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UNIVERSITY OF CALIFORNIA
Radiation Laboratory
Berkeley, California

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LECTURES ON "CONTINUOUS GROUPS AND REFLECTIONS IN QUANTUM MECHANICS" BY W. PAULI

Lecture 1

R. J. Riddell, Jr.

March 17, 1958

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LECTURES ON "CONTINUOUS GROUPS AND REFLECTIONS IN QUANTUM MECHANICS" BY W. PAULI

Notes by R. J. Riddell, Jr.

Radiation Laboratory
University of California
Berkeley, California

Spring 1958

LECTURE 1

Continuous Groups and Reflections in Quantum Theory

Interesting subjects:

a. Lorentz gr.

b. Canonical gr.

c. Linear Canonical transformations (Symplectic Gr.)

For (a), the spinors will be interesting, and the more recent applications concern the neutrinos. Also:

Majorana spinors

Charge conjugation - discrete gr.

Space reflections (parity) - discrete gr.

We will discuss the 2 x 2 matrices introduced by Gursey.

Then comes:

Isotopic spin

Quantization questions

Vacuum expectation values

Weak interactions

Analytic continuation (Wightman)

C P T theorem.

We will use free particles but only as examples, preparatory to the the interacting case.

Elementary properties of 2 x 2 matrices

We will have, in general,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

and now consider:

$$\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad ; \qquad \tilde{\omega} = -\omega, \quad \omega^2 = -1, \quad \omega^{-1} = -\omega.$$

Now:

$$D \equiv \det A = A_{11} A_{22} - A_{12} A_{21}$$

but:[1]

$$\omega A \omega^{-1} \equiv \bar{A} = D A^{-1} .$$

Thus the inverse can be specified as a rational construct of A if $D \neq 0$.

Also:[2]

$$A \omega \tilde{A} = D \omega$$

Remark on canonical transforms

Consider the pairs of variables $p_i, q_i \equiv X_{2i-1}, X_{2i}$

$$(p_1, q_1), \ldots, (p_n, q_n) \equiv (X_1, X_2, \ldots, X_{2n})$$

and we will treat the linear canon transforms of these variables.

Define:

$$\Omega = \begin{pmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 & \cdots \\ 0 & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 \cdots \\ \vdots & \vdots & \end{pmatrix}$$

A canonical transform will leave Ω invariant:[3]

Let: $X_i' = X_k S_{ki}$

Then: $S \Omega S = \Omega$

(If we had required orthogonality instead, we would omit the Ω and get $S\tilde{S} = 1$.)

Now consider:

$$X_i' = X_K S_{Ki}$$

$$y_i' = y_K S_{Ki}$$

and:

$$X' \Omega y' = X S \Omega S y = X \Omega y$$

Thus the form is invariant. $X \Omega y \equiv \sum_i \left[p_i^{(2)} q_i^{(1)} - p_i^{(1)} q_i^{(2)} \right]$

If we specialize to $n = 1$, then the condition is equivalent to $D = 1$. Thus a linear canonical transform is equivalent to a unimodular transform. For $n \neq 1$, D is still necessary, but not sufficient for the canonical form. This is clear by taking the det of eqn. det S = det $\tilde{S}$. This only proves $D = \pm 1$. The Liouville theorem says $D = +1$, and it can be proved independent of continuity arguments. There is a difference in the orthogonal group (reflections).

Spinors

Let us now introduce:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Except for an "i", these are the quaternions.

$$\sigma_1 \sigma_2 = i\sigma_3, \ldots; \qquad \sigma_i^2 = 1$$

The relations are retained under a unitary transform. Here: $\omega = -i\sigma_2$. In other representations, however, ω is invariant so the relation is only specially true. We also introduce:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_4 = i\sigma_0 \qquad \sigma_4^2 = -1.$$

Other relations:

$$\sigma_1\sigma_2 = -\sigma_2\sigma_1 \quad ,\ldots$$

[Note: $D\left\{\sigma_i\right\} = -1 \qquad i = 1,\ldots, 4.$

Thus:

$$\sigma_i = -\omega\tilde{\sigma}_i\omega^{-1} \qquad \tilde{\sigma}_i = \omega\sigma_i\omega^{-1} \qquad = 1, 2, 3$$

while
$$\sigma_{0,4} = +\omega\sigma_{0,4}\omega^{-1} \quad .]$$

Unitary Transforms.

U corresponds to the rotation gr. in 3-dimensions.

Now consider:

$$U(\alpha) = e^{i\alpha\sigma_i} = \cos\alpha + i\sigma_i \sin\alpha$$

(Expanding $e^{i\alpha\sigma_i}$ and using $\sigma_i^2 = 1$).

Here: $U(\alpha_1 + \alpha_2) = U(\alpha_1)\, U(\alpha_2)$.

Thus we have the one-dimension rotation group about the i axis.

(The connection with coordinates will come in later.)

U is of course unitary: $U\, U^+ = e^{i\alpha\sigma_i}\, e^{-i\alpha\sigma_i} = 1.$

$$U^+ = \cos\alpha - i \sin\alpha\cdot\sigma_i$$

If we want to rotate about an arbitrary axis, $\hat{n}$:[4]

$$U = e^{i\alpha(\hat{n}\cdot\vec{\sigma})} = \cos\alpha + i\sin\alpha\ (\vec{n}\cdot\vec{\sigma})$$

Thus: $\quad \mathrm{Det}\ U = 1$

since $\vec{n}\cdot\vec{\sigma}$ is invariant on rotations.[5]

To get $\mathrm{Det}\ U \neq 1$, we can introduce an arbitrary phase factor:

$$U' = e^{i\beta}\, U$$

If: $\left|\det U\right|^2 = 1$

$$\det U = e^{i\beta}$$

The most general U is:

$$U = e^{i\alpha(\hat{n}\cdot\vec{\sigma}) + i\beta}$$

FOOTNOTES

1. $$\omega\tilde{A}\omega^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -A_{21} & A_{11} \\ -A_{22} & A_{12} \end{pmatrix} = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

But:

$$\omega A \omega^{-1} A = \begin{pmatrix} A_{11}A_{22}-A_{12}A_{21} & 0 \\ 0 & A_{11}A_{22}-A_{12}A_{21} \end{pmatrix} = D \qquad \text{Q.E.D.}$$

2. $$A\omega\tilde{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} +A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -A_{11}A_{22} \quad A_{12}A_{21} \\ -A_{12}A_{21} + A_{11}A_{22} & 0 \end{pmatrix} = \omega D\{A\}$$

3. We have, as eqns. of motion:

$$\frac{dx_i}{dt} = \Omega_{iK} \frac{\partial H}{\partial x_K} \qquad \left[\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad , \quad \dot{q}_i = \frac{\partial H}{\partial p_i}\right]$$

and:

$$\frac{dx_i'}{dt} = \frac{dx_K}{dt} S_{Ki} = \Omega_{Ki} \frac{\partial H}{\partial x_\ell} S_{Ki}$$

$$= \Omega_{Ki} S_{Ki} \frac{\partial H}{\partial x'_m} \cdot \frac{\partial x'_m}{\partial x_\ell} = (\tilde{S}_{iK} \Omega_{K\ell} S_{\ell m}) \frac{\partial H}{\partial x'_m}$$

FOOTNOTES

3. (Cont)

$$\therefore \quad \Omega = \tilde{S}\,\Omega\,S$$

But:

$$\tilde{\Omega} = \tilde{S}\,\Omega\,S$$

$$\tilde{\Omega}^{-1} = S^{-1}\,\tilde{\Omega}^{-1}\,\tilde{S}^{-1} \qquad\qquad \tilde{\Omega}^{-1} = \Omega$$

$$\therefore \quad \Omega = S^{-1}\,\Omega\,\tilde{S}^{-1}$$

so: $\Omega = S\,\Omega\,\tilde{S}$

4. $$(\vec{n}\cdot\vec{\sigma})^2 = (n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)^2$$

$$= n_x^2 + n_y^2 + n_z^2 + n_x n_y \underbrace{(\sigma_x\sigma_y + \sigma_y\sigma_x)}_{\substack{11\\0}} + \ldots, = 1$$

5. Or:

$$U = \begin{pmatrix} \cos\alpha + i\sin\alpha\cdot n_z & i\sin\alpha\,(n_x - in_y) \\ i\sin\alpha\,(n_x + in_y) & \cos\alpha - i\sin\alpha\cdot n_z \end{pmatrix}$$

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Lecture 2

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Lecture 2

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Notes by R. J. Riddell, Jr.

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Berkeley, California

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LECTURE 2

Connection between the rotation group and Lorentz group with transformations of 2 complex variables.

The Lorentz group is a representation of the unimodular group of 2 x 2 complex matrices.

Consider:

$$\xi_\alpha' = \xi_\beta A_{\beta\alpha} \qquad A_{\beta\alpha} = \text{complex}$$

This will be isomorphic to the Lorentz group. Det A = 1. The subgroup of unitary transforms: $A\, A^+ = A^+\, A = 1$ is isomorphic to the 3-dimensional rotations.

The connection is most simply illustrated by considering a null-vector:

$$r^2 - t^2 = 0$$

Then we define:[1]

$$\frac{x - iy}{t + z} = \frac{t - z}{x + iy} = \frac{\xi_2}{\xi_1}$$

$$\frac{x + iy}{t + z} = \frac{t - z}{x - iy} = \frac{\xi_2^*}{\xi_1^*}$$

Then:

$$x - iy = C\xi_1^*\xi_2 \qquad x + iy = C\xi_1\xi_2^*$$

$$t + z = C\xi_1^*\xi_1 \qquad t - z = C\xi_2^*\xi_2$$

Now, we introduce the matrix

$$\chi_{\alpha\beta} = c\,\xi_\alpha^* \,\xi_\beta$$

$$\chi = \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix} = \chi^+$$

(Note that ξ^* is treated as a column vector; ξ, a row.)

Evidently:[2]

$$\det X = 0$$

which gives

$$|\vec{x}|^2 - t^2 = 0.$$

If we now let

$$\xi' = \xi A$$

then[3]

$$X' = A^+ X A,$$

and to each transform, A, of the ξ's we get a transform on the X's. The det is invariant if $|\det A| = 1$, so this corresponds to a Lorentz transformation, maintaining $|\vec{X}|^2 - t^2 = \text{const.}$ We use:

$$\det X' = \det X \cdot |\det A|^2 .$$

[Though X here is a null-vector, the relation is true in any case.]

We may generalize to a non-zero vector by choosing:

$$\chi_{\alpha\beta} = c\,(\xi_\alpha^* \,\xi_\beta \pm \eta_\alpha^* \,\eta_\beta)$$

where the $\pm$ choice gives a time-like or a space-like vector.

Then:[4]

$$\det x = \pm c^2 |\xi_1 \eta_2 - \xi_2 \eta_1|^2 = t^2 - |\vec{x}|^2$$

Now:

$$t = \tfrac{1}{2}\,\mathrm{tr}\,\chi = \tfrac{1}{2}\,c\,(\xi_1^* \xi_1 + \xi_2^* \xi_2)$$

so that the Trace-inv. unitarity restriction on A gives the sub-group of spatial-rotations. Note that if $C > 0$, $t > 0$; $C < C$, $t < 0$.

From the definition of $\chi_{\alpha\beta}$, we find:

$$\vec{\chi} = \xi \vec{\sigma} \xi^{*} \quad (\text{where } \chi_{\alpha\beta} = 2 \xi_{\alpha}^{*} \xi_{\beta})$$

or:[5]

$$\chi_k = \xi \sigma_k \xi^{*} \qquad k = 0, 1, 2, 3. \quad (\text{or } 1, \ldots, 4)$$

$$\sigma_0 = 1, \ldots .$$

Now, under a Lorentz transform

$$\chi'_k = \sum_i X_i L_{ik} (A) \qquad k = 1, \ldots, 4$$

and we readily find:[6]

$$A \sigma_k A^{+} = \sigma_i L_{ik} (A),$$

which thus relates A to L[7]. The transformation preserves the dets. of the σ_i's.

If we now consider: A and the related L (A) as well as B and the related L (B) then:

BA is related to $L(B A) = L(B) \cdot L(A)$.

this follows from:

$$\xi'' = \xi' A = \xi B A$$

Where we let B operate and then A. Then we get:

$$\chi'' = \chi' L(A) = \chi L(B) L(A) = \chi L(B A).$$

Also:

$$\sigma_i L_{ik} (B) L_{kj} (A) = (B \sigma_k B^{+}) L_{kj} (A)$$

$$= B A \sigma_j A^{+} B^{+} \qquad \text{Q.E.D.}$$

So far, we have proven that:

1° To every unitary A, there is associated a 3-D rot.

2° To every unimodular A, there is associated a Lorentz transform.

The relations are not unique, however, since by continuous transforms we can change

$$\xi_\alpha' \to -\xi_\alpha$$

This leaves the χ_k's unchanged. Consider a continuous rotation about χ_3:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \xi' = e^{i\alpha\sigma_3}\,\xi$$

This corresponds to a rotation by 2α. Thus:

$$\xi_1' = e^{i\alpha}\xi_1 \;,\; \xi_2' = e^{-i\alpha}\xi_2$$

and if $\alpha = \pi$

$$\xi_1' = -\xi_1 \;, \quad \xi_2' = -\xi_2$$

whereas:

$$x_{12}' = e^{2i\alpha}x_{12} = x_{12}$$

$$x_{21}' = e^{-2i\alpha}x_{21} = x_{21}$$

A Field Theory

Let us now identify ψ, a 2-component spinor, with ξ^* and ψ^* with ξ. Then we write:

$$\vec{j} = \psi^*\vec{\sigma}\,\psi \qquad j_{\alpha\beta} = 2\,\psi_\alpha\psi_\beta^*$$

The momentum vector is:

$$p_k = (\vec{p}, p_o) \qquad \text{(a null-vector)}$$

$$p_k\sigma_k = \vec{p}\cdot\vec{\sigma} - p_o\cdot 1$$

Further, we will define:

$$(p_k\sigma_k)\psi \equiv \phi \times \text{const.}$$

Now if: $\psi' = A^{+}\psi$ (since $\psi \sim \xi^*$) ,

then:[8]

$$\phi' = (p_k' \sigma_K)\psi' = (p_j L_{jK}\sigma_K) A^{+}\psi$$

$$= p_j L_{jK} A^{-1}\sigma_i L_{iK}\psi = A^{-1}(p_i\sigma_i)\psi = A^{-1}\phi.$$

In 3-D rotations, $A^{+} = A^{-1}$, so that ψ and ϕ transform in the same way. In the complete Lorentz case, however, they are different.

We may recall that

$$\omega\tilde{A}\omega^{-1} = (\det A)\ A^{-1} .$$

If: $\phi = \omega\psi^*$ (this is just one particular way to construct ϕ)

Then:

$$\phi' = \omega\psi'^* = \omega\tilde{A}\omega^* = A^{-1}\omega\psi^* = A^{-1}\phi .$$

since[9] $\det A = 1$.

FOOTNOTES

1. Since

$$(x+iy)(x-iy) - (t+z)(t-z) = 0.$$

$$\frac{x-iy}{t+z} = \frac{t-z}{x+iy} \qquad \frac{x-iy}{x+iy} = \frac{\xi_2\xi_1^*}{\xi_1\xi_2^*}$$

or:

$$x - iy = C\xi_2\xi_1^* \qquad x + iy = C\xi_1\xi_2^*$$

where C is a real constant (normalization)

Then:

$$t + z = \frac{\xi_1}{\xi_2}(x - iy) = C\xi_1^*\xi_1$$

$$t - z = \frac{\xi_2^*}{\xi_1^*}(x - iy) = C\ \xi_2^*\xi_2$$

Note that, if $t > 0$, $C > 0$ and if $t < 0$, $C < 0$.

FOOTNOTES

2. $$\chi = c \begin{pmatrix} \xi_1^* \xi_1 & \xi_1^* \xi_2 \\ \xi_2^* \xi_1 & \xi_2^* \xi_2 \end{pmatrix}$$

3. $$\chi_{\alpha\beta}' = \xi_\alpha^{*\prime} \xi_\beta' = \xi_\gamma^* A_{\gamma\alpha}^* \xi_\delta A_{\delta\beta}$$
$$= \tilde{A}_{\alpha\gamma}^* \xi_\gamma^* \xi_\delta A_{\delta\beta} = A_{\alpha\gamma}^+ \chi_{\gamma\delta} A_{\delta\beta} .$$

4. Since the ξ parts or η parts have det $x = 0$, we get:

$$\det \chi = \pm \Big[\xi_1^* \xi_1 \eta_2^* \eta_2 + \xi_2^* \xi_2 \eta_1^* \eta_1 - \xi_1^* \xi_2 \eta_2^* \eta_1 - \xi_2^* \xi_1 \eta_1^* \eta_2 \Big]$$
$$= \pm \Big\{ \xi_1 \eta_2 \big[\xi_1^* \eta_2^* - \xi_2^* \eta_1^* \big] - \xi_2 \eta_1 \big[\xi_1^* \eta_2^* - \xi_2^* \eta_1^* \big] \Big\} = \pm |\xi \omega \eta|^2$$

5. Here, again, ξ^* is on the right because of its role as a column vector.

6. $$\chi_K' = \xi A \sigma_K A^+ \xi^* = \xi \sigma_i \xi^* L_{iK}$$

and since L doesn't operate on the ξ, and ξ is arbitrary we get the stated result.

7. We can get another such relation from

$$X = \chi_K \sigma_K (1 - 2\delta_{K4}),$$

which is evident from the explicit form of X. Now:

$$\chi' = \chi_K' \sigma_K (1 - 2\delta_{K4}) = A^+ \chi A = A^+ \chi_K \sigma_K (1 - 2\delta_{K4}) A$$

FOOTNOTES

7. (Cont.)

Thus: $$X_i L_{iK} \sigma_K (1 - 2\delta_{K4}) = X_K A^+ \sigma_K (1 - 2\delta_{K4}) A$$

so:

$$L_{iK} \sigma_K (1 - 2\delta_{K4}) = A^+ \sigma_i (1 - 2\delta_{i4}) A$$

$$\underbrace{L_{i\ell} L_{iK} \sigma_K}_{} (1 - 2\delta_{K4}) = A^+ \sigma_i L_{i\ell} (1 - 2\delta_{i4}) A$$

$$\delta_{K\ell} \sigma_K (1 - 2\delta_{K4}) = A^+ \sigma_i L_{i\ell} (1 - 2\delta_{i4}) A$$

and finally:

$$(A^+)^{-1} \sigma_\ell (1 - 2\delta_{i4}) A^{-1} = \sigma_i L_{i\ell} (1 - 2\delta_{i4})$$

If we have a 3-D rotation, $A^+ = A^{-1}$ and the relation is the same as on page 3.

8. We use $$X_i' X_i' = X_j L_{ji} X_K L_{Ki}$$

$$= X_j X_K L_{ji} L_{Ki}$$

$$\therefore L_{ji} L_{Ki} = \delta_{jK} .$$

9. The requirement for a Lorentz transform is only the $|\det A| = 1$. However, if $\det A \neq 1$, we can always choose $A' = e^{i\phi} A$ such that $\det A' = 1$. This leaves X unchanged and so only serves to make A unique in this respect.

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Lecture 3

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LECTURE 3

Now, we will obtain the Dirac equation:

$$(p_k \sigma_k)\,\psi = \text{const.}\ \phi = -\,m\phi \quad .$$

(The const. m has the meaning of a rest-mass)

$$(-p_o + \vec{p}\cdot\vec{\sigma})\,\psi = -\,m\,\phi$$

so if we also choose:

$$(p_o + \vec{p}\cdot\vec{\sigma})\,\phi = m\,\psi$$

we find:

$$(-p_o{}^2 + (\vec{p})^2 + m^2)\psi = 0 \qquad \text{(Klein-Gordon Equation)}$$

[In this we have used:

$$(-p_o + \vec{p}\cdot\vec{\sigma})(p_o + \vec{p}\cdot\vec{\sigma}) = -p_o{}^2 + (\vec{p})^2$$

which follows from the commutation relations for the σ's for any vector p.]

Added remark:

Here, we have obtained the K.G. equation from the first two equations. We could have gone the other way just as well. If electromagnetic fields are present this method is readily generalized. Pauli doesn't see any advantage to this approach. Kramers developed it in his 2nd volume of quantum theory on Electrons and Radiation.

To get the coordinate space equation, we will introduce

$$p_o = -\frac{1}{i}\frac{\partial}{\partial x_o}\ ; \quad \vec{p} = i\,\frac{\partial}{\partial \vec{x}} \quad .$$

Then we have:

$$\left(\frac{\partial}{\partial x_o} + \vec{\sigma}\cdot\frac{\partial}{\partial \vec{x}}\right)\psi + i\, m\, \phi = 0$$

$$\left(\frac{\partial}{\partial x_o} - \vec{\sigma}\cdot\frac{\partial}{\partial \vec{x}}\right)\phi + i\, m\, \psi = 0 \quad .$$

These equations can be written more symmetrically if we introduce:

$$\vec{\tau} = -\vec{\sigma}$$

$$\tau_o = \sigma_o = 1$$

[Note that $\tau_1 \tau_2 = -i\,\tau_3, \ldots$] .

Then:

$$(\sigma_k p_k)\psi = -i\left(\sigma_k \frac{\partial}{\partial x_k}\right)\psi = -m\,\phi$$

$$(\tau_k p_k)\phi = -i\left(\tau_k \frac{\partial}{\partial x_k}\right)\phi = -m\,\psi \quad .$$

From the relation:

$$\omega A^T \omega^{-1} = A^{-1}\,(\det A)$$

we see immediately that

$$\tau_k = \omega \sigma_k^T \omega^{-1} \quad ; \qquad \sigma_k^T = \omega \tau_k \omega^{-1} \quad .$$

since $\det \sigma_i = -1$ $\quad i = 1, 2, 3$ and $\det \sigma_o = +1$.

Reflections

There is a fundamental connection between the two kinds of spinors ψ, ϕ and reflections in space. We assert that $\psi \to \phi$, $\phi \to \psi$ corresponds to a reflection. So far we have only considered a continuous group, but now we will consider the discrete group, P (parity):

$$x_i{}' = - x_i \qquad i = 1, 2, 3$$

$$x_o{}' = x_o$$

From the Dirac equations, it is clear that if $\psi \to \phi$, $\phi \to \psi$ and the coordinates transform as above, than the equations are invariant. Now:[1]

$$X' = \begin{pmatrix} x_o - x_3 & -(x_1 - ix_2) \\ -(x_1 + ix_2) & x_o + x_3 \end{pmatrix} = \omega X^T \omega^{-1} = X^{-1} (\det X)$$

$$= \overline{X} \quad \text{(as denoted by Gürsey)}$$

and if:

$$X' = A^+ X A$$

then:

$$(X^{-1})' = A^{-1} X^{-1} (A^+)^{-1} \quad .$$

In the subgroup of the pure rotations $A^+ = A^{-1}$ and so we see that X and X^{-1} or $\overline{X}$ all transform alike.

The transformation $\psi' = \phi$, $\phi' = \psi$ is invariant with respect to the 3-D rotations.[2] Thus, as long as we are dealing with theories which are P invariant it is natural to introduce the 4-component spinors as is done by Dirac. If the invariance is not present, then it is no longer natural.

The Dirac spinors:

$$\underline{\psi} = \begin{pmatrix} \psi \\ \phi \end{pmatrix}$$

we now introduce:

$$\vec{\alpha} = \begin{pmatrix} +\vec{\sigma} & 0 \\ 0 & +\vec{\tau} \end{pmatrix} = \begin{pmatrix} +\vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} ; \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \alpha_4$$

so: $$\left\{\alpha_i, \alpha_j\right\} = 2\delta_{ij}$$

$$\alpha_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv 1 \ .$$

The pair of equations are then combined to:

$$(\alpha_k p_k)\,\underline{\psi} + m\beta\,\underline{\psi} = 0$$

or:

$$\text{(A)} \qquad \frac{\partial \psi}{\partial t} + \vec{\alpha} \cdot \frac{\partial}{\partial \vec{x}}\,\underline{\psi} + i\,m\,\beta\,\underline{\psi} = 0 \ .$$

Taking the complex conjugate, we get:

$$\text{(B)} \qquad \frac{\partial \psi^*}{\partial t} + \frac{\partial \psi^*}{\partial \vec{x}} \cdot \vec{\alpha} - i\,m\,\psi^*\beta = 0 \ .$$

From (A) and (B) we readily construct:

$$\frac{\partial j_o}{\partial t} + \text{div}\,\vec{j} = 0 \quad \text{where} \quad \begin{aligned} j_o &= \text{const.}\ \underline{\psi}^*\underline{\psi} \\ \vec{j} &= \text{const.}\ (\underline{\psi}^*\vec{\alpha}\,\underline{\psi}) \end{aligned}$$

[$j_o \geq 0$ if the ψ's are c-numbers, but it is not necessary if the ψ's are q-numbers.]

We can now introduce the γ's:

$$\vec{\alpha} = i\gamma_4\vec{\gamma} \ ; \quad \beta = \gamma_4$$

$$\vec{\gamma} = -i\beta\vec{\alpha} = i\vec{\alpha}\beta \qquad \left\{\gamma_i, \gamma_j\right\} = 2\delta_{ij}$$

and then we can write ($x_4 = i\,t$):

$$\gamma_k \frac{\partial \psi}{\partial x_k} + m\,\underline{\psi} = 0$$

and if we let:[3]

$$\bar{\underline{\psi}} = \underline{\psi}^*\beta = \underline{\psi}^*\gamma_4$$

then we get:

$$\frac{\partial \bar{\Psi}}{\partial x_k} \gamma_k - \bar{\Psi} m = 0.$$

Now we see easily that

$$j_k = \text{const.} \; (\bar{\Psi} \gamma_k \Psi)$$

and

$$\frac{\partial j_k}{\partial x_k} = 0 .$$

[The $\bar{\psi}$ is more useful for studying the Lorentz properties, while ψ^* is more useful for studying the reality properties.]

Another important matrix is:

$$\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 = \gamma_5^+$$

and we can extend the commutation relations as:

$$\{\gamma_i, \gamma_j\} = 2\,\delta_{ij} \qquad i = 1, \ldots, 5.$$

[The origin of the 1, ..., 5 is a 5-dimensional theory, and will not be discussed here.]

In our representation:

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{\gamma} = \begin{pmatrix} 0 & -i\vec{\sigma} \\ i\vec{\sigma} & 0 \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

<u>Charge conjugation:</u>

The equations suggest defining:

$$\phi^c = -\omega \psi^*$$
$$\psi^c = +\omega \phi^*$$

[Then: $\psi^{cc} = +\omega(\phi^c)^* = \psi$, $\phi^{cc} = \phi$.]

Then:

$$\left(\frac{\partial}{\partial x_o} + \vec{\sigma}\cdot\frac{\partial}{\partial \vec{x}}\right)\psi^c + i\,m\,\phi^c = 0$$

$$\left(\frac{\partial}{\partial x_o} - \vec{\sigma}\cdot\frac{\partial}{\partial \vec{x}}\right)\phi^c + i\,m\,\psi^c = 0 \;.$$

We will show that in the presence of an electromagnetic field, the change will correspond to $e \to -e$.

Remark:

Majorana spinor:

$$\phi^c = \phi$$
$$\psi^c = \psi$$

Then:

$$(\sigma_k p_k)\psi - m\omega\psi^* = 0$$

$$(\tau_k p_k)\,\omega\psi^* - m\,\psi = 0.$$

These forms can only hold for non-electromagnetic particles, though m need not be zero. The equation is not gauge invariant and so cannot represent charged particles.

FOOTNOTES

[1] The relation between X' and X^{-1} is clear from the cofactors involved.

[2] It is not surprising that the full Lorentz transformations do not leave things invariant since the definition of P on the coordinates is manifestly non-covariant.

[3] We choose $\bar{\psi} = \psi^* \gamma_4$ since the anti-commutator of γ's then reestablishes the symmetry between x_1, x_2, x_3 and x_4 which was lost on complex conjugation.

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LECTURE ON "CONTINUOUS GROUPS AND REFLECTIONS IN QUANTUM MECHANICS" BY W. PAULI

Lecture 4

R. J. Riddell, Jr.

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LECTURE 4

Now, from the Dirac equations we find the relation[1]

$$\frac{\partial j_o}{\partial t} + \text{div}\,\vec{j} = 0$$

where:

$$j_o = \psi^*\psi + \phi^*\phi \quad ; \qquad \vec{j} = \psi^*\vec{\sigma}\psi - \phi^*\vec{\sigma}\phi \quad .$$

Though we have used the free particle equations to get the current, the latter are considerably more general than that. Pauli also considers that postulating the relations from $\vec{j}$, j_o is a justification for the definition of the P operation, since then:

$$\vec{j}' = -\vec{j} \qquad\qquad j'_o = j_o \quad .$$

The transformation is still not the most general one, since we could also have chosen as well:

$$\psi' = \psi e^{i\alpha}$$

$$\phi' = \phi\, e^{i\alpha}$$

where α will first be considered as constant. The equations are invariant under such a change also. The transformation is a gauge transformation and all charged particles allow this change, leaving the equations of motion invariant.[2] For neutral particles, the arbitrariness is not so clear, since then the transformation is not allowed. Here it would be necessary to consider the interactions to fix the phase, however, as the free particle equations will not suffice.

From experience, we believe that baryons (nucleons, hyperons) are conserved. In addition, leptonic charge may also be, though the evidence here is less certain. Pauli feels that the former and charge conservation are certain. The lepton case may be related to a discrete group rather than a continuous one, in which case the conservation would not be rigorous.

Charge conjugation (again):

We had:[3] $\phi^c = -\omega\psi^*$; $\psi^c = +\omega\phi^*$.

Now, in 4-component system, we might write:

$$\psi^c = E\,\psi^* \qquad \psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix}$$

where

$$E = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} = \begin{pmatrix} 0 & & & -1 \\ & & +1 & \\ & +1 & & \\ -1 & & & 0 \end{pmatrix}$$

so that:

$$\psi_1^c = -\psi_4^* \quad , \qquad \psi_2^c = \psi_3^*$$

$$\psi_3^c = \psi_2^* \quad , \qquad \psi_4^c = -\psi_1^* \quad .$$

Majorana introduced this concept of charge conjugation, and it was later generalized by Racah and Kramers.

Now: With our present choice of C, we would find that it does not commute with P ($[P, C] \neq 0$) but if we make a different choice of phase, it will. Thus we set:[4]

$$\phi^P = i\,\psi \qquad \phi^c = i\omega\,\psi^*$$

$$\psi^P = i\,\phi \qquad \psi^c = -i\omega\,\phi^*$$

[With the old phases, $\{C, P\} = 0$. This result is of importance.]

For E, we note that

$$\widetilde{E} = E \qquad EE^+ = 1.$$

These properties of E are essential. [Also, here, $E^+ = E$, $E^2 = 1$.] The "essential" features are seen in considering the transformations of E. We let:

$$\psi' = U\psi, \qquad UU = 1$$

which can also be considered as change of representation of the γ's:[5]

$$\gamma_k' = U \gamma_k U^{-1} \quad .$$

Then we find:

$$E' = U E \widetilde{U} \quad ,$$

and

$$E' = \widetilde{E'} \quad , \qquad E' E'^+ = 1.$$

Now:

$$\vec{\widetilde{\gamma}} = E^{-1} \vec{\gamma} E \quad , \qquad \widetilde{\gamma}_4 = -E^{-1} \gamma_4 E.$$

We would have started with these requirements on E, and then the essential properties will come out. The latter relations come from:

$$\gamma_i' = E^{-1} \vec{\gamma} E = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} i\sigma_i\omega & 0 \\ 0 & i\sigma_i\omega \end{pmatrix} = \begin{pmatrix} 0 & i\omega\sigma_i\omega \\ -i\omega\sigma_i\omega & 0 \end{pmatrix} .$$

Now:

$$\omega \widetilde{A} \omega^{-1} = -\omega \widetilde{A} \omega = A^{-1} (\det A)$$

so:

$$\omega \sigma_i \omega = \widetilde{\sigma}_i$$

since

$$\det \sigma_i = -1 \ , \quad \sigma_i^{-1} = \sigma_i \quad .$$

Thus:

$$\gamma_i{}' = \begin{pmatrix} 0 & i\tilde{\sigma}_i \\ -i\tilde{\sigma}_i & 0 \end{pmatrix} = \tilde{\gamma}_i$$

For γ_4, we have:

$$\omega \sigma_o \omega = -\sigma_o \quad ,$$

which proves the result.[6]

We might also proceed from:

$$\psi^c = E\,\psi^* = C^{-1}\bar{\psi} = -\bar{\psi}C^{-1} \quad .$$

[Then, again, E has simpler reality conditions and C, simpler transformation properties.]

$$\bar{\psi}^c = C\psi = -\psi C.$$

Properties of C:

$$\tilde{C} = -C \quad , \qquad C\,C^+ = 1$$

$$C^{-1} = -\gamma_4 E$$

$$C = -E^{-1}\gamma_4 = -E^+\gamma_4$$

and

$$\gamma_k{}^T = -C\gamma_k C^{-1} \qquad k = 1, \ldots, 4.$$

If we look at:

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 \qquad \tilde{\gamma}_5 = C\,\gamma_5\,C^{-1}$$

From C, we find:

C, $C\gamma_5$, $C\gamma_5\gamma_k$ are anti-symmetric (6)

$C\gamma_k$, $C\gamma_{k\ell}$ are symmetric (10)

$$\gamma_{k\ell} = \tfrac{1}{2}\left(\gamma_k\gamma_\ell - \gamma_\ell\gamma_k\right).$$

External Electromagnetic Fields

Rule:

$$\frac{\partial}{\partial \vec{x}} \rightarrow \frac{\partial}{\partial \vec{x}} - i e \vec{A}$$

$$\frac{\partial}{\partial x_0} \rightarrow \frac{\partial}{\partial x_0} + i e A_0$$

$$A_4 \equiv i A_0 \qquad\qquad x_4 \equiv i x_0 \quad .$$

Then the equations become:

$$\left(\frac{\partial}{\partial x_0} + i e A_0\right)\psi + \vec{\sigma}\cdot\left(\frac{\partial}{\partial \vec{x}} - i e \vec{A}\right)\psi + i m \phi = 0$$

$$\left(\frac{\partial}{\partial x_0} + i e A_0\right)\phi - \vec{\sigma}\cdot\left(\frac{\partial}{\partial \vec{x}} - i e A\right)\phi + i m \psi = 0 \quad .$$

If we do the same charge conjugation as before, we get:

$$\left(\frac{\partial}{\partial x_0} - i e A_0\right)\psi^c + \sigma\cdot\left(\frac{\partial}{\partial \vec{x}} + i e \vec{A}\right)\psi^c + i m \phi^c = 0$$

$$\left(\frac{\partial}{\partial x_0} - i e A_0\right)\phi^c - \vec{\sigma}\cdot\left(\frac{\partial}{\partial \vec{x}} + i e \vec{A}\right)\phi^c + i m \psi^c = 0$$

We will then find that the equations are invariant if $e' \rightarrow -e$, or $A_\mu' \rightarrow -A_\mu$. Thus we can truly call the transformation charge conjugation.

FOOTNOTES

1
$$\left(\frac{\partial}{\partial t} + \vec{\sigma}\cdot\frac{\partial}{\partial \vec{x}}\right)\psi + i m \phi = 0 \qquad \times\ \psi^*$$

$$\left(\frac{\partial}{\partial t} - \vec{\sigma}\cdot\frac{\partial}{\partial \vec{x}}\right)\phi^* - i m \psi^* = 0 \qquad \times\ \phi$$

$$\psi^* \frac{\partial}{\partial t}\psi + \psi^*\vec{\sigma}\cdot\frac{\partial}{\partial \vec{x}}\psi + \phi\frac{\partial \phi^*}{\partial t} - \phi\,\vec{\sigma}\cdot\frac{\partial}{\partial \vec{x}}\phi^* = 0 \quad .$$

Adding complex conjugate equation, we get the desired result.

2 In fact, this invariance leads directly to charge conservation.

3 $$\phi_1^c = \psi_2^*, \qquad \phi_2^c = -\psi_1^*$$

$$\psi_1^c = -\phi_2^*, \qquad \psi_2^c = -\phi_1^* .$$

4 We have,

$$\phi_p = e^{i\lambda}\psi \qquad \phi^c = -\omega\,\psi^*\, e^{i\lambda'}$$

$$\psi_p = e^{i\lambda}\phi \qquad \psi^c = \omega\,\phi^*\, e^{i\lambda'}$$

so

$$(\phi^p)^c = e^{i\lambda}\,\psi^c = e^{i(\lambda+\lambda')}\,\omega\phi^*$$

$$(\phi^c)^p = -\omega(\psi^*)^p\, e^{i\lambda'} = -\omega\, e^{i(\lambda'-\lambda)}\,\omega\phi^*$$

$$(\psi^p)^c = e^{i\lambda}\phi^c = -\omega\, e^{i(\lambda+\lambda')}\,\psi^*$$

$$(\psi^c)^p = \omega(\phi^*)^p\, e^{i\lambda'} = \omega\, e^{i(\lambda'-\lambda)}\,\psi^* .$$

Thus, we need:

$$e^{2i\lambda} = -1 \qquad \therefore \quad \lambda = \pm\frac{\pi}{2}$$

$$e^{i\lambda} = \pm i$$

λ' is arbitrary.

5 $$\gamma_k p_k \underline{\psi} + m\underline{\psi} = 0 \qquad \underline{\psi}' = U\underline{\psi} \qquad \underline{\psi} = U^{-1}\underline{\psi}'$$

$$\gamma_k p_k\; U^{-1}\underline{\psi}' + m\, U^{-1}\underline{\psi}' = 0 \qquad \underline{\psi}^{*\prime} = U^*\underline{\psi}^*$$

$$\gamma_k'\; p_k\underline{\psi}' + m\,\underline{\psi}' = 0 \qquad \gamma_k' = U\gamma_k U^{-1} .$$

5 (Cont.)

Now:

$$\psi^{c\prime} = E^{\prime} \underline{\psi}^{*\prime} = E^{\prime} U^{*} \underline{\psi}^{*} \quad .$$

We want to maintain the Dirac equations for ψ^c the same as for ψ, however, so we would have:

$$\psi^{c\prime} = U \psi^{c} = U E \psi^{*} \quad .$$

Thus:

$$E^{\prime} U^{*} = U E$$

$$E^{\prime} = U E (U^{*})^{-1} = U E \widetilde{U} \quad .$$

Also:

$$\widetilde{E}^{\prime} = U \widetilde{E} \widetilde{U} \qquad E^{\prime} E^{\prime +} = U E \underbrace{\widetilde{U} \cdot U^{*}}_{1} E^{+} U^{+} = 1 \quad .$$

6 We could have proceeded by:

$$\psi^{c} = E \psi^{*} \quad .$$

Then, from the Dirac equation:

$$\gamma_{\mu} \frac{\partial \psi}{\partial x_{\mu}} + m \psi = 0 \qquad \frac{\partial \overline{\psi}}{\partial x_{\mu}} \gamma_{\mu} - \overline{\psi} m = 0$$

$$\frac{\partial \psi^{*}}{\partial \vec{x}} \cdot \vec{\gamma} - \frac{\partial \psi^{*}}{\partial x_4} \cdot \gamma_4 + m \psi^{*} = 0$$

Ansatz:

$$\psi^{c} = E \psi^{*} \qquad \psi^{*} = E^{-1} \psi^{c}$$

Thus:

$$\widetilde{\vec{\gamma}} E^{-1} \frac{\partial \psi^{c}}{\partial \vec{x}} - \widetilde{\gamma}_4 E^{-1} \frac{\partial \psi^{c}}{\partial x_4} + m E^{-1} \psi^{c} = 0$$

and

$$E \widetilde{\vec{\gamma}} E^{-1} \frac{\partial \psi^{c}}{\partial \vec{x}} - E \widetilde{\gamma}_4 E^{-1} \frac{\partial \psi^{c}}{\partial x_4} + m \psi^{c} = 0.$$

6 (Cont.)

To make this agree with the ψ equation, we must have:

$$\vec{\gamma} = E\,\tilde{\vec{\gamma}}\,E^{-1} \qquad \gamma_4 = -E\,\tilde{\gamma}_4\,E^{-1}$$

Thus:

$$\tilde{\vec{\gamma}} = E^{-1}\,\vec{\gamma}\,E \;; \qquad \tilde{\gamma}_4 = -E^{-1}\,\gamma_4\,E.$$

Since the $\tilde{\gamma}_i$ satisfy the same commutation relations as the γ_i, E must be unitary. Now:

$$\vec{\gamma} = \tilde{E}\,\tilde{\vec{\gamma}}\,\tilde{E}^{-1} \qquad \gamma_4 = -\tilde{E}\,\tilde{\gamma}_4\,\tilde{E}^{-1}$$

$$\vec{\gamma} = \tilde{E}\,E^{-1}\,\vec{\gamma}\,\tilde{E}^{-1}\,E \qquad \gamma_4 = \tilde{E}\,E^{-1}\,\gamma_4\,E\,\tilde{E}^{-1}$$

Thus: $\tilde{E}\,E^{-1}$ commutes with all 4 γ_i, and hence with all 16 constructs of γ_i's and thus = const.

$\therefore\ \tilde{E}\,E^{-1} = \alpha \qquad \tilde{E} = \alpha E \qquad$ But taking transpose:

$$E = \alpha\tilde{E} = \alpha^2 E$$

$\therefore\ \alpha = \pm 1$.

If we let $\alpha = \pm 1$. Then:

$$\vec{\gamma}\tilde{E} = \tilde{E}\tilde{\vec{\gamma}} = \widetilde{\vec{\gamma}E} \quad \text{so} \quad \widetilde{\vec{\gamma}E} = \pm\,\vec{\gamma}E$$

Also

$$\tilde{E} = \pm E, \qquad \widetilde{\gamma_4 E} = \mp\,\gamma_4 E$$

$$\widetilde{\gamma_j\gamma_i E} = \pm\,\gamma_j\gamma_i E \qquad\qquad 4 \neq i \neq j \neq 4$$

$$\widetilde{\gamma_j\gamma_4 E} = \mp\,\gamma_j\gamma_4 E$$

$$\widetilde{\gamma_5 E} = \mp\,\gamma_5 E$$

$$\widetilde{\gamma_j\gamma_5 E} = \pm\,\gamma_j\gamma_5 E$$

$$\widetilde{\gamma_5\gamma_4 E} = \mp\,\gamma_5\gamma_4 E .$$

6 (Cont.)

For $\alpha = -1$, we find 10 independent anti-symmetric and 6 symmetric matrices. Since this is not possible for 4×4 matrices, $\alpha \neq -1$. If we choose $\alpha = +1$, the situation is reversed so

$$\tilde{E} = E$$

Since:

$$C^{-1} = -\gamma_4 E \quad \text{(antisymmetric)}$$

we find

$$\tilde{C} = -C.$$

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LECTURE 5

Remark on the signs of the transformations:

As we saw, the phase in the ϕ^c equation is arbitrary and Pauli now wants to go back to $\phi^c = -\omega\,\psi^*$, $\psi^c = \omega\,\phi^*$; $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$E = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \qquad C^{-1} = -\gamma_4 E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} = -C = \tilde{C} .$$

$$\psi^c = E\,\psi^* = C^{-1}\,\bar{\psi} \qquad\qquad \bar{\psi} = \psi^*\beta = \psi^*\gamma_4$$

$$\vec{\alpha} = i\,\vec{\gamma}_4\,\vec{\gamma} , \qquad \beta = \gamma_4 , \qquad \vec{\gamma} = i\,\vec{\alpha}\,\beta .$$

Now, in four-component language, the Dirac equation is:

$$\gamma_k\left(\frac{\partial}{\partial x_k} - i\,e\,A_k\right)\psi + m\,\psi + i\,\mu\cdot\frac{1}{2}\,F_{ik}\,\gamma_i\,\gamma_k\,\psi = 0$$

Here, we have added the Pauli term of the "anomalous" magnetic moment in the ~~$\mu\cdot\frac{1}{2}F_{ik}\gamma_i\gamma_k$.~~ We have:

$$F_{ik} = \frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k} ; \qquad F_{12} = H_3, \;\ldots; \qquad F_{41} = i\,E_1, \;\ldots .$$

The F_{ik} are the fields; $\vec{H}$, $\vec{E}$. The anomalous moment term is:[1]

$$i\,\frac{1}{2}\,F_{ik}\,\gamma_i\,\gamma_k = i(\vec{\alpha}\cdot\vec{E}) - (\vec{\Sigma}\cdot\vec{H})$$

where:

$$\gamma_1 \gamma_2 = \alpha_1 \alpha_2 = i \Sigma_3 \qquad\qquad \beta \vec{\Sigma}' = \vec{\Sigma} \beta .$$

Thus we can also write:

$$\left(\frac{\partial}{\partial x_o} + i e A_o\right) \psi + \vec{\alpha} \cdot \left(\frac{\partial}{\partial \vec{x}} - i e \vec{A}\right) \psi + i m \beta \psi$$

$$- \mu \beta \left\{ \vec{\alpha} \cdot \vec{E} + i \vec{\Sigma} \cdot \vec{H} \right\} \psi = 0 .$$

In the limit of small velocities, we get an additional magnetic moment beyond the "normal" $\frac{e\hbar}{2mc}$. The latter, anomalous part, gives the entire magnetic moment for the neutron while the proton has a large part also. (The correction for the electron is very small.)

More on charge conjugation:

If $\psi \rightarrow \psi^c$, then ψ^c satisfies the same Dirac equation as ψ if either $\begin{Bmatrix} e \rightarrow -e \\ \mu \rightarrow -\mu \end{Bmatrix}$ or $\begin{Bmatrix} A_\mu \rightarrow -A_\mu \\ F_{ik} \rightarrow -F_{ik} \end{Bmatrix}$

[We find $\tilde{\vec{\Sigma}} = -E^{-1} \vec{\Sigma} E$; (see footnote 1)

$$(\widetilde{\beta \vec{\Sigma}}) = E^{-1}(\beta \Sigma)E .]$$

In the Majorana theory (two components):

$$\psi^c = \psi .$$

Thus, for ψ^c, ψ to satisfy the same equation, it is necessary that $e = 0 = \mu$ (no electromagnetic interactions). This theory has been used to describe the neutrinos recently. [We note that $m = 0$ does not follow directly from the Majorana theory.]

We have been using a particular representation of

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

[These signs disagree with the previous choice for $\vec{\alpha}$, but to get the sign for γ_5 right, this choice for $\vec{\alpha}$ is necessary.]

Another representation is that using (Majorana representation

$$\gamma_1, \gamma_2, \gamma_3 = \text{real} \quad (\text{sym.})$$

$$\gamma_4 = \text{pure imaginary (anti-sym.)}$$

This is not unique. For example, we can permute the γ's chosen $[\gamma_1 \to \gamma_2, \ \gamma_2 \to \gamma_3, \ \gamma_3 \to \gamma_1]$ or we might also multiply the γ's by -1. These Majorana γ's may be obtained by a unitary transformation, for example:[2]

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i\omega \\ 1 & \omega \end{pmatrix} \qquad \vec{\alpha}' = U \vec{\alpha} U^{-1}$$

and then:

$$\alpha_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \alpha_2 = i \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

$$\beta = i \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}.$$

In the Majorana representation, $E = 1$; $\psi^c = \psi^*$; $C = -\gamma_4$. Thus in the abbreviated two-component theory $\psi^c = \psi = \psi^*$. [No change will appear in the form of the current.]

The change $\psi \to \psi^c$ corresponds to either the same particle in the opposite field or a particle with the opposite charge in the same field. Now to give physical meaning to the transformation it will be necessary to have:

$$\vec{j}^c = -\vec{j} .$$

If the ψ's are ordinary numbers, it is not possible to get a reversal since:

$$j_o = \psi^* \psi \qquad j_o^c = \psi \psi^* = j_o ,$$

and the sign has not changed. To continue, it is necessary to introduce the "hole" theory or second quantization.

In the first quantized theory, the charge density is ≥ 0 while the energy density is not. This is not satisfactory since there will be no lowest state, and so we proceed differently in the second quantized theory. The technique was developed by Jordan and Wigner using the anticommutation relations (exclusion principle). [It is certainly necessary to have spinors; whether one needs explicit Bose fields also is not yet clear.]

Second quantization:

We introduce a, a^* where:

$$a^2 = (a^*)^2 = 0 \qquad a\,a^* + a^*\,a = 1 .$$

If we set $N = a^* a$; $1 - N = a\,a^*$

$$N(1 - N) = 0 . \qquad \text{(Exclusion principle)}$$

Then a representation is:

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad a^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad 1 - N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

It is interesting to note that there is complete symmetry between a, a^*. Thus the theory is symmetric between N, $1 - N$.

[Notation: Anticommutator $\{A, B\} = A B + B A$

Commutator $[A, B] = A B - B A$].

Here, we have quantized according to the exclusion principle since N takes only the values 0, 1.

If we now consider a set of a's, we choose:

$$\left\{ a_r, \ a_s^* \right\} = \delta_{rs}$$

$$\left\{ a_r, \ a_s \right\} = \left\{ a_r^*, \ a_s^* \right\} = 0 .$$

"a" corresponds to absorption

"a*" corresponds to emission since:

$$a_r \ \psi(\ldots \ 1_r \ \ldots) = \epsilon_r \ \psi(\ldots \ 0_r \ \ldots)$$

$$a_r \ \psi(\ldots \ 0_r \ \ldots) = 0$$

$$a_r^* \ \psi(\ldots \ 1_r \ \ldots) = 0$$

$$a_r^* \ \psi(\ldots \ 0_r \ \ldots) = \epsilon_r \ \psi(\ldots \ 1_r \ \ldots) .$$

The ϵ_r's depend on the choice of ordering the r's, since

$$\epsilon^2 = 1$$

and

$$\epsilon_r(N_1, \ldots, N_r, \ldots) = (-1)^{\sum_{k=1}^{r} N_k} .$$

If we change the order of the states, the ϵ's will be changed though it would only correspond to a unitary transformation. No physical results can depend on the ordering.

We now introduce the complete set of eigenfunctions in a box V:

$$\psi_\rho(x) = \frac{1}{\sqrt{V}} \sum_{r=1,2} \sum_{\vec{k}} \left\{ a_r(\vec{k}) \, u_\rho^r(\vec{k}) \, e^{i(k\cdot x)} + b_r^*(\vec{k}) \, v_\rho^r(\vec{k}) \, e^{-i(k\cdot x)} \right\}$$

where:

$$k\cdot x = \vec{k}\cdot\vec{x} - \omega t , \qquad \omega = + \sqrt{\vec{k}^2 + m^2} .$$

We have separated ψ into two parts a, b^* (or u, v) because of the double-valued nature of ω.[3] We now have left the two possible spin choices (r=1,2). The u's and v's satisfy:

$$(i\,\gamma \cdot k + m)\, u^r(\vec{k}) = 0$$

$$(-i\,\gamma \cdot k + m)\, v^r(\vec{k}) = 0$$

or:

$$(\vec{\alpha} \cdot \vec{k} + \beta\, m - \omega)\, u^r(\vec{k}) = 0$$

$$(\vec{\alpha} \cdot \vec{k} + \beta\, m + \omega)\, v^r(-\vec{k}) = 0.$$

We can easily verify that:[4]

$$u^r(\vec{k})^*\ v^s(-\vec{k}) = 0$$

$$v^r(\vec{k})^*\ u^s(-\vec{k}) = 0$$

while we choose the normalization of the u's so that:

$$u^r(\vec{k})^*\ u^s(\vec{k}) = \delta_{rs}$$

$$v^r(\vec{k})^*\ v^s(\vec{k}) = \delta_{rs}$$

FOOTNOTES

1
$$\frac{1}{2} F_{ik}\, \gamma_i\, \gamma_k = \Big[F_{12}\, \gamma_1\, \gamma_2 + F_{13}\, \gamma_1\, \gamma_3 + F_{23}\, \gamma_2\, \gamma_3 + F_{14}\, \gamma_1\, \gamma_4 + F_{24}\, \gamma_2\, \gamma_4 + F_{34}\, \gamma_3\, \gamma_4 \Big]$$

$$= \Big[i\, \Sigma_3\, H_3 + \ldots + \vec{\alpha} \cdot \vec{E} \Big]$$

$$F_{i4} = -i\, E_i\,; \quad \gamma_i\, \gamma_4 = i\, \alpha_i\,; \quad \gamma_1\, \gamma_2 = i\, \Sigma_3\,; \quad F_{12} = H_3\,.$$

2
$$U^+ U = \frac{1}{2} \begin{pmatrix} -i & 1 \\ -i\omega & -\omega \end{pmatrix} \begin{pmatrix} i & -i\omega \\ 1 & \omega \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Footnote 2 cont.

2 (Cont.)

$$\vec{\alpha}' = \frac{1}{2}\begin{pmatrix} i & -i\omega \\ 1 & \omega \end{pmatrix}\begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\begin{pmatrix} -i & 1 \\ -i\omega & -\omega \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -\vec{\sigma} - \tilde{\vec{\sigma}} & -i\vec{\sigma} + i\tilde{\vec{\sigma}} \\ i\vec{\sigma} - i\tilde{\vec{\sigma}} & -\vec{\sigma} - \tilde{\vec{\sigma}} \end{pmatrix} \quad \text{since } \omega\vec{\sigma}\omega = \tilde{\vec{\sigma}} .$$

Thus:

$$\alpha_1' = \begin{pmatrix} -\sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \quad \alpha_2' = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}; \quad \alpha_3' = \begin{pmatrix} -\sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}$$

$$\beta' = \begin{pmatrix} 0 & -i\omega \\ -i\omega & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$$

and we get Pauli's choice by changing the sign of all α's, β.

3 Note that the $(^*)$ appears on the operator associated with the negative frequency, while that without one goes with positive frequency. This is also true for ψ^*. This is closely related to the "hole theory" of Dirac, since a is related to annihilation and a^* is related to creation.

4 Proof:

1. $(\vec{\alpha}\cdot\vec{k} + \beta m - \omega)\, u^r(\vec{k}) = 0$

   $(\vec{\alpha}\cdot\vec{k} + \beta m + \omega)\, v^r(-\vec{k}) = 0$

2. $(\vec{\alpha}^*\cdot\vec{k} + \beta^* m + \omega)\, v^r(-\vec{k})^* = 0.$

Multiplying 1. by $v^r(-\vec{k})$, and 2. by $-u^r(\vec{k})$ and adding, we get:

$$2\omega(v^r(-\vec{k})^*\, u^r(\vec{k})) = 0$$

while, using 1. for u^{r*}, u^s we get nothing of interest, but we may choose $u^r(\vec{k})$ and $u^s(\vec{k})$ orthonormal.

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LECTURE 6

We can introduce projection operators (Casimir) via:

$$P_{\rho\sigma}^{+}(\vec{k}) = \sum_{r=1,2} u_\rho^r(\vec{k})\, u_\sigma^r(\vec{k})^*$$

$$P_{\rho\sigma}^{-}(\vec{k}) = \sum_{r=1,2} v_\rho^r(-\vec{k})\, v_\sigma^r(-\vec{k})^* .$$

From the orthogonality relations:[1]

$$P^+(\vec{k}) + P^-(\vec{k}) = 1 .$$

Also:

$$(P^+)^2 = P^+ ; \qquad (P^-)^2 = P^-$$

$$P^+ P^- = P^- P^+ = 0 .$$

Since the P's are also constructed from u's, v's, they are solutions of the Dirac equations:

$$(\vec{\alpha}\cdot\vec{k} + \beta m \mp \omega) P^{\pm}(\vec{k}) = 0$$

and we can thus write:[2]

$$P^{\pm}(\vec{k}) = \pm \frac{1}{2\omega} (\vec{\alpha}\cdot\vec{k} + \beta m \pm \omega) .$$

Specifically, we can also obtain:

$$P^-(\vec{k})\, u^r(\vec{k}) = 0 \qquad \text{and} \qquad P^+(\vec{k})\, u^r(\vec{k}) = u^r(\vec{k})$$

$$P^+(\vec{k})\, v^r(-\vec{k}) = 0 \qquad\qquad P^-(\vec{k})\, v^r(-\vec{k}) = v^r(-\vec{k}) .$$

Remark on charge conjugation:

Our choice of v, u gives a particular connection of the charge conjugation. We have:

$$\psi^c = E\,\psi^* .$$

Now we can choose the v's as the charge conjugation states to the u's:

$$v^r(\vec{k})^c = E\,u^r(\vec{k})^* , \qquad u^r(\vec{k})^c = E\,v^r(\vec{k})^* ,$$

[Here, the k's are the same.]

and then:[3]

$$a_r^{\,c} \equiv b_r$$

$$b_r^{\,c} \equiv a_r .$$

We now pass on to the invariant functions, related to the anticommutators and the vacuum expectation values of free fields. We have:

$$\left\{ a_r^*, a_s \right\} = \delta_{rs} ; \qquad \left\{ b_r^*, b_s \right\} = \delta_{rs}$$

while all others are zero. We can generalize the use of these relations by considering the vacuum as the state of lowest energy. Then a, b must be considered to be absorption operators.

Definition of the vacuum:

$$a_r \left| 0 \right\rangle = 0 , \qquad b_r \left| 0 \right\rangle = 0 .$$

Thus for any set of operators:

$$\cdots\, a_s^* \, a_r \left| 0 \right\rangle = 0 , \quad \text{etc.}$$

Thus, from the anticommutator:

$$\langle 0 | a_r \, a_s^* | 0 \rangle = \delta_{rs}$$

$$\langle 0 | b_r \, b_s^* | 0 \rangle = \delta_{rs} \, .$$

We will see that then $| 0 \rangle$ corresponds to the state of lowest energy.

Let us now define the energy-momentum 4-vector. For this purpose we will make use of the Heisenberg rule relating operators in a c-number theory to those in a q-number theory:

$$\text{c-number:} \quad \psi_\rho^* \, \mathcal{O}_{\rho\sigma} \, \psi_\sigma \longrightarrow \frac{1}{2} \left[\psi^* \mathcal{O} \, \psi - \psi \, \tilde{\mathcal{O}} \, \psi^* \right] \quad : \quad \text{q-number.}$$

This rule makes expressions in the fields more symmetric, and is especially important for the current as will be seen.

Thus we introduce:

$$P_o = \int d^3x \, \frac{1}{2} \left(\psi^* \, i \, \frac{\partial \psi}{\partial t} - i \, \frac{\partial \psi}{\partial t} \, \psi^* \right)$$

$$= \sum_{r=1,2} \sum_{\vec{k}} \omega \left[\frac{1}{2} (a_r^* \, a_r - a_r \, a_r^*) + \frac{1}{2} (b_r^* \, b_r - b_r \, b_r^*) \right] ,$$

using the orthogonality relations of the u's, v's. Now we will interpret:

$$a_r^* \, a_r = N_r^+ , \qquad b_r^* \, b_r = N_r^-$$

and $\langle 0 | a_r \, a_r^* | 0 \rangle = 1, \quad \langle 0 | b_r \, b_r^* | 0 \rangle = 1.$

We find:

$$P_o = \sum_{r=1,2} \sum_{\vec{k}} \omega \left[N_r^+(\vec{k}) + N_r^-(\vec{k}) - 1 \right] .$$

[Pauli feels that the infinity associated with the (-1) is a real one, and indicates that the formalism is not yet entirely satisfactory. The fact that particles are never <u>really</u> free does not give a sufficient explanation yet, either. One begins to doubt all volume integrals.]

For the momentum:

$$\vec{P} = \int d^3x \; \frac{1}{2i} \left(\psi^* \frac{\partial}{\partial \vec{x}} \psi - \frac{\partial \psi}{\partial \vec{x}} \psi^* \right)$$

$$= \sum_{r=1,2} \sum_{\vec{k}} \vec{k} \left\{ N_r^+(\vec{k}) + N_r^-(\vec{k}) - 1 \right\}$$

In this case, the (-1) isn't such a problem, since we can sum $\vec{k}$ over the angles first and then over $|\vec{k}|$. In this way, the sum is set = 0. Now:

$$(\vec{P},\; i P_o) \equiv P_k \qquad (= \text{4-vector})$$

and

$$P_k = \int d^3x \; \frac{1}{2} \left(\psi^* \frac{1}{i} \frac{\partial \psi}{\partial x_k} - \frac{1}{i} \frac{\partial \psi}{\partial x_k} \psi^* \right)$$

where $x_4 = i\,t$. From the commutation relations:[4]

$$\left[N_r^+, a_r \right] = -a_r \; ; \qquad \left[N_r^+, a_r^* \right] = a_r^*$$

$$\left[N_r^-, b_r \right] = -b_r \; ; \qquad \left[N_r^-, b_r^* \right] = b_r^* ,$$

and for any operator, f, we have as a general property of the P_k:

$$i \left[P_k, \; f \right] = - \frac{\partial f}{\partial x_k} .$$

Here, we must restrict f to be an implicit function of x_k, and explicit function of the field operators only.[5]

Again using the Heisenberg rule, we find:

$$j_k(x) = \frac{i}{2} \left(\bar{\psi} \gamma_k \psi - \psi \tilde{\gamma}_k \bar{\psi} \right)$$

and using the charge conjugation[6]

$$\bar{\psi}^c \gamma_k \psi^c = \psi \tilde{\gamma}_k \bar{\psi}$$

and

$$j_k^{\,c} = - j_k .$$

[This is more satisfying than the c-number result.]

In addition:

$$e = \int d^3x \; \frac{1}{2} (\psi^* \psi - \psi \psi^*)$$

$$e = \sum_{r=1,2} \sum_{\vec{k}} (N_r^+(\vec{k}) - N_r^-(\vec{k})) .$$

Thus the a's describe particles of + charge, while the b's describe particles of - charge. Thus we have reached our goal of charge conjugation, in which the vacuum is the lowest energy state. This would not be so simple if we had used commutators instead. Feynman suggests that we would need to work with states of negative probability if we had chosen commutators.

FOOTNOTES

1 Suppose: $f_\sigma(\vec{k}) = \Sigma \left\{ c_r \, u_\sigma^{\,r}(\vec{k}) + d_r \, v_\sigma^{\,r}(-\vec{k}) \right\}$

Then:

$$P^+_{\rho\sigma} \, f_\sigma(\vec{k}) = \Sigma \, c_r \, u_\rho^{\,r}(\vec{k})$$

$$P^-_{\rho\sigma} \, f_\sigma(\vec{k}) = \Sigma \, d_r \, v_\rho^{\,r}(-\vec{k}) .$$

2 Since $(\vec{\alpha} \cdot \vec{k} + \beta m \mp \omega)(\vec{\alpha} \cdot \vec{k} + \beta m \pm \omega) = (k^2 + m^2 - \omega^2) = 0$

and $$(\vec{\alpha} \cdot \vec{k} + \beta m) \begin{Bmatrix} u^r(\vec{k}) \\ v^r(-\vec{k}) \end{Bmatrix} = \omega \begin{Bmatrix} u^r(\vec{k}) \\ -v^r(-\vec{k}) \end{Bmatrix}$$

FOOTNOTES

3 This gives

$$\psi^c = E\,\psi^* = \frac{1}{\sqrt{V}} \sum_{r=1,2} \sum_{\vec{k}} E \left\{ a_r^*(\vec{k})\, u^r(\vec{k})^*\, e^{-i\,k\cdot x} + b_r(\vec{k})\, v^r(\vec{k})^*\, e^{i\,k\cdot x} \right\}$$

$$= \frac{1}{\sqrt{V}} \sum_{r=1,2} \sum_{\vec{k}} \left\{ b_r^{*c}(\vec{k})\, v^r(\vec{k})^c\, e^{-i\,k\cdot x} + a_r^c(\vec{k})\, u^r(\vec{k})^c\, e^{i\,k\cdot x} \right\}$$

Here, we have used the properties:

$$\vec{\gamma} = E\,\tilde{\vec{\gamma}}\,E^{-1} \qquad \gamma_4 = -E\,\tilde{\gamma}_4\,E^{-1}$$

from which

$$\vec{\alpha} = E\,\tilde{\vec{\alpha}}\,E^{-1} \qquad \beta = E\,\tilde{\beta}\,E^{-1}.$$

Then, since

$$(\vec{\alpha}\cdot\vec{k} + \beta\,m - \omega)\, u^r(\vec{k}) = 0$$

$$(\tilde{\vec{\alpha}}\cdot\vec{k} + \tilde{\beta}\,m - \omega)\, u^r(\vec{k})^* = 0,$$

but $\psi^c = E\,\psi^*$, so if $v^r(\vec{k})^c = E\,u^r(\vec{k})^*$,

$$(\vec{\alpha}\cdot\vec{k} - \beta\,m - \omega)\, v^r(\vec{k})^c = 0$$

which is indeed the equation also satisfied by $v^r(\vec{k})$. We proceed in the same way for $u^r(\vec{k})^c$.

4 For example:

$$\left[a_r^*\, a_r, a_r\right] = \underbrace{a_r^*\, a_r\, a_r}_{=0} - a_r\, a_r^*\, a_r \qquad \text{since } a_r^2 = 0.$$

$$= -a_r\left[1 - a_r\, a_r^*\right] = -a_r.$$

FOOTNOTES

5 The relation follows immediately for ψ, ψ^* and then is readily generalized to arbitrary functions using power series.

$$\left[P_k, \psi \right] = \frac{1}{2i} \int d^3x \left[\psi^* \frac{\partial\psi}{\partial x_k} - \frac{\partial\psi}{\partial x_k} \psi^*, \psi \right] .$$

But we will see that

$$\left\{ \psi(x), \bar{\psi}(x') \right\} = -i\, S(x - x') .$$

Now, P_k will be invariant from the equations of motion so we may choose the times all to be the same. Then we will also see that:

$$S(\vec{x}, 0) = i\, \gamma_4\, \delta^{(3)}(\vec{x})$$

so:

$$\left\{ \psi, \psi^* \gamma_4 \right\} = \gamma_4\, \delta^{(3)}(\vec{x}) \qquad \left\{ \psi, \psi^* \right\} = \delta^{(3)}(\vec{x}) ,$$

Thus:

$$\left[P_k, \psi \right] = -\frac{1}{i} \int \delta^{(3)}(\vec{x}' - \vec{x}) \cdot \frac{\partial\psi(\vec{x}')}{\partial x'_k}\, d^3x' = -\frac{1}{i} \frac{\partial\psi(\vec{x})}{\partial x_k}$$

Also for ψ^*:

$$\psi^*(\vec{x}') \frac{\partial\psi}{\partial x'_k} \psi^*(\vec{x}) = -\psi^*(\vec{x}')\, \psi^*(\vec{x}) \frac{\partial\psi}{\partial x'_k} + \psi^*(\vec{x}') \frac{\partial\delta^{(3)}(\vec{x} - \vec{x}')}{\partial x_k}$$

and we again find on integrating by parts:

$$\left[P_k, \psi^* \right] = -\frac{1}{i} \frac{\partial\psi^*}{\partial x_k} .$$

These results are necessary, according to the definition of P_k as the infinitesimal translation generator, since then P must satisfy such a relation.

FOOTNOTES

6 Proof:

We set $j_k^c = \frac{i}{2}(\bar{\psi}^c \gamma_k \psi^c - \psi^c \tilde{\gamma}_k \bar{\psi}^c)$.

Now: $\psi^c = C^{-1}\bar{\psi} = C^{-1}\tilde{\gamma}_4 \psi^* = C^+ \tilde{\gamma}_4 \psi^*$

$\psi^{c*} = \tilde{C} \gamma_4 \psi$

so

$\bar{\psi}^c = \tilde{\gamma}_4 \psi^{c*} = \tilde{\gamma}_4 \tilde{C} \gamma_4 \psi$

but

$C^{-1} \tilde{\gamma}_k C = -\gamma_k \qquad \tilde{C} = -C$

$\tilde{\gamma}_k C = -C \gamma_k$

$\tilde{\gamma}_k \tilde{C} = C \gamma_k$

therefore:

$\bar{\psi}^c = C \psi$.

Then:

$$j_k^c = \frac{i}{2}(\psi \tilde{C} \gamma_k C^{-1} \bar{\psi} - \bar{\psi} \tilde{C}^{-1} \tilde{\gamma}_k C \psi)$$

$$= -\frac{i}{2}(\psi C \gamma_k C^{-1} \bar{\psi} - \bar{\psi} C^{-1} \tilde{\gamma}_k C \psi)$$

$$= -\frac{i}{2}(\bar{\psi} \gamma_k \psi - \psi \tilde{\gamma}_k \bar{\psi}) = -j_k . \qquad \text{Q.E.D.}$$

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LECTURE 7

Anticommutators and Vacuum Expectation Values:

We will now define:

$$\left\{ \psi_\rho(x); \overline{\psi}_\sigma(x') \right\} = -i\, S_{\rho\sigma}(x - x')$$

[The (-i) is purely conventional.] It is easy to calculate this quantity because of the assumption of free fields.

Similarly:

$$\left\langle \psi_\alpha(x)\, \overline{\psi}_\beta(x') \right\rangle_0 = -\frac{1}{2}(S^1 + i\, S)_{\alpha\beta}$$

$$\left\langle \overline{\psi}_\beta(x')\, \psi_\alpha(x) \right\rangle_0 = \frac{1}{2}(S^1 - i\, S)_{\alpha\beta}$$

These functions are all functions of $x - x'$ only as a result of the invariance under the translation group. That the anticommutator is a c-number is a consequence of the assumption of free particles.

Now, since S, S^1 as functions of x are given by ψ's:

$$(\gamma \frac{\partial}{\partial x} + m) \begin{Bmatrix} S \\ S^1 \end{Bmatrix} = 0$$

while

$$\begin{Bmatrix} S \\ S^1 \end{Bmatrix} (\gamma \overleftarrow{\frac{\partial}{\partial x}} - m) = 0$$

where the "←" indicates an operation on the x' associated variables. For $t = t'$, we have:[1]

equation, plus the boundary condition, we may readily find:

$$\Delta(\vec{x}, t) = -\frac{i}{2(2\pi)^3} \int \frac{d^3k}{\omega} \left[e^{i(\vec{k}\cdot\vec{x} - \omega t)} - e^{-i(\vec{k}\cdot\vec{x} - \omega t)} \right]$$

or:[3]

$$\Delta(\vec{x}, t) = -\frac{1}{(2\pi)^3} \int \frac{d^3k}{\omega} e^{i\,\vec{k}\cdot\vec{x}} \sin \omega t \ .$$

We will also define:

$$\Delta^1(\vec{x}, t) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{\omega} e^{i\,\vec{k}\cdot\vec{x}} \cos \omega t \ ,$$

and we see that:

$$\Delta^1(\vec{x}, t) = \Delta^1(\vec{x}, -t)$$

$$\Delta(\vec{x}, t) = -\Delta(\vec{x}, -t) \ .$$

There are some properties which will play a role later, and which we will later generalize. We may also write:

$$i\,\Delta = \frac{1}{(2\pi)^3} \int d^4k \quad \epsilon(k_o) \; e^{i\,k\cdot x} \; \delta(k\cdot k + m^2)$$

where

$$\epsilon(k_o) = \begin{cases} +1 & k_o > 0 \\ -1 & k_o < 0 \end{cases} \ .$$

If we carry out the integration on k_o first, we readily see that Δ is in agreement with the previous result. In this we use:

$$\int g(z)\,\delta(f(z))dz = \sum_{z_o} \frac{g(z_o)}{|f'(z_o)|}$$

where $f(z_o) = 0$. [The seemingly "harmless" factor $\epsilon(k_o)$ in k-space makes for a large change in the x-space, and vice-versa.]

Similarly:

$$\Delta^1 = \frac{1}{(2\pi)^3} \int d^4k \ e^{i\,k\cdot x} \ \delta(k\cdot k + m^2).$$

Let us now consider the combination $\Delta^1 + i\,\Delta$, which occurs in the vacuum expectation values:

$$\langle \psi(x)\,\bar{\psi}(x')\rangle = -\frac{1}{2}(S^1 + i\,S)$$

$$= -\frac{1}{2}(\gamma\frac{\partial}{\partial x} - m)(\Delta^1 + i\,\Delta)$$

$\Delta^1 + i\,\Delta$ has important analytic properties. We define:[4]

$$2\,i\,\Delta_+ = \Delta^1 + i\,\Delta = \frac{2}{(2\pi)^3}\int d^4k \ \theta(k_o)\ e^{i\,k\cdot x}\ \delta(k\cdot k + m^2)$$

where:

$$\theta(k_o) \equiv \frac{1+\epsilon(k_o)}{2} = \begin{cases} +1 & k_o > 0 \\ 0 & k_o < 0 \end{cases}.$$

If we consider the 3-dimensional forms, we get:

$$2i\,\Delta_+ = \frac{1}{(2\pi)^3}\int \frac{d^3k}{\omega}\ e^{i\,\vec{k}\cdot\vec{x}}\ e^{-i\,\omega\,t}$$

and, carrying out the angular integrations:

$$2i\,\Delta_+ = \frac{2}{(2\pi)^2}\ \frac{1}{r}\int_0^\infty \frac{k\,dk}{\omega}\ \sin kr\ e^{-i\,\omega\,t}.$$

This integral is not properly defined, since it is not properly behaved as $k\to\infty$. One device for defining the integral to take the limit:

$$\lim_{\epsilon\to 0}\int \ldots \times\ e^{-\epsilon\,\omega}\,dk.$$

[We could also consider the integral as a contour integral and then choose C in an appropriate way.]

Δ_+ is evidently a function of only one variable. This is a consequence of the Lorentz invariance. (Δ, Δ^1 are invariant also. The $\epsilon(k_o)$ is always invariant for time-like points, though it is not for space-like ones.)

We define:

$$s^2 = r^2 - t^2 = x \cdot x$$

and then:

$$2\, i\, \Delta^+ = \frac{1}{(2\pi)^2} \, \frac{2}{s} \int_0^\infty \frac{k dk}{\sqrt{k^2 + m^2}} \, \sin k s \ ,$$

and then:[5]

$$2\, i\, \Delta^+ = - \frac{m}{4\, \pi\, s} \, H_1^{(1)} (i\, m\, s).$$

For space-like points, we use the Lorentz invariance and then for $t = 0$, $r^2 = s^2$; since we may always choose a reference frame in which $t' = 0$. For time-like points, the situation is not so simple. We choose:

$$s = \begin{cases} \sqrt{r^2 - t^2} & \text{space-like} \\ + i \sqrt{t^2 - r^2} & \text{forward light cone} \\ - i \sqrt{t^2 - r^2} & \text{backward light cone.} \end{cases}$$

Now, since the integrand contains only <u>positive</u> frequencies, we must consider the integrals as analytic functions of s (Wightman). This result is more general than the assumption of <u>free particles</u>. [Since Δ, Δ^1 have both positive and negative frequencies, they do not satisfy the analyticity requirement.] If Δ_+ were to vanish for space-like points, for example, the analyticity requires it to vanish everywhere.

FOOTNOTES

1 Here,

$$\left\{ \psi_\rho(x), \bar{\psi}(x') \right\} = \frac{1}{V} \sum_{r=1,2} \sum_{\vec{k}} \sum_{r'=1,2} \sum_{\vec{k}'} \left\{ \left[a_r(\vec{k})\, u_\rho^{\,r}(\vec{k})\, e^{i\,k\cdot x} + b_r^{\,*}(\vec{k})\, v_\rho^{\,r}(\vec{k}) e^{-i\,k\cdot x} \right], \left[a^*_{r'}(\vec{k}')(\gamma_4)_{\sigma'\sigma}\, u_{\sigma'}^{\,r'}(\vec{k}')^*\, e^{-i\,k'\cdot x'} + b_{r'}(\vec{k}')(\gamma_4)_{\sigma'\sigma}\, v_{\sigma'}^{\,r'}(\vec{k})^*\, e^{i\,k'\cdot x'} \right] \right\} .$$

Using:

$$\left\{ a_r(\vec{k}),\ a_{r'}^{\,*}(\vec{k}') \right\} = \delta_{rr'}\ \delta^{(3)}(\vec{k} - \vec{k}'), \text{ etc.}$$

(The $\delta^{(3)}(\vec{k})$ is really a Kronecker δ ; since we are using a V.)

we get

$$\left\{ \psi_\rho(x), \bar{\psi}_\sigma(x') \right\} = \frac{1}{V} \sum_{r=1,2} \sum_{\vec{k}} \left[u_\rho^{\,r}(\vec{k})\, u_{\sigma'}^{\,r}(\vec{k})^*\, e^{i\,k\cdot(x - x')} + v_\rho^{\,r}(\vec{k})\, v_{\sigma'}^{\,r}(\vec{k})^*\, e^{-i\,k\cdot(x - x')} \right] (\gamma_4)_{\sigma'\sigma} .$$

If we now set $t' = t$, we get:

$$\left\{ \psi_\rho(\vec{x}),\ \bar{\psi}_\sigma(\vec{x}') \right\} = \frac{1}{V} (\gamma_4)_{\rho\sigma} \sum_{\vec{k}} (e^{i\,\vec{k}\cdot\vec{x}})$$

where we have the completeness relation of the u's and v's:

$$\sum_{r=1,2} \left\{ u_\rho^{\,r}(\vec{k})\, u_\sigma^{\,r}(\vec{k})^* + v_\rho^{\,r}(-\vec{k})\, v_\sigma^{\,r}(-\vec{k})^* \right\} = \delta_{\rho\sigma} .$$

This can only be done when $t = t'$.

But, in a volume, V,

$$\delta^{(3)}(\vec{x}) = \frac{1}{V} \sum e^{i\,\vec{k}\cdot\vec{x}}$$

so:

FOOTNOTES

1 (Cont.)

$$\left\{ \psi_\rho(\vec{x}), \bar{\psi}_\sigma(\vec{x}') \right\} = (\gamma_4)_{\rho\sigma} \; \delta^{(3)}(\vec{x} - \vec{x}') \; .$$

The completeness relation is

$$P^{+}_{\rho\sigma} + P^{-}_{\rho\sigma} = 1_{\rho\sigma}$$

2 These follow from:

$$\lim_{t \to 0} \left\{ \left(\frac{1}{i} \gamma_4 \frac{\partial}{\partial t} + \vec{\gamma} \cdot \frac{\partial}{\partial \vec{x}} - m \right) \Delta \right\} = i \, \gamma_4 \; \delta^{(3)}(\vec{x}) \; .$$

Since the γ's and 1 are independent, we see that the boundary condition is satisfied and required.

$$\frac{\partial}{\partial \vec{x}} \Delta(\vec{x}, 0) = 0 \qquad \text{and} \qquad \Delta(\vec{x}, 0) = 0 \quad \text{(from "m" term).}$$

3 These forms are easily obtained. From the differential equation

$$\Delta(\vec{x}, t) = \int d^3k \left\{ f_+(\vec{k}) \, e^{i(\vec{k}\cdot\vec{x} - \omega t)} + f_-(\vec{k}) \, e^{+i(\vec{k}\cdot\vec{x} + \omega t)} \right\}$$

and since

$$\frac{\partial \Delta}{\partial t}\bigg|_{t \to 0} = - \delta^{(3)}(\vec{x}) = \int d^3k \left[f_+(\vec{k}) - f_-(\vec{k}) \right] (-i\omega) e^{i\vec{k}\cdot\vec{x}} .$$

$$\therefore \quad f_+(\vec{k}) - f_-(\vec{k}) = \frac{-i}{(2\pi)^3 \omega}$$

Further

$$\Delta(\vec{x}, 0) = 0$$

so

$$f_+(\vec{k}) + f_-(\vec{k}) = 0$$

from which the result follows quickly.

FOOTNOTES

4
$$\Delta = \Delta_+ + \Delta_-$$
$$\Delta^1 = i(\Delta_+ - \Delta_-) .$$

From this, we see that S^1 indeed satisfies $\left\langle \left[\psi(x), \bar{\psi}(x') \right] \right\rangle_0$.

Calculating the commutator, we get:

$$\left\langle \left[\psi(x), \bar{\psi}(x') \right] \right\rangle_0 = -i(S_+ - S_-) = - S^1$$

since the positive frequency parts satisfy the same relation

$$\left\{ a_r(\vec{k}), a^*_{r'}(\vec{k}') \right\} = \delta_{rr'} \delta(\vec{k} - \vec{k}')$$

$$\left\langle \left[a_r(\vec{k}), a^*_{r'}(\vec{k}') \right] \right\rangle_0 = \delta_{rr'} \delta(\vec{k} - \vec{k}') .$$

However, the negative frequency parts come in with

$$\left\langle \left[b^*_r(\vec{k}), b_{r'}(\vec{k}') \right] \right\rangle_0 = - \delta_{rr'} \delta(\vec{k} - \vec{k}') .$$

5 We have (Watson, Bessel Functions, p. 183, Eq. (14)):

$$K_o(x) = \int_0^\infty \frac{\cos (xt)}{\sqrt{(t^2 + 1)}} dt$$

$$K'_o(x) = - \int_0^\infty \frac{t \sin(xt)}{\sqrt{t^2 + 1}} dt ,$$

But (p. 79, Eq. (4) of Watson)

$$K_o'(x) = - K_1(x) .$$

Now setting $k = mt$,

FOOTNOTES

5 (Cont.)

$$\int_0^\infty \frac{k\,dk}{\sqrt{k^2 + m^2}} \sin ks = m \int_0^\infty \frac{t\,dt}{\sqrt{t^2 + 1}} \sin(ms)t$$

$$= m\, K_1(ms).$$

However (p.78, Eq. (8) of Watson)

$$= - \frac{m\,\pi}{2} H_1^{(1)}(i\, m\, s).$$

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Lecture 8

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Radiation Laboratory
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LECTURE ON "CONTINUOUS GROUPS AND REFLECTIONS IN QUANTUM MECHANICS" BY W. PAULI

Lecture 8

R. J. Riddell, Jr.

May 1, 1958

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Notes by R. J. Riddell, Jr.

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Spring 1958

LECTURE 8

We will now define functions of positive class. We will restrict the discussion to Lorentz invariant functions. Then:

$$F_+(x) = \frac{1}{(2\pi)^3} \int d^4k \quad \theta(k_o)\ \rho(-k\cdot k)\ e^{i\ k\cdot x} .$$

The θ insures the presence of only positive frequencies, but F will only be Lorentz invariant if

$$\rho(-k\cdot k) = 0 \qquad \text{for} \quad k\cdot k > 0. \qquad \text{(space-like)}$$

We have

$$i\ \Delta_+(x;\lambda) = \frac{1}{(2\pi)^3} \int d^4k \quad \theta(k_o)\ \delta(k\cdot k + \lambda) e^{i\ k\cdot x}$$

where

$$\lambda \equiv m^2.$$

We can obtain F_+ from Δ_+ by introducing a "spectrum" of rest masses. Then:

$$F_+(x) = \int_0^\infty d\lambda \quad \rho(\lambda)\ i\Delta_+(x;\lambda).$$

[It is important that only m^2 appears.] Clearly:

if	$F(x)$	is in the positive class,
then	$F(-x)$	is in the negative class,
and	$F^*(x)$	is in the negative class,
so	$F^*(-x)$	is in the positive class.

Now, we go back to the problem of the analytic continuation. We have seen that for functions of positive class, the function will be analytic under the change:

$$t \rightarrow t - i\,\theta \quad , \qquad \theta > 0 \ ,$$

since

$$e^{-i k_o t} \rightarrow e^{-i k_o t} \times e^{-k_o \theta} \quad .$$

The integral will then still exist, and hence such functions can be continued into the lower half plane. Functions of negative class can be continued into the upper half plane, and those of mixed class cannot be continued at all.

Side remark: We are here reminded of the canonical ensemble, since there we also have $e^{-E/kT}$. Thus $\theta \rightarrow \frac{1}{kT}$. Pauli doesn't know of any deeper significance here. The analytic continuation may have physical significance, though he doesn't know what it is.

The continuation may be done in an invariant way:

$$x_k \rightarrow x_k - i\,\theta_k$$

where

$$(\theta \cdot \theta) < 0 \ , \qquad \theta_o > 0 \quad \text{(Forward cone)},$$

and then

$$e^{i\,k \cdot x} \rightarrow e^{i\,k \cdot x + k \cdot \theta} \quad .$$

Now, $(k \cdot \theta) < 0$, since $k \cdot \theta = \vec{k} \cdot \vec{\theta} - k_o \theta_o$, and

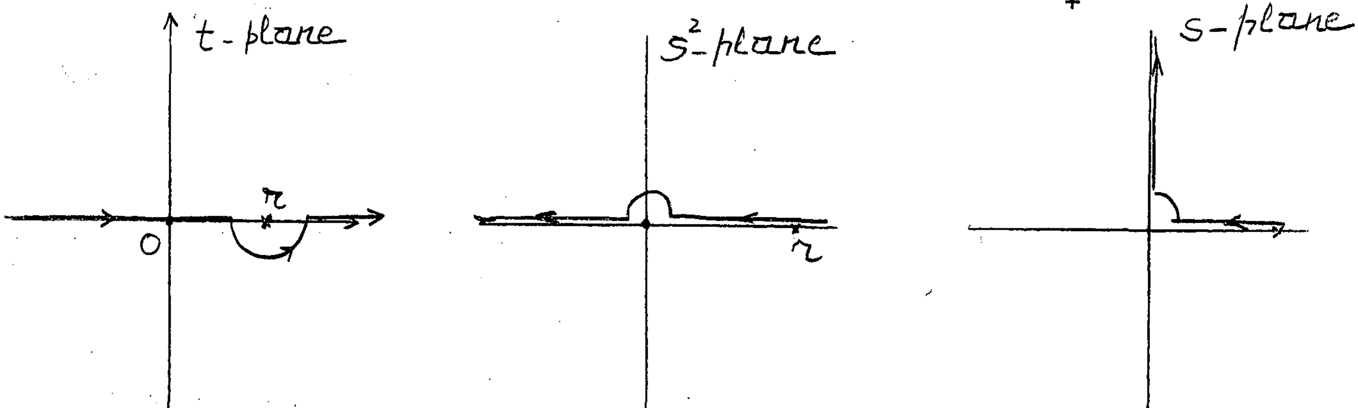
$$k_o > |\vec{k}| \ , \qquad \theta_o > |\vec{\theta}| \quad .$$

In this case, the canonical ensemble is extended to a distribution in momentum as well as energy.

Now, let us consider the analytic continuation. The singular points of S and $H_1^{(1)}(s)$ are those for:

$$t = r; \quad t = -r : \qquad s = 0.$$

If we follow t and s, we will obtain the continuation of Δ_+ .



$$s = \sqrt{r^2 - t^2} \qquad \text{(space-like)}.$$

Near the singularity, we choose $t = r - \epsilon e^{i\theta}$, to remain in the lower half plane. Then

$$s^2 = r^2 - t^2 \approx 2 r \epsilon e^{i\theta} \qquad 0 \leq \theta \leq \pi$$

$$s = \sqrt{2 r \epsilon} \; e^{i\theta/2} .$$

Thus, in the forward light cone $s = + i\sqrt{t^2 - r^2}$. For the backward cone we find[1]

$$s = - i \sqrt{t^2 - r^2} .$$

For negative class functions, we take the conjugate definition. For the mixed case, we can do nothing.

We now wish to apply this analysis to Δ_+ :

$$2i \, \Delta_+ = - \frac{m}{4\pi s} \; H_1^{(1)} (ims) .$$

Now

$$H_1^{(1)}(z) = J_1(z) + i \, N_1(z) ,$$

and

$$J_1(z) = \frac{z}{2}\left\{1 - \frac{z^2}{2^2 \cdot 2!} + \ldots\right\}$$

$$N_1(z) = \frac{2}{\pi}\left(\log\frac{z}{2} + \gamma\right) J_1(z) - \frac{2}{\pi z} + F_1(z)$$

where γ is Euler's constant, and $F_1(z)$ is an odd function of z with no singularities in the finite z plane. Everything is uniquely defined except for the $\log\frac{z}{2}$ term. We find[2]

$$H_1^{(1)}(z) = \left(1 - 4n - \frac{2\phi}{\pi}\right) J_1(z) + i\,\frac{2}{\pi}\left\{\left(\ln\frac{r}{2} + \gamma\right) J_1(z) - \frac{1}{z} + \frac{\pi}{2} F_1(z)\right\}$$

where

$$z = r\, e^{i(2\pi n + \phi)} , \qquad 0 \leq \phi < 2\pi .$$ n is an arbitrary integer, determined by the Riemann sheet which z is on.

Thus, the only unknown quantity is n. To determine it, we see that if x is space-like, $z = ir$, so

$$\phi = \frac{\pi}{2} , \quad \text{and } \frac{1}{z}, J_1(z), F_1(z) \text{ are pure imaginary.}$$

Now for space-like surfaces, $\Delta = 0$ and so

$$2i\,\Delta_+ = \Delta^1 = \text{pure real,}$$

so that $n = 0$.

Now, in the forward light cone, $s = i\,|s|$, and so $z = -|s|$, $\phi = \pi$. Thus we have

$$H_1^{(1)}(z) = -J_1(-m\,|s|) + i\cdot\frac{2}{\pi}\left\{\left(\ln\frac{|s|}{2} + \gamma\right) J_1(-m\,|s|) + \frac{1}{|s|} + \frac{\pi}{2} F_1(-m\,|s|)\right\}$$

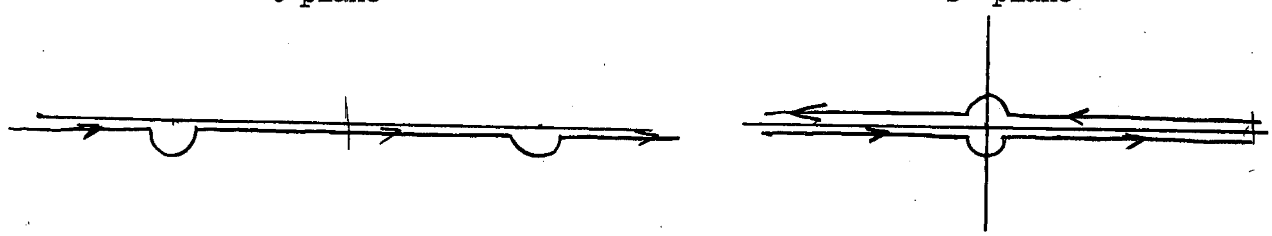
so[3]

$$2i\,\Delta_+ = \frac{m}{4\pi\,|s|}\left\{i\,J_1(m\,|s|) + N_1(m\,|s|)\right\} .$$

Similarly, in the backward cone $s = -i|s|$, $z = |s|$, and

$$2i\,\Delta_+ = \frac{m}{4\pi|s|}\left\{-i\,J_1(m|s|) + N_1(m|s|)\right\} .$$

Now we can obtain the Δ and Δ_1 by taking the real and imaginary parts of $2i\,\Delta_+ = \Delta_1 + i\,\Delta$. This must be done with care, since the $\frac{1}{s^2}$ singularity must be defined. The $\Delta_+(s^2)$ will generally occur in integrals over s^2, and thus we will set:



where we have chosen t in the way that preserves the analyticity of Δ_+. Thus, at the pole in the backward light-cone singularity, we get $\frac{1}{2}$ the usual integral around the pole. On the other hand, if we integrate s^2 in the increasing direction, we obtain $-\frac{1}{2}$, since we integrate clockwise. Thus we find

$$\frac{1}{s^2} = P\left(\frac{1}{s^2}\right) - \epsilon(t)\cdot i\pi\ \delta(s^2) ,$$

where P indicates that a principal value is meant for the integration. Thus we obtain[4]

$$4\pi\,\Delta = \begin{cases} \epsilon(t)\left[\frac{m}{|s|}\,J_1(m|s|) - 2\,\delta(s^2)\right] & s^2 \leq 0 \\ 0 & s^2 > 0 . \end{cases}$$

[Here, we have obtained the "famous" $\epsilon(t)$.]

Now, if we go back to the $F_+(s)$, we find for the singular parts[5] in s^2:

$$(2\pi)^2 F_+(s^2) = \frac{\ln s}{2} \cdot \int_0^\infty \rho(\lambda)\cdot\lambda \, d\lambda + \frac{1}{s^2} \int_0^\infty \rho(\lambda) d\lambda + \text{convergent parts.}$$

If we divide F_+ into its real and imaginary parts:

$$F_+ = F_1 + i F ,$$

we find:[6]

$$4\pi F = \begin{cases} \epsilon(t) \left[- \delta(s^2) \int_0^\infty \rho(\lambda) d\lambda + \frac{1}{2} \int_0^\infty \rho(\lambda)\lambda \, d\lambda + \mathcal{O}(s^2) \right] & (s^2 < 0) \\ 0 & (s^2 > 0) \end{cases}$$

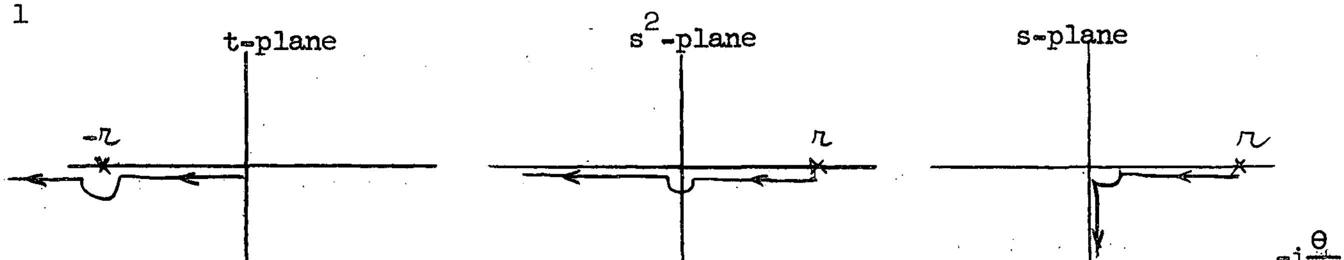
[In renormalization theory one assumes that

$$\int_{m_0^2}^\infty \frac{\rho(\lambda)}{\lambda} d\lambda$$

is convergent, though $\int_0^\infty \rho(\lambda) d\lambda = \infty$, and one then subtracts the divergences.[7]]

FOOTNOTES

1



$t = -r + \epsilon e^{-i\theta}$ $\qquad s^2 \approx 2 r \epsilon e^{-i\theta}$ $\qquad s \approx \sqrt{2 r \epsilon} \; e^{-i\frac{\theta}{2}}$

$0 \leq \theta \leq \pi$

2 Since $\ln z = \ln r + i \phi$ if $z = r e^{i\phi}$.

FOOTNOTES

3

$$2i\,\Delta_+ = \frac{i\,m}{4\pi\,|s|}\left\{-J_1(-m\,|s|) + \frac{i\,2}{\pi}\left[\left(\ln\frac{|s|}{2}+\gamma\right)J_1(-m\,|s|) + \frac{1}{|s|} + \frac{\pi}{2}F_1(-m\,|s|)\right]\right\}$$

$$= \frac{i\,m}{4\pi\,|s|}\left\{J_1(m\,|s|) - \frac{i\,2}{\pi}\left[\left(\ln\frac{|s|}{2}+\gamma\right)J_1(m\,|s|) - \frac{1}{|s|} + \frac{\pi}{2}F_1(m\,|s|)\right]\right\} .$$

4

$$2i\,\Delta_+ = -\frac{m}{4\pi s}\,H_1^{(1)}(ims) \sim -\frac{m}{4\pi s}\left[-\frac{i\,2}{\pi\cdot ims}\right] = \frac{1}{4\pi}\,\frac{2}{\pi s^2}$$

$$\sim \frac{1}{4\pi}\left(-2i\,\epsilon(t)\,\delta(s^2)\right) .$$

5

$$F_+(x) = \int_0^\infty d\lambda\;\rho(\lambda)\;i\,\Delta_+(x;\lambda)$$

$$= \int_0^\infty d\lambda\;\rho(\lambda)\left[-\frac{m}{8\pi s}\;H_1^{(1)}(ims)\right] .$$

But:

$$H_1^{(1)}(ims) = i\left(\frac{2}{\pi}\log s \times \frac{ism}{2} - \frac{2}{\pi i\,ms}\right) + \text{regular parts}$$

so

$$F_+(x) = \int_0^\infty d\lambda\;\rho(\lambda)\left[\frac{m^2}{8\pi^2}\log s + \frac{1}{4\pi^2}\,\frac{1}{s^2}\right]$$

$$(m^2 = \lambda) .$$

6 This comes immediately on expanding J_1 in the expression for Δ, and then using the linearity of the relation between Δ_+ and F_+.

7 See, for example, H. Lehmann, Nuovo Cimento 11, 342 (1954).

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LECTURE 9

Now, we have seen that

$$F_+(s) = \frac{1}{(2\pi)^3} \int d^4k \;\; \theta(k_o) \;\rho(-k\cdot k)\; e^{i\,k\cdot x}$$

$$= -\frac{1}{8\pi s} \int_0^\infty \rho(\lambda) \sqrt{\lambda}\;\; H_1^{(1)} (i \sqrt{\lambda} \;\; s) d\lambda \quad .$$

There is a possible generalization of this form if we allow an indefinite metric. There could be a complex value of the mass and the analyticity would be preserved. $(\lambda^2 = m^2)$ We may set:

$$m = m_o + i\mu \qquad\qquad m_o \geq 0$$

$$k_\mu = m\,\kappa_\mu \qquad\qquad \kappa_\mu \kappa_\mu = -1$$

where κ_μ has the ordinary reality properties. We may add a finite number of terms of this type and still preserve the analyticity theorem of Wightman, since $H_1^{(1)}$ is an analytic function of m. If we have a specific F_+, we have terms added of the form

$$F_+(s) = -\sum_m \left\{ A(m) \;\frac{m}{8\pi s}\; H_1^{(1)} (ims) + B(m) \frac{m^*}{8\pi s}\; H_1^{(1)}(im^* s) \right\} \quad .$$

If F_+ has a reality condition, we may find $B = A^*$, though in general this would not be necessary.[1]

Let us now go back to the case of free particles. We recall that

$$\langle \psi_\alpha(x)\, \bar\psi_\beta(x') \rangle_0 = -\frac{1}{2}(S^1 + i\, S)_{\alpha\beta}\,(x - x')$$

$$\langle \bar\psi_\beta(x')\, \psi_\alpha(x) \rangle_0 = \frac{1}{2}(S^1 - i\, S)_{\alpha\beta}\,(x - x')$$

where

$$\begin{Bmatrix} S \\ S^1 \end{Bmatrix} = \left(\gamma \frac{\partial}{\partial x} - m\right) \begin{Bmatrix} \Delta \\ \Delta^1 \end{Bmatrix} .$$

Now we wish to consider two kinds of operations: Charge conjugation and reflections.

<u>Charge conjugation:</u>

Now

$$\psi^c = C^{-1}\,\psi = -\bar\psi\, C^{-1}$$

$$\bar\psi^c = C\,\psi$$

and $C\,C^+ = 1$, $\quad C\,\gamma_k\, C^{-1} = -\tilde\gamma_k$. From these relations we find that

$$\langle \psi_\alpha{}^c(x)\, \bar\psi_\beta{}^c(x') \rangle_0 = \langle \psi_\alpha(x)\, \bar\psi_\beta(x') \rangle_0$$

and

$$\langle \bar\psi_\beta{}^c(x')\, \psi_\alpha{}^c(x) \rangle_0 = \langle \bar\psi_\beta(x')\, \psi_\alpha(x) \rangle_0 .$$

These follow from the properties of the C and S:[2]

$$\left(C(S^1 \mp i\, S)C^{-1}\right)_{\alpha\beta} = \left(S^1 \pm i\, S\right)_{\beta\alpha}(-x).$$

These relations may be generalized to the case of a mass spectrum as:

$$\langle \psi_\alpha(x)\, \bar\psi_\beta(x') \rangle_0 = -\left\{ (\gamma \frac{\partial}{\partial x})_{\alpha\beta}\; F(s) + \delta_{\alpha\beta}\, G(s) \right\}$$

$$\langle \bar\psi_\beta(x')\, \psi_\alpha(x) \rangle_0 = (\gamma \frac{\partial}{\partial x})_{\alpha\beta}\; F(s^*) + \delta_{\alpha\beta}\; G(s^*) .$$

Here s^* appears as the variable in the second equation, since as a function of $x - x'$, the function is of <u>negative</u> class. Thus, for the analytic continuation to apply, s^* rather than s appears. Here, even with this generalization of the commutation relations, the charge conjugation invariance holds. The minus sign in the first relation is conventional. In general there will be no relation between $\langle \psi_\alpha(x)\, \bar{\psi}_\beta(x') \rangle_0$ and $\langle \bar{\psi}_\beta(x')\, \psi_\alpha(x) \rangle_0$, but here the relation is so chosen that charge conjugation invariance occurs.[3] There is a reality condition on $F(s)$, $G(s)$, which requires that they be real, for real s.[4]

<u>Strong reflections</u>: (Equivalent to C P T).

We now go back to the case of a single free spinor field, of mass m. We define the strong reflections so that

$$x' = -x \qquad\qquad j'_\mu(x) = -j_\mu(-x) \quad .$$

[Under P, the space components of j_μ change sign, and so do they under T. Thus P T gives no change, while C changes the sign.]

We will now set:

$$\psi'(x) = e^{i\alpha}\, \gamma_5\, \psi(-x)$$

$$\bar{\psi}'(x) = -e^{-i\alpha}\, \bar{\psi}(-x)\gamma_5 \quad .$$

The value of α cannot be determined in a gauge invariant theory. This is very similar to the usual space reflections.

[In the Majorana theory:

$$\alpha = \frac{\pi}{2} \quad \text{since} \quad \psi^c = \psi \quad .]$$

We have a further rule to be considered. We have

$$j_\mu(x) = \frac{i}{2} (\bar{\psi} \gamma_\mu \psi - \psi \tilde{\gamma}_\mu \bar{\psi}) \quad .$$

The above transformation in ψ will only be correct if we add a third operational rule for computations.

Inversion

$$j_\mu'(x) = \frac{i}{2} (\psi' \tilde{\gamma}_\mu \bar{\psi}' - \bar{\psi}' \gamma_\mu \psi') = - j_\mu(-x)$$

where we have changed the order of the factors, or we read from right to left in the operations. The reason for this inversion is evident already in the equations of motion:

$$i \left[P_\mu, f \right] = - \frac{\partial f}{\partial x_\mu} \quad .$$

The total E must not change sign (and hence $\vec{P}$ must be invariant) because $E > E_{vac.}$ Thus:

$$P_\mu' = P_\mu$$

from Lorentz invariance. Thus

$$i \left[P_\mu, f \right]' = - \frac{\partial f'}{\partial x_\mu'} = \frac{\partial f'}{\partial x_\mu} = - i \left[P_\mu, f' \right] \quad .$$

Thus

$$\left[P_\mu, f \right]' = - \left[P_\mu, f' \right] \quad .$$

This can be brought about by the change of order.

Let us consider the effect of this transformation on the vacuum expectation value of the product of two free field operators. According to the above prescription:

$$\langle \psi_\alpha(x)\ \bar\psi_\beta(x')\rangle_0' = \langle \bar\psi'_\beta(x')\ \psi_\alpha'(x)\rangle_0$$

$$= -(\gamma_5)_{\alpha\alpha'}\ \langle \bar\psi_{\beta'}(-x')\ \psi_{\alpha'}(-x)\rangle_0\ (\gamma_5)_{\beta'\beta}$$

$$= -\frac{1}{2}\left[\gamma_5(S^1 - i\ S)\ \gamma_5\right]_{\alpha\beta}(x' - x) \quad .$$

But:

$$\gamma_5\ S_1(-x)\ \gamma_5 = S_1(x)$$

$$\gamma_5\ S(-x)\ \gamma_5 = -S(x)$$

since the γ_5's change the sign of the γ term in the relation between S, Δ and the $(-x)$ changes it again. Further $\Delta(x)$ is odd in x, and Δ_1 is even. Thus

$$\langle \psi_\alpha(x)\ \bar\psi_\beta(x')\rangle_0' = -\frac{1}{2}(S^1 + i\ S)_{\alpha\beta}\ (x - x') \quad .$$

Thus all of the functions (anticommutators, etc.) are invariant under the C P T transformation.

FOOTNOTES

1 References to this general question of analyticity:
A. S. Wightman, Phys. Rev. 101, 860 (1956);
Wightman and Hall, Kgl. Danske Vidensk 31, 5 (1957).

2 We have

$$\langle \psi_\alpha{}^c(x)\ \bar\psi_\beta{}^c(x')\rangle_0 = -C_{\beta\beta'}\ \langle \bar\psi_{\alpha'}(x)\ \psi_{\beta'}(x')\rangle_0\ C_{\alpha'\alpha}{}^{-1}$$

$$= -C_{\beta\beta'}\ \frac{1}{2}\left(\gamma\ \frac{\partial}{\partial x'} - m\right)_{\beta'\alpha'}(\Delta^1 - i\ \Delta)(x' - x)C_{\alpha'\alpha}{}^{-1}$$

$$= -\frac{1}{2}\left(\gamma\ \frac{\partial}{\partial x} - m\right)_{\beta\alpha}(\Delta^1 + i\ \Delta)(x - x')$$

$$= -\frac{1}{2}(S^1 + i\ S)_{\alpha\beta}\ (x - x')$$

FOOTNOTES

2 (Cont.)

and similarly for $\langle \overline{\psi}_\beta^{\,c}(x')\, \psi_\alpha^{\,c}(x) \rangle_0$. Here, we have used the fact that Δ^1 is even in $(x - x')$ and Δ is odd.

3 We see that

$$\langle \psi_\alpha^{\,c}(x)\, \overline{\psi}_\beta^{\,c}(x') \rangle_0 = -C_{\beta\beta'} \langle \overline{\psi}_{\alpha'}(x)\, \psi_{\beta'}(x') \rangle_0\, C_{\alpha'\alpha}^{-1}$$

$$= -C_{\beta\beta'} \left\{ (\gamma \frac{\partial}{\partial x'})_{\beta'\alpha'}\; F\big(s^*(x' - x)\big) + \delta_{\beta'\alpha'} G\big(s^*(x'-x)\big) \right\} C_{\alpha'\alpha}^{-1}$$

But $s(x' - x) \equiv s^*(x - x')$, since $| s(x' - x) | = | s(x - x') |$ and the only changes necessary are to change the future cone into the past in going from $x' - x$ to $x - x'$. Thus

$$\langle \psi_\alpha^{\,c}(x)\, \overline{\psi}_\beta^{\,c}(x') \rangle_0 = - \left\{ (\gamma \frac{\partial}{\partial x})_{\alpha\beta} F(s) + \delta_{\alpha\beta}\; G(s) \right\} \qquad \text{Q.E.D.}$$

Note here that we assume $\vec{r}$, t are real.

4 There is a reality condition on F, G. We have:

$$\langle \psi_\alpha(x)\, \overline{\psi}_\beta(x') \rangle_0 = \langle \psi_\alpha(x)\, \psi_{\beta'}^*(x') \rangle_0 (\gamma_4)_{\beta'\beta} = - \left\{ (\gamma \cdot \frac{\partial}{\partial x})_{\alpha\beta} F(s) + \delta_{\alpha\beta} G(s) \right\}.$$

In general, ψ^* will be the "adjoint" of ψ, and we will have

$$\langle A \mid M^* \mid B \rangle = \langle B \mid M \mid A \rangle^*, \text{ and } (AB)^* = B^* A^* .$$

Thus, here:

$$\langle \psi_\alpha(x)\, \psi_{\beta'}^*(x') \rangle_0^* (\gamma_4)_{\beta\beta'} = \langle \psi_{\beta'}(x')\, \psi_\alpha^*(x) \rangle_0 (\gamma_4)_{\beta\beta'}$$

$$= -(\gamma_4)_{\beta\beta'} \left\{ (\gamma \cdot \frac{\partial}{\partial x'})_{\beta'\beta''} F(s^*) + \delta_{\beta'\beta''} G(s^*) \right\} (\gamma_4)_{\beta''\alpha} .$$

FOOTNOTES

4 (Cont.)

But, since x_1, x_2, x_3 are real and x_4 is pure imaginary:

$$= - \left\{ \left(\gamma \cdot \frac{\partial}{\partial x^*}\right)_{\beta\alpha} F(s^*) + \delta_{\beta\alpha} G(s^*) \right\}$$

$$= - \left\{ \left(\gamma \cdot \frac{\partial}{\partial x}\right)_{\alpha\beta}^* F(s^*) + \delta_{\alpha\beta} G(s^*) \right\}$$

so

$$F^*(s) = F(s^*) \quad \text{and} \quad G^*(s) = G(s^*) \ .$$

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Lecture 10

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LECTURE 10

Now we consider the state vectors rather than the operators.[1] We bear in mind that in constructing expectation values we have to deal with two kinds of state vectors, one which appears on the right and one, on the left. Thus we have:

$$\langle a \mid \mathcal{O} \mid b \rangle = \Psi_a^* \, \mathcal{O} \, \Psi_b \quad .$$

In the case of a positive definite metric, we choose Ψ^* as the conjugate complex of Ψ. More generally, it will represent the adjoint.

Now, if $\mathcal{O}$ is the unit operator, $\mathcal{O} \equiv 1$

$$\langle a \mid b \rangle = \Psi_a^* \, \Psi_b \quad .$$

To compute expectation values, it is necessary to have a connection between the bras and the kets, though the eigenvalue problem can be solved independently of the correspondence. At this point, the bras and kets can be considered as completely different types of objects. Now, with Schwinger, we assume that they are in fact objects of the same type and that the same vector can be either a bra or a ket. The ket vector that is identical to bra vector ψ_a^* may be denoted by

$$\Psi_{\bar{a}} \equiv \Psi_a^* , \qquad \Psi_{\bar{b}}^* \equiv \Psi_b \quad . \tag{10.1}$$

Assuming further that the order of the vectors in a scalar product is not significant we have

$$\langle a \mid b \rangle = \langle \bar{b} \mid \bar{a} \rangle \quad .$$

We next define the "transposed operator," $\mathcal{O}^T$, by

$$\mathcal{O}\,\bar{\Psi} \equiv \Psi\,\mathcal{O}^T \quad .$$

Then one easily obtains the relations

$$\langle a \,|\, \mathcal{O} \,|\, b \rangle \equiv \langle \bar{b} \,|\, \mathcal{O}^T \,|\, \bar{a} \rangle \tag{10.2}$$

and

$$(A\ B)^T = B^T A^T . \tag{10.3}$$

If we have a metric preserving transformation then $U\,U^* = 1$. In the general case of an indefinite metric, we still write $U\,U^* = 1$, but U^* is now meant to be the adjoint (rather than conjugate) and U is not unitary, but rather "pseudo-unitary." Then if we have:

$$\Psi_a' = U\,\Psi_a \qquad\qquad \Psi'^*_a = \Psi^* U_a^*$$

and if:

$$\mathcal{O}' = U\,\mathcal{O}\,U^* \quad ,$$

the expectation values will be preserved:

$$\Psi^{*\,\prime}_a\,\mathcal{O}'\,\Psi'_a = \Psi^*_a\,\mathcal{O}\,\Psi_a \quad .$$

It is possible to combine a metric preserving transformation with the transposition operation. That is, we may generalize the relation (10.1) by setting:

$$\Psi_{\bar{a}} \equiv \Psi_a^*\,U^* \quad , \qquad\qquad \Psi_{\bar{a}}^* \equiv U\,\Psi_a \quad .$$

Then if we require:

$$\langle \bar{a} \,|\, \bar{\mathcal{O}} \,|\, \bar{b} \rangle \equiv \langle b \,|\, \mathcal{O} \,|\, a \rangle \quad ,$$

we find:

$$\bar{\mathcal{O}} \equiv U^*\,\mathcal{O}^T\,U$$

and:

$$\overline{A\,B} = \bar{B}\,\bar{A} \quad .$$

We have here defined a transposition operation which will be used in the definition of time reversal. This will avoid complex conjugation.

Remark: Wigner introduced the concept of "anti-linear operator" to discuss time reversal.

Linear operator: $\mathcal{O}(c_1 \Psi_1 + c_2 \Psi_2) = c_1 \mathcal{O} \Psi_1 + c_2 \mathcal{O} \Psi_2$

Anti-linear operator: $\bar{\mathcal{O}}(c_1 \Psi_1 + c \Psi_2) = c_1^* \bar{\mathcal{O}} \Psi_1 + c_2^* \bar{\mathcal{O}} \Psi_2$.

Let us now return to the strong reflections. [The CPT inversion is identical to strong reflections.]

One must <u>always</u> include the bra → ket inversion in some form to get the time reversal, T. For CPT inversion, we have taken,

$$\psi'(x) = e^{i\alpha} \gamma_5 \psi(-x)$$

and;

$$j_\mu'(x) = -j_\mu(-x) \quad .$$

In charge conjugation,

$$\psi^c \equiv C^{-1} \bar{\psi} = -\bar{\psi} C^{-1}$$

$$\bar{\psi}^c \equiv C \psi = -\psi C \quad .$$

We want to require that CPT have the same form for charge conjugated quantities, so that

$$\psi'^c(x) = e^{i\alpha} \gamma_5 \psi^c(-x) \quad .$$

Then we find[2] $\alpha = \frac{\pi}{2}$. Thus we have fixed α. ($e^{i\alpha} = i$)

Thus we find:

$$\psi'(x) = i \gamma_5 \psi(-x), \qquad \psi'^c(x) = i \gamma_5 \psi^c(-x) \quad .$$

We wish to consider now other classes of fields; specifically, we want to show that the transformation of quantities under CPT is already known from the continuous Lorentz transformation properties.

We have used certain assumptions in the development:

1. The vacuum is the state of lowest energy.
2. Spinors satisfy anticommutation relations.
3. The theory is local in character. [The current is a local quantity. The momentum, P_μ, is not local, and behaves differently.]

Let us consider, for instance, a quantity like:

$$R(x) \equiv \frac{1}{2}(\overline{\psi}(x)\, \mathcal{O}\, \phi(x) - \phi(x)\, \widetilde{\mathcal{O}}\, \overline{\psi}(x))$$

where ψ, ϕ transform in the same way.

If:

$\mathcal{O} = \gamma_\mu$ or $\gamma_\mu \gamma_5$ R = vector (V) axial vector (A)

$\mathcal{O} = 1$ or γ_5 R = scalar (S), or pseudoscalar (P)

$\mathcal{O} = \frac{1}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$ R = antisymmetric tensor (T).

Now, if we introduce:

$$\psi'(x) = i\,\gamma_5\, \psi(-x); \qquad \overline{\psi}'(x) = i\,\overline{\psi}(-x)\gamma_5 ,$$

$$\phi'(x) = i\,\gamma_5\, \phi(-x); \qquad \overline{\phi}'(x) = i\,\overline{\phi}(-x)\,\gamma_5 ;$$

and we take the inversion into account in computing $R'(x)$; that is:

$$R'(x) = \frac{1}{2}(\phi'(x)\, \widetilde{\mathcal{O}}\, \psi'(x) - \psi'(x)\, \widetilde{\mathcal{O}}\, \phi'(x)) ,$$

then we find the rules for CPT inversion:

$$(V, A)'(x) = -(V, A)(-x)$$

$$(S, P)'(x) = +(S, P)(-x)$$

$$T'(x) = +T(-x) .$$

This is in accord with the behavior of the current or a coordinate,

$$x_\mu' \rightarrow -x_\mu .$$

For the energy-momentum vector, however:

$$P_\mu = \int d^3x \ T_{\mu 4}(x) \quad ,$$

so that P_μ has the properties of part of a tensor under CPT. A scalar can be made from $T_{\nu\nu} = S$.

The theorem of CPT invariance was first formulated in a clear way by Luders.

NOTE: Pauli's Rule of Transposition and CPT

(1) In order to understand Pauli's prescription let us first consider time reversal in ordinary (first quantized) quantum mechanics. We suppose that the operators $q(t)$ and $p(t)$ represent a possible solution to the equations of motion. That is,

$$\left[H(p(t),\ q(t);\ t),\ q(t)\right] = -i \frac{dq(t)}{dt}$$

$$\left[H(p(t),\ q(t);\ t),\ p(t)\right] = -i \frac{dp(t)}{dt}$$

where

$$\left[p(t),\ q(t)\right] = -i \ .$$

In classical mechanics the "time reversed" physical process is defined by the equations:

$$q'(t) = q(-t) \qquad\qquad p'(t) = -p(-t) \ .$$

For a large class of hamiltonians this "time reversed" process will also be a solution of the equations of motion.

In quantum mechanics, the operators defined by the above equations do not obey the commutation relations. However it is easily verified that these relations are satisfied by

$$q_t(t) = q^T(-t) \qquad p_t(t) = -p^T(-t)$$

where here, as throughout, $\mathcal{O}^T$ represents the transpose of $\mathcal{O}$.

We may now inquire whether these transformed operators are also solutions to the (original) equations of motion. If one takes the transpose of the original equations of motion for $q(t)$ and $p(t)$ and replaces the dummy variable t by $(-t)$ then one obtains

$$-i\,\frac{dq^T(-t)}{d(-t)} = i\,\frac{dq^T(-t)}{dt} = \left[H(p(-t),\ q(-t);\ -t),\ q(-t)\right]^T$$

$$= \left[q^T(-t),\ H^{tr}(p^T(-t),\ q^T(-t);\ -t)\right] \quad .$$

The introduction of the new variables gives

$$\left[H^{tr}(-p_t(t),\ q_t(t);\ -t),\ q_t(-t)\right] = -i\,\frac{dq_t(t)}{dt} \quad .$$

The corresponding equation for $p_t(t)$ is obtained in the same way. [A superscript "tr" on an operator $\mathcal{O}(\xi, \eta)$ means the order of the operators ξ and η are to be inverted, but that these operators are not transposed; i.e., $\mathcal{O}(\xi, \eta)^T \equiv \mathcal{O}^{tr}(\xi^T, \eta^T)$]. These equations show that if

$$H^{tr}(-p,\ q,\ -t) \equiv H(p,\ q,\ t)$$

then the transformed operators $P_t(t)$ and $q_t(t)$ will satisfy the original equations of motion. [It might be remarked that for the Hermitian operators p and q the transpose and complex conjugate operators are identical so that one could have used complex conjugation instead of transposition in this case.]

(2) The operation of transposing an operator is somewhat subtle. To examine the properties of this operation, we follow Wigner who introduces the (antilinear) operator K which is required to satisfy the relations

$$\langle Ka \mid Kb \rangle = \langle a \mid b \rangle^*$$

$$K \left\{ \alpha \mid a \rangle + \beta \mid b \rangle \right\} = \alpha^* \mid K a \rangle + \beta^* \mid K b \rangle$$

$$K^2 = 1 .$$

The complex conjugate of an operator A, is defined as $A^k \equiv K A K$. One will observe that

$$(A B)^k = A^k B^k$$

and

$$\langle K a \mid A^k \mid K b \rangle = \langle K a \mid K A \mid b \rangle = \langle a \mid A \mid b \rangle^* .$$

The transpose of an operator is defined as the complex conjugate of the adjoint:

$$A^T = A^{*k} = K A^* K .$$

[An equivalent form is $A^T = (A^k)^*$.] We see that

$$(A B)^T = \left[(A B)^* \right]^k = (B^* A^*)^k = B^T A^T$$

and

$$\langle K a \mid A^T \mid K b \rangle = \langle a \mid A^* \mid b \rangle^* = \langle A a \mid b \rangle^* = \langle b \mid A \mid a \rangle .$$

These two properties of the transposed operator are those given in Eq. (10.2) and (10.3) of the text. One additional property we state as Lemma I:

If $\langle c \mid A \mid c \rangle = \langle K c \mid B \mid K c \rangle$ for all c, then

$$\langle a \mid A \mid b \rangle = \langle K b \mid B \mid K a \rangle$$

for all a and b. The proof is obtained by considering for a fixed a and b all c of the form $\mid c \rangle = \alpha \mid a \rangle + \beta \mid b \rangle$. This lemma allows one to deduce from the form of a transformation of expectation values the form of the corresponding transformation on matrix elements.

In order to calculate expectation values in a new or transformed system it is necessary to know not only the new operators but also the new state vectors. The transformation of the operators given above must be coupled with the transformation of the state vectors to their complex conjugates. Thus if a certain measurement on the original system gives an expectation value of

$$\langle \mathcal{O}(t) \rangle = \langle a \mid \mathcal{O}(p(t),\, q(t)) \mid a \rangle ,$$

then the expectation value of this measurement on the transformed system is given by

$$\langle \mathcal{O}(t) \rangle' = \langle K a \mid \mathcal{O}(p_t(t),\, q_t(t)) \mid K a \rangle .$$

One readily verifies that

$$\langle q(t) \rangle' = \langle q(-t) \rangle$$

and

$$\langle p(t) \rangle' = - \langle p(-t) \rangle$$

in accordance with the classical equations.

It should be noticed that the operator K is not uniquely defined by Wigner's three conditions. A particular operator, K, will be selected if, in addition to Wigner's conditions, we require for all of the vectors $\mid i \rangle$ of some complete orthogonal set the relationship $\mid K i \rangle = \mid i \rangle$. The operator K is then defined by

$$\langle i \mid K a \rangle = \langle K i \mid K a \rangle = \langle i \mid a \rangle^{*} .$$

In this "real" representation the matrices representing A, A^k, and A^T satisfy the simple relationships:

$$\langle i \mid A^k \mid j \rangle = \langle i \mid A \mid j \rangle^{*} ,$$

and

$$\langle i \mid A^T \mid j \rangle = \langle j \mid A \mid i \rangle .$$

The definition $\psi_{\bar{a}} = \psi_a^*$ used in the text can be understood as an equation in this real representation:

$$\langle i \mid \bar{a} \rangle = \langle a \mid i \rangle \quad .$$

Notice that an equation of this form cannot be valid in all representations; e.g., consider $\mid i' \rangle = e^{i\alpha} \mid i \rangle$.

(3) The case of quantum field theory may be treated in a manner similar to that of the first quantized case. Suppose $(\psi(x), \phi(x) \ldots)$ are a set of fields that satisfy the commutation relations and the equations of motion. We wish to determine a "new" set of fields $(\psi_N(x), \phi_N(x), \ldots)$ that represent the CPT-inverse physical situation and then inquire about the conditions under which the new physical situation can actually exist in nature (i.e., whether it also satisfies the equations of motion). According to the <u>physical meaning</u> of the operations C, P, and T expectation values in the original and CPT inverse systems must be related by

$$\langle Q \rangle' = -\langle Q \rangle = \int d\sigma_o \langle J_o(x) \rangle$$

$$\langle P_\mu \rangle' = \langle P_\mu \rangle = \int d\sigma_o \langle T_{o\mu}(x) \rangle$$

$$\langle J_{\mu\nu} \rangle' = -\langle J_{\mu\nu} \rangle = \int d\sigma_o \langle M_{o\mu\nu}(x) \rangle$$

where Q, P_μ, and $J_{\mu\nu}$ are the total charge, momentum, and angular momentum of the system. These relations must be satisfied also locally, where one is to compare contributions at x with those at $(-x)$. These more restrictive requirements are

$$\langle J_\mu(x) \rangle' = -\langle J_\mu(-x) \rangle$$

$$\langle T_{\mu\nu}(x) \rangle' = \langle T_{\mu\nu}(-x) \rangle$$

$$\langle M_{\mu\nu\lambda}(x) \rangle' = -\langle M_{\mu\nu\lambda}(-x) \rangle \quad .$$

[It is to be stressed that no change of coordinate system is contemplated here, although the same results could be obtained using this device. We are comparing two different physical situations.] The conditions on the local densities (i.e., $J_\mu(x)$, $T_{\mu\nu}(x)$, etc.) are of form

$$\langle \mathcal{O}(x) \rangle' = (-1)^n \langle \mathcal{O}(-x) \rangle ,$$

where n is the number of tensor indices on the operator $\mathcal{O}$. In order that the CPT inverse physics at x be related to the original physics at $(-x)$, the $(\psi_N(x),\ \phi_N(x),\ \ldots)$ must be functions of $(\psi(-x),\ \phi(-x),\ \ldots)$. A linear relationship between them will not be satisfactory, since, for one thing, the commutator equations of motion can never be satisfied. [We will require at least for some possible systems (e.g. free particles) that the CPT inverse fields will satisfy the equations of motion.] As in the first quantized case the satisfactory transformation involves the transpose:

$$\psi_N(x) = i\ \gamma_5\ \psi^T(-x) \qquad \bar{\psi}_N(x) = i\ \bar{\psi}^T(-x)\ \gamma_5$$

$$\phi_N(x) = (-1)^n\ \phi^T(-x) ,$$

where ϕ is a boson field of rank n (n tensor indices).

In addition to this transformation on the operators, it is necessary to transform the state vectors. In particular if an expectation value in the original system is

$$\langle \mathcal{O}(x) \rangle = \langle a \mid \mathcal{O}(\psi(x),\ \phi(x),\ \ldots;\ x) \mid a \rangle ,$$

then in the transformed system it will be given by

$$\langle \mathcal{O}(x) \rangle' = \langle K\ a \mid \mathcal{O}(\psi_N(x)\ \phi_N(x),\ \ldots;\ x) \mid K\ a \rangle .$$

In the calculation of $\langle \mathcal{O}(x) \rangle'$, the complex conjugation can be removed from the state vectors by using the relation $\langle K a | A | K b \rangle = \langle b | A^T | a \rangle$ then one obtains

$$\begin{aligned} \langle \mathcal{O}(x) \rangle' &= \langle a | \mathcal{O}^T(\psi_N(x), \phi_N(x), \ldots; x) | a \rangle \\ &= \langle a | \mathcal{O}^{tr}(i \gamma_5 \psi(-x), (-1)^n \phi(-x), \ldots; x) | a \rangle \\ &\equiv \langle a | \mathcal{O}^{tr}(\psi'(x), \phi'(x), \ldots; x) | a \rangle \\ &\equiv \langle a | \mathcal{O}'(\psi(x), \phi(x), \ldots; x) | a \rangle \end{aligned}$$

where $\psi'(x)$ is the new field as defined by Pauli, and $\mathcal{O}'(\psi(x), \phi(x), \ldots; x)$ is the operator computed using Pauli's prescription as given in the text.

In order to verify that the transformation we have given will actually generate the CPT inversion it is sufficient to show that

$$\mathcal{O}'(-x) = (-1)^n \mathcal{O}(x) \quad ,$$

for all tensor operators constructed from the field operators, their derivatives, and the vectors x_μ. To see this notice that for each tensor index attached to a boson field operator there is one (-1), by our definition. Also, for each derivative of a field operator and for each vector x_μ there is a minus sign associated with the change $(x) \to (-x)$. For fermions, we have

$$\begin{aligned} \bar{\psi}'(-x)\, \mathcal{O}\, \psi'(-x) &= -\bar{\psi}(x)\, \gamma_5\, \mathcal{O}\, \gamma_5\, \psi(x) \\ &= -(-1)^m\, \bar{\psi}(x)\, \mathcal{O}\, \psi(x) \end{aligned}$$

where m is the rank of the tensor $\mathcal{O}$. If the Lagrangian (and therefore all the operators derived from it) is antisymmetrized with respect to all fermion fields and symmetrized with respect to boson fields then the required transposition of the order of all the operators will introduce the necessary minus sign for each pair of fermion fields and the required transformation properties are indeed obtained. A detailed demonstration is given by Pauli in the Bohr anniversary volume.

We may now turn to the question of whether the equations of motion are satisfied for the new fields $\psi_N(x)$. Thus we ask whether the equation

$$\left[P_\mu(\psi_N(x), \phi_N(x), \dots x), \; f(\psi_N(x), \phi_N(x))\right] = -i \frac{\partial}{\partial x_\mu} f(\psi_N(x), \phi_N(x)) \qquad \text{(Eq. A)}$$

is satisfied. If in this equation we replace the dummy variables x by $(-x)$ and transpose, we obtain

$$\left[P_\mu^T(\psi_N(-x), \phi_N(-x), \dots; (-x), \; f^T(\psi_N(-x), \phi_N(-x)\right] = -i \frac{\partial}{\partial x_\mu} f^T(\psi_N(-x), \phi_N(-x)$$

which may be written

$$\left[P_\mu^{tr}(\psi'(-x), \phi'(-x), \dots; (-x), \; g(\psi'(x), \phi'(x))\right] = -i \frac{\partial}{\partial x_\mu} g(\psi'(x), \phi'(x))$$

where $g(\psi(x), \phi(x)) \equiv f^{tr}(i\,\gamma_5\,\psi(x), (-1)^n \phi(x))$. Comparing this equation to the known equation of motion for functions of $\psi(x)$ and $\phi(x)$, we see that Eq. (A) is satisfied if

$$P_\mu'(\psi(-x), \phi(-x), \dots, (-x)) = P_\mu(\psi(x), \phi(x), \dots; x)$$

If P_μ is obtained from a properly symmetrized Lorentz invariant Lagrangian, then this equation is always true as a consequence of the relationship

$$\theta'(-x) = (-1)^n \theta(x)$$

applied to the stress-energy tensor $T_{\mu\nu}(x)$.

The CPT transformation is peculiar in that the law of transformation with regard to each tensor index is the same, independently of the particular type of quantity involved. It is for this reason that the Lorentz invariance alone (see below) will guarantee the invariance under CPT. Under parity (P) and time reversed (TC) the possibility of having pseudoscalars, etc., destroys this connection, while charge conjugation alone (C) is not correlated to the tensor indices but to specific operators. Consequently for P, C and T separately the requirement that the new fields satisfy the original equations places additional restrictions on the Lagrangian.

Besides the assumption of proper Lorentz invariance [and a tacit assumption regarding the local nature of the fields and the Lagrangian] the fact that the Lagrangian, and consequently all operators, is symmetrized with respect to boson fields and antisymmetrized with respect to fermion-fields has been used in the above development. Since for commuting (anticommuting) fields the anti-symmetrized (symmetrized) forms would reduce to c-numbers and hence constitute a removeable normalization constant in the Lagrangian, the normal connection between spin and statistics is implicit in the symmetrization rule. Hence the usual connection was, in effect, assumed in the construction of our CPT transformation.

Conversely, if we are to represent the CPT-transformation in the way that we have, then we must require the usual connection between spin and statistics. This is Schwinger's deduction of that connection. This proof involves showing, independently of the connection between spin and statistics, that the CPT transformation can only be represented in a way equivalent to the one that was used here and hence that no CPT transformation could have been found if the Lagrangian had been symmetrized differently.

(4) It might be useful to point out that since the new fields and the old fields obey the same canonical commutation relations (i.e., on a space-like surface), they may be related by a unitary transformation:

$$\psi_N(x) = R^{-1}\,\psi(x)\,R$$

$$\phi_N(x) = R^{-1}\,\phi(x)\,R \quad .$$

If R is independent of the space-like surface then the commutator equations of motion are invariant. In terms of R the transformed expectation value is

$$\langle \mathcal{O}(x) \rangle' = \langle K a \mid \mathcal{O}(\psi_N(x), \phi_N(x) K a \rangle$$

$$= \langle K a \mid \mathcal{O}(R^{-1} \psi(x)R, R^{-1} \phi(x)R) \mid K a \rangle$$

$$= \langle R K a \mid \mathcal{O}(\psi(x) \phi(x)) \mid R K a \rangle$$

$$\equiv \langle a' \mid \mathcal{O}' \mid a' \rangle$$

In this form the transformation to the new physical situation is represented by a change in the state vectors alone. Pauli's transformation changes only the operators. If one wishes the transformation to change only the fundamental dynamical variables (i.e., p and q) but to maintain the functional form of the derived observables and also the commutation relations, then both operators and state vectors must be changed.

(5) It may be of interest to sketch briefly the connection of the transformations given above to the corresponding transformations in the Schrodinger representation. In that case, the state vectors are changed and the operators are left unaltered. In particular if

$$\langle \mathcal{O}(t) \rangle = \langle \Psi(t) \mid \mathcal{O}_S \mid \Psi(t) \rangle$$

then we may represent the expectation value in the transformed system by

$$\langle \mathcal{O}(t) \rangle' = \langle U K \Psi(-t) \mid \mathcal{O}_S \mid U K \psi(-t) \rangle$$

where U is a time independent unitary transformation. If we write $\Psi(t) = S(t)\Psi_H$ and define $S''(t) = U K S(-t) K U^{-1}$, then

$$\langle \mathcal{O}(t) \rangle' = \langle S''(t) U K \Psi_H \mid \mathcal{O}_S \mid S''(t) U K \Psi_H \rangle$$

$$= \langle U K \Psi_H \mid \bar{S}''(t) \mathcal{O}_S S''(t) \mid U K \Psi_H \rangle$$

$$\equiv \langle U K \Psi_H \mid \mathcal{O}''_H(t) \mid U K \Psi_H \rangle$$

$$= \langle K \Psi_H \mid U^{-1} \mathcal{O}''_H(t) U \mid K \Psi_H \rangle$$

$$= \langle \Psi_H \mid (U^{-1} \mathcal{O}''_H(t) U)^T \mid \Psi_H \rangle .$$

If $S''(t)$ is equal to $S(t)$, which is the condition here that the physics is invariant under the transformation, then $\mathcal{O}''_H(t)$ is the same as the usual Heisenberg operator $\mathcal{O}_H(t)$, and U is the same as the R defined in the preceding section.

Canonical commutation relations for a scalar field ($t = t'$):

These relations do not exhibit Lorentz invariance. For a complex scalar field, $\phi(x)$:

$$i\left[\frac{\partial\phi^*(\vec{x})}{\partial t},\ \phi(\vec{x}')\right] = i\left[\frac{\partial\phi(\vec{x})}{\partial t},\ \phi^*(\vec{x}')\right] = \text{Const. x}\ \delta^{(3)}(\vec{x}-\vec{x}')$$

where

$\delta^{(3)}(\vec{r})$ is a three dimensional δ function.

On strong reflections:

$$\phi'(x) = \phi(-x)\ , \qquad \phi^{*\prime}(x) = \phi^*(-x)\ .$$

Again we must take account of the inversion in the commutator to find

$$i\left[\frac{\partial\phi^*(x)}{\partial t},\ \phi(x')\right]' = i\left[\phi'(x'),\ \frac{\partial\phi^{*\prime}(x)}{\partial t}\right] = i\ \phi(-x'),\ -\frac{\partial\phi^*(-x)}{\partial(-t)}$$

$$= \text{Const. x}\ \delta^{(3)}(\vec{x}'-\vec{x})\ .$$

Thus, for scalar fields we have the same invariance in the commutator under CPT as for spinor fields.

FOOTNOTES

1 Pauli prefers the treatment of Schwinger: Phys. Rev. 82, 914 (1951), especially p. 925, ff.). Here the concept of the transposed operator appears. The concept of the conjugate complex does not so directly appear.

2 $\psi^{c\prime}(x) = C^{-1}\,\bar{\psi}'(-x) = -C^{-1}\,e^{-i\alpha}\,\tilde{\gamma}_5\,\bar{\psi}(-x)$

$$= e^{i\alpha}\,\gamma_5\,\psi^c(-x) = e^{i\alpha}\,\gamma_5\,C^{-1}\,\bar{\psi}(-x)$$

$$\therefore\quad e^{i\alpha}\,\gamma_5\,C^{-1} = -e^{-i\alpha}\,C^{-1}\,\tilde{\gamma}_5$$

$$e^{i\alpha}\,C\,\gamma_5 = -e^{-i\alpha}\,\tilde{\gamma}_5\,C = e^{-i\alpha}\,\widetilde{C\gamma_5} = -e^{-i\alpha}\,C\,\gamma_5$$

LECTURE ON "CONTINUOUS GROUPS AND REFLECTIONS IN QUANTUM MECHANICS" BY W. PAULI

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LECTURE 11

We will now generalize our results concerning the vacuum expectation values of field operators from free particles to interacting ones. The only assumption that will be made is that the vacuum is the state of lowest energy. We will follow the development of H. Lehman (Nuovo cimento **11**, 342 (1954)).

[Pauli regrets that spinor fields are ignored in later papers of this group. He feels that one should not make a set of assumptions or axioms if they are "empty"; i.e., they cannot describe our physical world. Physically, a system involving only scalar fields is not acceptable.]

We will begin with a scalar field, but will then pass to spinor fields. No explicit assumption about the interacting fields will be made, aside from assuming that the theory is invariant under the Lorentz translation group:

$$x'_\mu = x_\mu + a_\mu$$

and

$$A_\mu(x), \ldots, \psi_\alpha(x), \ldots$$

are assumed to transform according to a representation of the Lorentz group.

The equations of motion for a scalar field give:

$$i\left[P_\mu, A(x)\right] = -\frac{\partial A(x)}{\partial x_\mu}$$

where:

$$\left[P_\mu , P_\nu\right] = 0.$$

If we consider the eigenfunctions of P_μ :

$$P_\mu \left| k \right\rangle = k_\mu \left| k \right\rangle ,$$

then:[1]

$$\left\langle 0 \right| A(x) \left| \vec{k} \right\rangle = \left\langle 0 \right| A(0) \left| \vec{k} \right\rangle \quad e^{ik\cdot x} .$$

Thus we are explicitly using the concept of an energy-momentum four vector. In this formalism, however, it is not correct to express them in terms of the fields. The eigenvalue problem, whose solution would allow such an expression, is very important but it has not been fully treated as yet.

For simplicity, we will assume that the eigenvalues are real, although more generally we might have complex eigenvalues. We assume:

$$k_o > 0, \quad \text{and} \quad \lambda \equiv - k\cdot k > 0$$

(that is, $k_o > 0$ is all coordinate frames).

[For the general, non-real eigenvalues, we might write

$$k_\mu = m\ell_\mu \qquad \text{where } \ell_\mu \ell_\mu = -1, \text{ and } \operatorname{Re}\left\{ m \right\} > 0.]$$

Now we consider the expectation value in the vacuum of the product of two fields:[2]

$$\left\langle 0 \right| A(x)\, B(x') \left| 0 \right\rangle = \sum_{\substack{k_\mu \\ (k_o > 0)}} e^{ik\cdot(x - x')} \quad \left\langle 0 \right| A(0) \left| k \right\rangle \left\langle k \right| B(0) \left| 0 \right\rangle .$$

Thus we have a function of the positive class. [This can be generalized to a finite number of complex masses, with their complex conjugates as well.] We assume that this expectation value is invariant with respect to the continuous Lorentz group, so that the function satisfies our previous requirements about functions of a positive class. We now define a mass spectrum:

$$\rho_{AB}(\lambda)\frac{d^4k}{(2\pi)^3} = \sum_{(k_\mu;dk_\mu)} \langle 0|A(0)|k\rangle\langle k|B(0)|0\rangle$$

where the summation (or integration) is to be carried out over the 4-dimensional region in k of volume d^4k, centered on k_μ. The sum must only depend on λ, as a result of its Lorentz invariance.

Thus:

$$\langle A(x)B(x')\rangle_0 = \frac{1}{(2\pi)^3}\int d^4k\,\rho_{AB}(\lambda)\,e^{ik\cdot(x-x')} = F_{AB}(x-x'),$$

or:

$$F_{AB}(\xi) = \int_0^\infty i\,\Delta_+(\xi;\lambda)\,\rho_{AB}(\lambda)\,d\lambda = F_{AB}(S),$$

and the theorem of Wightman holds.

We have an important reality condition:[3]

$$\rho^*_{AB}(\lambda) = \rho_{B^*A^*}(\lambda)$$

[If the metric is not positive definite; B^*, A^* are the adjoint operators to B, A.] Thus we see that the expectation value of a self-adjoint operator is real.

If we choose $B = A^*$:

$$\rho^*_{AA^*} = \rho_{AA^*}$$

so ρ_{AA^*} is real. If the metric is also positive definite, ρ_{AA^*} is also $\geqslant 0$, since then:

$$\langle 0| A(0)|k\rangle\langle k|A^*(0)|0 \;=\; |\langle 0|A(0)|k\rangle|^2 \geqslant 0.$$

In the indefinite metric, these results will only be true for states of <u>positive norm</u>.[4]

We define a <u>real field</u> to be one which is identical to its adjoint ("self-adjoint.")

Now we return to the CPT theorem:

(1) Since F is of positive class, if F = 0 for space-like points, it will be zero everywhere.

(2) For <u>any</u> invariant function of a single 4-vector and a space-like argument:

$$F(\xi) = F(-\xi) \qquad\qquad (\xi\cdot\xi) > 0.$$

The latter statement is easily proven by choosing a Lorentz frame in which t = 0. Then it is always possible to rotate coordinates so that $\vec{\xi} \rightarrow -\vec{\xi}$. The theorem also holds for two vectors, but not for three. It will not be true for time-like arguments.

We will now prove the equivalence of the CPT inversion and microcausality. (A more general discussion is given by R. Jost, Helv. Phys. Acta <u>30</u>, 409 (1957)).

In the present case, the CPT theorem gives:

$$\langle A(x)B(x')\rangle'_0 = \langle B'(x')A'(x)\rangle_0$$

$$= \langle B(-x')A(-x)\rangle_0 = F_{BA}(\xi)$$

$$= \langle A(+x)B(+x')\rangle_0 = F_{AB}(\xi).$$

The last line only follows if the CPT theorem holds.[5] Note that the arguments, ξ, are the same for the two functions, since

$$(-x') - (-x) \equiv x - x' \quad .$$

The CPT equality is not an identity even if $B = A^*$, but only if $B = A$. [We cannot use $\rho_{AA^*} = \rho_{A^*A}$, even though both are real.]

We can show the equivalence of the CPT relation, which is:

$$F_{AB}(\xi) = F_{BA}(\xi)$$

to a relation for the commutator.

Microcausality:

A system will satisfy a microcausality condition if:

$$[A(x), B(x')] = 0 \qquad \text{for } (\xi \cdot \xi) > 0.$$

We will actually only need the much weaker assumption that

$$\langle [A(x), B(x')] \rangle_0 = 0 \qquad (\xi \cdot \xi) > 0$$

Now:

$$\langle [A(x), B(x')] \rangle_0 \equiv F_{AB}(\xi) - F_{BA}(-\xi)$$

$$(= 0 \quad \text{for} \quad (\xi \cdot \xi) > 0 \;).$$

However, $F_{BA}(\xi) = F_{BA}(-\xi)$ on this space-like region, so if

$$G_+(\xi) \equiv F_{AB}(\xi) - F_{BA}(\xi),$$

then G_+ is a function of positive class, zero on a space-like surface, and hence is zero everywhere. Thus, for all ξ:

$$F_{AB}(\xi) = F_{BA}(\xi) \quad .$$

Thus microcausality is equivalent to the CPT theorem.

FOOTNOTES

1 We can integrate the equations of motion to obtain

$$A(x+a) = e^{-iaP} A(x) e^{iaP} \quad .$$

from which the result follows immediately, since $P_\mu |0\rangle = 0$, if $|0\rangle$ is the vacuum.

2 If the metric is not positive definite, $\sum_k |k\rangle\langle k|$ will not in general represent the unit operator, even if k is a complete, orthogonal set of eigenvectors, since they would not be normalized to + 1. In general, the unit operator would be

$$\sum_k \frac{|k\rangle\langle k|}{\langle k|k\rangle} = \sum_k |k\rangle \, \eta(k) \, \langle k| \quad .$$

It should be pointed out that "expectation value" as used in the text only has the conventional meaning if the metric is positive definite. Otherwise, the matrix element $\langle o|\mathcal{O}|o\rangle$ must be divided by the norm of the state $\langle o|o\rangle$.

3 Since $\langle 0|A^*(0)|k\rangle = \langle k|A(0)|0\rangle^*$.

4 In the more general case, we have:

$$\langle 0|A(0)|k\rangle\langle k|A^*(0)|0\rangle = |\langle 0|A(0)|k\rangle|^2 \, \eta(k).$$

5 As we have seen in the long note, lecture 10, the CPT inversion may be either identified with the operators exclusively, as is done by Pauli, or partly with the operators and partly with the state vectors, as was done in the note. The latter has the advantage that operator functions of the field operators need not be changed in the inversion. It would also be possible to change only the state

FOOTNOTES

5 (Cont.)

vectors. Then we would have:

$$\langle A(x)\ B(x')\rangle_0' = (\Psi_0'\ |\ A(x)\ B(x')\ |\ \Psi_0')$$

However, if we assume that the vacuum is unique, the CPT inversion can at most lead to a trivial phase change from the original vacuum, Ψ_0. Thus:

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LECTURE ON "CONTINUOUS GROUPS AND REFLECTIONS IN QUANTUM MECHANICS" BY W. PAULI

Notes by R. J. Riddell, Jr.

Radiation Laboratory
University of California
Berkeley, California

LECTURE 11

We will now generalize our results concerning the vacuum expectation values of field operators from free particles to interacting ones. The only assumption that will be made is that the vacuum is the state of lowest energy. We will follow the development of H. Lehman (Nuovo cimento 11, 342 (1954)).

[Pauli regrets that spinor fields are ignored in later papers of this group. He feels that one should not make a set of assumptions or axioms if they are "empty"; i.e., they cannot describe our physical world. Physically, a system involving only scalar fields is not acceptable.]

We will begin with a scalar field, but will then pass to spinor fields. No explicit assumption about the interacting fields will be made, aside from assuming that the theory is invariant under the Lorentz translation group:

$$x'_\mu = x_\mu + a_\mu$$

and

$$A_\mu(x), \ldots, \psi_\alpha(x), \ldots$$

are assumed to transform according to a representation of the Lorentz group.

The equations of motion for a scalar field give:

$$i\left[P_\mu, A(x)\right] = -\frac{\partial A(x)}{\partial x_\mu}$$

where:

$$\left[P_\mu , P_\nu\right] = 0.$$

If we consider the eigenfunctions of P_μ:

$$P_\mu \left| k \right\rangle = k_\mu \left| k \right\rangle ,$$

then:[1]

$$\left\langle 0 \right| A(x) \left| \vec{k} \right\rangle = \left\langle 0 \right| A(0) \left| \vec{k} \right\rangle \quad e^{ik \cdot x} \quad .$$

Thus we are explicitly using the concept of an energy-momentum four vector. In this formalism, however, it is not correct to express them in terms of the fields. The eigenvalue problem, whose solution would allow such an expression, is very important but it has not been fully treated as yet.

For simplicity, we will assume that the eigenvalues are real, although more generally we might have complex eigenvalues. We assume:

$$k_o > 0, \quad \text{and} \quad \lambda \equiv - k \cdot k > 0$$

(that is, $k_o > 0$ is all coordinate frames).

[For the general, non-real eigenvalues, we might write

$$k_\mu = m\ell_\mu \qquad \text{where} \quad \ell_\mu \ell_\mu = -1, \text{ and Re } \{m\} > 0.]$$

Now we consider the expectation value in the vacuum of the product of two fields:[2]

$$\left\langle 0 \right| A(x) \; B(x') \left| 0 \right\rangle = \sum_{\substack{k_\mu \\ (k_o > 0)}} e^{ik \cdot (x - x')} \qquad \left\langle 0 \right| A(0) \left| k \right\rangle \left\langle k \right| B(0) \left| 0 \right\rangle .$$

Thus we have a function of the positive class. [This can be generalized to a finite number of complex masses, with their complex conjugates as well.] We assume that this expectation value is invariant with respect to the continuous Lorentz group, so that the function satisfies our previous requirements about functions of a positive class. We now define a mass spectrum:

$$\rho_{AB}(\lambda)\frac{d^4k}{(2\pi)^3} = \sum_{(k_\mu ; dk_\mu)} \langle 0|A(0)|k\rangle\langle k|B(0)|0\rangle$$

where the summation (or integration) is to be carried out over the 4-dimensional region in k of volume d^4k, centered on k_μ. The sum must only depend on λ, as a result of its Lorentz invariance.

Thus:

$$\langle A(x)B(x')\rangle_0 = \frac{1}{(2\pi)^3}\int d^4k\,\rho_{AB}(\lambda)\, e^{ik\cdot(x-x')} = F_{AB}(x-x').$$

or:

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Lecture 12

R. J. Riddell, Jr.

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Radiation Laboratory
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Berkeley, California

Spring 1958

LECTURE 12

Remark on the stronger physical requirement of microcausality:

The statement

$$[A(x), B(x')] = 0 \qquad (\xi \cdot \xi) > 0$$

can be formulated in terms of the vacuum expectation values, or:

$$\left\langle [A(x), B(x')] \quad A(x'')\dots B(x''')\dots \right\rangle_0 = 0.$$

(Wightman)

We now generalize the previous discussion to the case of a single spinor field (which anticommutes.) [For several fields, one can always find by suitable transformations that they can be made to anticommute. See the paper of Lüders.]

We will not assume C or P invariance, but only the continuous Lorentz transformations and the assumption of the vacuum as the lowest state. Then we can build up quantities like

$$(k\cdot k), \quad (k\cdot\gamma), \quad (k\cdot k)\gamma_5, \quad (k\cdot\gamma)\gamma_5, \quad \dots$$

Then we find:[1]

$$-\sum_{k_\mu;dk_\mu} \langle 0|\psi_\alpha(0)|k\rangle \langle k|\bar\psi_\beta|0\rangle = \frac{d^4k}{(2\pi)^3}\Big\{ i(\gamma\cdot k)\rho^{I}_{\psi\bar\psi} + 1\cdot\sigma^{I}_{\psi\bar\psi} + i\gamma_5(\gamma\cdot k)\rho^{II}_{\psi\bar\psi} + i\gamma_5\,\sigma^{II}_{\psi\bar\psi}\Big\}_{\alpha\beta}$$

where

$$\bar{\Psi} = \Psi^* \gamma_4 \ .$$

From this follows that:

$$\left.\begin{matrix} F_{\psi\bar{\psi}}^{()} & (\xi) \\ G_{\psi\bar{\psi}}^{()} & (\xi) \end{matrix}\right\} = \int_0^\infty \frac{1}{2} (\Delta_1 + i\Delta)\ (\xi;\lambda) \left\{\begin{matrix} \rho_{\psi\bar{\psi}}^{()} & (\lambda) \\ \sigma_{\psi\bar{\psi}}^{()} & (\lambda) \end{matrix}\right\} d\lambda \ .$$

and we have functions of the positive class as before. The () superscript means that either I or II may be inserted throughout the equation. Thus:

$$- \left\langle \Psi_\alpha (x) \bar{\Psi}_\beta (x') \right\rangle_0 = \Big\{ (\gamma \frac{\partial}{\partial \xi})\ F_{\psi\bar{\psi}}^{I} (\xi) + 1\ G_{\psi\bar{\psi}}^{I}\ (\xi)$$

$$+ \gamma_5 (\gamma \frac{\partial}{\partial \xi})\ F_{\psi\bar{\psi}}^{II}\ (\xi) + i\gamma_5\ G_{\psi\bar{\psi}}^{II}\ (\xi) \Big\}_{\alpha\beta} \ .$$

There is a reality condition on these quantities, since:

$$\left\langle \Psi_\beta\ (x') \Psi_\alpha^*\ (x) \right\rangle_0^* = \left\langle \Psi_\alpha\ (x) \Psi_\beta^* (x') \right\rangle_0$$

Then we find[2]

$$\rho_{\psi\bar{\psi}}^{*()} = \rho_{\psi\bar{\psi}}^{()} \ , \quad \sigma_{\psi\bar{\psi}}^{*()} = \sigma_{\psi\bar{\psi}}^{()}$$

We must again obtain an equivalence between CPT and microcausality. We set

$$\left\langle \bar{\Psi}_\beta (x') \Psi_\alpha (x) \right\rangle_0 = \Big\{ (\gamma \frac{\partial}{\partial \xi})\ F_{\bar{\psi}\psi}^{I}\ (-\xi) + 1 \cdot G_{\bar{\psi}\psi}^{I}\ (-\xi)$$

$$+ \gamma_5 (\gamma \frac{\partial}{\partial \xi})\ F_{\bar{\psi}\psi}^{II}\ (-\xi) + i\gamma_5\ G_{\bar{\psi}\psi}^{II} (-\xi) \Big\}_{\alpha\beta}$$

or:

$$\langle \bar{\psi}_\beta(x)\psi_\alpha(x')\rangle_0 = \Big\{ -(\gamma \frac{\partial}{\partial \xi})\, F^{I}_{\bar{\psi}\psi}\ (\xi) + 1\cdot G^{I}_{\bar{\psi}\psi}\ (\xi)$$

$$-\gamma_5\ (\gamma \frac{\partial}{\partial \xi})\, F^{II}_{\bar{\psi}\psi}\ (\xi) + i\gamma_5\, G^{II}_{\bar{\psi}\psi}\ (\xi) \Big\}_{\alpha\beta} \ .$$

Again:

$$\left.\begin{array}{l} F^{()}_{\bar{\psi}\psi}\ (\xi) = F^{()}_{\bar{\psi}\psi}\ (s) \\ G^{()}_{\bar{\psi}\psi}\ (\xi) = G^{()}_{\bar{\psi}\psi}\ (s) \end{array}\right\} = \int \frac{1}{2}\,(\Delta_1 + i\Delta)\ (\xi;\lambda) \left\{\begin{array}{l} \rho^{()}_{\bar{\psi}\psi}\ (\lambda) \\ \sigma^{()}_{\bar{\psi}\psi}\ (\lambda) \end{array}\right\} d\lambda$$

$$F^{()}_{\bar{\psi}\psi}\ (-\xi) = F^{()}_{\bar{\psi}\psi}\ (s^*)\ , \qquad G^{()}_{\bar{\psi}\psi}\ (-\xi) = G^{()}_{\bar{\psi}\psi}\ (s^*)$$

We cannot get any connection between the F's without a further assumption (either CPT or microcausality).

We assume:

$$\langle \{\psi_\alpha(x), \bar{\psi}_\beta(x')\} \rangle_0 = 0 \qquad \text{if } (\xi\cdot\xi) > 0.$$

Then:[3]

$$\left.\begin{array}{l} F^{()}_{\bar{\psi}\psi}\ (\xi) = F^{()}_{\bar{\psi}\psi}\ (-\xi) = F^{()}_{\bar{\psi}\psi}\ (\xi) \\ G^{()}_{\psi\bar{\psi}}\ (\xi) = G^{()}_{\bar{\psi}\psi}\ (-\xi) = G^{()}_{\bar{\psi}\psi}\ (\xi) \end{array}\right\} \text{if } (\xi\cdot\xi) > 0.$$

Thus:

$$F^{()}_{\psi\bar{\psi}}\ (\xi) = F^{()}_{\bar{\psi}\psi}\ (\xi)$$

$$G^{()}_{\psi\bar{\psi}}\ (\xi) = G^{()}_{\bar{\psi}\psi}\ (\xi)$$

everywhere.

The CPT relation is:

$$\langle \bar{\psi}_\beta(x') \psi_\alpha(x) \rangle_o' \equiv \langle \psi_\alpha'(x) \bar{\psi}_\beta'(x') \rangle_o = \langle \bar{\psi}_\beta(x') \psi_\alpha(x) \rangle_o \quad .$$

This gives the same relation as does microcausality.[4]

This is all that can be obtained, since we don't assume any specific invariance. If one assumes P invariance, for example, the II parts vanish.

Now, if we consider the vacuum expectation value of two ψ's, the Lorentz invariance leads to the form:[5]

$$\langle \psi_\alpha(x) \psi_\beta(x') \rangle_o = \Big[\Big\{ (\gamma \frac{\partial}{\partial \xi}) F^I_{\psi\psi}(\xi) + 1 \cdot G^I_{\psi\psi}(\xi) + \gamma_5 (\gamma \frac{\partial}{\partial \xi}) F^{II}_{\psi\psi}(\xi) + i\gamma_5 G^{II}_{\psi\psi}(\xi) \Big\} C^{-1} \Big]$$

where

$$C \gamma_\mu C^{-1} = -\tilde{\gamma}_\mu , \quad \tilde{\gamma}_5 = C\gamma_5 C^{-1} , \quad C^*C = 1.$$

If we change the order of the operators:

$$-\langle \psi_\beta(x') \psi_\alpha(x) \rangle_o = \Big[\Big\{ (\gamma \frac{\partial}{\partial \xi}) F^I_{\psi\psi}(-\xi) + 1 \cdot G^I_{\psi\psi}(-\xi) - \gamma_5 (\gamma \frac{\partial}{\partial \xi}) F^{II}_{\psi\psi}(-\xi) + i\gamma_5 G^{II}_{\psi\psi}(-\xi) \Big\} C^{-1} \Big]_{\alpha\beta} ,$$

since C, $C\gamma_5$, $C\gamma_5\gamma_\mu$ are antisymmetric, and $C\gamma_\mu$, $C\{\gamma_\mu, \gamma_\nu\}$ are symmetric matrices (and from this, C^{-1}, $\gamma_5 C^{-1}$, $\gamma_5\gamma_\mu C^{-1}$ are antisymmetric and $\gamma_\mu C^{-1}$, $\{\gamma_\mu, \gamma_\nu\} C^{-1}$ are symmetric).

If we now apply microcausality:

$$\langle \{ \psi_\alpha(x), \psi_\beta(x') \} \rangle_o = 0 \qquad \text{for } (\xi \cdot \xi) > 0$$

and CPT gives:[6]

$$\langle \psi_\beta (x') \psi_\alpha (x) \rangle_0' \equiv \langle \psi_\alpha' (x) \psi_\beta' (x') \rangle_0 = \langle \psi_\beta (x') \psi_\alpha (x) \rangle_0 .$$

In each case, we find an identity with the exception of $F^{II}_{\psi\psi}$, which must vanish. Thus, only

$$F^{II}_{\psi\psi} = 0,$$

to satisfy either condition.

If we consider states of only positive norm, we find other relations. To get such relations, we must have ψ^* rather than $\bar{\psi}$, so that if we multiply by γ_4 and take the trace, we find:

$$(-i) \frac{1}{4} \operatorname{Tr} (\gamma \cdot k) \gamma_4 = k_0 > 0$$

so that[7]

$$\rho^{I}_{\psi\bar{\psi}} > 0 .$$

FOOTNOTES

1 The most general relation would be:

$$M_{\alpha\beta} = -\sum \langle 0 | \psi_\alpha (0) | k \rangle \langle k | \bar{\psi}_\beta (0) | 0 \rangle = \sum_A (\gamma_A)_{\alpha\beta} f_A (k)$$

where A is summed over the 16 independent 4 x 4 matrices. Now, the Lorentz invariance gives relations like:

$$M_{\alpha\beta} (\gamma_\mu)_{\beta\alpha} = 4 f_\mu (k),$$

and since this term must transform as part of a four-vector

$$f_\mu (k) = k_\mu F(k)$$

where $F'(k) = F'(\lambda)$. The other terms are obtained similarly, except that

FOOTNOTES

1 (Cont.)

$\frac{1}{2}[\gamma_\mu, \gamma_\nu]$ is antisymmetric and no antisymmetric tensor can be formed from k_μ's.

The k_μ's give the most general form, since the Lorentz invariance requires that:

$$k_\mu f(\lambda) = \frac{\partial}{\partial k_\mu} g(\lambda),$$

$[f(\lambda) = g'(\lambda)]$ and the $\frac{\partial}{\partial k_\mu}$ gives no more generality.

2 Using the adjoint relation and $\vec{k}^* = \vec{k}$, $k_4^* = - k_4$.

3 We use the same arguments as in the scalar case. For the F's, we obtain

$$(\gamma \frac{\partial}{\partial \xi}) \left[F_{\bar{\psi}\psi}^{I}(-\xi) - F_{\psi\bar{\psi}}^{I}(\xi)\right] = 0.$$

but this is equivalent to the 4 equations

$$\frac{\partial}{\partial \xi}\left[F_{\bar{\psi}\psi}^{I}(-\xi) - F_{\psi\bar{\psi}}^{I}(\xi)\right] = 0 .$$

Thus, the F's can only differ by a constant, which is of no significance in the commutation relation. A similar result holds for F^{II}.

4

$$\langle \psi'_\alpha (x) \bar{\psi}'_\beta (x') \rangle_0 = -(\gamma_5)_{\alpha\alpha'} \langle \psi_{\alpha'} (-x) \bar{\psi}_{\beta'} (-x') \rangle_0 (\gamma_5)_{\beta'\beta}$$

$$= \left\{ \gamma_5 \left[(\gamma \cdot \frac{\partial}{\partial(-\xi)} F_{\psi\bar{\psi}}^{I} (-\xi) + 1 \cdot G_{\psi\bar{\psi}}^{I} (-\xi) \right.\right.$$

FOOTNOTES

4 (Cont.)

$$+ \gamma_5 \left(\gamma \frac{\partial}{\partial(-\xi)}\right) F^{II}_{\psi\bar{\psi}}(-\xi) + i\gamma_5 G^{II}_{\psi\bar{\psi}}(-\xi)\Big] \gamma_5 \Big\}_{\alpha\beta}$$

$$= \Big\{ \left(\gamma \cdot \frac{\partial}{\partial \xi}\right) F^{I}_{\psi\bar{\psi}}(-\xi) + 1 \cdot G^{I}_{\psi\bar{\psi}}(-\xi) + \gamma_5 \left(\gamma \cdot \frac{\partial}{\partial \xi}\right) F^{II}_{\psi\bar{\psi}}(-\xi)$$

$$+ i\gamma_5 G^{II}_{\psi\bar{\psi}}(-\xi) \Big\}_{\alpha\beta}$$

$$= \langle \bar{\psi}_\beta(x') \psi_\alpha(x) \rangle_0 .$$

5 The C enters to give the proper transformation properties. Since

$$\bar{\psi}^c = -\psi C,$$

then

$$\psi_\alpha (\psi C)_\beta$$

transforms as does $\psi_\alpha \bar{\psi}_\beta$. The C is removed by multiplication by C^{-1}, to get the stated relation.

6

$$\langle \psi^c_\alpha(x) \psi^c_\beta(x') \rangle_0 = -(\gamma_5)_{\alpha\alpha'} \langle \psi_{\alpha'}(-x) \psi_{\beta'}(-x') \rangle_0 (\gamma_5)_{\beta\beta'}$$

$$= -(\gamma_5)_{\alpha\alpha'} \Big[\Big\{ -\left(\gamma \frac{\partial}{\partial \xi}\right) F^{I}_{\psi\psi}(-\xi) + 1 \cdot G^{I}_{\psi\psi}(-\xi) - \gamma_5 \left(\gamma \frac{\partial}{\partial \xi}\right) F^{II}_{\psi\psi}(-\xi)$$

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$$= \Big[(\gamma_5) \Big[\Big\{ -\left(\gamma \frac{\partial}{\partial \xi}\right) F^{I}_{\psi\psi}(-\xi) + 1 \cdot G^{I}_{\psi\psi}(-\xi) - \ldots \Big\} \tilde{C}^{-1} \Big] \tilde{\gamma}_5 \Big]_{\alpha\beta}$$

FOOTNOTES

6 (Cont.)

$$= -\Big[\gamma_5 \Big\{ (-\gamma \tfrac{\partial}{\partial \xi})\, F^{I}_{\psi\psi}\ (-\xi) + 1 \cdot G^{I}_{\psi\psi}\ (-\xi) - \gamma_5\ (\gamma \tfrac{\partial}{\partial \xi})\, F^{II}_{\psi\psi}\ (-\xi)$$

$$+ i\gamma_5\, G^{II}_{\psi\psi}\ (-\xi) \Big\}\ \gamma_5\ C^{-1} \Big]_{\alpha\beta}$$

$$= -\Big[\Big\{ (\gamma \tfrac{\partial}{\partial \xi})\, F^{I}_{\psi\psi}\ (-\xi) + 1 \cdot G^{I}_{\psi\psi}\ (-\xi) + \gamma_5 (\gamma \tfrac{\partial}{\partial \xi})\, F^{II}_{\psi\psi}\ (-\xi)$$

$$+ i\gamma_5 G^{II}_{\psi\psi}\ (-\xi) \Big\}\ C^{-1} \Big]_{\alpha\beta} .$$

7 We have:

$$- \sum_{k_\mu ; dk_\mu} \langle 0 | \psi_\alpha\ (0) |\ k \rangle\ \langle k\ |\ \psi^*_{\beta'} (0)\ |\ 0 \rangle\ (\gamma_4)_{\beta'\beta} = \frac{d^4k}{(2\pi)^3} \Big\{ i(\gamma \cdot k) \rho^{I}_{\psi\bar{\psi}}$$

$$+ \cdots \Big\}$$

Thus if we multiply by $(\gamma_4)_{\beta\alpha}$ and sum on α also, to get the trace, we find

$$- \sum_{k_\mu ; dk_\mu} |\ \langle 0\ |\ \psi_\alpha (0)\ |\ k \rangle\ |^2 = \frac{d^4k}{(2\pi)^3}\ \ \mathrm{Tr}\ \ i(\gamma \cdot k)\ \gamma_4 \rho^{I}_{\psi\bar{\psi}} + \cdots$$

$$= -4k_0\ \frac{d^4k}{(2\pi)^3}\, \rho^{I}_{\psi\bar{\psi}} \quad .$$

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Lecture 13

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R. J. Riddell, Jr.

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LECTURE 13

Generalization to more fields:

If we have several spinor fields which anticommute (as can always be accomplished), we obtain very similar conditions to the single-field case:

$$\rho_{\psi\bar{\phi}}^{()}(\lambda) = \rho_{\bar{\psi}\phi}^{()}(\lambda), \qquad \sigma_{\psi\bar{\phi}}^{()}(\lambda) = \sigma_{\bar{\phi}\psi}^{()}(\lambda)$$

$$F_{\psi\bar{\phi}}^{()}(\xi) = F_{\bar{\phi}\psi}^{()}(\xi), \qquad G_{\psi\bar{\phi}}^{()}(\xi) = G_{\bar{\phi}\psi}^{()}(\xi) .$$

These are the conditions both for weak microcausality and for CPT. The reality condition relates different quantities:

$$\rho^*_{\phi\bar{\psi}} = \rho_{\psi\bar{\phi}}$$

$$\sigma^*_{\phi\bar{\psi}} = \sigma_{\psi\bar{\phi}}$$

In the paper of Jost, one finds the generalization to more than two factors in the expectation value. In such cases, the CPT and microcausality always connect factors like:

$$\left\langle \psi^{(1)}(x_1)\ldots\psi^{(n)}(x_n) \right\rangle_0 = \pm \left\langle \psi^{(n)}(x_n)\ldots\psi^{(1)}(x_1) \right\rangle_0 \quad .$$

We will probably come back to this in a different connection. To develop the theory would require the theory of analytic functions of several complex variables. We may give an explicit proof for some interaction hamiltonians later. Jost does not use the positive definite metric.

Inequalities:

The various quantities $\rho^{I}, \ldots$ satisfy certain inequalities. First we let $\phi = \psi$:

Then,

$$\rho^{I}_{\psi\bar\psi}\;,\; \sigma^{I}_{\psi\bar\psi}\;,\; \rho^{II}_{\psi\bar\psi}\;,\; \sigma^{II}_{\psi\bar\psi}\;,$$

are all real. In addition, if the metric is positive definite, we have seen that:

$$\rho_{AA^*} > 0 \qquad \text{(scalars)}$$

$$\rho^{I}_{\psi\bar\psi} \quad 0 \qquad \text{(spinors)}$$

In addition, there are other relations. Consider:

$$\frac{1}{4}\sum_{\alpha,\beta} \sum_{k_\mu;dk_\mu} \left\langle k \left| \psi^*_\alpha(0) \right| 0 \right\rangle M_{\alpha\beta} \left\langle 0 \left| \psi_\beta(0) \right| k \right\rangle \equiv \frac{1}{(2\pi)^3}\, d^4k \cdot k_0 \sum\nolimits_M \;,$$

which defines $\sum_M$. We see that $\sum_1 = \rho^{I}_{\psi\bar\psi}$. (see note No. 7, Lecture 12)

Now, if $M^2 = 1$, the eigenvalues of M are ± 1, so:

$$-\rho^{I} \le \sum\nolimits_M \le \rho^{I} \;.$$

Also:

$$k_o\left(\sum_1 + \sum_M\right) \geq 0 \;; \qquad k_o\left(\sum_1 - \sum_M\right) \geq 0 \;.$$

Since:

$$0 < \begin{Bmatrix}(1+M)\\(1-M)\end{Bmatrix} < 2 \;.$$

If:

$$M = \frac{i}{\sqrt{\lambda}}\,(\gamma\cdot k)\;;\; M^2 = -\frac{1}{\lambda}\,(k\cdot k) = 1 \;.$$

Now:

$$-\sum_{k_\mu;dk_\mu} \langle 0 \,|\, \psi_\alpha(0) \;|k\rangle \;\langle k \,|\, \bar{\psi}_\beta(0) \,|\; 0\rangle = \frac{d^4k}{(2\pi)^3}\left\{ i(\gamma\cdot k)\rho^{I}_{\psi\bar{\psi}} + 1\cdot\sigma^{I}_{\psi\bar{\psi}} \right.$$

$$\left. + i\,\gamma_5\,(\gamma\cdot k)\rho^{II}_{\psi\bar{\psi}} + i\,\gamma_5\,\sigma^{II}_{\psi\bar{\psi}} \right\} \;.$$

If we now multiply this equation by $\left[-\frac{1}{4}\,(\gamma_4 M)_{\beta\alpha}\right]$, we find[1]

$$k_o \sum_M = \frac{1}{\sqrt{\lambda}}\; k_o\,\sigma^{I}_{\psi\bar{\psi}} \;,$$

so:

$$\sigma^{I} \leq \sqrt{\lambda}\,\rho^{I} \;.$$

This relation was derived by Lehmann.

Now we may generalize:

$$M = A\gamma_5 + B\,\frac{i}{\sqrt{\lambda}}\,(\gamma\cdot k) + C\,\frac{1}{\sqrt{\lambda}}\,(\gamma\cdot k)\,\gamma_5$$

$$M^2 = A^2 + B^2 + C^2 = 1 \;.$$

If we proceed as above, we get:[2]

$$-\rho_I < \sum_M = (A\rho^{II} + B\frac{1}{\sqrt{\lambda}}\sigma^{I} + C\frac{1}{\sqrt{\lambda}}\sigma^{II}) < \rho^{I} ,$$

for all A, B, C.

We will get the best result if we maximize the bracket:[3]

$$A = \alpha\rho^{II} , \quad B = \alpha\frac{1}{\sqrt{\lambda}}\sigma^{I} , \quad C = \alpha\frac{1}{\sqrt{\lambda}}\sigma^{II}$$

where

$$\alpha = \frac{1}{\sqrt{(\rho^{II})^2 + \frac{(\sigma^{I})^2 + (\sigma^{II})^2}{\lambda}}} .$$

Thus we find that

$$-\rho_I < (\sum_M)_{max.} = \sqrt{(\rho^{II})^2 + \frac{(\sigma^{I})^2 + (\sigma^{II})^2}{\lambda}} < \rho^{I}$$

so:

$$(\rho^{II})^2 + \frac{(\sigma^{I})^2 + (\sigma^{II})^2}{\lambda} < (\rho^{I})^2 .$$

This result was not obtained by Lehmann since he assumed reflection invariance.

The spinor equations of Gürsey:

We now begin a new discussion, in which we will write the spinor equations in a form which plays a role in the theory of the neutrino. We will consider, however, the proton-neutron system in the non-electric approximation. We assume

$$m_P = m_N = m .$$

Then:

$$\gamma_\mu \frac{\partial\psi_P}{\partial x_\mu} = m\psi_P \qquad \gamma_\mu \frac{\partial\psi_N}{\partial x_\mu} = m\psi_N .$$

We now introduce two new fields, which satisfy:

$$\gamma_\mu \frac{\partial \xi}{\partial x_\mu} = m\gamma_5 \chi$$

$$\gamma_\mu \frac{\partial \chi}{\partial x_\mu} = -m\gamma_5 \xi$$

The system of (χ, ξ) has a certain group associated with it:[4]

$$\begin{pmatrix}\chi\\ \xi\end{pmatrix} \longrightarrow a\begin{pmatrix}\chi\\ \xi\end{pmatrix} + b\gamma_5 \begin{pmatrix}\chi\\ \xi\end{pmatrix}^c$$

where:

$$|a|^2 + |b|^2 = 1.$$

A second group is:[5]

$$\begin{pmatrix}\chi\\ \xi\end{pmatrix} \rightarrow \begin{pmatrix}e^{i\alpha\gamma_5} & \chi\\ e^{-i\alpha\gamma_5} & \xi\end{pmatrix} \equiv \cos\alpha\begin{pmatrix}\chi\\ \xi\end{pmatrix} \pm i \sin\alpha\cdot\gamma_5 \begin{pmatrix}\chi\\ \xi\end{pmatrix}$$

This is equivalent to the transformation in ψ_P, ψ_N of:[6]

$$\psi \rightarrow e^{i\alpha}\,\psi \quad . \quad \text{(Baryon conservation)}$$

The former group is the isospin group.

The connection between ψ's and χ, ξ is:[7]

$$\chi = \tfrac{1}{2}(1+\gamma_5)\psi_P + \frac{i}{2}\left[(1+\gamma_5)\psi_N\right]^c$$

$$\xi = \tfrac{1}{2}(1-\gamma_5)\psi_P - \frac{i}{2}\left[(1-\gamma_5)\psi_N\right]^c$$

If we choose the representation for γ_5 as

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & 0 \end{pmatrix}$$

Then:

$$\left[\tfrac{1}{2}(1+\gamma_5)\psi\right]^c = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix}^c = \begin{pmatrix} 0 \\ 0 \\ \psi_2^* \\ -\psi_1^* \end{pmatrix}$$

$$\left[\tfrac{1}{2}(1-\gamma_5)\psi\right]^c = \begin{pmatrix} 0 \\ 0 \\ \psi_3 \\ \psi_4 \end{pmatrix}^c = \begin{pmatrix} -\psi_4^* \\ \psi_3^* \\ 0 \\ 0 \end{pmatrix}$$

Thus we find:

$$\chi_1 = \psi_1^{P} \qquad \chi_2 = \psi_2^{P} \qquad \chi_3 = i\psi_2^{*N} \qquad \chi_4 = -i\,\psi_1^{*N}$$

$$\xi_1 = +i\psi_4^{*N} \qquad \xi_2 = -i\psi_3^{*N} \qquad \xi_3 = \psi_3^{P} \qquad \xi_4 = \psi_4^{P}$$

It is of interest to write down the vacuum expectation values.

$$\left\langle \xi_\alpha(x)\,\bar{\xi}_\beta(x')\right\rangle_0 = \left\langle \bar{\xi}_\beta(x)\,\xi_\alpha(x')\right\rangle_0 =$$

$$= \left\langle \chi_\alpha(x)\,\bar{\chi}_\beta(x')\right\rangle_0 = \left\langle \bar{\chi}_\beta(x)\,\chi_\alpha(x')\right\rangle_0 =$$

$$= -\gamma_\mu^{\alpha\beta} \frac{\partial}{\partial x_\mu} (i\Delta_+(s)) \quad .$$

and:[8]

$$+\left\langle \xi_\alpha(x)\bar{\chi}_\beta(x')\right\rangle_0 = -\left\langle \chi_\alpha(x)\bar{\xi}_\beta(x')\right\rangle_0 = m\gamma^5_{\alpha\beta}(i\Delta_+(s)) .$$

The group is isomorphic to the spinor group of three dimensional rotations.

FOOTNOTES

1

$$\frac{1}{4}\sum_{k_\mu;dk_\mu}\sum_{\alpha,\beta}\left\langle 0 \,|\, \psi_\alpha(0) \,|\, k\right\rangle \left\langle k \,|\, \psi^*_\beta(0) \,|\, 0\right\rangle \frac{i}{\sqrt{\lambda}} (\gamma\cdot k)_{\beta\alpha} =$$

$$\frac{d^4k}{(2\pi)^3} \mathrm{Tr}\left\{\left[i(\gamma\cdot k)\rho^{I}_{\psi\bar\psi} + 1\cdot\sigma^{I}_{\psi\bar\psi} + i\gamma_5(\gamma\cdot k)\rho^{II}_{\psi\bar\psi} + i\gamma_5\sigma^{II}_{\psi\bar\psi}\right]\left(-\frac{1}{4}\gamma_4\frac{i}{\sqrt{\lambda}}(\gamma\cdot k)\right)\right\}$$

$$= \frac{d^4k}{(2\pi)^3}\frac{k_0}{\sqrt{\lambda}}\sigma^{I}_{\psi\bar\psi}$$

2 The new terms lead to

$$\mathrm{Tr}\left\{ i\gamma_5(\gamma\cdot k)\rho^{II}_{\psi\bar\psi}\left(-\frac{1}{4}\gamma_4\gamma_5\right)\right\} = k_0\rho^{II}_{\psi\bar\psi}$$

$$\mathrm{Tr}\left\{ i\gamma_5\sigma^{II}_{\psi\bar\psi}\left(-\frac{1}{4}\gamma_4\frac{1}{\sqrt{\lambda}}(\gamma\cdot k)\gamma_5\right)\right\} = \frac{k_0}{\sqrt{\lambda}}\sigma^{II}_{\psi\bar\psi}$$

and the other traces all vanish.

FOOTNOTES

3 We must require $A^2 + B^2 + C^2 = 1$. α then enters as a Lagrange multiplier.

4 We must show that the equations remain invariant with regard to this group. First, we develop the equation for ξ^c:

$$\xi^c = C^{-1}\bar{\xi} = -\bar{\xi}C^{-1} \qquad \xi = C\xi^c$$

so:

$$\gamma_\mu \frac{\partial \xi^c}{\partial x_\mu} = \gamma_\mu C^{-1}\frac{\partial \bar{\xi}}{\partial x_\mu} = \frac{\partial \bar{\xi}}{\partial x_\mu}\gamma_\mu C^{-1} = \frac{\partial \xi^*}{\partial x_\mu}\gamma_4\gamma_\mu C^{-1}$$

$$= -\frac{\partial \xi^*}{\partial x_\mu^*}\gamma_\mu\gamma_4 C^{-1} = -m\chi^*\gamma_5\gamma_4 C^{-1} = +m\chi^*\gamma_4\gamma_5 C^{-1}$$

$$= -m\gamma_5 C^{-1}\bar{\chi} = -m\gamma_5\chi^c \quad .$$

Here, we have used the facts that C^{-1}, $\gamma_5 C^{-1}$ are antisymmetric, $\gamma_\mu C^{-1}$ is symmetric, x = real, x_4 = pure imaginary, plus the equation for ξ. We note that the equations are not C invariant.

$$\xi' = a\xi + b\gamma_5\xi^c \qquad \chi' = a\chi + b\gamma_5\chi^c$$

so that:

$$\gamma_\mu \frac{\partial \xi'}{\partial x_\mu} = a\gamma_\mu\frac{\partial \xi}{\partial x_\mu} - b\gamma_5\gamma_\mu\frac{\partial \xi^c}{\partial x_\mu}$$

$$= a\, m\gamma_5\chi + b\, m\chi^c$$

$$= m\gamma_5\chi'$$

A similar development holds for χ' .

FOOTNOTES

5
$$\gamma_\mu \frac{\partial \chi'}{\partial x_\mu} = -\cos\alpha_\circ m\gamma_5 \xi + i \sin\alpha_\circ \gamma_5 \circ m \gamma_5 \xi$$

$$= -m\gamma_5 \xi' ,$$

and similarly for ξ' .

6
See below, for the representation $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

7 We have

$$\gamma_\mu \frac{\partial \chi}{\partial x_\mu} = \gamma_\mu \tfrac{1}{2}(1+\gamma_5)\frac{\partial \psi P}{\partial x_\mu} + \frac{i}{2}\gamma_\mu \frac{\partial}{\partial x_\mu}\left[(1+\gamma_5)\psi_N\right]^c$$

Now:

$$\psi^c = C^{-1}\bar{\psi} = -\tilde{\bar{\psi}} C^{-1} = -\psi^* \gamma_4 C^{-1} = -\gamma_4 C^{-1}\psi^*$$

so:

$$\left[(1 \pm \gamma_5)\psi_N\right]^c = -\gamma_4 C^{-1}(1 \pm \gamma_5^*)\psi_N^*$$

But:

$$C^{-1}\gamma_5^* = -\tilde{C}^{-1}\tilde{\gamma}_5 = -\widetilde{\gamma_5 C^{-1}} = \gamma_5 C^{-1}$$

so:

$$\left[(1 \pm \gamma_5)\psi_N\right]^c = -\gamma_4 (1 \pm \gamma_5) C^{-1}\psi_N^*$$

$$= -(1 \mp \gamma_5)\gamma_4 C^{-1}\psi_N^* = -(1 \mp \gamma_5)(\psi_N^* \gamma_4 C^{-1})$$

$$= -(1 \mp \gamma_5)\bar{\psi}_N C^{-1} = (1 \mp \gamma_5)\psi_N^c .$$

Thus:

FOOTNOTES

7 (Cont.)

$$\gamma_\mu \frac{\partial \chi}{\partial x_\mu} = \tfrac{1}{2}(1-\gamma_5)\, m\psi_P + \frac{i}{2}\gamma_\mu (1-\gamma_5)\frac{\partial}{\partial x_\mu}\psi_N^{\,c}$$

$$= \tfrac{1}{2}(1-\gamma_5)\, m\psi_P + \frac{i}{2}(1+\gamma_5)\, m\psi_N^{\,c}$$

$$= m\left\{\tfrac{1}{2}(1-\gamma_5)\psi_P + \frac{i}{2}\left[(1-\gamma_5)\psi_N\right]^c\right\} .$$

But:

$$\gamma_5 \xi = -\tfrac{1}{2}(1-\gamma_5)\psi_P - \frac{i}{2}(1+\gamma_5)\psi_N^{\,c}$$

so:

$$\gamma_\mu \frac{\partial \chi}{\partial x_\mu} = -m\gamma_5 \xi .$$

A similar development holds for the ξ equation.

8 We need:

$$\xi = \tfrac{1}{2}(1-\gamma_5)\psi_P - \frac{i}{2}\left[(1-\gamma_5)\psi_N\right]^c$$

$$= \tfrac{1}{2}(1-\gamma_5)\psi_P - \frac{i}{2}(1+\gamma_5)\psi_N^{\,c}$$

$$\bar{\xi} = \tfrac{1}{2}\psi_P^{\,*}(1-\gamma_5)\gamma_4 + \frac{i}{2}\psi_N^{\,c*}(1+\gamma_5)\gamma_4$$

$$\bar{\xi} = \tfrac{1}{2}\bar{\psi}_P(1+\gamma_5) + \frac{i}{2}\bar{\psi}_N^{\,c}(1-\gamma_5).$$

Thus, for example:

FOOTNOTES

8 (Cont.)

$$\left\langle \xi_\alpha(x)\bar{\xi}_\beta(x')\right\rangle_o = \tfrac{1}{4}(1-\gamma_5)_{\alpha\alpha'}\left\langle \psi^P_{\alpha'}(x)\bar{\psi}^P_{\beta'}(x')\right\rangle_o (1+\gamma)_{5\beta'\beta}$$

$$+\tfrac{1}{4}(1+\gamma_5)_{\alpha\alpha'}\left\langle \psi^N_{\alpha'}(x)^c\,\psi^N_{\beta'}(x')^c\right\rangle_o (1-\gamma_5)_{\beta'\beta}$$

$$=\tfrac{1}{4}(1-\gamma)_{5\alpha\alpha'}\left[-(\gamma_\mu\frac{\partial}{\partial x_\mu}-m)i\,\Delta_+(s)\right]_{\alpha'\beta'}(1-\gamma)_{5\beta'\beta}$$

$$+\tfrac{1}{4}(1+\gamma)_{5\alpha\alpha'}\left[-(\gamma_\mu\frac{\partial}{\partial x_\mu}-m)i\,\Delta_+(s)\right]_{\alpha'\beta'}(1-\gamma)_{5\beta'\beta}$$

$$=\left\{-\tfrac{1}{2}(\gamma_\mu\frac{\partial}{\partial x_\mu}-m)i\,\Delta_+(s)-\tfrac{1}{2}(\gamma_\mu\frac{\partial}{\partial x_\mu}+m)i\,\Delta_+(s)\right\}_{\alpha\beta}$$

$$\left\langle \xi_\alpha(x)\bar{\xi}_\beta(x')\right\rangle_o = -\gamma_\mu^{\alpha\beta}\frac{\partial}{\partial x_\mu}i\,\Delta_+(s)$$

We also find:

$$\bar{\chi} = \tfrac{1}{2}\bar{\psi}_P(1-\gamma_5)-\tfrac{i}{2}\bar{\psi}^c_N(1+\gamma_5) .$$

Also:

$$-\left\langle \xi_\alpha(x)\bar{\chi}_\beta(x')\right\rangle_o = \Big\{\tfrac{1}{4}(1-\gamma_5)(\gamma_\mu\frac{\partial}{\partial x_\mu}-m)(1-\gamma_5)$$

$$-\tfrac{1}{4}(1+\gamma_5)(\gamma_\mu\frac{\partial}{\partial x_\mu}-m)(1+\gamma_5)\Big\}_{\alpha\beta}\,i\,\Delta_+(s)$$

$$= -m\,\gamma^5_{\alpha\beta}\,i\,\Delta_+(s).$$

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LECTURE 14

Another possibility for introducing new fields is to set

$$\gamma_\mu \frac{\partial}{\partial x_\mu} \psi = m \phi$$

$$\gamma_\mu \frac{\partial}{\partial x_\mu} \phi = m \psi \ .$$

These show an invariance under the group of transformations:

$$\psi' = a \psi + b \gamma_5 C^{-1} \bar{\psi}$$

$$\phi' = a \phi - b \gamma_5 C^{-1} \bar{\phi}$$

$$\psi' = e^{i\alpha\gamma_5} \psi \qquad \phi' = e^{-i\alpha\gamma_5} \phi$$

$$\bar{\psi}' = \bar{\psi} e^{i\alpha\gamma_5} \qquad \bar{\phi}' = \bar{\phi} e^{i\alpha\gamma_5} \ .$$

The connection with ψ_P, ψ_N is now:[1]

$$\phi = \frac{1}{2}(1 + \gamma_5) \psi_P - \frac{i}{2} \left[(1 + \gamma_5)\psi_N\right]^c$$

$$\psi = \frac{1}{2}(1 - \gamma_5) \psi_P - \frac{i}{2} \left[(1 - \gamma_5)\psi_N\right]^c \ .$$

Here, the vacuum expectation values satisfy:

$$< \psi_\alpha(x) \bar{\psi}_\beta(x') >_o = < \bar{\psi}_\beta(x) \psi_\alpha(x') >_o = < \phi_\alpha(x) \bar{\phi}_\beta(x') >_o =$$

$$= < \bar{\phi}_\beta(x) \phi_\alpha(x') >_o = - \gamma_{\alpha\beta}{}^\mu \frac{\partial}{\partial x_\mu} \left(i \Delta_+ (s)\right)$$

and

$$< \phi_\alpha(x)\, \bar{\psi}_\beta(x') >_o \;=\; < \psi_\alpha(x)\, \bar{\phi}_\beta(x') >_o \;=\; - < \bar{\psi}_\beta(x)\, \phi_\alpha(x') >_o \;=$$

$$= \; - < \bar{\phi}_\beta(x)\, \psi_\alpha(x') >_o \;=\; \delta_{\alpha\beta} \left[i\, \Delta_+ (s) \right] \quad .$$

Remark: There is an interesting problem here. One might ask for the most general expressions for the vacuum expectation values, with the Lorentz and isospin groups present. First, one can replace one Δ_+ by F and the second by G (the free particle Dirac equation would connect them). This theory is gauge invariant since $< \psi\, \psi >_o = 0$. This would not be the most general possibility. Pauli does not know the most general result compatible with the group. A particular case is that in which there is only one spinor, ψ. ($\phi = 0$). Derivatives can also appear, though they will be restricted.

The system of Gürsey is slightly different to that just given. The relation is

$$\xi = \psi \qquad\qquad \chi = \gamma_5\, \phi$$

where the sign of χ related to ϕ is just a convention. The γ_5 is the essential difference.

There is a little more symmetry in the Gürsey choice, since

$$\begin{pmatrix} \chi \\ \xi \end{pmatrix}' = a \begin{pmatrix} \chi \\ \xi \end{pmatrix} + b\, \gamma_5\, C^{-1} \begin{pmatrix} \bar{\chi} \\ \bar{\xi} \end{pmatrix}$$

$$\xi' = e^{i\alpha\gamma_5}\xi \;, \qquad\qquad \chi' = e^{-i\alpha\gamma_5}\chi \;.$$

However,

$$\gamma_\mu \frac{\partial}{\partial x_\mu}\, \xi = m\, \gamma_5\, \chi \;, \qquad\qquad \gamma_\mu \frac{\partial}{\partial x_\mu}\, \chi = - m\, \gamma_5\, \xi \;.$$

[The signs of the two equations must be different, to keep the second order equation correct.]

Also:

$$\gamma_\mu \frac{\partial}{\partial x_\mu} \xi^c = - m \gamma_5 \chi^c , \qquad \gamma_\mu \frac{\partial}{\partial x_\mu} \chi^c = m \gamma_5 \xi^c .$$

Gürsey has a particular way of writing these equations that is not fundamental, but is instructive. He introduces matrices for ψ, and chooses γ_5 diagonal:

$$\Psi = \begin{pmatrix} \psi_1 & \psi_1{}^c \\ \psi_2 & \psi_2{}^c \end{pmatrix} = \begin{pmatrix} \psi_1 & -\psi_4{}^* \\ \psi_2 & +\psi_3{}^* \end{pmatrix} .$$

The Dirac equation is obtained, using

$$D = \frac{\partial}{\partial t} - \vec{\sigma}\cdot\frac{\partial}{\partial \vec{x}}$$

and

$$\bar{\Psi} = \omega \tilde{\Psi} \omega^{-1} ,$$

where, for ordinary 2 x 2 matrices (not q-numbers),

$$\bar{\Psi} = \text{Det.} \left\{ \psi \right\} \times \psi^{-1} .$$

Also:

$$\bar{\Psi} = + \begin{pmatrix} \psi_3{}^* & \psi_4{}^* \\ -\psi_2 & \psi_1 \end{pmatrix} .$$

With respect to the groups:

$$\Psi' = \Psi U$$

where U is the most general unitary transformation of two variables.[2] For the first transformation

$$\text{Det } U = 1,$$

while for the second

$$U = e^{i\alpha} .$$

Replacing

$$X \longleftrightarrow X$$

$$\Xi \longleftrightarrow \xi$$

by going from column vector to matrix, we find:[3]

$$D \overline{\Xi}^* = i\, m\, X$$

$$D \overline{X}^* = -i\, m\, \Xi$$

and

$$\Xi \rightarrow \Xi U = \Xi R e^{i\alpha}$$

$$X \rightarrow X \overline{U}^+ = X R e^{-i\alpha}$$

where R is the rotation group (Det = 1).

[Pauli doesn't like this too well.]

The existence of the group, isomorphic to baryon conservation and isospin conservation, is the important thing.

$$\psi_P' = \psi_P e^{i\alpha} ; \qquad \psi_N' = \psi_N e^{i\alpha} \quad \text{(baryon conservation)}$$

The transformation R does not commute with C or P, but it does with CP. In the case of weak interactions we can write it in such a way that only one field (X or ξ) occurs. This was introduced first in the case of the free neutrino. The equation of a four-component neutrino has the full group. However, if it is a Majorana two-component neutrino, then only the second group is preserved as compatible with

$$\psi^c = \psi$$

$$\psi' = e^{i\alpha\gamma_5}\psi$$

$$\psi'^c = e^{i\alpha\gamma_5}\psi^c .$$

FOOTNOTES

1 We have:

$$\gamma_\mu \frac{\partial \phi}{\partial x_\mu} = \frac{1}{2}(1-\gamma_5)\gamma_\mu \frac{\partial \psi_P}{\partial x_\mu} - \frac{i}{2}(1+\gamma_5)\gamma_\mu \frac{\partial \psi_N{}^c}{\partial x_\mu}$$

$$= \frac{1}{2}(1-\gamma_5) m\, \psi_P - \frac{i}{2}\left[(1-\gamma_5) m\, \psi_N\right]^c$$

$$= m\,\psi \; .$$

Similarly for the ψ equation.

The invariance under the first group comes from the fact that the ϕ, ψ satisfy C invariant equations (the lack of a γ_5 as compared to Gürsey accounts for this), so that:

$$\gamma_\mu \frac{\partial \psi'}{\partial x_\mu} = a\,\gamma_\mu \frac{\partial \psi}{\partial x_\mu} - b\,\gamma_5\,\gamma_\mu \frac{\partial \psi^c}{\partial x_\mu}$$

$$= m\left[a\,\phi - b\,\gamma_5\,\phi^c\right] = m\,\phi' \; .$$

We find:

$$\frac{\partial \bar{\phi}}{\partial x_\mu}\gamma_\mu = \frac{\partial \phi^*}{\partial x_\mu}\gamma_4\,\gamma_\mu = -\frac{\partial \phi^*}{\partial x_\mu{}^*}\gamma_\mu\,\gamma_4 = -m\,\psi^*\,\gamma_4 = -m\,\bar{\psi}$$

and

$$\gamma_\mu \frac{\partial \phi^c}{\partial x_\mu} = \gamma_\mu\, C^{-1}\frac{\partial \bar{\phi}}{\partial x_\mu} = \frac{\partial \bar{\phi}}{\partial x_\mu}\gamma_\mu\, C^{-1} = -m\,\bar{\psi}\; C^{-1} = m\,\psi^c \; .$$

2 For the first group of transformations we have:

$$\begin{pmatrix} \chi_1 \\ \chi_2 \\ 0 \\ 0 \end{pmatrix} = \frac{(1+\gamma_5)}{2}\,\chi \qquad \text{and}$$

2 (Cont.)

$$\begin{pmatrix} \chi_1{}' \\ \chi_2{}' \\ 0 \\ 0 \end{pmatrix} \rightarrow \frac{(1 + \gamma_5)}{2} \chi' = a \frac{(1+\gamma_5)}{2} \chi + b \frac{(1+\gamma_5)}{2} \gamma_5 \chi^c$$

$$= \frac{(1+\gamma_5)}{2} (a \chi + b \chi^c)$$

Also:

$$\frac{(1+\gamma_5)}{2} \chi^{c'} = a^* \frac{(1+\gamma_5)}{2} \chi^c - b^* \gamma_5 \frac{(1+\gamma_5)}{2} \chi \quad .$$

Thus we have;

$$\begin{pmatrix} \chi_1{}' & \chi_1{}'^c \\ \chi_2{}' & \chi_2{}'^c \end{pmatrix} = \begin{pmatrix} a \chi_1 + b \chi_1{}^c & -b^* \chi_1 + a^* \chi_1{}^c \\ a \chi_2 + b \chi_2{}^c & -b^* \chi_2 + a^* \chi_2{}^c \end{pmatrix}$$

$$= \begin{pmatrix} \chi_1 & \chi_1{}^c \\ \chi_2 & \chi_2{}^c \end{pmatrix} \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}$$

so that, since $|a|^2 + |b|^2 = 1$, det U = 1.

For the second group,

$$\chi' = e^{i\alpha\gamma_5} \chi \quad .$$

Now

$$(\chi')^c = \left[(\cos\alpha + i \sin\alpha \cdot \gamma_5)\chi\right]^c$$

$$= \cos\alpha \cdot \chi^c - i \sin\alpha \left[\gamma_5 \chi\right]^c$$

$$= (\cos\alpha + i \sin\alpha\, \gamma_5)\chi^c$$

$$= e^{i\alpha\gamma_5} \chi^c$$

2 (Cont.)

Thus

$$\begin{pmatrix} \chi_1{}' & \chi_1{}^{c'} \\ \chi_2{}' & \chi_2{}^{c'} \end{pmatrix} = e^{i\alpha} \begin{pmatrix} \chi_1 & \chi_1{}^{c} \\ \chi_2 & \chi_2{}^{c} \end{pmatrix} .$$

3 In this representation,

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_k = \begin{pmatrix} 0 & i\sigma_k \\ -i\sigma_k & 0 \end{pmatrix}$$

where the "elements" are 2 x 2 matrices. Thus:

$$\frac{1}{i}\frac{\partial}{\partial t} + i\,\sigma_k \cdot \frac{\partial}{\partial x_k} \begin{pmatrix} \xi_3 \\ \xi_4 \end{pmatrix} = m \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$$

and

$$\frac{1}{i}\frac{\partial}{\partial t} - i\,\sigma_k \cdot \frac{\partial}{\partial x_k} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = -m \begin{pmatrix} \chi_3 \\ \chi_4 \end{pmatrix}$$

or

$$\frac{1}{i}\frac{\partial}{\partial t} - i\,\tilde{\sigma}_k \frac{\partial}{\partial x_k} \begin{pmatrix} \xi_1{}^* \\ \xi_2{}^* \end{pmatrix} = m \begin{pmatrix} \chi_3{}^* \\ \chi_4{}^* \end{pmatrix} .$$

If we choose the usual representation, $\tilde{\sigma}_x = \sigma_x$, $\tilde{\sigma}_y = -\sigma_y$, $\tilde{\sigma}_z = \sigma_z$, so if we multiply the equation by σ_y, we get:

$$\left(\frac{1}{i}\frac{\partial}{\partial t} + i\,\sigma_k \frac{\partial}{\partial x_k}\right) \sigma_y \begin{pmatrix} \xi_1{}^* \\ \xi_2{}^* \end{pmatrix} = m\,\sigma_y \begin{pmatrix} \chi_3{}^* \\ \chi_4{}^* \end{pmatrix}$$

or

$$\left(\frac{1}{i}\frac{\partial}{\partial t} + i\,\sigma_k \frac{\partial}{\partial x_k}\right) \begin{pmatrix} -\xi_2{}^* \\ \xi_1{}^* \end{pmatrix} = m \begin{pmatrix} -\chi_4{}^* \\ \xi_3{}^* \end{pmatrix} .$$

3 (Cont.)

Thus, since

$$(\,\Xi\,)^* = \begin{pmatrix} \xi_3^* & \xi_4^* \\ -\xi_2 & \xi_1 \end{pmatrix}^* = \begin{pmatrix} \xi_3 & -\xi_2^* \\ \xi_4 & \xi_1^* \end{pmatrix}$$

the equations satisfy:

$$D \;\; \Xi^* = i\, m\, X \; .$$

The development is similar for the X equation.

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LECTURE 15

We now come to a discussion of the weak interactions. For these, the symmetry is less than in the strong interactions or the electromagnetic ones. For the weak interactions, the symmetry can be no more than CP. The symmetries involve:

C	P	T
$Q' = -Q$	$\vec{x}' = -\vec{x}$	$t' = -t$
Particles → Antiparticles	$t' = t$	$\vec{x}' = \vec{x}$
	$Q' = Q$	$Q' = Q$

In the P, T cases "charge" is preserved. Here "charge" means more than electromagnetic charge, and includes leptonic charge plus the particle → antiparticle concept for nucleons, etc.

The experiments are not yet good enough to prove the CP invariance. Now, there are interesting postulates which can be made about the weak interactions, though none is yet proved. Different definitions can be made for the leptonic charges. One particular choice is:

$$Q_{e\ell.} = \frac{1}{2}\int dV \left(\left[\psi_P^*, \psi_P\right] - \left[\psi_e^*, \psi_e\right] \right)$$

$$Q_{lept.} = \frac{1}{2}\int dV \left(\left[\psi_e^*, \psi_e\right] + \left[\psi_\nu^*, \psi_\nu\right] \right)$$

where ψ_P is the proton field; ψ_e, the electron; and ψ_ν, the neutrino. The leptonic charge is then:

+1	(negaton $e-$)	(neutrinos ν)
-1	(positon e_+)	(anti-neutrinos $\bar{\nu}$)
o	nucleons	

If we have conservation of leptonic charge then possible reactions are

$P \rightarrow N + e_+ + \nu$ (bound)

$N \rightarrow P + e- + \bar{\nu}$ (free or bound)

Also:

$P + \bar{\nu} \rightarrow N + e_+$ (Cowan and Reines experiment)

$N + \nu \rightarrow P + e-$

We only have strong sources for negaton decay, not positon (neutrons).

A check on the conservation of leptonic charge would be gained from observation of the sequence

$$N \rightarrow P + e- + \bar{\nu},$$

and

$$N + \bar{\nu} \rightarrow P + e-,$$

which would be forbidden if leptons are conserved. There is an experiment by Davis, but it is not good enough in accuracy. The absence of double beta decay is of the same character.[1] We will assume the conservation of leptons.

The thing which is definitely established is that C, P are violated.

Other processes of interest are:

$$\pi_+ \to \mu_+ + \nu \qquad\qquad \pi_+ \to \mu_+ + \bar{\nu}$$

$$\pi_- \to \mu_- + \bar{\nu} \qquad\qquad \pi_- \to \mu_- + \nu$$

or

$$\mu_+ \to e_+ + \nu + \bar{\nu} \qquad\qquad \mu_+ \to e_+ + \nu + \nu$$

$$\mu_- \to e_- + \nu + \bar{\nu} \qquad\qquad \mu_- \to e_- + \bar{\nu} + \bar{\nu}.$$

We can exclude the second one (experimentally), which gives the μ_+ the same leptonic charge as e_+.[2]

There are also weak processes in which ν's do not play a role as well:

$$\Lambda_o \to P + \pi_-$$

$$\Lambda_o \to N + \pi_o$$

$$K \to 3\pi$$

$$K \to 2\pi$$

$$K \to \mu + \pi + \nu$$

$$K \to e + \pi + \nu$$

$$P \to N + \mu_+ + \nu \qquad\qquad N + \mu_+ \to P + \bar{\nu}$$

$$N \to P + \mu_- + \bar{\nu} \qquad\qquad P + \mu_- \to N + \nu$$

It is important that non-neutrino C and P violating interactions also occur. The neutrino ones are just a special case.

We will now consider ordinary beta decay. The interaction will be written as:

$$H_{int.} = \sum C_i (\bar{\psi}_P \sigma_i \psi_N)(\bar{\psi}_e \sigma_i \psi_\nu) + h.c.$$

Leptonic and electromagnetic charges are conserved. The conservation laws correspond to the invariance under:

E.M. Charge:

$$\psi_P \to \psi_P e^{i\alpha}, \quad \psi_e \to \psi_e e^{-i\alpha}, \quad \psi_N \to \psi_N, \quad \psi_\nu \to \psi_\nu$$

Leptonic charge:

$$\psi_e \to \psi_e e^{i\beta}, \quad \psi_\nu \to \psi_\nu e^{i\beta}, \quad \psi_P \to \psi_P, \quad \psi_N \to \psi_N$$

Baryonic Charge:

$$\psi_P \to \psi_P e^{i\gamma}, \quad \psi_N \to \psi_N e^{i\gamma}, \quad \psi_e \to \psi_e, \quad \psi_\nu \to \psi_\nu .$$

Pauli believes that only leptonic charge conservation is not certain.

The baryonic charge is defined by:

$$N = \tfrac{1}{2} \int dV \left([\psi_P^*, \psi_P] + [\psi_N^*, \psi_N] \right).$$

The θ_i's are:

S	P	V	A	T
1	γ_5	γ_μ	$i\gamma_5\gamma_\mu$	$i\gamma_\mu\gamma_\nu$ $\mu<\nu$
(1)	(1)	(4)	(4)	(6)

The θ_i's are hermitian: $\theta_i^+ = \theta_i$, and $\theta_i^2 = 1$. (S, P, V, A, T are scalar, pseudoscalar, vector, axial vector, and tensor, respectively.) These combinations will preserve parity. We can also add other terms:

$$H_{int.} = \sum (\bar{\psi}_P \theta_i \psi_N) \left\{ \bar{\psi}_e \theta_i (C_i + C_i' \gamma_5) \psi_\nu \right\} + h.c.$$

If <u>both</u> C_i and C_i' are different from zero, one violates parity.

The coexistence of both is necessary (Wightman, Wigner, Wick).

If leptonic charge were not conserved, we could still be more general:

$$H_{int.} = \sum_i (\bar{\psi}_P \sigma_i \psi_N) \Big\{ C_i (\bar{\psi}_e \sigma_i \psi_\nu) + C_i' \bar{\psi}_e \sigma_i \gamma_5 \psi_\nu$$

$$+ D_i (\bar{\psi}_e \sigma_i \psi_\nu^c) + D_i' (\bar{\psi}_e \sigma_i \gamma_5 \psi^c) \Big\} + h.c.$$

where:

$$\psi_\nu^c = C^{-1} \bar{\psi}_\nu, \quad \tilde{C} = -C, \; CC^* = 1, \; \tilde{\gamma}_\mu = -C \gamma_\mu C^{-1} .$$

We will see that permutations of the fields will not give anything new, since algebraic relations exist between the C's for different choices.

The parity transformation is:

$$\vec{x}' = -\vec{x}, \; t' = t, \; \vec{\sigma}' = \vec{\sigma}, \; \vec{P}' = -\vec{P}$$

where $\vec{P}$ is the momentum. If A indicates any of the fields P, N, e, or ν, then parity exists if there is a transformation of the form:

$$\psi_A' (\vec{x}, t) = e^{i\alpha_A} \gamma_4 \psi_A (-\vec{x}, t)$$

Which leaves $H_{int.}$ invariant, and also leaves the free particle equations invariant.

We also have:

$$\bar{\psi}_A' (\vec{x}, t) = \bar{\psi}_A (-\vec{x}, t) \gamma_4 e^{-i\alpha_A}$$

Now:

$$\gamma_4 \sigma_i = \pm \sigma_i \gamma_4 ,$$

and if we have

$$C' = D' = 0: \quad \psi_A'(\vec{x},t) = i\gamma_4\psi_A(-\vec{x},t)$$

$$\psi_A'^{c}(\vec{x},t) = i\gamma_4\psi_A^{c}(-\vec{x},t),$$

then $H_{int.}$ is invariant. On the other hand, if

$C = D = 0$, we can choose $\alpha_P = \alpha_N = \alpha_e = \pi/2, \alpha_\nu = -\pi/2$,

which again leaves $H_{int.}$ unchanged.

FOOTNOTES

1. The experiment of Davis uses neutrinos from a pile to react in

$$\bar{\nu} + Cl^{37} \rightarrow e_- + A^{37},$$

which is not found. In double beta decay, one looks for

$$N + N \rightarrow P + P + e_- + e_-.$$

2. In the μ-decay, the ρ value indicates that the two neutrinos are different; while the polarization of the μ in $\pi-\mu$ decay indicates that $\pi_+ \rightarrow \mu_+ + \bar{\nu}$. This, then, is another piece of evidence for lepton conservation.

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LECTURE 16

Formulas regarding permutations.

The invariants are

$$\mathcal{J}_i = (\overline{\psi}_P \sigma_i \psi_N)(\overline{\psi}_e \sigma_i \psi_\nu).$$

We could permute the fields to get:

$$\mathcal{J}_i{}' = (\overline{\psi}_P \sigma_i \psi_\nu)(\overline{\psi}_e \sigma_i \psi_N).$$

Among such permutations there will be linear relations:

$$4\begin{pmatrix} \mathcal{J}_S{}' \\ \mathcal{J}_V{}' \\ \mathcal{J}_T{}' \\ \mathcal{J}_A{}' \\ \mathcal{J}_P{}' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & 0 & 2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & 2 & 0 & -2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{J}_S \\ \mathcal{J}_V \\ \mathcal{J}_T \\ \mathcal{J}_A \\ \mathcal{J}_P \end{pmatrix}$$

The signs chosen here are for c-number fields. For anticommuting fields, the matrix will be multiplied by (-1).

There is also an identity:

$$\mathcal{J}_V' - \mathcal{J}_A' = \mathcal{J}_V - \mathcal{J}_A ,$$

for q-number fields. Thus:

$$(\bar{\psi}_P \gamma_\mu \psi_N)(\bar{\psi}_e \gamma_\mu \psi_\nu) - (\bar{\psi}_P i\gamma_5 \gamma_\mu \psi_N)(\bar{\psi}_e i\gamma_5 \gamma_\mu \psi_\nu) =$$

$$= (\bar{\psi}_P \gamma_\mu \psi_\nu)(\bar{\psi}_e \gamma_\mu \psi_N) - (\bar{\psi}_P i\gamma_5 \gamma_\mu \psi_\nu)(\bar{\psi}_e i\gamma_5 \gamma_\mu \psi_N) .$$

This is an algebraic identity. Since it is true as an identity, it will still be true if:

$$\psi_\nu \rightarrow \gamma_5 \psi_\nu .$$

$$(\bar{\psi}_P \gamma_\mu \psi_N)(\bar{\psi}_e \gamma_\mu \gamma_5 \psi_\nu) + (\bar{\psi}_P i\gamma_5 \gamma_\mu \psi_N)(\bar{\psi}_e i\gamma_\mu \psi_\nu) =$$

$$= (\bar{\psi}_P \gamma_\mu \gamma_5 \psi_\nu)(\bar{\psi}_e \gamma_\mu \psi_N) + (\bar{\psi}_P i\gamma_\mu \psi_\nu)(\bar{\psi}_e i\gamma_5 \gamma_\mu \psi_N) .$$

These particular relations will be important later. From these two identities we obtain:

$$\left(\bar{\psi}_P \gamma_\mu (1 \pm \gamma_5)\psi_N\right)\left(\bar{\psi}_e \gamma_\mu (1 \pm \gamma_5)\psi_\nu\right) =$$

$$\left(\bar{\psi}_P \gamma_\mu (1 \pm \gamma_5)\psi_\nu\right)\left(\bar{\psi}_e \gamma_\mu (1 \pm \gamma_5)\psi_N\right) .$$

These identities are derived by Pauli (1935), and are used by Fierz, Zs. f. Phys. 104, 553 (1937). They are derived using the Schur identities:[1]

$$\sum_{A=1}^{16} \gamma_{\alpha\beta}^{A} \gamma_{\rho\sigma}^{A} = 4\, \delta_{\beta\rho}\, \delta_{\alpha\sigma}$$

By multiplying with a suitable γ^B, we find:

$$\sum \gamma_{\alpha\beta}^{A} (\gamma^B \gamma^A \gamma^B)_{\rho\sigma} = 4 \gamma_{\rho\beta}^{B} \gamma_{\alpha\sigma}^{B}$$

From this one can get the table.[2] [See also the book of Umezawa.]

We will now introduce a different notation, which is convenient.

$$\psi^R = \frac{1-\gamma_5}{2}\psi, \quad \psi^L = \frac{1+\gamma_5}{2}\psi,$$

so

$$a^R = \frac{1-\gamma_5}{2}, \quad a^L = \frac{1+\gamma_5}{2}$$

are projection operators:

$$a^R \cdot a^L = 0 \qquad (a^R)^2 = a^R \qquad (a^L)^2 = a^L .$$

Then we find:[3]

$$(\gamma^\mu \frac{\partial}{\partial x_\mu})\psi^R + m\psi^L = 0 .$$

$$(\gamma^\mu \frac{\partial}{\partial x_\mu})\psi^L + m\psi^R = 0.$$

If $m = 0$, the ψ^L, ψ^R are uncoupled. The canonical commutation relations are:[4]

$$\left\{\psi_\alpha^R(\vec{x},t), \psi_\beta^{*R}(\vec{x}',t)\right\} = a_{\alpha\beta}^R\, \delta^{(3)}(\vec{x}-\vec{x}')$$

$$\left\{\psi_\alpha^L(\vec{x},t), \psi_\beta^{*L}(\vec{x}',t)\right\} = a_{\alpha\beta}^L\, \delta^{(3)}(\vec{x}-\vec{x}') .$$

The others vanish. Here, m is not necessarily = 0.

We must now consider the charge conjugate operators.

$$\psi^c = C^{-1}\overline{\psi}, \ \overline{\psi}^c = \overline{\psi^c} = -\psi C$$

$$(\psi^c)^R = \frac{1-\gamma_5}{2}\psi^c = C^{-1}\overline{\psi^L} = (\psi^L)^c \ .$$

To be complete, we need

$$\overline{\psi^R} = \overline{\psi}^R = \overline{\psi}\left(\frac{1+\gamma_5}{2}\right)$$

$$\overline{\psi^L} = \overline{\psi}^L = \overline{\psi}\left(\frac{1-\gamma_5}{2}\right)$$

The sign on γ_5 changes for $\overline{\psi}$ because of the γ_4 in the definition of $\overline{\psi}$. Also:

$$(\psi^c)^L = C^{-1}\overline{\psi}_R = (\psi^R)^c \ .$$

Thus L and C do not commute, nor do R and C.

We will see that for positive energy states ψ_ν^R corresponds to polarization of the spin in the direction of the momentum, while for ψ_ν^L it is opposite.

We can write:

$$H_{int.} = \sum_i (\overline{\psi}_P \sigma_i \psi_N)\left\{\overline{\psi}_e \sigma_i \times \left[C_{R_i}\psi_\nu^R + C_{L_i}\psi_\nu^L + D_{R_i}(\psi_i^c)^R + D_{L_i}(\psi_i^c)^L\right]\right\} + h.c.$$

Here,

$$C_R = (C - C') \qquad C_L = (C + C')$$

$$D_R = (D - D') \qquad D_L = (D + D') \ .$$

For the "R-Model", $C_L = D_L = 0$; while for the "L-Model", $C_R = D_R = 0$.

FOOTNOTES

1 This identity is easily derived using the property of the completeness of the 16 γ^A's, and the property that

$$\mathrm{Tr}\ (\gamma^A \gamma^B) = 4\delta_{AB}.$$

Thus, any matrix $X_{\alpha\beta}$ can be written

$$X_{\alpha\beta} = C^A \gamma_{\alpha\beta}{}^A$$

and

$$\mathrm{Tr}(X\gamma^B) = 4C^B$$

so

$$X_{\alpha\beta} = \tfrac{1}{4}(X_{\rho\sigma}\gamma_{\sigma\rho}{}^A)\gamma_{\alpha\beta}{}^A$$

Since X is arbitrary, we may choose $X_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta j}$, to get

$$4\,\delta_{\alpha i}\delta_{\beta j} = \gamma_{ji}{}^A \gamma_{\alpha\beta}{}^A \qquad \text{Q.E.D.}$$

2 For example, we calculate the $\mathcal{J}_V$. We have:

$$\sum_{A,\mu} \gamma_{\alpha\beta}{}^A (\gamma^\mu \gamma^A \gamma^\mu)_{\rho\sigma} = \sum_\mu 4\ \gamma_{\rho\beta}{}^\mu \gamma_{\alpha\sigma}{}^\mu$$

The right side is clearly that involved in the vector elements. Now, for $A = 1$, we just get $\sum (\gamma^\mu)^2 = 4$, and for $A = 16(\gamma^5)$ we get

$$\sum_\mu \gamma^\mu \gamma^5 \gamma^\mu = -4\gamma^5 .$$

For the vector parts:

$$\sum_\mu \gamma^\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu.$$

FOOTNOTES

2 (Cont.)

For tensor:

$$\sum_{\mu<\nu} \gamma^{\mu}\gamma^{\nu}\gamma^{\sigma}\gamma^{\mu}\gamma^{\nu} = 0.$$

For axial vector:

$$\sum_{\mu} \gamma^{\mu}\gamma^{5}\gamma^{\nu}\gamma^{\mu} = +2\gamma^{5}\gamma^{\nu}$$

Thus we find:

$$4\gamma_{\rho\beta}^{\mu}\gamma_{\alpha\sigma}^{\mu} = 4\delta_{\rho\beta}\delta_{\alpha\sigma} - 2\gamma_{\rho\beta}^{\nu}\gamma_{\alpha\sigma}^{\nu} + 2(i\gamma_5\gamma^{\nu})_{\rho\beta}(i\gamma_5\gamma^{\nu})_{\alpha\sigma}$$

$$-4\gamma_{\rho\beta}^{5}\gamma_{\alpha\sigma}^{5}.$$

The remainder of the table may be constructed similarly.

3

$$(\gamma_{\mu}\frac{\partial}{\partial x_{\mu}})\psi = m\psi$$

$$(\gamma_{\mu}\frac{\partial}{\partial x_{\mu}})\gamma_5\psi = -m\gamma_5\psi$$

4 We have:
$$\left\{\psi_{\rho}(x), \psi_{\sigma}^{*}(x')\right\} = \delta_{\rho\sigma}\delta^{(3)}(\vec{x}-\vec{x}')$$
so:

$$(a_R)_{\rho\rho'}\left\{\psi_{\rho'}(x), \psi_{\sigma'}^{*}(x')\right\}(a_R)_{\sigma'\sigma} = (a_R^{\,2})_{\rho\sigma}\delta^{(3)}(\vec{x}-\vec{x}')$$

$$= (a_R)_{\rho\sigma}\,\delta^{(3)}(\vec{x}-\vec{x}')$$

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LECTURE 17

Continuation on polarization.

This holds for positive frequencies:

$$\psi \sim e^{-i\omega t} \, , \qquad \omega > 0 \, .$$

$$\psi \sim e^{i(\vec{p}\cdot\vec{x} - \omega t)} \qquad \omega = |\vec{p}| > 0 \, .$$

Now:

$$\gamma_\nu \frac{\partial \psi}{\partial x_\nu} = 0 \, ,$$

$$\psi^R = \frac{1 - \gamma_5}{2} \psi \, , \qquad \psi^L = \frac{1 + \gamma_5}{2} \psi \, .$$

The polarization definition depends only on $\omega > 0$, not the representation.

Now in the 4 x 4 matrices

$$\Sigma_1 \Sigma_2 = i \Sigma_3 \, , \quad \Sigma_2 \Sigma_3 = i \Sigma_1 \, , \quad \text{---}$$

and

$$\gamma_1 \gamma_2 = i \Sigma_3 \, \ldots\ldots$$

There is no alternative to this choice for $\sum$. Then:

$$\vec{\alpha} = i\gamma_4\vec{\gamma} = -\gamma_5\vec{\sum} = -\sum\gamma_5 .$$

Thus:

$$\frac{\partial\psi}{\partial t} + i\gamma_4\vec{\gamma}\cdot\frac{\partial\psi}{\partial\vec{x}} = 0.$$

$$\frac{\partial}{\partial\vec{x}} = i\vec{p} \quad ; \quad \frac{\partial}{\partial t} = -i\omega$$

so:

$$-\omega\psi - \gamma_5(\vec{p}\cdot\vec{\sum})\psi = 0.$$

$$\left(\frac{\vec{p}\cdot\vec{\sum}}{\omega}\right)\psi = -\gamma_5\psi.$$

But,

$$\gamma_5\psi^R = \gamma_5\left(\frac{1-\gamma_5}{2}\right)\psi = -\psi^R$$

$$\gamma_5\psi^L = \gamma_5\left(\frac{1+\gamma_5}{2}\right)\psi = \psi^L$$

so:

$$\left(\frac{\vec{p}\cdot\vec{\sum}}{\omega}\right)\psi^R = \psi^R$$

$$\left(\frac{\vec{p}\cdot\vec{\sum}}{\omega}\right)\psi^L = -\psi^L .$$

Thus, whether γ_5 is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

is not relevant. This would only determine whether positive energy parts appear above or below. In either case:

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

but in the first choice

$$\vec{\alpha} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix},$$

while in the second,

$$\vec{\alpha}' = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}$$

Now, we had

$$H_{int.} = \sum_i (\bar{\psi}_P \sigma_i \psi_N) \left\{ \bar{\psi}_e \sigma_i \left(C_{Ri} \psi_\nu^R + C_{Li} \psi_\nu^L \right.\right.$$

$$\left.\left. + D_{Ri} (\psi_\nu^c)^R + D_{Li} (\psi_\nu^c)^L \right) \right\} + h.c.$$

If $m_\nu = 0$, the free particle neutrino equations admit a group (canonical transformation):

$$\text{(I)} \qquad \psi_\nu' = a\psi_\nu + b\gamma_5 \psi_\nu^c; \qquad \psi_\nu^{c\,\prime} = -b^* \gamma_5 \psi_\nu + a^* \psi_\nu^c$$

where $|a|^2 + |b|^2 = 1$. [The γ_5 makes the transformation canonical.]

In terms of R and L:

$$(\psi_\nu{}')^R = a\psi_\nu{}^R - b(\psi_\nu{}^c)^R \ ; \quad (\psi_\nu{}')^L = a\psi_\nu{}^L + b(\psi_\nu{}^c)^L$$

$$(\psi_\nu{}^{c'})^R = b^*\psi_\nu{}^R + a^*(\psi_\nu{}^c)^R \ ; \quad (\psi_\nu{}^{c'})^L = -b^*\psi_\nu{}^L + a^*(\psi_\nu{}^c)^L \ .$$

Thus the transformation is isomorphic to the unitary group[1] (a duplication of rotations in three dimensions).

Now if we consider transformations of ψ_ν, then $H_{int.}$ will be invariant if we also change the coupling constants at the same time, in such a way that

$$\begin{pmatrix} C_R{}' \\ D_R{}' \end{pmatrix} = \begin{pmatrix} a^* & -b^* \\ b & a \end{pmatrix} \begin{pmatrix} C_R \\ D_R \end{pmatrix} ; \quad \begin{pmatrix} C_L{}' \\ D_L{}' \end{pmatrix} = \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix} \begin{pmatrix} C_L \\ D_L \end{pmatrix} .$$

Then $H_{int.}$ will be invariant.[2]

We note that $\begin{pmatrix} D_L{}^* \\ C_L{}^* \end{pmatrix}$ transforms as $\begin{pmatrix} C_R \\ D_R \end{pmatrix}$.

There is also a second group:

$$\text{(II)} \qquad \psi_\nu{}' = e^{i\alpha\gamma_5}\psi_\nu \equiv (\cos\alpha + i\gamma_5 \sin\alpha)\psi_\nu$$
$$\bar{\psi}_\nu{}' = \bar{\psi}_\nu e^{i\alpha\gamma_5}$$

Thus:

$$\psi_\nu{}^{R'} = e^{-i\alpha}\psi_\nu{}^R \ ; \quad \psi_\nu{}^{L'} = e^{i\alpha}\psi_\nu{}^L ,$$

and:

$$\psi_\nu{}^{'c} = e^{i\alpha\gamma_5}\psi_\nu{}^c$$

The latter relation is important, since it indicates that the Majorana abbreviation is compatible with this group (II). It is not compatible with the first group (I).

To provide invariance for $H_{int.}$ we must transform the C's as:

$$\begin{pmatrix} C_R' \\ D_R' \end{pmatrix} = e^{i\alpha} \begin{pmatrix} C_R \\ D_R \end{pmatrix}; \qquad \begin{pmatrix} C_L' \\ D_L' \end{pmatrix} = e^{-i\alpha} \begin{pmatrix} C_L \\ D_L \end{pmatrix}$$

The most general canonical transformation is one that combines I and II.

[Pauli doesn't feel that it is natural to combine these two groups. If there is a natural divisor to a group, he feels that it should be split off.]

Let us now consider the problem of determining the $H_{int.}$ from the experiments. In any particular experiment we will generally make averages over the Hilbert space. The a^R, a^L will correspond to projection operators in this space. The transition probability can only depend on such combinations as appear in $H_{int.}$ We cannot distinguish between a neutrino or an anti-neutrino, nor can we distinguish the polarization of the neutrino alone. We <u>can</u> discriminate between the R and L states.

We have the following as the invariants of the unitary group.[3]

$$A_{ij}^R = C_{Ri}C_{Rj}^* + D_{Ri}D_{Rj}^*$$

$$A_{ij}^L = C_{Li}C_{Lj}^* + D_{Li}D_{Lj}^*$$

$$B_{ij} = C_{Ri}D_{Lj} + D_{Ri}C_{Lj} \quad .$$

B_{ij} only occurs in double beta-decay processes.

There are certain relations among these invariants:

$$A_{ij} = A_{ji}^{*} , \qquad A_{ii} = A_{ii}^{*} \geq 0$$

and[4]

$$\left| A_{ij}^{R} \right|^{2} \leq A_{ii}^{R} A_{jj}^{R}$$

$$\left| A_{ij}^{L} \right|^{2} \leq A_{ii}^{L} A_{jj}^{L}$$

$$\left| B_{ij} \right|^{2} \leq A_{ii}^{R} A_{jj}^{L} .$$

It is often convenient to introduce the combinations:

$$K_{ij} = \tfrac{1}{2}(A_{ij}^{R} + A_{ij}^{L}) \qquad L_{ij} = \tfrac{1}{2}(-A_{ij}^{R} + A_{ij}^{L})$$

$$I_{ij} = \tfrac{1}{2}(B_{ij}^{*} + B_{ji}^{*}) \qquad J_{ij} = \tfrac{1}{2}(-B_{ij}^{*} + B_{ji}^{*})$$

[The $(\ )^{*}$ is a convention in I and J.]

Thus:

$$K_{ij} = K_{ji}^{*} , \ L_{ij} = L_{ji}^{*} , \ I_{ij} = I_{ji} , \ J_{ij} = - J_{ji} .$$

There has been a lot of confusion about the C, P, and T invariance. One could not empirically determine more than the unitary invariants, though it was customary to use certain "normal forms." The experiments always depend only on these invariant combinations. This settles certain controversies in a simple way. One H that can be obtained from another by such transformations in equivalent as far as the experiment is concerned.

For instance:

A consequence of lepton conservation is

$$D_R = D_L = 0 \qquad \underline{\text{Normal form}}$$

or:

$$B_{ij} = 0 \quad \underline{\text{and}} \quad A_{ij}A_{k\ell} = A_{i\ell}\,A_{kj} \qquad \underline{\text{Invariant condition}}$$

FOOTNOTES

1 ψ_ν^R transforms according to

$$\Psi_\nu'^R = U^R \Psi_\nu^R$$

where

$$\Psi_\nu^R = \begin{pmatrix} \psi_\nu^R \\ (\psi_\nu^c)^R \end{pmatrix} \quad \text{and} \quad U^R = \begin{pmatrix} a & -b \\ b^* & a^* \end{pmatrix},$$

while

$$\Psi_\nu'^L = U^L \Psi_\nu^L$$

where

$$U^L = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}.$$

Clearly, $U^* = U^{-1}$ and $\det U = 1$.

2 In terms of the two component Ψ_ν introduced previously, we may write the neutrino part of $H_{int.}$ as:

$$C_{Ri}\,\Psi_\nu^R + C_{Li}\,\Psi_\nu^L$$

where:

$$C_{Ri} = (C_{Ri} \quad D_{Ri}), \qquad C_{Li} = (C_{Li} \quad D_{Li}).$$

FOOTNOTES

2 (Cont.)

Then

$$\Psi_{\nu}^{\prime R} = U^R \Psi_{\nu}^{R}, \text{ and } H_{\text{int.}}$$

will be invariant if

$$C_{Ri}^{\prime} = \mathcal{C}_{Ri}(U^R)^{-1} = \mathcal{C}_{Ri}(U^R)^{*} .$$

If we write the transformation on the column vectors instead, they will transform as the complex conjugate matrix rather than the adjoint. This is the situation in the text.

3 Since the free particle equations are invariant under the unitary group, the momentum, energy, and $\vec{\sum}\cdot\vec{p}$ carried by the neutrino will not differentiate between a neutrino of ψ_{ν}^{R}, for example, and $\psi_{\nu}^{R'}$. Thus if we keep $H_{\text{int.}}$ invariant, the experiments will only give information concerning the quantities formed from the coupling constants which are invariant under the group. These invariants are easily constructed using the two component notation. If we consider:

$$\mathcal{C}' = \mathcal{C}U^{-1} ,$$

then:

$$\mathcal{C}^{*\prime} = U\mathcal{C}^{*}$$

where:

$$\mathcal{C}^{*} = \begin{pmatrix} C^{*} \\ D^{*} \end{pmatrix}, \text{ and } U^{*} = U^{-1} .$$

Thus:

$$\mathcal{C}'\mathcal{C}^{*\prime} = (\mathcal{C}U^{-1})(U\mathcal{C}^{*}) = \mathcal{C}\mathcal{C}^{*}$$

FOOTNOTES

3 (Cont.)

so that

$$\mathcal{C}_{Ri}\,\mathcal{C}_{Rj}^{*} = \text{invariant} = A_{ij}^{R}$$

$$\mathcal{C}_{Li}\,\mathcal{C}_{Lj}^{*} = \text{invariant} = A_{ij}^{L}$$

We have seen that $\begin{pmatrix} D_L^* \\ C_L^* \end{pmatrix}$ transforms as $\begin{pmatrix} C_R \\ D_R \end{pmatrix}$, however,

so also:

$$C_{Ri}\,D_{Lj} + D_{Ri}\,C_{Lj} = \text{invariant} = B_{ij}.$$

[We note that group (I) also has

$C_{Ri}\,D_{Rj} - C_{Rj}\,D_{Ri} = \text{invariant}$, but this is not invariant under group (II).]

4 These relations follow simply from the two component notation, since for any vector, Ψ (in a positive metric):

$$(\Psi_i, \Psi_i)\,(\Psi_j, \Psi_j) \geq \left|(\Psi_i, \Psi_j)\right|^2 .$$

LECTURE ON "CONTINUOUS GROUPS AND REFLECTIONS IN QUANTUM MECHANICS" BY W. PAULI

Notes by R. J. Riddell, Jr.

Radiation Laboratory
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LECTURE 18

We will now consider the normal forms and invariant conditions for various conservation laws. The normal form can be reached if the corresponding invariant condition is satisfied. We have already seen the conditions for lepton conservation, which may be fulfilled. There is as yet no disagreement with experiment. Now:

Parity conservation (certainly violated):

$$L_{ij} = 0 \quad \text{or} \quad A_{ij}^{L} = A_{ij}^{R}$$

and:

$$I_{ij}^{*} J_{k\ell}^{*} = 0 \quad \text{or} \quad B_{ij} = \pm B_{ji}$$

Invariant cond.

(The + and − are not equivalent.)

Then:

$$C_{Ri} = C_{Li} \ ; \quad D_{Ri} = D_{Li}$$

or:

$$C_i' = D_i' = 0.$$

Normal form.

Charge conjugation invariance (certainly violated):

$$A_{ij}^{L} = A_{ij}^{R*} \quad \text{and} \quad \mathrm{Re}(I_{ij} J_{k\ell}^{*}) = 0$$

$$\mathrm{Im}\, K_{ij} = \mathrm{Re}\, L_{ij} = 0$$

Inv. cond.

Normal form:

$$C_{Li} = C_{Ri}^{*}, \quad D_{Li} = D_{Ri}^{*}; \quad \text{Im}\, C_i = \text{Im}\, D_i = \text{Re}\, C_i' = \text{Re}\, D_i' = 0.$$

Thus, for the invariants:

$$K_{ij} = K_{ji}, \quad L_{ij} = -L_{ji}$$

$$I_{ij} = \left| I_{ij} \right| e^{i\alpha} \qquad J_{k\ell} = \pm i \left| J_{k\ell} \right| e^{i\alpha} \qquad 0 \leq \alpha < \pi.$$

Time reversal invariance (no disagreement with experiment; $T \sim CP$):

A_{ij}^{R}, A_{ij}^{L}, B_{ij} are all real.
$\text{Im}(I_{ij} J_{k\ell}^{*}) = 0$
} Invariant conditions

C_R, C_L, D_R, D_L are all real.
C, C', D, D' are all real
} Normal form.

Majorana abbreviation:

$$\psi^{c}(x) = \psi(x) \qquad \text{Normal form.}$$

[All electromagnetic forces must be zero, but m is not necessarily zero.]

Now, the combination:

$$\psi^{c}(x) + \psi(x)$$

will have an interaction with matter, while

$$\psi^{c}(x) - \psi(x)$$

will not. (Thus the latter cannot be said to exist in this connection.)

This then leads to a two state theory:

$$C_R = D_R , C_L = D_L .$$

The invariants can only be used if $m = 0$.[1] Then we find:

$$A_{ij}^{R} A_{k\ell}^{L} = B_{ik} B_{j\ell}^{*} .$$

The Majorana condition is compatible with parity. There is no disagreement with experiment then, but if leptons are conserved there will be.

If we consider more restricted two component theories, we can be more specific. We consider:

R-model: $A_{ij}^{L} = B_{ij} = 0$ Inv. cond.

$C_L = D_L = 0$ Normal form.

The experimentally observed polarization of electrons indicates that the couplings in this model must be S, T, (P).

L-model: $A_{ij}^{R} = B_{ij} = 0$ Inv. cond.

$C_R = D_R = 0$ Normal form.

In this case, the interactions must be V, A.[2] In both cases the Majorana condition is fulfilled.

From these alone, lepton charge conservation does not follow. It is something new:

$$\left| K_{ij} \right|^{2} = K_{ii} K_{jj} \qquad \text{(No disagreement)}$$

These developments are found in the papers of Pursey and Lüders in Nuovo cimento.

We pass now to the spectrum, to see what information can be obtained. We give the results of the pertinent calculations. The beta spectrum is given as:

$$N_I(E_e)dE_e = \frac{1}{(2\pi)^3}\rho(E_e)\ F(\pm Z, E_e)\ \xi\ (1 \pm b\frac{m_e}{E_e})dE_e ,$$

where the upper sign corresponds to negatons, and the lower to positons. $F(Z,E_e)$ is the coulomb correction, while $(1 \pm b\ m_e/E_e)$ is the Fierz term (1937), and ξ is the relevant matrix element. The statistical factor, ρ, was computed independently of Fermi by F. Perrin, who correctly concluded that $m_\nu \sim 0$. We find a detailed account in the Handbook of Beta and Gamma Spectroscopy, edited by Siegbahn (see expecially the articles of C. S. Wu, and M. E. Rose).

We will give the result for non-relativistic nucleons, no coulomb corrections $\left(F\ (Z,E_e) = 1\right)$, and allowed transitions.

For the Fermi-type of transitions, (S, V) $\Delta I = 0$. For Gamow-Teller (T, A) $\Delta I = 0, \pm 1$, with $0 \to 0$ forbidden. The pseudo-scalar only appears in the relativistic correction to nucleons. (P "escapes.")

Now, one finds:

$$\xi = \left| M_F \right|^2 (K_{SS} + K_{VV}) + \left| M_{G.T.} \right|^2 (K_{TT} + K_{AA})$$

$$b\xi = 2\left\{ \left| M_F \right|^2 \mathrm{Re}K_{SV} + \left| M_{G.T.} \right|^2 \mathrm{Re}\ K_{TA} \right\}$$

where the K's are invariant K_{ij}'s. The most important factor in the spectrum is $\rho(E_e)$.

To compute $\rho(E_e)$, we need:

$$\int d\Omega\, p_e^2\, dp_e\, p_\nu^2\, dp_\nu\, \delta(E_o + m_\nu - E_e - E_\nu)$$

where $d\Omega$ is the element of solid angle between $\vec{p}_e$ and $\vec{p}_\nu$, and $E_o + m_\nu$ will be the change of energy in the nuclear configuration, so that E_o will be the maximum energy available to E_e. Now:

$$E_e dE_e = p_e dp_e, \quad E_\nu dE_\nu = p_\nu dp_\nu;$$

so that

$$\rho \sim p_e E_e p_\nu E_\nu = p_e E_e \sqrt{(E_o + m_\nu - E_e)^2 - m_\nu^2} \times$$

$$\times (E_o + m_\nu - E_e).$$

If $m_\nu = 0$:

$$\rho \sim p_e E_e (E_o - E_e)^2 .$$

Thus we find a quadratic dependence near $E_e = E_o$. If $m_\nu \neq 0$, there will be a region for which

$$E_o - E_e \ll m_\nu$$

and here:

$$\rho \sim p_e E_e\, m_\nu \sqrt{2m_\nu (E_o - E_e)} .$$

If $m_\nu \neq 0$, there is also another factor in the spectrum:

$$(1 - C \frac{m_e m_\nu}{E_e E_\nu}). \qquad -1 \leq C \leq 1$$

(See J. R. Pruitt, Phys. Rev. 73, 1219 (1948).)

Langer - Muffat [Phys. Rev. 88, 689 (1952) find for the neutrino mass:

$$m_\nu < 250 \text{ e.v.}$$

FOOTNOTES

1 Otherwise the free neutrino equation is not invariant under the unitary group.

2 The A^{35} experiment, and electron capture in Eu^{152} seem to indicate that the R-model is ruled out.

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LECTURE 19

We now set $m_\nu = 0$, and the spectrum for β-decay is given by:

$$N_\mp(E_e)dE_e = \frac{1}{(2\pi)^3} \rho(E_e) \xi \left(1 \pm b \frac{m_e}{E_e}\right) d E_e ,$$

where:

$$\rho(E_e) = p_e E_e p_\nu E_\nu = p_e E_e (E_o - E_e)^2 ,$$

$$\xi \equiv |M_F|^2 (K_{SS} + K_{VV}) + |M_{G.T.}|^2 (K_{TT} + K_{AA}),$$

$$b\xi \equiv 2 \left\{ |M_F|^2 \text{ Re } K_{SV} + |M_{G.T.}|^2 \text{ Re } K_{TA} \right\} .$$

The "b" terms are the Fierz terms.[1] These are for allowed transitions. The pseudoscalar, P, escapes detection in this limit since the non-relativistic limit for its nuclear matrix element is zero. The Fermi terms are obtained from S,V and the Gamow-Teller from T,A. The Fierz terms were given in a paper: Zs. f. Phys. 104, 553 (1937). These would influence the shape of the spectrum for small energies. The experiments indicate that such terms are either zero or very small. Thus $\text{Re } K_{SV} = \text{Re } K_{TA} = 0$. If there is T (or PC) invariance, the Re can be omitted, so that $K_{SV} = K_{TA} = 0$.

We must now consider other β-experiments. We will only give the results of perturbation calculations in terms of the invariants. This is not only agreeable, but practically useful.

Polarization (of electron)

We define:

$$P \equiv \frac{I_1 - I_2}{I_1 + I_2}$$

where 1 and 2 correspond to spins aligned parallel and anti-parallel to Z, the direction of motion. One finds[2] for β^I:

$$P = \pm \frac{p_e}{E_e} \frac{G}{(1 \pm b \frac{m_e}{E_e})} ,$$

where:

$$G \xi \equiv \left| M_F \right|^2 (L_{SS} - L_{VV}) + \left| M_{G.T.} \right|^2 (L_{TT} - L_{AA}).$$

Experimentally one finds $G = -1.0 \pm 0.1$ both for Fermi and G.T. : although the error is large, $G = -1$ is not in disagreement with the experiments. Then for G.T. :

$$- L_{TT} + L_{AA} = K_{TT} + K_{AA} ,$$

or

$$A_{TT}^L = - A_{AA}^R = 0 ,$$

since $A_{ii} \geq 0$. Thus we find:

$$L_{TT} = - K_{TT} , \quad L_{AA} = K_{AA} .$$

In the same way for Fermi:

$$-L_{SS} + L_{VV} = K_{SS} + K_{VV} \;,$$

$$A_{SS}^{L} = -A_{VV}^{R} = 0 .$$

and:

$$L_{SS} = -K_{SS} \;, \quad L_{VV} = K_{VV} \;.$$

According to these formulas, both P and C are violated, since from Lecture 18, P requires $L_{ij} = 0$, and C requires $L_{ij} = -L_{ji}$ so that $L_{ii} = 0$.

The Coulomb corrections require a more detailed discussion, but one still finds that C,P are both violated.

Much more can be said. We have already obtained inequalities of the Schwarz type:

$$\left| A_{ij} \right|^2 \leq A_{ii} A_{jj} \qquad \text{(for either R or L).}$$

There follows:

$$A_{Si}^{L} = A_{Ti}^{L} = A_{Ai}^{R} = A_{Vi}^{R} = 0 \quad \text{for all } i.$$

$$L_{Si} = -K_{Si} \;, \; L_{Ti} = -K_{Ti} \;; \; L_{Ai} = K_{Ai} \;, \; L_{Vi} = K_{Vi} \;.$$

Thus:

$$K_{TA} = -L_{TA} \;, \quad K_{AT} = L_{AT} \;.$$

Therefore, since $K_{ij} = K_{ji}^{*}$, $L_{ij} = L_{ji}^{*}$;

$$K_{TA} = L_{TA} = 0.$$

We also find for the pairs TV, SV, SA, TA :

$$K_{ij} = L_{ij} = 0 \;.$$

Finally, the only terms which can be different from zero are:

$$L_{AV} = K_{AV} \;; \quad L_{ST} = -K_{ST} \;.$$

We have not yet used lepton conservation or T invariance, but only $G = -1$. Thus the system of possible couplings decomposes into 2 classes:

$$(A, V)$$

or

$$(S,\ T,\ (P)\ )\ .$$

From the allowed spectra, we cannot distinguish them.

The choice $G = -1$ would imply a two component theory for the neutrino.

This development, however, will not permit one to conclude that the two-component model <u>must</u> be true. We can make two choices:

R-model: all $A_{ij}^{L} = 0$, $K_{ij} = - L_{ij}$.

This model is compatible with the preceding evidence for S,T.

L-model: all $A_{ij}^{R} = 0$, $K_{ij} = + L_{ij}$.

This is compatible with A,V. On the other hand, the polarization experiments say more than one obtains from an arbitrary 2-component theory. Not only are $K_{TA} = L_{TA}$,...., but they are also zero. This gives a different aspect of the situation.

We can now conclude for the B's that

$$B_{ij} = 0 ,$$

since

$$\left| B_{ij} \right|^2 \leq A_{ii}^{R} A_{jj}^{L} .$$

There are still other experiments to be considered. These include the $\beta^{\mp}$ asymmetry in ejection from nuclei with aligned spins. If θ is the angle between $\vec{p}_e$ and the nuclear spin $\vec{I}$, the distribution of electrons is given by[3]

$$1 + \alpha \cos \theta$$

where:

$$\alpha = \frac{\langle I_z \rangle}{I} \frac{p_e}{E_e} \frac{A}{(1 \pm b \frac{m_e}{E_e})}$$

and:

$$A\xi = \pm \lambda_{II'} \left| M_{G.T.} \right|^2 (L_{TT} - L_{AA})$$

$$+ \delta_{II'} \left(\frac{I}{I+1}\right)^{\frac{1}{2}} \left| M_F \right| \left| M_{G.T.} \right| \cdot 2 \mathrm{Re} (L_{ST} - L_{VA})$$

where I, I' are the initial and final nuclear spin states. $\langle I_z \rangle / I$ is the polarization of the nucleus, and p_e/E_e is of order v/c.

For pure G.T. we get the same evidence as before:

$$A = -1.$$

The quantity λ is given by:

$$\lambda_{II'} = \begin{cases} 1 & I' = I - 1 \\ \frac{1}{I+1} & I' = I \\ -\frac{I}{I+1} & I' = I + 1 \end{cases}$$

FOOTNOTES

1 It might be of interest to indicate how they arise. In computing the spectrum, one needs the square of the matrix element of $H_{int.}$ between the initial and final states, since the transition probability is

$$\omega = \frac{2\pi}{\hbar} \left| (i \left| H_{int.} \right| f) \right|^2 \rho(E)$$

where $\rho(E)$ = density of final states. We must average over initial states and sum over final states which are not distinguished in β-decay spectra.

FOOTNOTES

1 (Cont.)

Thus we sum over all neutrino states and over positive energy electron states. Now, in the non-relativistic limit for the nuclear matrix element, only

$$M_F = (\bar{\psi}_P, 1\,\psi_N) = (\bar{\psi}_P, \gamma_4 \psi_N)$$

$$(\bar{\psi}_P, T_{ij} \psi_N) = -\varepsilon'_{ijk}(\bar{\psi}_P, \Sigma_k \psi_N)$$

$$= (\bar{\psi}_P, i\gamma_5\gamma_k \psi_N) = (\bar{\psi}_P, \gamma_4 \Sigma_k \psi_N)$$

are different from zero. In terms of the two large components, we thus have:

$$M_F = (\psi_P^*, \psi_N)$$

$$\left| M_{G.T.} \right|^2 = \left| (\psi_P^*, \vec{\sigma}\,\psi_N) \right|^2 .$$

Terms involving M_F cannot interfere with ones involving $M_{G.T.}$, since the nuclear states must be different in the two cases. Let us now consider the Fermi transitions in a simple case for which $C_R = C_L$ and $D_R = D_L = 0$. Then on appropriate averaging and summing we get:

$$\left| (f \left| H_{int.} \right| i) \right|^2 = \left| M_F \right|^2 \times \sum \Big\{ \bar{\psi}_e \, (C_S \psi_\nu + C_V \gamma_4 \psi_\nu) \times$$

$$(\psi_\nu^* C_S^* + C_V^* \; \psi_\nu^* \gamma_4) P_+ (\bar{\psi}_e)^* \Big\}$$

Now $(\bar{\psi}_e)^* = \gamma_4 \psi_e$, and the projection operator for positive energies is

$$P_+ = \frac{\vec{\alpha}\cdot\vec{k} + \beta m + \omega}{2\omega}$$ (see Lecture 5, p. 6),

and on taking the sums, we obtain:

FOOTNOTES

1 (Cont.)
$$(C_S C_S^* + C_V C_V^*)\ \mathrm{Tr}\left\{\gamma_4 \left(\frac{\vec{\alpha}\cdot\vec{k} + \beta m + \omega}{2\omega}\right)\gamma_4\right\}$$

$$+ (C_S C_V^* + C_V C_S^*)\ \mathrm{Tr}\left\{\frac{\vec{\alpha}\cdot\vec{k} + \beta m + \omega}{2\omega}\ \gamma_4\right\}$$

$$= 2\left[(K_{SS} + K_{VV}) + 2\ \mathrm{Re}\ K_{SV}\left(\frac{m_e}{\omega_e}\right)\right] .$$

Thus we obtain the form indicated. The treatment is similar in the more general case. For positrons, the projection operator would change and the opposite sign would occur in the second term.

2 In this case, it is necessary to discriminate between the neutrino spin states. For simplicity, let us consider a Fermi transition, with only R-type neutrinos. These will introduce a projection operator $(1 - \gamma_5)/2$ (see Lec. 17). In addition, we must introduce σ_z, since P is given in terms of the expectation value of σ_z. Then we must obtain:

$$\mathrm{Tr}\left\{\gamma_4 (C_S^R + C_V^R \gamma_4)\left(\frac{1-\gamma_5}{2}\right)(C_S^{R*} + C_V^{R*}\gamma_4)\ (-i\gamma_1\gamma_2)\gamma_4\left(\frac{\vec{\alpha}\cdot p + \beta m - \omega}{2\omega}\right)\right.$$

since $(-i\gamma_1\gamma_2) = \Sigma_z$. On evaluating the trace, we obtain

$$-(C_S^R C_S^{R*} - C_V^R C_V^{R*})\frac{p_e}{E_e} = \frac{p_e}{E_e}(L_{SS} - L_{VV}),$$

which is the indicated form. The other terms are treated similarly.

3 This result is obtained similarly to the preceding. The principal difference is that the nuclear elements must be considered for their I dependence. On carrying out the Tr as previously for summing over electron and neutrino states, we are left with a form for a pure G.T. transition like:

FOOTNOTES

3 (Cont.)

$$\left|(f\left|H_{int.}\right|i)\right|^2 \sim (f\left|\psi_P^{*}\sigma_k\psi_N\right|i)\,(f\left|\psi_P^{*}\sigma_\ell\psi_N\right|i)^{*}\times$$

$$\times\left(\delta_{k\ell} - \frac{i(p_e)_m}{\omega_e}\,\varepsilon_{k\ell m}\right).$$

Using the rotational invariance of the system, the result can only depend on cos Θ, so that only $(p_e)_z$ may enter. The sum over k,ℓ may be readily carried out using the general expressions for the matrix elements of a vector to give the desired result.

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Lecture 20

R. J. Riddell, Jr.

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LECTURE 20

The first experiments in Co^{60}, which is a pure G.T., give the asymmetry. In Co^{58}, which is mixed, one anticipated seeing the effect of the second term, but it was not found, presumably because $|M_F| << |M_{G.T.}|$. In the $\beta - \gamma$ angular correlation experiments, however, such a term was observed. In this case, if θ is angle between the momentum of the photon, $\vec{p}_k$, and the electron, $\vec{p}_e$, and

$$\tau = \begin{cases} +1 & \text{for right circular polarized } \gamma\text{'s} \\ -1 & \text{for left circular polarized } \gamma\text{'s,} \end{cases}$$

then for electric dipole radiation and allowed β-decay, we obtain an angular distribution of the form

$$1 + \gamma \cos \theta$$

where:

$$\gamma = -\tau \frac{p_e}{E_e} \frac{C}{(1 \pm b \frac{m_e}{E_e})}$$

and:

$$C\xi = \frac{1}{3} \left(\frac{I' + 1}{I'}\right)^{\frac{1}{2}} \left\{ \pm \frac{I' + 1}{\sqrt{I(I+2)}} \lambda_{I'I} |M_{G.T.}|^2 (L_{TT} - L_{AA}) \right.$$

$$- \delta_{I'I} |M_F| \cdot |M_{G.T.}| \cdot 2 \operatorname{Re} (L_{ST} - L_{VA}) .$$

The second term has been observed in Sc^{46} at Pasadena. If one assumes the two component theory, then this relation can be used to get the ratio $|M_F| / |M_{G.T.}|$.

From the polarization experiments, we deduced that

$$\mathrm{Re}\,(L_{ST} - L_{VA}) = -\,\mathrm{Re}\,(K_{ST} + K_{VA})$$

We can get a further relation if we invoke:

(1) Time reversal: This leads to K_{ij} = real.

and

(2) Lepton conservation: We then obtain[1]

$$|K_{AV}| = \sqrt{K_{AA} \cdot K_{VV}}$$

$$|K_{ST}| = \sqrt{K_{SS} \cdot K_{TT}}\ .$$

Although all of the concepts developed up to now are consistent with the experiments, this does not constitute a proof of their validity.

There are also some strange, heuristic principles which have been introduced, and although they are not understood, they seem to work. These can be traced back to a paper of B. Steck and J. H. D. Jensen: Zs. f. Phys. 141, 175 (1955). This paper was written before the parity violation was known, and was based on the lack of Fierz terms in β decay.

They considered the transformations:

$$\psi_e' = \gamma_5 \psi_e, \quad \psi_\nu' = \gamma_5 \psi_\nu$$

and introduced the principle that $H_{int.}$ should be invariant. This leads to a vanishing of the Fierz terms, since one can then either have [S,T,(P)] or (V,A) in $H_{int.}$, but not both. [An overall sign change on transformation is irrelevant.] This principle is not easy to understand, since it is not a principle of nature. The mass m_e is an obstacle since the free particle equation is not invariant under the γ_5 transformation. This paper anticipated the μ-decay, since it was predicted that $\rho = 3/4$ or 0 for this theory, even with parity conservation. The Yang-Lee neutrino was given by invariance under

$$\psi_\nu' = \gamma_5 \psi_\nu$$

alone. This gives the parity violation. When combined with Steck-Jensen, this gives $\psi_e' = \pm \gamma_5 \psi_e$.

These principles were considered further by Salam, Sakurai, and Feynman and Gell-Mann.

If we assume invariance under $\psi' \rightarrow \gamma_5 \psi$, we find that to obtain non-zero results:

$$\overline{\left(\frac{1+\gamma_5}{2}\right)\psi_e} \quad [S,T,P] \quad \left(\frac{1-\gamma_5}{2}\right)\psi_\nu$$

$$= \overline{\left(\frac{1+\gamma_5}{2}\right)\psi_e} \quad [V,A] \quad \left(\frac{1+\gamma_5}{2}\right)\psi_\nu.$$

The first choice is the R-model, and the second, the L-model. Recent experiments seem to pick V,A rather than S,T,P.

Thus, we seem to have:

$$H_{int.} = G(\bar{\psi}_2 \gamma_\mu (1 \pm \gamma_5)\psi_1)(\bar{\psi}_4 \gamma_\mu (1 \pm \gamma_5)\psi_3) + h.c.$$

The sign is difficult. If electromagnetic effects are not considered, the choice is only a convention. The connection between R,L, and the electric charge is the question.

That the coupling constants are equal does not follow in this formulation. (See Sakurai.) This is not in a universal form. Pauli feels that we don't understand the transformation because we don't understand the source of the weak interactions.

For further work, see:

Feynman and Gell-Mann: Phys. Rev. 109, 193 (1958).

Sakurai: Nuovo cimento 7, 649 (1958).

Another way of looking at the situation (perhaps equivalent to the first) begins with the possibility of eliminating 2 components in the Dirac equation by going to a second order equation (see H. A. Kramers: Quantum Mechanics - German ed. 1938, p. 280, English ed. (ter Haar) 1957, p. 272.)

We find that the Dirac equation can be written:

$$(\gamma_\mu \frac{\partial}{\partial x_\mu}) \psi_e^L + m\psi_e^R = \omega_e^R$$

$$(\gamma_\mu \frac{\partial}{\partial x_\mu}) \psi_e^R + m\psi_e^L = \omega_e^L \quad .$$

Then, we find, on elimination:[2]

$$(\Box - m^2) \psi_e^L = m\,\omega_e^L + (\gamma_\mu \frac{\partial}{\partial x_\mu}) \omega_e^R$$

$$(\Box - m^2) \psi_e^R = m\,\omega_e^R + (\gamma_\mu \frac{\partial}{\partial x_\mu}) \omega_e^L \quad .$$

Thus, if we assume that the derivative terms vanish, the equations split, and if $\omega_e^R = 0$, only ψ_e^L will be coupled:

$$(\Box - m^2) \psi_e^L = m\,\omega_e^L \quad .$$

There is a remark by Heisenberg which is of interest here. He considers what a theory might be like if it were truly γ_5 invariant. Then for all fields satisfying

$$(\gamma_\mu \frac{\partial \psi}{\partial x_\mu} + m\,\psi) = 0$$

we would also have fields with

$$(\gamma_\mu \frac{\partial \psi}{\partial x_\mu} - m\,\psi) = 0 \quad .$$

The real world, however, would only incorporate half of these.

FOOTNOTES

1 These follow from the invariant condition

$$A_{ij}\, A_{k\ell} = A_{i\ell}\, A_{kj} \quad ,$$

and

$$A_{Si}^{\ L} = A_{Ti}^{\ L} = A_{Vi}^{\ R} = A_{Ai}^{\ R} = 0 \quad .$$

2 These are readily obtained. From the Dirac equation,

$$(\gamma_\mu \frac{\partial}{\partial x_\mu})\, \psi_e + m\psi_e = \omega_e \quad ,$$

we get the first equations by multiplying by $(1 \pm \gamma_5)/2$, where

$$\omega_e^{\ R} = \frac{1 - \gamma_5}{2}\, \omega_e \quad , \qquad \omega_e^{\ L} = \frac{1 + \gamma_5}{2}\, \omega_e \quad .$$

If the first equation is multiplied by $\gamma_\mu \frac{\partial}{\partial x_\mu}$, and the second is used to eliminate the terms in $\psi_e^{\ R}$, we get:

$$(\gamma_\mu \frac{\partial}{\partial x_\mu})\, (\gamma_\mu \frac{\partial}{\partial x_\mu})\, \psi_e^{\ L} - m^2 \psi_e^{\ L} = \gamma_\mu \frac{\partial}{\partial x_\mu}\, \omega_e^{\ R} + m\,\omega_e^{\ L}$$

which gives the indicated result, since $(\gamma_\mu \frac{\partial}{\partial x_\mu})\, (\gamma_\mu \frac{\partial}{\partial x_\mu}) = \Box$.

UCRL-8213
Lecture 21

UNIVERSITY OF CALIFORNIA
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LECTURE 21

μ-meson decay:

For μ meson decay, we have several choices:

A) $\mu_- \to e_- + \bar{\nu} + \nu$ — μ_-, e_- have equal leptonic charge.

B) $\mu_- \to e_- + \bar{\nu} + \bar{\nu}$ — μ_-, e_- have opposite lepton charges. e_-, ν are the same - (from β decay).

C) $\mu_- \to e_- + \nu + \nu$ — No leptonic charge conservation.

The various possible interactions are:

A: $$H_{int.} = \sum_i (\bar{\psi}_e \theta_i \psi_\mu) \left[C_i (\bar{\psi}_\nu \theta_i \psi_\nu) + C_i' (\bar{\psi}_\nu \theta_i \gamma_5 \psi_\nu) \right] + h.c.$$

B: $$H_{int.} = \sum_i (\bar{\psi}_e \theta_i \psi_\mu) \left[C_i (\bar{\psi}_\nu^c \theta_i \psi_\nu) + C_i' (\bar{\psi}_\nu^c \theta_i \gamma_5 \psi_\nu) \right] + h.c.$$

C: $$H_{int.} = \sum_i (\bar{\psi}_e \theta_i \psi_\mu) \left[C_i (\bar{\psi}_\nu \theta_i \psi_\nu^c) + C_i' (\bar{\psi}_\nu \theta_i \gamma_5 \psi_\nu^c) \right] + h.c.$$

It is convenient to introduce the abbreviations:

$$k_{ij} = C_i C_j^* + C_i' C_j'^* \quad ; \quad k_{ij} = k_{ji}^*$$

$$\ell_{ij} = C_i C_j'^* + C_i' C_j^* \quad ; \quad \ell_{ij} = \ell_{ji}^*$$

These are not invariants here.

The computations of: Kinoshita and Sirlin: Phys. Rev. 107, 533 (1957)
Bouchiat and Michel: Phys. Rev. 106, 171 (1957)

give the spectrum for both cases A and B. One finds:

$$N(E_e)\, dE_e = \frac{1}{3(2\pi)^3}\, m_\mu E_e p_e \lambda \phi(E_e)\, dE_e \quad ,$$

where:

$$\phi(E_e) = 3(E_o - E_e) + 2\rho\left(\frac{4}{3} E_e - E_o - \frac{1}{3}\frac{m_e^{\,2}}{E_e}\right)$$

$$+ 3\eta \frac{m_e}{E_e}(E_o - E_e) \quad ,$$

$$E_o = \frac{m_\mu^{\,2} + m_e^{\,2}}{2m_\mu} \quad ,$$

and

$$\lambda = (k_{SS} + k_{pp}) + 4(k_{VV} + k_{AA}) + 6k_{TT}$$

$$\rho\lambda = 3(k_{VV} + k_{AA} + 2k_{TT})$$

$$\eta\lambda = (k_{SS} - k_{pp}) - 2(k_{VV} - k_{AA}) \quad .$$

ρ is the famous Michel parameter. If $\rho = 0$, $N(E_e)$ disappears at the limit of the spectrum $E_e = E_o$.

The 2-component models (Case A):

Both models give pure V,A interaction.

R: $C_i' = -C_i$ or $C_i^R \equiv C_i - C_i'$

L: $C_i' = C_i$ or $C_i^L \equiv C_i + C_i'$

Then:

$$H_{int.}^{(R)} = \sum_i C_i^R (\bar{\psi}_e O_i \psi_\mu) \left[\overline{(a^R \psi_\nu)}\, O_i (a^R \psi_\nu)\right]$$

$$H_{int.}^{(L)} = \sum_i C_i^L (\psi_e O_i \psi_\mu) \left[\overline{(a^L \psi_\nu)}\, O_i\, (a^L \psi_\nu)\right] \quad ,$$

where a^R, a^L are R,L projection operators. They satisfy:

$$\overline{a^R\psi_\nu} = \bar{\psi}_\nu a^L \quad , \quad \overline{a^L\psi_\nu} = \bar{\psi}_\nu a^R \quad .$$

We also find that

$$a^R\mathcal{O}_{S,T,P} = \mathcal{O}_{S,T,P}\, a^R \quad , \qquad a^R\mathcal{O}_{V,A} = \mathcal{O}_{V,A}\, a^L \quad ;$$

and similarly for $a^L\mathcal{O}$. Thus, in <u>case A:</u>

$$S, T, P \equiv 0,$$

and only V, A are not $\equiv$ 0. For the R-model: $\ell_{ij} = -k_{ij}$, while for the L-model: $\ell_{ij} = k_{ij}$. Thus:

$$H_{int.} = \sum_\rho (\bar{\psi}_e \gamma_\rho (C_V{}^k \pm C_A{}^k \gamma_5)\psi_\mu) \left\{ \bar{\psi}_\nu{}^k \gamma_\rho \psi_\nu{}^k \right\} + h.c.$$

where K = R or L. From the Steck - Jensen transformation, if we replace $\psi_e \rightarrow \pm \gamma_5 \psi_e$ and require invariance, we find:

$$C_V{}^R = \pm C_A{}^R \quad \text{and} \quad C_V{}^L = \mp C_A{}^L \quad .$$

Further, for pure V,A the Michel formula gives

$$\rho = 3/4 \ .$$

The experiments give a value slightly less even if radiative corrections are included, but the disagreement is within the errors.

<u>Case B.</u> Here, there is definite disagreement with the 2-component model.

Independently of the model, only S,A,P are possible, since $\left\{(\psi_\nu)_\alpha , (\psi_\nu)_\beta\right\} = 0$. This requires that $C\mathcal{O}_i$ be skew-symmetric.[1] Symmetric terms will vanish. This eliminates V,T.

If we also require the R or L model, only S,P are left,[2] and for them

$$C_i{}' = \pm C_i \ ,$$

where $i,j = S,P$. Then:

$$\ell_{ij} = - k_{ij} \quad : \quad R$$

$$\ell_{ij} = k_{ij} \quad : \quad L$$

The Steck - Jensen requirement gives further:

$$C_S{}^R = \pm C_P{}^R \ , \quad C_S{}^L = \mp C_P{}^L \ .$$

The two component model, with only S,P, gives $= 0$, and so is in disagreement with the experiments. If we do not require the model we can say little.

The value, $\rho = 3/4$, was predicted by Steck - Jensen, and is independent of the 2-component model.

The asymmetry experiments for μ decay are also in good agreement with the model. If we let θ = angle between the spin of the μ and momentum of the ejected electron, the distribution of electrons is given by:

$$1 \pm a \cos \theta$$

where:

$$a \, \phi \, (E_e) = \zeta \xi \frac{P_e}{E_e} \left\{ E_o - E_e + 2\delta \left(\frac{4}{3} E_e - E_o - \frac{1}{3} \frac{m_e^2}{E_e} \right) \right\}$$

$$\xi \lambda = 6 \text{ Re } \ell_{SP} - 8 \text{ Re } \ell_{VA} - 14 \ell_{TT}$$

$$\delta \xi \lambda = - 6 \ (\text{Re } \ell_{VA} + \ell_{TT}).$$

ζ gives the polarization of the μ meson, $0 < \zeta < 1$.

In Case A for either the R or the L model:

$$\delta = 3/4 \ , \ \eta = -\tfrac{1}{2} \frac{k_{VV} - k_{AA}}{k_{VV} + k_{AA}} \ ,$$

$$\xi = -2 \frac{\text{Re}\ \ell_{VA}}{k_{VV} + k_{AA}} = \pm 2 \frac{\text{Re}\ k_{VA}}{k_{VV} + k_{AA}} \ .$$

The Steck - Jensen condition in addition gives:

$$\eta = 0 \ , \ \xi = \pm 1 .$$

In the actual experiments, Θ is not measured, but rather Θ', the angle between the momentum of the μ, $\vec{p}_\mu$, and $\vec{p}_e$ is measured. The distribution is:

$$1 \pm a' (E_o) \cos \Theta' \ .$$

For μ^+:

$$a' \sim -1/3 \ .$$

The spin of the μ is not directly measured, so $a = \pm a'$ since on the basis of rotational invariance alone, the μ from π decay can be polarized either in the direction of or oppositely to the direction of μ. If we believe the 2-component theory, then $|C_A| = |C_V|$ is not in disagreement with the experiment.

FOOTNOTES

1 This arises from:

$$(\bar{\psi}_\nu^c \ \sigma_i \ \psi_\nu) = ((C\ \psi_\nu)\ \sigma_i\ \psi_\nu)$$

$$= -(\psi_\nu\ C\ \sigma_i\ \psi_\nu)$$

$$= -((\psi_\nu)_\alpha (\psi_\nu)_\beta)\ (C\ \sigma_i)_{\alpha\beta}$$

$$= -\tfrac{1}{2}((\psi_\nu)_\alpha (\psi_\nu)_\beta) \left[(C\sigma_i)_{\alpha\beta} - (C\sigma_i)_{\beta\alpha} \right]$$

2 For example, with the R model, we have:

FOOTNOTES

2 (Cont.)

$$\left(\overline{(a^R \psi_\nu)}^{\,c} \; \theta_i a^R \psi_\nu \right) = \left((C\, a^R \psi_\nu)\; \theta_i \; a^R \psi_\nu \right)$$

$$= (\psi_\nu \, C \, a^R \, \theta_i a^R \psi_\nu)$$

since Ca^R is skew symmetric, because both C and $C\gamma_5$ are. Thus, for θ_i to contribute, $a^R \theta_i = \theta_i a^R$. This is true only for S,T,P.

A similar discussion holds for L.

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Lecture 22

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LECTURE 22

The Majorana form for the neutrino.

In the Majorana theory:

$$\psi_\nu = \psi_\nu^c .$$

For the μ-decay, this requires that[1]

$$C_V = C_A' = C_T = C_T' = 0 .$$

For such a system, the canonical commutation relations are modified. We have:

$$\left\{\psi_\nu(\vec{x}),\ \psi_\beta^*(\vec{x}')\right\} = \delta_{\alpha\beta}\,\delta^{(3)}(\vec{x}-\vec{x}')$$

and using the projection operators

$$a^R \equiv \frac{1-\gamma_5}{2} , \qquad a^L \equiv \frac{1+\gamma_5}{2}$$

we obtain:

$$\left\{\psi_\alpha^R(\vec{x}),\ \psi_\beta^{*R}(\vec{x}')\right\} = a_{\alpha\beta}^R\ \delta^{(3)}(\vec{x}-\vec{x}')$$

$$\left\{\psi_\alpha^L(\vec{x}),\ \psi_\beta^{*L}(\vec{x}')\right\} = a_{\alpha\beta}^L\ \delta^{(3)}(\vec{x}-\vec{x}')$$

$$\left\{\psi_\alpha^L(\vec{x}),\ \psi_\beta^{*R}(\vec{x}')\right\} = 0.$$

However,

$$\left\{\psi_\alpha(\vec{x}),\ \psi_\beta(\vec{x}')\right\} = E_{\alpha\beta}\,\delta^{(3)}(\vec{x}-\vec{x}') , \qquad \text{(not zero)}$$

since:

$$\psi^c = E\psi^* = \psi \ ,$$

where:

$$E = -\gamma_4 C^{-1} \ , \quad E = \tilde{E} \ , \quad E\,E^* = 1 \ .$$

Also:[2]

$$\psi^R = E\,(\psi^L)^* \ , \quad \psi^L = E\,(\psi^R)^* ,$$

and so we obtain:

$$\left\{\psi_\alpha^R(\vec{x}) \ , \ \psi_\beta^L(\vec{x}')\right\} = (a^R E)_{\alpha\beta}\, \delta^{(3)}(\vec{x}-\vec{x}')$$

$$\left\{\psi_\alpha^L(\vec{x}), \ \psi_\beta^R(\vec{x}')\right\} = (a^L E)_{\alpha\beta}\, \delta^{(3)}(\vec{x}-\vec{x}') \ .$$

In the Majorana theory one cannot obtain an ordinary vector, but only a pseudo-vector:

$$j_\mu = i\,\bar{\psi}\,\gamma_5\,\gamma_\mu\psi \, .$$

Since

$$\gamma_5 a^R = -a^R \ , \quad \gamma_5 a^L = a^L \ ,$$

we find for the R-model:

$$j_\mu = i\,\bar{\psi}\,\gamma_\mu\,\psi$$

while for the L-model

$$j_\mu = -i\,\bar{\psi}\,\gamma_\mu\psi \, .$$

In particular[3]:

$$j_0 = \frac{1}{i}\,j_4 = \psi^{R*}\psi^R - \psi^{L*}\psi^L$$

Now:

$$\psi^{R*}\psi^R = \psi^L\psi^{L*}$$

and

$$\psi^R\psi^{*R} = \psi^{L*}\psi^L$$

so that we see that j_0 is already written as a commutator, as it should be. Thus:

$$j_0 = \left[\psi^{R*}, \psi^{R}\right] = -\left[\psi^{L*}, \psi^{L}\right] .$$

If we write

$$\text{Lepton charge} = \mp Q_\nu = \tfrac{1}{2}\int j_0 dV ,$$

we will have the same for electrons:

$$Q_e^{(\text{lept.})} = \tfrac{1}{2}\int\left[\psi_e^{*}, \psi_e\right] dV$$

[The sign is not definite. If we choose the minus sign, R carries a minus charge, while L carries a plus charge. The situation is reversed for the plus sign.]

The Majorana form plus lepton conservation is entirely equivalent to the two-component theory. There is still the right-left freedom. This situation was stated by Touschek: Nuovo cimento 5, 1281 (1957).

In β decay, we set:

$$Q^{\text{lept.}} = Q_e^{\text{lept.}} \mp Q_\nu .$$

We can get a Majorana theory with no lepton conservation by mixing the R and L.

We might now ask: What is the gauge group associated with lepton charge conservation?

If Q_ν has minus sign: $\psi_e' = e^{i\alpha}\psi_e$; $\psi_\nu' = e^{i\alpha\gamma_5}\psi_\nu$

If Q has plus sign: $\psi_e' = e^{i\alpha}\psi_e$; $\psi_\nu' = e^{-i\alpha\gamma_5}\psi_\nu$.

These are then the gauge transformations for lepton charge. The γ_5 is needed in the ψ_ν equation, since R and L must be separated:

$$\psi_\nu^R \to e^{-i\alpha}\psi_\nu^R \qquad \psi_\nu^L \to e^{i\alpha}\psi_\nu^L \quad .$$

In the μ decay:

A. $\mu_- \to e_- + \bar{\nu} + \nu$ (Corresponds to experiments)

B. $\mu_- \to e_- + \bar{\nu} + \bar{\nu}$

The gauge groups are given by:

A. $\psi_e' = e^{i\alpha}\psi_e$; $\psi_\mu' = e^{i\alpha}\psi_\mu$; $\psi_\nu' = e^{\pm i\alpha\gamma_5}\psi_\nu$.

B. $\psi_e' = e^{i\alpha}\psi_e$; $\psi_\mu' = e^{-i\alpha}\psi_\mu$; $\psi_\nu' = e^{\pm i\alpha\gamma_5}\psi_\nu$.

The transformations are always given by:

$$\psi' = e^{-i\alpha Q}\,\psi\, e^{i\alpha Q}$$

and if α is infinitesmal:

$$[Q, \psi_e] = -\psi_e \quad ; \qquad [Q, \psi_e^*] = +\psi_e^*$$

$$[Q, \psi_\nu^R] = \pm\psi_\nu^R \quad ; \qquad [Q, \psi_\nu^L] = \mp\psi_\nu^L \; .$$

The preceding method of writing the current is instructive, since j_ν is a pseudovector while j_e is a vector. This lepton conservation must bring a parity violation, since the sum of an ordinary and a pseudo quantity is conserved. This is the case in β decay. In the μ meson case the situation is different since we have two neutrinos and there is conservation for μ plus e, and for the ν's separately. Thus we can't be sure that P is violated here.

Finally, we might mention the π decay. If we believe in a two-component theory <u>and</u> lepton conservation, then in the decay

$$\pi \to \mu + \nu ,$$

the μ and ν have opposite lepton charges, and so for π^- decay, the neutrino must be an antiparticle, since the muon is a particle. Experiments of the asymmetry and polarization of the electrons in μ decay taken together indicate that the μ^- has right handed polarization. We conclude that the $\bar{\nu}$ is also right handed in the decay:

$$\pi^- \rightarrow \mu^- + \bar{\nu} \ .$$

Since the ν in β decay seems to be left handed there is no disagreement with assuming that the neutrinos in π decay are the same as in β decay.

FOOTNOTES

1 This follows from

$$\bar{\psi}_\nu = C\,\psi_\nu^{\;c} = C\,\psi_\nu$$

so that:

$$(\bar{\psi}_\nu\, \Theta_i\, \psi_\nu) = -\,(\psi_\nu\, C\Theta_i\, \psi_\nu)$$

and so $C\Theta_i$ must be skew symmetric, thus eliminating V, T; or $\gamma_5 A$, $\gamma_5 T$. Since $\psi_\nu = \psi_\nu^{\;c}$, all three cases are alike.

2 From Lecture 16: $(\psi^R)^c = (\psi^c)^L$.

Thus:

$$E(\psi^R)^* = \psi^L \ .$$

Finally:

$$\left\{\psi_\alpha^{\;R}(\vec{x}),\ \psi_\beta^{\;L}(\vec{x}')\right\} = \left\{\psi_\alpha^{\;R}(\vec{x}),\quad E_{\beta\gamma}\ (\psi^R)_\gamma^*\right\}$$

$$= E_{\beta\gamma}\ a_{\alpha\gamma}^{\;\;R}\ \delta^{(3)}\ (\vec{x} - \vec{x}')$$

$$= (a^R E)_{\alpha\beta}\ \ \delta^{(3)}\ (\vec{x} - \vec{x}'),$$

since $E = \widetilde{E}$. The other case is similar.

FOOTNOTES

3

$$j_0 = - \overline{\left[(a^R + a^L)\psi\right]} \quad \gamma_5\gamma_4 \, (a^R + a^L)\psi$$

$$= - \psi^* \, (a^R + a^L) \quad \gamma_4\gamma_5\gamma_4 \, (a^R + a^L)\psi$$

$$= - \psi^* \, (a^R + a^L) \quad \gamma_5 \, (a^R + a^L)\psi$$

$$= \psi^{*R}\psi^R - \psi^{*L}\psi^L \quad .$$

Now:

$$(\psi^*)_\alpha^{\ R} \psi_\alpha^{\ R} = (E^{-1})_{\alpha\beta} \, \psi_\beta^{\ L} \, E_{\alpha\beta} \, (\psi^L)_\gamma^* \quad = \quad \psi_\alpha^{\ L} \, (\psi^L)_\alpha^* \quad .$$

UCRL-8213
Lecture 23

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Some further remarks on weak interactions.

The π decay has been discussed extensively. In principle, we might have:

(1) $\pi \rightarrow e + \nu$

(2) $\pi \rightarrow \mu + \nu$

The former has not been observed, and one has a limit for the rates:

$$R_1/R_2 < 10^{-5} \quad .$$

This is not easy to understand. One can introduce intermediate states for the decays:

$$\left.\begin{array}{l} \pi^- \rightarrow n + \bar{p} \\ \pi^+ \rightarrow p + \bar{n} \end{array}\right\} \rightarrow \left\{\begin{array}{l} e_{\mp} + \nu \\ \mu_{\mp} + \nu \end{array}\right. ,$$

where the first is a strong interaction, and the second a weak one. A pseudoscalar (P) interaction is ruled out since it gives predominantly (1), while an axial vector (A) interaction gives (2). Thus A is better, but one still finds too much of (1). The calculated ratio is:

$$\frac{R_1}{R_2} = \frac{m_e^2 (m_\pi^2 - m_e^2)^2}{m_\mu^2 (m_\pi^2 - m_\mu^2)^2} \sim 10^{-4}$$

This is larger than the experimental limit.

This difficulty is an open problem. It is not entirely excluded that there is a direct interaction, $\pi \rightarrow \mu + \nu$, but probably there would be other difficulties. The nature of the masses may be involved in solving the problem.

m_e is probably of electromagnetic nature, and the mechanical mass may be zero (non-electromagnetic approximation.). The mass of the μ seems to be partly non-electromagnetic. It is possible that the difficulty here is related to the difference in the origin of the masses. There are as yet unpublished reports of Feynman, and of Ruderman and Gatto on the problem.

One must also compare the decays in

$$\pi \rightarrow e + \gamma + \nu$$
$$\pi \rightarrow \mu + \gamma + \nu$$

Again the ratio is less than 10^{-5}. This is also difficult to explain. J. C. Taylor [Nuovo cimento 6, 1226 (1957)] has found that if the π is coupled to nucleons by

$$g_A \frac{\partial \phi_\pi}{\partial x_\mu} (\bar{\psi}_P i \gamma_5 \gamma_\mu \psi_N) \quad ,$$

then the ratio is 10^{-5}. The coupling

$$g_P \quad \phi_\pi \quad (\bar{\psi}_P \gamma_5 \psi_N) \quad ,$$

is disfavored by a larger ratio. Pauli doesn't believe that perturbation theory is permitted for the first stage of the intermediate state calculations. Although the part of the calculations dependent on the strong interactions drops out in the ratios, hyperons, for example, could upset the calculations.

Brief remarks on Λ decay.

(See J. J. Sakurai: Phys. Rev. 108, 491 (1957).)

The production of Λ's is believed to follow a strong interaction like

$$\pi^- + p \rightarrow \Lambda^0 + K^0 \quad ,$$

while the decay proceeds via

$$\Lambda^0 \rightarrow p + \pi^- \quad ,$$

in which parity is violated. The spin of the Λ is $\frac{1}{2}$, and its isospin is zero. The angular dependence of the decay (asymmetry) is given by:

$$I(\theta) \sim 1 + \alpha \langle \vec{\sigma}_\Lambda \rangle \frac{\vec{p}_\pi}{|p_\pi|} \quad ,$$

where

$$\alpha = \frac{2 \,\mathrm{Re}\, (A_S A_P^*)}{|A_S|^2 + |A_P|^2} \quad .$$

This is a purely phenomenological formula in which A_S and A_P are the amplitudes of the emitted S and P states for the π's. The experiments give (from Berkeley and Venice conference)

$$\alpha \langle \vec{\sigma}_\Lambda \rangle = (0.44 \pm 0.11)\, \vec{n} \quad ,$$

where $\vec{n}$ is a unit vector perpendicular to the plane of production:

$$\vec{n} = \frac{[\vec{p}_{inc} \times \vec{p}_\Lambda]}{|[\vec{p}_{inc} \times \vec{p}]|} \quad .$$

Here, the asymmetry is better with gradient coupling. For

$$H_{int} = \frac{1}{m_\pi} \frac{\partial \phi_\pi}{\partial x_\mu} \bar{\psi}_P (g_v + g_v' \gamma_5)\, i\gamma_\mu \psi_\Lambda ,$$

one finds

$$\alpha = \frac{2p_\pi (M_\Lambda^2 - M_p^2)\, (E_p(\vec{p}_\pi) + M_p) \cos \lambda}{[E_p(\vec{p}_\pi) + M_p]^2 (M_\Lambda - M_p)^2 + (M_\Lambda + M_p)^2 p_\pi^2} \quad .$$

Here, λ is given by $g_v' = g_v e^{i\lambda}$. For $\cos \lambda = \pm 1$,

$$\alpha = \pm 0.89 .$$

A non-gradient coupling,

$$H_{int} = \phi_\pi \bar{\psi}_p (g_S + g_S' \gamma_5) \psi_\Lambda$$

gives

$$\alpha = -\frac{2p_{\pi}\left[E_P(\vec{p}_{\pi}) + M_P\right]\cos\lambda}{\left[E_P(\vec{p}_{\pi}) + M_P\right]^2 + p_{\pi}^2}$$

and for $\cos\lambda = \pm 1$,

$$\alpha = \pm 0.10 \quad .$$

This disagrees with the experiments. It is interesting that the gradient coupling fits better.

It is very significant that the neutrino does not occur here, so that it cannot be the <u>cause</u> of parity violation. The original P violation observed was in the K decay. The Dalitz analysis showed that P was violated there (Phil. Mag. <u>44</u>, 1068 (1953); Phys. Rev. <u>94</u>, 1046 (1954)). The θ and τ have the same lifetime and mass, and identifying them we get a parity violation.

<u>General formalism of the expectation value of two fields.</u>

We will now return to the general theory of the expectation value of two fields. There is an old problem of the connection between spin and statistics: Integral spin is connected to Bose statistics; half-integral, to Fermi statistics. The question was treated by Pauli in many papers, with the stress on free particles. It is better to consider interacting particles, and this case was taken up again by Lüders and Zumino. The postulate of microcausality was also made in that field operators at spacelike positions were assumed to either commute or anti-commute. There is one further point: If we permit any arbitrary kind of metric, then the connection between spin and statistics would not follow. Feynman has shown that with a very indefinite metric, one can have spin $\frac{1}{2}$ particles of Bose statistics. Thus the positive metric plays a role. To obtain the connection between spin and statistics, we require:

(1) Inhomogeneous Lorentz group.

(2) Vacuum is the state of lowest energy.

(3) Microcausality.

(4) Positive definite metric.

In addition, Luders and Zumino postulate:

(5) The vacuum cannot be identically annihilated.

This seems to Pauli a little artificial.

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LECTURE 24

The Lüders-Zumino method for the connection between spin and statistics.

As stated previously, we assume:

(1) Lorentz invariance under the inhomogeneous Lorentz group. No reflection assumption is made.

(2) The vacuum is the state of lowest energy. Then:

$$\langle A(x)\; A^*(x')\rangle_0 = F\;(\xi),$$

where $F(\xi)$ is a function of the positive class. A^* is the adjoint field to A, and $\xi = x - x'$. As previously

$$F(\xi) = (2\pi)^{-3}\int d^4k\;\; \theta(k_o)\,\rho(-k\cdot k)\; e^{ik\cdot\xi}$$

$$= \int d\lambda\,\rho(\lambda)\; i\,\Delta_+(x;\lambda),$$

where $\theta(k_o) = \begin{cases} 0 & k_o < 0 \\ 1 & k_o > 0 \end{cases}$.

Then $F(-\xi) = F(\xi)$ for spacelike ξ, and thus

$$\langle A(x')A^*(x)\rangle_0 = \langle A(x)A^*(x')\rangle_0$$

for spacelike ξ. The connection is now essentially derived from a postulate that microcausality holds.

(3) Microcausality requires either

$$\left.\begin{array}{ll} \text{(a)} & \langle [A(x),\; A^*(x')]\rangle_0 = 0 \\ \text{or} & \\ \text{(b)} & \langle \{A(x),\; A^*(x')\}\rangle_0 = 0 \end{array}\right\}\;\xi\ \text{spacelike.}$$

The question is, which one holds? We will show that (a) holds, while (b) leads to a contradiction.

For hermitian fields (self-adjoint), the proof is simple. For non-hermitian fields it is not. In the hermitian case, (a) holds as a consequence of (1) and (2) (see Lecture 11). That (b) does not hold follows from the positive definite character of the metric.

(4) If (a) and (b) both hold:

$$\langle A(x)\, A(x') \rangle_0 = 0 \ , \quad \text{for spacelike } \xi .$$

In a positive definite metric, this would require that

$$A(x) \, | \, 0 \rangle \equiv 0 \ .$$

This is not allowed under the Lüders-Zumino postulate #5. For an indefinite metric, the proof is not so trivial, and is still an open question.

For a non-hermitian field, we have either

(a) $\langle [A(x), A^*(x')] \rangle_0 = \langle [A(x), A(x')] \rangle_0 = \langle [A^*(x), A^*(x')] \rangle_0 = 0$

or

(b) $\langle \{A(x), A^*(x')\} \rangle_0 = \langle \{A(x), A(x')\} \rangle_0 = \langle \{A^*(x), A^*(x')\} \rangle_0 = 0$

for spacelike ξ. This can be carried back to the hermitian case, via:

$$A = \frac{1}{\sqrt{2}} (A_1 + i\, A_2)$$

$$A^* = \frac{1}{\sqrt{2}} (A_1 - i\, A_2) \ ,$$

where A_1, A_2 are hermitian. This is not quite the same as Lüders-Zumino. They require gauge invariance instead of the additional postulate about $[A, A]$ and $[A^*, A^*]$. It then follows directly that $\langle A(x) A(x') \rangle_0 =$ $= \langle A^*(x) A^*(x') \rangle_0 = 0$, for all ξ. Then one can use (3) in the original form.

The reason for the Lüders-Zumino argument is that they wanted to consider the possibility that A_1 commutes and A_2 commutes, but A_1 and A_2 anti-commute with each other. Lüders showed that a trivial transformation can always bring one to the commuting form.

The most interesting point seems to be the entering of the positive definite metric. Whether it can be eliminated is not known.

Let us now consider spinors. We begin a Majorana field. Now:[1]

$$\langle \psi_\beta^*(x)\,\psi_\beta(x')\rangle_0 = E^{-1}_{\beta\alpha}\langle \psi_\alpha(x)\;\psi_\beta(x')\rangle_0 = \frac{1}{i}\frac{\partial}{\partial \xi_0}\;g(\xi)$$

since:

$$\psi^c \equiv E\psi^* = \psi,$$

where $g(\xi)$ is a function of the positive class. We make postulate (4), that

$$\rho(\xi) \geq 0.$$

Now we define:[2]

$$\frac{1}{4}\sum_{k_i;k_i+dk_i}\;\sum_\alpha\langle 0\,|\,\psi_\alpha(0)\,|\,k\rangle\langle k\,|\,\psi_\alpha^*\;(0)\,|\,0\rangle \equiv k_0\rho(\lambda)\,\frac{d^4k}{(2\pi)^3}$$

and we find:[3]

$$-\frac{1}{4}\sum_\alpha\langle\psi_\alpha(x)\;\psi_\alpha^*\;(x')\rangle_0 = \frac{1}{i}\frac{\partial}{\partial t}\,F(\xi)$$

$$+\frac{1}{4}\sum_\alpha\langle\psi_\alpha^*(x')\,\psi_\alpha(x)\rangle_0 = \frac{1}{i}\frac{\partial}{\partial t}\;F(-\xi)\quad.$$

Thus, since $F(\xi) = F(-\xi)$ for a spacelike ξ, we see that (b) holds in this case. (a) and (b) both holding leads to a contradiction with the positive definite metric, since we would get

$$\psi_\alpha(x)\,|\,0\rangle \equiv 0.$$

In the non-Majorana case, we must either postulate gauge invariance, so that

$$\langle \psi_\alpha(x)\,\psi_\beta(y)\rangle_0 = \langle \bar{\psi}_\alpha(x)\,\bar{\psi}_\beta(y)\rangle_0 = 0,$$

or make the analogous assumption to the scalar case above. We then get to the Majorana fields by:

$$\psi_I = \frac{1}{\sqrt{2}}\,(\psi + \psi^c),$$

$$\psi_{II} = \frac{1}{\sqrt{2}i}\,(\psi - \psi^c).$$

FOOTNOTES

1 From Lecture 12, we have:

$$\langle \psi_\alpha(x)\,\psi_\beta(x')\rangle_0 = \left[\left\{\gamma\frac{\partial}{\partial\xi}F^I + 1\cdot G^I + \gamma_5\gamma\frac{\partial}{\partial\xi}F^{II} + \gamma_5 G^{II}\right\} C^{-1}\right]_{\alpha'\beta}$$

and from Lecture 4:

$$E^{-1} = -C\gamma_4 .$$

Thus, for the F^I term, we have:

$$-(C\gamma_4\gamma_\mu C^{-1})_{\beta\beta}$$

If $\mu = 4$, we get -4, while if $\mu \neq 4$

$$\mathrm{Tr}\,(C\gamma_4\gamma_\mu C^{-1}) = \mathrm{Tr}(\widetilde{\gamma_4\gamma_\mu}\,\tilde{C}\,C^{-1}) = -\mathrm{Tr}\,(\widetilde{\gamma_4\gamma_\mu}\,C\,C^{-1})$$

$$= -\mathrm{Tr}\,(\widetilde{\gamma_4\gamma_\mu}) = 0 .$$

Similarly, we find that all other terms vanish.

2 See Lecture 12.

3 We had:

$$\langle \psi_\beta^*(x)\,\psi_\beta(x')\rangle_0 = E_{\beta\alpha}^{-1}\langle \psi_\alpha(x)\,\psi_\beta(x')\rangle = \frac{1}{i}\frac{\partial}{\partial\xi_0}\,g(\xi)$$

FOOTNOTES

3 (Cont.)

Now:

$$\langle \psi_\beta (x') \psi_\beta^* (x) \rangle_0 = \langle \psi_\beta (x') \psi_\alpha (x) \rangle_0 \; E_{\alpha\beta}^{-1}$$

$$= E_{\beta\alpha}^{-1} \langle \psi_\alpha (x') \psi_\beta (x) \rangle_0$$

$$= -\frac{1}{i} \frac{\partial}{\partial \xi_0} \quad g(-\xi)$$

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LECTURE 25

An essential point in the preceding development was the requirement of a positive definite metric. To illustrate, we will consider some "anormal" cases of an indefinite metric. For example, we can obtain Bose statistics and spin $\frac{1}{2}$, or Fermi statistics and spin zero. The case of an indefinite metric has been treated in an interesting way by Feynman: Phys. Rev. 76, 749 (1949). This is very short. The discussion was shown to be essentially complete by Pauli: Prog. Theor. Phys. 5, 526 (1950).

We consider first "anormal" scalar fields. For free particles:

$$A(x) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\omega}} \left\{ a(\vec{k})\, e^{ik\cdot x} + b^*(\vec{k})\, e^{-ik\cdot x} \right\}$$

where:

$$\left\{a, a^*\right\} = 1 \qquad \left\{b, b^*\right\} = -1$$

and all other anticommutators are zero. Evidently b^* cannot be the hermitian conjugate to b. Rather, $b^* = -b^H$, where b^H = hermitian conjugate. The states will now have norms of oscillating sign, according to the number of b particles present. $\eta \sim (-1)^{\Sigma N_b}$. The vacuum is defined by

$$a\,|\,0\rangle = b\,|\,0\rangle = 0.$$

Thus:

$$\langle b\, b^* \rangle_0 = -1.$$

This last sign is the important one, as it leads to the Fermi statistics.

Now we find for the free fields:

$$\left\{A(x),\ A^*(x')\right\} = i\Delta(x - x') \ ,$$

while the commutator is given by the Δ_1 function.

For non-free fields, we set:

$$\langle A(x)A^*(x')\rangle_0 = F(s) \ .$$

If we now require microcausality for the anticommutator, it is necessary that:

$$\langle A^*(x)A(x')\rangle_0 = - F(s)$$

since:

$$\langle \left\{A(x),\ A^*(x')\right\}\rangle_0 = F(s) - F(s^*),$$

and this will be zero for spacelike points (s = real.)

It is interesting to decompose A into its self-adjoint parts:

$$A(x) = \frac{1}{\sqrt{2}}\,(A^I + i\ A^{II}) \qquad\qquad A^{I*} = A^I$$

$$A^*(x) = \frac{1}{\sqrt{2}}\,(A^I - i\ A^{II}) \qquad\qquad A^{II*} = A^{II}$$

Let us assume gauge invariance. Then:

$$\langle A(x)A(x')\rangle_0 = \langle A^*(x)A^*(x')\rangle_0 = 0 \ .$$

From this follows that

$$\langle A^I(x)A^I(x') - A^{II}(x)A^{II}(x')\rangle_0 = 0 \ ,$$

$$\langle A^I(x)A^{II}(x') + A^{II}(x)A^I(x')\rangle_0 = 0 \ ,$$

Thus:

$$\langle A^I(x)A^{II}(x')\rangle_0 = -\langle A^{II}(x)A^I(x')\rangle_0 = - F(s)$$

If we now require microcausality, for anticommutators, we get:

$$\langle A^I(x)A^I(x')\rangle_0 + \langle A^{II}(x)A^{II}(x')\rangle_0 + \langle A^I(x')A^I(x)\rangle_0 + \langle A^{II}(x')A^{II}(x)$$

$$= G(s) + G(s) + G(s^*) + G(s^*) = 0 \quad \text{for } \xi \text{ spacelike, where}$$

$$\langle A^I(x)A^I(x') \rangle_0 = G(s) \; .$$

Thus, $G(s) \equiv 0$, and $\langle A^I(x)A^I(x') \rangle_0 = \langle A^{II}(x)A^{II}(x') \rangle_0 = 0$

for all ξ. For the cross terms, microcausality gives:

$$\langle A^I(x)A^{II}(x') \rangle_0 - \langle A^{II}(x)A^I(x') \rangle_0 - \langle A^I(x')A^{II}(x) \rangle_0 + \langle A^{II}(x')A^I(x) \rangle_0$$

$$= - F(s) - F(s) + F(s^*) + F(s^*) = 0 \quad \text{for } \xi \text{ spacelike.}$$

This is automatically satisfied.

The relations for gauge invariance are in the Lüders-Zumino paper, and they can be satisfied for an indefinite metric, as is seen by the specific example of Feynman.

The spinor case is quite analogous. For free particles:

$$\psi_\rho(x) = \frac{1}{\sqrt{V}} \sum_{r=1,2} \sum_k \left\{ a_r(\vec{k}) u_\rho^{\,r}(\vec{k})\, e^{ik\cdot x} + b_r^{\,*}(\vec{k})\, v_\rho^{\,r}(\vec{k})\, e^{-ik\cdot x} \right\}$$

In the "anormal" case, we choose

$$\left[a_r, a_r^{\,*} \right] = 1 \qquad \left[b_r, b_r^{\,*} \right] = -1 \; .$$

The vacuum is given by:

$$a \,|\, 0 \rangle = b \,|\, 0 \rangle = 0$$

so:

$$\langle b_r\, b_r^{\,*} \rangle_0 = -1 \; ,$$

and again the b states have oscillating norm. Again $b^* = - b^H$.

The commutation relation alone does not lead to the indefinite metric, but we must include the definition of the vacuum, since otherwise we could reverse the meaning of b and b* as annihilation and creation operators.

Now:

$$S = (\gamma \frac{\partial}{\partial x} - m)\Delta$$

and for free particles, we find:

Normal		Anormal
$\langle \psi_\alpha(x)\bar{\psi}_\beta(x') \rangle_0$	$= - i S^+_{\alpha\beta}(x - x')$	$= - i S^+_{\alpha\beta}(x - x')$
$\langle \bar{\psi}_\beta(x')\psi_\alpha(x) \rangle_0$	$= - i S^-_{\alpha\beta}(x - x')$	$= + i S^-_{\alpha\beta}(x - x')$
$\{\psi_\alpha(x), \bar{\psi}_\beta(x')\}$	$= - i S_{\alpha\beta}(x - x')$	$\langle\{\ \}\rangle_0 = - S^{(1)}_{\alpha\beta}(x - x')$
$\langle [\psi_\alpha(x), \psi(x')] \rangle_0$	$= - S^{(1)}_{\alpha\beta}(x - x')$	$[\ \ ] = - i S_{\alpha\beta}(x - x')$

For non-free spinors, we set:

Normal		Anormal
$-\langle \psi_\alpha(x)\bar{\psi}_\beta(x') \rangle_0$	$= \gamma \frac{\partial}{\partial \xi} F(s) + G(s)$	$= \gamma \frac{\partial}{\partial \xi} F(s) + G(s)$
$\langle \bar{\psi}_\beta(x)\psi_\alpha(x') \rangle_0$	$= -\gamma \frac{\partial}{\partial \xi} F(s) + G(s)$	$= \gamma \frac{\partial}{\partial \xi} F(s) - G(s)$
$\langle\{\psi_\alpha(x), \bar{\psi}_\beta(x')\}\rangle_0$	$= 0$	$\langle [\psi_\alpha(x), \psi_\beta(x')] \rangle_0 = 0$

where ξ is spacelike in the last line. We can again separate the fields into their Majorana parts, with the same results as in the Lüders-Zumino paper. In the anormal case, it is necessary to define charge conjugation with an opposite sign:

$$\psi^c = C^{-1}\bar{\psi}$$

$$\bar{\psi}^c = - C\psi$$

The latter is necessary since ψ^* is no longer the hermitian conjugate. The decomposition into Majorana parts is carried out using

$$\psi^I = \frac{1}{\sqrt{2}}(\psi + \psi^c)$$

$$\psi^{II} = \frac{1}{\sqrt{2}\ i} (\psi - \psi^{c}) \quad .$$

One can then easily obtain:

$$\left\langle \psi_{\alpha}^{*I} (x) \, \psi_{\beta}^{I} (x') \right\rangle_{o} = 0$$

$$\left\langle \psi_{\alpha}^{*II} (x) \, \psi_{\beta}^{II} (x') \right\rangle_{o} = 0 \quad .$$

These are the relations from which one obtains a contradiction with a positive definite metric. They can be fulfilled with an indefinite metric.

Pauli does not see whether other metrics exist which retain the spin and statistics connection.

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LECTURE 26

Generalization of the CPT theorem to higher spins.

The generalization of the CPT theorem to higher spins has a certain mathematical interest.

We must consider the irreducible representations of 3 dimensional rotations and of the Lorentz group. The 3D rotations have irreducible representations of degree $2j + 1$, where $j = 0, \frac{1}{2}, 1, \ldots$. We must distinguish between infinitesimal and finite transformations. The former

In 3D, we have:

$$[J_1, J_2] = i J_3 .$$

The 4D rotation group splits into the direct product of two 3D groups. In the 4D case, we have the operators[1]

$$L_{ik} = - L_{ki} \qquad i,k = 1, 2, 3, 4.$$

L_{ik}, the angular momentum operator, is a 6-vector. If we define:

$$M_1 \equiv \tfrac{1}{2}(L_{23} + L_{14}) \qquad N_1 \equiv \tfrac{1}{2}(L_{23} - L_{14})$$

$$M_2 \equiv \tfrac{1}{2}(L_{31} + L_{24}) \qquad N_2 \equiv \tfrac{1}{2}(L_{31} - L_{24})$$

$$M_3 \equiv \tfrac{1}{2}(L_{12} + L_{34}) \qquad N_3 \equiv \tfrac{1}{2}(L_{12} - L_{34}) .$$

Then:

$$[M_1, M_2] = iM_3 , \ldots$$

$$[N_1, N_2] = iN_3, \ldots$$

$$[M_i, N_k] = 0$$

Thus the 4D group is nothing essentially new. The principal difference between the Lorentz group and the 4D rotations appears in the reality conditions.

In 3D or 4D rotations, the J's (or L's) will be hermitian.

For Lorentz transformations, the L_{ik} ($i, k = 1, 2, 3$) will be hermitian, while L_{4i}, L_{i4} will be antihermitian. This has the consequence that the representations of the Lorentz group of finite degree $(2j + 1)$ are non-unitary.

We consider the 4D rotations. They will be characterized by two numbers (m, n), and the degree of a representation will be $(2m + 1)(2n + 1)$. We do not consider reflections. We find the representations:

Type	Degree
Scalar $(0,0)$	1
Spinor $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$	2
(Space reflections permute m and n)	
Vector $(\frac{1}{2},\frac{1}{2})$	4
Self-dual Tensor $(1,0)$ and $(0,1)$	3
(The 34 element is, apart from a factor, equal to the 12 element. In the case of light $\vec{E} + i\vec{H}$, $\vec{E} - i\vec{H}$ correspond to self-dual tensors (plane waves)).	
Symmetric tensor, zero spur $(1,1)$	9 .

If we consider the multiplication of 2 quantities, we observe that the direct product is irreducible:

$$(m_1, n_1) \times (m_2, n_2) = \sum \alpha_{m,n} (m,n)$$

where:

$$|m_1 - m_2| \leq m \leq (m_1 + m_2)$$

$$\left| n_1 - n_2 \right| \leq n \leq (n_1 + n_2).$$

If we consider now the 3D subgroup, we see that since the L_{i4} operators distinguish between M and N, in the subgroup m and n will be equivalent. Thus, we will have the irreducible representations in 3D of

$$\left| m - n \right| \leq j \leq m + n$$

and the m and n will be mixed. The values of j give a distinction between the Fermi and the Bose classes. It is only important that j is an integer or half an odd integer, but not m or n separately. Thus:

Fermions: $m \pm n =$ integer $+ \frac{1}{2}$

Bosons: $m \pm n =$ integer.

Pauli found it useful[1] to divide the representations into two further parts:

Fermion: 1 (a) $m =$ int., $n =$ int. $+ \frac{1}{2}$
1 (b) $m =$ int. $+ \frac{1}{2}$, $n =$ int.

Boson: 2(a) $m =$ int., $n =$ int.
2(b) $m =$ int. $+ \frac{1}{2}$, $n =$ int. $+ \frac{1}{2}$.

Thus we have 4 classes. It is not essential that we deal with the irreducible representations.

The multiplication of these classes corresponds to the "4-group" of mathematics; i.e.

$$1(a) \times 2(b) = 1(b)$$

and so on.

Now, we assign the characters to these classes for the CPT transformation:

2(a)	2(b)	
+1	-1	$= (-1)^{2m} = (-1)^{2n}$

1(a)	1(b)	
i	-i	$= i\ (-1)^{2m} = -i\ (-1)^{2n}$

The factor "i" is introduced in this choice to allow for the possibility of reality conditions on the fields. Such conditions will be of the form:[2]

$$(u\ (n,m)\ )^* = v\ (m,n).$$

Reality conditions will then be preserved under the transformations with the "i" present. The reality conditions are important, for example, in constructing a vector from spinors.

If we apply this transformation particularly to the case $u'(\frac{1}{2},0)^* = u\ (0,\frac{1}{2})$, this is equivalent to $\psi' = i\gamma_5\psi$.

In the Feynman case of spinors with Bose quantization, the reality conditions are abandoned, and the i's do not appear. There, the character is simply:

$$(-1)^{2m},$$

and $(m,n)^*$ is not the Hermitian conjugate, but the adjoint to (m,n).

Now the connection between spin and statistics enters. At first glance, the above relations do not seem to be satisfied for products of fermions. Consider a product:

$$\prod_k (m_k n_k)$$

and

$$n = \sum_{k=1}^{N} n_k \qquad m = \sum_{k=1}^{N} m_k .$$

Then we obtain for the character of the product:

$$(-i)^N (-1)^{2n},$$

whereas the character as given above should be

$$(-i)\ (-1)^{2n} \qquad \text{N odd (fermion)}$$

$$(-1)^{2n} \qquad \text{N even (boson)}$$

Thus they don't agree. We have an extra factor:

$$(-i)^{N-1} \qquad N \text{ odd.}$$

$$(-i)^{N} \qquad N \text{ even.}$$

If N = even = 2V, we have $(-i)^{2V} = (-1)^{V} = (-1)^{\frac{N(N-1)}{2}}$, and we get the same result for N = odd. Thus, the definition of the character is not true for C-numbers, but for q-numbers which are quantized according to Fermi statistics for half an odd integer spins and with Bose statistics for integral spin the definition is consistent, since the $(-1)^{N(N-1)/2}$ factor represents the sign associated with the anticommutation of the fermion fields.

Thus we must add the rule of inversion to the usual multiplication law, and we must assume that all products are symmetrized or antisymmetrized according to the Bose or Fermi statistics. Thus, for example, a vector would be constructed as:

$$u_1(\tfrac{1}{2},0)\ u_2(0,\tfrac{1}{2}) - u_2(0,\tfrac{1}{2})\ u_1(\tfrac{1}{2},0)$$

and the transformation would include an extra (-1) because of the inversion.

FOOTNOTES

[1] W. Pauli, Phys. Rev. 58, 716 (1940).

[2] This follows from the fact that, if u(n,m) transforms according to an operator Λ

$$u'(n,m) = \Lambda(n,m;\ n',m')\ u(n',m'),$$

then with a suitable ordering of (n,m), $u'(m,n)^*$ transforms according to Λ^*.

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In addition to the field quantities, we also have the coordinates, and derivatives with respect to coordinates. The transformation laws also hold for them. Thus:

$$x_\nu' = -x_\nu \quad .$$

The transformations which have been obtained are the ones used for CPT. It is not essential whether the quantities used are irreducible or not. Thus we find that any Lorentz invariant equation will remain invariant under a CPT inversion.[1]

In the above development we have used the local character of the fields. If one introduces form factors (non-local interactions), the situation is not so simple.

The connection between spin and statistics enters the development with the symmetrization of products of fields. In the Feynman anormal case with an indefinite metric CPT also holds, but not in the form given here.

Representation Theory

Let us now consider the representations of the various groups associated with fields: 3D rotations, homogeneous and inhomogeneous; and Lorentz transformations. This discussion will be a summary of a series of lectures which were given at CERN.

We will first consider the infinitesmal transformations. (This means that the Lie algebra will be involved.) For an n-dimensional rotation group, we will introduce the operators which generate the infinitesmal rotations:

$$e_{\lambda\mu} \equiv - e_{\mu\lambda} = \chi_\lambda \frac{\partial}{\partial x_\mu} - \chi_\mu \frac{\partial}{\partial x_\lambda} \quad \lambda\mu = 1, \ldots, n.$$

Then we find:

$$\left[e_{\lambda\mu}, e_{\rho\sigma}\right] = \delta_{\lambda\sigma} e_{\mu\rho} + \delta_{\mu\rho} e_{\lambda\sigma} - \delta_{\lambda\rho} e_{\mu\sigma} - \delta_{\mu\sigma} e_{\lambda\rho} .$$

The particular relation for the "e" operators is special, but the commutation relations associated with the Lie ring are general. The latter may of course be derived using the special choice for e.

In addition to the rotations, we may wish to include the inhomogeneous group (translations). A particular choice is:

$$d_\lambda = \frac{\partial}{\partial x_\lambda}$$

The general commutation relations are then:

$$\left[d_\lambda , e_{\mu\nu}\right] = \delta_{\lambda\mu} d_\nu - \delta_{\lambda\nu} d_\mu$$

$$\left[d_\lambda , d_\mu\right] = 0.$$

There is always a particular representation in which the d's are zero.

We can define:

$$e_{\mu\nu} = - e_{\nu\mu} = i J_{\mu\nu} ; \quad d_\mu = i p_\mu$$

The $J_{\mu\nu}$, p_λ are then hermitian for a unitary representation of the continuous group. There is an important theorem due to Lie: From the representations of the Lie ring, we get the representations of the entire group. For the representations, an important concept is that of invariance.

One may take two points of view.

(1) Abstract - The commutator is considered as an abstract product.

(2) The operators, a, are related to matrices, A, in which

$$a \rightarrow A$$

$$[a, b] \rightarrow AB - BA.$$

Then we get the Jacobi identity:

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0.$$

We see that, in the original abstract approach to the Lie ring quantities such as A^2, B^2, ... are not defined.

An invariant is a quantity which commutes with all elements of the ring.

<u>3D rotations</u>.

We set $J_{12} = J_3, \ldots$

Then:

$$[J_1, J_2] = i J_3, \ldots$$

For the homogeneous group,

$$J^2 = J_1{}^2 + J_2{}^2 + J_3{}^2$$

is an invariant, since it commutes with J_1, J_2, and J_3.

In the inhomogeneous group, we have the relations:

$$[J_1, p_2] = [p_1, J_2] = i\ p_3, \ldots ; [p_1, J_1] = 0, \ldots .$$

The invariants are[2]

$$p^2 = p_1{}^2 + p_2{}^2 + p_3{}^2$$

$$\vec{p} \cdot \vec{J} = p_1 J_1 + p_2 J_2 + p_3 J_3.$$

<u>4D rotations</u>.

We set:

$$J_{23} = M_1 \,,\ J_{31} = M_2,\ J_{12} = M_3$$

$$J_{41} = N_1 \,,\ J_{42} = N_2,\ J_{43} = N_3$$

(These are not the same as the relations in lecture 26.)

Then:

$$F = \tfrac{1}{2} \sum_{\mu<\nu} J_{\mu\nu}^{\ 2} = \tfrac{1}{2}\,(\vec{M}^2 + \vec{N}^2)$$

$$G = J_{41}J_{23} + J_{42}J_{31} + J_{43}J_{12} = \vec{M}\cdot\vec{N}$$

are invariants. The $\vec{M},\vec{N}$ can be decomposed as:

$$\vec{K} = \tfrac{1}{2}(\vec{M} + \vec{N}) \qquad\qquad \vec{L} = \tfrac{1}{2}\,(\vec{M} - \vec{N}).$$

Then:

$$\left[K_i,\ L_j\right] = 0$$

$$\left[K_1,\ K_2\right] = i\ K_3,\ \ldots$$

$$\left[L_1,\ L_2\right] = i\ L_3,\ \ldots$$

For the Lorentz group, the reality conditions lead to

$$\vec{N} \rightarrow i\ \vec{N}'$$

Then $\vec{N}'$ is hermitian and the invariants are:

$$F = \tfrac{1}{2}\,(\vec{M}^2 - \vec{N}^2)\ .$$

$$\frac{1}{i}\,G = G' = (\vec{M}\cdot\vec{N}')\ .$$

This change in reality conditions leads to the result that the unitary representations of the Lorentz group are of infinite degree.

Finally, we have the inhomogeneous Lorentz group:

$$\left[p_1,\ N_1\right] = i\ p_0,\ \ldots \qquad \left[p_2,\ N_1\right] = \left[p_1,\ N_2\right] = 0,\ \ldots$$

$$\left[p_0,\ N_1\right] = i\ p_1,\ \ldots \qquad \left[p_0,\ M_1\right] = 0,\ \ldots$$

The invariants of the inhomogeneous group include:

$$P = - p_\nu p_\nu = p_0^2 - \vec{p}^2 .$$

We will introduce

$$v_{k\lambda\mu} = p_k J_{\lambda\mu} + p_\lambda J_{\mu k} + p_\mu J_{k\lambda} ,$$

and using a dual notation to define:

$$(\omega_1, \omega_2, \omega_3, \omega_4) = - i \, (v_{234}, v_{314}, v_{124}, v_{321}) ,$$

we get:

$$[\omega_\lambda, J_{\mu\nu}] = i \, (\delta_{\lambda\nu} \omega_\mu - \delta_{\lambda\mu} \omega_\nu)$$

$$[\omega_\lambda, p_\mu] = 0, \qquad \omega_\nu \, p_\nu = 0$$

and

$$[\omega_\mu, \omega_\nu] \neq 0 \qquad \text{for} \quad \mu \neq \nu$$

The second invariant is thus:

$$W = \omega_\nu \omega_\nu = - \frac{1}{6} \, v_{k\lambda\mu} v_{k\lambda\mu}$$

$$= \tfrac{1}{2} \, (p_\lambda p_\lambda) \, (J_{\mu\nu} J_{\mu\nu}) - J_{k\mu} J_{k\nu} p_\mu p_\nu .$$

In the rest system $(P > 0)$,

$$\vec{p} = 0, \; p_4 = i \, m$$

so the first invariant is:

$$P = m^2 .$$

For the second, we have:

$$(\omega_1, \omega_2, \omega_3, \omega_4) = m(J_{23}, J_{31}, J_{12}, 0)$$

Hence:

$$\vec{\omega} = i \, m \times \text{(Angular momentum in rest system)}.$$

and

$$-\mathcal{W} = m^2 S(S+1) , \qquad -\frac{\mathcal{W}}{P} = S(S+1)$$

where S is the spin, and $m \neq 0$.

If $P = 0$, there are two cases:

(a) $\omega = 0$. Then: $\omega_\nu = \lambda p_\nu$, and λ is essentially the spin.

$$\lambda = \pm \frac{(\vec{p}.\vec{M})}{p} = \pm j_0$$

There are two such representations according to the ± . They correspond to right and left.

(b) $\omega = 0$. ω then has continuous eigenvalues. (Wigner.)

FOOTNOTES

[1] For a fuller account of the above developments, see:

Niels Bohr and the Development of Physics, W. Pauli, ed., p.30 ff.

[2] J^2 is no longer an invariant. For example:

$$\left[J^2, p_1\right] = 2i \left[p_2 J_3 - J_2 p_3\right] .$$

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We give the explicit representations of J, p. We define:

$$J_1 \pm i\, J_2 \equiv J_\pm \;, \qquad p_1 \pm i\, p_2 \equiv p_\pm \;.$$

Then:

$$\left[J_3, J_+\right] = J_+ \;; \quad \left[J_3, J_-\right] = -J_- \;; \quad \left[J_+, J_-\right] = 2\, J_3 \;.$$

$$\left[J_+, p_-\right] = \left[J_-, p_+\right] = 2\, p_3 \;; \quad \left[J_3, p_\pm\right] = \left[p_3, J_\pm\right] = \pm\, p_\pm \;.$$

In the homogeneous group (3D),

$$J^2 = j\,(j+1) \qquad \text{where} \quad j = 0, \tfrac{1}{2}, 1, \ldots$$

We can choose J_3 diagonal. Then:

$$(m \,|\, J_3 \,|\, m') = m\, \delta_{m\, m'} \qquad -j \leq m \leq j$$

$$(m' \,|\, f_\pm \,|\, m'') \neq 0 \qquad \text{only if} \quad m'' = m' \pm 1 \,,$$

where f is either J or p. Then one finds:

$$(j, m \,|\, J_3 \,|\, j, m) = m = \frac{(j, m \,|\, p_3 \,|\, j, m)}{(j \,|\, p \,|\, j)}$$

$$(j, m \,|\, J_+ \,|\, j, m-1) = \sqrt{(j+m)(j-m+1)} = \frac{(j, m \,|\, p_+ \,|\, j, m-1)}{(j \,|\, p \,|\, j)}$$

$$(j, m \,|\, J_- \,|\, j, m+1) = \sqrt{(j-m)(j+m+1)} = \frac{(j, m \,|\, p_- \,|\, j, m+1)}{(j \,|\, p \,|\, j)} \;.$$

Also:

$$(j, m | p_3 | j+1, m) = (j | p | j+1) \sqrt{(j+m+1)(j-m+1)}$$

$$(j, m | p_+ | j+1, m-1) = (j | p | j+1) \sqrt{(j-m+2)(j-m+1)}$$

$$(j, m | p_- | j+1, m+1) = -(j | p | j+1) \sqrt{(j+m+2)(j+m+1)}$$

$$(j, m | p_3 | j-1, m) = (j | p | j-1) \sqrt{(j+m)(j-m)}$$

$$(j, m | p_+ | j-1, m-1) = -(j | p | j-1) \sqrt{(j+m)(j+m-1)}$$

$$(j, m | p_- | j-1, m+1) = (j | p | j-1) \sqrt{(j-m)(j-m+1)}$$

(These had already been guessed before quantum mechanics.)

Up to this point, the commutation relations among the J's and between J's and p's have been used. The relations among the p's were not employed. If these are also taken into account, we get a complete representation of the inhomogeneous group. We define:

$$(j | p | j+1)(j+1 | p | j) \quad (2j+3)(2j+1) \equiv \phi(j)$$

Then:

$$(j | p | j-1)(j-1 | p | j) \quad (2j+1)(2j-1) = \phi(j-1)$$

$$\left\{\phi(-1) = 0\right\} \ .$$

Then one gets, using simple algebra,

1. $$(j, m | [p_+ , p_-] | j, m) = 2m \left\{ \frac{\phi(j+1) - \phi(j)}{2j+1} + (j | p | j)^2 \right\}$$

2. $$(\vec{p})^2 \equiv p_1{}^2 + p_2{}^2 + p_3{}^2 = \phi(j) \frac{j+1}{2j+1} + \phi(j-1) \frac{j}{2j+1}$$

$$+ |(j | p | j)|^2 \ j(j+1)$$

3. $$(\vec{J} \cdot \vec{p}) \equiv C = (j | p | j) \ j(j+1)$$

(Pauli has not found these in the literature.)

Now there are various cases:

(a) $C = 0$ $\qquad (j \mid p \mid j) = 0 \qquad \phi(j) = \phi(j-1) = \text{const.}$
$= p^2$

$$\left| (j \mid p \mid j+1) \right|^2 = \frac{p^2}{(2j+3)(2j+1)}.$$

There is still the question of the range of j. One must prove that the lowest value of j is zero. If we assume that $j \geq j_0$, where $j_0 \neq 0$, then we get a contradiction in equation 1, since if $j_0 \neq 0$, $m \neq 0$ is allowed also. However, if $j_0 = 0$, only $m = 0$ and we cannot conclude that $\phi(-1) \neq 0$.

(b) The situation is different if $C \neq 0$. Then:

$$\frac{-\phi(j) + \phi(j+1)}{2j+1} + \frac{C^2}{j^2(j+1)^2} = 0$$

and:

$$\phi(j) - \phi(j-1) = C^2 \left(\frac{1}{j^2} - \frac{1}{(j+1)^2} \right)$$

so:

$$\phi(j) + \frac{C^2}{(j+1)^2} = \text{const.} = (\vec{p})^2 ,$$

as one finds from substituting into the expression for the eigenvalue for p^2.

If we now attempt to obtain the minimum $j = j_0$, we set:

$$\phi(j_0 - 1) = 0 .$$

Thus:

$$C^2 = j_0^2 p^2$$

$$C = \pm \sqrt{p^2}\, j_0$$

or:

$$J_{11} = \frac{\vec{J} \cdot \vec{p}}{\sqrt{p^2}} = \pm j_0 .$$

Thus as an immediate result of this algebra the angular momentum parallel to the momentum is quantized. As $j \to \infty$, $\phi(j)$ increases monotonically, and so ϕ is not zero for any other j. j_0 can be either an integer or half an odd integer.

Lorentz group[1]

Now we have:

$$J_{0k} = \frac{1}{i} J_{4k}$$

and:

$$\vec{M} = (J_{23}, J_{31}, J_{12})$$

$$\vec{N} = (J_{01}, J_{02}, J_{03}) \qquad \left[\vec{N}_1, \vec{N}_2\right] = -i\,\vec{M}_3, \ \ldots$$

The invariants are:

$$F = \tfrac{1}{2} (\vec{M}^2 - \vec{N}^2)$$

$$G = (\vec{M} \cdot \vec{N})$$

The development is again pure algebra. There are again two possibilities.

(1) Principal series.

$$2F = j_0^2 - 1 - \nu^2 \qquad G = j_0 \cdot \nu$$

ν is real, $j_0 = 0, 1, 2, \ldots$

or $j_0 = \frac{1}{2}, \frac{3}{2}, \ldots$

There is a special case: $j_0 = 0$, $G = 0$.

$$2F = -1 - \nu^2 < -1.$$

All representations of the 3D rotations for $j > j_0$ are contained in this series.

(2) Complementary series (or "critical strip" - Pauli).

$$G = j_0 = 0$$

$$2F = -1 + \alpha^2 \qquad 0 \leq \alpha \leq 1.$$

This representation is not contained in 1 .

The principal series are oscillating and bounded in the group manifold. The complementary series are not bounded in the group manifold.

Every function of the group manifold can be expanded in the representation. There is a theorem due to Weyl:

The Lorentz group is not finite and a complete set is given by only a part of the unitary representation; one can discard the other. The principal series is all that is needed. This only holds for finite transformations, not for the Lie ring and infinitesmal transformations. The physicist usually obtains only (1), while the purely algebraic method does not distinguish (1) and (2).

There is a connection with the hydrogen spectrum. In it there is more degeneracy than in a general central force field. The principle quantum number defines a set of j's which give the same eigenvalues. The degree of degeneracy is $2n^2$ (spin = 2x). It was shown by Hulthen, Fock, and Bargmann that the commutation relations are the same in the discrete spectrum as in the 4D rotation group. In the continuous spectrum the equivalence is with the Lorentz group, although only the (n, n) states are realized in the hydrogen atom. In the continuous group only case (1) is realized: $j_0 = 0$, $G = 0$.

FOOTNOTES

1 Gelf and Neumark, Journal of Phys. U.S.S.R. 10, pp. 93-4.