit on a given sheet, since (10) guarantees that the number of kinematically independent M^c-function components is the same at all points on the orbit.[37] On any $r = n = 2$ orbit, there is a point k such that the spatial components of the vectors $k_i \in k$ are all parallel to the 3 axis. Then, from (A.9),

$$\sigma \cdot k_i = \begin{pmatrix} (k_i)^0 + (k_i)^3 & 0 \\ 0 & (k_i)^0 - (k_i)^3 \end{pmatrix} , \quad i=1,\cdots,4 . \tag{18}$$

Let us now make the following choice for the matrices A and $B \in SL(2,C)$, where λ is any complex parameter:

$$A = \begin{pmatrix} \exp(-\lambda/2) & 0 \\ 0 & \exp(\lambda/2) \end{pmatrix} , \tag{19a}$$

$$B = \begin{pmatrix} \exp(\lambda/2) & 0 \\ 0 & \exp(-\lambda/2) \end{pmatrix} . \tag{19b}$$

From (C.1) it follows that $\Lambda(A,B)\, k_i = k_i$, for all $i = 1,\cdots,4$, when A and B are given by (19) and k is a point of the form in (18), so $\Lambda(A,B)$ belongs to the little group $\mathscr{L}_+(k)$.

Because of (A.27), we have, when A and B are given by (19),

$$D^{(j_i)}(A) = \exp[-\lambda\, J_3^{(j_i)}] \, , \tag{20a}$$

$$D^{(j_i)}(B) = \exp[\lambda\, J_3^{(j_i)}] \, . \tag{20b}$$

Suppose that the particles are ordered such that the first ℓ have lower undotted spinor indices and the remaining $4-\ell$ have lower dotted ones, the assignment of index types to individual particles being completely arbitrary. Then, because of (17) and (20), we have, at any point k of the form in (18) on any sheet,

$$M^c_{(\alpha)(\dot\beta)}(K) = \exp\left[-\lambda\left(\sum_{s=1}^{\ell} \alpha_s - \sum_{t=\ell+1}^{4} \beta_t\right)\right] M^c_{(\alpha)(\dot\beta)}(K) \, , \tag{21}$$

where $(\alpha) = \alpha_1 \cdots \alpha_\ell$ and $(\dot\beta) = \dot\beta_{\ell+1} \cdots \dot\beta_4$ are not to be confused with the outgoing and incoming spinor indices of the preceding section. Equation (21) requires that

$$M^c_{(\alpha)(\dot\beta)}(K) = 0 \quad \text{if} \quad \sum_s \alpha_s - \sum_t \beta_t \neq 0 \, . \tag{22}$$

Therefore, the M^c-function components have "kinematical zeroes" at any point of the form in (18). At other $r = n = 2$ points, the relationship among the values of the given components, as given by (17), will be more complicated, but the number of such linear relationships will be the same as the number of "zeroes" in (22). It is convenient to continue to use the term "kinematical zeroes" to refer to the restrictions at these latter points.

Now consider any $r = 2$, $n = 3$ orbit on the same sheet having the same $\mathcal{L}_+$ invariants as the point k in (18). Again, it is sufficient to consider a single point on the orbit to determine the number of kinematically independent M^c-function components anywhere on the orbit. There is a point $k^{(+)}$ or $k^{(-)}$ on the orbit such that $k_i^{(\pm)} \in k^{(\pm)}$ has the form $k_i^{(\pm)} = k_i + C_i \omega_\pm$, where k_i is given by (18), C_i is a real or complex number, and $\omega_\pm = \hat{e}^1 \pm i\, \hat{e}^2$ is a complex light-like vector in the space orthogonal to the k_i. Choosing the real space-like vectors $\hat{e}^1$ and $\hat{e}^2$ to be parallel to the 1 and 2 axes respectively, we have from (18) and (A.9),

$$\sigma \cdot k_i^{(\pm)} = \begin{pmatrix} (k_i)^0 + (k_i)^3 & (1 \pm 1) C_i \\ (1 \mp 1) C_i & (k_i)^0 - (k_i)^3 \end{pmatrix} . \tag{23}$$

Using $\Lambda(A,B)$ defined by (19) and (C.1), but restricting ourselves to real values of λ, we find that

$$\lim_{\lambda \to \pm\infty} \Lambda(A,B) k_i^{(\pm)} = k_i ; \qquad i = 1, \cdots, 4 , \tag{24}$$

with k_i given by (18). Since $M^c(K)$ and $M^c\left(K^{(\pm)}\right)$, where the obvious notation $K_i^{(\pm)} = \left\{k_i^{(\pm)}, t_i\right\}$ is introduced, have both been assumed to lie on the same sheet, and since the holomorphy of $M(K)$ at the point k yields its continuity there, we have, using (10), (20), and (24),

$$\lim_{\lambda\to\pm\infty} M^c_{(\alpha)(\dot\beta)}\left(\Lambda(A,B)K^{(\pm)}\right) = M^c_{(\alpha)(\dot\beta)}(K)$$

$$= \lim_{\lambda\to\pm\infty} \exp\left[-\lambda\left(\sum_s \alpha_s - \sum_t \beta_t\right)\right] M^c_{(\alpha)(\dot\beta)}\left(K^{(\pm)}\right) \quad . \tag{25}$$

In particular, (25) implies that

$$M^c_{(\alpha)(\dot\beta)}\left(K^{(\pm)}\right) = M^c_{(\alpha)(\dot\beta)}(K) \qquad \text{if} \quad \sum_s \alpha_s = \sum_t \beta_t \ , \tag{26}$$

and, in order that the limiting procedure in (25) be consistent with the results for the $r = n = 2$ point k in (18) and (22), we must have

$$M^c_{(\alpha)(\dot\beta)}\left(K^{(+)}\right) = 0 \quad \text{if} \quad \sum_s \alpha_s < \sum_t \beta_t \ , \tag{27a}$$

$$M^c_{(\alpha)(\dot\beta)}\left(K^{(-)}\right) = 0 \quad \text{if} \quad \sum_s \alpha_s > \sum_t \beta_t \ . \tag{27b}$$

There is, however, no kinematical restriction on the values of the M^c-function components with $\sum_s \alpha_s > \sum_t \beta_t$ at the point $k^{(+)}$, and no such restriction on those with $\sum_s \alpha_s < \sum_t \beta_t$ at the point $k^{(-)}$. Either of Eqs. (27) is therefore sufficient to determine the number of kinematical zeroes at an $r = 2$, $n = 3$ point, this number being exactly half the number given by (22) for an $r = n = 2$ point.[38]

By adding up the number of M^c-function components not restricted by (22) or (27), we can get the number of M^c-function components whose values are independent at any $r = 2$ point on the I_+-saturated kernel. The result is given in Table I, which is actually valid for M^c functions with any number of particles, but for which at most four have spin. The last remark is a consequence of the fact that (22) and (27) are valid for any number of incoming and outgoing particles with arbitrary spins.[24]

Table I. Number of independent M^c-function components at an $r = 2$ point.

These results are valid for the case when at most four particles have spins, although the total number of particles can be arbitrary, and are valid on the I_+-saturated kernel of the domain of regularity. Here r = rank of Gram determinant. We take $j_1 + j_2$ and $j_3 + j_4$ both to be integers, with $j_1 + j_2 \geq j_3 + j_4$ and $j_1 \geq j_2$, $j_3 \geq j_4$.

	Number of linearly independent 4-vectors, n	
	$n = 2$	$n = 3$
Case I $j_1 - j_2 \geq j_3 + j_4$	$(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)$	$(j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)$
Case II $j_3 + j_4 \geq j_1 - j_2$ $j_1 - j_2 \geq j_3 - j_4$	$(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)$ $-\frac{1}{3}(j_2 + j_3 + j_4 - j_1)(j_2 + j_3 + j_4 - j_1 + 1)$ $\cdot (j_2 + j_3 + j_4 - j_1 + 2)$	$(j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)$ $-\frac{1}{6}(j_2 + j_3 + j_4 - j_1)(j_2 + j_3 + j_4 - j_1 + 1)$ $\cdot (j_2 + j_3 + j_4 - j_1 + 2)$
Case III $j_3 - j_4 \geq j_1 - j_2$	$(2j_4 + 1)[(2j_2 + 1)(2j_3 + 1)$ $-\frac{4}{3}j_4(j_4 + 1) - (j_2 + j_3 - j_4 - j_1)$ $\cdot (j_2 + j_3 + j_4 - j_1 + 1)]$	$(2j_4 + 1)[(j_1 + 1)(2j_2 + 1)(2j_3 + 1)$ $-\frac{2}{3}j_4(j_4 + 1) - \frac{1}{2}(j_2 + j_3 - j_4 - j_1)$ $\cdot (j_2 + j_3 + j_4 - j_1 + 1)]$

4. DISCRETE SYMMETRIES

A. Identical Particles, PCT, P, T, and C

Our present purpose is to discuss the restrictions on the number of independent M^c-function components at various points on the domain of regularity, in addition to those mentioned in the last section, if the functions have definite signature under the various discrete symmetry operations. Most of the results are well known, and our main goal is to find the results at points where the number of linearly independent four vectors, n, is 3, but the rank of the Gram determinant, r, is 2.[24]

Invariance under PCT and the connection between spin and statistics are both consequences of the basic principles of S-matrix theory[10] and of Field Theory.[34,39] In terms of the connected parts of the S matrix in canonical form, the PCT identity reads

$$\text{PCT} : S^c_{\alpha_b;\dot{\alpha}_a}(K_b; K_a) = \eta_{PCT} \prod_{i=1}^{4} D^{(j_i)}(C)_{\alpha_i}{}^{\alpha'_i} \, S^c_{\alpha'_a;\dot{\alpha}'_b}(\bar{K}_a; \bar{K}_b) , \tag{28}$$

where η_{PCT} is a phase factor. The spin-statistics connection states that exchanging the order of the momentum variables and spin components of any two identical initial or any two identical final particles of spin j changes the sign of the scattering function by $(-1)^{2j}$.

The assumption that transition probabilities are invariant under a change of direction of all spatial components of the four-momenta leads to the relation[40]

$$P : S^c_{\alpha_b;\dot{\alpha}_a}(K_b; K_a) = \eta_p \, S^c_{\alpha_b;\dot{\alpha}_a}(\tilde{K}_b; \tilde{K}_a) . \tag{29}$$

Here $\tilde{K}_i = \{\tilde{k}_i, t_i\}$, with $\tilde{k}_i = (k_i^0, -\underset{\sim}{k}_i)$, and $\eta_p = \pm 1$ is the "process intrinsic parity."[40,41] Provided all processes allowed by additive conservation laws occur in nature and are invariant under spatial inversion, the cluster decomposition law[42] enables one to assign a real "particle intrinsic parity" to each particle.[40] For a given process, η_ρ is then just the product of the "particle intrinsic parities" of the particles occurring in that process.

It is straightforward to show that if transition probabilities are invariant under time reversal, which involves exchanging initial and final states and changing the sign of all three vectors, one may write

$$T : S^c_{\alpha_b;\dot{\alpha}_a}(K_b; K_a) = \eta_T \, D^{(j_a)}(C)_{\alpha_a}{}^{\alpha'_a} \, D^{(j_b)}(C)_{\dot{\alpha}_b}{}^{\dot{\alpha}'_b} \, S^c_{\alpha'_a;\dot{\alpha}'_b}(\tilde{K}_a; \tilde{K}_b) \quad , \qquad (30)$$

where η_T is an arbitrary constant phase factor for $t_a \neq t_b$, but $\eta_T = +1$ is required for an elastic process, i.e., one in which $t_a = t_b$.

It may happen that PT is a symmetry of the scattering amplitude even if P and T are not. In this case, one has

$$PT : S^c_{\alpha_b;\dot{\alpha}_a}(K_b; K_a) = \eta_{PT} \, D^{(j_a)}(C)_{\alpha_a}{}^{\alpha'_a} \, D^{(j_b)}(C)_{\dot{\alpha}_b}{}^{\dot{\alpha}'_b} \, S^c_{\alpha'_a;\dot{\alpha}'_b}(K_a; K_b) \ , \qquad (31)$$

where $\eta_{PT} = +1$ when $t_a = t_b$.

When (31) is valid, the PCT identity (28) requires charge conjugation invariance,

$$C : \underset{\alpha_b;\, \dot{\alpha}_a}{S^c(K_b;\, K_a)} = \eta_C \underset{\alpha_b;\, \dot{\alpha}_a}{S^c(\bar{K}_b;\, \bar{K}_a)} , \tag{32}$$

where η_C is a constant phase factor.

The relevant symmetry relations for M^c functions with spinor indices of the types introduced in (4) and (6) follow easily from (28)-(32) and are given in Table II. These relations are valid over the entire domain of regularity of the M^c functions if they are valid in any real neighborhood of some physical region. Thus, if a given symmetry is valid for a certain physical process, it is valid for the processes related by crossing. Let us understand that by $-K_i$ we mean $\{-k_i, t_i\}$. Then the crossing relation[6,10] says, for example, that when all the k_i's take on real values with positive energies and when the point under consideration is on the boundary of the "physical" sheet,

$$\underset{\alpha_2\alpha_4;\, \alpha_1\alpha_3}{M^c(\bar{K}_2, K_4;\, K_1, \bar{K}_3)} = \eta_t \underset{\alpha_3\alpha_4;\, \alpha_1\alpha_2}{M^c(-K_3, K_4;\, K_1, -K_2)} \tag{33}$$

describes a physical process in which $\bar{t}_2$ represents an outgoing antiparticle of the particle t_2 and $\bar{t}_3$ an incoming antiparticle of the particle t_3. The constant phase factor η_t has been discussed elsewhere.[6,10]

Table II. Effect of invariance under various discrete symmetry operations on the M^c functions

N_a = number of fermions in state a

Symmetry	$M^c\underset{\alpha_b;\,\alpha_a}{(K_b;\,K_a)}$	$M^c\underset{\alpha_b;\,\dot\alpha_a}{(K_b;\,K_a)}$
PCT	$= (-1)^{N_a}\,\eta_{PCT}\,M^c\underset{\alpha_a;\,\alpha_b}{(\bar K_a;\,\bar K_b)}$	$= (-1)^{N_a}\,\eta_{PCT}\,M^c\underset{\dot\alpha_a;\,\alpha_b}{(\bar K_a;\,\bar K_b)}$ $= \eta_{PCT}\,D^{(j_a)}\left(C\,\frac{\tilde\sigma\cdot k_a}{m_a}\right)_{\dot\alpha_a}^{\alpha'_a} D^{(j_b)}\left(\frac{\sigma\cdot k_b}{m_b}\,C\right)_{\alpha_b}^{\dot\alpha'_b} M^c\underset{\alpha'_a;\,\dot\alpha'_b}{(\bar K_a;\,\bar K_b)}$
P	$= \eta_P\,M^c\overset{\dot\alpha_b;\,\dot\alpha_a}{(\tilde K_b;\,\tilde K_a)}$ $= \eta_P\,D^{(j_b)}\left(\frac{\sigma\cdot k_b}{m_b}\right)_{\alpha_b\dot\alpha'_b} D^{(j_a)}\left(\frac{\sigma\cdot k_a}{m_a}\right)_{\alpha_a\dot\alpha'_a} M^c\underset{\alpha'_b;\,\alpha'_a}{(\tilde K_b;\,\tilde K_a)}$	$= (-1)^{N_a}\,\eta_P\,M^c\overset{\dot\alpha_b;\,\alpha_a}{(\tilde K_b;\,\tilde K_a)}$ $= \eta_P\,D^{(j_b)}\left(\frac{\sigma\cdot k_b}{m_b}\right)_{\alpha_b\dot\alpha'_b} D^{(j_a)}\left(\frac{\sigma^T\cdot k_a}{m_a}\right)_{\dot\alpha_a\alpha'_a} M^c\underset{\alpha'_b;\,\dot\alpha'_a}{(\tilde K_b;\,\tilde K_a)}$
T	$= (-1)^{N_a}\,\eta_T\,M^c\overset{\dot\alpha_a;\,\dot\alpha_b}{(\tilde K_a;\,\tilde K_b)}$ $= (-1)^{N_a}\,\eta_T\,D^{(j_a)}\left(\frac{\sigma\cdot k_a}{m_a}\right)_{\alpha_a\dot\alpha'_a} D^{(j_b)}\left(\frac{\sigma\cdot k_b}{m_b}\right)_{\alpha_b\dot\alpha'_b} M^c\underset{\alpha'_a;\,\alpha'_b}{(\tilde K_a;\,\tilde K_b)}$	$= \eta_T\,D^{(j_a)}(C)_{\alpha_a}^{\alpha'_a}\,D^{(j_b)}(C)_{\dot\alpha_b}^{\dot\alpha'_b}\,M^c\underset{\alpha'_a;\,\dot\alpha'_b}{(\tilde K_a;\,\tilde K_b)}$
PT	$= (-1)^{N_a}\,\eta_{PT}\,M^c\underset{\alpha_a;\,\alpha_b}{(K_a;\,K_b)}$	$= (-1)^{N_a}\,\eta_{PT}\,M^c\underset{\dot\alpha_a;\,\alpha_b}{(K_a;\,K_b)}$ $= \eta_{PT}\,D^{(j_a)}\left(C\,\frac{\tilde\sigma\cdot k_a}{m_a}\right)_{\dot\alpha_a}^{\alpha'_a} D^{(j_b)}\left(\frac{\sigma\cdot k_b}{m_b}\,C\right)_{\alpha_b}^{\dot\alpha'_b} M^c\underset{\alpha'_a;\,\dot\alpha'_b}{(K_a;\,K_b)}$
C	$= \eta_c\,M^c\underset{\alpha_b;\,\alpha_a}{(\bar K_b;\,\bar K_a)}$	$= \eta_c\,M^c\underset{\alpha_b;\,\dot\alpha_a}{(\bar K_b;\,\bar K_a)}$

B. Functions With Definite Parity Signature

Even when spatial inversion is not a symmetry of the process under consideration, it can be useful to decompose the M^c functions for two incoming and two outgoing particles into parts having positive and negative parity signature. Let us recall from the discussion of Sec. 3 that, if the I_+-saturated kernel, $\mathcal{U}_a^{(I_+)}$, of a sheet $\mathcal{U}_a$ on the domain of regularity of such M^c functions contains a point $k = \{k_b; k_a\}$, it also contains the spatially inverted point $\tilde{k} = \{\tilde{k}_b; \tilde{k}_a\}$.[43] Consequently, the functions

$$M_{\pm}{}^{c}{}_{\alpha_b;\,\alpha_a}(K_b;\, K_a)$$

$$= \frac{1}{2}\left[M^c{}_{\alpha_b;\,\alpha_a}(K_b;\, K_a) \pm D^{(j_b)}\left(\frac{\sigma\cdot k_b}{m_b}\right)_{\alpha_b\dot{\alpha}_b'} D^{(j_a)}\left(\frac{\sigma\cdot k_a}{m_a}\right)_{\alpha_a\dot{\alpha}_a'} M^c{}_{\alpha_b';\,\alpha_a'}(\tilde{K}_b;\, \tilde{K}_a)\right] \tag{34}$$

are holomorphic everywhere on $\mathcal{U}_a^{(I_+)}$.

From (34), we may make the following decomposition on the I_+-saturated kernel of the domain of regularity:

$$M^c{}_{\alpha_b;\,\alpha_a}(K_b;\, K_a) = M_+{}^c{}_{\alpha_b;\,\alpha_a}(K_b;\, K_a) + M_-{}^c{}_{\alpha_b;\,\alpha_a}(K_b;\, K_a)\ . \tag{35}$$

With the aid of (B.8), we find from (34) that, for $\epsilon = \pm 1$,

$$M_\epsilon{}^c{}_{\alpha_b;\,\alpha_a}(K_b;\, K_a) = \epsilon\, D^{(j_b)}\left(\frac{\sigma\cdot k_b}{m_b}\right)_{\alpha_b\dot{\alpha}_b'} D^{(j_a)}\left(\frac{\sigma\cdot k_a}{m_a}\right)_{\alpha_a\dot{\alpha}_a'} M_\epsilon{}^c{}_{\alpha_b';\,\alpha_a'}(\tilde{K}_b;\, \tilde{K}_a)\ , \tag{36}$$

which means that the functions defined by (34) have definite signature under spatial inversion. By comparison of (35) and (36) with Table II, we see that, if spatial inversion symmetry is valid with $\eta_p = \pm 1$, then $M_{\mp\,\alpha_b;\,\alpha_a}(K_b;\,K_a) \equiv 0$.

The crossing property in (33) yields, with the aid of (34) and the fact that

$$D^{(j_i)}\left(\frac{-\sigma\cdot k_i}{m_i}\right) = (-1)^{2j_i}\, D^{(j_i)}\left(\frac{\sigma\cdot k_i}{m_i}\right) \quad ,$$

the relation

$$M^{c}_{\epsilon\;\alpha_2\,\alpha_4;\,\alpha_1\,\alpha_3}(\bar{K}_2,K_4;\,K_1,\bar{K}_3) = \eta_t\, M^{c}_{(-1)^{2j}\epsilon\;\alpha_3\,\alpha_4;\,\alpha_1\alpha_2}(-K_3,K_4;\,K_1,-K_2) \quad , \tag{37}$$

where $j = j_2 + j_3$. Thus, if a single fermion is crossed, the sign of the signature under spatial inversion is changed. In particular, this means that, when spatial inversion invariance holds, the "process intrinsic parities" of two processes related by crossing a single fermion have opposite signs.

At any $r = n = 3,2$, or 1 points, $\{k_b;\,k_a\}$ and $\{\tilde{k}_b;\,\tilde{k}_a\}$ lie on the same $\mathcal{L}_+$ orbit, so (36) and the covariance properties of the M^c_ϵ functions lead to a linear relation between the values of the components at such a point. This fact is well known, but we will review it, because our goal is to extend the analysis to $r = 2$, $n = 3$ points, in which case two points related by spatial inversion lie on different $\mathcal{L}_+$ orbits. Because of the $D^{(j_i)}\left(\frac{\sigma\cdot k_i}{m_i}\right)$

matrices that act on the $M_\epsilon{}^c$ functions under the parity operation defined by (36), it is much simpler to use the connected parts of the S-matrix elements to count independent components. The S^c functions have kinematical singularities arising from the "boost" matrices that relate them to the M^c functions, as in (4). These singularities, unlike those of the M^c functions, are not Lorentz invariant, so we can always find points on any orbit at which the S^c functions are analytic if the M^c functions are holomorphic.[44]

By application of the proper boost matrices to (34), we obtain

$$S_{\pm}{}^c\underset{\alpha_b;\,\dot\alpha_a}{(K_b;\,K_a)} = \frac{1}{2}\,[S^c\underset{\alpha_b;\,\dot\alpha_a}{(K_b;\,K_a)} \pm S^c\underset{\alpha_b;\,\dot\alpha_a}{(\tilde K_b;\,\tilde K_a)}] \ , \qquad (38)$$

where

$$S_\epsilon{}^c\underset{\alpha_b;\,\dot\alpha_a}{(K_b;\,K_a)} = \epsilon\, S_\epsilon{}^c\underset{\alpha_b;\,\dot\alpha_a}{(\tilde K_b;\,\tilde K_a)} \ . \qquad (39)$$

The last relation is equivalent to (29) when spatial inversion invariance holds.

To see the restrictions at $r = n = 3$ or 2 points, it is best to choose a point on a given orbit such that all spatial components are normal to the 2 axis; then a rotation of π about this axis carries $\{\tilde k_b;\,\tilde k_a\}$ into $\{k_b;\,k_a\}$. Since the matrix $A(k_i)$ in (A.33) corresponding to the Wigner rotation is equal to A when A is unitary, and since the required rotation matrix for spin j_i is now simply the matrix $D^{(j_i)}(C)$ in (A.29-A.30), (3) gives us, at the type of point indicated,

$$S_{\epsilon}{}^{c}\begin{pmatrix}K_b;\ K_a\\ \alpha_b;\ \dot{\alpha}_a\end{pmatrix} = \epsilon(-1)^{\sum_{i=1}^{4}(j_i-\alpha_i)}\ S_{\epsilon}{}^{c}\begin{pmatrix}K_b;\ K_a\\ -\alpha_b;\ -\dot{\alpha}_a\end{pmatrix} . \tag{40}$$

Equation (40) allows us to conclude that at any $r = 3$ point on the domain of regularity of the M^c functions, disregarding possible restrictions due to other symmetry operations, the number of $M_{\epsilon}{}^{c}$-function components whose values are independent is

$$N_{\epsilon} = \frac{1}{2}\prod_{i=1}^{4}(2j_i + 1) \quad , \tag{41a}$$

if there are fermions involved in the process. On the other hand, the number is

$$N_{\epsilon} = \frac{1}{2}\left[\prod_{i=1}^{4}(2j_i + 1) + \epsilon(-1)^{\sum_{i=1}^{4} j_i}\right] \quad , \tag{41b}$$

if all the particles are bosons.

In order to obtain the restrictions at $r = n = 2$ points we note that, in terms of the $S_{\epsilon}{}^{c}$ functions, (22) becomes

$$S_{\epsilon}{}^{c}\begin{pmatrix}K_b;\ K_a\\ \alpha_b;\ \dot{\alpha}_a\end{pmatrix} = 0 \qquad \text{if} \quad \alpha_1 + \alpha_2 \neq \alpha_3 + \alpha_4 \ . \tag{42}$$

Then (40) restricts the number of $M_{\epsilon}{}^{c}$ functions whose values are independent to exactly half the number allowed by (42) alone, if some of the particles are fermions, and to that number plus

$\frac{1}{2}\,\epsilon(-1)^{\sum_{i=1}^{4} j_i}$, if all the particles are bosons. The number allowed by (42) alone has already been given in Table I.

From remark H2(b) in Sec. 3, we know that an $r = 2$, $n = 3$ point is related to its spatial inverse only by an improper Lorentz transformation. However, on the I_+-saturated kernel of the domain of regularity, the limiting procedure of (25) must be valid for the M_ϵ^c functions, and we do get restrictions at $r = 2$, $n = 3$ points. In particular, in place of (26) we have

$$M_{\epsilon\ \alpha_b;\,\alpha_a}^{c}(K_b;\,K_a) = M_{\epsilon\ \alpha_b;\,\alpha_a}^{c}(K_b^{(\pm)};K_a^{(\pm)}), \quad \text{for} \sum_{i=1}^{4} \alpha_i = 0\ , \qquad (43)$$

if $\{k_b;\,k_a\}$ represents an $r = n = 2$ point of the form in (18) and $\{k_b^{(\pm)},\,k_a^{(\pm)}\}$ an $r = 2$, $n = 3$ point on the same sheet having the form in (23).

Similarly, in place of (27) we get

$$M_{\epsilon\ \alpha_b;\,\alpha_a}^{c}\left(K_b^{(+)};\,K_a^{(+)}\right) = 0 \quad \text{if} \sum_{i=1}^{4} \alpha_i < 0\ , \qquad (44a)$$

$$M_{\epsilon\ \alpha_b;\,\alpha_a}^{c}\left(K_b^{(-)};\,K_a^{(-)}\right) = 0 \quad \text{if} \sum_{i=1}^{4} \alpha_i > 0\ , \qquad (44b)$$

but there are no restrictions on the components not accounted for by (43) or (44).

By inspection of (43), we see that at the $r = 2$, $n = 3$ point under consideration, the values of the $M_\epsilon{}^c$-function components satisfying $\sum_i \alpha_i = 0$ are subject to the same restrictions due to the parity operation as they are subject to at the $r = n = 2$ limit point of the orbit, so the number of these that are independently determined is the same as the number mentioned in the sentence following (42). Adding to this the number of components not restricted by either (43) or (44), we find that the number of $M_\epsilon{}^c$-function components whose values are not subject to any linear restriction among themselves is exactly the same at any $r = 2$, $n = 3$ point on the I_+-saturated kernel of the domain of regularity as the number at any $r = 3$ point, given by either (41a) or (41b).

C. Restrictions in Special Cases

Invariance under any discrete symmetry other than spatial inversion can restrict the number of independent M^c-function components in the physical region of a process with two incoming and two outgoing particles only in special cases where the point in momentum space resulting from the symmetry operation lies on the same $\mathcal{L}_+$ orbit as the original point. We will consider the restrictions for $r = 3$ points by working in the physical region of a particular center-of-mass system with the 1 and 3 axes orientated as in Fig. 1 and with the 2 axis pointing out of the paper. The restrictions thus obtained are easily extended to all $r = 3$ points and the extension of the results to $r = 2$ points is straightforward.

For example, for a scattering process of the form $t_1 + t_1 \to t_3 + t_3$, the exchange symmetry for identical particles leads to a restriction, as may be seen by simultaneously exchanging the orders of the initial particles among themselves and of the final particles among themselves. Then, in the reference system of Fig. 1, a rotation of π about the 2 axis carries the transformed momenta back into the original orientation and we have

$$E : S^c\begin{pmatrix}K_3,K_4;\ K_1,K_2\\ \alpha_3,\alpha_4;\ \dot{\alpha}_1,\dot{\alpha}_2\end{pmatrix} = (-1)^{\sum_{i=1}^{4}(j_i+\alpha_i)} S^c\begin{pmatrix}K_3,K_4;\ K_1,K_2\\ -\alpha_4,-\alpha_3;-\dot{\alpha}_2,-\dot{\alpha}_1\end{pmatrix}, \quad \text{for } t_1 = t_2 \text{ and } t_3 = t_4 . \tag{45}$$

By crossing, one finds that the functions for the process $t_1 + \bar{t}_3 \to \bar{t}_1 + t_3$ have the same number of independent components as those allowed by (45). This restriction could also have been obtained by applying the PCT relation (28) in the new channel. For a process of the form $t + t \to t + t$ with $t = \bar{t}$, the PCT relation and the symmetry under the exchange of identical particles simultaneously lead to a restriction in the same channel. In the reference frame of Fig. 1 a rotation of π about the 3 axis carries the point on the right-hand side of (28) into that on the left and we obtain, in this frame,

$$PCT : S^c\begin{pmatrix}K_3,K_4;\ K_1,K_2\\ \alpha_3,\alpha_4;\ \dot{\alpha}_1,\dot{\alpha}_2\end{pmatrix} = S^c\begin{pmatrix}K_3,K_4;\ K_1,K_2\\ -\alpha_1,-\alpha_2;-\dot{\alpha}_3,-\dot{\alpha}_4\end{pmatrix},$$

$$\text{for } t_1 = t_2 = t_3 = t_4 = t = \bar{t} . \tag{46}$$

Time-reversal invariance restricts the number of independent components only in an elastic two-particle process. In the reference frame of Fig. 1, the required exchange of the four-momenta is brought about by a rotation of π about the 1 axis and (30) becomes

$$T : S^c_{\alpha_3,\alpha_4;\, \dot\alpha_1,\dot\alpha_2}(K_3,K_4;\, K_1,K_2)$$

$$= (-1)^{\alpha_1+\alpha_2-\alpha_3-\alpha_4}\, S^c_{\alpha_1,\alpha_2;\, \dot\alpha_3,\dot\alpha_4}(K_3,K_4;\, K_1,K_2), \qquad \text{for } t_1 = t_3 \text{ and } t_2 = t_4, \tag{47}$$

where we have put $\eta_T = +1$, as required for an elastic process when time-reversal invariance is valid.

The relation corresponding to (45), (46), and (47) for the parity operation has already been given in (40). By combining (40) with (47) one can get the restriction due to PT symmetry, as it appears in the frame of Fig. 1.

Equations (40) and (45)-(47) are sufficient to find all possible restrictions for a two-particle scattering process at $r = 3$ points. In Tables III, IV, and V, we have listed the number of independent $M_\epsilon{}^c$-function components having definite signature under the relevant symmetry operations, for those cases where spatial inversion is not the only possible symmetry that can give a restriction. In Tables IV and V we have allowed only those terms satisfying $(-1)^{2(j_1+j_3)} = 1$ when taking (45) into account. This is because the additive quantum numbers independent of the Lorentz group must be the same for

t_1 and t_3 in a process of the form $t_1 + t_1 \rightarrow t_3 + t_3$, and in nature all strongly interacting particles with half odd-integer spin apparently have odd baryon number, while those with integer spin have even baryon number.

Of course, when spatial inversion is a symmetry, for all processes in nature, only terms with $\epsilon = P = +1$ are nonvanishing in any of the tables,[40] and when time reversal is a symmetry for the elastic processes in Tables III and V only terms with $T = +1$ are nonvanishing. If neither P nor T is a symmetry, but PT is, then only terms with $PT = +1$ are nonvanishing in Tables III and V.[45]

Any process not accounted for in Tables III, IV, and V for which a discrete symmetry other than spatial inversion can restrict the number of independent M^c-function components is related through crossing to one in the table. For example, in the process $t_1 + \bar{t}_1 \rightarrow t_2 + \bar{t}_2$ charge conjugation invariance can yield a restriction, but this is exactly the same restriction given by PT symmetry in the process $t_1 + t_2 \rightarrow t_1 + t_2$.

Although Tables III, IV, and V were derived for $r = 3$ points, they also tell us how many independent M_ϵ^c-function components having a given symmetry there are at any $r = 2$, $n = 3$ point on the I_+-saturated kernel. This is because the restrictions due to exchange symmetry, PCT, T, and PT, are all a result of the symmetry of the scattering functions under permutations of certain variables. If such a restriction holds at $r = 3$ points, which form a dense subdomain of the domain of regularity, it holds everywhere. Before the symmetry under these permutations is considered, the number of $M_\epsilon{}^c$-function components whose values are independent is the same at all $n = 3$ points and the number must continue to be the same when the permutations are accounted for.

Table III. Number of independent $M_\epsilon^{\;c}$-function components having definite signature under T and PT for the process $t_1 + t_2 \to t_1 + t_2$.

These results hold at any point on the I_+-saturated kernel of the domain of regularity where there are three linearly independent four-momenta. Exceptional cases where exchange symmetry and/or PTC must give a restriction are: (1) $t_1 = t_2$; (2) $t_1 = \bar{t}_2$; (3) both $t_1 = \bar{t}_1$ and $t_2 = \bar{t}_2$. These exceptions are either given directly by the processes in Tables IV and V or related to them through crossing

(a) If at least one of the incoming particles is a fermion:

ϵ = P	T	PT	Number of terms
+	+	+	$\frac{1}{4}(2j_1+1)(2j_2+1)(4j_1j_2+2j_1+2j_2+3)$
+	-	-	$\frac{1}{4}(2j_1+1)(2j_2+1)(4j_1j_2+2j_1+2j_2-1)$
-	+	-	$\frac{1}{4}(2j_1+1)^2(2j_2+1)^2$
-	-	+	$\frac{1}{4}(2j_1+1)^2(2j_2+1)^2$

(b) If both particles are bosons:

ϵ = P	T	PT	Number of terms
+	+	+	$\frac{1}{4}[(2j_1+1)(2j_2+1)(4j_1j_2+2j_1+2j_2+3)+1]$
+	-	-	$\frac{1}{4}[(2j_1+1)(2j_2+1)(4j_1j_2+2j_1+2j_2-1)+1]$
-	+	-	$\frac{1}{4}[(2j_1+1)^2(2j_2+1)^2-1]$
-	-	+	$\frac{1}{4}[(2j_1+1)^2(2j_2+1)^2-1]$

Table IV. Number of independent M_ϵ^c-function components for the process $t_1 + t_1 \rightarrow t_2 + t_2$ with $t_1 \neq t_2$.

These results hold at any point on the I_+-saturated kernel of the domain of regularity where there are three linearly independent four-momenta. Since the additive quantum numbers that are independent of the Lorentz group must be the same for t_1 and t_2, such a process apparently occurs in nature only when t_1 and t_2 are both fermions or both bosons. The table is valid even when $t_1 = \bar{t}_1$.

(a) When particles t_1 and t_2 are fermions:	
$\epsilon = P$	Number of terms
+	$\frac{1}{4}(2j_1+1)(2j_2+1)(4j_1j_2+2j_1+2j_2+3)$
-	$\frac{1}{4}(2j_1+1)^2(2j_2+1)^2$
(b) When t_1 and t_2 are bosons:	
$\epsilon = P$	Number of terms
+	$(2j_1j_2+j_1+j_2+1)^2$
-	$(2j_1j_2+j_1+j_2)(2j_1j_2+j_1+j_2+1)$

Table V. Number of independent M_ϵ^c-function components having definite signature under T and PT for the process $t + t \to t + t$.

These results hold at any point on the I_+-saturated kernel of the domain of regularity where there are three linearly independent four-momenta.

(a) When particle t is a fermion:			
ϵ = P	T	PT	Number of terms
+	+	+	$\frac{1}{8}(2j + 1)(8j^3 + 12j^2 + 18j + 7)$
+	-	-	$\frac{1}{8}(2j + 1)(8j^3 + 12j^2 + 2j - 1)$
-	+	-	$\frac{1}{8}(2j + 1)^4$
-	-	+	$\frac{1}{8}(2j + 1)^4$

(b) When particle t is a boson. If, in addition $t = \overline{t}$, then only terms with PT = + are allowed by the PCT theorem.			
ϵ = P	T	PT	Number of terms
+	+	+	$2j^4 + 4j^3 + 6j^2 + 4j + 1$
+	-	-	$2j^2(j + 1)^2$
-	+	-	$j(j + 1)(2j^2 + 2j + 1)$
-	-	+	$j(j + 1)(2j^2 + 2j + 1)$

5. INVARIANT AMPLITUDES

A. The Theorem of Hepp and Williams

In the preceding sections we have considered the M^c functions for two incoming and two outgoing particles as functions of the four-momenta on the mass shell. For many purposes one would like to be able to describe the analytic properties in terms of the invariants formed from the four-momenta, and in this section we consider how this may be done. Because of the mass shell conditions and four-momenta conservation, one can form only two independent $\mathcal{L}_+$ invariants, which can be taken to be two of the three Mandelstam invariants: $s = (k_1 + k_2)^2$, $t = (k_1 - k_3)^2$, and $u = (k_1 - k_4)^2$.

Let us first consider a process for which all four particles are spinless; that is, one for which the M^c functions are $\mathcal{L}_+$ invariant:

$$M^c(K) = M^c(\Lambda K) . \tag{48a}$$

Hepp[11] and Williams[12] showed that such a function can be expressed as a holomorphic function of the independent $\mathcal{L}_+$ invariants on the I_+-saturated kernel of its domain of regularity. On that subdomain one may write

$$M^c(K) = A(s,t) . \tag{48b}$$

The domain of regularity of the function of $\mathcal{L}_+$ invariants on the right-hand side of (48b) is a union of sheets, each sheet being the image of the I_+-saturated kernel of some sheet of the original function.

Consequently, the relation (48b) is not valid at an $r = 2$, $n = 3$ point $k = \{k_b; k_a\}$ on the domain of regularity if there is a singularity at an $r = n = 2$ point that is a limit point of the $r = 2$, $n = 3$ orbit. Although this is a mathematically interesting complication, it is of no significance as far as physical considerations are concerned, since $r = 2$, $n = 3$ points are not physical, as mentioned in Sec. 3.[46]

B. Standard Covariants

We would like to generalize the preceeding considerations of this section to the cases where one or more of the four particles has spin. That is, we would like to be able to express the M_ϵ^c functions for any process with two incoming and two outgoing particles in terms of invariant functions that are holomorphic in the $\mathcal{L}_+$ invariants, s and t, everywhere on the I_+-saturated kernel of the domain of regularity. The spin dependence and, consequently, all the kinematical properties of the M_ϵ^c functions will be accounted for by globally holomorphic functions of the four-momenta referred to as "standard covariants." First we will precisely define the latter.

<u>Definition</u>. Consider the M^c functions for a scattering process involving two incoming particles of spins j_1 and j_2 and two outgoing particles of spins j_3 and j_4. A set of spinor functions $Y_{+1}^{(g)}(k)$, for $g = 1,\cdots,N_+$, and $Y_{-1}^{(g)}(k)$, for $g = 1,\cdots,N_-$, where

$N_+ + N_- = \prod_{i=1}^{4}(2j_i + 1)$, is said to be <u>a set of standard covariants</u>

for this process (and the processes related by crossing) if they satisfy the following five properties:

SC1. They are globally holomorphic functions of the four-momenta for the process, subject to the mass shell and four-momentum conservation constraints.

SC2. They are $\mathcal{L}_+$ covariant, i.e.,

$$Y_{\epsilon}{}^{(g)}{}_{\alpha_b;\,\alpha_a}\big(\Lambda(A,B)k\big) = D^{(j_b)}(A)_{\alpha_b}{}^{\alpha_b'}\, D^{(j_a)}(A)_{\alpha_a}{}^{\alpha_a'}\, Y_{\epsilon}{}^{(g)}{}_{\alpha_b';\,\alpha_a'}(k) \; . \tag{49}$$

SC3. They have definite signature under the spatial inversion operation:

$$Y_{\epsilon}{}^{(g)}{}_{\alpha_b;\alpha_a}(k) = \epsilon\, D^{(j_b)}\left(\frac{\sigma\cdot k_b}{m_b}\right)_{\alpha_b\dot{\alpha}_b'} D^{(j_a)}\left(\frac{\sigma\cdot k_a}{m_a}\right)_{\alpha_a\dot{\alpha}_a'} Y^{(g)}{}_{\alpha_b';\,\alpha_a'}(\tilde{k}) \; . \tag{50}$$

SC4. Those of the same signature ϵ under spatial inversion are linearly independent of each other in the space of the spinor indices at all $n = 3$ points. In other words, if we form the functions

$$\Gamma_{\epsilon}(k)_{\alpha_b;\,\alpha_a} = \sum_{g=1}^{N_\epsilon} \gamma_{\epsilon}{}^{(g)}\, Y_{\epsilon}{}^{(g)}{}_{\alpha_b;\,\alpha_a}(k) \quad , \tag{51}$$

where the $\gamma_\epsilon{}^{(g)}$'s are real or complex numbers, then at any $n = 3$ point $k = \{k_b;\, k_a\}$, the only solution to the equations $\Gamma_\epsilon(k)_{\alpha_b;\alpha_a} = 0$ for all values of $\{\alpha_b;\, \alpha_a\}$ is $\gamma_\epsilon{}^{(g)} = 0$, for all $g = 1,\cdots,N_\epsilon$.

SC5. If the number of M^c-function components whose values are independent is restricted by any discrete symmetry other than parity, then each of the spinor functions $Y_\epsilon^{(g)}(k)$ has definite signature, +1 or -1, under this operation, where the form of the symmetry operation is the same as that for the M^c functions in Table II.

The above properties of the standard covariants lead to the following lemma, the proof of which is given in Appendix D.

Lemma. Consider a set of standard covariants for the M^c functions describing a given two particle scattering process. Then,

L1. At any $r = 3$ point, the standard covariants of spatial inversion signature +1 are linearly independent of those of spatial inversion signature -1.

L2. For $\epsilon = +1$ or -1, N_ϵ is given by (41a) if there are some fermions involved in the process and by (41b) if all the particles are bosons.

L3. In those cases in which property SC5 holds, the number of standard covariants having a given signature under any of the applicable symmetry operations is in agreement with the number in Tables III, IV, and V.

The choice of a set of standard covariants for any process is by no means unique. If we have found a set $Y_\epsilon^{(g)}(k)$, for $g = 1,\cdots,N_\epsilon$, satisfying the required properties, and if we can write

$$Y_\epsilon^{(g)}(k) = \sum_{g'=1}^{N_\epsilon} f_\epsilon^{gg'}(s,t)\, \bar{Y}_\epsilon^{(g')}(k) , \tag{52}$$

such that the coefficients $f_\epsilon^{gg'}(s,t)$ are globally holomorphic functions of the Mandelstam invariants with $\det f_\epsilon^{gg'}(s,t)$ nowhere zero, then the $\overline{Y}_{\epsilon}^{(g)}(k)$'s also form a set of standard covariants. Obviously, there are an infinite number of suitable choices for the coefficients $f_{\epsilon}^{gg'}(s,t)$--in particular, when they are simply constants.

The same relations that change M-function spinor index types are valid for the standard covariants, as, for example, in (6). Thus, the use of all lower undotted spinor indices in this section, rather than some other choice, has no real significance.

C. Invariant Amplitudes for Scattering Functions With Spin

We are now ready to consider the possibility of expanding the M_{ϵ}^{c} functions for a given process in terms of standard covariants having the same discrete symmetry signatures, using the properties of the standard covariants in the definition and lemma of Part B of this section. The problem of actually constructing such standard covariants will be left for the following paper and we assume here that we already know a suitable set. Our results are expressed by the following theorem:

Theorem. Consider the M^{c} functions describing a process with two incoming particles of spins j_1 and j_2 and two outgoing particles of spins j_3 and j_4. Then, on the I_+-saturated kernel of the domain of regularity of the M^{c} functions one may write the following global decomposition involving the standard covariants for the process:

$$M_{\epsilon}^{c}(K)_{\alpha_b;\,\alpha_a} = \sum_{g=1}^{N_{\epsilon}} A_{\epsilon}^{(g)}(s,t)\, Y_{\epsilon}^{(g)}(k)_{\alpha_b;\,\alpha_a} \quad . \tag{53}$$

The "invariant amplitudes" $A_\epsilon^{(g)}(s,t)$, for $g = 1,\cdots,N_\epsilon$, are holomorphic functions of the Mandelstam invariants s and t everywhere on this domain. Those invariant amplitudes whose corresponding standard covariants in the sum (53) have the wrong signature under whatever discrete symmetry operations cause the number of independent M_ϵ^c-function components to be less than the number N_ϵ in (41a) or (41b) are identically zero, but there is no point at which some linear combination involving the remaining ones is required to vanish in order to assure that the M_ϵ^c functions have the kinematical behavior required by the considerations of Secs. 3 and 4.

The above theorem says that, given a set of standard covariants for the process under consideration, the decomposition (53) is possible such that the invariant amplitudes are free of kinematical singularities and zeroes. The absence of kinematical singularities is equivalent to saying that, if any of the invariant amplitudes has a singularity at a point in the space of the $\mathcal{L}_+$ invariants, then there is a point k with the same values of the invariants at which the M_ϵ^c functions are also singular. The absence of kinematical zeroes means that there is no point at which some linear combination of the values of the invariant amplitudes or of any of their derivatives is required to vanish in order to assure the correct kinematical behavior of the M_ϵ^c functions; kinematical zeroes are a complicating factor when one works with c.m. helicity amplitudes.[21] Thus, the only singularities and zeroes that occur in our invariant amplitudes are the dynamical ones that appear in the M_ϵ^c functions themselves. We are free to make any reasonable-looking dynamical approximation for the invariant amplitudes without destroying the covariance properties of the M_ϵ^c functions.

If the M_ϵ^c functions for a particular value of ϵ are identically zero, then (53) is trivial with vanishing $A_\epsilon^{(g)}$'s. In our proof of the theorem for nontrivial cases we will first completely ignore those cases in which the number of independent M_ϵ^c-function components is less than the number in (41a) or (41b) at $n = 3$ points. The necessary modifications for the exceptional cases will be easy to make.

For our proof, let us first try writing

$$M_\epsilon{}^c{}_{\alpha_b;\,\alpha_a}(K) = \sum_{g=1}^{N_\epsilon} \mathcal{a}_\epsilon^{(g)}(k)\, Y_\epsilon{}^{(g)}{}_{\alpha_b;\,\alpha_a}(k) \quad , \qquad (54)$$

which corresponds to (53), except that we regard the invariant amplitudes $\mathcal{a}_\epsilon^{(g)}(k)$ as functions of the four-momenta for the present. We will show that (54) is invertible; that is, we will solve for the amplitudes $\mathcal{a}_\epsilon^{(g)}(k)$ in terms of the M_ϵ^c functions and show that this does not introduce any singularities not present in the M_ϵ^c functions themselves. Finally, the theorem of Hepp[11] and Williams[12] allows one to re-express (54) in the form of (53). Remember that, according to the last paragraph, we are ignoring the exceptional cases for the present.

First we consider the scalars

$$\mathcal{Y}^{(g',g)}_{\epsilon',\epsilon}(k) = Y_{\epsilon'}{}^{(g')}(k)_{\alpha_b;\alpha_a} \; Y_{\epsilon}{}^{(g)}(k)^{\alpha_b;\alpha_a} \tag{55a}$$

and

$$\mathcal{M}^{(g)}_{\epsilon',\epsilon}(k) = M_{\epsilon'}{}^{c}(k)_{\alpha_b;\alpha_a} \; Y_{\epsilon}{}^{(g)}(k)^{\alpha_b;\alpha_a} \; . \tag{55b}$$

Each of the scalars defined by (55a) is holomorphic everywhere, and the determinant formed from them, det $\mathcal{Y}^{(g',g)}_{\epsilon',\epsilon}$, cannot vanish at any $r = 3$ point. The only way the determinant at such a point could vanish would be for some of the standard covariants to be linearly dependent, contradicting statements SC4 and L1. Since the standard covariants are not all linearly independent at an $r = 2$ point, det $\mathcal{Y}^{(g',g)}_{\epsilon',\epsilon}$ must vanish at such a point.

The determinant just introduced consists of four blocks, the upper left-hand one having components of the form $\mathcal{Y}^{(g',g)}_{+1,+1}$, the lower right-hand one $\mathcal{Y}^{(g',g)}_{-1,-1}$, the upper right-hand one $\mathcal{Y}^{(g',g)}_{+1,-1}$, and the lower left-hand one $\mathcal{Y}^{(g',g)}_{-1,+1}$. However, it follows from (50) that $\mathcal{Y}^{(g',g)}_{\epsilon,-\epsilon} = 0$, since one has an invariant function of three independent four-vectors that has negative signature under spatial inversion, and such a function vanishes identically. Consequently, the determinant is factorizable,

$$\det \mathcal{Y}^{(g',g)}_{\epsilon',\epsilon} = [\det \mathcal{Y}^{(g',g)}_{+1,+1}][\det \mathcal{Y}^{(g',g)}_{-1,-1}] \; , \tag{56}$$

and neither of the determinants det $\mathcal{Y}^{(g',g)}_{\epsilon,\epsilon}$, for $\epsilon = \pm 1$, can vanish at any $r = 3$ point. Since (56) is known to vanish at

$r = 2$ points, we must have $\det \mathcal{Y}_{\epsilon,\epsilon}^{(g',g)} \propto G^N(k_1,k_2,k_3)$, where k_1, k_2, and k_3 are any three of the momenta and N is some integer.

In (55b) $\mathcal{M}_{\epsilon,-\epsilon}^{(g)}$ vanishes identically for the same reason as $\mathcal{Y}_{\epsilon,-\epsilon}^{(g',g)}$ does, and from (54) and (55) we obtain the set of equations

$$\sum_{g'} \mathcal{A}_\epsilon^{(g')}(k)\, \mathcal{Y}_{\epsilon,\epsilon}^{(g',g)} = \mathcal{M}_{\epsilon,\epsilon}^{(g)}(k) \tag{57}$$

where, because of (55b), each of the functions $\mathcal{M}_{\epsilon,\epsilon}^{(g)}$ is holomorphic in the four-momenta on the domain of regularity of the M^c functions. At least at $r = 3$ points, where $\det \mathcal{Y}_{\epsilon,\epsilon}^{(g',g)}$ cannot vanish, (57) is soluble for the invariant functions $\mathcal{A}_\epsilon^{(g)}(k)$ in terms of the $\mathcal{M}_{\epsilon,\epsilon}^{(g)}(k)$'s. The solution of (57) has the form

$$\mathcal{A}_\epsilon^{(g)}(k) = \frac{\mathcal{R}_\epsilon^{(g)}(k)}{G^N(k_1,k_2,k_3)} , \tag{58}$$

where $\mathcal{R}_\epsilon^{(g)}(k)$ is some combination of the $\mathcal{Y}_{\epsilon,\epsilon}^{(g'g)}(k)$'s and the $\mathcal{M}_{\epsilon,\epsilon}^{(g)}(k)$'s in (55).

The numerator $\mathcal{R}_\epsilon^{(g)}(k)$ on the right-hand side of (58) is a holomorphic function of the four-momenta on the I_+-saturated kernel of the domain of regularity of the M^c functions. The only possible kinematical singularities on this domain in the invariant amplitudes on the left-hand side of (58) are poles where the rank of the Gram determinant is less than three--it turns out, as we will see below, that it is sufficient to consider only $r = 2$ points.

Let us look at (54) once more. We have already seen that the invariant amplitudes on the right-hand side of the equation are holomorphic at all $r = 3$ points on the domain of regularity of the M^c functions. Since these points form a dense subdomain of the domain of regularity, as we approach any $r = 2$ point on the I_+-saturated kernel the limit of the right-hand side of (54) must exist and be equal to the value of the left-hand side at the given $r = 2$ point. If the limit point is an $r = 2$, $n = 3$ point, the standard covariants are all linearly independent at the point and there can be no cancelling singularities in the invariant amplitudes; i.e., the limit at such a point must exist for each invariant amplitude in (58) separately and not just for the right-hand side of (54) as a whole.

The above considerations show that there exists some neighborhood of any given $r = 2$, $n = 3$ point on the I_+-saturated kernel such that $\mathcal{R}_\epsilon^{(g)}(k)$ in (58) has the form $\mathcal{R}_\epsilon^{(g)}(k) = \mathcal{X}_\epsilon^{(g)}(k)\, G^N$, where $\mathcal{X}_\epsilon^{(g)}(k)$ is holomorphic in the given neighborhood. From (58), $\mathcal{A}_\epsilon^{(g)}(k) = \mathcal{X}_\epsilon^{(g)}(k)$ is holomorphic in that neighborhood.

Now let us consider the limit in (54) and (58) as we approach any $r = n = 2$ point on the domain of regularity. If $\mathcal{R}_\epsilon^{(g)}(k)$ in (58) were to vanish more slowly than G^N at this point, there would be a pole there. One would then have, in some neighborhood of this point, $\mathcal{R}_\epsilon^{(g)}(k) = \mathcal{J}_\epsilon^{(g)}(k)\, G^M$ and $\mathcal{A}_\epsilon^{(g)}(k) = \mathcal{J}_\epsilon^{(g)}(k)/G^{N-M}$, where $M < N$ is some positive integer and $\mathcal{J}_\epsilon^{(g)}(k)$ is holomorphic and nonzero in the given neighborhood. This means that $G^{N-M}\mathcal{A}_\epsilon^{(g)}(k) = \mathcal{J}_\epsilon^{(g)}(k)$ is holomorphic and nonzero in that neighborhood, which, according to remark H2(c) in Sec. 3, contains points of every $r = 2$, $n = 3$ orbit with the same $\mathcal{L}_+$ invariants as the $r = n = 2$ orbit under consideration. We have already seen that there cannot be a pole in any of the invariant amplitudes at any $r = 2$, $n = 3$ point on the I_+-saturated kernel, so $G^{N-M}\mathcal{A}_\epsilon^{(g)}(k)$ must vanish at such a point, giving a contradiction. Consequently, there is no pole at any $r = n = 2$ point.

We may conclude that there are no kinematical singularities in the invariant amplitudes at any $r = 2$ or 3 point on the I_+-saturated kernel and that, at such points, we can express them as holomorphic functions of $\mathcal{L}_+$ invariants. Thus, there can remain only $r = 1$ points with which to be concerned, but, as we point out below, they present no problem.

The argument just given is in many respects similar to that of Williams,[12] who found a decomposition into invariant amplitudes for functions obtained by summing over the spinor indices with Clebsch-Gordan coefficients. We now point out, as Williams did, that it is impossible for our invariant amplitudes to have a singularity at any $r = 1$ point, since such points, at which $k_i \cdot k_j = \pm m_i m_j$, are isolated in the space of the invariants, and an analytic function of several complex variables cannot have isolated singularities.[47] Thus, our invariant amplitudes are holomorphic functions of $\mathcal{L}_+$ invariants everywhere on the I_+-saturated kernel of the domain of regularity of the M^c functions, and the decomposition (53) is possible on that domain; at least for those cases in which the number of independent M^c-function components is given by one of the Eqs. (41).

In the exceptional cases in which properties SC5 and L3 hold, the scalars $\mathcal{Y}_{\epsilon,\epsilon}^{(g',g)}$ in (55a) formed from standard covariants that have opposite signatures under any of the applicable discrete symmetries vanish identically. Then the determinant in (56) splits up further; that is, det $\mathcal{Y}_{\epsilon,\epsilon}^{(g',g)}$ for either value of ϵ, can itself be written as a product of smaller determinants. Furthermore, when the M_{ϵ}^{c} functions have definite signature under the symmetries in question, the scalars in (55b) involving standard covariants with different symmetry properties also vanish identically. The solution of (54) proceeds very much as before, except that now only the standard covariants having the correct symmetry properties need be used in the expansion, and the set of equations to be solved is of smaller order. Since the standard covariants continue to be linearly independent at $r = 2$, $n = 3$ points, which was the crucial factor in our previous proof, we have no kinematical singularities at points where the Gram determinant vanishes.

The point in the theorem regarding the absence of any kinematical zeroes in the invariant amplitudes is trivial, since the properties of the standard covariants guarantee that, on the I_+-saturated kernel of the domain of regularity, the right-hand side of (53) has the same kinematical properties as the left-hand side. It is also clear why we did not bother to analyze the restrictions that Lorentz covariance places on the derivatives of the M^c function when we analyzed the kinematical restrictions on their values at certain points in Secs. 3 and 4. The properties of the standard covariants given in SC1-SC5 automatically guarantee this behavior. One may therefore state that all the kinematical properties of the M_ϵ^c functions are in the standard covariants and all the dynamical properties in the invariant amplitudes.

By considering the crossing relation, as in (37), one can see that our standard covariants will automatically have the correct signature under the various discrete symmetry operations in the crossed channels. In particular, the spatial inversion signature will change when an odd number of fermions are crossed, for the same reason this happens for the M_ϵ^c functions, as discussed in connection with (37). Of course, the standard covariants chosen with the original channel in mind may not be the most natural ones for the crossed channels, particularly with regard to the calculation of pole terms and the angular momentum decomposition.

D. Comparison With c.m. Helicity Amplitudes

The several methods that have been employed to learn the nature of the kinematical singularities and zeroes of c.m. helicity amplitudes[21] all reduce essentially to considerations similar to our foregoing discussion. In an arbitrary reference frame one might write the connected parts of the helicity amplitudes for two incoming and two outgoing particles in the form

$$H^c_{\alpha_b;\dot{\alpha}_a}(K) = \sum_{\beta_b,\beta_a} \mathcal{H}^\ell_{\beta_b;\beta_a}(s,t)\, \mathcal{Z}^{\ell\,\beta_b;\beta_a}_{\alpha_b;\dot{\alpha}_a}(k) \quad , \tag{59}$$

where ℓ indicates the channel whose c.m. energy-squared is the Mandelstam invariant s,t, or u. Here α_a and α_b are the helicity indices, which undergo a unitary transformation that depends on the four-momenta when one goes from one Lorentz reference frame to another. On the right-hand side of (59), the transformation matrices act on these helicity subscripts, while β_b and β_a are labels distinguishing the various "invariant amplitudes" $\mathcal{M}^{\ell}_{\beta_b\beta_a}$ from one another and also the various "spin functions" $\mathcal{Z}^{\ell\beta_b;\beta_a}$. The latter are chosen such that, as the four-momenta approach their values in the center-of-mass system of the ℓ channel, one has

$$\lim_{k\to k^{c.m.}} \mathcal{Z}^{\ell\,\beta_b;\beta_a}_{\alpha_b;\alpha_a}(k) = \delta_{\alpha_b\beta_b}\,\delta_{\alpha_a\beta_a}\,.$$

The same boosts that relate the helicity amplitudes in an arbitrary frame to the M^c functions change the "spin functions" on the right-hand side of (59) into covariants $Z^{\ell\,\beta_b;\beta_a}_{\alpha_b;\alpha_a}(k)$, so that we obtain

$$M^c_{\alpha_b;\alpha_a}(K) = \sum_{\beta_b;\beta_a} \mathcal{M}^{\ell}_{\beta_b;\beta_a}(s,t)\; Z^{\ell\,\beta_b;\beta_a}_{\alpha_b;\alpha_a}(k)\,. \tag{60}$$

Equation (60) resembles (53), but, in contrast to the properties of the standard covariants in the earlier equation, the covariants on the right-hand side of (60) are not holomorphic functions everywhere and have a different kinematical behavior than the M^c functions. These unnatural properties are reflected by the presence of "kinematical" singularities and zeroes in the "invariant amplitudes" $\mathcal{H}^{\ell}_{\beta_b\beta_a}$, alias ℓ channel c.m. helicity amplitudes, a fact that becomes obvious when one attempts to invert (60) as we did for the expansion in (53).

6. SUMMARY AND DISCUSSION

In this paper we have discussed those properties of two-particle scattering amplitudes with spin that are relevant in obtaining invariant amplitudes free of kinematical singularities and zeros everywhere on the I_+-saturated kernel of the domain of regularity of the corresponding M^c functions. Even though we have used arguments based on analytic S-matrix theory to introduce many basic properties, our treatment of the kinematical properties of scattering amplitudes and the decomposition into invariant amplitudes is of general applicability.[48]

The main points that have been introduced or reviewed in this paper are the following:

(a) There is a dense subset of the domain of regularity of the M^c functions for a given scattering process such that, if this subset, referred to as the I_+-saturated kernel, contains a point in complex four-momenta space, it contains all points with the same invariants formed from the four-momenta.[8,11-13,20]

(b) At points on the I_+-saturated kernel where the rank of the Gram determinant, r, formed from the independent momenta in a scattering process is less than three, covariance under proper complex Lorentz transformations ($\mathcal{L}_+$-covariance), which is a consequence of Stapp's Theorem,[8,20] can restrict the number of M^c-function components whose values are independent to be less than the number at points where the rank of the Gram determinant is three.

(c) Invariance of a scattering process under spatial inversion limits the number of M^c-function components whose values are independent at points where the rank of the Gram determinant, r, is three or less. However, the number of independent components having a definite signature, +1 or -1, under spatial inversion is the same at all points in complex four-momenta space where the number of linearly independent four-momenta, n, is three, even though some of these points have the same invariants as other points where only two of the momenta are linearly independent.

(d) Invariance under any other discrete symmetry restricts the number of independent components only for a two-particle scattering process where the process resulting from its application is the same as the original one. The number of independent components with a given parity signature is the same at all points where there are three linearly independent momenta, including the exceptional points mentioned in (c).

(e) Given "standard covariants" satisfying the properties SC1 through SC5 in Sec. 5, Part B, one may express M^c functions with definite signature under spatial inversion as a sum involving these standard covariants and invariant amplitudes as in (53)--because of a theorem of Hepp[11] and Williams,[12] these invariant amplitudes are holomorphic functions of the Mandelstam invariants on the I_+-saturated kernel of the domain of regularity of the M^c functions. The invariant amplitudes are free of any kinematical zeroes.

(f) The difficulties arising from the use of c.m. helicity amplitudes are a result of the fact that one is actually doing a decomposition of the form (60), where the "Z-covariants" do not have the natural properties of the standard covariants in Sec. 5, Part B.

The considerations of (a) through (c) above are easily extended to the case of an arbitrary number of particles with spin, where the restrictions in (d) do not apply.[24]

Having completely avoided any use of arguments based on perturbation theory to obtain invariant amplitudes free of kinematical singularities and zeroes for a two-particle scattering process, we hope to have set forth a prescription in Sec. 5 that accomplishes for more general theories what Hearn[15] did for perturbation theory. Although Hepp's analysis[11] guarantees that the results obtained by the two prescriptions will agree,[19] we believe that our approach, particularly with regard to the specification of the properties of the standard covariants, is more relevant. The actual construction of the standard covariants will be discussed in the following paper.

The fact that our discussion involved spinor amplitudes is of no practical consequence, since anyone who prefers to use the S-matrix elements in the canonical or the helicity form has merely to apply the appropriate "boosts." These boosts affect only the standard covariants and leave the invariant amplitudes, which contain all the dynamics, unchanged. The use of spinor amplitudes serves to greatly simplify the considerations required to find invariant amplitudes, since in other cases the properties labelled SC in Sec. 5 would have to be modified to incorporate the complicated kinematical behavior of the S-matrix elements.

With regard to the content of Part D of Sec. 5, we note that the removal of kinematical singularities from and the classification of the constraints of c.m. helicity amplitudes has received considerable attention.[21] If one could succeed in finding combinations of such amplitudes from which all constraints can be removed, one would end up with invariant amplitudes of the same nature as those we considered in Part C of Sec. 5. However, the various types of singularities and constraints in the space of the scalar invariants with which one is involved in the helicity amplitude approach are not associated with the actual kinematical behavior of the scattering functions in the space of the four-momenta.[49] Our approach, on the other hand, is directly based on this behavior, which, as we saw in Secs. 3, 4, and 5, actually causes complications only at points where the rank of the Gram determinant is less than three. For real values of the four-momenta, such points lie on the boundaries of the various physical scattering regions.[50]

We conclude by mentioning that, although the M^c functions were introduced in Sec. 2 for stable particles that are observed in initial and final states in the laboratory, one is justified in introducing M^c functions with complex masses. The existence of such M^c functions appears to be essential in S-matrix theory if one is to correctly treat the singularities of the stable particle M^c functions on their unphysical sheets, particularly the poles and cuts lying close to the physical region and having experimentally observable effects.[51] Most functions involving particles of higher spin have complex masses associated with those particles and the analysis of this paper is applicable to them.

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APPENDIX A: SPINOR CALCULUS AND REAL LORENTZ TRANSFORMATIONS

We discuss here the group of two-dimensional unimodular matrices, SL(2,C), and its relation to the real proper Lorentz group.[52] In addition, some of the notational conventions employed in the main part of the paper are established.

It is easy to see that the replacement of every matrix $A \in SL(2,C)$ by A^*, by $A^{T^{-1}}$, or by $A^{\dagger^{-1}}$ yields a representation of the same group. Here we use the superscript T to indicate the transpose of any matrix and $\dagger$ to indicate the Hermitian conjugate. Introducing the matrix

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \tag{A.1}$$

it is easy to verify that

$$CAC^{-1} = A^{T^{-1}} , \tag{A.2}$$

so that there are actually only two inequivalent representations of SL(2,C). In particular, there is only one irreducible representation of the unitary unimodular two-dimensional matrices, which form the subgroup SU(2) of SL(2,C).

Any $A \in SL(2,C)$ may be written

$$A = HV , \tag{A.3}$$

where H is Hermitian and $V \in SU(2)$.

A two-dimensional spinor transforming under the matrix $A \in SL(2,C)$ may be written as ξ_α, with the transformation property

$$\xi_\alpha \to \xi'_\alpha = A_\alpha{}^\beta \xi_\beta . \qquad \text{(A.4a)}$$

We will write a spinor transforming under A^* as $\eta_{\dot\alpha}$, with the property

$$\eta_{\dot\alpha} \to \eta'_{\dot\alpha} = A^*{}_{\dot\alpha}{}^{\dot\beta} \eta_{\dot\beta} . \qquad \text{(A.4b)}$$

Let us now introduce

$$\xi^\alpha = C^{-1^{\alpha\beta}} \xi_\beta , \qquad \text{(A.5a)}$$

$$\eta^{\dot\alpha} = C^{-1^{\dot\alpha\dot\beta}} \eta_{\dot\beta} , \qquad \text{(A.5b)}$$

where, because of (A.1),

$$C^{-1^{\alpha\beta}} = C^{-1^{\dot\alpha\dot\beta}} = (-1)^{\frac{1}{2}-\alpha} \delta^{\alpha,-\beta} , \qquad \text{(A.6a)}$$

$$C_{\alpha\beta} = C_{\dot\alpha\dot\beta} = (-1)^{\frac{1}{2}+\alpha} \delta_{\alpha,-\beta} \qquad \text{(A.6b)}$$

are, respectively, the raising and lowering operators for spinor indices.

Then, from (A.2), (A.4), and (A.5),

$$\xi^\alpha \to \xi'^\alpha = A^{T^{-1}\alpha}{}_\beta \xi^\beta , \qquad \text{(A.7a)}$$

$$\eta^{\dot\alpha} \to \eta'^{\dot\alpha} = A^{\dagger^{-1}\dot\alpha}{}_{\dot\beta} \eta^{\dot\beta} . \qquad \text{(A.7b)}$$

We now consider the Hermitian Pauli matrices

$$\sigma_\mu = (\sigma_0, \underset{\sim}{\sigma}) , \tag{A.8}$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.9}$$

In addition we introduce

$$\tilde{\sigma}_\mu = C \, \sigma_\mu{}^T \, C^{-1} , \tag{A.10}$$

which yields

$$\tilde{\sigma}_\mu = \sigma^\mu = (\sigma_0, -\underset{\sim}{\sigma}) . \tag{A.11}$$

As is customary, tensor indices are raised and lowered by the metric tensor components $g^{\mu\nu} = g_{\mu\nu}$, with $g^{00} = +1$ and $g^{rr} = -1$ for $r = 1,2,3$, and $g^{\mu\nu} = 0$ otherwise. The convention for any four-vector x is that

$$x^\nu = (x^0, \underset{\sim}{x}) . \tag{A.12}$$

For any real four-vector x, we may define the Hermitian matrix

$$\sigma \cdot x = x^0 + \underset{\sim}{\sigma} \cdot \underset{\sim}{x} , \tag{A.13}$$

with

$$\det(\sigma \cdot x) = x^\mu x_\mu = (x^0)^2 - \underset{\sim}{x} \cdot \underset{\sim}{x} . \tag{A.14}$$

If we now consider $x'^{\mu} = \Lambda^{\mu}{}_{\nu}(A,A^*)\, x^{\nu}$, defined by

$$\sigma \cdot x' = A\, \sigma \cdot x\, A^{\dagger} = \sigma \cdot \Lambda(A,A^*)\, x \,, \tag{A.15}$$

for any $A \in SL(2,C)$, then $\sigma \cdot x'$ is also Hermitian, with

$$\det(\sigma \cdot x') = x'^{\nu} x'_{\nu} = \det(\sigma \cdot x) \,. \tag{A.16}$$

Consequently, $\Lambda(A,A^*)$ defined by (A.15) is a real Lorentz transformation. Since any $A \in SL(2,C)$ is continuously connected to the identity, $\Lambda(A,A^*) \in L_{+}^{\uparrow}$; that is, it is a proper orthochronous Lorentz transformation.

Equation (A.13) may be written as

$$x_{\alpha\dot{\beta}} = \sigma_{\alpha\dot{\beta}} \cdot x \,, \tag{A.17}$$

and, because of (A.15), we then have

$$x'_{\alpha\dot{\beta}} = A_{\alpha}{}^{\gamma} A^{*}{}_{\dot{\beta}}{}^{\dot{\delta}}\, x_{\gamma\dot{\delta}} = \sigma_{\alpha\dot{\beta}} \cdot \Lambda(A,A^*)x \,. \tag{A.18}$$

Thus, any four-vector is equivalent to a spinor transforming like the direct product $A \otimes A^*$.

Likewise, because of (A.10) and (A.15),

$$\tilde{\sigma} \cdot x' = A^{\dagger^{-1}}\, \tilde{\sigma} \cdot x\, A^{-1} = \tilde{\sigma} \cdot \Lambda(A,A^*)x \,, \tag{A.19}$$

and also

$$x^{\dot{\alpha}\beta} = \tilde{\sigma}^{\dot{\alpha}\beta} \cdot x \,. \tag{A.20}$$

It is clear that, for every $A \in SL(2,C)$ there is a $\Lambda \in L_+^\uparrow$; in fact, $\pm A$ give the same Λ. One may also show that for any $\Lambda \in L_+^\uparrow$ one can find $\pm A \in SL(2,C)$ such that (A.15) is satisfied.

Let k be any real four-vector on the mass shell; that is, $k^2 = m^2$, where $m \neq 0$ is the mass of the particle under consideration. Its rest-frame value is $\bar{k} = (m,0)$ and the "boost" $L(k)$ is the Hermitian matrix in $L_+^\uparrow$ defined by

$$k = L(k)\,\bar{k} . \tag{A.21}$$

We now define the Hermitian matrix $\mathcal{B}(k) \in SL(2,C)$ by

$$\mathcal{B}(k)\,\sigma\cdot\bar{k}\,\mathcal{B}^\dagger(k) = \sigma\cdot k = \sigma\cdot L(k)\,\bar{k} . \tag{A.22}$$

Thus we have

$$L(k) = \Lambda\Big(\mathcal{B}(k),\mathcal{B}^*(k)\Big) , \tag{A.23}$$

with

$$\mathcal{B}(k) = (\sigma\cdot k/m)^{\frac{1}{2}} = [2m(m+k^0)]^{-\frac{1}{2}}[m + k^0 + \underset{\sim}{\sigma}\cdot\underset{\sim}{k}] . \tag{A.24}$$

It may be verified that there always exist real parameters $\underset{\sim}{\phi}$ and $\underset{\sim}{\lambda}$ such that in (A.3) one may write

$$V = \exp(-i\,\underset{\sim}{\sigma}\cdot\underset{\sim}{\phi}/2) , \tag{A.25a}$$

$$H = \exp(\underset{\sim}{\lambda}\cdot\underset{\sim}{\sigma}/2) . \tag{A.25b}$$

Then, by the usual methods, we get representations of higher order with properties corresponding to (A.3) and (A.25). For any integer or half odd integer j we have the $2j+1$ by $2j+1$ matrices

$$D^{(j)}(A) = D^{(j)}(H)\, D^{(j)}(V) , \qquad (A.26)$$

with

$$D^{(j)}(V) = \exp\left(-i\, \underset{\sim}{\phi} \cdot \underset{\sim}{J}^{(j)}\right) , \qquad (A.27a)$$

$$D^{(j)}(H) = \exp\left(\underset{\sim}{\lambda} \cdot \underset{\sim}{J}^{(j)}\right) . \qquad (A.27b)$$

The matrices $\underset{\sim}{J}^{(j)}$ are the usual $2j+1$ by $2j+1$ matrices satisfying the commutation rules:

$$[J_r^{(j)}, J_s^{(j)}] = i \sum_t \epsilon_{rst}\, J_t^{(j)} . \qquad (A.28)$$

The matrix C in (A.1) and (A.6) can be written $C = \exp(-i\pi\, \sigma_2/2)$, and the generalization is

$$D^{(j)}(C) = \exp\left(-i\pi\, J_2^{(j)}\right) , \qquad (A.29)$$

or

$$D^{(j)}(C^{-1})^{\alpha\beta} = D^{(j)}(C^{-1})^{\dot{\alpha}\dot{\beta}} = (-1)^{j-\alpha}\, \delta^{\alpha,-\beta} , \qquad (A.30a)$$

$$D^{(j)}(C)_{\alpha\beta} = D^{(j)}(C)_{\dot{\alpha}\dot{\beta}} = (-1)^{2j}\, D^{(j)}(C^{-1})^{\alpha\beta} . \qquad (A.30b)$$

Corresponding to (A.2) and (A.10-11) we have

$$D^{(j)}(C)\, D^{(j)}(A)\, D^{(j)}(C^{-1}) = D^{(j)}(A^{T^{-1}}) , \qquad (A.31a)$$

$$D^{(j)}(C)\, \underset{\sim}{J}^{(j)*}\, D^{(j)}(C^{-1}) = -\underset{\sim}{J}^{(j)} . \qquad (A.31b)$$

The transformation matrices just introduced act on higher-order spinors, analogously to Eqs. (A.4) and (A.7) for the two-dimensional case.

Corresponding to any $\Lambda \in L_+^{\uparrow}$ and any real four-momentum k on the mass shell, one may define the "Wigner rotation"

$$R(k,\Lambda) = L^{-1}(\Lambda k)\, \Lambda L(k) , \qquad (A.32)$$

which is well known to those familiar with the unitary representations of the inhomogeneous Lorentz group. One may write $R(k,\Lambda) = \Lambda\big(A(k), A^*(k)\big)$, where $A(k) \in SU(2)$ is given by

$$A(k) = \mathcal{B}^{-1}(\Lambda k)\, A\, \mathcal{B}(k) . \qquad (A.33)$$

The generalization to the matrices $D^{(j)}\big(A(k)\big)$ is then obvious.

APPENDIX B: CLEBSCH-GORDAN COEFFICIENTS IN SPINOR FORM

For many applications, it is convenient to follow Williams[53] by expressing the Clebsch-Gordan coefficients in spinor form. One writes

$$(j_1\alpha_1\ j_2\alpha_2/j_1j_2j\alpha) = [j;\ j_1\ j_2]_{\alpha}{}^{\alpha_1\alpha_2} = [j;\ j_1\ j_2]_{\dot{\alpha}}{}^{\dot{\alpha}_1\dot{\alpha}_2}, \tag{B.1a}$$

$$= \{j;\ j_1\ j_2\}^{\alpha}{}_{\alpha_1\alpha_2} = \{j;\ j_1\ j_2\}^{\dot{\alpha}}{}_{\dot{\alpha}_1\dot{\alpha}_2}, \tag{B.1b}$$

where, on the left-hand side, we have employed the notation of Edmonds[54] for the Clebsch-Gordan coefficients. Because of the form of the raising and lowering operators in (A.30), one easily verifies that

$$[j;\ j_1\ j_2]^{\alpha}{}_{\alpha_1\alpha_2} = (-1)^{2j}\ [j;\ j_1\ j_2]_{\alpha}{}^{\alpha_1\alpha_2}. \tag{B.2}$$

To add n spin j indices to get total spin nj one defines by repeated addition, the generalized Clebsch-Gordan coefficients

$$[nj;\ j\cdots j]_{\alpha}{}^{\alpha_1\cdots\alpha_n} = [nj;\ (n-1)j\ j]_{\alpha}{}^{\beta\alpha_n}\ [(n-1)j;\ j\cdots j]_{\beta}{}^{\alpha_1\cdots\alpha_{n-1}}, \tag{B.3}$$

which are completely symmetric in their upper spinor indices.

The matrices $D^{(j)}(A)$ in (A.26) can be expressed in the form

$$D^{(j)}(A)_{\alpha}{}^{\beta} = [j;\ \tfrac{1}{2}\cdots\tfrac{1}{2}]_{\alpha}{}^{\alpha_1\cdots\alpha_{2j}}\{j;\ \tfrac{1}{2}\cdots\tfrac{1}{2}\}^{\beta}{}_{\beta_1\cdots\beta_{2j}}\ A_{\alpha_1}{}^{\beta_1}\cdots A_{\alpha_{2j}}{}^{\beta_{2j}}, \tag{B.4}$$

and the generalization of the matrix $\sigma \cdot k$ and (A.22) is

$$D^{(j)}(\sigma \cdot k/m)_{\alpha\dot{\beta}} = \left\{ D^{(j)}\left(\mathcal{B}(k)\right) D^{(j)\dagger}\left(\mathcal{B}(k)\right) \right\}_{\alpha\dot{\beta}} \tag{B.5a}$$

$$= [j; \tfrac{1}{2} \cdots \tfrac{1}{2}]_{\alpha}{}^{\alpha_1 \cdots \alpha_{2j}} [j; \tfrac{1}{2} \cdots \tfrac{1}{2}]_{\dot{\beta}}{}^{\dot{\beta}_1 \cdots \dot{\beta}_{2j}}$$

$$\sigma_{\alpha_1 \dot{\beta}_1} \cdot \frac{k}{m} \cdots \sigma_{\alpha_{2j} \dot{\beta}_{2j}} \cdot \frac{k}{m} , \tag{B.5b}$$

with $\mathcal{B}(k)$ defined by (A.24).

The generalization of (A.15) is

$$D^{(j)}\left(\sigma \cdot \Lambda(A,A^*)k/m\right)_{\alpha\dot{\beta}} = D^{(j)}(A)_{\alpha}{}^{\alpha'} D^{(j)}(A^*)_{\dot{\beta}}{}^{\dot{\beta}'} D^{(j)}(\sigma \cdot k/m)_{\alpha'\dot{\beta}'} . \tag{B.6}$$

Because of (A.11),

$$D^{(j)}(\tilde{\sigma} \cdot k/m)^{\alpha\dot{\beta}} = D^{(j)}(\sigma \cdot \tilde{k}/m)_{\alpha\dot{\beta}} , \tag{B.7}$$

where we use $\tilde{k} = (k^0, -\underset{\sim}{k})$. Furthermore,

$$D^{(j)}(\sigma \cdot k/m)\, D^{(j)}(\tilde{\sigma} \cdot k/m) = I . \tag{B.8}$$

The last relation is true because it is valid for the simplest case as a consequence of the relation

$$\sigma_\mu \tilde{\sigma}_\nu + \sigma_\nu \tilde{\sigma}_\mu = 2\, g_{\mu\nu} , \tag{B.9}$$

and the generalization follows from the construction in (B.5), or from the group properties.

APPENDIX C: COMPLEX LORENTZ TRANSFORMATIONS

In (A.13-14), let us replace the real four-vector x by a complex four-vector z. For any matrices A and $B \in SL(2,C)$ we define $z' = \Lambda(A,B)z$ by

$$\sigma \cdot z' = \sigma \cdot \Lambda(A,B)z = A\,\sigma \cdot zB^T , \tag{C.1}$$

and one has

$$\det(\sigma \cdot z') = \det(\sigma \cdot z) = z'^{\nu} z'_{\nu} = z^{\nu} z_{\nu} . \tag{C.2}$$

Thus $\Lambda(A,B) \in \mathcal{L}_+$, the group of proper complex Lorentz transformations with $\det \Lambda = 1$, a group that is continuously connected to the identity. The full complex Lorentz group is $\mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_-$, where any $\Lambda_I \in \mathcal{L}_-$ is an improper Lorentz transformation with $\det \Lambda_I = -1$. In contrast to the real Lorentz group, which has four components because the unit matrix, $I \in L_+^{\uparrow}$, and the simultaneous reflection of all four coordinate axes, $-I \in L_+^{\downarrow}$, are not related by any continuous transformation, the complex Lorentz group has only two components, $\mathcal{L}_+$ and $\mathcal{L}_-$, because I and $-I$ are connected by a continuous path in $\mathcal{L}_+$. For example, in (C.1) put

$$A = I \tag{C.3a}$$

$$B = \begin{pmatrix} \exp(i\psi/2) & 0 \\ 0 & \exp(-i\psi/2) \end{pmatrix} \tag{C.3b}$$

Then, as ψ varies continuously from 0 to 2π, z' changes continuously from z to $-z$.

Note that one can still define $D^{(j)}(\sigma\cdot k/m)$ for complex four-vectors k on the mass shell by means of (B.5b) and that (B.6) is replaced by

$$D^{(j)}\big(\sigma\cdot\Lambda(A,B)k/m\big)_{\alpha\dot{\beta}} = D^{(j)}(A)_{\alpha}{}^{\alpha'}\, D^{(j)}(B)_{\dot{\beta}}{}^{\dot{\beta}'}\, D^{(j)}(\sigma\cdot k/m)_{\alpha'\dot{\beta}'} \quad . \tag{C.4}$$

APPENDIX D: PROOF OF THE LEMMA IN SECTION 5

To prove statement L1 of the lemma, we will first assume that at a given $r = 3$ point, the standard covariants of signature +1 are not linearly independent of those of signature -1, and we will then show that this assumption is not consistent with property SC4 of the standard covariants. In other words, we assume that, for some $r = 3$ point k, there exists a set of nonzero γ_+'s and γ_-'s such that, for the functions defined by (51);

$$\Gamma_{+}(k)_{\alpha_b;\alpha_a} + \Gamma_{-}(k)_{\alpha_b;\alpha_a} = 0 , \tag{D.1}$$

for all choices of $\{\alpha_b; \alpha_a\}$.

But, since any $r = 3$ point k lies on the same $\mathcal{L}_+$ orbit as the point $\tilde{k}$ obtained by spatial inversion, there exists some $A_k \in SL(2,C)$ such that (49) and (51) give

$$\Gamma_{\epsilon}(k)_{\alpha_b;\alpha_a} = D^{(j_b)}(A_k)_{\alpha_b}{}^{\alpha'_b} D^{(j_a)}(A_k)_{\alpha_a}{}^{\alpha'_a} \Gamma_{\epsilon}(\tilde{k})_{\alpha'_b;\alpha'_a} . \tag{D.2}$$

After substituting (D.2) into (D.1) and multiplying each spinor index in the result from the left by $D^{(j_i)}(A_k^{-1})$, we get

$$\Gamma_{+}(\tilde{k})_{\alpha_b;\alpha_a} + \Gamma_{-}(\tilde{k})_{\alpha_b;\alpha_a} = 0 . \tag{D.3}$$

Thus, if (D.1) is valid for all $\{\alpha_b; \alpha_a\}$ at any $r = 3$ point, (D.3) is also true.

Because of (50) and (51),

$$\Gamma_{\epsilon}(k)_{\alpha_b;\alpha_a} = \epsilon\, D^{(j_b)}\left(\frac{\sigma\cdot k_b}{m_b}\right)_{\alpha_b\dot{\alpha}_b'} D^{(j_a)}\left(\frac{\sigma\cdot k_a}{m_a}\right)_{\alpha_a\dot{\alpha}_a'} \Gamma(\tilde{k})_{\alpha_b';\alpha_a'} \quad . \qquad (D.4)$$

If we multiply each spinor index in (D.3) from the left by $D^{(j_i)}\left(\frac{\sigma\cdot k_i}{m_i}\right)$ and then make use of (D.4), we get

$$\Gamma_{+}(k)_{\alpha_b;\alpha_a} - \Gamma_{-}(k)_{\alpha_b;\alpha_a} = 0 \ . \qquad (D.5)$$

The consistency of (D.1) and (D.5) requires that

$$\Gamma_{\epsilon}(k)_{\alpha_b;\alpha_a} = 0 \ , \qquad (D.6)$$

for both $\epsilon = +1$ and $\epsilon = -1$, for all choices of $\{\alpha_b;\ \alpha_a\}$. But, as mentioned after (51) in property SC4, the linear independence of the standard covariants of the same parity signature at any $n = 3$ point means that (D.6) cannot be true there for nonzero $\gamma_{\epsilon}^{(g)}$'s. Consequently, (D.1) cannot be true and statement L1 of the lemma is valid.[55]

We now consider statement L2 of the lemma. First note that each component $Y_{\epsilon}^{(g)}(k)_{\alpha_b;\alpha_a}$, for a fixed value of g, but different values of $\{\alpha_b;\ \alpha_a\}$, is actually a different function. However, as was the case for the M_{ϵ}^{c} functions in Sec. 4, (50) means that at most N_{ϵ} of their values, where N_{ϵ} is given by (41.a) or (41.b), whichever is appropriate, can actually be chosen independently at any $r = 3$ point. Thus, the number of standard covariants $Y_{\epsilon}^{(g)}(k)$ that are linearly independent for a given ϵ cannot be greater than N_ϵ. Since $N_+ + N_- = \prod_{i=1}^{4}(2j_i + 1)$, and since, according to L1, the covariants of signature +1 are linearly independent of those of signature -1, the number must be N_ϵ.

It remains to prove statement L3 of the lemma, which applies to those special cases in which the number of independent M_ϵ^c-function components at an $r = 3$ point can be less than the number in (41a) or (41b). Property SC5 says that the standard covariants are symmetric or antisymmetric under each of the applicable symmetries, and we want to show that the results agree with Tables III, IV, and V for the number of independent M_ϵ^c-function components having such symmetry properties.

If, for example, $m_1 = m_3$ and $m_2 = m_4$, we require that the standard covariants have definite signature under the simultaneous exchanges $(k_1,\alpha_1) \longleftrightarrow (k_3,\alpha_3)$ and $(k_2,\alpha_2) \longleftrightarrow (k_4,\alpha_4)$, which is the same as the PT operation for the M^c functions in Table II. Then the same considerations that led to Table III tell us how many standard covariants at most can have a particular signature under this operation, and their linear independence at $r = 3$ points means that this equals the actual number. Note that, by choosing our covariants to have definite PT signature, we automatically assure that they have definite signature under T. The above considerations are easily extended to the case that $m_1 = m_2$ and $m_3 = m_4$, when the covariants are chosen to have definite signature under the simultaneous exchanges $(k_1,\alpha_1) \longleftrightarrow (k_2,\alpha_2)$ and $(k_3,\alpha_3) \longleftrightarrow (k_4,\alpha_4)$, and to the case when all masses are equal, when definite signature under both types of exchanges mentioned in this paragraph is chosen.

FOOTNOTES AND REFERENCES

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+ This paper is based on a Ph.D. Thesis from Yale University.

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30. More rigorous definitions are to be found in Refs. 12 and 20. See also A. S. Wightman, in Dispersion Relations and Elementary Particles, edited by C. de Witt and R. Omnes (John Wiley and Sons, Inc., New York, 1960), p. 227.

31. In its most general form, Stapp's Theorem guarantees $\mathcal{L}_+$ covariance on the entire multisheeted domain of regularity of a set of tensor-valued functions originally defined to be holomorphic and either $L_+^\uparrow$-covariant or $\mathcal{L}_+$-covariant on some arbitrarily small real or complex domain that does not have to be on the mass shell. It is a generalization of earlier work by Hall and Wightman, (Ref. 13), Jost, (Ref. 32), and Wightman (Refs. 33,34) who showed that functions originally defined to be holomorphic and $L_+^\uparrow$-covariant on the so-called "future tube," such as the Wightman Functions in axiomatic field theory, have a single-sheeted $\mathcal{L}_+$-covariant extension to the so-called "extended tube," which is the union of the $\mathcal{L}_+$ orbits of all points on the "future tube."

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35. This statement is actually true only when the number of interacting particles described by the M^c functions is four. If the number of particles is greater than four, there is an infinite number of $r = 2$, $n = 3$ orbits having the same $\mathcal{L}_+$ invariants as a given $r = n = 2$ orbit.

36. The vanishing of a linear combination, with constant coefficients, of the components in some full neighborhood of the domain of regularity would not be compatible with Lorentz covariance. A general discussion of linear relationships for physical values of the four-momenta, in terms of helicity amplitudes, is given by J. Daboul, Linear Symmetries of Scattering Amplitudes, Temple University preprint, November 1969.

37. We stress once more that we mean the number of M^c-function components whose values are not subject to any linear relation of kinematical origin at the given point. The M^c functions themselves, as already mentioned, are independent functions (i.e., their derivatives at the given point cannot all satisfy the same linear relationship that their values do).

38. Our discussion of the kinematical behavior is not really complete because we have not discussed the restrictions that Lorentz covariance places on the derivatives of the M^c functions at $r = 2$ points. However, the considerations here turn out to be sufficient for specifying the properties of the "standard covariants" introduced in Sec. 5 when the decomposition of the M^c functions into invariant amplitudes is considered.

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42. See the references mentioned in footnote 28.

43. This remark is not necessarily true for functions involving more than four particles, in which case the rank of the Gram determinant, r, formed from the four-momenta is 4 on a dense subdomain of the domain of regularity. When $r = 4$, k and $\tilde{k}$ lie on different $\mathcal{L}_+$ orbits and have pseudoscalar invariants of opposite sign, and Stapp's Theorem does not then guarantee that, if a given $\mathcal{U}_a^{(I_+)}$ contains the point k, it also contains the point $\tilde{k}$.

44. Note that, because of the kinematical branch points in the "boost" matrices, the mapping from the M^c functions to the S^c functions at any point on the domain of regularity of the former is not single-valued. This fact is not of any importance for the application we have in mind here.

45. We call attention to the fact that in Table III the total number of terms is $(2j_1 + 1)^2(2j_2 + 1)^2$, in agreement with (16), but the total number is less in Tables IV and V, because the basic postulates of S-matrix theory, or of field theory, forbid the occurrence of terms with the wrong signature under exchange symmetry and under PCT.

46. In its more general form, the theorem of Hepp and Williams says that one can express any multisheeted invariant function of any number of four-vectors (regardless of whether or not mass-shell constraints are present) as a function of the independent $\mathcal{L}_+$ invariants on the I_+-saturated kernel of its domain of

regularity. However, with more than four functionally independent vectors (e.g., a scattering amplitude with more than five particles), it is not possible to use the same set of $\mathcal{L}_+$ invariants globally; in other words one must use "local" $\mathcal{L}_+$-invariant coordinates. The result of Hepp and Williams is a generalization to arbitrary domains of a theorem of Hall and Wightman (Ref. 13) that enables one to express an $\mathcal{L}$-invariant single-sheeted function regular everywhere on the "extended tube" as a function of "local" $\mathcal{L}$-invariant coordinates (see also note 31).

47. See any book on several complex variables, e.g., S. Bochner and W. T. Martin, Several Complex Variables (Princeton University Press, Princeton, 1948), p. 71.

48. A comparison of the S-matrix and field-theoretical approaches is given by R. Seiler in "On the Covariance and Analyticity of Scattering Amplitudes," University of Pittsburgh preprint No. NYO-3829-26. [To be published in Lectures in Theoretical Physics (Gordon and Breach), vol. XI.]

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52. Further information on the matters discussed here can be obtained from Refs. 3, 6, 9, 12, and 34. See also A. S. Wightman in Dispersion Relations and Elementary Particles, edited by C. de Witt and R. Omnes (John Wiley and Sons, Inc., New York, 1960), p. 159.

53. D. N. Williams, in Lectures in Theoretical Physics, edited by W. E. Brittin and A. O. Barut (University of Colorado Press, Boulder, Colorado, 1965), vol. VII A, p. 139. Also see Ref. 12.

54. A. Edmonds, Angular Momentum in Quantum Mechanics (Princeton University Press, Princeton, New Jersey, 1957).

55. It is worth noting that if k is an $r = 2$, $n = 3$ point, then $\tilde{k}$ lies on a different $\mathcal{L}_+$ orbit, according to H2.(b), and (D.2) is not valid for any choice of $A_k \in SL(2,C)$. In this case it is possible to satisfy (D.1) with $\Gamma_+(k)_{\alpha_b;\alpha_a} \neq 0$ and $\Gamma_-(k)_{\alpha_b;\alpha_a} \neq 0$, in contrast to the result (D.6) for an $r = 3$ point. The standard covariants of signature +1 are not linearly independent of those of signature -1 at any $r = 3$, $n = 3$ point; the total number of standard covariants in the two sets that are linearly independent at such a point is the same as the number given in Table I.

FIGURE CAPTIONS

Fig. 1. Special reference frame used for obtaining discrete symmetry restrictions. The 2 axis points out of the paper.

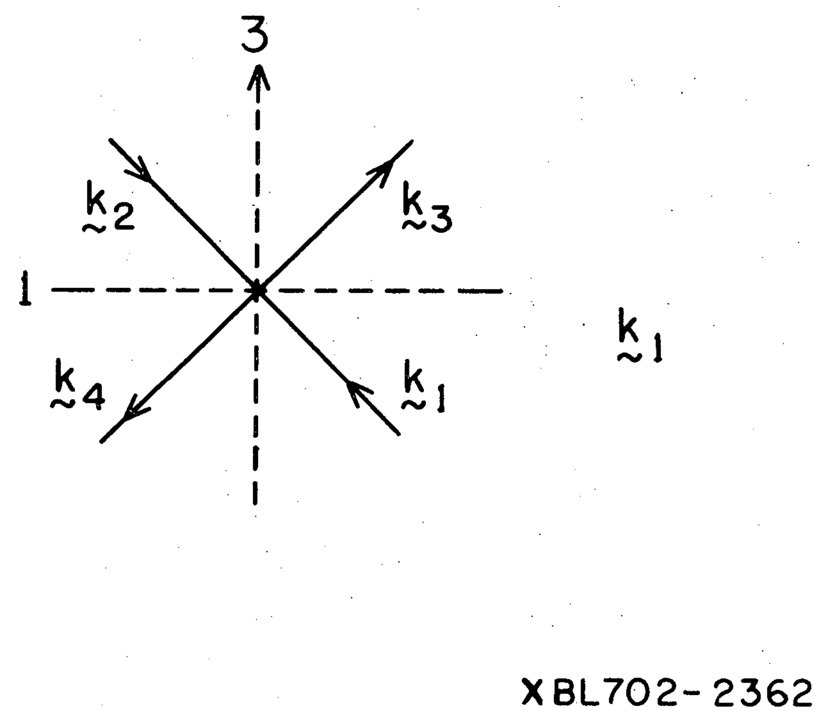

Fig. 1

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