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Conformal-Geodesics-Preserving Local Diffeomorphisms and Their Holographic Interpretation

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CONFORMAL-GEODESICS-PRESERVING LOCAL DIFFEOMORPHISMS AND THEIR HOLOGRAPHIC INTERPRETATION

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by

Tzu-Mo Kuo

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Contents

Abstract

Conformal-Geodesics-Preserving Local Diffeomorphisms and Their Holographic Interpretation

by

Tzu-Mo Kuo

We study unparametrized conformal geodesics, or called conformal circles, and study local diffeomorphisms mapping conformal geodesics to conformal geodesics in pseudo-Riemannian conformal manifolds. We show that such local diffeomorphisms are conformal local diffeomorphisms. Our result extends the result of Yano and Tomonaga. We also present a holographic interpretation for our result on Poincaré-Einstein manifolds. The proofs take suitable variations of conformal geodesics.

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Chapter 1

Introduction

Riemannian geodesics, as fundamental geometric objects, are often considered when studying Riemannian structures. One classic problem concerning geodesics and Riemannian structures is the following: If there is a diffeomorphism that maps geodesics to geodesics, is it an isometry? The answer is negative due to affine transformations on Euclidean spaces. In general, one may need to further assume irreducible Riemannian manifolds for the diffeomorphism to be an isometry [1, 2]. The parallel problems for CR manifolds [3] and conformal manifolds [4, 5] are affirmative in some sense.

In the context of a pseudo-Riemannian conformal manifold $(M^n, [g])$ with $n \geq 2$, a distinguished family of curves known as conformal geodesics or called conformal circles emerges. These curves satisfy a third-order differential equation for non-null conformal geodesics [6]. The derivation of these conformal geodesics is based on various perspectives of conformal manifolds, including Cartan geometry [7], the standard tractor bundle [8], and the Poincaré-Einstein manifold [9]. The persepectives are based on the conformal model, Möbius sphere $S^{(p,q)}$. For the Riemann signature, by identifying the sphere with the Euclidean space, the conformal geodesics are either straight lines or planar circles [10]. In the context of the Poincaré ball B^{n+1}, each circle on the boundary $S^n = \partial B^{n+1}$ can be orthogonally extended to form a totally geodesic surface within B^{n+1}. Based on the case of the Riemannian conformally flat model, each conformal geodesic in a Riemannian conformal manifold $(M, [g])$ can be formally extended to an asymptotically totally geodesic surface in the Poincaré-Einstein space (M_+, g_+) by holographic construction [9] where the term holography comes from physics, e.g., [11]. We use the term "holography" to mean the geometry from Poincaré-Einstein space (M_+, g_+).

In this dissertation, we solve a classical problem: if a local diffeomorphism f between pseudo-Riemannian conformal manifolds $(M^n, [g])$, $(N^n, [h])$ with the same metric signature maps conformal geodesics to conformal geodesics, then it is a conformal local diffeomorphism provided some conditions on f. We also consider the parallel problem in the holographic settings $F\colon M_+ \longrightarrow N_+$. The problem for a Riemannian conformal manifold can be traced back to Carathéodory [4]. He showed a bijection on $\mathbb{R}^2$, which doesn't need to be continuous, that maps straight lines (resp. circles) to straight lines (resp. circles) is a conformal transformation. Later on, K. Yano and Y. Tomonaga [5, 12] showed that an infinitesimal transformation on a Riemannian conformal manifold is a conformal killing vector field if and only if it carries unit-speed conformal circles to unit-speed conformal circles where the unit speed is with respect to a metric in the conformal class. The problem is also discussed in terms of distinguished curves by Cartan geometry [13].

The conformal geodesic equation can be derived from the integral curve equation of a vector field on the normal Cartan geometry modeled on $S^{(p,q)}$. The Riemannian geodesic equation can also be derived in the sense of Cartan geometry. In Chapter 2, we starts from considering model manifolds of Riemannian manifolds and conformal manifolds. We introduce geodesics as the projection of integral curves of constant horizontal vector fields derived from the Maurer-Cartan form. In Chapter 3, we review the backgrounds of principal bundles, Cartan geometry, and the category equivalence between Riemannian manifolds, conformal manifolds and Cartan geometries. In subsection 3.4.1, we derive the integral equation for Riemannian geodesics. In subsection 3.6.1, we review the integral equation for conformal geodesics and review their geometric properties. In Chapter 4, we review some background of Poincaré-Einstein manifolds and extend the results from Fine and Herfray to pseudo-Riemannian setting. In Chapter 5, we show our main theorems, Theorem 5.1.1 for the classical problem of conformal geodesics, and Theorem 5.2.5 for the parallel problem in the holographic settings. Our proof is motivated by the work from Yano and Tomonaga.

Chapter 2

Introduction to Model Manifolds

In this chapter, we review model manifolds and geodesics in the Riemannian model and the conformal model. As mentioned in Chapter 1, we consider geodesics as the quotient of integral curves in the transformation group of the model. Particularly, Theorem 2.1.1 shows constant velocity straight lines in the Euclidean space is the quotient of integral curves, and Theorem 2.2.7 is the parallel result for conformal geodesics in the Riemannian sphere S^n. The concepts from model manifolds will be further generalized to Cartan geometry in Chapter 3.

A manifold M is called a **model** if it's a homogeneous G-space G/H where the manifold M is of some geometric structure, the Lie group G preserves the structure, and H is a subgroup of G.

Example 2.0.1. For Riemannian Manifolds, the automorphism group G is the isometry group.

1. The Euclidean space $(\mathbb{R}^n, g_{std})$. Its isometry group is the Euclidean group $\text{Euc}(n) = \mathbb{R} \rtimes O(n)$ and $\mathbb{R}^n \cong \mathbb{R} \rtimes O(n)/O(n)$.

2. The hyperbolic space (H^n, g_H). Its isometry group is $O_+(n+1,1)$ and $H^n \cong O_+(n+1,1)/O(n+1)$.

Example 2.0.2 ([14, 1.6.2; 15, Chapter 2])**.** For conformal manifolds, the group G is the conformal automorphism group. The model is the Möbius sphere $S^{(p,q)} := C/\mathbb{R}^*$ with the conformal class induced from the standard inner product of $\mathbb{R}^{p,q}$ where C is the light cone in $\mathbb{R}^{p+1,q+1}$ and $n = p+q \geq 3$. The conformal transformation group of $S^{p,q}$ is $PO(p+1,q+1)$ and $S^{(p,q)} \cong PO(p+1,q+1)/PP_{line}$ where $O(p+1,q+1)$ is the orthogonal group of $\mathbb{R}^{p+1,q+1}$ and P_{line} is the stabalizer of a null line in $\mathbb{R}^{p+1,q+1}$. The letter P before both of the groups O and P_{line} is the quotient by $\pm id_{\mathbb{R}^{p,q}}$. Particularly, $S^{(n,0)} \cong S^n$ and $PO(n+1,1) \cong O_+(n+1,1)$ where the group O_+ is the time preserving orthogonal group on $\mathbb{R}^{n+1,1}$; that is, with respect to the standard basis $(t, \mathbf{x})$, the $(0,0)$ components of $O_+(n+1,1)$ matrices are all positive.

Let $\pi\colon G \longrightarrow G/H = M$ be the quotient map of a model M. Recalling that the Maurer-Cartan form w on a Lie group G is defined by $\forall p \in G$

$$\begin{aligned} w_p\colon T_pG &\longrightarrow \mathfrak{g} \\ V &\longmapsto dL_{p^{-1}}(V) \end{aligned} \tag{2.0.1}$$

where $dL_{p^{-1}}$ is the differential of the left action $L_{p^{-1}}\colon G \longrightarrow G$, $g \longmapsto p^{-1}g$. Since the inverse of $L_{p^{-1}}$ is L_p, the map (2.0.1) is an isomorphism. The isomorphism property lets us introduce a vector field $V_X = w^{-1}(X)$ on G for any given element $X \in \mathfrak{g}$. The following Propositions 2.0.3 and 2.0.4 give us the preliminary notions of geodesics on model manifolds.

Proposition 2.0.3. *Let $\mathfrak{h}$ be the Lie algebra of H. Then,*

$$d\pi(V_X) = 0 \qquad \forall X \in \mathfrak{h}. \tag{2.0.2}$$

Proof. Recalling the fundamental vector field is defined by $\zeta_X(p) := \frac{d}{dt}\Big|_{t=0} p \cdot e^{tX}$ for $X \in \mathfrak{h}$, the result follows from $w(\zeta_X) = X$ and $d\pi(\zeta_X) = 0$. □

Proposition 2.0.4. *Let $X \in \mathfrak{g}$ and $\gamma_X : I \longrightarrow G$ be the integral curve of $V_X = w^{-1}(X)$ with the initial condition $g \in G$. Then,*

$$\gamma_X(t) = g \cdot e^{tX} \tag{2.0.3}$$

where $\cdot$ is the group action of G.

Proof. Note that

$$\gamma_X'(t) = \frac{d}{ds}\Big|_{s=t} g \cdot e^{sX} = \frac{d}{ds}\Big|_{s=0} g \cdot e^{(t+s)X} = g \cdot e^{tX} \frac{d}{ds}\Big|_{s=0} e^{sX}. \tag{2.0.4}$$

Recalling the definition of the Maurer-Cartan form, then

$$w_{ge^{tX}}(\gamma_X'(t)) = \frac{d}{ds}\Big|_{s=0} L_{(ge^{tX})^{-1}} \left(g \cdot e^{tX} \cdot e^{sX}\right) = X. \tag{2.0.5}$$

□

Assume the Lie algebra $\mathfrak{g}$ is of the form $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$, where $\mathfrak{n}$ is a subalgebra. Because of Proposition 2.0.3, we consider the integral curve $\gamma_X \subseteq G$ introduced from $X \in \mathfrak{n}$ in Section 2.1 and Section 2.2.

§ 2.1 Riemannian Geodesics in the Euclidean Space

Theorem 2.1.1. *Let* $\pi\colon Euc(n) \to \mathbb{R}^n$ *be the quotient map. Let* $X \in \mathbb{R}^n \subseteq \mathfrak{euc}(n)$ *where* $\mathfrak{euc}(n) = \mathbb{R}^n \oplus \mathfrak{o}(n)$ *is the Lie algebra of* $Euc(n)$*. Let* $\gamma_X\colon I \to Euc(n)$ *be the integral curve of* $w^{-1}(X)$*. The projection* $\pi(\gamma(t))$ *is a constant velocity straight line in* $\mathbb{R}^n$*. Conversely, any constant velocity straight line is the projection of an integral curve induced from some* $X \in \mathbb{R}^n \subseteq \mathfrak{euc}(n)$*.*

Proof. Assume $g = \begin{pmatrix} 1 & 0 \\ \mathbf{b} & A \end{pmatrix}$. Then,

$$\gamma_X(t) = g \cdot e^{tX} = \begin{pmatrix} 1 & 0 \\ \mathbf{b} & A \end{pmatrix} \left[\mathbb{I} + t \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 \\ \mathbf{b} + tAX & A \end{pmatrix}. \tag{2.1.1}$$

So, $\pi(\gamma_X(t)) = \mathbf{b} + tAX$. The converse direction can be easily seen from the proof we just did. □

§ 2.2 Conformal Geodesics in the Euclidean Space

Let $n = p + q \geq 3$. We've seen Example 2.0.2 that $S^{(p,q)}$ is a $PO(p+1, q+1)$-homogeneous space and its conformal transformation group is $PO(p+1, q+1)$. In this section, we review the structure of the Lie algebra $\mathfrak{o}(p+1, q+1)$ of PO and the structure of PP_{line}. We also review the conformal transformation group of $\mathbb{R}^{p,q}$. Finally, while $n = p$ and identifying $\mathbb{R}^n$ and $S^n \setminus \{pt\}$, we derive that the conformal geodesics in $\mathbb{R}^n$ are either straight lines or circles. In this section, $\langle v, w\rangle$ means the standard inner product of v, w in $\mathbb{R}^{p,q}$.

Proposition 2.2.1 ([14] Proposition 1.6.2)**.** *The Möbius sphere $S^{(p,q)}$ is diffeomorphic to $PO(p+1,q+1)/PP_{line}$.*

Proof. (Sketched) Consider the quotient map of the light cone

$$\pi : C \xrightarrow{\pi} \mathbb{R}P^{n+1}$$

where $C = \{v \in \mathbb{R}^{p+1,q+1} \mid \langle v, v\rangle = 0\}$. Since $\text{Im}(\pi) = S^{p,q}$ and the group action of $PO(p+1,q+1)$ is transitively on the $\text{Im}(\pi)$, then $S^{(p,q)} \cong PO(n+1,1)/PP_{line}$. □

The Lie algebra structure of $\mathfrak{o}(p+1,q+1)$ can be precisely described if we introduce a new coordinate on $\mathbb{R}^{p+1,q+1}$ such that its standard inner product is of the following form with respect to the new coordinate

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbb{I}_{p,q} & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{where } \mathbb{I}_{p,q} = \begin{pmatrix} \mathbb{I}_p & 0 \\ 0 & -\mathbb{I}_q \end{pmatrix}. \tag{2.2.1}$$

For instance, let

$$e_- = \frac{1}{\sqrt{2}}(\hat{e}_0 - \hat{e}_{n+1}), \quad e_i = \hat{e}_i \quad \text{for } 1 \le i \le n, \quad e_+ = \frac{-1}{\sqrt{2}}(\hat{e}_0 + \hat{e}_{n+1})$$

where $(\hat{e}_i)_{i=0}^{n+1}$ is the standard basis for $\mathbb{R}^{p+1,q+1}$. We call (e_-, e_i, e_+) the **light cone basis**.

Proposition 2.2.2 ([14] 1.6.3)**.** *Denote the Lie algebra $\mathfrak{o}(p+1,q+1)$ by $\mathfrak{g}$. Let $n = p+q$.*

Then, any element in $\mathfrak{g}$ *with respect to the light cone basis is of the form*

$$\begin{pmatrix} a & Z & 0 \\ X & A & -Z^t \\ 0 & -X^t & -a \end{pmatrix},$$

where $a \in \mathbb{R}$, $X \in \mathbb{R}^n$, $Z \in \mathbb{R}^{n*}$ *and* $A \in \mathfrak{o}(p,q)$. *Therefore,* $\mathfrak{g} \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *where* $\mathfrak{g}_{-1} \cong \mathbb{R}^n$, $\mathfrak{g}_0 \cong \mathfrak{o}(p,q) \oplus \mathbb{R}$ *and* $\mathfrak{g}_1 \cong \mathbb{R}^{n*}$. *Their Lie bracket relations are as follows*

$$\mathfrak{g}_0 \times \mathfrak{g}_{-1} \longrightarrow \mathfrak{g}_{-1} \qquad [(A,a),X] = AX - aX$$

$$\mathfrak{g}_0 \times \mathfrak{g}_0 \longrightarrow \mathfrak{g}_0 \qquad [(A,a),(B,b)] = ([A,B],0)$$

$$\mathfrak{g}_0 \times \mathfrak{g}_1 \longrightarrow \mathfrak{g}_1 \qquad [(A,a),Z] = aZ - ZA$$

$$\mathfrak{g}_{-1} \times \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0 \qquad [X,Z] = (XZ - (XZ)^t - ZX).$$

Definition 2.2.3. The group P_{line} is defined to stabalize the null line generated by e_-.

Proposition 2.2.4 ([14] Proposition 1.6.3)**.** *With respect to the light cone basis, any element in* P_{line} *is of the form*

$$\begin{pmatrix} \lambda & \lambda Z & \frac{-\lambda}{2}\langle Z^t, Z^t\rangle \\ 0 & C & -CZ^t \\ 0 & 0 & \lambda^{-1} \end{pmatrix}, \tag{2.2.2}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$, $C \in O(p,q)$ *and* $Z \in \mathbb{R}^{n*}$. *Furthermore,* $PP_{line} = G_0 \ltimes P_+$ *where* G_0 *is the group for* $Z = 0$ *and* $\lambda > 0$, *and* $P_+ = exp\,(\mathfrak{g}_1)$ *which corresponds to* $\lambda = 1$ *and* $C = \mathbb{I}_n$. *Their Lie algebras are*

$$\mathfrak{p}_{line} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \quad \mathfrak{g}_0, \quad \mathfrak{p}_+ = \mathfrak{g}_1$$

for PP_{line}, G_0 and P_+ respectively. Moreover, the adjoint action of G_0 on $\mathfrak{g}_{-1}$ is $X \overset{(\lambda,C)}{\mapsto} \lambda^{-1}CX$ and G_0 preserves the grading of $\mathfrak{g}$ under the same action.

Remark 2.2.5. Due to the G_0 action on $\mathfrak{g}_{-1}$, the group G_0 is isomorphic to conformal orthogonal group $CO(p,q)$ [14, proof of Proposition 1.6.3] where

$$CO(p,q) = \{A \in GL(n) |\ \exists c > 0 \text{ such that } \langle Av, Aw\rangle = c\langle v, w\rangle\ \forall v, w \in \mathbb{R}^{p,q}\}$$

and the inner product $\langle , \rangle$ is the standard one on $\mathbb{R}^{p,q}$. Henceforth, its Lie algebra is

$$\mathfrak{g}_0 \cong \mathfrak{co}(p,q) \cong \mathbb{R}^n \oplus \mathfrak{o}(p,q).$$

To know the conformal geodesic in the model case, we need to review the conformal transformation group of $\mathbb{R}^{p,q}$ for $n = p + q \geq 3$.

Proposition 2.2.6. *Let $\mathcal{U}$ be an open set in $\mathbb{R}^{p,q}$. Then, each conformal transformation $\phi\colon \mathcal{U} \to \mathbb{R}^{p,q}$ is a composition of the following conformal transformations*

1. *Translation $Y \mapsto Y + X$, $X \in \mathbb{R}^n \cong \mathfrak{g}_{-1}$;*
2. *Conformal orthogonal transformation $X \mapsto \lambda^{-1}CX$, $C \in O(p,q)$ and $\lambda > 0$;*
3. *Special conformal transformation*

$$X \mapsto \frac{1}{1 + ZX + \frac{1}{4}\langle X, X\rangle\langle Z, Z\rangle}\left(X + \frac{\langle X, X\rangle}{2} Z^t\right) \tag{2.2.3}$$

   where $Z \in \mathbb{R}^{n*} \cong \mathfrak{g}_1$.

The 3 is the composition of inversion $X \mapsto Y = \frac{X}{||X||^2}$, translation $Y \mapsto W = Y + \frac{1}{2}Z^t$ and inversion $W \mapsto \frac{W}{||W||^2}$ when X is not null, where $||X||$ is the standard inner product of $\mathbb{R}^{p,q}$ on X.

Identify $\mathfrak{g}_{-1} \cong \mathbb{R}^{p,q}$ *and* $S^{p,q} \setminus \{pt\}$ *by*

(2.2.4)
$$\begin{aligned} i\colon \mathfrak{g}_{-1} &\longrightarrow S^{p,q} = C/\mathbb{R}^* \\ X &\mapsto [exp(X)e_-] = [(1, X, -\frac{1}{2}\langle X, X\rangle)] \end{aligned}$$

where C *is the light cone in* $\mathbb{R}^{p+1,q+1}$, $(1, X, -\frac{1}{2}\langle X, X\rangle)$ *is with respect to the light cone basis, and* [ ] *denotes an equivalence class in* $C/\mathbb{R}^*$. *Then,*

1. $exp(Y) \in exp(\mathfrak{g}_{-1}) \subseteq O(p,q)$ *induces a translation on* $\mathbb{R}^{p,q}$*;*
2. $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \in O(p,q)$ *with* $C \in O(p,q)$ *and* $\lambda > 0$ *induces a conformal orthogonal transformation on* $\mathbb{R}^{p,q}$*;*
3. $exp(Z) \in exp(\mathfrak{g}_1) \subseteq O(p,q)$ *induces a special conformal transformation on an open set of* $\mathbb{R}^{p,q}$.

Proof. The first 1, 2, 3 are in [15, Theorem 1.9 and Theorem 2.9]. The matrix $exp(Y)$ in $O(p+1, q+1)$ for $Y \in \mathfrak{g}_{-1}$ is

$$exp(Y) = exp\begin{pmatrix} 0 & 0 & 0 \\ Y & 0 & 0 \\ 0 & -Y & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ Y & \mathbb{I} & 0 \\ -\frac{1}{2}\langle Y, Y\rangle & -Y & 1 \end{pmatrix}.$$

Let $Y \in \mathfrak{g}_{-1}$. Therefore, for $X \in \mathfrak{g}_{-1}$

$$exp(Y)exp(X) = exp(Y + X)$$

which implies the translation. For 5, we have

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} (exp(X))e_- = [(\lambda, CX, -\frac{1}{2\lambda}\langle X, X\rangle)]$$

$$= [(1, \lambda^{-1}CX, -\frac{1}{2\lambda^2}\langle X, X\rangle]$$

which gives a conformal orthogonal transformation. Finally,

(2.2.5)

$$\begin{pmatrix} 1 & Z & -\frac{1}{2}\langle Z^t, Z^t\rangle \\ 0 & \mathbb{I} & -Z^t \\ 0 & 0 & 1 \end{pmatrix} (exp(X))e_- = \left[\left(1 + ZX + \frac{1}{4}X||^2||Z||^2,\ X + \frac{||X||^2}{2}Z^t,\ \frac{-1}{2}||X||^2\right)\right]$$

$$= \left[\left(1, \frac{1}{1 + ZX + \frac{1}{4}||X||^2||Z||^2}\left(X + \frac{||X||^2}{2}Z^t\right), *\right)\right]$$

where $||X||^2$ and $||Z||^2$ mean the inner products of X and Z themselves, and the middle component is in the form of (2.2.3). □

Theorem 2.2.7. *Let $\pi\colon O_+(n+1,1) \longrightarrow S^n$ be the quotient map. Given $X \in \mathfrak{g}_{-1}$ and let $\gamma_X\colon I \longrightarrow O_+(n+1,1)$ be an integral curve of $V_X = w^{-1}(X)$ where w is the Maurer-Cartan form of $O_+(n+1,1)$. Identify $\mathbb{R}^n$ and $S^n \setminus \{pt\}$ by $i\colon R^n \longrightarrow S^n \setminus \{pt\}$ from (2.2.4). Then, $\pi(\gamma_X(t))$ is either a planar circle or a straight line in $\mathbb{R}^n$.*

Proof. Recall the diffeomorphism of a homogeneous space (see e.g. [16, Theorem 21.18])

(2.2.6)
$$\mathcal{I}: O_+(n+1,1)/P_{ray} \longrightarrow S^n = C/\mathbb{R}^*$$
$$g \cdot P \mapsto [ge_-].$$

Assume $\gamma_X(0) = \mathbb{I}$ the identity matrix in O_+. Then, $\gamma_X(t) = e^{tX}$ by Proposition 2.0.4. Therefore,

$$\mathcal{I}(\gamma_X(t)) = \left[\left(1, tX, -\frac{1}{2}\langle X, X\rangle t^2\right)\right] = i(tX)$$

which is a straight line through the origin in $\mathbb{R}^n$. Now consider $\gamma_X(t) = e^Z \cdot e^{tX}$ where $Z \in \mathfrak{g}_1$ with $ZX = 0$, $\langle X, X\rangle = 1$ and $\langle Z, Z\rangle = 1$. Then,

$$\mathcal{I}(\gamma_X(t)) = i\left(\frac{1}{1+\frac{t^2}{4}}\left(tX + \frac{t^2}{2}Z^t\right)\right)$$

where the equation is because of (2.2.5). The curve inside the argument of i is the circle centered at Z^t with the radius 1 and it is in the plane spanned by X and Z^t. It is because

$$\begin{aligned}\left|Z^t - \frac{1}{1+\frac{t^2}{4}}\left(tX + \frac{t^2}{2}Z^t\right)\right|^2 &= |Z^t|^2\left(1 - \frac{t^2}{2\left(1+\frac{t^2}{4}\right)}\right)^2 + \frac{t^2}{1+\frac{t^2}{4}}|X|^2 \\ &= 1.\end{aligned}$$

Observe that $s(t) = g \cdot \gamma_X(t)$ is still an integral curve of $w^{-1}(X)$ for all given $g \in O_+(n+1, 1)$. Since $O_+(n+1, 1)$ is the conformal transformation group of $\mathbb{R}^n$ by Proposition 2.2.6, we know the conformal geodesics of $\mathbb{R}^n$ are conformal transformations of straight lines or planar circles which are still straight lines or planar circles. □

Chapter 3

Cartan Geometry

In Section 3.1, we review principal bundles, principal connection, and the curvature forms defined from the connections. In Section 3.2, we review the notions of an associated bundle. We also review a linear connection and its curvature tensor on an associated vector bundle through a bijective relation Proposition 3.2.5 from a principal connection and the curvature form. In Section 3.3, we review Cartan geometries, Cartan curvatures, and torsion functions.

In Section 3.4, we review the category equivalence between Riemannian manifolds and torsion-free Cartan geometries modeled on the Euclidean space $\mathbb{R}^n$. We also review that the Cartan curvature is composed of the torsion tensor and the curvature tensor on the tangent bundle of a Riemannian manifold. In subsection 3.4.1, we derive the Riemannian geodesic equation by considering an integral curve equation introduced by the torsion-free Cartan connection. The same consideration for the Riemannian model $\mathbb{R}^n$ is already derived in Theorem 2.1.1.

In Section 3.5, we review affine Cartan connections and introduce Weyl connections in the end of the section. In Section 3.6, we review the category equivalence between conformal manifolds and normal Cartan geometries modeled on the Möbius sphere $S^{(p,q)}$. We also review how Weyl connections induce normal Cartan connection. In the end of the section, we briefly describe how the normal Cartan geometry derives the standard tractor bundle. In subsection 3.6.1, we review the integral curve equation for conformal geodesics. We also review their geometric properties.

§ 3.1 Principal Bundle, Connection and Curvature

In the following, E, M, F are smooth manifolds and H is a Lie group with the Lie algebra $\mathfrak{h}$. The main reference of this section are [17, 18]

Definition 3.1.1 (Fiber Bundle)**.** Let $\pi\colon E \longrightarrow M$ be a smooth map. The quadruple (E, π, M, F) is called a **fiber bundle** with fiber F if each $x \in M$ has an open neighborhood $U \subseteq M$ such that there is a diffeomorphism $\phi\colon \pi^{-1}(U) \longrightarrow U \times F$ to make the following diagram commute

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad\phi\quad} & U \times F \\ & {\scriptstyle\pi}\searrow \qquad \swarrow{\scriptstyle\pi_1} & \\ & U & \end{array}$$

The pair (U, ϕ) is called a **chart** and a collection $\{(U_\alpha, \phi_\alpha)\}$ is called an **atlas** if $\{U_\alpha\}$ covers M.

Assume $\xi = (E, \pi, M, F)$ to be a fiber bundle and the fiber F to have a smooth H-left-action. Then,

Definition 3.1.2 (H-atlas, H-structure and H-bundle)**.** A H**-atlas** on ξ is an atlas

$\{(U_\alpha, \phi_\alpha)\}$ such that transition maps are of the form

$$\phi_\alpha \circ \phi_\beta^{-1} \colon (U_\alpha \cap U_\beta) \times F \longrightarrow (U_\alpha \cap U_\beta) \times F$$
$$(x, f) \mapsto (x, \phi_{\alpha\beta}(x) \cdot f),$$

where $\phi_{\alpha\beta} \colon U_\alpha \cap U_\beta \longrightarrow H$ is a smooth map. Two H-atalses are equivalent if their union is a H-atlas.

An equivalence class of H-atlases is called a H-structure on ξ and H-bundle means a specified H-structure on ξ.

Definition 3.1.3 (Principal Bundle)**.** A fiber bundle (P, π, M, H) is called a **principal H-bundle** if it is a H-bundle and the Lie group H, as the fiber, acts on itself by the left translation.

Remark 3.1.4. Every principal bundle has a free and transitive right action on its fibers defined by H-structure $\{(U_\alpha, \phi_\alpha)\}$:

$$\forall\, p \in U_\alpha \text{ with } \phi_\alpha(p) = (x, a), \quad p \cdot h := \phi_\alpha^{-1}((x, ah)).$$

So, the above charts are H-equivariant

$$\phi_\alpha(p \cdot h) = (x, ah) = (x, a) \cdot h = \phi_\alpha(p) \cdot h.$$

Conversely, if $\pi \colon P \longrightarrow M$ is a surjective submersion and H acts on P from the right freely and transitively on its fibers, then (P, π, M) is a principal H-bundle ([19] 10.3 Lemma).

Proposition 3.1.5. *Given a principal H-bundle $\pi \colon P \longrightarrow M$. There is a bijective* *relation between local sections of $P \to M$ and H-equivariant local charts*

$$(s\colon U \to P) \mapsto \begin{pmatrix} \phi_s\colon \pi^{-1}(U) & \to U \times H \\ s(x)h & \mapsto (x,h) \end{pmatrix}$$
$$\phi^{-1}(x, id) \mapsfrom \left(\phi\colon \pi^{-1}(U) \to U \times H\right)$$

Definition 3.1.6 (Reduction)**.** Let $H_0 \subseteq H$ be a Lie subgroup and $P \to M$ be a principal H-bundle. Then, a submanifold $P_0 \subseteq P$ is called a H_0**-reduction** of P if $P_0 \to M$ is a principal H_0-bundle and its H_0-right-action is the restriction of H-action on P.

Example 3.1.7. The projection $H \to *$ from a Lie group H to a point $*$ is a principal H-bundle.

Example 3.1.8 (Frame Bundle)**.** Given a vector bundle $E \to M$ of the rank r and $\mathrm{Fr}(E_x)$ is the collection of bases of E_x. Then, the bundle $\mathrm{Fr}(E) = \bigsqcup_{x\in M} \mathrm{Fr}(E_x)$ is a principal $GL(r,\mathbb{R})$-bundle on M with the right action

$$(e_i)\cdot A = (e_k[A]^k_i), \quad \forall \text{ basis } (e_i) \in \mathrm{Fr}(E_x) \quad \forall A \in GL(r,\mathbb{R}).$$

If there is a metric on E, then the orthonormal frame bundle $\mathrm{Fr}_{O(r)}(E)$ is defined by collecting orthonormal bases.

Definition 3.1.9 (Fundamental Vector Field and Vertical Subbundle)**.** Given a principal H-bundle $\xi = (P, \pi, M, H)$. Let $A \in \mathfrak{h}$. The associated vector field on P is given by

$$\zeta_A(p) = \left.\frac{d}{dt}\right|_{t=0} p \cdot e^{tA} \in T_pP,$$

called fundamental vector field. The kernel $VP = \ker d\pi$ is called the **vertical subbundle**.

Remark 3.1.10. Note that every fundamental vector field is vertical, so

$$\ker d\pi_p \cong \mathfrak{h} \quad \forall p \in P.$$

Definition 3.1.11 (Principal Connection)**.** A smooth $\mathfrak{h}$-valued 1-form γ on P is called a **principal connection** if it satisfies the followings

1. $\gamma_p(\zeta_A(p)) = A$ for all $A \in \mathfrak{h}$
2. H-equivariance: $R_h^* \gamma = \mathrm{Ad}(h^{-1})\,\gamma$ for all $h \in H$, where R_h is the right action on P as mentioned in the Remark 3.1.4 and Ad is the adjoint action on $\mathfrak{h}$.

Example 3.1.12 (Maurer-Cartan Form)**.** Recall Example 3.1.7, the projection $H \longrightarrow *$. Let $h \in H$ and denote the left action on H by $L_h \colon H \longrightarrow H,\ a \mapsto ha$. Then, the Maurer-Cartan form w is a principal connection on H defined by

$$\begin{aligned} w \colon TH &\longrightarrow \mathfrak{h} \\ (h, V) &\mapsto dL_{h^{-1}}V. \end{aligned}$$

Sometimes, we need the local description of γ.

Proposition 3.1.13. [19, Chapter 3, 11.4, Lemma (6)] *Let $\xi = (P, \pi, M, H)$ be a principal H-bundle with a principal connection γ. Let $\phi \colon \pi^{-1}(U) \longrightarrow U \times G$ be a H-equivariant local chart and $s \colon U \longrightarrow P$ be its corresponding local section in Proposition 3.1.5. Then, for all $(x, h) \in U \times H$*

$$(\phi^{-1^*}\gamma)_{(x,h)} = w_H(h) + Ad(h^{-1})(s^*\gamma)_x \tag{3.1.1}$$

where $w_H(h)$ is the Maurer-Cartan form of H at h.

Remark 3.1.14. One can define a principal connection in an equivalent way by considering a horizontal smooth distribution $HP \subseteq TP$ such that

1. Right-invariant: $dR_h(H_pP) = H_{ph}P \quad \forall h \in H$
2. $TP = VP \oplus HP$.

Their relation is $HP = \ker \gamma$. Note that $H_pP \cong T_{\pi(p)}M \quad \forall p \in P$.

Definition 3.1.15 (Curvature)**.** Assume there is a principal connection γ on a principal H-bundle $\xi = (P, \pi, M, H)$. The curvature form Ω is a $\mathfrak{h}$-valued two-form on P defined by

$$\Omega(X, Y) = d\gamma(X, Y) + [\gamma(X), \gamma(Y)] \quad X, Y \in TP.$$

Proposition 3.1.16. [17, Theorem 30.4] *The curvature form Ω satisfies two properties*

1. *Horizontal: $\Omega(X, Y) = \Omega(X^{hor}, Y^{hor})$ where X^{hor} is the projection of X onto HP along VP;*
2. *Equivariant: $R_h^* \Omega = Ad(h^{-1})\, \Omega$ for all $h \in H$ where Ad is the adjoint action on $\mathfrak{h}$.*

Example 3.1.17. Example 3.1.12 defines the Maurer-Cartan form w on H. Its curvature Ω turns out to be 0.

Definition 3.1.18 (Category of a Principal H-bundle)**.** One can define category on principal H-bundles.

- Objects: Principal H-bundles (P, π, M, H)
- Morphisms: $(P, \pi, M) \xrightarrow{\phi} (P', \pi', M')$ where $\phi \colon P \to P'$ is fiberwise with $\phi(p \cdot h) = \phi(p) \cdot h, \quad p \in P$ and $h \in H$.

Note that if $\phi\colon P \to P'$ is fiberwise, then there is a unique smooth map $\underline{\phi}\colon M \to M'$ such that the following diagram commutes

$$\begin{array}{ccc} P & \xrightarrow{\phi} & P' \\ \pi\downarrow & & \downarrow\pi' \\ M & \xrightarrow{\underline{\phi}} & M' \end{array}.$$

§ 3.2 Associated Bundle

The concepts of associated bundles bring the bridge between principal bundles and vector bundles. Take the tangent bundle of a manifold as an example, it's an associated bundle of its frame bundle. Due to this relation, we'll see how the connections and curvatures we defined earlier on principle bundles relate to associated vector bundles.

Let S be a manifold with a smooth H-left-action and $\pi\colon P \to M$ be a principal H-bundle. Define $P \times_H S = (P \times S)/\sim$ where each equivalence class is of the relation

$$[p \cdot h, h^{-1} \cdot s] = [p, s] \quad , \ h \in H\ , \ [p, s] \in P \times_H S.$$

Define $\pi_S\colon P \times_H S \to M,\ [p, s] \mapsto \pi(p)$.

Definition 3.2.1 (Associated Bundle)**.** The fiber bundle $(P \times_H S, \pi_S, M, S)$ is called an **associated bundle** for the left H-action on S if it is a H-bundle with the induced H-structure from the principal bundle (P, π, M, H).

Remark 3.2.2. The induced H-structure is the following. Let $\{(U_\alpha, \phi_\alpha)\}$ be a H-atlas from the principal bundle (P, π, M, H)

$$\begin{aligned} \phi_\alpha \colon \pi^{-1}(U_\alpha) &\to U_\alpha \times H \\ p &\mapsto (\pi(p), \psi_\alpha(p)) \,. \end{aligned}$$

The induced H-atlas on the associated bundle $(P \times_H S, \pi_S, M, S)$ is

$$\begin{aligned} \pi_S^{-1}(U_\alpha) &\to U_\alpha \times S \\ [\, p, s\,] &\mapsto (\pi(p), \psi_\alpha(p) \cdot s) \,. \end{aligned}$$

Example 3.2.3. Given a smooth manifold M^n. Its tangent bundle is in fact an associated vector bundle $TM \cong \mathrm{Fr}(M^n) \times_{GL(n)} \mathbb{R}^n$ where $\mathrm{Fr}(M^n)$ is the frame bundle of the tangent bundle TM. The isomorphism is

$$\begin{aligned} TM &\to \mathrm{Fr}(M^n) \times_{GL(n)} \mathbb{R}^n \\ v^i e_i &\mapsto [\, (e_i), v^i \hat{e}_i \,], \end{aligned}$$

where $(e_i) \in \mathrm{Fr}(M^n)$ and $(\hat{e}_i)$ is the standard basis of $\mathbb{R}^n$.

The curvature form Ω we defined earlier on a principal bundle by a principal connection satisfies horizontal and equivariant properties in Proposition 3.1.16. With an associated vector bundle, the principal connection's curvature should give rise to the curvature form with corresponding linear connection on the vector bundle. In the following, we'll deal with the correspondent relations.

Given a principal H-bundle $P \xrightarrow{\pi} M$ and a representation $H \xrightarrow{\rho} GL(V)$ with $E = P \times_H V$.

Definition 3.2.4. Define $\Omega^k_H(P, V)$ to be the set of all smooth V-valued k-forms on P

such that $\forall\phi \in \Omega^k_H(P,V)$,

- Equivariant: $R_h^*\phi = \rho(h^{-1})\cdot\phi$;
- Horizontal: ϕ vanishes if one of its arguments is vertical.

The form $\phi \in \Omega^k_H(P,V)$ is called a **tensorial form** and $\Gamma_H(P,V) := \Omega^0_H(P,V)$.

Proposition 3.2.5 ([14] Corollary 1.2.7)**.** *Define* $\Omega^k(M,E) := \Gamma\left(\left(\Lambda^k T^*M\right)\otimes E\right)$. *There is a bijective relation between* $\Omega^k(M,E)$ *and* $\Omega^k_G(P,V)$*. The relation is* $\forall p \in P$

$$\alpha_{\pi(p)}(\xi_1,\dots,\xi_k) = [p, \tilde{\alpha}_p(\tilde{\xi}_1,\dots,\tilde{\xi}_k)],$$

where $\alpha \in \Omega^k(M,E)$*,* $\tilde{\alpha} \in \Omega^k_G(P,V)$ *and* $d\pi(\tilde{\xi}_j) = \xi_j \in TM$*.*

Example 3.2.6. Recall Proposition 3.1.16. The curvature form Ω on a principal bundle with a principal connection is a tensorial form.

Given a principal H-bundle $P \xrightarrow{\pi} M$ with a principal connection γ. Assume there is a representation $H \xrightarrow{\rho} V$. In the following, we will see how principal connection induces a linear connection and corresponding curvature on the associated vector bundle $E = P\times_H V$.

Definition 3.2.7 (Covariant Derivative)**.** The covariant derivative D on a tensorial form $\phi \in \Omega^k_G(P,V)$ is defined by $D\phi := (d\phi)^{hor}$, where

$$(d\phi)^{hor}(X_1,\dots,X_k) = (d\phi)(X_1^{hor},\dots,X_k^{hor})$$

for $X_i \in TP$ and X_i^{hor} is the projection of X_i onto HP along VP.

Proposition 3.2.8 ([17] Proposition 31.16)**.** *The covariant derivative D has the property,*

$$D\colon \Omega_G^k(P,V) \to \Omega_G^{k+1}(P,V).$$

Corollary 3.2.9. *The covariant derivative D on a principal bundle $P \xrightarrow{\pi} M$ with a principal connection gives rise to a linear connection ∇ on the associated vector bundle $E = P \times_H V$. For any $x \in M$, $X \in T_xM$ and $u \in \pi^{-1}(x)$, the linear connection at x on $\phi \in \Gamma(E)$ is*

$$\nabla_X \phi(x) := [u, D_{\tilde{X}} \tilde{\phi}(u)]$$

where $d\pi(\tilde{X}) = X$ and $\tilde{\phi} \in \Gamma_H(P,V)$ by Proposition 3.2.5

Proposition 3.2.10. *Assume the previous settings in Corollary 3.2.9. Since the principal connection's curvature Ω is tensorial, it induces a two-form on M, $R \in \Omega^2(M, \mathfrak{g})$. In fact, $\forall X, Y \in \Gamma(TM)$ and $\phi \in \Gamma(E)$ with the corresponding $\tilde{\phi} \in \Gamma_H(P,V)$*

$$\begin{aligned}(R(X,Y)\phi)(\pi(u)) &:= [u, \Omega_u(\tilde{X}, \tilde{Y}) \cdot \tilde{\phi}(u)] \\ &= (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})\phi,\end{aligned}$$

where $u \in P$, $\tilde{X}$ and $\tilde{Y}$ are uniquely horizontal lifts of X and Y, the action $\cdot$ for $\Omega_u(\tilde{X}, \tilde{Y}) \cdot \tilde{\phi}$ is induced by infinitesimal representation $d\rho\colon \mathfrak{h} \to \mathfrak{gl}(V)$.

To prove the proposition, we need some knowledge on horizontal lifts of vector fields.

Proposition 3.2.11 ([20] Proposition 1.2 in Chapter 2)**.** *Given a principal H-bundle $P \xrightarrow{\pi} M$ and a principal connection γ. Then, for any vector field X on M, there is a* *uniquely horizontal lift* $\tilde{X}$ *such that* $dR_h\tilde{X} = \tilde{X}$ *for all* $h \in H$. *Precisely, the lift comes from the isomorphism of* $d\pi$ *between the horizontal distribution HP and* TM.

Proposition 3.2.12. *Assuming the previous settings and assuming there is a representation* $\rho\colon H \to GL(V)$. *Let* $\tilde{\phi} \in \Omega^k_H(P,V)$ *and* $X_i \in \Gamma(TM)$. *Then,* $\tilde{\phi}(\tilde{X}_1, \ldots, \tilde{X}_k) \in \Gamma_H(P,V)$ *where* $\tilde{X}_i$ *are the uniquely horizontal lifts.*

Proof. Let $h \in H$ and $u \in P$. Then,

$$\begin{aligned} R_h^*\left(\tilde{\phi}_u(\tilde{X}_1(u), \ldots, \tilde{X}_k(u)\right) &= \tilde{\phi}_{uh}\left(\tilde{X}_1(uh), \ldots, \tilde{X}_k(uh)\right) \\ &= \tilde{\phi}_{uh}\left(dR_h(\tilde{X}_1(u)), \ldots, dR_h(\tilde{X}_k(u))\right) \\ &= (R_h^*\phi)_u(\tilde{X}_1, \ldots, \tilde{X}_k) \\ &= Ad(h^{-1})\left(\tilde{\phi}_u(\tilde{X}_1, \ldots, \tilde{X}_k)\right). \end{aligned}$$

□

Proposition 3.2.13 ([20] Proposition 1.3 in Chapter 2)**.** *Let* $\tilde{X}$ *and* $\tilde{Y}$ *be the horizontal lifts of* $X, Y \in \Gamma(TM)$. *Then, the horizontal component of* $[\tilde{X}, \tilde{Y}]$ *is the horizontal lift of* $[X, Y]$.

Let's begin to prove Proposition 3.2.10.

Proof. Recall γ is a principal connection on the principal bundle $P \xrightarrow{\pi} M$ and $H \xrightarrow{\rho} GL(V)$ is a representation. Let $\phi \in \Gamma(E)$ with the corresponding $\tilde{\phi} \in \Gamma_H(P,V)$. Due to

Propositions 3.2.8 and 3.2.12, we know $D_{\tilde{X}} D_{\tilde{Y}} \tilde{\phi} \in \Gamma_H(P, V)$. So,

$$\begin{aligned}
&\left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) \phi(\pi(u)) = \left[u, \left(D_{\tilde{X}} D_{\tilde{Y}} - D_{\tilde{Y}} D_{\tilde{X}} - D_{[\tilde{X},\tilde{Y}]}\right) \tilde{\phi}(u)\right] \\
&= \left[u, \left(\tilde{X}\tilde{Y} - \tilde{Y}\tilde{X} - [\tilde{X},\tilde{Y}]^{hor}\right) \tilde{\phi}(u)\right] = \left[u,\ \zeta_{\gamma([\tilde{X},\tilde{Y}])}(\tilde{\phi}(u))\right] \\
&= \left[u, \left.\frac{d}{dt}\right|_{t=0} \tilde{\phi}(u \cdot e^{t\gamma([\tilde{X},\tilde{Y}])})\right] = \left[u, \left.\frac{d}{dt}\right|_{t=0} \rho\left(e^{-t\gamma([\tilde{X},\tilde{Y}])}\right)(\tilde{\phi}(u))\right] \\
&= \left[u, -\left(\gamma([\tilde{X},\tilde{Y}])\right) \cdot \tilde{\phi}(u)\right]
\end{aligned}$$

where the action $\cdot$ in the last equality is $d\rho$ action. Also, since $\tilde{X}$ and $\tilde{Y}$ are horizontal, the curvature form Ω becomes

$$\begin{aligned}
\Omega(\tilde{X},\tilde{Y}) \cdot \tilde{\phi}(u) &= \left(d\gamma(\tilde{X},\tilde{Y}) + [\gamma(\tilde{X}),\gamma(\tilde{Y})]\right) \cdot \tilde{\phi}(u) \\
&= -\left(\gamma([\tilde{X},\tilde{Y}])\right) \cdot \tilde{\phi}(u)
\end{aligned}$$

□

Sometimes, it may be helpful to have the following formula for covariant derivative.

Proposition 3.2.14 ([17] Proposition 31.19)**.** *For all $\tilde{\phi} \in \Gamma_H(P, V)$,*

$$D_{\tilde{X}}\tilde{\phi} = \tilde{X}(\tilde{\phi}) + d\rho(\gamma(\tilde{X}))\tilde{\phi}, \quad \tilde{X} \in TP.$$

Note that $\tilde{X}$ here is not horizontal in general.

The local description of Christoffel symbols in terms of a principal connection is in the following proposition.

Proposition 3.2.15. *Let $\pi_H \colon Fr_H M \to M$ be a H-reduction of the frame bundle FrM and γ a principal connection on $Fr_H M$ with the induced linear connection ∇ on $TM \cong Fr_H M \times_H \mathbb{R}^n$. Given a H-equivariant chart $\phi\colon \pi_H^{-1}(U) \to U \times H$ with the* *corresponding section $s\colon U \longrightarrow Fr_H M$ in Proposition 3.1.5. Define the Christoffel symbols Γ_{ij}^k for $(e_i)_i \in Fr_H M$ by $\nabla_{e_i} e_j = \Gamma_{ij}^k e_k$. Then,*

$$(s^*\gamma)(e_i) \cdot \hat{e}_j = \Gamma_{ij}^k \hat{e}_k$$

where $\cdot$ is the Lie algebra action $\mathfrak{gl}(n)$ on $\mathbb{R}^n$ and $(\hat{e}_i)_i$ is the standard basis of $\mathbb{R}^n$.

Proof. Let $\tilde{\phi}_j \in \Gamma_H(\mathrm{Fr}_H M)$ such that $\tilde{\phi}_j((e_i)_i) = \hat{e}_j$. Using Corollary 3.2.9,

$$\begin{aligned}\Gamma_{ij}^k e_k = \nabla e_i e_j &= [(e_i)_i, D_{ds(e_i}\tilde{\phi}_j((e_i)_i)] \\ &= [(e_i)_i, 0 + (s^*\gamma)(e_i) \cdot \hat{e}_j]\end{aligned}$$

where the 2nd equality is due to Proposition 3.2.14. Using the identification $TM \cong \mathrm{Fr}_H \times_H \mathbb{R}^n$, $v^i e_i \mapsto [(e_i)_i, v^i \hat{e}_i]$, we get the result. □

§ 3.3 Cartan Connection and Curvature

Let $H \subseteq G$ be a closed Lie subgroup of a Lie group G with the corresponding Lie algebras $\mathfrak{h}$ and $\mathfrak{g}$. The main reference of this section is [14].

Definition 3.3.1 ([14])**.** Given a principal H-bundle $P_H \longrightarrow M$. It is called Cartan geometry of type (G, H, w) where $w \in \Omega^1(P_H, \mathfrak{g})$ satisfies

1. $w(\zeta_A(u)) = A, \quad \forall A \in \mathfrak{h}$;
2. H-equivariant: $R_h^* w = \mathrm{Ad}(h^{-1})\, w, \quad \forall h \in H$;
3. $w_u\colon T_u P_H \longrightarrow \mathfrak{g}$ is a linear isomorphism $\forall u \in P_H$.

Proposition 3.3.2. *Assume $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ and denote w by $w = (w)_{\mathfrak{n}} + (w)_{\mathfrak{h}}$ with respect to the Lie algebra decomposition. Then, $(w)_{\mathfrak{h}}$ is a principal connection if and only if $\mathfrak{n}$ is H-stable under the Adjoint action of H.*

Proof. By the equivariance property of w, we have

$$\begin{aligned} R_h^* w &= R_h^*(w)_{\mathfrak{n}} + R_h^*(w)_{\mathfrak{h}} \\ &= Ad(h^{-1})w = Ad(h^{-1})(w)_{\mathfrak{n}} + Ad(h^{-1})(w)_{\mathfrak{h}}. \end{aligned} \tag{3.3.1}$$

If $\mathfrak{n}$ is H-stable, then $R_h^*(w)_{\mathfrak{n}} = Ad(h^{-1})(w)_{\mathfrak{n}}$ and $R_h^*(w)_{\mathfrak{h}} = Ad(h^{-1})(w)_{\mathfrak{h}}$.
Let $A \in \mathfrak{h}$. Because w generates A, we get

$$A = w(\zeta_A) = (w)_{\mathfrak{h}}(\zeta_A).$$

Therefore, $(w)_{\mathfrak{h}}$ is a principal connection if $\mathfrak{n}$ is H-stable.
Conversely, since $R_h^*(w)_{\mathfrak{h}} = Ad(h^{-1})(w)_{\mathfrak{h}}$, then $R_h^*(w)_{\mathfrak{n}} = Ad(h^{-1})(w)_{\mathfrak{n}}$ from (3.3.1).
With the linear isomorphism property $w_u : T_u P_H \cong \mathfrak{n} \oplus \mathfrak{h}$, we have
$Ad(h^{-1})\mathfrak{n} \subseteq \mathfrak{n}$. □

In general, the $\mathfrak{n}$ subspace for $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ may not be H-stable. One may induce a principal connection on an associated principal bundle from a given Cartan geometry.

Proposition 3.3.3 ([14] special case in Theorem 1.5.6)**.** *Given a Cartan geometry $(\mathcal{G}, w) \to M$ in the type (G, H). Consider the canonical injective map*

$$\begin{aligned} i\colon \mathcal{G} &\to \mathcal{G} \times_H G \\ u &\mapsto [u, id], \end{aligned}$$

where $\mathcal{G} \times_H G$ is a principal G-bundle due to the Remark 3.1.4. Then, there exists a *unique principal connection γ on $\mathcal{G} \times_H G$ such that $i^*\gamma = w$. In particular,*

$$\gamma_{i(u)}\left(di(\xi) + \zeta_A(i(u))\right) = w_u(\xi) + A \quad \forall u \in \mathcal{G},\ \xi \in T_u\mathcal{G},\ A \in \mathfrak{g}.$$

Note that the above assignment is functorial; that is, any morphism of Cartan geometries in Definition 3.3.12 induces a principal bundle morphism preserving principal connections.

Apart from inducing principal connections, one may sometimes have an induced Cartan geometry from a given one. The following proposition will be considered in Section 3.6.

Proposition 3.3.4. [14, spcial case in Lemma 1.5.15] *Given a Cartan geometry $(\mathcal{G}_0, \sigma) \to M^n$ in the type $(\mathbb{R}^n \rtimes G_0, G_0)$ and a G_0-equivariant global section $s\colon \mathcal{G}_0 \to \mathcal{G}_0 \times_{G_0} PP_{line}$ for $\mathcal{G}_0 \times_{G_0} PP_{line} \to \mathcal{G}_0$. Then, $\exists!$ Cartan connection w^σ on $\mathcal{G}_0 \times_{G_0} PP_{line}$ in the type $(O_+(n+1,1), PP_{line})$ such that $s^*w^\sigma = \sigma$. In particular,*

$$w^{\sigma}_{s(u)}(ds(\xi) + \zeta_A(s(u))) = \sigma_u(\xi) + A \quad \forall u \in \mathcal{G}_0,\ \xi \in T_u\mathcal{G}_0,\ A \in \mathfrak{p}. \tag{3.3.2}$$

Given a Cartan geometry of the type (G, H, w) with $P_H \xrightarrow{\pi} M$. Recalling $\mathfrak{h} \cong \ker d\pi_u$ for all $u \in P_H$, so we have $\dim \mathfrak{g}/\mathfrak{h} = \dim T_xM$ for $x \in M$. In fact,

Proposition 3.3.5. *There is the isomorphsim for TM by the Cartan connection*

$$\begin{aligned} P_H \times_{Ad} (\mathfrak{g}/\mathfrak{h}) &\simeq TM \\ [p, V + \mathfrak{h}] &\mapsto d\pi_p w_p^{-1}(V) \\ [p, w_p(\tilde{X}) + \mathfrak{h}] &\leftarrow\!\shortmid X, \qquad \textit{where } d\pi_p(\tilde{X}) = X. \end{aligned}$$

The action Ad is the adjoint action $Ad\colon H \to \mathfrak{g}/\mathfrak{h}$.

Definition 3.3.6 (Curvature Form and Curvature Function)**.** The **Cartan curvature** $K \in \Omega^2(P_H, \mathfrak{g})$ is defined by

$$K(\xi, \eta) = dw(\xi, \eta) + [w(\xi), w(\eta)].$$

The corresponding curvature function $\kappa\colon P_H \to \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$ is defined by

$$\kappa(X, Y) = K(w^{-1}(X), w^{-1}(Y)), \quad \forall X, Y \in \mathfrak{g}/\mathfrak{h}.$$

The above definition for κ is well-defined. In fact, we have

Proposition 3.3.7 ([14] Lemma 1.5.1)**.** *The Cartan curvature K is horizontal and H-equivariant. Therefore, for the curvature function $\kappa\colon P_H \to \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$,*

$$\kappa \circ R_h = h^{-1} \cdot \kappa;$$

that is, for all $p \in P_H$ and $h \in H$

$$\kappa_{p\cdot h}(X, Y) = Ad(h^{-1})\, \kappa_p(Ad(h)\, X,\ Ad(h)\, Y), \quad \forall X, Y \in \mathfrak{g}/\mathfrak{h}.$$

So, $\kappa \in \Gamma_H(P_H, \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g})$.

Assume $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$. Compared to the curvature form from a principal connection, there is one extra $\mathfrak{n}$-valued component for a Cartan curvature.

Definition 3.3.8 (Torsion)**.** Let $\pi_{\mathfrak{g}/\mathfrak{h}}\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ be the quotient map. Define the **torsion function**

$$\tau = \pi_{\mathfrak{g}/\mathfrak{h}} \circ \kappa \in \Gamma_H(P_H, \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes (\mathfrak{g}/\mathfrak{h})).$$

It thus defines a torsion tensor $T \in \Omega^2(M, TM)$ by the bijective relation Proposition 3.2.5 between $\Gamma_H(P_H, \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes (\mathfrak{g}/\mathfrak{h}))$ and $\Gamma(M, \Lambda^2 T^*M \otimes TM) = \Omega^2(M, TM)$. If τ

vanishes, then it's called **torsion-free**.

Remark 3.3.9. The torsion function turns out to be the torsion tensor in Riemannian geometry (See Section 3.4).

Example 3.3.10 (Riemannian Model)**.** As in the Example 2.0.1, the quotient map

$$(\mathbb{R}^n \rtimes O(n), w_{O(n)}) \longrightarrow \mathbb{R}^n \rtimes O(n)/O(n) \cong \mathbb{R}^n$$

is the Cartan geometry of the type $(\mathbb{R}^n \rtimes O(n), O(n), w_{O(n)})$ where $w_{O(n)}$ is the Maurer-Cartan form on $O(n)$.

Example 3.3.11 (Conformal Model)**.** Recall Example 2.0.2, the quotient map

$$(PO(p+1, q+1), w_{PO}) \longrightarrow PO(p+1, q+1)/PP_{line} \cong S^{p,q}$$

is the Cartan geometry of the type $(PO(p+1, q+1), PP_{line}, w_{PO})$ where w_{PO} is the Maurer-Cartan form on $PO(p+1, q+1)$.

Definition 3.3.12 (Category)**.** Let $H \subseteq G$ be a Lie subgroup of a Lie group G. The category of the Cartan geometry of the type (G, H) is defined by

- Objects: (P, π, M, w), which is a principal H-bundle with a Cartan connection w.
- Morphisms: $(P, \pi, M, w) \xrightarrow{\phi} (P', \pi', M', w')$ where ϕ is a principal bundle morphism with the property

$$\phi^* w' = w.$$

Remark 3.3.13. Given a point $p \in P$. Because w_p and $w'_{\phi(p)}$ are linear isomorphisms, we know $d\phi$ is invertible. Therefore, ϕ and its base map $\underline{\phi}\colon M \to M'$ are local diffeomorphisms.

§ 3.4 Riemannian Manifold in Cartan Geometry

The Riemannian model in Example 3.3.10 motivates that there should be an induced Cartan geometry in the type $(\mathbb{R}^n \rtimes O(n), O(n))$ for each Riemannian manifold (M^n, g). That is, there should be a principal $O(n)$-bundle with a $\mathbb{R}^n \oplus \mathfrak{o}(n)$-valued Cartan connection. In this section, we focus on the construction of torsion-free Cartan geometry and briefly introduce the equivalence between Riemannian manifold and torsion-free Cartan geometry. In the following, we only consider Cartan geometry in the type $(\mathbb{R}^n \rtimes O(n), O(n))$.

Recall a Riemannian manifold (M^n, g) is a smooth manifold M with a positive definite metric tensor g. The induced Levi-Civita connection ∇ is a unique linear connection satisfying: $\forall X, Y, Z \in \Gamma(TM)$

1. Torsion-free: $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$.
2. Metric connection: $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ where $\langle , \rangle$ is the metric g.

The 2nd property gives rise to a principal connection γ on the orthonormal frame bundle $\mathrm{Fr}_{O(n)}M$. We need a $\mathbb{R}^n$-valued form on $\mathrm{Fr}_{O(n)}M$ to get a Cartan connection. It's called a **canonical form, or called a soldering form**.

Definition 3.4.1 (Canonical form)**.** Let $\pi\colon \mathrm{Fr}\,M \to M$ be the frame bundle of M.

Recall that any element $(e_i) \in \mathrm{Fr}_x M$ can be considered as a linear isomorphism

$$(e_i) : T_x M \to \mathbb{R}^n, \quad v^i e_i \mapsto v^i \hat{e}_i$$

where $(\hat{e}_i)$ is the standard basis of $\mathbb{R}^n$. Define the **canonical form** $\Theta \in \Omega^1(\mathrm{Fr}\, M, \mathbb{R}^n)$ by

$$\Theta_{(e_i)}(\tilde{\xi}) = (e_i) \circ d\pi(\tilde{\xi})$$

where (e_i) is the linear isomorphism we just defined.

The defintion of the canonical form directly gives the following proposition.

Proposition 3.4.2. *The canonical form Θ is in $\Omega^1_{GL(n)}(Fr\, M, \mathbb{R}^n)$ and it's strictly horizontal; that is $\Theta(\tilde{\xi}) = 0$ if and only if $\tilde{\xi} \in VFr\, M$.*

The one form $\theta \in \Omega^1_{O(n)}(\mathrm{Fr}_{O(n)} M)$ by pulling back Θ to $\mathrm{Fr}_{O(n)} M$ through the reduction $\mathrm{Fr}_{O(n)} M \to \mathrm{Fr}\, M$ is also strictly horizontal. We also call θ the canonical form on $\mathrm{Fr}_{O(n)} M$.

Proposition 3.4.3. *Let $\xi = (Fr_{O(n)} M, \pi, M, O(n))$ be the orthonormal frame. Given a Cartan connection ω in the type $(\mathbb{R}^n \rtimes O(n), O(n))$ on $Fr_{O(n)} M$. Assume $\omega = \theta + \gamma$ with respect to the Lie algebra decomposition $\mathbb{R}^n \oplus \mathfrak{o}(n)$ where the $\mathbb{R}^n$-valued of ω is equal to the canonical form θ. Then,*

1. *The $\mathfrak{o}(n)$-valued γ is a principal $O(n)$-connection. Denote the induced linear connection by ∇ on TM, recalling $TM \cong Fr_{O(n)} M \times_{O(n)} \mathbb{R}^n$.*

2. *The $\mathbb{R}^n$-valued part of the Cartan curvature K is*

(3.4.1)
$$\begin{aligned} K_{\mathbb{R}^n}(\tilde{\xi},\tilde{\eta}) &= \tilde{\xi}(\theta(\tilde{\eta})) + \gamma(\tilde{\xi})\cdot\theta(\tilde{\eta}) - [\tilde{\eta}(\theta(\tilde{\xi})) + \gamma(\tilde{\eta})\cdot\theta(\tilde{\xi})] - \theta([\tilde{\xi},\tilde{\eta}]) \\ &= d\theta(\tilde{\xi},\tilde{\eta}) + \gamma(\tilde{\xi})\cdot\theta(\tilde{\eta}) - \gamma(\tilde{\eta})\cdot\theta(\tilde{\xi}) \end{aligned}$$

where $\tilde{\xi},\tilde{\eta} \in \Gamma(TFr_{O(n)}M)$ and $\cdot$ is the usual $\mathfrak{gl}(n)$ action on $\mathbb{R}^n$. It gives the torsion tensor by Proposition 3.2.5

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

3. *The $\mathfrak{o}(n)$-valued part of the Cartan curvature K is*

(3.4.2)
$$K_{\mathfrak{o}(n)}(\tilde{\xi},\tilde{\eta}) = d\gamma(\tilde{\xi},\tilde{\eta}) + [\gamma(\tilde{\xi}),\gamma(\tilde{\eta})].$$

It gives the curvature tensor of ∇ on M.

Proof. Since $\mathbb{R}^n \subseteq \mathbb{R}^n \oplus \mathfrak{o}(n)$ is $O(n)$-stable, the $\mathfrak{o}(n)$-valued γ of the Cartan connection ω is a principal $O(n)$-connection by Proposition 3.3.2. Equations (3.4.1) and (3.4.2) are due to the definition of Cartan curvature. Let $\tilde{\xi},\tilde{\eta}$ be the uniquely horizontal lifts of $X, Y \in \Gamma(TM)$. Because the horizontal lifts are $O(n)$-right-invariant as Proposition3.2.11, then $\theta(\tilde{\xi}), \theta(\tilde{\eta}) \in \Gamma_{O(n)}(\mathrm{Fr}_{O(n)}M, \mathbb{R}^n)$. By the tangent bundle isomorphism in Example 3.2.3, we know $\forall u \in \mathrm{Fr}_{O(n)}M$

$$X(\pi(u)) = [u, \theta(\tilde{\xi})], \quad Y(\pi(u)) = [u, \theta(\tilde{\eta})] \in \quad TM \cong \mathrm{Fr}_{O(n)}M \times_{O(n)} \mathbb{R}^n.$$

By the covariant derivative properties in Propositions 3.2.8 and 3.2.14, we get $K_{\mathbb{R}^n} \in$

$\Omega^2_{O(n)}(\mathrm{Fr}_{O(n)}M, \mathbb{R}^n)$ and $\forall u \in \mathrm{Fr}_{O(n)}M$

$$\begin{aligned} T(X,Y)(\pi(u)) &= (\nabla_X Y - \nabla_Y X - [X,Y])\,(\pi(u)) \\ &= \left[u, D_{\tilde{\xi}}(\theta(\tilde{\eta})) - D_{\tilde{\eta}}(\theta(\tilde{\xi})) - \theta([\tilde{\xi},\tilde{\eta}])\right] = \left[u, K_{\mathbb{R}^n}(\tilde{\xi},\tilde{\eta})(u)\right]. \end{aligned}$$

The $\mathfrak{o}(n)$-valued part is already proven in Proposition 3.2.10 □

Let γ be the principal connection from the Levi-Civita connection of ∇. By 2 in Proposition 3.4.3, we know (M,g) induces a torsion-free Cartan geometry on $\mathrm{Fr}_{O(n)}M \longrightarrow M$ in the type $(\mathbb{R}^n \rtimes O(n), O(n))$. Furthermore, we have the following theorem.

Theorem 3.4.4 ([14] Theorem 1.6.1)**.**

1. *Given an $O(n)$-reduction $P_{O(n)} \hookrightarrow Fr\,M$ and any Cartan connection w in the form $w = \theta + \gamma$ where θ is the canonical form and γ is the $\mathfrak{o}(n)$-value of w. Then, w is uniquely determined by its torsion function. Besides, there exists a uniquely torsion-free Cartan connection on $P_{O(n)}$.*

2. *The category of torsion-free Cartan geometries in the form $w = \theta + \gamma$ from 1 on $O(n)$-reductions $P_{O(n)} \hookrightarrow Fr\,M$ is equivalent to the category of $O(n)$-reductions of $Fr\,M$ where the morphism between $O(n)-$reductions are principal bundle morphisms preserving canonical forms.*

In Section 3.5, we will consider reduction $P_H \longrightarrow \mathrm{Fr}\,M$ for a closed subgroup $H \subseteq GL(n)$ and review the corresponding theorem for Theorem 3.4.4, particularly for $H = O(n)$ and $H = CO(p,q)$.

3.4.1 Riemannian Geodesics

Recall in Theorem 2.1.1 that Riemannian geodesics in the Euclidean are the projection of integral curves of constant horizontal vector fields on the Euclidean group $\mathrm{Euc}(n)$. The integral curve interpretation of Riemannian geodesics in the Euclidean model can be generalized to Riemannian manifolds [20, Proposition 6.3 in Chapter III]. In this subsection, we derive the integral curve equations from the torsion-free Cartan connection w in the type $(\mathbb{R}^n \rtimes O(n), O(n))$ of the Riemannian manifold (M, g).

Theorem 3.4.5. *Given an orthonormal frame (e_i) of $\pi\colon Fr_{O(n)}M \to M$ defined on an open set $U \subseteq M$. The frame induces a local trivialization*

$$\begin{aligned} \phi\colon \pi^{-1}(U) &\to U \times O(n) \\ (x, (e_i)\cdot C) &\mapsto (x, C). \end{aligned}$$

Let $\mathcal{X} = \mathcal{X}^i \hat{e}_i \in \mathbb{R}^n \subseteq \mathbb{R}^n \oplus \mathfrak{o}(n)$ where $(\hat{e}_i)$ is the standard basis of $\mathbb{R}^n$. Let $\tilde{\gamma}\colon I \to Fr_{O(n)}M$ be an integral curve of $w^{-1}(\mathcal{X})$ with $\pi(\tilde{\gamma}) \subseteq U$. The local coordinate of $\tilde{\gamma}$ is denoted by $(x(t), C^i{}_j(t)) \in U \times O(n)$. Identifying T_xM and $\mathbb{R}^n$ by $v^i e_i \mapsto v^i \hat{e}_i$ for all $x \in U$. Then,

$$\begin{cases} \dot{x}^i &= \mathcal{X}^j C^i{}_j; \\ \dot{C}^i{}_j &= -\mathcal{X}^k C^l{}_k \Gamma^i_{la} C^a{}_j, \end{cases} \tag{3.4.3}$$

where Γ^i_{jk} is the Christoffel symbols defined from the Levi-Civita connection $\nabla_{e_i} e_j = \Gamma^k_{ij} e_k$ of g where (e_i) is the given orthonormal frame. It can be shown that $x(t)$ is a Riemmanian geodesic. Conversely, any Riemannian geodesics are locally the projection of an integral curve of $w^{-1}(\mathcal{X})$ for some $\mathcal{X} \in \mathbb{R}^n \subseteq \mathfrak{euc}(n)$.

Proof. We first show that (3.4.3) implies $x(t)$ is a Riemannian geodesic.

$$\begin{aligned}\ddot{x}^i &= \mathcal{X}^j \dot{C}^i{}_j = -\mathcal{X}^j \mathcal{X}^k C^l{}_k \Gamma^i_{la} C^a{}_j \\ &= -\dot{x}^a \dot{x}^l \Gamma^i_{la}\end{aligned}$$

where the last equality is because of the 1st equation of (3.4.3).

Next, we want to show

$$\mathcal{X} = (\phi^{-1^*} w)_{\tilde{\gamma}(t)}(\dot{\tilde{\gamma}}(t)) \tag{3.4.4}$$

implies (3.4.3) where $\dot{\tilde{\gamma}}(t)$ is the derivative of $\tilde{\gamma}(t)$ with respect to t. Recall $w = \theta + \gamma$ in Theorem 3.4.4 where θ is the canonical form from FrM and γ is the principal connection from the Levi-Civita connection ∇ of g. Since $\mathcal{X} \in \mathbb{R}^n$, (3.4.4) is equal to

$$\begin{cases} (\phi^{-1^*}\theta)_{\tilde{\gamma}(t)}(\dot{\tilde{\gamma}}(t)) & = (\phi^{-1^*}\theta)_{\tilde{\gamma}(t)}(\dot{x}(t)) = \mathcal{X} \\ (\phi^{-1^*}\gamma)_{\tilde{\gamma}(t)}(\tilde{\gamma}(t)) & = 0 \end{cases}. \tag{3.4.5}$$

The first equality is because θ is strictly horizontal in Proposition 3.4.2. Let $\dot{x}(t) = \dot{x}^i(t)e_i(t) = \dot{\tilde{x}}^i(t)u_i(t)$ where $u_i(t) = e_k(t)C^k{}_i(t)$. Using Definition 3.4.1 of θ, the 1st equation in (3.4.5) implies

$$\begin{aligned}\mathcal{X}^i \hat{e}_i &= \theta_{(u_i)_i}(d\phi^{-1}(\dot{x}(t))) = (u_i(t))_i \cdot d(\pi \circ \phi^{-1})(\dot{\tilde{x}}^i(t)u_t(t)) \\ &= \dot{\tilde{x}}^i(t)\hat{e}_i\end{aligned}$$

where $\pi\colon \mathrm{Fr}_{O(n)} \to M$. So, $\mathcal{X}^i = \dot{\tilde{x}}^i(t)$ which implies the 1st equation of (3.4.3). Recall the local description of a principal connection in Proposition 3.1.13, the 2nd equation in (3.4.5) implies

$$0 = \gamma_{(e_i)\dot{C}}(d\phi^{-1}(\dot{\tilde{\gamma}}(t)) = w_{O(n)}(C(t))(\dot{C}(t)) + \mathrm{Ad}(C^{-1}(t))(s^*\gamma)(\dot{x}(t)) \tag{3.4.6}$$

where $s\colon U \longrightarrow \mathrm{Fr}_{O(n)}M$ is the corresponding section of ϕ in Proposition 3.1.5 and $w_{O(n)}(C(t))$ is the Maurer-Cartan form of $O(n)$ at $C(t)$. Note that

$$\begin{aligned} w_{O(n)}(C(t))(\dot{C}(t)) &= \left(R^*_{C(t)} w_{O(t)}\right)_{\mathbb{I}} (\dot{C}(t) \cdot C^{-1}(t)) \\ &= \mathrm{Ad}(C^{-1}(t)) w_{O(n)}(\mathbb{I})(\dot{C}(t) \cdot C^{-1}(t)) \\ &= \mathrm{Ad}(C^{-1}(t))(\dot{C}(t) \cdot C^{-1}(t)) \end{aligned} \tag{3.4.7}$$

where $\cdot$ is the matrix multiplication and $\mathbb{I}$ is the identity matrix. Also,

$$[(s^*\gamma)(\dot{x}(t))]^i{}_a = [(s^*\gamma)(\mathcal{X}^j C^l{}_j(t) e_l(t))]^i{}_a = \mathcal{X}^k C^l{}_k(t)\Gamma^i_{la}(t) \tag{3.4.8}$$

where the last equality comes from Proposition 3.2.15. Combining (3.4.7) and (3.4.8) into (3.4.6), we get the 2nd equation of (3.4.3). □

§ 3.5 Affine Cartan Connections

Let $(\mathbf{b}, g) \in \mathbb{R}^n \rtimes GL(n)$. The affine transformation $\mathbb{R}^n \rtimes GL(n)$ on $\mathbb{R}^n$ is defined by

$$\mathbf{v} \mapsto g \cdot \mathbf{v} + \mathbf{b}, \quad .$$

If $\mathbb{R}^n$ is considered as the hyperplane in $\mathbb{R}^{n+1}$, $\mathbb{R}^n \cong \{(1, \mathbf{v}) | \ \mathbf{v} \in \mathbb{R}^n\}$, then the affine transformations can be identified as $n+1$-square matrices

$$\mathbb{R}^n \rtimes GL(n) \cong \left\{ \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{b} & g \end{pmatrix} \middle| \ \mathbf{b} \in \mathbb{R}^n, \ g \in GL(n) \right\}.$$

Let $H \subseteq GL(n)$ be a Lie subgroup with Lie algebra $\mathfrak{h}$. Define the affine extension $\mathbb{R}^n \rtimes H$ by

$$\left\{ \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{b} & h \end{pmatrix} \middle| \ \mathbf{b} \in \mathbb{R}^n, \ h \in H \right\}.$$

It's direct to prove the following proposition.

Proposition 3.5.1. *The adjoint action of* $\mathbb{R}^n \rtimes H$ *on* $\mathbb{R}^n \subseteq \mathbb{R}^n \oplus \mathfrak{h}$ *is just usual linear transformations* $h \cdot \boldsymbol{v}$ *on* $\mathbb{R}^n$. *Henceforth,* $\mathbb{R}^n \subseteq \mathbb{R}^n \oplus \mathfrak{h}$ *is* H*-stable under the adjoint action.*

Given a smooth manifold M^n and assume there is a H-reduction of $\operatorname{Fr} M$, $P_H \hookrightarrow \operatorname{Fr} M$. Consider Cartan connections on P_H in the type $(\mathbb{R}^n \rtimes H, H)$ with the form $w = \theta + \gamma$ where the $\mathbb{R}^n$-value of w is equal to the canonical form θ and γ is the $\mathfrak{h}$-value of w. Because of Propositions 3.5.1 and 3.3.2, we know γ is a principal connection. Denote the collection of all such Cartan connections by S.

Definition 3.5.2. Any $w \in S$ is called **affine Cartan connection** of H on P_H.

Proposition 3.5.3. *The affine Cartan connections form an affine space; that is,* $\forall \hat{w}$ *and* $w \in S$, *there exists an unique* H*-equivariant smooth map* $\psi \colon P_H \longrightarrow \mathfrak{h} \otimes \mathbb{R}^{n^*}$ *such that*

$$\hat{w} = w - \psi \circ \theta,$$

and conversely, $\forall w \in S$ *and* $\forall \psi \in \Gamma_H(P_H, \mathfrak{h} \otimes \mathbb{R}^{n^*})$*,* $\hat{w} = w - \psi \circ \theta$ *is an affine Cartan connection.*

Proof. Since $\hat{w}, w \in \Omega^1_H(P_H, \mathfrak{h})$, define $\psi \circ \theta$ on TP_H

$$-\psi \circ \theta = \hat{\gamma} - \gamma$$

where $\hat{w} = \theta + \hat{\gamma}$ and $w = \theta + \gamma$. □

Corollary 3.5.4. *Given* $\hat{w}$ *and* $w \in S$*. Their torsion functions are in the relation*

$$\hat{\tau}_u = \tau_u - \partial\psi_u \quad \forall u \in P_H, \tag{3.5.1}$$

where $\partial \colon L(\mathbb{R}^n, \mathfrak{h}) \longrightarrow L(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$ *is* H*-equivariant linear map defined by*

$$\partial\psi_u(X, Y) = [X, \psi_u(Y)] - [Y, \psi_u(X)],$$

and $L(V, W)$ *is the set of linear maps from* V *to* W*. The brackets in (3.5.1) are Lie algebra brackets of* $\mathbb{R}^n \oplus \mathfrak{h}$*.*

Proof. The formula of the $\mathbb{R}^n$-valued Cartan curvature of an affine Cartan connection w is same as (3.4.1)

$$K_{\mathbb{R}^n}(\tilde{\xi}, \tilde{\eta}) = d\theta(\tilde{\xi}, \tilde{\eta}) + \gamma(\tilde{\xi}) \cdot \theta(\tilde{\eta}) - \gamma(\tilde{\eta}) \cdot \theta(\tilde{\xi}) \qquad \forall \tilde{\xi}, \tilde{\eta} \in \Gamma(TP_H).$$

Because $\gamma(\tilde{\xi}) = 0$ if $\tilde{\xi}$ is horizontal, we have for all $X, Y \in \mathbb{R}^n \subseteq \mathbb{R}^n \oplus \mathfrak{h}$

$$\hat{\tau}(X, Y) - \tau(X, Y) = d\theta(\hat{w}^{-1}(X), \hat{w}^{-1}(Y)) - d\theta(w^{-1}(X), w^{-1}(Y)). \tag{3.5.2}$$

By Proposition 3.5.3, we have

$$w(\hat{w}^{-1}(X)) = X - \psi(X)$$

which implies

$$\hat{w}^{-1}(X) = w^{-1}(X) + \zeta_{-\psi(X)}$$

where $\zeta_{-\psi(X)}$ is the fundamental vector field generated by $-\psi(X)$. After direct computation, (3.5.2) is equal to

(3.5.3)

$$\hat{\tau}_u(X,Y) - \tau_u(X,Y) = \zeta_{-\psi(X)}(u)(\theta(w^{-1}(Y))) - \zeta_{-\psi(Y)}(u)(\theta(w^{-1}(X))) \quad u \in P_H.$$

The first term is

(3.5.4)
$$\begin{aligned}\zeta_{-\psi(X)}(u)(\theta(w^{-1}(Y))) &= \frac{d}{dt}\bigg|_{t=0} \theta_{ue^{-t\psi(X)}}(w^{-1}(Y)) \\ &= \frac{d}{dt}\bigg|_{t=0} Ad(e^{t\psi(X)})\left(\theta_u(dR_{e^{t\psi(X)}}w^{-1}(Y))\right) \\ &= [\psi(X), Y]\end{aligned}$$

where R is the H-right action of P_H, and the last equality is because $Ad(e^{t\psi(X)}) = e^{ad(t\psi(X))}$ and $\theta_u(dR_{e^{t\psi(X)}}w^{-1}(Y)) = \theta_u(w^{-1}(Y)) = Y$. Using similar arguments, one gets the second term in (3.5.3). □

Definition 3.5.5. The kernel of ∂, denoted by $\mathfrak{h}^{(1)} = ker\ \partial \subseteq \mathfrak{h} \otimes \mathbb{R}^{n^*}$, is called the **first prolongation** of $\mathfrak{h}$.

With the previous settings, we have the general version of Theorem 3.4.4.

Theorem 3.5.6 ([14] Theorem 1.6.1)**.** *Identify* $L(\Lambda^2\mathbb{R}^n, \mathbb{R}^n) = \mathbb{R}^n \otimes \Lambda^2\mathbb{R}^{n^*}$. *Assume* $\mathbb{R}^n \otimes \Lambda^2\mathbb{R}^{n^*} = \mathcal{R} \oplus Im\ \partial$ *where* $\mathcal{R}$ *is* H*-invariant. Then,*

1. *There exists* $w \in S$ *such that its torsion function's value is in* $\mathcal{R}$.

2. *If* $\mathfrak{h}^{(1)} = \{0\}$, *then any* $w \in S$ *is uniquely defined by its torsion.*

3. *With* $\mathfrak{h}^{(1)} = \{0\}$*, the following categories are equivalent*

   A. • *Objects: Affine Cartan geometries* $P_H \to M$ *with torsions' values in* $\mathcal{R}$ *where each* P_H *is a* H*-reduction of* $Fr\, M$ *.*

      • *Morphisms: Cartan geometry morphisms.*

   B. • *Objects:* H*-reductions of* $Fr\, M$*,* $(P_H, \theta) \to M$ *where each* θ *is the canonical form.*

      • *Morphisms: Principal bundle morphisms preserving canonical forms.*

   In particular, the bottom smooth maps between base manifolds in the category B are local diffeomorphisms.

Example 3.5.7 ([14], Example 1.6.1)**.** When $H = O(n)$, then $\mathfrak{o}(n)^{(1)} = \{0\}$ and $\mathbb{R}^n \otimes \Lambda^2 \mathbb{R}^{n^*} = Im\ \partial$.

Example 3.5.8 ([14], 1.6.4)**.** Consider the conformal orthogonal group $H = CO(p,q) \cong G_0$ in Remark 2.2.5. Its first prolongation $\mathfrak{g}_0^{(1)} \subseteq L(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ is identified to $\mathfrak{g}_1$ by the following map

$$\mathfrak{g}_1 \to \mathfrak{g}_0^{(1)}, \quad Z \mapsto ad\ Z.$$

Also, $Im\ \partial = \mathbb{R}^n \otimes \Lambda^2 \mathbb{R}^{n^*}$.

Example 3.5.8 shows a torsion-free Cartan connection in the type $(G_0 \rtimes \mathbb{R}^n, G_0)$ is not unique. Such a Cartan connection is called a **Weyl connection** (See Definition 3.6.2).

§ 3.6 Conformal Manifold in Cartan Geometry

Let $n = p + q \geq 3$. As what was done in Section 3.4, the conformal model in Example 2.0.2 implies that we need to find out a Cartan geometry in the type $(PO(p+1, q+1), PP_{line})$ for a pseudo-Riemannian conformal manifold $(M^n, [g])$. In this section, we review the equivalence between pseudo-Riemannian conformal manifold $(M^n, [g])$ with signature (p, q) and normal Cartan geometry in the type $(PO(p+1, q+1), PP_{line})$ in Theorem 3.6.11. In the end, we briefly mention how to have the standard tractor bundle with the linear connection from the normal Cartan geometry. The reference of this section is [14].

Recall $PP_{line} = G_0 \ltimes exp\,(\mathfrak{g}_1)$ in Proposition 2.2.4 and note that G_0 is identified as the conformal orthogonal group $CO(p, q)$. So, to have a principal PP_{line}-bundle, one can first get a principal G_0-bundle and then associate it with PP_{line} to become a principal PP_{line}-bundle. The G_0-bundle can be got from the conformal structure $[g]$.

Definition 3.6.1. Define the conformal frame bundle $\mathrm{Fr}_{G_0}M$ for $(M^n, [g])$. Its fiber is

$$\left(\mathrm{Fr}_{G_0}M\right)_x = \{(e_i) \in \mathrm{Fr}_x M \mid \exists g \in [g] \text{ such that } g(e_i, e_j) = \delta_{ij}\}.$$

Since it's a principal G_0-bundle, in the following, we denote it by $\mathcal{G}_0$.

Example 3.5.8 implies that there are lots of torsion-free G_0-affine Cartan connections. We use these connections to help us to find out the unique Cartan connection on the principal PP_{line}-bundle.

Definition 3.6.2. The torsion-free G_0-affine Cartan connections on $\mathcal{G}_0$ are called **Weyl connections** where affine Cartan connections are in Definition 3.5.2.

Proposition 3.6.3. *Weyl connections form an affine space; that is, given two Weyl connections $\hat{\sigma} = \theta + \hat{\gamma}$ and $\sigma = \theta + \gamma$. Then, $\exists!$ G_0-equivariant $\Upsilon : \mathcal{G}_0 \to \mathfrak{g}_1$ such that*

$$\hat{\sigma}_u(\tilde{\xi}) = \sigma_u(\tilde{\xi}) + [\theta_u(\tilde{\xi}), \Upsilon(u)] \quad \forall u \in \mathcal{G}_0,\ \tilde{\xi} \in T_u\mathcal{G}_0.$$

Conversely, for any Weyl connection σ and $\Upsilon \in \Gamma_{G_0}(\mathcal{G}_0, \mathfrak{g}_1)$, $\hat{\sigma} = \sigma + [\theta, \Upsilon]$ is a Weyl connection.

Proof. Recalling Proposition 3.5.3, we have an unique $\psi \in \Gamma_{G_0}(\mathcal{G}_0, \mathfrak{g}_0 \otimes \mathfrak{g}_1)$ such that

$$\hat{\sigma} = \sigma - \psi \circ \theta.$$

Since their torsion functions are both zero, we have $\partial\psi = 0$ from Corollary 3.5.4. By Example 3.5.8, we know $\psi = -ad(\Upsilon)$ where $\Upsilon \in \Gamma_{G_0}(\mathcal{G}_0, \mathfrak{g}_1)$. □

Remark 3.6.4. [14, 1.6.5] A Levi-Civita connection of $g \in [g]$ induces a Weyl connection on $\mathcal{G}_0$ by using a similar approach from (3.3.2). Note that not all Weyl connections are Levi-Civita connections.

Definition 3.6.5. Define the principal P_{line}-bundle by $\mathcal{G} = \mathcal{G}_0 \times_{G_0} P_{line} \to M$.

We first show a Weyl connection σ on $\mathcal{G}_0$ induces a Cartan connection w^σ in the type $(PO(p+1, q+1), PP_{line})$ on $\mathcal{G}$. Fix a Weyl connection σ. Then, every Weyl connection $\hat{\sigma} = \sigma + [\theta, \Upsilon]$ from Proposition 3.6.3 has a corresponding uniquely defined G_0-equivariant global section $s_{\hat{\sigma}} : \mathcal{G}_0 \to \mathcal{G}$ of $\mathcal{G} \to \mathcal{G}_0$:

(3.6.1) $$s_{\hat{\sigma}}(u) = s_\sigma(u) \cdot exp(\Upsilon(u)) \quad \forall u \in \mathcal{G}_0,$$

where $s_\sigma(u) = [u, id]$.

Remark 3.6.6. In fact, for fixed σ, all G_0-equivariant global sections are defined by (3.6.1) [14, Proposition 1.6.4 and Proposition 5.1.1].

Proposition 3.6.7. *According to Proposition 3.3.4, every Weyl connection $\hat{\sigma}$ induces a unique Cartan connection $w^{\hat{\sigma}}$ on $\mathcal{G}$ in the type $(PO(p+1,q+1), PP_{line})$ such that $s_{\hat{\sigma}}^* w^{\hat{\sigma}} = \hat{\sigma}$.*

Proposition 3.6.8. [14, Proposition 1.6.4] *Let $\pi^{\mathcal{G}}_{\mathcal{G}_0} \colon \mathcal{G} \to \mathcal{G}_0$, $[u,p] \mapsto u$. Given any Weyl connections $\hat{\sigma}$ and σ with respective Cartan connection $w^{\hat{\sigma}}$ and w^{σ} on $\mathcal{G}$. Denote $w^{\sigma} = \theta^{\sigma}_{-1} + \theta^{\sigma}_0 + \theta^{\sigma}_1$ with respect to the Lie algebra decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ in Proposition 2.2.2. Then,*

$$\theta^{\hat{\sigma}}_{-1} + \theta^{\hat{\sigma}}_0 = \theta^{\sigma}_{-1} + \theta^{\sigma}_0.$$

Proof. Let $s_\sigma \colon \mathcal{G}_0 \to \mathcal{G}, u \mapsto [u, id]$ be the map defined by σ in (3.6.1). Given $Z \in \mathfrak{g}_1$ and $u \in \mathcal{G}_0$. We first show

$$(\theta^{\sigma}_{-1} + \theta^{\sigma}_0)_{s_\sigma(u)\cdot e^Z} \circ d\pi^{\mathcal{G}}_{\mathcal{G}_0} = (\sigma - (ad\, Z) \circ \theta)_u \tag{3.6.2}$$

where θ is the canonical form on $\mathcal{G}_0$ from FrM. From the right equivariant property of w^{σ}, we have

$$w^{\sigma}_{s_\sigma(u)\cdot e^Z} = Ad(e^{-Z}) w^{\sigma}_{s_\sigma(u)} dR_{e^{-Z}}. \tag{3.6.3}$$

Noting that for all $A \in \mathfrak{g}_1$ and $p \in \mathcal{G}$,

$$w^{\sigma}(dR_{e^{-Z}} \zeta_A(p)) = Ad(e^Z) w^{\sigma}(\zeta_A(p)) = e^{ad(Z)} w^{\sigma}(\zeta_A(p)) = A.$$

Therefore,

$$(\theta^{\sigma}_{-1} + \theta^{\sigma}_0)_{s_\sigma(u)} dR_{e^{-Z}} = (\theta^{\sigma}_{-1} + \theta^{\sigma}_0)_{s_\sigma(u)}. \tag{3.6.4}$$

Recall from Proposition 3.3.4

$$w^{\sigma}_{s(u)}(ds(\xi)+\zeta_A(s(u))) = \sigma_u(\xi)+A \quad \forall A \in \mathfrak{g}_1.$$

Because $Ad(e^{-Z}) = e^{-ad(Z)} \equiv 1 - ad(Z) \mod \mathfrak{g}_1$, (3.6.3), (3.6.4), and the above equation, we have (3.6.2). Let $\hat{\sigma} = \sigma - ad(\Upsilon)\circ\theta$ from Proposition 3.6.3. Then,

$$\begin{aligned}
&(\theta^{\hat{\sigma}}_{-1}+\theta^{\hat{\sigma}}_{0})_{s_\sigma(u)\cdot e^Z} = (\theta^{\hat{\sigma}}_{-1}+\theta^{\hat{\sigma}}_{0})_{s_{\hat{\sigma}}(u)\cdot e^{-\Upsilon(u)+Z}}\\
&= (\hat{\sigma} - ad(-\Upsilon+Z)\circ\theta)_u \circ d\pi^{\mathcal{G}}_{\mathcal{G}_0}\\
&= (\sigma - ad(Z)\circ\theta)_u \circ d\pi^{\mathcal{G}}_{\mathcal{G}_0}\\
&= (\theta^{\sigma}_{-1}+\theta^{\sigma}_{0})_{s_\sigma(u)\cdot e^Z}
\end{aligned}$$

where the first equality is from (3.6.1) and the second equality is from (3.6.2). □

Definition 3.6.9. Let S be the collection of all Cartan connections w on $\mathcal{G}$ in the type $(PO(p+1,q+1), PP_{line})$ of the form $w = \theta_{-1}+\theta_0+w_1$ where $\theta_{-1}+\theta_0$ is defined from Proposition 3.6.8 and w_1 is the $\mathfrak{g}_1$-value of w.

In the following, we identify the Lie algebra components of $\mathfrak{g} = \mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ as $\mathfrak{g}_i \cong \mathfrak{g}^i/\mathfrak{g}^{i+1}$ where $\mathfrak{g}^i = \bigoplus_{k\geq i}\mathfrak{g}_k$ and $\mathfrak{g}_k = \{0\}$ if $|k|>1$.

Proposition 3.6.10. *The collection S forms an affine space; that is,* $\forall\, \hat{w}, w \in S$, $\exists! P \in \Gamma_{PP_{line}}(\mathcal{G}, \mathfrak{g}_1\oplus\mathfrak{g}^*_{-1})$ *such that*

$$\hat{w} = w - P\circ\theta_{-1}. \tag{3.6.5}$$

Their curvature functions in $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0$ values are

$$\hat{\kappa}_{-1} = \kappa_{-1} = 0 \qquad , \hat{\kappa}_0 = \kappa_0 - \partial P, \tag{3.6.6}$$

where $\partial\colon L(\mathfrak{g}_{-1}, \mathfrak{g}_1) \to L(\Lambda^2\mathfrak{g}_{-1}, \mathfrak{g}_0)$, $\quad \partial P(X,Y) = [X, P(Y)] - [Y, P(X)]$. *The map* ∂ *is* PP_{line}*-equivariant map and* $L(V,W)$ *is the set of linear maps from* V *to* W.

Proof. Let $w = \theta_{-1} + \theta_0 + \theta_1 \in S$ where θ_1 is the $\mathfrak{g}_1$-value of w. From the definition of the Cartan curvature K, its corresponding $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0$ values are

(3.6.7) $\quad K_{\mathfrak{g}_{-1}}(\tilde{\xi}, \tilde{\eta}) = d\theta_{-1}(\tilde{\xi}, \tilde{\eta}) + [\theta_0(\tilde{\xi}), \theta_{-1}(\tilde{\eta})] + [\theta_{-1}(\tilde{\xi}), \theta_0(\tilde{\eta})].$

(3.6.8) $\quad K_{\mathfrak{g}_0}(\tilde{\xi}, \tilde{\eta}) = d\theta_0(\tilde{\xi}, \tilde{\eta}) + [\theta_1(\tilde{\xi}), \theta_{-1}(\tilde{\eta})] + [\theta_0(\tilde{\xi}), \theta_0(\tilde{\eta})] + [\theta_{-1}(\tilde{\xi}), \theta_1(\tilde{\eta})].$

Since every element in S shares same $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0$ values, let's assume that w is the Cartan connection induced by a Weyl connection σ on $\mathcal{G}_0$ with the corresponding map $s\colon \mathcal{G}_0 \to \mathcal{G}, u \mapsto [u, id]$. Note that $s^*(\theta_{-1}+\theta_0) = \sigma = \theta + \gamma$ by Proposition 3.3.4 where the decomposition of σ is due to the arguments after Proposition 3.5.1. Considering the action of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ by the Lie bracket, then $s^*K_{\mathfrak{g}_{-1}} = K^{\sigma}_{\mathbb{R}^n} = 0$ where $K^{\sigma}_{\mathbb{R}^n}$ is the $\mathbb{R}^n$-value of the Cartan curvature of σ.

Let $P \circ \theta_{-1} = -(\hat{w} - w)$. Similar to the proof of Proposition 3.5.3, one can get the second identity in (3.6.6) from (3.6.8). To show ∂ is PP_{line}-equivariant, first note that if $P \in L(\mathfrak{g}_{-1}, \mathfrak{g}_1)$, then $\forall g \in PP_{line}$

$$(g \cdot P)(X) := Ad(g)(P(Ad(g^{-1})X)).$$

Then, the equivariancy means $(Ad(g)\partial P) = \partial(g \cdot P)$. The identity comes from direct computations on both sides of the equivariancy identity. □

Similar to the conditions in Theorem 3.5.6, it can be shown that $L(\Lambda^2\mathfrak{g}_{-1}, \mathfrak{g}_0)$ has the PP_{line}-invariant decomposition

$$L(\Lambda^2\mathfrak{g}_{-1}, \mathfrak{g}_0) = \mathcal{R} \oplus Im\,\partial.$$

Then, there exists a unique $w \in S$ such that the $\mathfrak{g}_0$-value of its Cartan curvature function is in $\mathcal{R}$. In the following, we still identify $\mathfrak{g}_i \cong \mathfrak{g}^i/\mathfrak{g}^{i+1}$ and denote the components of $\kappa \in L(\Lambda^2\mathfrak{g}_{-1}, \mathfrak{g}_0)$ by

$$\kappa(\hat{e}_i, \hat{e}_j)(\hat{e}_l) = [\kappa(\hat{e}_i, \hat{e}_j), \hat{e}_l] := \kappa_{ij}{}^k{}_l\, \hat{e}_k, \tag{3.6.9}$$

where $\mathfrak{g}_{-1} \cong \mathbb{R}^n$ in Proposition 2.2.2 and $(\hat{e}_i)$ is the standard basis of $\mathbb{R}^n$. We also denote $P(X) = X^i P_{ij}\, \hat{e}^j$ where $X \in \mathfrak{g}_{-1}$ and $P \in L(\mathfrak{g}_{-1}, \mathfrak{g}_1)$.

Theorem 3.6.11. *Let $\mathcal{R} = \{\kappa \in L(\Lambda^2\mathfrak{g}_{-1}, \mathfrak{g}_0) |\ \kappa_{ki}{}^k{}_j = 0\}$. Then,*

1. *The subspace $\mathcal{R}$ is P_{ray}-invariant.*

2. *Let $\kappa \in L(\Lambda^2\mathfrak{g}_{-1}, \mathfrak{g}_0)$. Then, $\exists! P_{ij} \in L(\mathfrak{g}_{-1}, \mathfrak{g}_1)$ such that $L = \kappa - \partial P$ satisfying $L_{ki}{}^k{}_j = 0$. The corresponding P is*

$$P_{ij} = \frac{1}{n-2}\left(\kappa_{ki}{}^k{}_j + \frac{1}{n}(\kappa_{kj}{}^k{}_i - \kappa_{ki}{}^k{}_j) - \frac{1}{2(n-1)}\delta^{ab}\kappa_{ka}{}^k{}_b\delta_{ij}\right) \tag{3.6.10}$$

   where δ_{ij} is the Kronecker delta.

3. *$L(\Lambda^2\mathfrak{g}_{-1}, \mathfrak{g}_0) = \mathcal{R} \oplus Im\, \partial$.*

Proof. For 1, note that if $\kappa \in L(\Lambda^2\mathfrak{g}_{-1}, \mathfrak{g}_0)$, then $\forall\, p = g_0 e^Z \in PP_{line}$ where $g_0 \in G_0$ and $Z \in \mathfrak{g}_1$ by Proposition 2.2.4

$$(p \cdot \kappa)(X, Y) := Ad(p)(\kappa(Ad(p^{-1})X, Ad(p^{-1})Y)) \qquad \forall X, Y\mathfrak{g}_{-1}.$$

Denote $\kappa' = p \cdot \kappa$. Then,

$$\begin{aligned}\kappa'^{\ \ k}_{ij\ \ l}\hat{e}_k &= (p \cdot \kappa)(\hat{e}_i, \hat{e}_j) \cdot \hat{e}_l \\ &= [Ad(p)\kappa(Ad(p^{-1})\hat{e}_i, Ad(p^{-1})\hat{e}_j), \hat{e}_l] \\ &= Ad(g_0)[\kappa(Ad(g_0^{-1})\hat{e}_i, Ad(g_0^{-1})\hat{e}_j), Ad(g_0^{-1})\hat{e}_l].\end{aligned}$$

Since $\kappa_{ki}{}^k{}_j = 0$ with respect to any basis of $\mathfrak{g}_{-1}$, we have $\kappa'_{ki}{}^k{}_j = 0$. The 2 is in [14, Lemma 1.6.6] and 3 is the result of 2. □

Remark 3.6.12. From Remark 3.6.4, if σ corresponds to a Levi-Civita connection ∇ of $g \in [g]$, then P_{ij} in (3.6.10) corresponds to the Schouten tensor

$$P_{ij} = \frac{1}{n-2}\left(Ric_{ij} - \frac{R}{2(n-1)}g_{ij}\right)$$

where Ric is the Ricci curvature and R is the scalar curvature.

Definition 3.6.13. A Cartan geometry $(\mathcal{G}, w) \to M^n$ in the type $(PO(p+1, q+1), PP_{line})$ is called normal if its Cartan curvature function is torsion-free and Ricci-trace-free; that is, $\kappa_{-1} = 0$ and $(\kappa_0)_{ki}{}^k{}_j = 0$.

Using Proposition 3.6.10 and Theorem 3.6.11, it can be shown normal Cartan connection is uniquely defined in S where S is from Definition 3.6.9.

Corollary 3.6.14. *There exists a unique Cartan connection $w \in S$ on $\mathcal{G}$ in the type $(PO(p+1, q+1), PP_{line})$ such that w is normal.*

Proposition 3.6.15. [14, 1.6.8] *Given a Weyl connection σ and let w^σ be the induced Cartan connection on $\mathcal{G}$ due to Proposition 3.6.7. Let $w^{nor} = w^\sigma - P \circ \theta_{-1}$ be the normal Cartan connection by Corollary 3.6.14. Then, the respective values $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$*

and $\mathfrak{g}_1$ of the curvature function κ^{nor} of w^{nor} are

$$\kappa_{-1}^{nor} = \kappa_{-1}^{\sigma} = 0, \quad \kappa_0^{nor} = \kappa_0^{\sigma} - \partial P \in \Gamma_{P_{ray}}(\mathcal{G}, \mathfrak{g}_0 \otimes \Lambda^2 \mathfrak{g}_{-1}^*), \quad \kappa_1^{nor} \in \Gamma_{G_0}(\mathcal{G}, \mathfrak{g}_1 \otimes \Lambda^2 \mathfrak{g}_{-1}^*)$$

The components of κ_0^{nor} are

$$(\kappa_0^{nor})_{ij}{}^l{}_k = (\kappa_0^{\sigma})_{ij}{}^l{}_k - [\delta_i^l P_{jk} - \delta_j^l P_{ik} - P_j{}^l \delta_{ik} + P_i{}^l \delta_{jk} - (P_{ij} - P_{ji})\delta_k^l] \tag{3.6.11}$$

where δ_j^i is the Kronecker delta. If σ corresponds to a Levi-Civita connection ∇ on M, then κ_0^{nor} induces the Weyl curvature tensor and κ_0^{σ} induces the Riemann curvature tensor of ∇. Let $s\colon \mathcal{G}_0 \to \mathcal{G}$, $u \mapsto [u, id]$ be the map by σ. Then, $s^*\kappa_1^{nor}$ induces the Cotton-York tensor Y on M

$$Y_{ijk} = -\nabla_{[i} P_{j]k}. \tag{3.6.12}$$

It turns out we have the following theorem.

Theorem 3.6.16. [14, Theorem 1.6.7] *Let $n = p + q \geq 3$. The following categories are equivalent.*

1. *The category of normal Cartan geometries in the type $(PO(p+1, q+1), PP_{line})$.*

2. *The category of $CO(p, q)$-reductions of FrM.*

Remark 3.6.17. Note that $\mathfrak{g}_{-1} \subseteq \mathfrak{g}$ is not PP_{line}−stable, so the $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$-value of w^{nor} is not a principal PP_{line} connection by Proposition 3.3.2. To have a principal connection from w^{nor}, one can associate $\mathcal{G}$ with $PO(p+1, q+1)$ to have a principal PO-bundle $\mathcal{G}_{PO}$ with the induced principal connection γ from w^{nor} by Proposition 3.3.3. For $n = p \geq 3$, recalling $PO(n+1, 1) \cong O_+(n+1, 1)$ in Example 2.0.2, the associated vector bundle $\mathcal{T} = \mathcal{G}_{O_+} \times_{O_+} \mathbb{R}^{n+1,1}$ has the induced linear connection ∇ from γ. The $\mathcal{T}$ is called the **standard tractor bundle**, and ∇ is called the **normal tractor connection** [21]. The $O_+(n+1,1)$-representations are special cases of $(\mathfrak{g}, P)$-modules whose associated vector bundles are called **tractor bundles** [22]. For indefinite signature (p,q), since $PO(p+1,q+1)$ doesn't have a standard representation, another natural choice of normal Cartan geometries is in the type $(O(p+1,q+1), P_{ray})$. The associated vector bundle with the standard representation of $O(p+1,q+1)$ gives the standard tractor bundle with the normal tractor connection [23].

Recall Proposition 3.1.5, a choice of a local conformal frame $(e_i) \in \Gamma(\mathcal{G}_0)$ and a Weyl connection σ determine a local chart of $\mathcal{G}$

$$U \xrightarrow{(e_i)} \mathcal{G}_0 \xrightarrow{\sigma} \mathcal{G}. \tag{3.6.13}$$

One can therefore derive the formula of the normal tractor connection ∇ on the standard tractor bundle with respect to a Weyl connection σ by using Corollary 3.2.9. The formula for ∇ is in [24, Equation (22)].

3.6.1 Conformal Geodesics

Given a conformal manifold $(M^n, [g])$ and the corresponding Cartan geometry in the type $(PO(p+1,q+1), PP_{line}, w)$ with the corresponding principal-PP_{line} bundle $\pi\colon \mathcal{G} \to M$ where $\mathcal{G} = \mathrm{Fr}_{G_0}M \times_{G_0} PP_{line}$. Let $g \in [g]$ and ∇ be its Levi-Civita connection. Recalling Remark 3.6.6, there is a corresponding global section $s\colon \mathrm{Fr}_{G_0}M \to \mathcal{G}$. Recall Proposition 2.2.4, $PP_{line} \cong G_0 \ltimes \exp(\mathfrak{g}_{-1})$ where $\mathfrak{g}_{-1} \cong \mathbb{R}^n$. Let (e_i) be an orthonormal frame of TM with respect to g defined on an open set $U \subseteq M$. Then, (e_i)

and s induce a local chart of $\mathcal{G}$:

$$\pi^{-1}(U) \to U \times G_0 \times \mathfrak{g}_{-1}$$
$$s((e_i)_x \cdot C) \cdot \exp(b) \mapsto (x, C, b).$$

Identify T_xM and $\mathbb{R}^n$ by $v^i e_i \mapsto v^i \hat{e}_i$ where (e_i) is the chosen orthonormal frame.

Theorem 3.6.18. [7] *Let $\mathcal{X} = \mathcal{X}^i \hat{e}_i \in \mathfrak{g}_{-1} \cong \mathbb{R}^n$ and $\tilde{\gamma}_X : I \to \mathcal{G}$ be an integral curve of $w^{-1}(\mathcal{X})$ so that $\pi(\tilde{\gamma}_X) \subseteq U$. Denote the coordinate of $\tilde{\gamma}_X(t)$ by $(x(t), C^i{}_j(t), b_k(t))$. Then,*

$$\begin{aligned} \dot{x}^i &= C^i{}_j \mathcal{X}^j; \\ \dot{C}^i{}_j &= -(\Gamma^i_{kl} + b^i_{kl}) C^k{}_n \mathcal{X}^n C^l{}_j; \\ \dot{b}_k &= (b_j \Gamma^j_{ik} + \frac{1}{2} b_j b^j_{ik} + P_{ik}) C^i{}_n \mathcal{X}^n, \end{aligned} \tag{3.6.14}$$

where $\dot{x} = \dot{x}^i e_i$, Γ^k_{ij} is the Christoffel symbols of ∇ defined by $\nabla_{e_i} e_j = \Gamma^k_{ij} e_k$, and b^j_{ik} is defined by

$$b^j_{ki} = \delta^j_k b_i + \delta^j_i b_k - g_{ki} g^{jl} b_l. \tag{3.6.15}$$

where $\delta^j_i = g^{jk} g_{ki}$. It turns out the integral curve equations of (3.6.14) can be written as

$$\begin{aligned} (\nabla_{\dot{x}} \dot{x})^i &= -b^i_{jk} \dot{x}^j \dot{x}^k; \\ (\nabla_{\dot{x}} c_j)^i &= -b^i_{lk} \dot{x}^l C^k{}_j; \\ (\nabla_{\dot{x}} b)_i &= (\tfrac{1}{2} b_j b^j_{ki} + P_{ki}) \dot{x}^k, \end{aligned} \tag{3.6.16}$$

where $c_j = e_k C^k{}_j$.

Since $x(t)$ and $b(t)$ can be determined by the first and third equations of (3.6.16).

Then, one can define the conformal geodesic equation as in the following.

Definition 3.6.19. Let $g \in [g]$ and ∇ be the corresponding Levi-Civita connection. A conformal geodesic with respect to g is a curve $\gamma \colon I \longrightarrow M$ and a 1-form b along γ such that they satisfy

$$\nabla_{\dot\gamma}\dot\gamma^i = -S_{jk}{}^{il} b_l \dot\gamma^j \dot\gamma^k, \tag{3.6.17}$$

$$\nabla_{\dot\gamma} b_i = (\frac{1}{2} b_j b_l S_{ki}{}^{jl} + P_{ki})\dot\gamma^k \tag{3.6.18}$$

where $S_{ij}{}^{kl}$ is defined by

$$S_{ik}{}^{jl} = \delta_i^j \delta_k^l + \delta_i^l \delta_k^j - g_{ik} g^{jl}. \tag{3.6.19}$$

The equations (3.6.17) and (3.6.18) are conformally invariant for conformal change $\hat g = \Omega^2 g$ if $\dot{\hat\gamma} = \dot\gamma$ and $\hat b_i = b_i - \Omega^{-1}\nabla_i \Omega$.

Denote $g_{ij}v^i w^j$ by $\langle v, w\rangle$ for $g \in [g]$. We call a curve $\gamma \colon I \longrightarrow M$ *null* if $\langle \dot\gamma, \dot\gamma\rangle = 0$ on I and it is called *non-null* if it's not null. Direct computation from (3.6.17) shows

$$\nabla_{\dot\gamma}\langle \dot\gamma, \dot\gamma\rangle = -2\langle \dot\gamma, \dot\gamma\rangle b_i \dot\gamma^i. \tag{3.6.20}$$

Therefore, if a conformal geodesic has a null velocity at some point, then it's a null conformal geodesic. For completeness, we recall null pseudo-Riemannian geodesics are necessary and sufficient to be null conformal geodesics with some reparametrization [10, 25].

Assume a conformal geodesic γ is non-null. By solving b_i in (3.6.17)

$$b_i = \frac{1}{\langle \dot\gamma, \dot\gamma\rangle}\left(\nabla_{\dot\gamma}\dot\gamma_i - 2\frac{\langle \dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle}{\langle \dot\gamma, \dot\gamma\rangle}\dot\gamma_i\right), \tag{3.6.21}$$

one can have a third-order differential equation from (3.6.18). The third-order differential equation is equivalent to the system of the equations (3.6.17) and (3.6.18) if one defines the one form b_i back.

Definition 3.6.20. Given $g \in [g]$ with corresponding Levi-Civita connection ∇. A parametrized non-null conformal geodesic γ is defined to satisfy

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma^i = 3\frac{\langle\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle}{\langle\dot\gamma,\dot\gamma\rangle}\nabla_{\dot\gamma}\dot\gamma^i - \frac{3\langle\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle}{2\langle\dot\gamma,\dot\gamma\rangle}\dot\gamma^i + \langle\dot\gamma,\dot\gamma\rangle\dot\gamma^j P_j{}^i - 2P_{jk}\dot\gamma^j\dot\gamma^k\dot\gamma^i \tag{3.6.22}$$

with initial conditions $\gamma(0),\ \dot\gamma(0),\ \nabla_{\dot\gamma}\dot\gamma(0)$. The equation is invariant under conformal change $\hat g = \Omega^2 g$.

Since the induced metric on a non-null curve γ is nondegenerate, we can consider the orthogonal decomposition of the pullback bundle of TM by γ [26]. We call γ satisfies the *tangential (resp. normal) part* of (3.6.22) if it is a solution of the equation that is the orthogonal projection of (3.6.22) to the tangent (resp. normal) bundle of γ. It is known [6] that any regular curve can be reparametrized to satisfy the tangential part of (3.6.22). The normal part of (3.6.22) is invariant under reparametrization of γ and it is only satisfied by non-null conformal geodesics. Since (3.6.22) is derived from (3.6.18), it's convenient to introduce a vector field along an arbitrary curve γ

$$E^i(\gamma, v, g) = \nabla_{\dot\gamma}v^i - (\tfrac{1}{2}v^j v^l S_k{}^i{}_{jl} + P_k{}^i)\dot\gamma^k \tag{3.6.23}$$

where v is a vector field along γ. If v satisfies the right-hand side of (3.6.21) by descending index, then we denote the vector field $E^i(\gamma, v, g)$ by $E^i(\gamma, g)$.

For completeness, we recall another third-order differential equation for a non-null conformal geodesic γ. If $\gamma : I \to M$ is reparametrized so that it is of unit tangent velocity with respect to $g \in [g]$, then it satisfies [10]

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma^i = -\left(\langle\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle + P_{jk}\dot\gamma^j\dot\gamma^k\right)\dot\gamma^i + P^i_j\dot\gamma^j \quad \text{if } \langle\dot\gamma,\dot\gamma\rangle = 1; \tag{3.6.24}$$

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma^i = \left(\langle\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle - P_{ik}\dot\gamma^i\dot\gamma^k\right)\dot\gamma^j + P^j_i\dot\gamma^i \quad \text{if } \langle\dot\gamma,\dot\gamma\rangle = -1; \tag{3.6.25}$$

The first line is the equation introduced by Yano [12]. Recalling conformal geodesics in the Euclidean space are either straight lines or planar circles in Theorem 2.2.7, P. Tod has derived the same result by considering (3.6.24) in [10].

Chapter 4

Poincaré-Einstein Manifold

This chapter reviews a Poincaré-Einstein manifold and the asymptotic isothermal coordinate expansion of a surface therein.

Let $(M^n, [g])$ be a pseudo-Riemannian conformal manifold with signature (p, q) and $n \geq 2$. There exists a pseudo-Riemannian manifold (M_+^{n+1}, g_+) with boundary $\partial M_+ = M$ [27]. The signature of g_+ is $(p+1, q)$. Let $r \in C^\infty(M_+)$ be a defining function for M, that is $r > 0$ on the interior M_+^0, $r = 0$ on M and $dr \neq 0$ on M. Then, g_+ and r satisfy

(i) $r^2 g_+$ can be smoothly extended to be a metric on M_+ so that

$$\bar{g}|_M := r^2 g_+|_M \in [g];$$

(ii) $Ric(g_+) + n g_+ = O(r)$.

The pair (M_+, g_+) is called a *Poincaré-Einstein manifold* for $(M, [g])$ and the pair $(M, [g])$ is called the *conformal infinity* to (M_+, g_+).

Let $g \in [g]$. There exists a unique defining function r, called *geodesic defining function*, such that $|dr|^2_{\bar{g}} = 1$ near $M \subseteq M_+$. The function r makes an identification between a neighborhood U of M in M_+ and a neighborhood $\mathcal{U}$ of $M \times \{0\}$ in $M \times [0, \infty)$. By the identification, g_+ is in *normal form* relative to g; that is, $g_+ = \frac{dr^2 + g_r}{r^2}$ on $\mathcal{U}$ where $g_r = g - r^2 P + O(r^3)$. The tensor P is the Schouten tensor of g for $n \geq 3$. When $n = 2$, P is a symmetric two-tensor on M satisfying $P_i{}^i = \frac{1}{2} R$ and $P_{ij,}{}^j = \frac{1}{2} R_{,i}$ where R is the scalar curvature of g. By the pulling back of an even diffeomorphism between neighborhoods of $M \times \{0\} \subseteq M \times [0, \infty)$ which restricts to the identity map on $M \times \{0\}$, the normal forms g_+ relative to conformal related metrics are identical modulo $O(r)$.

Remark 4.0.1. Note that the orders of r above in the Ricci condition and in g_r can be further refined to higher orders depending on dimension n. See details in [27].

Remark 4.0.2. For $n = 2$, the trace and the divergence conditions of P are conformally invariant. ([27], arguments after Theorem 3.7)

Surfaces in the Poincaré-Einstein Manifold

Let (M_+^{n+1}, g_+) be a Poincaré-Einstein manifold in normal form relative to $g \in [g]$ where r is its corresponding geodesic defining function. Let $\gamma \colon I \to M$ be a non-null curve. The interval I can be shrunk if necessary. Choose a local coordinate $\{x^i | \, 1 \leq i \leq n\}$ on an open set $\mathcal{W}$ in M containing $\gamma(t)$ for $t \in I$. The coordinate of $\gamma(t)$ is denoted by $\gamma^i(t)$. If $\Sigma \subseteq \overline{M_+}$ is an embedded surface orthogonal to M with $\Sigma \cap M = \gamma$, then one can have an **asymptotic isothermal coordinate** of Σ near γ; that is, there is a diffeomorphism $\sigma \colon (t, \lambda) \mapsto (x^i(t, \lambda), r(t, \lambda))$ from $I \times I$ to $\Sigma \subseteq \overline{M_+}$ such that

$$\begin{cases} \sigma(t,0) = \gamma^i(t) \\ \sigma^*\bar{g} = \begin{pmatrix} (-1)^\varepsilon & 0 \\ 0 & 1 \end{pmatrix} c(t,\lambda) + O(\lambda^3) \end{cases}, \tag{4.0.1}$$

where $c(t,0) \neq 0$ and $\varepsilon = 0$ if $\langle \dot\gamma, \dot\gamma \rangle > 0$ and $\varepsilon = 1$ otherwise. In fact, to satisfy (4.0.1) for Σ orthogonal to M, the expansions of x^i, r and $\sigma^* g_+$ with respect to λ are in the following forms.

Proposition 4.0.3. [9] *Let g_+ be a Poincaré-Einstein metric and $\Sigma \subseteq \overline{M_+}$ a surface as above. Then, the asymptotic isothermal coordinate in (4.0.1) satisfies*

(4.0.2)

$$\begin{array}{rcccccccc} x^i(t,\lambda) & \equiv & \gamma^i(t) & + & 0 & + & \frac{|\dot\gamma|^2 v^i}{2}\lambda^2 & + & \frac{u^i}{3}\lambda^3 \\ r(t,\lambda) & \equiv & 0 & + & |\dot\gamma|\lambda & + & 0 & + & \frac{(-1)^\varepsilon |\dot\gamma|}{6}\left[\kappa(\gamma,v,g) - \frac{3}{2}\langle\dot\gamma,\dot\gamma\rangle\langle v,v\rangle\right]\lambda^3 \end{array}$$

and

$$\sigma^* g_+ = \frac{1}{\lambda^2}\begin{pmatrix} (-1)^\varepsilon & 0 \\ 0 & 1 \end{pmatrix}\left(1 + \frac{(-1)^\varepsilon\, 2}{3}\kappa(\gamma,v,g)\lambda^2\right) + O(\lambda), \tag{4.0.3}$$

where the expansions of x^i and r are modulo $O(\lambda^4)$, the $|\dot\gamma|$ denotes the square root of $|\langle\dot\gamma,\dot\gamma\rangle|$, the v^i satisfies $\langle\dot\gamma, v\rangle = \left\langle \dot\gamma, \nabla_{\dot\gamma}\left(\frac{\dot\gamma}{\langle\dot\gamma,\dot\gamma\rangle}\right)\right\rangle$, the u^i satisfies $\langle\dot\gamma, u\rangle = 0$ and $\kappa(\gamma,v,g) = \langle E(\gamma,v,g),\dot\gamma\rangle$ which is the tangential part of (3.6.23). Note that $\nabla_{\dot\gamma}\left(\frac{\dot\gamma}{\langle\dot\gamma,\dot\gamma\rangle}\right)$ is equal to the right-hand side of the b-form in (3.6.21) by lowering index.

Since $\sigma^*\bar{g}$ is pseudo-Riemannian for small λ, the tangent bundle TM_+ has the orthogonal decomposition along $T\Sigma$ near M [26]. Let $(e_{\alpha'})$ be a local orthonormal frame of the normal bundle of Σ with respect to $\bar{g}$ near $\Sigma \cap M$. After direct computation, the Taylor expansions of $e_{\alpha'}$ with respect to λ are [9]

$$
\begin{aligned}
(4.0.4)\quad e_{\alpha'}(t,\lambda) =& \phi_{\alpha'}(\lambda) - \frac{(-1)^{\varepsilon}\lambda^2}{2}\left[\langle \phi_{\alpha'}(\lambda), \partial_t v\rangle_g + \left(v^i \partial_i g - 2P^g\right)(\phi_{\alpha'}(\lambda), \dot{\gamma})\right]\dot{\gamma} \\
& - \langle \phi_{\alpha'}(\lambda), |\dot{\gamma}| v\rangle_g \ \lambda \ \partial_r + O(\lambda^3),
\end{aligned}
$$

where $\phi_{\alpha'}(\lambda)$ is a family of sections of the normal bundle of γ in M.

Proposition 4.0.4. [9] *Considering the projection of the second fundamental form of Σ on $e_{\alpha'}$ with respect to g_+, then, its asymptotic expansion is*

(4.0.5)

$$
\frac{1}{\lambda^2}\begin{pmatrix} (-1)^{\varepsilon}\left\langle \nabla_{\dot{\gamma}}\left(\frac{\dot{\gamma}}{\langle \dot{\gamma},\dot{\gamma}\rangle}\right) - v, \phi_{\alpha'}(\lambda)\right\rangle - \frac{\langle u, \phi_{\alpha'}(\lambda)\rangle}{\langle \dot{\gamma},\dot{\gamma}\rangle}\lambda & * \\ \langle E(\gamma, v, g), \phi_{\alpha'}(\lambda)\rangle\lambda & (-1)^{\varepsilon}\frac{\langle u, \phi_{\alpha'}(\lambda)\rangle}{\langle \dot{\gamma},\dot{\gamma}\rangle}\lambda \end{pmatrix} + O(1).
$$

Due to Proposition 4.0.3, the asymptotic minimal condition $H = O(r^2)$ of Σ is equivalent to $v = \nabla_{\dot{\gamma}}\left(\frac{\dot{\gamma}}{\langle \dot{\gamma},\dot{\gamma}\rangle}\right)$ which is exactly the same as the b-form in (3.6.21). The asymptotic totally geodesic condition $K = O(r^2)$ is equivalently satisfied when $v = \nabla_{\dot{\gamma}}\left(\frac{\dot{\gamma}}{\langle \dot{\gamma},\dot{\gamma}\rangle}\right)$, $u = 0$ and γ being an unparametrized conformal circle.

Note that if one considers the first variation of renormalized area of Σ with free boundary in M, then the critical point is a minimal surface with $u = 0$ [9, 28] (need to rewrite this last statement).

Definition 4.0.5. Let $\Sigma \subseteq \overline{M_+}$ be an embedded surface orthogonal to M so that $\Sigma \cap M$ is a non-null curve. It is called a proper surface if it is asymptotic totally geodesic $K = O(r^2)$ where K is the second fundamental form of Σ with respect to g_+.

Chapter 5

Preserving Geodesics

§ 5.1 Geodesics in Riemannian Manifolds and Pseudo-Riemannian Conformal Manifolds

In this section, we derive in Theorem 5.1.1 that if a local diffeomorphism f between manifolds M^n and N^n maps conformal geodesics to conformal geodesics, then f is a conformal local diffeomorphism. We also mention in Remark 5.1.4 that the idea of the proof of the theorem can apply to the parallel problem for parametrized Riemannian geodesics.

Given a local diffeomorphism $f\colon M \to N$ between pseudo-Riemannian conformal manifolds $(M^n, [g])$ and $(N^n, [h])$. Assume both of the conformal classes have same signature (p, q). If f is a conformal local diffeomorphism, it's direct to see f maps unparametrized conformal geodesics to unparametrized conformal geodesics. The converse direction is also true if the map f preserves some nullity condition.

Theorem 5.1.1. *Let $(M^n, [g])$ and $(N^n, [h])$ be pseudo-Riemannian conformal manifolds with same signature (p, q). Assume a local diffeomorphism $f\colon M \to N$ satisfying* *(i)* $df(v)$ *is non-null (resp. null)* $\forall$*non-null (resp. null)* $v \in TM$ *if* $p \neq q$*;*

(ii) $sgn\langle v, v\rangle_g = sgn\langle df(v), df(v)\rangle_h$ $\forall v \in TM$ *if* $p = q$*.*

If f *maps unparametrized non-null conformal geodesics to unparametrized non-null conformal geodesics, then* f *is a conformal local diffeomorphism.*

Proof. Given $y \in M$. Let $g \in [g]$ and $h \in [h]$. Choose a normal coordinate of g centered at y, $\{x^i | 1 \leq i \leq n\}$. Since f is a local diffeomorphism, we can identify the coordinate system near $f(y)$ as $\{x^i\}_{i=1}^n$. Let $\gamma(t)$ be a parametrized non-null conformal geodesic satisfying (3.6.22) with initial conditions $\dot{\gamma}_0^k$, $\ddot{\gamma}_0^k$ at $y = \gamma(0)$ where $\ddot{\gamma}_0^k$ is the coordinate of $\nabla_{\dot{\gamma}}\dot{\gamma}(0)$ with respect to g. Since $f \circ \gamma$ is an unparametrized non-null conformal geodesic, it satisfies the normal part of (3.6.22) with the given parameter t, which is the following in the coordinate we chose.

$$E^k(\gamma, h) - \frac{\langle E(\gamma, h), \dot{\gamma}\rangle_h}{\langle \dot{\gamma}, \dot{\gamma}\rangle_h}\dot{\gamma}^k = 0 \tag{5.1.1}$$

where $E(\gamma, h)$ is a vector field along γ defined from (3.6.23). In the following, we are considering $t = 0$ for (5.1.1). Observe that (5.1.1) is a degree-two polynomial of $\ddot{\gamma}_0^k$ with coefficients depending on the derivatives of g and h. It is because $\gamma(t)$ satisfies the third order differential equation (3.6.22) with respect to g. Now let $\dot{\gamma}_0^k = V^k$ and $\ddot{\gamma}_0^k = \epsilon A^k$ where $V, A \in \mathbb{R}^n$ are fixed. The variable $\epsilon \in \mathbb{R}$ is an arbitrary number in an open interval containing 1. Because the non-null conformal geodesic equation is an autonomous ODE, (5.1.1) depends smoothly on ϵ. Based on the arguments we just made, we know (5.1.1) at $t = 0$ is a polynomial of ϵ with degree two. Therefore, the coefficient of ϵ^2 vanishes. After direct computation, the coefficient gives

$$\frac{\langle V,A\rangle_g}{\langle V,V\rangle_g}\left(A^k-\frac{\langle V,A\rangle_h}{\langle V,V\rangle_h}V^k\right)=\frac{\langle V,A\rangle_h}{\langle V,V\rangle_h}A^k-\frac{\langle V,A\rangle_h^2}{\langle V,V\rangle_h^2}V^k. \tag{5.1.2}$$

If $\langle V,A\rangle_g=0$, we get

$$0=\frac{\langle V,A\rangle_h}{\langle V,V\rangle_h}A^k-\frac{\langle V,A\rangle_h^2}{\langle V,V\rangle_h^2}V^k. \tag{5.1.3}$$

Assume the normal coordinate we chose is $g_{ii}(y)>0$ for $1\le i\le p$ and $g_{ii}(y)<0$ for $p+1\le i\le p+q$. If $V^k=\delta^{ik}$ and $A^k=\delta^{jk}$ for $i\neq j$, then $h_{ij}(y)=0$ from (5.1.3). If $g_{ii}=g_{jj}$ for $i\neq j$, we then let $V^k=\delta^{ik}+\delta^{jk}$ and $A^k=\delta^{ik}-\delta^{jk}$; so, we have $h_{ii}=h_{jj}$ from (5.1.3). Therefore, the pullback metric $(f^*h)_{ij}$ is of the form

$$\begin{pmatrix} B\mathbb{I}_p & \\ & -C\mathbb{I}_q\end{pmatrix}$$

for some B, $C\neq 0$. If $p\neq q$, then B and C are positive because h is of the signature (p,q). If $p=q$, then B and C can be both positive or both negative. Recalling that the f preserves the nullity of null vectors, we know $(\partial_i+\partial_j)$ is null at y with respect to f^*h when $g_{ii}\neq g_{jj}$ which implies $B=C$. The sign of B and C is positive for $p=q$ since the sign of h_{ii} is the same as the sign of g_{ii}. □

Remark 5.1.2. Note that if $(M,[g])$ is Riemannian, that is $p=n$ and $q=0$, then any local diffeomorphism $f\colon M\longrightarrow N$ automatically satisfies (i) and (ii) in Theorem 5.1.1.

Remark 5.1.3. The particular case for Theorem 5.1.1 has been studied in the literature if one assumes the f to be a bijection (no need to be continuous) and $M=N=\mathbb{R}^n$ with the standard Riemannian conformal structure [4, 29]. Note that the proof in [4, 29] needs the global property of conformal geodesics; that is, f maps straight lines (resp.

circles) to straight lines (resp. circles). However, by assuming additional regularity of f in this paper, we only need the local condition of conformal geodesics, namely the conformal geodesic equation, to establish Theorem 5.1.1.

Remark 5.1.4. One can give a different proof from ([30], Chapter 6, Addendum 1) for the parallel problem of parametrized Riemannian geodesics by following the proof idea of Theorem 5.1.1.

Remark 5.1.5. Though the initial condition $\ddot{\gamma}_0^k = \epsilon A^k$ in the proof gives an ϵ-family of conformal geodesics which induce a Jacobi field, we do not need the Jacobi field equation introduced by [31, 32] to prove the theorem.

We follow similar arguments of Theorem 5.1.1 to prove Theorem 5.2.5 in §5.2.

§ 5.2 Holorgraphic Interpretation to Preserving Conformal Geodesics

In this section, we consider a local diffeomorphism $F\colon M_+ \to N_+$ which smoothly extends a local diffeomorphism $f\colon M \to N$. We introduce the definition of asymptotic local isometry and cosider its local conditions. We also introduce an adapted coordinate for a surface Σ in M_+. We prove Theorem 5.2.5 in this section.

Let $(N^n, [h])$ be a pseudo-Riemannian conformal manifold with same signature (p, q) as $(M, [g])$ and with a Poincaré-Einstein space (N_+, h_+). We keep $\{x^i\}_{i=1}^n$ as a coordinate system on an open set $\mathcal{W}$ in M.

Definition 5.2.1. A local diffeomorphism $F\colon M_+ \to N_+$ is called an asymptotic local isometry if

$$F^* h_+ - g_+ = O(r). \tag{5.2.1}$$

It's useful to realize Definition 5.2.1 in terms of local coordinates. Let r and s be geodesic defining functions for $g \in [g]$ and $h \in [h]$ respectively. Identifying some neighborhoods of $M \subseteq \overline{M_+}$ and $N \subseteq \overline{N_+}$ to neighborhoods $\mathcal{U}$ of $M \times \{0\} \subseteq M \times [0, \infty)$ and $\mathcal{V}$ of $N \times \{0\} \subseteq N \times [0, \infty)$ respectively, a local diffeomorphism $F\colon M_+ \to N_+$ can be identified near $M \subseteq \overline{M_+}$ and $N \subseteq \overline{N_+}$ as a local diffeomorphism from $\mathcal{U}$ to $\mathcal{V}$

$$(x, r) \mapsto (\mathscr{F}(x, r), F^s(x, r)) \tag{5.2.2}$$

where $\mathscr{F}(x, 0) = f(x)$, $s \circ F = F^s$ on $\mathcal{U}$ and $F^s(x, 0) = 0$. Then, (5.2.1) is equivalent to

$$F^* \bar{h} - (F^s/r)^2 \, \bar{g} = O(r^3), \tag{5.2.3}$$

where $\bar{g} = dr^2 + g_r$ and $\bar{h} = ds^2 + h_s$. Since f is a local diffeomorphism, we denote the coordinate of $\mathscr{F}(x, r)$ by $F^i(x, r)$ with $F^i(x, 0) = x^i$. In terms of the coordinates (x^i, r) on M_+ and (x^i, s) on N_+, (5.2.3) is given by

$$O(r^3) = F^s_{,i} F^s_{,j} + F^k_{,i} F^l_{,j} (h_s \circ F)_{kl} - (F^s/r)^2 \, (g_r)_{ij}, \tag{5.2.4}$$

$$O(r^3) = F^s_{,r} F^s_{,r} + F^k_{,r} F^l_{,r} \, (h_s \circ F)_{kl} - (F^s/r)^2 \,, \tag{5.2.5}$$

$$O(r^3) = F^s_{,r} F^s_{,i} + F^k_{,r} F^l_{,i} (h_s \circ F)_{kl} \tag{5.2.6}$$

where we have used commas to express partial derivatives with respect to the coordinates (x^i, r) on M_+. At $r = 0$, (5.2.4) and (5.2.6) give

$$f^*h = e^{2\sigma}g, \quad F^s_{,r} = e^{\sigma} \quad \text{for some } \sigma \in C^\infty(M), \qquad (5.2.7)$$
$$F^i_{,r} = 0 \quad \text{for } 1 \le i \le n,$$

If we choose $g \in [g]$ and $h \in [h]$ suitably such that f is homothetic, that is σ constant, then (5.2.4)-(5.2.6) further imply at $r = 0$

$$F^i_{,rr} = F^i_{,rrr} = 0 \quad \text{for } 1 \le i \le n, \qquad (5.2.8)$$
$$F^s_{,rr} = 0,$$

and at $r = 0$

$$F^s_{,rrr} = 0. \qquad (5.2.9)$$

Conversely, if F satisfies (5.2.7)-(5.2.9), then F is an asymptotic local isometry.

Let $\Sigma \subseteq M_+$ be a surface orthogonal M and (t, λ) its asymptotic isothermal coordinate (4.0.2). Since we are considering (5.2.7) to (5.2.9), it's better to introduce a change of variables on Σ, $(t, \lambda) \mapsto (t, \underline{r}(t, \lambda))$, where $\underline{r}(t, \lambda)$ is equal to the right-hand side of $r(t, \lambda)$ in (4.0.2) modulo $O(\lambda^4)$. Then, the expansions of x^i and r in terms of $\underline{r}$ are

$$\begin{aligned} x^i(t, \underline{r}) &= \gamma^i(t) + 0 + \tfrac{v^i}{2}\underline{r}^2 + \tfrac{u^i}{3|\dot\gamma|^3}\underline{r}^3 + O(\underline{r}^4), \\ r(t, \underline{r}) &= \underline{r} + O(\underline{r}^4) \end{aligned} \qquad (5.2.10)$$

where v^i and u^i remain the same conditions as in Proposition 4.0.3 and in Proposition 4.0.4. We call $(t, \underline{r})$ the *adapted coordinate* of Σ.

Proposition 5.2.2. *Let $F\colon M_+ \to N_+$ be a local diffeomorphism in terms of (5.2.2).*

Assume F maps non-null vectors in TM to non-null vectors in TN. Then, $\partial_r\mathscr{F} = 0$ at $r = 0$ if and only if $F(\Sigma)$ is orthogonal to N for any surface Σ in $\overline{M_+}$ that is orthogonal to M and intersects M along a non-null curve. In addition, if F maps proper surfaces in $\overline{M_+}$ to proper surfaces in $\overline{N_+}$, then $\partial_r\mathscr{F} = 0$ at $r = 0$.

Proof. Let $\gamma(t)$ be a non-null curve in a coordinate open set $\mathcal{W} \subseteq M$ with the coordinate $\gamma^i(t)$. Then, the adapted coordinate $(t, \underline{r})$ in (5.2.10) locally defines a surface $\Sigma \subseteq \overline{M_+}$ orthogonal to M with $\Sigma \cap M = \gamma$. Since $F(\Sigma)$ is orthogonal to N, then

$$dF(X_{\underline{r}}) - \frac{\langle dF(X_{\underline{r}}), dF(X_t)\rangle_{\bar{h}}}{\langle dF(X_t), dF(X_t)\rangle_{\bar{h}}} dF(X_t) \tag{5.2.11}$$

is orthogonal to $TN \subseteq TM_+$ where $\{X_t, X_{\underline{r}}\}$ is the coordinate basis for the adapted coordinate $(t, \underline{r})$. The orthogonal condition gives at $r = 0$

$$0 = F^k_{,r} - \frac{F^i_{,r}\dot{\gamma}^j h_{ij}}{\langle\dot{\gamma}, \dot{\gamma}\rangle_h}\dot{\gamma}^k. \tag{5.2.12}$$

Since $\dot{\gamma}$ is an arbitrary non-null vector at $t = 0$, we get $F^k_{,r} = 0$ at $r = 0$. Conversely, let $(t, \underline{r})$ be the adapted coordinate of Σ. Projecting (5.2.11) to $TN \subseteq T\overline{N_+}$ orthogonally, it gives the right-hand side of (5.2.12) which turns to be 0 due to $F^i_{,r} = 0$ at $r = 0$.

If $\gamma(t)$ is a non-null conformal geodesic, then the formula of the adapted coordinate locally extends $\gamma(t)$ to a proper surface in $\overline{M_+}$. One can follow the same arguments just made to get $\partial_r\mathscr{F} = 0$ at $r = 0$. □

As mentioned at the end of Section 5.1, the idea for proving Theorem 5.2.5 is to consider a suitable family of proper surfaces Σ_ϵ. Then, the dependence of ϵ in the second fundamental forms of proper surfaces $F(\Sigma_\epsilon)$ may imply F is an asymptotic local isometry where its local conditions are (5.2.7)-(5.2.9). However, recalling Proposition

4.0.3, Proposition 4.0.4 and the adapted coordinate (5.2.10), a proper surface Σ is characterized by v^i, u^i and γ being an unparametrized conformal geodesic. Therefore, we can utilize the adapted coordinate of $F(\Sigma_\epsilon)$ to avoid the tedious computation of the second fundamental forms. The following Lemma 5.2.3 and Proposition 5.2.4 respectively give the coordinate change of $F(\Sigma)$ to its adapted coordinate and provide that F satisfies (5.2.7) and (5.2.8), except for (5.2.9), for F preserving proper surfaces.

Lemma 5.2.3. *Let (M_+, g_+) be a Poincaré-Einstein space in the normal form relative to $g \in [g]$ and $\tilde{\Sigma} \subseteq M_+$ be a surface orthogonal to M with $\tilde{\Sigma} \cap M = \gamma$ being a non-null curve. Assume it has a parametrization $(t, \underline{r}) \mapsto (\tilde{x}^i(t, \underline{r}), \tilde{r}(t, \underline{r}))$ from $I \times I$ to $\tilde{\Sigma} \subseteq M_+$ where $\tilde{x}^i_{,\underline{r}} = 0$ and $\tilde{r} = 0$ both at $\underline{r} = 0$. The existence of the adapted coordinate (η, ρ) of $\tilde{\Sigma}$ implies there is a coordinate change $t = t(\eta, \rho)$, $\underline{r} = \underline{r}(\eta, \rho)$ with $t(\eta, 0) = \eta$. The coordinate change is in the following modular higher orders.*

$$t(\eta, \rho) = \eta + t_{(2)}\rho^2 + t_{(3)}\rho^3, \qquad (5.2.13)$$
$$\underline{r}(\eta, \rho) = r_{(1)}\rho + r_{(2)}\rho^2 + r_{(3)}\rho^3$$

where

$$r_{(1)} = \frac{1}{\tilde{r}_{,\underline{r}}},\ r_{(2)} = -\frac{\tilde{r}_{,\underline{rr}}}{2(\tilde{r}_{,\underline{r}})^3},\ r_{(3)} = -\frac{1}{6r_{,\underline{r}}}\left(2\tilde{r}_{,t\underline{r}}\, r_{(1)}t_{(2)} + \tilde{r}_{,\underline{rrr}}(r_{(1)})^3 + 6\tilde{r}_{,\underline{rr}}r_{(1)}r_{(2)}\right),$$
$$t_{(2)} = \frac{1}{2\langle\dot{\gamma}, \dot{\gamma}\rangle}\left(v^i\dot{\gamma}_i - \tilde{x}^i_{,\underline{rr}}\dot{\gamma}_i(r_{(1)})^2\right),\ t_{(3)} = \frac{1}{6\langle\dot{\gamma}, \dot{\gamma}\rangle}\left((r^3_{(1)})\tilde{x}^i_{,\underline{rrr}}\dot{\gamma}_i + 6r_{(1)}r_{(2)}\tilde{x}^i_{,\underline{rr}}\dot{\gamma}_i\right).$$

The partial derivatives of $\tilde{x}^i$ and $\tilde{r}$ above are at $\underline{r} = 0$. The term v in $t_{(2)}$ is $\nabla_{\dot{\gamma}}\left(\frac{\dot{\gamma}}{\langle\dot{\gamma},\dot{\gamma}\rangle}\right)$. Note that $\dot{\gamma}^i = \tilde{x}^i_{,t}(t, 0) = \tilde{x}^i_{,\eta}(\eta, 0)$ and $\tilde{x}^i_{,tt}(t, 0) = \tilde{x}^i_{,\eta\eta}(\eta, 0)$.

Proposition 5.2.4. *Let $F\colon M_+ \to N_+$ be a local diffeomorphism such that*

(i) $dF(v)$ is non-null (resp. null) $\forall$non-null (resp. null) $v \in TM$ if $p \neq q$;

(ii) $sgn\langle v, v\rangle_g = sgn\langle dF(v), dF(v)\rangle_h \ \forall v \in TM$ *if* $p = q$.

The F *satisfies (5.2.7) and (5.2.8) if and only if* F *maps proper surfaces in* $\overline{M_+}$ *to proper surfaces in* $\overline{N_+}$.

Proof. Let $\gamma(t)$ be a non-null conformal geodesic in a coordinate open set of M with the initial conditions $\dot{\gamma}_0^k$ and $\ddot{\gamma}_0^k$ at $\gamma(0) = p$. Let $\Sigma \subseteq \overline{M_+}$ be its extended proper surface defined by the adapted coordinate $(t, \underline{r})$ in (5.2.10). Since $F(\Sigma) \cap N$ is still an unparametrized conformal geodesic from Proposition 4.0.4, we know $f : M \to N$ is a conformal local diffeomorphism due to Theorem 5.1.1. Without loss of generality, we assume that f is the identity map on (M, g) where $g \in [g]$ and that the local coordinate of its extended local diffeomorphism is $F : (x^i, r) \mapsto (F^i(x, r), F^r(x, r))$ where r is the geodesic defining function of g. From Proposition 5.2.2 and Lemma 5.2.3, we let $(\eta(t, \underline{r}), \rho(t, \underline{r}))$ be the coordinate change of $F(\Sigma)$ to its adapted coordinate.

Let $\dot{\gamma}_0^k = V^k$, $\ddot{\gamma}_0^k = \epsilon A^k$ with $|V| = 1$ and $\langle V, A\rangle = 0$ where $\epsilon \in \mathbb{R}$ is arbitrary near 1. The variable ϵ gives an ϵ-family of non-null conformal geodesics $\gamma_\epsilon(t)$. The extended proper surfaces Σ_ϵ is defined by the formula of the adapted coordinate (5.2.10) where the coefficients in the expansion of $x^i(t, \underline{r})$ depend on $\gamma_\epsilon(t)$. Because Σ_ϵ depends smoothly on ϵ, we know the adapted coordinate for $F(\Sigma_\epsilon)$ defined from Lemma 5.2.3 depends smoothly on ϵ. Since $F(\Sigma_\epsilon)$ are proper surfaces, we have

$$\nabla_{\dot{\gamma}_\epsilon}\left(\frac{\dot{\gamma}_\epsilon}{\langle\dot{\gamma}_\epsilon, \dot{\gamma}_\epsilon\rangle}\right)^i = F^i_{,\rho\rho} \qquad \text{at } \rho = 0 \tag{5.2.14}$$

and

$$0 = F^i_{,\rho\rho\rho} \qquad \text{at } \rho = 0. \tag{5.2.15}$$

Following the formulas and the conventions from (5.2.13), we know $t_{(2)} = -\frac{V_j F^j_{,rr}}{(F^r_{,r})^2}$ at the point p. Using chain rule on $F^i_{,\rho\rho}$ for $(t(\eta,\rho), \underline{r}(\eta,\rho))$, the straightforward computation for (5.2.14) at p is

$$\epsilon A^i = \frac{1}{(F^r_{,r})^2}\left(\epsilon A^i + F^i_{,rr} - V_j F^j_{,rr} V^i\right). \tag{5.2.16}$$

Therefore, $F^r_{,r} = 1$ and $F^i_{,rr} = 0$ at $r = 0$. The results we got imply $r_{(2)} = -F^r_{,rr}$ at $r = 0$ and $t_{(3)} = \frac{1}{6}F^j_{,rrr}V_j$ at p from (5.2.13). Doing chain rule again, (5.2.15) is equal to the following at p.

$$0 = -3\epsilon\, F^r_{,rr}\, A^i + F^i_{,rrr} + F^j_{,rrr} V_j V^i. \tag{5.2.17}$$

So, $F^r_{,rr} = 0$ and $F^i_{,rrr} = 0$ at $r = 0$.

Conversely, assume $(F^i(x,r), F^r(x,r)) = (x^i + O(r^4), r + O(r^3))$ where we assume f is the identity map on (M, g). Let $\Sigma \subseteq \overline{M_+}$ be a proper surface and $(t, \underline{r})$ be its adapted coordinate. From Proposition 5.2.2, we know $F(\Sigma)$ orthogonal to N. Considering the coordinate change $(\eta(t,\underline{r}), \rho(t,\underline{r}))$ of $F(\Sigma)$ to its adapted coordinate and following the formula from (5.2.13), we have

$$r_{(1)} = 1,\ r_{(2)} = 0,\ t_{(2)} = 0,\ t_{(3)} = 0.$$

Computing $F^i_{,\rho\rho}$ and $F^i_{,\rho\rho\rho}$ directly from chain rule, we have at $\rho = 0$

$$F^i_{,\rho\rho} = \nabla_{\dot\gamma}\left(\frac{\dot\gamma}{\langle\dot\gamma,\dot\gamma\rangle}\right)^i,\ F^i_{,\rho\rho\rho} = 0$$

which implies $F(\Sigma)$ is proper by (5.2.10) and Proposition 4.0.4. □

Theorem 5.2.5. *Let (M_+, g_+) and (N_+, h_+) be Poincaré-Einstein manifolds for $(M, [g])$ and $(N, [h])$ respectively with same signature $(p+1, q)$. Given a local diffeomorphism $F\colon M_+ \to N_+$ such that it smoothly extends a local diffeomorphism $f\colon M \to N$. Assume*

F satisfies

(i) $dF(v)$ is non-null (resp. null) $\forall$non-null (resp. null) $v \in TM$ if $p \neq q$,

(ii) $sgn\langle v, v\rangle_g = sgn\langle dF(v), dF(v)\rangle_h$ $\forall v \in TM$ if $p = q$.

If the F maps proper surfaces in M_+ to proper surfaces in N_+, then there is a local diffeomorphism G on an open neighborhood W of $M \subseteq \overline{M_+}$,

$$G\colon W \longrightarrow M_+,$$

where G smoothly extends the identity map on M, such that $\tilde{F} = F \circ G$ is an asymptotic local isometry.

Proof. From Proposition 5.2.4, we can choose $g \in [g]$ and $h \in [h]$ suitably to let f be a local isometry. Consider the identification of F in (5.2.2)

$$F\colon \mathcal{U} \to \mathcal{V}, \quad (x, r) \mapsto (\mathscr{F}(x, r), F^s(x, r)).$$

We aim to find out an open set $\mathcal{W} \subseteq \mathcal{U}$ such that it contains $M \times \{0\}$ and the following map is well-defined

$$\begin{array}{rccc} G\colon & \mathcal{W} & \to & \mathcal{U} \\ & (x, r) & \mapsto & (x, r - \mathscr{R}(x)r^3) = (x, p_x(r)) \end{array} \tag{5.2.18}$$

where $\mathscr{R}(x) = \partial_r^3 F^s(x, 0)/6$ and $p_x(r) = r - \mathscr{R}(x)r^3$.

For any $x \in M$, it has open neighborhoods $\mathcal{B}_x$, $\mathcal{N}_x$ in M such that

$$x \in \mathcal{B}_x \subseteq\subseteq \mathcal{N}_x \quad \text{and } \mathcal{N}_x \times [0, \epsilon) \subseteq \mathcal{U} \quad \exists \epsilon > 0$$

where $\mathcal{B}_x \subseteq\subseteq \mathcal{N}_x$ means its closure $\overline{\mathcal{B}_x}$ is compact in $\mathcal{N}_x$. Here we choose ϵ small enough such that the polynomial $p_y(r)$ of r on $[0,\epsilon)$ is strictly increasing for all $y \in \overline{\mathcal{B}_x}$. Hence, there exists $0 < \epsilon'_x < \epsilon$ so that

$$\mathcal{B}_x \times [0,\epsilon'_x) \to \mathcal{B}_x \times [0,\epsilon) \subseteq \mathcal{U}$$

$$(y,r) \mapsto (y, p_y(r)).$$

Let $\mathcal{W} = \bigcup_{x\in M} \mathcal{B}_x \times [0,\epsilon'_x)$. Since we know the asymptotic expansion of F from Proposition 5.2.4, we get $\tilde{F} = F \circ G \colon \mathcal{W} \to \mathcal{V}$ is an asymptotic local isometry. □

Corollary 5.2.6. *Let $F\colon M_+ \to N_+$ and $G\colon W \to M_+$ be local diffeomorphisms as stated in Theorem 5.2.5. Assume there is a geodesic defining function r for some $g \in [g]$ and $C \geq 0$ such that*

$$|\partial_r^3 (s \circ F)(p)| \leq C \quad \forall p \in M \subseteq M_+, \tag{5.2.19}$$

where $\partial_r = {}^{\bar{g}}\nabla r$ is the gradient of r with respect to $\bar{g} = r^2 g_+$ and s is the geodesic defining function of $h \in [h]$ to make $f\colon M \to N$ be a local isometry.

(i) If $C > 0$, then G can be chosen as an embedding with its image while W is small enough.

(ii) If $C = 0$, then G can be chosen to be the identity map on W. Particularly, F is an asymptotic local isometry.

Proof. Recall the definition of G in (5.2.18),

$$\begin{aligned} G\colon \quad \mathcal{W} \quad &\to \quad \mathcal{U} \\ (x,r) \quad &\mapsto \quad (x, r - \mathcal{R}(x) r^3) = (x, p_x(r)). \end{aligned}$$

Let $\mathcal{W}$ be small enough so that $\mathcal{W} \subseteq M \times [0, \epsilon)$ for some $\epsilon > 0$. If ϵ is small enough, then $p_x(r)$ is strictly increasing because for $r \in [0, \epsilon)$

$$\partial_r p_x(r) = 1 - 3\mathscr{R}(x)r^2 > 1 - 3Cr^2.$$

Hence, G is an open injective immersion for $C > 0$. The case for $C = 0$ is straightforward due to the definition of G. □

The following proposition gives geometric conditions to satisfy the presumptions of Corollary 5.2.6.

Proposition 5.2.7. *Let $F\colon M_+ \to N_+$ and $G\colon W \to M_+$ be local diffeomorphisms as stated in Theorem 5.2.5. Assume there is a geodesic defining function r for some $g \in [g]$ and $a \geq 0$ such that*

$$(F^*h_+ - g_+)({}^{\bar{g}}\nabla r, {}^{\bar{g}}\nabla r) = O(r^a). \tag{5.2.20}$$

Let s be a geodesic defining function of $h \in [h]$ to make f be a local isometry.

(i) If $a = 0$, there exists $C > 0$ such that $|\partial_r^3(s \circ F)| \leq C$ on M.

(ii) If $a = 1$, we have $\partial_r^3(s \circ F) = 0$ on M.

Proof. Consider the identification of F near $M \subseteq \overline{M_+}$ and $N \subseteq \overline{N_+}$ in (5.2.2), $\mathcal{U} \to \mathcal{V}$, $(x, r) \mapsto (\mathscr{F}(x, r), F^s(x, r))$. Then, (5.2.20) is equivalent to

$$O(r^{a+2}) = F^s_{,r}F^s_{,r} + F^i_{,r}F^j_{,r}(h_s \circ F)_{ij} - (F^s/r)^2,$$

where the right-hand side above is exactly from (5.2.5) while considering local conditions of an asymptotic local isometry. Then, the Taylor expansion for the right-hand side gives the following.

$$O(r^{a+2}) = \sum_{c=0}^{2} \Bigg\{ \sum_{c\geq b\geq 0} \frac{(b+1)(c-b+1)}{(b+1)!\,(c-b+1)!} F^s_{(b+1)} F^s_{(c-b+1)} + \sum_{c\geq b+d\geq 0} \frac{(b+1)(c-b-d+1)}{(b+1)!\,(c-b-d+1)!\,d!} F^i_{(b+1)} F^j_{(c-b-d+1)} (h_{(d)})_{ij} - \sum_{c\geq b\geq 0} \frac{1}{(b+1)!\,(c-b+1)!} F^s_{(b+1)} F^s_{(c-b+1)} \Bigg\} r^c,$$

where $F^s_{(b)}$, $F^i_{(b)}$ and $h_{(b)}$ mean the bth-order partial derivative of r at $r = 0$ of F^s, F^i and $h_s \circ F$ respectively. So, we have from above

$$\begin{aligned} 0 &= F^i_{,r} F^j_{,r} h_{ij} \ , \\ 0 &= F^s_{,r} F^s_{,rr} + 3 F^i_{,r} F^j_{,rr} h_{ij} \ , \\ O(r^a) &= \frac{2}{3} F^s_{,r} F^s_{,rrr} + \frac{3}{4} (F^s_{,rr})^2 + F^i_{,rr} F^j_{,rr} h_{ij}, \end{aligned} \tag{5.2.21}$$

where the third equality above is when $F^i_{,r} = 0$ at $r = 0$. We know $F^s_{,r} = 1$, $F^s_{,rr} = 0$ and $F^i_{,rr} = 0$ at $r = 0$ from Proposition 5.2.4. This completes the proof. □

Remark 5.2.8. It is straightforward to observe $F^i_{,r} = 0$ and $F^s_{,rr} = 0$ at $r = 0$ from (5.2.21) when considering Riemannian conformal classes $[g]$ and $[h]$. However, (5.2.21) alone still can't simply imply $F^s_{,rrr} = 0$ at $r = 0$.

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