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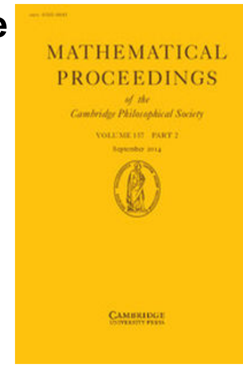

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On tubular neighbourhoods of manifolds. I

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1. *Introduction.* Let X be a submanifold of Y, in either the topological, smooth, or piecewise linear ($=PL$) categories. A *normal cell bundle* on X in Y is a bundle $\xi = (p, E, X)$ in the category whose fibre is a closed cell, and such that E is a neighbourhood of X in Y and $p: E \to X$ is a retraction. The triple (Y, X, ξ) is a *tubular neighbourhood*, or briefly, a *tube*. For convenience we may refer to a tube by its cell bundle.

Two tubes (Y_i, X_i, ξ_i) $(i=0,1)$, are *isomorphic* if there is an isomorphism of the category $h: (Y_0, X_0) \to (Y_1, X_1)$ inducing a bundle map from ξ_0 to ξ_1. That is, h takes fibres of ξ_0 onto fibres of ξ_1. If $Y_0 = Y_1 = Y$, the two tubes are *I-cobordant* if there exists a tube $(Y \times I, Z, \xi)$ such that Z is isomorphic to $X_0 \times I$, and $Z \cap (Y \times i) = X_i \times i$, and $\xi | X_i \times i = \xi_i \times i$. A tube is *trivial* if its cell bundle is a trivial bundle.

The purpose of this article is to prove the following theorem.

THEOREM A. *Let* (Σ^{11}, Σ^7) *be the standard pair of PL spheres. There are PL trivial normal cell bundles* ξ_0 *and* ξ_1 *on* Σ^7 *such that the tubes* $(\Sigma^{11}, \Sigma^7, \xi_0)$ *and* $(\Sigma^{11}, \Sigma^7, \xi_1)$ *are not PL I-cobordant, nor are they topologically isomorphic.*

A consequence of Theorem A and Hudson's isotopy extension theorem ((5)) is:

COROLLARY B. *There is a PL embedding* $f: \Sigma^7 \times I^4 \to \Sigma^7 \times I^4$ *which leaves* $\Sigma^7 \times 0$ *fixed, and which is not PL isotopic to any embedding taking each fibre* $y \times I^4$ *onto a fibre* $z \times I^4$.

2. *Terminology.* Three categories will be used:

(1) the *topological category* $\mathscr{T}$ of spaces and continuous maps;
(2) the *piecewise linear category* $\mathscr{P}$ of polyhedra and piecewise linear maps;
(3) the *smooth category* $\mathscr{S}$ smooth manifolds and smooth maps.

Euclidean n-space R^n, the closed half space E^n and the closed interval $I = [0, 1]$ may be considered as objects in each of the three categories. An *n-manifold* in one of the categories $\mathscr{T}$ or $\mathscr{P}$ is a paracompact object that is locally isomorphic to E^n. The *n-disc* is the closed unit ball $D^n \subset R^n$, and is an object in $\mathscr{T}$ and $\mathscr{S}$. Its boundary is $\partial D^n = S^{n-1}$. The n-cube I^n is the Cartesian product of n copies of $[-1, 1]$; its boundary is Σ^{n-1}. A *cell* is an isomorph of I^n or D^n, depending on the category, for some appropriate n. Similarly, an *n-sphere* is an isomorph of S^n or Σ^n.

Embedding, isotopy, etc., have their usual interpretations in each category. We assume submanifolds to be closed as subsets.

Fix one of the categories; let Y be an object and $p: E \to B$ a map. The triple (p, E, B) is a *Y-bundle* if it is locally trivial in the category, with fibre Y. That is, B is covered by open sets U for each of which there exists an isomorphism of the category $p^{-1}U \to U \times Y$ making commutative the usual diagram

$$\begin{array}{ccc} p^{-1}U & \longrightarrow & U \times Y \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

For the theory of such bundles, see (4).

The notions of *tubular neighbourhood* and of *isomorphism* and *I-cobordism* of tubular neighbourhoods, defined in section 1, make sense in each of the three categories. Another equivalence relation that may hold between two tubes (Y, X_i, ξ_i) $(i = 0, 1)$, is that of *h-cobordism*: there exists an h-cobordism $Z \subset Y \times I$ between the manifolds $X_0 \times 0$ and $X_1 \times 1$ and a normal cell bundle ξ on Z reducing to $\xi_i \times i$ on $X_i \times i$.

Let (Y, X, ξ) be a tube and $h\colon Y \to Y'$ an embedding. There is a unique normal cell bundle $h(\xi)$ on $h(X)$ in $h(Y)$ making $h\colon (Y, X, \xi) \to (h(Y), h(X), h(\xi))$ an isomorphism of tubes.

Let (Y, X, ξ) be a smooth tube; let $\xi = (p, E, X)$. A *triangulation* of ξ is a PL cell bundle $p\colon \hat{E} \to \hat{X}$, where $\hat{E}$ and $\hat{X}$ are smooth triangulations of E and X. The triangulation $\hat{E}$ always extends to a smooth triangulation $\hat{Y}$ of Y; the resulting PL tube $(\hat{Y}, \hat{X}, \hat{\xi})$ is called a *triangulation of* (Y, X, ξ). It follows from the triangulation theorems for smooth bundles proved in (4) that triangulations of smooth tubes always exist, and are unique up to PL isomorphism. In this paper all that is needed is the existence of triangulations for trivial tubes, and this is obvious.

3. *Geometrical facts.* We list in this section various theorems in differential and piecewise linear topology that are needed to prove Theorem A.

PROPOSITION 1 (Whitehead ((11))). *Let C be a collapsible polyhedron in a piecewise linear manifold M^m. Then C has a regular neighbourhood, and any regular neighbourhood of C is a PL m-cell.*

PROPOSITION 2 (Munkres ((8))). *A smooth manifold which can be smoothly triangulated by a PL cell is a smooth cell.*

PROPOSITION 3 (Hirsch ((1,2))). *Let M^m be a smooth manifold with a fixed compatible PL structure. Let $X \subset M$ be a PL $m-1$ dimensional submanifold lying on the boundary of a PL m-dimensional submanifold. Let $A \subset X$ be a closed subset having a neighbourhood in X which is a smooth submanifold of M. Then there exists a smooth submanifold $X' \subset M$ which can be smoothly triangulated by X, and such that $X \cap X'$ is a neighbourhood of A in X.*

Let $M \subset S^q$ be a smooth submanifold. Call M *depressible* if there exists a smooth embedding $F\colon M \times I \to S^q \times I$ such that $F(M \times 0) = M \times 0$, and $F(M \times 1) \subset S^{q-1} \times 1$.

PROPOSITION 4 (Levine ((7))). *There is a smoothly embedded* 7*-sphere in R^{11} which is not depressible, and which has a trivial smooth normal bundle.*

Proof. In (7) it is proved that $\tilde{\Sigma}^{11,7}$ has order 60, while $\tilde{\Sigma}^{10,7}$ has order 3 (where $\tilde{\Sigma}^{n,m}$ is the group of embedded m-spheres in S^n modulo the subgroup bounding framed manifolds). A non-zero $\alpha \in \Sigma^{11,7}$ of order 5 is represented by an indepressible embedded

$M^7 \subset R^{11}$. Moreover, M must have trivial normal bundle ν since ν is represented by the image of α under a homomorphism $\tilde{\Sigma}^{11,7} \to \Pi_6(SO_4) = Z_{12} \oplus Z_{12}$.

PROPOSITION 5. *A smooth submanifold $M \subset S^q$ is depressible provided there exists a smooth q-cell $D \subset S^q \times I$ with $M \times 0 \subset \partial D$.*

Proof. Deform D diffeotopically in $S^q \times I$, leaving ∂D fixed, until a small concentric q-cell D' coincides with a hemisphere of $S^q \times 1$ bounded by $S^{q-1} \times 1$. Since $D - \text{int} D'$ is diffeomorphic to $S^{q-1} \times I$, it is easy to construct the required embedding of $M \times I$.

PROPOSITION 6. *A smooth submanifold $M \subset S^q$ is depressible provided there exists a smooth triangulation of $S^q \times I$, a PL submanifold $X \subset S^q \times I$ of dimension q, and a PL q-cell $B \subset X$, such that:*

(*a*) *X lies on the boundary of a PL submanifold of dimension $q+1$;*

(*b*) *$M \times 0 \subset \partial B$;*

(*c*) *$M \times 0$ has neighbourhoods in B and X that are smooth submanifolds.*

Proof. Apply Proposition 3 twice, first to get a smooth triangulation $f: X \to Y =$ a smooth submanifold of $S^q \times I$, and then to obtain a smooth triangulation $g: f(B) \to D$ $=$ a smooth submanifold of Y, such that f and g leave fixed a neighbourhood of $M \times 0$ in X. By Proposition 2, D must be a smooth q-cell, and the proof is completed by applying Proposition 5.

PROPOSITION 7 (Hirsch–Zeeman; see also Irwin ((6))). *Let W be a contractible PL m-manifold and $K \subset \partial W$ a compact polyhedron. If* $\dim K \leqslant m-4$, *there exists a PL m-cell C such that $K \subset C \subset W$.*

Proof. See the Engulfing theorem in (3).

PROPOSITION 8 (Zeeman ((12))). *Let $A, B \subset \Sigma^n$ be PL embedded k-spheres. If $n \geqslant k+3$, there exists a PL homeomorphism of Σ^n taking A onto B.*

PROPOSITION 9 (Smale ((10))). *A PL manifold of dimension $\geqslant 6$ is a PL cell if it is compact and contractible and has a simply connected boundary.*

4. *Sections of tubes.* Let (Y, X, ξ) be a tube, with $\xi = (p, E, X)$. A *section* of ξ is a cross-section $f: X \to \partial E$ of the sphere bundle $(p, \partial E, X)$. The section f is *engulfable* provided there exists a contractible open set W of $Y - \text{int} E$ containing $f(X)$. If the tube is *PL* and f is a *PL* section, then f is *shrinkable* if there exists a collapsible polyhedron C such that $f(X) \subset C \subset Y - \text{int} E$.

PROPOSITION 10. *The existence of an engulfable section is an isomorphism invariant of tubes. The existence of a shrinkable section is a PL isomorphism invariant of PL tubes.*

Proof. Trivial.

PROPOSITION 11. *Let (Y, X, ξ) be a PL tube with X compact.*

(*a*) *Every shrinkable section is engulfable.*

(*b*) *If* $\dim Y - \dim X \geqslant 4$, *every PL engulfable section is shrinkable.*

Proof. Since a regular neighbourhood of a collapsible set is contractible, (*a*) is obvious. To prove (*b*), let $f: X \to \partial E$ be a *PL* engulfable section, and $W \subset Y - \text{int} E$ a contractible open set containing $f(X)$. The required collapsible polyhedron exists by virtue of Proposition 7.

PROPOSITION 12. *Let* $M \subset S^q$ *be a smooth submanifold with a smooth normal cell bundle* ν. *Let* $(\hat{S}^q, \hat{M}, \hat{\nu})$ *be a triangulation of* (S^q, M, ν). *If* ν *has an engulfable section, and*

$$q - \dim M \geqslant 4,$$

then M *is depressible.*

Proof. Assume that ν has an engulfable section. By Proposition 10, $\hat{\nu}$ has a shrinkable section $f: \hat{M} \to \partial\hat{E}$, where $\nu = (p, E, M)$ and $\hat{\nu} = (p, \hat{E}, \hat{M})$. By Proposition 7 there is a PL q-cell $C \subset S^q - \operatorname{int} E$ with $f(M) \subset C$. We may assume that $C \cap \partial E$ is a neighbourhood in ∂E of $f(M)$. Since E is a smooth submanifold of S^q, C is smooth in a neighbourhood of $f(M)$. Proposition 12 follows from Proposition 6 (or 3).

Remark. Using Proposition 3 it is easy to show that M is actually diffeotopic to a submanifold of S^{q-1} (i.e. M is *compressible* in the sense of (3)).

5. *Proof of Theorem A.* Let $M \subset S^{11}$ be the smoothly embedded 7-sphere of Proposition 4, and let ν be its smooth normal 4-cell bundle. Let $(\hat{S}^{11}, \hat{M}, \hat{\nu})$ be a triangulation (see (2)) of (S^{11}, M, ν). By the smooth triangulation theorems of Whitehead ((11)) and Zeeman's unknotting theorem (Proposition 8), there is a PL normal cell bundle ξ_0 on $\Sigma^7 \subset \Sigma^{11}$ and a PL isomorphism $f: (\hat{S}^{11}, \hat{M}, \hat{\nu}) \to (\Sigma^{11}, \Sigma^7, \xi_0)$. By Proposition 12, $(\hat{S}^{11}, \hat{M}, \hat{\nu})$ does not have an engulfable section. Therefore neither has $(\Sigma^{11}, \Sigma^7, \xi_0)$, by Proposition 10. On the other hand Σ^7 obviously has another PL normal cell bundle in Σ^{11}, say ξ_1, which does have an engulfable section, and even a shrinkable one. *Therefore* $(\Sigma^{11}, \Sigma^7, \xi_0)$ *and* $(\Sigma^{11}, \Sigma^7, \xi_1)$ *are not topologically isomorphic.* Moreover, both ξ_0 and ξ_1 are trivial bundles.

It remains to prove that ξ_0 is not PL I-cobordant to ξ. This will follow from:

PROPOSITION 13. *Let* (S^q, M, ν) *be a smooth tube with a triangulation* $(\hat{S}^q, M_0, \xi_0)$ *which is* PL *h-cobordant to a tube having a shrinkable section. If either* (*a*) $\dim M \leqslant q-3$, *or* (*b*) *the h-cobordism is a* PL *I-cobordism, then* M *is depressible.*

Proof. Let $(\hat{S}^q, M_1, \xi_1)$ be a PL tube with a shrinkable section $f: M_1 \to \partial E_1$, where $\xi_i = (p_i, E_i, M_i)$ for $i = 0, 1$. Let $Z \subset \hat{S}^q \times I$ be a PL h-cobordism between $M_0 \times 0$ and $M_1 \times 1$, and let ξ be a PL normal cell bundle on Z which extends $\xi_0 \times 0$ and $\xi_1 \times 1$. Put $\xi = (p, E, Z)$. Thus $Z \cap (S^q \times i) = M_i \times i$. Without loss of generality we assume that $E \cap (S^q \times [0, \frac{1}{2}]) = E_0 \times [0, \frac{1}{2}]$, so that E is smooth in a neighbourhood of $E_0 \times 0$. The shrinkable section $f: M_1 \times 1 \to \partial E_1 \times 1$ extends to a PL section $F: Z \to \partial E$, because $M_1 + 1$ is assumed to be a deformation retract of Z, and the covering homotopy theorem is valid for PL bundles. Since f is shrinkable, there is a collapsible $C \subset S^q \times 1$ such that $C \cap \partial E = F(M_1 \times 1)$. Let $K = C \cup F(Z)$. It is clear that K is contractible, and that K is collapsible if Z is PL homeomorphic to $M_1 \times I$.

Let $T \subset E$ be the total space of the open cell bundle corresponding to ξ. That is, $T = E - cl(S^q \times I - E)$. Consider the submanifold $X \subset \hat{S}^q \times I$ defined by $X = (E \cup S^q \times 1) - T$. Observe that $K \subset X$. Let $B \subset X$ be a regular neighbourhood of K meeting $\partial X = \partial E_0 \times 0$ in a neighbourhood of $K \cap \partial E = F(M_0 \times 0)$. If K is collapsible, then B is a PL cell. If K is contractible, so is B, and if $\dim M \leqslant q-3$, then $\dim K \leqslant \dim X - 3$ and hence ∂B is simply connected. In this case if $\dim X \geqslant 6$ then B is a PL cell by Proposition 9, while if $\dim X \leqslant 5$ the theorem is trivial using standard embedding theorems. Thus we have a q-cell $B \subset X$; X clearly lies on the boundary of $(S^q \times I) - T$. Moreover, both B and X are smooth in a neighbourhood U of $F(M_0 \times 0)$ in B. Let $g: M_0 \times 0 \to \partial E_0 \times 0$ be a smooth section of ν_0 whose image lies in U. Then $g(M_0 \times 0)$ lies in the smooth part of B. Proposition 6 now implies that $g(M_0)$, and hence also M_0, is depressible, proving Proposition 11.

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