Title
The KH-Theory of Complete Simplicial Toric Varieties and the Algebraic K-Theory of Weighted Projective Spaces

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by

Adam Lucas Massey

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Abstract of the Dissertation

The KH-Theory of Complete Simplicial Toric Varieties and the Algebraic K-Theory of Weighted Projective Spaces

by

Adam Lucas Massey

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2012

Professor Christian Haesemeyer, Chair

We show that, for a complete simplicial toric variety X, we can determine its homotopy K-theory (denoted KH-theory) entirely in terms of the torus pieces of open sets forming an open cover of X. We accomplish this by constructing a simplicial scheme BOT_X and constructing a relationship between the spectrum $\mathrm{KH}(X)$ and a certain spectrum determined by BOT_X. Using our construction of BOT_X, we construct conditions under which, given two complete simplicial toric varieties with the same simplicial structure, we can induce a morphism from BOT_X to BOT_Y that is, in each degree, component-wise an isogeny. This allows us to show that, under these conditions, the two spectra $\mathrm{KH}(X) \otimes \mathbb{Q}$ and $\mathrm{KH}(Y) \otimes \mathbb{Q}$ are weakly equivalent. We then apply this result to determine the rational KH-theory of weighted projective spaces. We next turn our attention to calculating the $\mathcal{F}_{\mathrm{K}}$ groups for complete toric surfaces and 2-dimensional weighted projective spaces. This allows us to determine $\mathrm{K}_n(\mathbb{P}(a, b, c)) \otimes \mathbb{Q}$ for $n \leq 0$, and allows us to conclude that complete toric surfaces and 2-dimensional weighted projective spaces are K_0-regular. We conclude by determining conditions under which our approach for dimension 2 works in arbitrary dimensions.

The dissertation of Adam Lucas Massey is approved.

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1 Introduction

One of the most interesting invariants that objects in algebraic geometry have is their algebraic K-theory (see Definition 3.77). Just as in the differential geometry setting, K-theory allows us to classify many important properties of our space (such as isomorphism classes of vector bundles), and because it is an invariant, this information does not change when we alter how we view the space (such as by using a different coordinate system, for example). As a subject, algebraic K-theory is often very challenging; even completely calculating the algebraic K-theory of a single point is still an open problem. Algebraic K-theorists usually settle on calculating the K-theory of an object in terms of the K-theory of a point (equivalently, in terms of the K-theory of the underlying field, since we define $\mathrm{K}_n(\mathrm{Spec}(k)) = \mathrm{K}_n(k)$ and $\mathrm{Spec}(k)$ is the single point). However, even after doing this, calculating algebraic K-theory is still extremely difficult, especially when the geometric object we are considering is not smooth.

One alternative that mathematicians have discovered (introduced in [Wei1], and formalized in [TT]) is Homotopy Algebraic K-Theory, which we denote by KH-theory (see Definition 3.85). Homotopy Algebraic K-Theory has many nice properties that do not hold for general K-theory, such as satisfying the Bass Fundamental Theorem ($\mathrm{KH}_n(A[t, t^{-1}]) \cong \mathrm{KH}_n(A) \oplus \mathrm{KH}_{n-1}(A)$ for all n) and respecting $\mathbb{A}^1$ homotopy ($\mathrm{KH}_n(X \times \mathbb{A}^1) \cong \mathrm{KH}_n(X)$ for all n); see [Wei1] or see Theorem 3.84. It turns out that KH-theory agrees with K-theory on smooth geometric objects (see [Wei1, Example 1.4] and [Wei1, Proposition 6.10]), and even when dealing with non-smooth objects, these two theories are still related.

In this paper, we examine the basic properties of complete simplicial toric varieties, and use these properties to attempt to compute their KH-theory. It turns out that just the basic knowledge of their simplicial structure is not enough to calculate the KH-theory. The reason for this is that even if two complete simplicial toric varieties have the same simplicial structure (see Definition 4.6), these structures might not themselves yield any relationship between the corresponding varieties. Indeed, given two complete simplicial toric varieties X

and Y, there doesn't need to be any morphism $X \longrightarrow Y$ even if X and Y have the same simplicial structure.

The solution to this problem is to add additional conditions to force a relationship between X and Y. This is done via the construction of the simplicial schemes BOT_X and BOT_Y (see Definition 4.15). It turns out that these two simplicial schemes will have a relationship if we assume certain relationships between the fans of X and Y. The main goal of the first half of this paper is to construct and examine this relationship.

Using this relationship, we can construct a relationship between the KH-theory of ordinary projective space and the KH-theory of any weighted projective space. In particular, if $\mathbb{P}(q_0, ..., q_d)$ is a weighted projective space of dimension d, then the rank of $\mathrm{KH}_0(\mathbb{P}(q_0, ..., q_d))$ is $d + 1$.

Following the ideas in [CHWW], the K-theory of any toric variety (in particular, any weighted projective space) over a field of characteristic 0 is obtained as a direct sum of the KH-theory of that toric variety and another invariant called $\mathcal{F}_{\mathrm{K}}$ which will be defined later; see Definition 3.90 and Theorem 3.94. We obtain the former by the construction discussed above; the latter turns out to be much more difficult and is handled separately. Using these ideas, our goal throughout this paper will be to prove the following theorem:

Theorem 1.1. *Let $\mathbb{P}(q_0, ..., q_d)$ be any weighted projective space of dimension d over a field k such that the characteristic of k does not divide the product $q_0 \cdot q_1 \cdots q_d$. Then*

(a) For every n, we have

$$\mathrm{KH}_n(\mathbb{P}(q_0, ..., q_d)) \otimes \mathbb{Q} \cong \mathrm{KH}_n(\mathbb{P}^d) \otimes \mathbb{Q}. \tag{1.1}$$

If the characteristic of k is 0, then we can conclude the following additional results:

(b) Any 2-dimensional weighted projective space $\mathbb{P}(a, b, c)$ is K_0-regular, and for every $n \leq 0$, we have

$$\mathrm{K}_n(\mathbb{P}(a, b, c)) \otimes \mathbb{Q} \cong \mathrm{K}_n(\mathbb{P}^2) \otimes \mathbb{Q}. \tag{1.2}$$

(c) If our weighted projective space is of the form $\mathbb{P}(1,1,a)$, then for $n \leq 0$ we can conclude the stronger statement

$$\mathrm{K}_n(\mathbb{P}(1,1,a)) \cong \mathrm{K}_n(\mathbb{P}^2). \tag{1.3}$$

If the characteristic of k is 0 and we assume that our weighted projective space has only isolated singular points, then we have the following additional results:

(d) Any d-dimensional weighted projective space $\mathbb{P}(q_0, ..., q_d)$ whose singular set consists of only isolated singular points is K_0-regular, and for every $n \leq 0$, we have

$$\mathrm{K}_n(\mathbb{P}(q_0, ..., q_d)) \otimes \mathbb{Q} \cong \mathrm{K}_n(\mathbb{P}^d) \otimes \mathbb{Q}. \tag{1.4}$$

(e) If our weighted projective space is of the form $\mathbb{P}(1,1,...,1,a)$, then for $n \leq 0$ we can conclude the stronger statement

$$\mathrm{K}_n(\mathbb{P}(1,1,...,1,a)) \cong \mathrm{K}_n(\mathbb{P}^d). \tag{1.5}$$

Theorem 1.1 is proven in several stages. Most of our work towards proving Theorem 1.1 is actually done by proving Theorem 4.9; as such, much of this paper will focus on the proof of (and applications of) Theorem 4.9.

We begin in Section 2 by establishing the basic notation and terminology that we will use throughout the paper, before moving on to discuss the necessary background work in Section 3. In Section 3.1, we provide a brief review of the necessary commutative algebra and algebraic geometry concepts that will be used throughout this paper. In Section 3.2, we briefly introduce Grothendieck Topologies, and discuss the particular examples that we will be interested in. We then move on to study simplicial and cosimplicial categories in Section 3.3, model categories in Section 3.4, and homotopy limits in Section 3.5. We then give the precise definitions and basic properties of algebraic K-theory and KH-theory in Section 3.6 before concluding our background work by introducing transfer arguments in Section 3.7 and descent arguments in Section 3.8. Each of these play a vital role in our construction, and must all be well understood.

We then, using the background work constructed in Section 3, begin our construction of the relationship between BOT_X and BOT_Y, alluded to above, in Section 4. In Section 4.1, we introduce an example that illustrates our expectations for the general approach. In Section 4.2, we introduce the simplicial scheme associated to a complete simplicial toric variety, and the basic notion of the simplicial structure of a complete simplicial toric variety; we use these notions to construct the simplicial scheme BOT_X in Section 4.3. Using BOT_X, we determine conditions under which BOT_X and BOT_Y are related, and use that relationship to construct a relationship between the KH-theories of X and Y, in Section 4.4. We apply these ideas to weighted projective spaces in Section 4.5.

Finally, we attempt to calculate as many of the $\mathcal{F}_{\mathrm{K}}$ groups as possible in Section 5. Focusing primarily on the case where the dimension of the toric variety is 2 and the underlying field has characteristic 0, we begin by calculating the $\mathcal{F}_{\mathrm{K}}$ groups for the weighted projective space $\mathbb{P}(1,1,2)$ in Section 5.1 before generalizing the approach to $\mathbb{P}(1,1,a)$ in Section 5.2. In Section 5.3, we prove the $\mathcal{F}_{\mathrm{K}}$ Decomposition Theorem for complete toric surfaces and use it to prove that any complete toric surface (in particular, any dimension 2 weighted projective space) is K_0-regular. We conclude in Section 5.4 by examining the failure of the $\mathcal{F}_{\mathrm{K}}$ Decomposition Theorem to hold in higher dimensions, before determining additional conditions on the toric variety that allow a variant of the $\mathcal{F}_{\mathrm{K}}$ Decomposition Theorem to hold. Using this variant, we can again show that, in this case, higher dimensional complete toric varieties satisfying this extra condition are also K_0-regular.

2 Notations and Terminology

We begin by establishing some notation that we will use throughout the paper to avoid any confusion, as the terms mentioned here arise in many different forms in the literature (particularly in the discussion of toric varieties). Note that throughout this paper, unless otherwise mentioned, we make no assumption on the characteristic of k.

2.1 Toric Varieties

In this section we establish the notation and basic definitions/results about toric varieties that we will assume throughout the paper. Much of what we discuss can be found in [Ful], which is the standard reference on the subject, and [Cox]. One can also find many of the basics that we assume for this paper discussed in the early sections of [CHWW].

We call X a toric variety if it is a normal variety along with a split algebraic torus $T \cong \mathbb{G}_m^n$ embedded as a Zariski dense open subset and an action of T on X extending the obvious action of T on itself. The case in which the torus is not split will not be covered in this paper.

Let N be a lattice of finite rank, and $M = \mathrm{Hom}(N, \mathbb{Z})$ be the dual lattice. We let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{Hom}(N_{\mathbb{R}}, \mathbb{Z})$. For $m \in M_{\mathbb{R}}$ and $n \in N_{\mathbb{R}}$, we let $\langle m, n\rangle$ denote the value of m applied to n.

A rational, strongly convex polyhedral cone $\sigma \subset N_{\mathbb{R}}$ is a subset of the form

$$\sigma = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_k \tag{2.1}$$

for some $v_1, ..., v_k \in N$, and if both u and $-u$ are in σ, then $u = 0$. In other words, σ contains no lines through the origin. Throughout this paper, when we say cone, we mean a rational, strongly convex polyhedral cone. We define the dimension of σ, denoted $\dim(\sigma)$, to be the dimension of the subspace $\mathrm{Span}\{v_1, ..., v_k\} \subset N_{\mathbb{R}}$.

Given a cone σ, we define the dual cone $\sigma^\vee = \{m \in M_{\mathbb{R}} | \langle m, n\rangle \geq 0 \; for \; n \in \sigma\}$. Much of our focus will be on $\sigma^\vee \cap M$, which is an abelian monoid under addition of functions, and is finitely generated by Gordan's Lemma (see [Ful, Section 1.2, Proposition 1]). We define the affine toric variety associated to σ, which we denote U_σ, to be $U_\sigma = \mathrm{Spec}\,(k[\sigma^\vee \cap M])$. Here we need to write $\sigma^\vee \cap M$ multiplicatively; to do that, we write elements of $k[\sigma^\vee \cap M]$ as k-linear combinations of formal symbols $\{\chi^m | m \in \sigma^\vee \cap M\}$.

Given a cone σ, we say that τ is a face of σ, and write $\tau \prec \sigma$, if

$$\tau = \{n \in \sigma | \langle m, n \rangle = 0\} \tag{2.2}$$

for some $m \in \sigma^\vee$. Notice that any face of a strongly convex polyhedral cone is again a strongly convex polyhedral cone. We define a facet of σ to be a face of codimension 1. A *fan* Δ in $N_\mathbb{R}$ is a collection of cones such that if $\tau \prec \sigma$ and $\sigma \in \Delta$, then $\tau \in \Delta$ and such that if $\sigma_1, \sigma_2 \in \Delta$ then $\sigma_1 \cap \sigma_2 \prec \sigma_1$ and $\sigma_1 \cap \sigma_2 \prec \sigma_2$.

If $\tau \prec \sigma \subset N_\mathbb{R}$, then we get an inclusion $U_\tau \longrightarrow U_\sigma$, and in fact U_τ is a principal open subset of U_σ. This is because, taking the morphism $N \longrightarrow N$ to be the identity map and $\tau \longrightarrow \sigma$ to be inclusion induces a map $k[\sigma^\vee \cap M] \longrightarrow k[\tau^\vee \cap M]$ which turns out to be given by inverting finitely many χ^m; see [Ful, page 18]. Therefore, given any fan Δ, we can construct a variety $X(\Delta)$ by taking affine opens U_σ for all $\sigma \in \Delta$ and then, for all $\sigma_1, \sigma_2 \in \Delta$, gluing U_{σ_1} and U_{σ_2} along $U_{\sigma_1 \cap \sigma_2}$. Thus, we get that $U_{\sigma_1} \cap U_{\sigma_2} = U_{\sigma_1 \cap \sigma_2}$. $X(\Delta)$ is in fact a toric variety, with U_0 as its torus (here 0 denotes the cone consisting only of the origin in $N_\mathbb{R}$). There is a converse, due to Sumihiro, which says that any toric variety is given by a fan inside some lattice; see [Sum]. When referring to the toric variety associated to a fan Δ, we write $X(\Delta)$; conversely, when given a toric variety X, we denote by Δ_X its associated fan and we denote by N^X the lattice in which Δ_X lives.

Since any toric variety is determined by a fan, and since any fan determines a toric variety, one would suspect that a morphism of toric varieties is equivalent to a lattice morphism on the respective fans, and that is almost the case. Suppose X and Y are two toric varieties, and Δ_X and Δ_Y are their respective fans; if we have a lattice morphism that induces a map $\varphi : \Delta_X \longrightarrow \Delta_Y$ such that for any cone $\sigma \in \Delta_X$, $\varphi(\sigma)$ is contained in a cone inside Δ_Y, then φ induces a morphism of toric varieties $X \longrightarrow Y$. To construct the map, notice that because $\varphi(\sigma)$ is contained in a cone inside Δ_Y (which we will call σ' for convenience), we induce a map $U_\sigma \longrightarrow U_{\sigma'}$ for every cone $\sigma \in \Delta_X$. Then we glue them together. Throughout this paper, we will use the notion of toric morphism and lattice/fan morphism interchangeably.

We also note that the diagonal map $U_{\sigma_1 \cap \sigma_2} \longrightarrow U_{\sigma_1} \times U_{\sigma_2}$ is a closed embedding; see

[Ful, page 21]. As a consequence, any toric variety must be separated. Similarly, since any variety is Noetherian and quasi-compact in the Zariski topology, any toric variety must be Noetherian and quasi-compact also.

We define the *support* of a fan Δ, denoted $|\Delta|$, to be the union of all cones in Δ as viewed inside $N_{\mathbb{R}}$. We say a toric variety X is *complete* if $|\Delta_X| = N_{\mathbb{R}}$. If X is a toric variety over $\mathbb{C}$, the X is complete if and only if X (as a complex manifold) is compact in the usual topology.

We say that a cone σ is *simplicial* if its minimal set of generators $\{v_1, ..., v_k\}$ is linearly independent over $\mathbb{R}$. We say that a fan Δ is simplicial if every cone $\sigma \in \Delta$ is simplicial, and we say that a toric variety X is simplicial if its associated fan Δ_X is simplicial. Similarly, we say that a cone σ is *smooth* if its minimal set of generators $\{v_1, ..., v_k\}$ is part of a $\mathbb{Z}$-basis of N. We say that a fan Δ is smooth if every cone $\sigma \in \Delta$ is smooth, and we say that a toric variety X is smooth if its associated fan Δ_X is smooth. Notice that any smooth cone/fan/toric variety must also automatically be simplicial.

Let N be a lattice, and $\sigma \subset N_{\mathbb{R}}$ be a cone. We define $N_\sigma = (\sigma \cap N) + (-\sigma \cap N)$ to be the sublattice of N generated by σ. Similarly, we define $\widetilde{N}_\sigma = N/N_\sigma$.

One important fan that we can create from a given fan Δ (inside the lattice N) is the star of a given cone τ, denoted $\mathrm{Star}(\tau)$. The star of the cone τ is the fan consisting of all cones $\sigma \in \Delta$ such that $\tau \prec \sigma$, but considered inside the lattice $\widetilde{N}_\tau$. The toric variety $X(\mathrm{Star}(\tau))$ is a closed subvariety of $X(\Delta)$, and the dimension of the cone τ is the codimension of $X(\mathrm{Star}(\tau))$ inside $X(\Delta)$; in particular, if τ is a maximal cone, then $X(\mathrm{Star}(\tau))$ is a point inside $X(\Delta)$.

We conclude this section by presenting a result from [Ful] that will form the basis of our arguments throughout the paper. First, we need a lemma.

Lemma 2.1. *Let $\sigma \subset N_{\mathbb{R}}$ be a cone. Then $\sigma \cap N$ and N_σ are saturated.*

Proof. Given any point $x \in \sigma \cap N$, we have that the line segment joining x to the origin lies completely in $\sigma \subset N_{\mathbb{R}}$. Then any point of N that lies on this line segment is also in $\sigma \cap N$; in particular, the minimal lattice point (i.e. the point of N on the line segment joining 0 and x with shortest distance to 0) on this line is also in $\sigma \cap N$. Denote this minimal lattice point by y. Since y and x lie on the same line in $N_{\mathbb{R}}$, x must be a multiple of y. Since both x and y live in σ, x must be a positive multiple of y (otherwise σ contains a non-trivial line). Finally, since x and y both live in N, x must be a positive integer multiple of y. Thus, we have shown that given any element $p \cdot y \in \sigma \cap N$, we have $y \in \sigma \cap N$ also. So $\sigma \cap N$ is saturated. Since $N_\sigma = (\sigma \cap N) + (-\sigma \cap N)$, the fact that it is also saturated is immediate.

□

We can now prove our main proposition, which may be found in [Ful].

Proposition 2.2. *Let σ be a p-dimensional cone in a lattice N with* $\dim N_{\mathbb{R}} = n$. *Then $U_\sigma \cong U_{\sigma'} \times T$, where $T \cong \mathbb{G}_m^{n-p}$ is a split algebraic torus of rank $n-p$.*

Proof. By Lemma 2.1, N_σ is saturated. Choose a splitting $N \cong N_\sigma \oplus N''$ (note that $N'' \cong \widetilde{N}_\sigma$, but that this splitting is not canonical). In this splitting, we have $\sigma = \sigma' \oplus \{0\}$, where σ' is a maximal cone in N_σ. Now taking duals gives us the splitting $M \cong M' \oplus M''$. Then $\sigma^\vee \cap M = (\sigma'^\vee \cap M') \oplus M''$, and $k[\sigma^\vee \cap M] = k[(\sigma'^\vee \cap M') \oplus M'']$. Taking Spec of both sides gives us $U_\sigma \cong U_{\sigma'} \times T_{N''}$ and observe that $T_{N''} \cong \mathbb{G}_m^{n-p}$ as desired.

□

Remark 2.3. We call the torus $T_{N''}$ constructed in Proposition 2.2 the *torus part* of U_σ. Observe that, as $N'' \cong \widetilde{N}_\sigma$, we can construct the torus part of U_σ by taking $\mathrm{Spec}(k[\mathrm{Hom}(\widetilde{N}_\sigma, \mathbb{Z})])$. In the remainder of the paper, when we say "torus part of U_σ", we mean $\mathrm{Spec}(k[\mathrm{Hom}(\widetilde{N}_\sigma, \mathbb{Z})])$.

2.2 Weighted Projective Spaces

In this section we study a particular class of toric varieties that we will be interested in as the paper progresses: the class of weighted projective spaces. To understand weighted projective spaces, we first recall ordinary projective spaces. Ordinary projective space, denoted $\mathbb{P}^d$, is constructed by

$$\mathbb{P}^d = \mathbb{A}^{d+1} \setminus \{0\}/\sim \tag{2.3}$$

where $(a_0, ..., a_d) \sim (b_0, ..., b_d)$ if and only if there is a $\lambda \in k$ such that $a_i = \lambda \cdot b_i$ for all $i = 0, ..., d$. This is a toric variety where the torus is

$$T_{\mathbb{P}^d} = \{(a_0, ..., a_d)|\ a_i \neq 0\ for\ i = 0, ..., d\}. \tag{2.4}$$

As this is a toric variety, we should be able to find a fan $\Delta_{\mathbb{P}^d}$ and indeed we can. Consider the lattice $\mathbb{Z}^{d+1}$ with basis $\{e_0, ..., e_d\}$; then we construct a lattice N by

$$N = \mathbb{Z}^{d+1}/\langle e_0 + \cdots + e_d\rangle. \tag{2.5}$$

In other words, we impose the relation $e_0 + \cdots + e_d = 0$. This gives us a lattice of dimension d. Consider the set $\{x_0, ..., x_d\}$ of residues of $\{e_0, ..., e_d\}$ in this quotient. Then $\Delta_{\mathbb{P}^d}$ is the fan consisting of all cones generated by proper subsets of $\{x_0, ..., x_d\}$. If we follow the method described in Section 2.1 for the construction of $X(\Delta_{\mathbb{P}^d})$, we recover the usual construction for $\mathbb{P}^d$ by gluing together copies of $\mathbb{A}^d$.

Observe that all the usual properties of $\mathbb{P}^d$ can be seen in the fan $\Delta_{\mathbb{P}^d}$. Indeed, we notice that $\Delta_{\mathbb{P}^d}$ is smooth (and therefore simplicial) since every proper subset of $\{x_0, ..., x_d\}$ is part of a basis for N. Therefore, we get that $\mathbb{P}^d$ is smooth (and therefore simplicial). Since it is a toric variety, $\mathbb{P}^d$ is automatically separated, Noetherian, and quasi-compact. Finally, $\mathbb{P}^d$ is complete, as the union of all the maximal cones in $\Delta_{\mathbb{P}^d}$ give us $N_{\mathbb{R}}$. None of these facts should be surprising, as they are all known properties of projective space; see [Hart].

Our goal in defining weighted projective spaces is to generalize the above example. Let $q_0, ..., q_d$ be positive integers with $\gcd(q_0, ..., q_d) = 1$. Weighted projective space, denoted $\mathbb{P}(q_0, ..., q_d)$, is constructed by

$$\mathbb{P}(q_0, ..., q_d) = \mathbb{A}^{d+1} \setminus \{0\}/\sim \tag{2.6}$$

where $(a_0, ..., a_d) \sim (b_0, ..., b_d)$ if and only if there is a $\lambda \in k$ such that $a_i = \lambda^{q_i} \cdot b_i$ for all

$i = 0, ..., d$. This is a toric variety where again the torus is

$$T_{\mathbb{P}(q_0,...,q_d)} = \{(a_0, ..., a_d)|\ a_i \neq 0\ for\ i = 0, ..., d\}. \tag{2.7}$$

As this is a toric variety, we should be able, just as in the ordinary case, to find a fan $\Delta_{\mathbb{P}(q_0,...,q_d)}$ and indeed we can. Consider the lattice $\mathbb{Z}^{d+1}$ with basis $\{f_0, ..., f_d\}$; then we construct a lattice N by

$$N = \mathbb{Z}^{d+1}/\langle q_0 f_0 + \cdots + q_d f_d\rangle. \tag{2.8}$$

In other words, we impose the relation $q_0 f_0 + \cdots + q_d f_d = 0$. This gives us a lattice of dimension d. Consider the set $\{y_0, ..., y_d\}$ of residues of $\{f_0, ..., f_d\}$ in this quotient. Then $\Delta_{\mathbb{P}(q_0,...,q_d)}$ is the fan consisting of all cones generated by proper subsets of $\{y_0, ..., y_d\}$. Observe that ordinary projective space is the weighted projective space where all the weights are 1; namely, $\mathbb{P}^d = \mathbb{P}(1, ..., 1)$.

Before continuing, we make an important observation about weighted projective spaces that will implicitly be used everywhere, and is useful to have explicitly stated. A priori, the only weighted projective spaces that are toric varieties are those where the weights are all relatively prime, as mentioned above. However, we'll see below that we can always adjust the weights to make this the case. The full details of this argument can be found in [Reid]. We begin with the notion of a *well formed weighted projective space.*

Definition 2.4. *A weighted projective space $\mathbb{P}(q_0, ..., q_d)$ is called well formed if no d of the $d+1$ weights have a common factor.*

The important proposition below says that every weighted projective space satisfies this property.

Proposition 2.5. *Let $\mathbb{P}(q_0, ..., q_d)$ be a weighted projective space. Then we have the following:*

(a) If m is a common divisor of $q_0, ..., q_d$, then $\mathbb{P}(q_0, ..., q_d) = \mathbb{P}(\frac{q_0}{m}, ..., \frac{q_d}{m})$.

(b) Suppose $q_0, ..., q_d$ have no common factors, but that m divides q_i for all $i \neq j$. Then we have $\mathbb{P}(q_0, ..., q_d) = \mathbb{P}(\frac{q_0}{m}, ..., \frac{q_{j-1}}{m}, q_j, \frac{q_{j+1}}{m}, ..., \frac{q_d}{m})$.

For the proof of this proposition, see [Reid, Proposition 3.6]. As a corollary of this proposition, we see that we can always assume that the weights are relatively prime (making every weighted projective space a toric variety after all), and we can assume that every weighted projective space is well formed.

Observe that, just as with ordinary projective spaces, weighted projective spaces are complete, separated and quasi-compact. However, they are not, in general, smooth. In fact, it turns out that the only smooth weighted projective spaces are the ordinary projective spaces. Nevertheless, weighted projective spaces are simplicial. Indeed, suppose there was a non-simplicial cone in $\Delta_{\mathbb{P}(q_0,...,q_d)}$; then some proper subset of $\{y_0, ..., y_d\}$ is linearly dependent over $\mathbb{R}$, which would mean some proper subset of $\{f_0, ..., f_d\}$ is linearly dependent over $\mathbb{R}$, which is a clear contradiction. As much of our work in this paper will be concerned with simplicial toric varieties, our results will apply to weighted projective spaces as well.

2.3 Resolution of Singularities for Toric Varieties

Toric varieties have several very nice features that make studying their geometry considerably easier than other kinds of spaces. But one of the most convenient features is that it is relatively easy to resolve their singularities. Recall from Section 2.1 that every toric variety is given by a fan, and that a toric variety is smooth if and only if every cone in the fan is smooth. The way in which we obtain a resolution of singularities for a given toric variety is by *refining* the fan; in other words, by adding additional cones to the fan until each cone in the fan is smooth. In general, one must first simplicialize all cones (that is, one must first make the fan simplicial) before one can resolve singularities. However, as our focus in this paper is on simplicial toric varieties, we will not need this first step, and therefore we skip it. Again, for the full details, we refer the reader to the standard sources, such as [Ful] and [Cox]. We will mostly follow the presentation given in [Cox].

So suppose we have a simplicial fan Δ. We say a fan Δ' is a refinement of Δ if each cone of Δ is a union of cones in Δ'. Notice that, if we take the identity map on lattices and its induced map $f : \Delta' \longrightarrow \Delta$, then we can construct a toric morphism

$$f_* : X(\Delta') \longrightarrow X(\Delta). \tag{2.9}$$

Recalling that $|\Delta|$ denotes the support of Δ (see Section 2.1 for the definition), one can show that f_* is proper if and only if $f_{\mathbb{R}}^{-1}(|\Delta|) = |\Delta'|$ (see [Cox]). From the perspective of supports, notice that saying Δ' is a refinement of Δ is equivalent to saying that $|\Delta| = |\Delta'|$ and that every cone of Δ' is contained in a cone of Δ. Our goal is to prove the following theorem.

Theorem 2.6. *If $X(\Delta)$ is the toric variety coming from the fan Δ, then there exists a refinement Δ' of Δ such that the toric morphism*

$$f_* : X(\Delta') \longrightarrow X(\Delta) \tag{2.10}$$

is a resolution of singularities.

Sketch of proof. We sketch the proof of this theorem; see [Cox, Theorem 5.1] for the full details. Note again that usually one begins by refining to obtain a simplicial fan, and then further refining to get a smooth fan. We omit the first step, as our focus throughout this entire paper is only on simplicial toric varieties (making this first step unnecessary).

Notice that the set of all nonsingular cones, denoted Δ^0, forms a subfan of Δ and that $X(\Delta^0)$ is the smooth locus of $X(\Delta)$. Our goal will be to refine Δ without changing Δ^0. Then the induced morphism f_* will be proper and because Δ^0 is unchanged, f_* will be an isomorphism away from the singular locus of $X(\Delta)$. This will give us precisely the properties we desire in a resolution of singularities.

In order to resolve singularities, we need to assign a measure to the singularity. Let $\sigma \in \Delta$ be any cone, generated by the primitive elements $x_1, ..., x_d$. Then we define the multiplicity of σ to be the group index

$$\text{mult}(\sigma) = [N_\sigma : \mathbb{Z}x_1 + \mathbb{Z}x_2 + \cdots + \mathbb{Z}x_d] \tag{2.11}$$

where N_σ is as defined in Section 2.1. Recall that σ is smooth if its primitive generators form part of a $\mathbb{Z}$-basis for N. This is equivalent to the generators forming a $\mathbb{Z}$-basis for N_σ so obviously σ is smooth if and only if $\mathrm{mult}(\sigma) = 1$. Similarly, we define the multiplicity of the fan Δ to be

$$\mathrm{mult}(\Delta) = \max_{\sigma \in \Delta} \mathrm{mult}(\sigma). \tag{2.12}$$

Again, it is obvious that the fan Δ is smooth (and hence the toric variety $X(\Delta)$ is smooth) if and only if $\mathrm{mult}(\Delta) = 1$. The proof works by showing that if $\mathrm{mult}(\Delta) > 1$, we can find a refinement Δ_1 such that either $\mathrm{mult}(\Delta_1) < \mathrm{mult}(\Delta)$ or $\mathrm{mult}(\Delta_1) = \mathrm{mult}(\Delta)$ and has fewer cones of that multiplicity. Obviously if one shows this, then the result follows.

The idea is to take a nonsmooth cone σ of minimal dimension, generaed by the primitive elements $x_1, ..., x_d$, and then find an element $u = a_1 x_1 + \cdots + a_d x_d$ with $0 < a_i < 1$ for all i (otherwise u would lie in a proper face of σ and thus lie in a smooth cone by minimality). Taking τ to be in the star of σ, we see that we can write $\tau = \sigma + \tau'$ with $\tau' \cap \sigma = \{0\}$. We then consider the cone

$$\tau_i = \langle u, x_1, ..., \widehat{x}_i, ..., x_d \rangle. \tag{2.13}$$

Then one can show that $\mathrm{mult}(\tau_i) = a_i \,\mathrm{mult}(\tau) < \mathrm{mult}(\tau)$. So to construct Δ_1, let τ_0 be a nonsmooth cone of largest multiplicity, and let σ be a nonsmooth face of τ_0 of minimal dimension. Performing the above operations on σ and τ_0, we get the new fan

$$\Delta_1 = (\Delta \setminus \mathrm{Star}(\sigma)) \cup \left(\bigcup_{\tau \in \mathrm{Star}(\sigma)} \{\tau_1, ..., \tau_d\} \right). \tag{2.14}$$

Notice that all cones in $\mathrm{Star}(\sigma)$ are replaced by cones of strictly smaller multiplicity, and that we have deleted at least one cone (namely τ_0) of maximal multiplicity. Therefore, either $\mathrm{mult}(\Delta_1) < \mathrm{mult}(\Delta)$ or $\mathrm{mult}(\Delta_1) = \mathrm{mult}(\Delta)$ and has fewer cones of that multiplicity, as desired. Repeatedly applying this procedure gives us the desired resolution.

□

The proof of Theorem 2.6 tells us explicitly how to find the correct refinement to get a resolution of singularities, but in practice it is often easier to find the desired refinement, especially in low dimensions. We will now look at an example, which we will return to many times throughout this paper, that illustrates the techniques we will use.

Example 2.7. *Consider the weighted projective space* $\mathbb{P}(1,1,2)$*. As we have already seen, this is a complete simplicial toric variety. The fan is generated by the one-dimensional cones* $(1,0)$*,* $(0,1)$*, and* $(-1,-2)$ *in the lattice* $\mathbb{Z}^2$*. Clearly the two-dimensional cone generated by the rays* $(1,0)$ *and* $(-1,-2)$ *is the only singular cone. To resolve the singularity, we simply add in additional rays until all cones are smooth. In this case, we need only add one additional cone, namely the cone generated by* $(0,-1)$*. This new fan is (isomorphic to) the fan of the Hirzebruch surface* $\mathcal{H}_2$*, which is the* $\mathbb{P}^1$*-bundle over* $\mathbb{P}^1$ *associated to the sheaf* $\mathcal{O}(0)+\mathcal{O}(-2)$*.*

Next, we need to determine the exceptional curve for this resolution. In general, we do this by taking $\mathrm{Star}(\tau)$*, where* τ *is the cone added in for the resolution. Recall that* $\mathrm{Star}(\tau)$ *is simply the set of all cones containing* τ *as a face, but is considered in the lattice* $\widetilde{N_\tau}$ *(see Section 2.1 for definition). It is easy to see that, in this case, the exceptional curve turns out to be* $\mathbb{P}^1$*. So this resolution gives us the blow-up square*

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r] \arrow[d] & \mathcal{H}_2 \arrow[d] \\ \{*\} \arrow[r] & \mathbb{P}(1,1,2) \end{tikzcd}$$

and we obtain a resolution of singularities for $\mathbb{P}(1,1,2)$*, as desired.*

3 Background

In this section we build the necessary machinery that will be used throughout the paper. Much of the material presented in this section is very general and can be found in several other sources. However, we present them here as they will play a key role in the sections to follow.

3.1 Commutative Algebra and Algebraic Geometry

There will come a point in the proof of our main theorem where we will need to understand properties of morphisms of the form $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$. This can be done primarily by understanding the properties of prime ideals. In this section, we collect the results that will be needed when analyzing these properties in the context of our problem. Many of our results can be found in the standard sources such as [AM], [Mil], [Hart], [Sha1], and [Sha2].

We recall that if A is a ring, we can construct a scheme called $\mathrm{Spec}(A)$ by taking the topological space to consist of prime ideals of A as points and closed sets $V(I)$ (the set of all prime ideals of A containing the ideal I), and taking the sheaf to be the standard structure sheaf; see [Hart, page 70]. We call such a scheme an *affine scheme*. Recall also that if $f : A \longrightarrow B$ is a morphism of rings, then this induces a morphism of schemes $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$; conversely, if we have a morphism of schemes $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$, then there is a ring homomorphism $f : A \longrightarrow B$ inducing it; see [Hart, Chapter II, Proposition 2.3].

Definition 3.1. *We say that a morphism of affine schemes* $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ *is surjective if the morphism on the underlying topological spaces is surjective; that is, for any prime ideal $P \subset A$, $f(P)$ is a prime ideal of B.*

We would like to establish conditions under which a morphism of affine schemes will be surjective. As the following theorems will show, that condition turns out to be that $f : A \longrightarrow B$ is injective and integral.

We begin by stating some basic definitions and results on integrality from Commutative Algebra; see [AM, Chapter 5] for the standard proofs of each of these statements.

Definition 3.2. *Let B be a ring, and $A \subset B$ be a subring. An element $x \in B$ is said to be integral over A if x satisfies some monic polynomial*

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \tag{3.1}$$

where $a_i \in A$ for all i. We say that B is integral over A if every $x \in B$ is integral over A.

We have the following important proposition.

Proposition 3.3. *The following are equivalent:*

(a) $x \in B$ *is integral over* A.

(b) $A[x]$ *is a finitely generated* A*-module.*

(c) $A[x]$ *is contained in a subring* C *of* B *such that* C *is a finitely generated* A*-module.*

(d) There exists a faithful $A[x]$*-module* M *which is finitely generated as an* A*-module.*

Proof. See [AM, Proposition 5.1].

□

Corollary 3.4. *Let $x_1, ..., x_n$ be elements of B, each integral over A. Then the ring $A[x_1, ..., x_n]$ is a finitely generated A-module.*

Proof. Proceed by induction. The case $n = 1$ is immediate by Proposition 3.3. Let $A_{n-1} = A[x_1, ..., x_{n-1}]$. Then $A[x_1, ..., x_n] = A_{n-1}[x_n]$; by our inductive hypothesis, A_{n-1} is a finitely generated A-module. Since x_n is integral over A, it is integral over A_{n-1} trivially. Thus, $A_{n-1}[x_n]$ is a finitely generated A_{n-1}-module by Proposition 3.3, and thus a finitely generated A-module since if $y_1, ..., y_m$ generate $A_{n-1}[x_n]$ as an A_{n-1}-module, and $z_1, ..., z_k$ generate A_{n-1} as an A-module, then the mk products $\{y_i z_j\}$ generate $A_{n-1}[x_n]$ as an A-module.

□

Definition 3.5. *Let $f : A \longrightarrow B$ be a ring homomorphism.*

(a) There is an A-module structure on B induced by $a \cdot x = f(a)x$; we call this structure the canonical A-module structure induced by f. Thus, f makes B an A-algebra. More generally, if M is any B-module, then M is also an A-module, induced by $a \cdot x = f(a)x$.

(b) *We say that f is of finite-type if B is finitely generated as an A-algebra via f; in other words, if there exists $x_1, ..., x_n \in B$ such that the morphism $A[t_1, ..., t_n] \longrightarrow B$ with*

$$\sum a_{i_1,\ldots,i_n} t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n} \mapsto \sum f(a_{i_1,\ldots,i_n}) x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \tag{3.2}$$

is a surjection.

(c) *We say that f is integral if B is integral, in the sense of Definition 3.2, over the subring $f(A)$.*

(d) *We say that f is finite if B is finitely generated as an A-module under the canonical A-module structure induced by f.*

Theorem 3.6. *Let $f : A \longrightarrow B$. Then f is finite if and only if it is both finite type and integral.*

Proof. If f is finite, then B is finitely generated over $f(A)$. That means there exists $x_1, ..., x_n$ that generate B as an $f(A)$-module (and equivalently, an A-module). Then take $A[t_1, ..., t_n] \longrightarrow B$ by $t_i \mapsto x_i$ and

$$\sum a_{i_1,\ldots,i_n} t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n} \mapsto \sum f(a_{i_1,\ldots,i_n}) x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}. \tag{3.3}$$

Then this map hits every element of $f(A)$ and it hits $x_1, ..., x_n$ so it must hit all of B. So f is of finite type.

To see that f is integral, observe that, for any $x \in B$, $f(A) \subset f(A)[x] \subset B$. Since B is finitely generated over $f(A)$, $f(A)[x]$ must be finitely generated over $f(A)$ also. By Proposition 3.3, x is integral over $f(A)$. Since this can be done for any $x \in B$, B is integral over $f(A)$; hence, f is integral.

Now suppose that f is finite type and integral. Since f is finite type, there exists $x_1, ..., x_n$ such that $B = f(A)[x_1, ..., x_n]$. Since B is integral over $f(A)$, Corollary 3.4 says that $f(A)[x_1, ..., x_n]$ is finitely generated as an $f(A)$-module; hence, B is finitely generated as an $f(A)$-module. So f is finite.

□

What follows are the properties of integrality that we will use.

Proposition 3.7. *Let $A \subset B$ in Rings, with B integral over A.*

(a) If J is an ideal of B and $I = J \cap A$ is an ideal of A, then B/J is integral over A/I.

(b) If S is a multiplicatively closed subset of A, then $S^{-1}B$ is integral over $S^{-1}A$.

Proof. Any $x \in B$ satisfies $x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$, where $a_i \in A$ for all i.

For (a), simply reduce this equation modulo J; then the coefficients will be reduced modulo I since $I = J \cap A$ and the result is immediate.

For (b), notice that, for any $s \in S$, the element $\frac{x}{s} \in S^{-1}B$ satisfies

$$\left(\frac{x}{s}\right)^n + \frac{a_1}{s}\left(\frac{x}{s}\right)^{n-1} + \cdots + \frac{a_{n-1}}{s^{n-1}}\left(\frac{x}{s}\right) + \frac{a_n}{s^n} = 0, \tag{3.4}$$

where $a_i \in A$ for all i. So $\frac{x}{s}$ is integral over $S^{-1}A$ and we are done.

□

Corollary 3.8. *Let $A \subset B$ in Rings, with B integral over A. If P is a prime ideal of A, then B_P is integral over A_P.*

Proof. This is immediate from Proposition 3.7 part (b), with $S = A \setminus P$.

□

Proposition 3.9. *Let $A \subset B$ be integral domains, and B integral over A. Then B is a field if and only if A is a field.*

Proof. Suppose A is a field, and let $x \in B$ with $x \neq 0$. Then let the polynomial $x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$ be the polynomial of smallest degree that x satisfies. Then, since $x \neq 0$, $a_n \neq 0$ also; otherwise we would have $x(x^{n-1} + a_1x^{n-2} + \cdots + a_{n-1}) = 0$, which would imply $x^{n-1} + a_1x^{n-2} + \cdots + a_{n-1} = 0$ contradicting the fact that our initial polynomial was the smallest degree polynomial over A that x satisfies. Then we simply have $x^{-1} = -a_n^{-1}(x^{n-1} + a_1x^{n-2} + \cdots + a_{n-1})$, which makes B a field also.

Conversely, suppose that B is a field. Let $x \in A$, $x \neq 0$. Then we have $x \in B$ also; since B is a field, that gives us $x^{-1} \in B$. Since B is integral over A, there exists a polynomial $(x^{-1})^n + a_1(x^{-1})^{n-1} + \cdots + a_{n-1}(x^{-1}) + a_n = 0$. Multiplying both sides by x^n and putting all the terms involving x on one side gives us $1 = -x(a_1 + \cdots + a_{n-1}x^{n-2} + a_nx^{n-1})$, or $x^{-1} = -(a_1 + \cdots + a_{n-1}x^{n-2} + a_nx^{n-1})$, making A a field also.

□

Corollary 3.10. *Let $A \subset B$ be rings, and B integral over A. Let Q be a prime ideal of B, and $P = Q \cap A$. Then Q is maximal if and only if P is maximal.*

Proof. By Proposition 3.7 part (a), B/Q is integral over A/P. Since Q (and hence P) are prime ideals, A/P and B/Q are both integral domains. Thus, by Proposition 3.9, A/P is a field if and only if B/Q is a field, which gives the result.

□

Finally, we can prove the following important theorem.

Theorem 3.11. *Let $A \subset B$ in Rings, with B integral over A, and let P be a prime ideal of A. Then there exists a prime ideal Q of B such that $Q \cap A = P$. In other words, the morphism* $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ *induced by the inclusion map $A \longrightarrow B$ is surjective.*

Proof. By Proposition 3.7 part (b), B_P is integral over A_P. Let $\alpha : A \longrightarrow A_P$ and $\beta : B \longrightarrow B_P$ be the canonical localization maps. Then we get the commuting square:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A_P \arrow[r] & B_P \end{tikzcd}$$

where the horizontal maps are injections. Let N be a maximal ideal of B_P; then $N \cap A_P = M$ is maximal by Corollary 3.10. Since the ring A_P is local, M is the unique maximal ideal of

A_P. Then we have that $\alpha^{-1}(M) = P$ and by commutativity $\alpha^{-1}(M) = \beta^{-1}(N) \cap A$. So $Q = \beta^{-1}(N)$ is our desired prime ideal.

□

Corollary 3.12. *If $f : A \longrightarrow B$ is injective and integral, then the induced morphism* $\operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A)$ *is surjective.*

Proof. Factor f into $A \longrightarrow f(A) \longrightarrow B$, where the first map is an isomorphism (every injective map is an isomorphism onto its image) and the second map is the obvious inclusion map. Then $\operatorname{Spec}(f(A)) \longrightarrow \operatorname{Spec}(A)$ is an isomorphism and hence surjective. The morphism $\operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(f(A))$ is surjective by Theorem 3.11; composing these two gives the result.

□

Corollary 3.13. *If $f : A \longrightarrow B$ is injective and finite, then the induced morphism* $\operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A)$ *is surjective.*

Proof. By Theorem 3.6, f must be integral. Now the result is immediate from Corollary 3.12.

□

We also want to recall some basic notions from algebraic geometry. In particular, we want to recall the notions of flat, unramified, and étale morphisms. We begin with flatness.

Definition 3.14. *Let A be a ring, and N an A-module. We say that N is* flat *if the functor $M \mapsto M \otimes_A N$ is an exact functor. We say that N is* faithfully flat *if, whenever we have an exact sequence*

$$0 \longrightarrow M' \otimes_A N \longrightarrow M \otimes_A N \longrightarrow M'' \otimes_A N \longrightarrow 0$$

the sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

must have already been exact to start with. A ring homomorphism $f : A \longrightarrow B$ is called flat (faithfully flat) if B is flat (faithfully flat) as an A-module via the module structure given in part (a) of Definition 3.5. We say a morphism of schemes $f : X \longrightarrow Y$ is flat if, for every point $x \in X$, $y = f(x) \in Y$, the induced morphism $f^{\#} : \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$ is a flat morphism of rings.

Proposition 3.15. *Flat morphisms satisfy the following very important properties:*

(a) An open immersion is flat.

(b) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are flat, then so is $g \circ f : X \longrightarrow Z$.

(c) Flat morphisms are closed under base extension.

Proof. See [Mil, Chapter 1, Proposition 2.4].

□

Corollary 3.16. *If $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ are both flat, then so is $f \times f' : X \times X' \longrightarrow Y \times Y'$.*

Proof. This follows from properties (b) and (c) of Proposition 3.15. To see this, consider the following diagram:

$$\begin{tikzcd}
X \times X' \arrow[r, "\alpha"] \arrow[d] \arrow[dr, "f\times f'"] & X \times_Y (Y \times Y') \arrow[r] \arrow[d, "\beta"] & X \arrow[d, "f"] \\
X \times_{Y'} (Y \times Y') \arrow[r] \arrow[d] & Y \times Y' \arrow[r] \arrow[d] & Y \arrow[d] \\
X' \arrow[r, "f'"'] & Y' \arrow[r] & \mathrm{Spec}(k)
\end{tikzcd}$$

Observing that $Y' = Y \times_Y Y'$, we realize that α is obtained by the base extension of f'

$$X \times X' \longrightarrow X \times (Y \times_Y Y') = X \times_Y (Y \times Y')$$

where the latter equality is by elementary properties of fiber products. Since f' is flat, so is α by property (c) of Proposition 3.15. Observe that

$$\alpha(x, x') = (x, f'(x')). \tag{3.5}$$

Similarly, we realize that β is a base extension of f. Indeed, observing that $X = X \times_Y Y$, we can base extend f to get

$$X \times_Y (Y \times Y') = (X \times_Y Y) \times Y' \times X' \longrightarrow Y \times Y'$$

where the initial equality is by elementary properties of fiber products. Since f is flat, so is β by property (c) of Proposition 3.15. Observe that

$$\beta(x, y') = (f(x), y'). \tag{3.6}$$

So we immediately see that $\beta \circ \alpha = f \times f'$. Since both α and β are flat, by property (b) of Proposition 3.15, $f \times f'$ is flat also.

□

We have the following important propositions, which provides us with some basic properties of flatness that allows us to relate being flat to being projective.

Proposition 3.17. *Let $f : A \longrightarrow B$ be a flat morphism of rings. If N is flat as a B-module, then N is also flat as an A-module, via the A-module structure given in part (a) of Definition 3.5.*

Proof. See [AM, Chapter 2, Exercise 8].

□

Proposition 3.18. *Every finitely presented flat A-module N is projective.*

Proof. See [Wei2, Theorem 3.2.7].

□

Corollary 3.19. *If A is Noetherian, then any finitely generated flat A-module N is projective. In particular, if we have a ring map $f : A \longrightarrow B$ of Noetherian rings, with f both flat and finite, then B is projective as an A-module via the module structure given in part (a) of Definition 3.5.*

Proof. Recall that if A is Noetherian, then any finitely generated A-module is also Noetherian; therefore, N is Noetherian. Since N is finitely generated, there is a surjection from a finite rank free module to N; $\varphi : A^n \longrightarrow N$ surjective for some n. Recall that we say N is finitely presented if $\ker \varphi$ is also finitely generated. However, since A is Noetherian, A^n is Noetherian (since it is finitely generated); by definition of Noetherian module, every submodule of a Noetherian module is finitely generated. Since $\ker \varphi$ is a submodule of the Noetherian module A^n, it is finitely generated, making N finitely presented. Now the result follows from Proposition 3.18. In the case of $f : A \longrightarrow B$ being a map of Noetherian rings, with f is flat and finite, this is just a special case of the first part of this corollary, since f flat and finite means that B is a finitely generated, flat A-module and therefore projective since A is Noetherian.

□

Now that we have recalled what it means for a morphism of schemes to be flat, we now want to recall what it means for a morphism to be unramified. We follow the presentation given in [Mil]. Recall that if $x \in X$, then $k(x)$ denotes the residue field of the point x.

Definition 3.20. *Let $f : A \longrightarrow B$ be a ring homomorphism of finite type. We say that f is* unramified *at a point $Q \in \operatorname{Spec}(B)$ if and only if $P = f^{-1}(Q)$ generates the maximal ideal in B_Q and $k(Q)$ is a finite separable field extension of $k(P)$. We say that f is unramified if it is unramified for every $Q \in \operatorname{Spec}(B)$. Similarly, given a morphism of schemes $F : Y \longrightarrow X$ that is locally of finite type, we say that F is unramified at $y \in Y$ if $\mathcal{O}_{Y,y}/M_x\mathcal{O}_{Y,y}$ is a finite separable field extension of $k(x)$, where $x = f(y)$. We say F is unramified if it is unramified at every $y \in Y$.*

The following result about unramified morphisms gives us a good way to test whether a morphism of schemes is unramified.

Proposition 3.21. *Let $f : Y \longrightarrow X$ be a morphism of schemes that is locally of finite type. The following are equivalent:*

(a) f is unramified.

(b) For all $x \in X$, the fiber $Y_x \longrightarrow \mathrm{Spec}(k(x))$ over x is unramified.

(c) All geometric fibers of f are unramified; in other words, for all morphisms $\mathrm{Spec}(k) \longrightarrow X$ with k separably closed, $Y \times_X \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k)$ is unramified.

(d) For all $x \in X$, Y_x has an open covering by spectra of finite separable $k(x)$-algebras.

(e) For all $x \in X$, Y_x is a sum $\coprod \mathrm{Spec}(k_i)$, where the k_i are finite separable field extensions of $k(x)$.

Proof. See [Mil, Chapter 1, Proposition 3.2].

□

Just as we had with flat morphisms, we have the following results for unramified morphisms.

Proposition 3.22. *Unramified morphisms satisfy the following very important properties:*

(a) An open immersion is unramified.

(b) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are unramified, then so is $g \circ f : X \longrightarrow Z$.

(c) Unramified morphisms are closed under base extension.

Proof. See [Mil, Chapter 1, Proposition 3.3].

□

Corollary 3.23. *If $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ are both unramified, then so is $f \times f' : X \times X' \longrightarrow Y \times Y'$.*

Proof. This follows from properties (b) and (c) of Proposition 3.22. To see this, consider the following diagram:

$$\begin{tikzcd}
X \times X' \arrow[r, "\alpha"] \arrow[d] \arrow[dr, "f\times f'"] & X \times_Y (Y \times Y') \arrow[r] \arrow[d, "\beta"] & X \arrow[d, "f"] \\
X \times_{Y'} (Y \times Y') \arrow[r] \arrow[d] & Y \times Y' \arrow[r] \arrow[d] & Y \arrow[d] \\
X' \arrow[r, "f'"'] & Y' \arrow[r] & \mathrm{Spec}(k)
\end{tikzcd}$$

Observing that $Y' = Y \times_Y Y'$, we realize that α is obtained by the base extension of f'

$$X \times X' \longrightarrow X \times (Y \times_Y Y') = X \times_Y (Y \times Y')$$

where the latter equality is by elementary properties of fiber products. Since f' is unramified, so is α by property (c) of Proposition 3.22. Observe that

$$\alpha(x, x') = (x, f'(x')). \tag{3.7}$$

Similarly, we realize that β is a base extension of f. Indeed, observing that $X = X \times_Y Y$, we can base extend f to get

$$X \times_Y (Y \times Y') = (X \times_Y Y) \times Y' \times X' \longrightarrow Y \times Y'$$

where the initial equality is by elementary properties of fiber products. Since f is unramified, so is β by property (c) of Proposition 3.22. Observe that

$$\beta(x, y') = (f(x), y'). \tag{3.8}$$

So we immediately see that $\beta \circ \alpha = f \times f'$. Since both α and β are unramified, by property (b) of Proposition 3.22, $f \times f'$ is unramified also.

□

We are now ready to recall the definition of an étale morphism.

Definition 3.24. *Let $f : A \longrightarrow B$ be a homomorphism of rings. We say that f is étale if f is both flat and unramified. Similarly, given a morphism of schemes $F : Y \longrightarrow X$, we say that F is étale if F is both flat and unramified.*

An étale morphism is the algebraic geometry analog of a morphism satisfying the conditions of the Implicit Function Theorem from ordinary geometry. Recall that a morphism satisfying the conditions of the Implicit Function Theorem is a local diffeomorphism in ordinary geometry; however, due to the fact that Zariski open sets are often very large, étale morphisms are not necessarily local isomorphisms. However, they retain many of the important properties of local analytic isomorphisms, which makes their study extremely important.

Just as with flat morphisms and unramified morphisms, we next have the following very useful result.

Proposition 3.25. *Étale morphisms satisfy the following very important properties:*

(a) An open immersion is étale.

(b) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are étale, then so is $g \circ f : X \longrightarrow Z$.

(c) Étale morphisms are closed under base extension.

(d) If $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ are both étale, then so is the map $f \times f' : X \times X' \longrightarrow Y \times Y'$.

Proof. Parts (a), (b), and (c) are immediate from Propositions 3.15 and 3.22. Part (d) is immediate from Corollaries 3.16 and 3.23.

□

We now have the tools to determine if a given morphism is étale. We can now put these tools to use.

Proposition 3.26. *Let A be a ring, and let $B = A[T]/(P(T))$, where $P(T)$ is an irreducible, separable, monic polynomial in $A[T]$. Then the morphism $A \longrightarrow B$ is étale.*

Proof. Following the proof given in [Mil, Example 3.4], we need to check that B is both flat and unramified over A. Flatness is immediate, as B is actually free as an A-module, with rank equal to the degree of $P(T)$. To check that this mapping is unramified, recall that a polynomial is separable if the ideal $(P(T), P'(T)) = A[T]$, where $P'(T)$ denotes the formal derivative of $P(T)$. This is equivalent to saying that $P'(T)$ is invertible in B. Let $\mathcal{P}$ be any prime ideal in $\mathrm{Spec}(A)$, and let $k(\mathcal{P})$ denote its residue field. Then $P(T)$ is separable if and only if $\overline{P(T)} \in k(\mathcal{P})[T]$ is separable for all $\mathcal{P}$. One direction is obvious; if $(P(T), P'(T)) = A[T]$, then tensoring both sides with $k(\mathcal{P})$ over A gives the claim. For the other direction, let $I = (P(T), P'(T))$ and suppose that $\overline{I}_{\mathcal{P}} = I \otimes_A k(\mathcal{P})$ denotes the image of I in $k(\mathcal{P})[T]$ for all $\mathcal{P}$. If $\overline{I}_{\mathcal{P}} = k(\mathcal{P})[T]$, then since we have that $\mathcal{P}A_{\mathcal{P}}[T] \subset I_{\mathcal{P}} \subset A_{\mathcal{P}}[T]$ by the correspondence theorem, we conclude that $I_{\mathcal{P}} = A_{\mathcal{P}}[T]$. In other words, the inclusion $I \longrightarrow A[T]$ induces a surjection (and hence an equality) when tensoring with $A_{\mathcal{P}}$ over A. But [AM, Proposition 3.9] says that any A-module homomorphism $f : M \longrightarrow N$ such that the induced morphisms $f_{\mathcal{P}} : M_{\mathcal{P}} \longrightarrow N_{\mathcal{P}}$ are surjective for all $\mathcal{P}$ is itself surjective as well. So the inclusion $I \longrightarrow A[T]$ is a surjection and hence an equality, giving the other direction.

So now we check that $A \longrightarrow B$ is unramified by using Proposition 3.21, part (b). We do so by looking at the morphism of schemes $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$. For a point $\mathcal{P} \in \mathrm{Spec}(A)$, the fiber is

$$\mathrm{Spec}(B)_{\mathcal{P}} = \mathrm{Spec}(B) \times_A \mathrm{Spec}(k(\mathcal{P})) = \mathrm{Spec}(B \otimes_A k(\mathcal{P})) \tag{3.9}$$

and since $B \otimes_A k(\mathcal{P}) = k(\mathcal{P})[T]/(\overline{P(T)})$, this question reduces to whether or not the extension of fields $k(\mathcal{P})[T]/(\overline{P(T)})$ over $k(\mathcal{P})$ is separable. But since $P(T)$ is a separable monic polynomial in $A[T]$, $\overline{P(T)}$ is separable for every $\mathcal{P}$, and therefore the extension of fields is indeed separable and we are done.

$\square$

Corollary 3.27. *If $B = A[T]/(T^r - a)$, then the morphism $A \longrightarrow B$ is étale if and only if $ra \in A^*$.*

Proof. We want to show that these conditions are equivalent to the polynomial $T^r - a$ being separable. First notice that $ra \in A^*$ if and only if ra is not contained in any prime ideal of A, which is the same as saying that $r(a_{\mathcal{P}})$ is not contained in the unique maximal ideal $\mathcal{P}A_{\mathcal{P}}$ of $A_{\mathcal{P}}$, which is in turn the same as saying that $r\overline{a}$ is not 0 in the field $k(\mathcal{P})$. So $ra \in A^*$ if and only if $r \neq 0$ and $\overline{a} \neq 0$ in $k(\mathcal{P})$. The statement $r \neq 0$ is the same as saying that the characteristic of $k(\mathcal{P})$ does not divide r, so we have that for all $\mathcal{P} \in \operatorname{Spec}(A)$, $\overline{a} \neq 0$ and the characteristic of $k(\mathcal{P})$ does not divide r. This means that the polynomial $T^r - \overline{a}$ is separable in $k(\mathcal{P})[T]$ for all $\mathcal{P} \in \operatorname{Spec}(A)$. By the claim we established during the proof of Proposition 3.26, this means that $T^r - a$ is separable in $A[T]$. By Proposition 3.26, the map $A \longrightarrow B$ is étale and we are done.

□

In addition to the basic commutative algebra and algebraic geometry results we've presented above, we will need to know something about groups schemes; in particular, since our toric varieties are assumed to have only split tori, we need to say something about diagonalizable group schemes. Much of the material we present here can be found in [KMRT]. In what follows, let Alg_k denote the category of untial commutative (associative) k-algebras with k-algebra homomorphisms as morphisms.

Definition 3.28. *Let H be an abstract abelian group, written multiplicatively, and let k be a field. We have a Hopf algebra structure on the group algebra $k\langle H\rangle$ over k given by comultiplication $c(h) = h \otimes h$, co-inverse $i(h) = h^{-1}$, and co-unit $u(h) = 1$. The group scheme represented by $k\langle H\rangle$ is said to be* diagonalizable *and is denoted H_{diag}. By definition, $H_{diag}(R) = \operatorname{Hom}_{Alg_k}(k\langle H\rangle, R) = \operatorname{Hom}(H, R^\times)$ for any $R \in Alg_k$.*

We can find many examples of diagonalizable group schemes. For instance, if $H = \mathbb{Z}$, then $H_{diag} = \mathbb{G}_m$; similarly, if $H = \mathbb{Z}^n$, then $H_{diag} = \mathbb{G}_m^n$, the split algebraic torus of rank n. Since our toric varieties are all split toric varieties, this is the viewpoint that we want to take. We now give the following proposition.

Proposition 3.29. *There is an anti-equivalence of categories between the category of diagonalizable group schemes over k, denoted* Diag*, and abelian groups, denoted* Ab*, where the* *functors are given by* $F : \mathsf{Diag} \longrightarrow \mathsf{Ab}$*,* $F(G) = G^*$ *(here* G^* *denotes the group of characters) and* $M : \mathsf{Ab} \longrightarrow \mathsf{Diag}$*,* $M(H) = H_{diag}$*.*

Proof. See [KMRT, Proposition 20.17].

□

Remark 3.30. This is not the proposition that actually appears in [KMRT]; the true proposition proves that there is an equivalence of categories between the category of group schemes of multiplicative type over k and the category of abelian groups with continuous Γ-action. Here $\Gamma = \mathrm{Gal}(k_{sep}/k)$. However, as we work exclusively with diagonalizable groups schemes, this added generality will not be helpful to us so we omit its presentation.

We conclude this section with one final observation about diagonalizable group schemes.

Definition 3.31. *We say that a diagonalizable group scheme is* finite *if the Hopf algebra representing the group scheme is a finite* k*-algebra (see part (d) of Definition 3.5 with* f *the canonical* k*-algebra structure map).*

Proposition 3.32. *If* H *is a finite abelian group, then* H_{diag} *is a finite diagonalizable group scheme.*

Proof. The Hopf algebra representing H_{diag} is the group algebra $k\langle H \rangle$. Since the basis (as a k-module) of the group algebra is in one-to-one correspondence with the elements of H, the result is immediate.

□

3.2 Grothendieck Topologies

In the sections that follow, we will be concerned with several kinds of topologies. Section 3.8, in particular, looks at descent properties with respect to our various topologies, and we then use these descent properties in Section 4. To that end, we want to discuss Grothendieck topologies and to identify the main topologies that we will examine in later sections. We begin with the following definition, as seen in [Art].

Definition 3.33. *A Grothendieck topology T consists of a category $\mathcal{C}$ and a set* $\mathrm{Cov}(T)$ *of families $\{\phi_i : U_i \longrightarrow U\}_{i \in I}$ of maps in $\mathcal{C}$ called* coverings *(where in each covering the target space U of the maps ϕ_i is fixed) satisfying the following conditions:*

(a) If ϕ_i is an isomorphism then the set $\{\phi_i\}$ is an element of $\mathrm{Cov}(T)$*; in other words, an isomorphism forms a cover consisting of one element.*

(b) If $\{U_i \longrightarrow U\}$ is in $\mathrm{Cov}(T)$ *and $\{V_{ij} \longrightarrow U_i\}$ is in* $\mathrm{Cov}(T)$ *then $\{V_{ij} \longrightarrow U\}$, formed by taking composition of maps, is in* $\mathrm{Cov}(T)$ *as well.*

(c) If $\{U_i \longrightarrow U\}$ is in $\mathrm{Cov}(T)$ *and $V \longrightarrow U$ in $\mathcal{C}$ is arbitrary, then $U_i \times_U V$ exists and $\{U_i \times_U V \longrightarrow V\}$ is in* $\mathrm{Cov}(T)$*.*

We discuss the following examples, which will each play a role in our work.

Example 3.34 (Zariski Topology)**.** *Let us consider a scheme X and let $\mathcal{U}$ be the set of all Zariski open subsets of X. Define a covering to be $\{\phi_i : U_i \longrightarrow U\}_{i \in I}$ where each of the maps ϕ_i are open immersions and $U = \cup_i U_i$. One can immediately see that all three conditions of Definition 3.33 are trivially satisfied. So the Zariski topology forms a Grothendieck topology on the category* Schemes*.*

Example 3.35 (Étale Topology)**.** *Let us consider a scheme X over a field k. Define a covering to be $\{\phi_i : U_i \longrightarrow U\}_{i \in I}$ where each of the maps ϕ_i are étale morphisms and $U = \cup_i U_i$. One can immediately see that all three conditions of Definition 3.33 are trivially satisfied. So the Étale topology forms a Grothendieck topology on the category* Schemes/k*.*

Example 3.36 (Nisnevich Topology)**.** *Let us consider a scheme X over a field k. We define the Nisnevich Topology to be the Grothendieck topology generated by coverings of the form $\{\phi_1 : U \longrightarrow X, \phi_2 : V \longrightarrow X\}$ where $\phi_1 : U \longrightarrow X$ is an open immersion, $\phi_2 : V \longrightarrow X$ is an étale morphism such that the induced morphism $V \times_X (X \setminus U) \longrightarrow X \setminus U$ is a split surjection, and $X = \phi_1(U) \cup \phi_2(V)$. One can check that all three conditions of Definition 3.33 are satisfied, showing that the Nisnevich Topology is indeed a Grothendieck topology on the category* Schemes/k*.*

The next topology is extremely useful, especially in the calculation of KH groups. But to present it, we need to first give a definition.

Definition 3.37. *An abstract blow-up square of* Schemes/k *is a Cartesian square of* Schemes/k

$$\begin{tikzcd} E \arrow[r] \arrow[d] & X' \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

where $X' \longrightarrow X$ is a proper morphism and $Z \longrightarrow X$ is a closed embedding such that $(X' \setminus E)^{red} \longrightarrow (X \setminus Z)^{red}$ is an isomorphism. When X, X', and Z fit into such a square, we say that $X' \longrightarrow X$ is an abstract blow-up with center Z.

Example 3.38 (cdh-Topology)**.** *Let us consider a scheme X over a field k. We define the cdh-Topology to be the Grothendieck topology generated by coverings of the form $\{\phi_1 : U \longrightarrow X, \phi_2 : V \longrightarrow X\}$ where $\phi_1 : U \longrightarrow X$ is an open immersion, $\phi_2 : V \longrightarrow X$ is an étale morphism such that the induced morphism $V \times_X (X \setminus U) \longrightarrow X \setminus U$ is a split surjection, and $X = \phi_1(U) \cup \phi_2(V)$, and by coverings of the form $\{\phi_1 : Z \longrightarrow X, \phi_2 : X' \longrightarrow X\}$, where $\phi_1 : Z \longrightarrow X$ is a closed immersion, $\phi_2 : X' \longrightarrow X$ is an abstract blow up with center Z, and $X = \phi_1(Z) \cup \phi_2(X')$. In other words, the cdh-Topology is generated by Nisnevich covers and by abstract blowup squares. One can check that all three conditions of Definition 3.33 are satisfied, showing that the cdh-Topology is indeed a Grothendieck topology on the category* Schemes/k.

There are various uses for looking at Schemes and Schemes/k in different topologies. The most common example is Étale Cohomology, which is a cohomology constructed using the Étale Topology instead of the Zariski Topology (see [Mil]). However, we will use them in a different way; namely, we will examine various descent properties of the functors K, KH, $\mathrm{KH}(-) \otimes \mathbb{Q}$, and $\mathcal{F}_{\mathrm{K}}$ in these different topologies. We will revisit this in Section 3.8.

3.3 Simplicial and Cosimplicial Objects over a Category

In the sections that follow, we will be interested in relating the simplicial structures between two complete simplicial toric varieties. The idea will be to look at the simplicial scheme structure that is created by a complete simplicial toric variety. In order to make this a reality, we need to first define what these terms mean. We will also be interested in cosimplicial objects as they allow us to define the holim functor in Section 3.5 and will occur naturally in our arguments in Section 4 since taking KH of a simplicial object will give rise to a cosimplicial object. To that end, we present the basic definitions as they appear in [BK] in this section, beginning with the notion of a simplicial object over a category $\mathcal{C}$ (see [BK, Chapter VIII]).

Definition 3.39. *Let $\mathcal{C}$ be a category. We say X is a simplicial object over $\mathcal{C}$ if*

(a) For every $n \geq 0$, we have an object $X_n \in \mathcal{C}$.

(b) For every $0 \leq i \leq n$, we have face maps

$$d_i : X_n \longrightarrow X_{n-1} \tag{3.10}$$

and degeneracy maps

$$s_i : X_n \longrightarrow X_{n+1} \tag{3.11}$$

satisfying the usual simplicial identities:

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i \text{ for } i < j \\
d_i s_j &= s_{j-1} d_i \text{ for } i < j \\
d_i s_j &= \text{id} \quad\ \text{for } i = j, j+1 \\
d_i s_j &= s_j d_{i-1} \text{ for } i > j+1 \\
s_i s_j &= s_j s_{i-1} \text{ for } i > j.
\end{aligned} \tag{3.12}$$

In particular, we say that X is a simplicial scheme if X is a simplicial object over the category of schemes.

As one can see in [BK, Chapter X], we can dualize Definition 3.39 to define the notion of a cosimplicial object over $\mathcal{C}$.

Definition 3.40. *Let $\mathcal{C}$ be a category. We say X is a cosimplicial object over $\mathcal{C}$ if*

(a) For every $n \geq 0$, we have an object $X^n \in \mathcal{C}$.

(b) For every $0 \leq i \leq n$, we have coface maps

$$d^i : X^{n-1} \longrightarrow X^n \tag{3.13}$$

and codegeneracy maps

$$s^i : X^{n+1} \longrightarrow X^n \tag{3.14}$$

satisfying the usual cosimplicial identities:

$$\begin{aligned} d^j d^i &= d^i d^{j-1} \text{ for } i < j \\ s^j d^i &= d^i s^{j-1} \text{ for } i < j \\ s^j d^i &= \mathrm{id} \qquad \text{for } i = j, j+1 \\ s^j d^i &= d^{i-1} s^j \text{ for } i > j+1 \\ s^j s^i &= s^{i-1} s^j \text{ for } i > j. \end{aligned} \tag{3.15}$$

Now that we have defined simplicial and cosimplicial objects, we need to understand morphisms between these kinds of objects. In particular, we want to understand isomorphisms of such objects.

Definition 3.41. *Let X and Y be two simplicial objects over $\mathcal{C}$. We say that $f : X \longrightarrow Y$ is a morphism of simplicial objects if $f_n : X_n \longrightarrow Y_n$ is a morphism in $\mathcal{C}$ for every $n \geq 0$*

and for all pairs (i, n), we have that $s_i f_n = f_{n+1} s_i$ and $d_i f_n = f_{n-1} d_i$. We say that f is an isomorphism if $f_n : X_n \longrightarrow Y_n$ is an isomorphism for all $n \geq 0$

Let X and Y be two cosimplicial objects over $\mathcal{C}$. We say that $f : X \longrightarrow Y$ is a morphism of cosimplicial objects if $f^n : X^n \longrightarrow Y^n$ is a morphism in $\mathcal{C}$ for every $n \geq 0$ and for all pairs (i, n), we have that $s^i f^{n+1} = f^n s^i$ and $d^i f^{n-1} = f^n d^i$. We say that f is an isomorphism if $f^n : X^n \longrightarrow Y^n$ is an isomorphism for all $n \geq 0$

Since we have defined objects and morphisms, one could now easily check that we get two new categories: the category of simplicial objects over $\mathcal{C}$ and the category of cosimplicial objects over $\mathcal{C}$.

3.4 Model Categories

In much of our work, we will need to use the fact that various categories we study are, in fact, model categories. Working within a model category allows us to derive very nice results; the challenge is often finding a good model structure to use in the first place. In our case, the model structure will be vital to our main results, so for convenience we state the definitions and basic results here. We will follow mostly the presentation in [Hov], although we will also occasionally use material from [Hir] and [GJ]. We begin by defining model structures and model categories. To do that, we first need the notions of retracts, functorial factorizations, and lifting properties, as model structures are defined by these properties.

Definition 3.42. *Suppose $\mathcal{C}$ is a category, and let $\mathrm{Map}(\mathcal{C})$ denote the category whose objects are morphisms in $\mathcal{C}$ and whose morphisms are commutative squares.*

(a) A map f in $\mathcal{C}$ is a retract *of a map g in $\mathcal{C}$ if f is a retract of g as objects of $\mathrm{Map}(\mathcal{C})$. That is, f is a retract of g if and only if there is a commutative diagram of the form*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & B \end{tikzcd}$$

where the horizontal compositions are the respective identity maps.

(b) *A* functorial factorization *is an ordered pair of functors* $\mathrm{Map}(\mathcal{C}) \longrightarrow \mathrm{Map}(\mathcal{C})$*, which we denote* (α, β)*, such that* $f = \beta(f) \circ \alpha(f)$ *for all maps* f *in* $\mathrm{Map}(\mathcal{C})$*.*

Definition 3.43. *Suppose* $i : A \longrightarrow B$ *and* $p : X \longrightarrow Y$ *are maps in* $\mathcal{C}$*. Then* i *has the left lifting property with respect to* p *and* p *has the right lifting property with respect to* i *if, for every commutating square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

there is a lift $h : B \longrightarrow X$ *such that* $h \circ i = f$ *and* $p \circ h = g$*.*

We are now ready to define a model structure on a category. The following definition is as it appears in [Hov].

Definition 3.44. *A* model structure *on a category* $\mathcal{C}$ *is three subcategories of* $\mathcal{C}$ *called weak equivalences, cofibrations, and fibrations, and two functorial factorizations* (α, β) *and* (γ, δ) *satisfying the following properties:*

(a) (2-out-of-3) *If* f *and* g *are morphisms such that* $g \circ f$ *is defined and any two of* f*,* g*, and* $g \circ f$ *are weak equivalences, then so is the third.*

(b) *(Retracts) If* f *is a retract of* g *and* g *is a weak equivalence, fibration, or cofibration, then so is* f*.*

(c) *(Lifting) Define a map to be a trivial cofibration if it is both a cofibration and a weak equivalence. Similarly, define a map to be a trivial fibration if it is both a fibration and a weak equivalence. Then cofibrations have the left lifting property with respect to trivial fibrations, and fibrations have the right lifting property with respect to trivial cofibrations.*

(d) *(Functorial factorization) For any morphism* f*, the factorization* (α, β) *has* $\alpha(f)$ *a cofibration and* $\beta(f)$ *a trivial fibration, while the factorization* (γ, δ) *has* $\gamma(f)$ *a trivial cofibration and* $\delta(f)$ *a fibration.*

Definition 3.45. *A model category is a category $\mathcal{M}$ with all small limits and colimits together with a model structure on $\mathcal{M}$.*

For the remainder of the paper, we will use the letter $\mathcal{M}$ to refer to a model category. We state one very important fact about model categories, as it appears in [Hov]:

Lemma 3.46. *Let $\mathcal{M}$ be a model category. Then a map is a cofibration (resp. trivial cofibration) if and only if it has the left lifting property with respect to all trivial fibrations (resp. fibrations). Dually, a map is a fibration (resp. trivial fibration) if and only if it has the right lifting property with respect to all trivial cofibrations (resp. cofibrations).*

Proof. See [Hov, Lemma 1.1.10].

□

Remark 3.47. Notice that Lemma 3.46 says that the definition of a model category really only requires that we define what weak equivalences are and what either our fibrations or our cofibrations are; then the remaining class is determined solely by the appropriate lifting property. This means that if we want to prove that a map is a fibration, for example, we can always do so by showing that it has the right lifting property with respect to all trivial cofibrations.

We now present a very useful result about fibrations that we will use later when constructing the homotopy limit.

Proposition 3.48. *Suppose $\mathcal{M}$ is a model category. If $\alpha_i : X_i \longrightarrow Y_i$ is a fibration (trivial fibration) in $\mathcal{M}$ for every $i \in I$ (for I some indexing set), then the obvious map $\alpha : \prod_{i\in I} X_i \longrightarrow \prod_{i\in I} Y_i$ is also a fibration (trivial fibration) in $\mathcal{M}$.*

Proof. Let $f : A \longrightarrow B$ be any trivial cofibration. Then for all $i \in I$, we have a lift in the following square:

$$\begin{tikzcd} A \arrow[r, "k_i"] \arrow[d, "f"'] & X_i \arrow[d, "\alpha_i"] \\ B \arrow[r, "h_i"'] \arrow[ur, dashed, "g_i"] & Y_i \end{tikzcd}$$

which makes both triangles commute. Notice that, by the universal property of products, we have a map $g : B \longrightarrow \prod_{i\in I} X_i$ which is in each coordinate g_i and we have a map $h : B \longrightarrow \prod_{i\in I} Y_i$ which is in each coordinate h_i. This gives us a diagram

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "f"'] & \prod_{i\in I} X_i \arrow[d, "\alpha"] \\ B \arrow[r, "h"'] \arrow[ur, dashed, "g"] & \prod_{i\in I} Y_i \end{tikzcd}$$

so now we need to check the two triangles commute.

Since the morphism $\prod_{i\in I} X_i \longrightarrow \prod_{i\in I} Y_i$ is, in each coordinate, the map α_i, we get that $\alpha \circ g = (\alpha_i \circ g_i)_{i\in I} = (h_i)_{i\in I} = h$ so the first triangle commutes. Similarly, for the second triangle we get $g \circ f = (g_i \circ f)_{i\in I} = (k_i)_{i\in I} = k$ so the second triangle commutes. So $g : B \longrightarrow \prod_{i\in I} X_i$ gives us a lift. As this was done with any trivial cofibration, Lemma 3.46 says that $\alpha : \prod_{i\in I} X_i \longrightarrow \prod_{i\in I} Y_i$ is a fibration as desired. The proof for the trivial fibration case is analogous, except that we replace "fibration" with "trivial fribration" and "trivial cofibration" with "cofribration" in the above proof.

□

Definition 3.49. *Let $\mathcal{M}$ be a model category, and X an object of $\mathcal{M}$. Let $*$ denote the final object in $\mathcal{M}$, and let $\emptyset$ denote the initial object of $\mathcal{M}$. Then we say that X is fibrant if the morphism $X \longrightarrow *$ is a fibration. Similarly, we say that X is cofibrant if the morphism $\emptyset \longrightarrow X$ is a cofibration.*

Definition 3.50. *Let $\mathcal{M}$ be a model category, and X an object of $\mathcal{M}$. Let $*$ denote the final object in $\mathcal{M}$. We define the functor $X \mapsto R(X)$ by splitting the morphism $X \longrightarrow *$ (according to the functorial factorization given in Definition 3.44) into a trivial cofibration followed by a fibration. In other words, we get an object $R(X)$ in $\mathcal{M}$ such that the morphism $X \longrightarrow *$ factors through $R(X)$, the morphism $X \longrightarrow R(X)$ is a trivial cofibration, and the morphism $R(X) \longrightarrow *$ is a fibration. Then according to Definition 3.49, $R(X)$ is a fibrant object in $\mathcal{M}$. We call this functor the* fibrant replacement functor. *Dually, let $\emptyset$ denote the initial object of $\mathcal{M}$. We define the functor $X \mapsto Q(X)$ by splitting the morphism $\emptyset \longrightarrow X$ into a* *cofibration followed by a trivial fibration. In other words, we get an object $Q(X)$ in $\mathcal{M}$ such that the morphism $\emptyset \longrightarrow X$ factors through $Q(X)$, the morphism $Q(X) \longrightarrow X$ is a trivial fibration, and the morphism $\emptyset \longrightarrow Q(X)$ is a cofibration. Then according to Definition 3.49, $Q(X)$ is a cofibrant object in $\mathcal{M}$. We call this functor the* cofibrant replacement functor.

Notice that Definition 3.50 says we can, up to applying a trivial cofibration (which is, in particular, a weak equivalence), assume that our objects are fibrant objects. Similarly, Definition 3.50 also says we can, up to applying a trivial fibration (which is, in particular, a weak equivalence), assume that our objects are cofibrant objects.

Proposition 3.51. *Let $\mathcal{M}$ be a model category, and $X \longrightarrow Y$ a weak equivalence in $\mathcal{M}$. Then $R(X) \longrightarrow R(Y)$ is a trivial fibration; in particular, it is also a weak equivalence.*

Proof. Applying the functorial factorization to Y to construct $R(Y)$, we get the diagram

$$\begin{tikzcd} X \arrow[d] & \\ Y \arrow[r] & R(Y) \end{tikzcd}$$

where as always, $R(Y)$ is fibrant. Taking the composition of these two maps, we get get a morphism $X \longrightarrow R(Y)$, which can be split, using the functorial factorization, into $X \longrightarrow Z \longrightarrow R(Y)$, where $X \longrightarrow Z$ is a trivial cofibration and $Z \longrightarrow R(Y)$ is a fibration. Since a composition of fibrations is a fibration, and since both $Z \longrightarrow R(Y)$ and $R(Y) \longrightarrow *$ are fibrations, we have $Z \longrightarrow *$ is a fibration. In other words, the Z we construct above is fibrant. Since we constructed it using the functorial factorization, as we did in Definition 3.50, we see that $Z = R(X)$. Since $Z \longrightarrow R(Y)$ is a fibration, we get $R(X) \longrightarrow R(Y)$ is a fibration, which is the first half of the proof. Now our diagram becomes:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & R(X) \arrow[d] \\ Y \arrow[r] & R(Y) \end{tikzcd}$$

and this diagram commutes by construction. Now suppose that $X \longrightarrow Y$ is a weak equivalence. Since $Y \longrightarrow R(Y)$ is a trivial cofibration (and hence a weak equivalence), the two-out-of-three axiom for model categories says that $X \longrightarrow R(Y)$ is a weak equivalence. Since $X \longrightarrow R(Y)$ is the composite of $X \longrightarrow R(X)$ and $R(X) \longrightarrow R(Y)$, and since $X \longrightarrow R(X)$ is a trivial cofibration (and hence a weak equivalence), using the two-out-of-three axiom again gives us that $R(X) \longrightarrow R(Y)$ is a weak equivalence, and thus a trivial fibration as desired.

□

At this point we are now prepared to present a very useful lemma in model category theory.

Lemma 3.52 (Ken Brown's Lemma)**.** *Suppose that $\mathcal{M}$ is a model category and $\mathcal{C}$ is a category with a subcategory of weak equivalences that satisfies the 2-out-of-3 axiom. Suppose $F : \mathcal{M} \longrightarrow \mathcal{C}$ is a functor that takes trivial cofibrations between cofibrant objects to weak equivalences. Then F takes all weak equivalences between cofibrant objects to weak equivalences. Dually, if F takes all trivial fibrations between fibrant objects to weak equivalences, then F takes all weak equivalences between fibrant objects to weak equivalences.*

Proof. See [Hov, Lemma 1.1.12].

□

We conclude this section by presenting the model categories (and their model structures) that we will be using in this paper. We begin by looking at the category of spectra. We will, throughout this paper, assume that the category of spectra, which we denote $\mathsf{Spectra}$, comes with the stable model category structure as presented in [BF]. For completeness, we present the definition and model structure here. Recall that a category is called *pointed* if the initial and final objects are the same.

Definition 3.53. *We define the category $\mathsf{Spectra}$ as follows. An object X, called a spectrum, is a sequence of pointed simplicial sets X_n (for $n \geq 0$) and maps of pointed simplicial sets $\sigma^n : S^1 \wedge X^n \longrightarrow X^{n+1}$ where $S^1 = \Delta[1]/\partial\Delta[1]$ is the simplicial circle formed by identifying* *the two vertices of* $\Delta[1]$. *A morphism in* $\mathsf{Spectra}$, *denoted* $f : X \longrightarrow Y$, *consists of maps* $f^n : X^n \longrightarrow Y^n$ *of pointed simplicial sets (for* $n \geq 0$*) such that* $\sigma^n(\mathrm{id}_{S^1} \wedge f^n) = f^{n+1}\sigma^n$.

Theorem 3.54. *Let* $\mathsf{Spectra}$ *be as in Definition 3.53. Then the category* $\mathsf{Spectra}$ *has a model structure with the following classes of morphisms:*

(a) The weak equivalences are stable weak equivalences; that is, $f : X \longrightarrow Y$ *is a weak equivalence if* $f_* : \pi_* X \longrightarrow \pi_* Y$ *is an isomorphism of groups for all values of* $*$. *Here* $\pi_* X = \varinjlim \pi_{*+n}|X^n|$, *where* $|X^n|$ *denotes the geometric realization of* X^n.

(b) The cofibrations are the stable cofibrations; that is, $f : X \longrightarrow Y$ *is a cofibration if* $f^0 : X^0 \longrightarrow Y^0$ *is a cofibration of pointed simplicial sets (which is just an injection) and the morphisms*

$$X^{n+1} \coprod_{S^1 \wedge X^n} S^1 \wedge Y^n \longrightarrow Y^{n+1} \tag{3.16}$$

are cofibrations of pointed simplicial sets (i.e. injections) for all $n \geq 0$.

(c) Applying Lemma 3.46, the fibrations are the morphisms $f : X \longrightarrow Y$ *that satisfy the right lifting property with respect to all trivial cofibrations.*

Proof. See [BF]. Note that we can give an explicit construction for the fibrations (as opposed to using Lemma 3.46); however, we won't use this explicit description so we skip it. The interested reader can see the explicit construction in [BF].

□

One nice result that we plan to use implicitly throughout this paper is that the category of cosimplicial objects over some model category $\mathcal{M}$ also forms a model category. Before doing so, we need another definition

Definition 3.55. *Let* X *be a cosimplicial object over* $\mathcal{M}$ *and let* $n \geq -1$. *We let*

$$M^n X = \{(x^0, x^1, ..., x^n) \in X^n \times \cdots \times X^n | \ s^i x^j = s^{j-1} x^i\} \tag{3.17}$$

(for $0 \leq i < j \leq n$). Observe that if

$$\alpha_1 : \prod_{k=0}^{n} X^n \longrightarrow \prod_{0 \leq i < j \leq n} X^{n-1} \tag{3.18}$$

is given by

$$\alpha_1(x^0, x^1, ..., x^n) = (s^i x^j)_{0 \leq i < j \leq n} \tag{3.19}$$

and

$$\alpha_2 : \prod_{k=0}^{n} X^n \longrightarrow \prod_{0 \leq i < j \leq n} X^{n-1} \tag{3.20}$$

is given by

$$\alpha_1(x^0, x^1, ..., x^n) = (s^{j-1} x^i)_{0 \leq i < j \leq n} \tag{3.21}$$

then $M^n X$ is the equalizer of α_1 and α_2. We call $M^n X$ the matching space *of X. Observe that the matching spaces all come with natural maps $s_n^X : X^{n+1} \longrightarrow M^n X$ given by $s_n^X(a) = (s^0 a, ..., s^n a)$. In particular, we have $M^{-1} X = *$ and $M^0 X = X^0$.*

Theorem 3.56. *Let $\mathcal{M}$ be a model category. Then the category of cosimplicial objects over $\mathcal{M}$ forms a model category with the following classes of morphisms:*

(a) The weak equivalences are morphisms $f : X \longrightarrow Y$ such that, for every $n \geq 0$, $f^n : X^n \longrightarrow Y^n$ is a weak equivalence in $\mathcal{M}$.

(b) The fibrations are morphisms $f : X \longrightarrow Y$ such that

$$(f^{n+1}, s_n^X) : X^{n+1} \longrightarrow Y^{n+1} \times_{M^n Y} M^n X \tag{3.22}$$

are all fibrations in $\mathcal{M}$ for all $n \geq -1$.

(c) Applying Lemma 3.46, the cofibrations are morphisms $f : X \longrightarrow Y$ that satisfy the left

lifting property with respect to all trivial fibrations.

Proof. See [BK, Chapter X, Section 4]. Note that we could give an explicit construction of the cofibrations; they are morphisms $f : X \longrightarrow Y$ such that f is one-to-one and induces an isomorphism on the maximal augmentation. However, the maximal augmentation is a construction that we will not use in the paper, so we omit its presentation. The interested reader may find its construction in [BK, Chapter X, 4.2].

□

The final model structure we consider is a model structure on a category of diagrams.

Theorem 3.57. *Let $\mathcal{M}$ be a model category, I a small category, and $\mathcal{M}^I$ the category of diagrams. Then $\mathcal{M}^I$ has a model structure with the following classes of morphisms:*

(a) $X \longrightarrow Y$ is a weak equivalence if and only if $X_i \longrightarrow Y_i$ is for every i.

(b) $X \longrightarrow Y$ is a fibration if and only if $X_i \longrightarrow Y_i$ is for every i.

(c) By Lemma 3.46, $X \longrightarrow Y$ is a cofibration if and only if it satisfies the left lifting property with respect to trivial fibrations.

Proof. See [Hov, Theorem 5.1.3].

□

3.5 Homotopy Limits

At this point we want to give the general definition and basic properties of homotopy limits, as they will be crucial in our proof; for additional details, see [BK, Chapter XI].

In order to define homotopy limits, we need to define two new functors: the total object functor and the cosimplicial replacement functor. These are denoted Tot and Π^*, respectively. These will be important, as the holim functor will be defined based on these. We will begin with the total object functor, which requires that we begin by defining function objects.

Definition 3.58. *Let X and Y be two cosimplicial objects over a simplicial model category $\mathcal{M}$. We define the function object* $\mathrm{Hom}(X, Y)$ *to be the object where the n-simplices are the maps*

$$\Delta[n] \otimes X \longrightarrow Y, \tag{3.23}$$

with faces

$$\Delta[n-1] \otimes X \longrightarrow \Delta[n] \otimes X \longrightarrow Y \tag{3.24}$$

(where $\Delta[n-1] \otimes X \longrightarrow \Delta[n] \otimes X$ by $d^i \otimes X$), and degeneracies

$$\Delta[n+1] \otimes X \longrightarrow \Delta[n] \otimes X \longrightarrow Y \tag{3.25}$$

(where $\Delta[n+1] \otimes X \longrightarrow \Delta[n] \otimes X$ by $s^i \otimes X$).

Example 3.59. *Consider the category* $\mathsf{Spectra}$. *This is a simplicial model category, where the action is given by*

$$\Delta[n] \otimes X = (\Delta[n] \times X_m)_m. \tag{3.26}$$

In other words, the spectrum given by $\Delta[n] \otimes X$ is the spectrum whose sequence of simplicial sets is given by $\Delta[n] \times X_m$ for all m. With this simplicial model category structure, one can define the function spectrum $\mathrm{Hom}(X, Y)$ *for two cosimplicial spectra X and Y using the construction in Definition 3.58.*

We in particular have a very important property about function objects in categories of cosimplicial objects over a simplicial model category, which we state as the next theorem; for the proof, see [BK, Chapter X, Section 5].

Theorem 3.60 (Axiom SM7)**.** *With the notion of function objects defined in Definition 3.58, the category of cosimplicial objects over a simplicial model category $\mathcal{M}$ satisfies axiom*

SM7; in other words, if $i : A \longrightarrow B$ is a cofibration and $p : X \longrightarrow Y$ is a fibration, then the map

$$(i, p) : \mathrm{Hom}(B, X) \longrightarrow \mathrm{Hom}(A, X) \times_{\mathrm{Hom}(A,X)} \mathrm{Hom}(B, Y) \tag{3.27}$$

is a fibration, and is a weak equivalence if either i or p is a weak equivalence.

Corollary 3.61. *If $f : X \longrightarrow Y$ is a weak equivalence with X and Y fibrant, and B is cofibrant, then f induces a weak equivalence* $\mathrm{Hom}(B, X) \longrightarrow \mathrm{Hom}(B, Y)$.

Proof. This is immediate from Theorem 3.60 by letting $A = \emptyset$.

□

Now we define the total object of a cosimplicial object over a simplicial model category $\mathcal{M}$, and we prove an important basic property.

Definition 3.62. *Let $\widetilde{\Delta}$ denote the cosimplicial standard simplex. Let X be a cosimplicial object over a simplicial model category $\mathcal{M}$. We define the* total object *of X, denoted* $\mathrm{Tot}(X)$, *to be* $\mathrm{Hom}(\widetilde{\Delta}, X)$. *Note that* $\mathrm{Tot}(X)$ *is an object of $\mathcal{M}$.*

Corollary 3.61 has the following very important consequence for $\mathrm{Tot}(X)$ that we plan to use.

Corollary 3.63. *If $X \longrightarrow Y$ is a weak equivalence between fibrant objects, then* $\mathrm{Tot}(X) \longrightarrow \mathrm{Tot}(Y)$ *is a weak equivalence.*

Proof. The cosimplicial standard simplex $\widetilde{\Delta}$ is cofibrant (see [BK, Chapter X, Example 4.3]). Therefore this is immediate by Corollary 3.61 and the definition of Tot.

□

Now that we have defined Tot, we proceed to define Π^*, the cosimplicial replacement functor. To do this, we need to first consider the nerve (also called underlying space) of a small category; then, with that understanding, we can define a cosimplicial object constructed from a category of diagrams associated to that small category.

Definition 3.64. *Let I be a small category. We denote by $\mathcal{I}$ the nerve of I, which consists of simplices*

$$i_0 \xleftarrow[\alpha_1]{} i_1 \xleftarrow[\alpha_2]{} \cdots \xleftarrow[\alpha_n]{} i_n$$

which we denote by the shorthand u. The face maps d_j are given by deleting the i_0 term if $j = 0$, deleting the i_n term if $j = n$, and composing α_j with α_{j+1} for all other j. In other words, this gives us

$$i_1 \xleftarrow[\alpha_2]{} \cdots \xleftarrow[\alpha_n]{} i_n$$

for $d_0(u)$,

$$i_0 \xleftarrow[\alpha_1]{} i_1 \xleftarrow[\alpha_2]{} \cdots \xleftarrow[\alpha_{n-1}]{} i_{n-1}$$

for $d_n(u)$, and

$$i_0 \xleftarrow[\alpha_1]{} i_1 \xleftarrow[\alpha_2]{} \cdots \xleftarrow[\alpha_{j-1}]{} i_{j-1} \xleftarrow[\alpha_j\alpha_{j+1}]{} i_{j+1} \cdots \xleftarrow[\alpha_n]{} i_n$$

for $d_j(u)$, with $0 < j < n$. For $s_j(u)$, we simply add in an identity map at the j^{th} spot. In other words, we get

$$i_0 \xleftarrow[\alpha_1]{} i_1 \xleftarrow[\alpha_2]{} \cdots \xleftarrow[\alpha_j]{} i_j \xleftarrow[id_{i_j}]{} i_j \cdots \xleftarrow[\alpha_n]{} i_n$$

for $s_j(u)$.

Definition 3.65. *Let $\mathcal{M}$ be a simplicial model category, I a small category, and $\mathcal{M}^I$ the category of diagrams. Let X be an object in $\mathcal{M}^I$. Let X_{i_j} denote the object in our diagram corresponding to the object i_j in I, and let X_{α_j} denote the morphism in our diagram X that is induced by the morphism α_j in $\mathcal{I}$. Let u be as defined in Definition 3.64, and let $\mathcal{I}_n$ denote the set of all n-simplices in the nerve of I. We define $\Pi^*(X)$, the cosimplicial replacement of X, to be the cosimplicial object (over $\mathcal{M}$) where*

$$(\Pi^*(X))^n = \prod_{u \in \mathcal{I}_n} X_{i_0} \tag{3.28}$$

with coface and codegeneracy maps induced by $s^j = \mathrm{id}_{X_{i_0}}$ *for* $0 \leq j \leq n$*,* $d^j = \mathrm{id}_{X_{i_0}}$ *for* $0 < j \leq n$*, and* $d^0 = X_{\alpha_1}$*.*

The advantage of viewing our situation from the perspective of diagrams is that, from Theorem 3.57, a weak equivalence/fibration is just a map of diagrams where $X_i \longrightarrow Y_i$ is a weak equivalence/fibration for every i. We would like to see what happens to a weak equivalence or a fibration if we apply the cosimplicial replacement functor. To do this, we need the following lemma.

Lemma 3.66. *Letting* $d(\mathcal{I}_n)$ *denote the set of all degenerate simplicies in* $\mathcal{I}_n$*, we have that*

$$\prod_{u \in d(\mathcal{I}_n)} X_{i_0} \cong M^{n-1}(\Pi^*(X)) \tag{3.29}$$

where $M^n X$ *denotes the matching space of Definition 3.55. Consequently, we have*

$$(\Pi^*(X))^n \cong Z^n(\Pi^*(X)) \times M^{n-1}(\Pi^*(X)) \tag{3.30}$$

where

$$Z^n(\Pi^*(X)) = \prod_{u \in \mathcal{I}_n \setminus d(\mathcal{I}_n)} X_{i_0} \tag{3.31}$$

is the "cofree" part of $(\Pi^*(X))$ *in degree* n*.*

Proof. See [GJ, Chapter VII, Example 4.2] and [GJ, Chapter VIII, Section 2].

□

Theorem 3.67. *Suppose that* $X \longrightarrow Y$ *is a fibration/trivial fibration in* $\mathcal{M}^I$*. Then the map* $\Pi^*(X) \longrightarrow \Pi^*(Y)$ *is a fibration/trivial fibration in the category of cosimplicial objects over* $\mathcal{M}$*.*

Proof. Recall from Theorem 3.56 that $\Pi^*(X) \longrightarrow \Pi^*(Y)$ is a fibration/trivial fibration in the category of cosimplicial objects over $\mathcal{M}$ if and only if

$$(\Pi^*(X))^{n+1} \longrightarrow (\Pi^*(Y))^{n+1} \times_{M^n(\Pi^*(Y))} M^n(\Pi^*(X)) \tag{3.32}$$

are all fibrations/trivial fibrations in $\mathcal{M}$ for all $n \geq -1$. Applying Lemma 3.66 to $(\Pi^*(X))^{n+1}$ and $(\Pi^*(Y))^{n+1}$, and canceling the fiber product, the map reduces to

$$Z^{n+1}(\Pi^*(X)) \times M^n(\Pi^*(X)) \longrightarrow Z^{n+1}(\Pi^*(Y)) \times M^n(\Pi^*(X)) \tag{3.33}$$

This map is the identity on $M^n(\Pi^*(X))$ and is induced by the given map $X \longrightarrow Y$ (which is a vertex-wise fibration/trivial fibration) on $Z^{n+1}(\Pi^*(X)) \longrightarrow Z^{n+1}(\Pi^*(Y))$. In other words, this map is a product of fibrations/trivial fibrations. But by Proposition 3.48, we conclude that this product of maps is a fibration/trivial fibration as well, completing the proof.

□

Corollary 3.68. *If X is fibrant in $\mathcal{M}^I$, then $\Pi^*(X)$ is fibrant in the category of cosimplicial objects over $\mathcal{M}$.*

Proof. If $*$ is the final object of $\mathcal{M}$, then the constant diagram of $*$ is the final object of $\mathcal{M}^I$ and so $\Pi^*(X) \longrightarrow \Pi^*(*)$ is a fibration by Theorem 3.67. The final object in the category of cosimplicial objects over $\mathcal{M}$ is the cosimplicial object consisting of $*$ in each degree, with the obvious coface and codegeneracies. However, $\Pi^*(*)$ is, in each degree, a product of copies of $*$. As such a product is always canonically isomorphic to $*$ itself, $\Pi^*(*)$ is canonically isomorphic to the cosimplicial object consisting of $*$ in each degree. Thus, $\Pi^*(X)$ is fibrant.

□

Corollary 3.69. *Suppose that $X \longrightarrow Y$ is a weak equivalence between fibrant objects in $\mathcal{M}^I$. Then the map $\Pi^*(X) \longrightarrow \Pi^*(Y)$ is a weak equivalence in the category of cosimplicial objects over $\mathcal{M}$.*

Proof. By Lemma 3.52, it's enough to show that if $X \longrightarrow Y$ is a trivial fibration between fibrant objects in $\mathcal{M}^I$, then the map $\Pi^*(X) \longrightarrow \Pi^*(Y)$ is a weak equivalence in the category of cosimplicial objects over $\mathcal{M}$. But this is immediate from Theorem 3.67, since cosimplicial replacement sends trivial fibrations to trivial fibrations (which are weak equivalences).

$\square$

Definition 3.70. *Let X be an object in $\mathcal{M}^I$, where $\mathcal{M}$ is a simplicial model category. Then we define*

$$\operatorname{holim}(X) = \operatorname{Tot}(\Pi^*(R(X))), \tag{3.34}$$

where $R(X)$ denotes the fibrant replacement of X. In particular, as the category of cosimplicial objects over $\mathcal{M}$ can also be realized as the diagram category $\mathcal{M}^{\mathbf{\Delta}}$, where $\mathbf{\Delta}$ denotes the cosimplicial indexing category, we define $\operatorname{holim}(X)$ *as above for any X in the category of cosimplicial objects over $\mathcal{M}$. Note that* $\operatorname{holim}(X)$ *is an object of $\mathcal{M}$.*

Now we can begin to prove some very important results related to the homotopy limits of our cosimplicial objects. We begin with a general observation.

Proposition 3.71. *Let X and Y be objects in $\mathcal{M}^I$, where $\mathcal{M}$ is a simplicial model category, and suppose $f : X \longrightarrow Y$ is a weak equivalence. Then* $\operatorname{holim}(X) \longrightarrow \operatorname{holim}(Y)$ *is a weak equivalence. In particular, if X and Y are cosimplicial objects over $\mathcal{M}$ (again, $\mathcal{M}$ is a simplicial model category) and $f : X \longrightarrow Y$ is a weak equivalence, then the induced morphism* $\operatorname{holim}(X) \longrightarrow \operatorname{holim}(Y)$ *is a weak equivalence also.*

Proof. As we saw in Proposition 3.51, $R(f) : R(X) \longrightarrow R(Y)$ is a trivial fibration. By using either Theorem 3.67 or Corollary 3.69, we have that the morphism $\Pi^*(R(X)) \longrightarrow \Pi^*(R(Y))$ is a weak equivalence, and by Corollary 3.68, it is a weak equivalence between fibrant objects. Finally, by Corollary 3.63, the map $\operatorname{Tot}(\Pi^*(R(X))) \longrightarrow \operatorname{Tot}(\Pi^*(R(Y)))$ is a weak equivalence. By Definition 3.70, that means that $\operatorname{holim}(X) \longrightarrow \operatorname{holim}(Y)$ is a weak equivalence, as desired.

$\square$

We conclude this section by stating the following theorem.

Theorem 3.72. *Given a homotopy cartesian square of diagrams in $\mathcal{M}^I$, where $\mathcal{M}$ is a simplicial model category, applying the* holim *functor at each vertex gives us a homotopy cartesian square in $\mathcal{M}$.*

Proof. This is an immediate application of [BK, Chapter XI, Example 4.3].

□

3.6 Background on Algebraic K-Theory and KH-Theory

We are now ready to discuss Algebraic K-Theory and KH-theory. The origins of K-theory are due to Grothendieck, who, for a scheme X, constructed $\mathrm{K}_0(X)$ as the group of isomorphism classes of locally free coherent sheaves on X modulo exact sequences. The group $\mathrm{K}_0(X)$ is often called the *Grothendieck Group* because of this. This later inspired a topological construction that was analogous to $\mathrm{K}_0(X)$, which yielded a theory of higher topological K-theory; that is, it yielded groups $\mathrm{K}_n(X)$, for $n \geq 0$. However, algebraists were unable to discover a suitable analog of higher topological K-theory until Quillen did so in the landmark paper [Qui1]. Since then, much work has gone into expanding Quillen's ideas. Waldhausen showed in [Wal] how to build K-theory out of a complicial biWaldhausen category (which is very similar to a Model Category). Weibel showed in [Wei1] that there is a homotopy-invariant version of K-theory, which he called KH-theory, and showed that KH satisfies the Mayer-Vietoris property from [Tho]. Thomason collected much of the work on Higher Algebraic K-theory into the paper [TT], where he constructs (among other things) a more flexible definition of K-theory using perfect complexes, a Projective Bundle formula, a Localization sequence, and a non-connective spectrum K^B with K as its -1-cover (this spectrum K^B is often called *non-connective* K-theory). The construction of K^B, in particular, allows us to extend K-theory to negative degrees, and for $n \geq 0$, $\pi_n \mathrm{K}^B = \mathrm{K}_n$. In later sections of this paper, when we say K, we are really referring to the spectrum K^B.

In regards to Thomason's construction of $\mathrm{K}(X)$ (that is, the K-theory spectrum for the scheme X), we recall the following definitions.

Definition 3.73. *For any integer m, a chain complex $E^\cdot$ of $\mathcal{O}_X$-modules on a scheme X*

is said to be strictly m-pseudo-coherent if E^i is an algebraic vector bundle (that is, E^i is a locally free $\mathcal{O}_X$-module of finite type) for all $i \geq m$ and $E^i = 0$ for all i sufficiently large. The complex $E^{\cdot}$ is called strictly pseudo-coherent if it is strictly m-pseudo-coherent for all m; that is, if it is a bounded above complex of algebraic vector bundles.

Definition 3.74. *A complex $E^{\cdot}$ of $\mathcal{O}_X$-modules on a scheme X is said to be strictly perfect if it is strictly pseudo-coherent and strictly bounded below. In other words, a strict perfect complex is a strict bounded complex of algebraic vector bundles.*

Remark 3.75. If $X = \operatorname{Spec}(A)$ is affine, recall that algebraic vector bundles on X correspond to finitely generated projective A-modules. This follows from the fact that the category of A-modules is equivalent to the category of quasi-coherent $\mathcal{O}_X$-modules, via the map $M \mapsto \widetilde{M}$ (see [Hart, Chapter II, Corollary 5.5]). So over $X = \operatorname{Spec}(A)$, strict perfect complexes can be viewed as strict bounded complexes of finitely generated projective A-modules.

Definition 3.76. *We say that a complex $E^{\cdot}$ of $\mathcal{O}_X$-modules on a scheme X is pseudo-coherent if it is locally quasi-isomorphic to a strict pseudo-coherent complex. We say that a complex $E^{\cdot}$ of $\mathcal{O}_X$-modules on a scheme X is perfect if it is pseudo-coherent and has locally finite* Tor*-amplitude.*

For the precise definitions of locally quasi-isomorphic and locally finite Tor-amplitude, see [TT]. Beyond their presence in this definition, they will not be important to our work. We now present Thomason's definition of the K-theory spectrum for X.

Definition 3.77. *For a scheme X, $\mathrm{K}(X)$ is the* K*-theory spectrum of the complicial bi-Waldhausen category of perfect complexes of globally finite* Tor*-amplitude in the category of $\mathcal{O}_X$-modules. The spectrum $\mathrm{K}(X)$ has the property that its stable homotopy groups give us the* K*-theory of X; in other words*

$$\pi_n \mathrm{K}(X) = \mathrm{K}_n(X). \tag{3.35}$$

For a scheme X, $\mathrm{K}^{naive}(X)$ is the K*-theory spectrum of the complicial biWaldhausen category of strict perfect complexes in the category of $\mathcal{O}_X$-modules. The spectrum $\mathrm{K}^{naive}(X)$*

has the property that its stable homotopy groups give us the naive K*-theory of* X*; in other words*

$$\pi_n \mathrm{K}^{naive}(X) = \mathrm{K}_n^{naive}(X). \tag{3.36}$$

The functors $\mathrm{K}(-)$ *and* $\mathrm{K}^{naive}(-)$ *are both contravariant functors from the category of schemes to the category of spectra; if* $f : X \longrightarrow Y$*, we write* $f^* : \mathrm{K}(Y) \longrightarrow \mathrm{K}(X)$ *for the induced morphism, and similarly for* $\mathrm{K}^{naive}(-)$*.*

Remark 3.78. The spectrum $\mathrm{K}^{naive}(X)$ of Definition 3.77 is the construction for higher algebraic K-theory given in [Qui1]; see [TT, Proposition 3.10].

We now present a proposition that we will implicitly use during the course of our proof.

Proposition 3.79. *For a scheme* X *with an ample family of line bundles (in particular, for* X *affine), there is a natural homotopy equivalence of spectra* $\mathrm{K}^{naive}(X) \cong \mathrm{K}(X)$*. In particular, the* K*-theory of such a scheme can be calculated using either theory.*

Proof. See [TT, Corollary 3.9].

□

Our next concern is morphisms between K-theories.

Proposition 3.80. *Given two Waldhausen categories* $\mathcal{A}$ *and* $\mathcal{B}$*, an exact functor* $F : \mathcal{A} \longrightarrow \mathcal{B}$ *induces a map of spectra* $F_* : \mathrm{K}(\mathcal{A}) \longrightarrow \mathrm{K}(\mathcal{B})$*.*

Proof. See [TT] or [Wal].

□

Thomason shows in [TT, Section 3] that if $E^{\cdot}$ and $F^{\cdot}$ are strict perfect complexes, then $E^{\cdot} \otimes_{\mathcal{O}_X} F^{\cdot}$ is also strict perfect. Similarly, he shows the same result holds if $E^{\cdot}$ and $F^{\cdot}$ have finite Tor amplitude or are both pseudo-coherent. Thus we get a pairing

$$\mathrm{K}(X) \wedge \mathrm{K}(X) \longrightarrow \mathrm{K}(X) \tag{3.37}$$

induced by the tensor product, which gives a graded ring structure to $\oplus\,\mathrm{K}_*(X)$, and a pairing

$$\mathrm{K}^{naive}(X) \wedge \mathrm{K}^{naive}(X) \longrightarrow \mathrm{K}^{naive}(X) \tag{3.38}$$

induced by the tensor product, which gives a graded ring structure to $\oplus\,\mathrm{K}_*^{naive}(X)$; for the details of the proof, see [TT], [Wal], and [GroSGA6]. Under either of these pairings, we denote the product of two elements a and b by $a \cdot b$, and we call this the cup product of a and b. Note that when a and b are both elements of $\mathrm{K}_1(X)$, authors often write $\{a, b\}$ to denote their cup product. In order to avoid confusion with the literature, we will adopt this notation as well.

Proposition 3.81. *If $g : R \longrightarrow S$ is a ring homomorphism of commutative rings, then $g^* : \oplus\,\mathrm{K}_*(R) \longrightarrow \oplus\,\mathrm{K}_*(S)$ is a graded ring homomorphism. In other words, $g^*(a \cdot b) = g^*(a) \cdot g^*(b)$.*

Proof. As always, both maps are induced by what happens on the level of finitely generated projective modules. Let P and Q be two finitely generated projective R-modules. Then the cup product of P and Q is induced by the tensor product $P \otimes_R Q$, and so $g^*(a \cdot b)$ is induced by $(P \otimes_R Q) \otimes_R S$. On the other hand, if we first apply g^*, this corresponds to mapping P and Q to $P \otimes_R S$ and $Q \otimes_R S$; then taking the cup product says that $g^*(a) \cdot g^*(b)$ is induced by taking $(P \otimes_R S) \otimes_S (Q \otimes_R S)$. But notice that, as modules, we have $(P \otimes_R S) \otimes_S (Q \otimes_R S) \cong (P \otimes_R Q) \otimes_R S$. So on the level of projective modules, and hence on the level of strict perfect complexes, these are canonically isomorphic. This means that $g^*(a \cdot b) = g^*(a) \cdot g^*(b)$, and therefore that g^* is a graded ring homomorphism as desired. □

Another important theorem is the Bass Fundamental Theorem. It will prove extremely useful to us later, when we examine maps on the K-theory of algebraic tori.

Theorem 3.82 (Bass Fundamental Theorem)**.** *If R is a ring, there is a canonical split exact* *sequence*

$$0 \longrightarrow \mathrm{K}_n(R) \xrightarrow{\Delta} \mathrm{K}_n(R[t]) \oplus \mathrm{K}_n\left(R[\tfrac{1}{t}]\right) \xrightarrow{\pm} \mathrm{K}_n\left(R[t,\tfrac{1}{t}]\right) \underset{\longleftarrow}{\xrightarrow{\partial}} \mathrm{K}_{n-1}(R) \longrightarrow 0$$

where the splitting of ∂ is given by multiplication by $t \in \mathrm{K}_1(R[t,\frac{1}{t}])$. If R is regular, this splitting yields an isomorphism

$$\mathrm{K}_n\left(R\left[t,\frac{1}{t}\right]\right) \cong \mathrm{K}_n(R) \oplus \mathrm{K}_{n-1}(R) \tag{3.39}$$

where again the splitting is given by multiplication by $t \in \mathrm{K}_1(R[t,\frac{1}{t}])$.

Proof. For the proof, see [Wei3, Theorem 8.2] or [TT, Theorem 6.6].

□

Remark 3.83. In Theorem 3.82, when we say multiplication by t, we mean multiplication under the cup product; i.e., under the grading on $\oplus\,\mathrm{K}_*(R[t,\frac{1}{t}])$ induced by the tensor product, as described above.

The results presented so far in this section also extend to negative degrees when you replace K with K^B; see [TT, Section 6] for the details.

We are now ready to define homotopy K-theory (which we denote by KH-theory for the remainder of the paper). Weibel defines KH for a ring A by setting $\mathrm{KH}(A)$ to be the (fibrant) geometric realization of the simplicial spectrum $\mathrm{K}^B(\Delta A)$, where ΔA is the simplicial ring so that, for any n, $\Delta_n A = A[t_0, ..., t_n]/\left(\sum t_i - 1\right) A$. Then KH satisfies the following important properties:

Theorem 3.84. *Let A be an associative ring.*

(a) (Homotopy Invariance) For every set X, let $A[X]$ denote the polynomial ring in the commuting variables X. Then

$$\mathrm{KH}_n(A) \cong \mathrm{KH}_n(A[X]). \tag{3.40}$$

(b) (Bass Fundamental Theorem) For all $n \in \mathbb{Z}$ we have

$$\mathrm{KH}_n(A[x, x^{-1}]) \cong \mathrm{KH}_n(A) \oplus \mathrm{KH}_{n-1}(A). \tag{3.41}$$

(c) (Graded Rings) If $A = A_0 \oplus A_1 \oplus \cdots$ is a graded ring, then

$$\mathrm{KH}_n(A) \cong \mathrm{KH}_n(A_0). \tag{3.42}$$

Proof. See [Wei1, Theorem 1.2, parts (a), (c), and (d)]. Note that he proves part (a) by showing the stronger statement that, as spectra, $\mathrm{KH}(A) \simeq \mathrm{KH}(A[X])$.

□

Weibel then goes on to show that KH satisfies excision for ideals and the Mayer-Vietoris property for ideals. He next extends his construction of KH to quasi-projective schemes by using Jouanolou's Device, before finally defining $\mathrm{KH}(X)$, where X is a scheme, to be $\mathrm{holim}(\mathrm{KH}(\mathcal{U}))$, where $\mathcal{U}$ denotes any cover of X by affine open subschemes.

This definition was later modified in [TT] to the following, more useful definition:

Definition 3.85. *Let X be a scheme, and let $\Delta^{\cdot}$ denote the standard simplicial object, where $\Delta^n = \mathrm{Spec}\left(\mathbb{Z}[T_0, ..., T_n]/(\sum_{i=0}^n T_i = 1)\right)$. We define*

$$\mathrm{KH}(X) = \mathrm{hocolim}_{\Delta^{op}} \mathrm{K}^B(X \times \Delta^{\cdot}). \tag{3.43}$$

The spectrum $\mathrm{KH}(X)$ has the property that its stable homotopy groups give us the KH-theory of X; in other words

$$\pi_n \mathrm{KH}(X) = \mathrm{KH}_n(X). \tag{3.44}$$

We will see in Section 3.8 that these two definitions for KH agree when X is assumed quasi-compact and quasi-separated. Since all toric varieties are quasi-compact and separated, this is enough for our purposes. The fact that these definitions agree for all Noetherian schemes is due to Thomason; see [Tho, Exercise 2.5] and [TT, Theorem 10.3]. The original intuition is due to Brown and Gersten; see [BG, Theorem 4]. See Remark 3.120 for further discussion of this.

Definition 3.86. *Let X be a scheme. We define $\mathrm{K}(X) \otimes \mathbb{Q}$ to be the spectrum whose stable homotopy groups are $\mathrm{K}_n(X) \otimes \mathbb{Q}$. In other words,*

$$\pi_n \mathrm{K}(X) \otimes \mathbb{Q} = \mathrm{K}_n(X) \otimes \mathbb{Q}. \tag{3.45}$$

Just as K is contravariant, $\mathrm{K}(-) \otimes \mathbb{Q}$ is contravariant, and given any morphism

$$f : X \longrightarrow Y \tag{3.46}$$

the induced morphism $(\mathrm{K}(-) \otimes \mathbb{Q})(f)$ is, in each degree n, given by

$$f^* \otimes \mathrm{id}_{\mathbb{Q}} : \mathrm{K}_n(Y) \otimes \mathbb{Q} \longrightarrow \mathrm{K}_n(X) \otimes \mathbb{Q} \tag{3.47}$$

where f^* is the induced map from Definition 3.77. To simplify notation, we will denote $(\mathrm{K}(-) \otimes \mathbb{Q})(f)$ by $(f^*)_{\mathbb{Q}}$.

Similarly, we define $\mathrm{KH}(X) \otimes \mathbb{Q}$ to be the spectrum whose stable homotopy groups are $\mathrm{KH}_n(X) \otimes \mathbb{Q}$. In other words,

$$\pi_n \mathrm{KH}(X) \otimes \mathbb{Q} = \mathrm{KH}_n(X) \otimes \mathbb{Q}. \tag{3.48}$$

Again, just as KH is contravariant, $\mathrm{KH}(-) \otimes \mathbb{Q}$ is contravariant, and given any morphism

$$f : X \longrightarrow Y \tag{3.49}$$

the induced morphism $(\mathrm{KH}(-) \otimes \mathbb{Q})(f)$ is, in each degree n, given by

$$\mathrm{KH}(f) \otimes \mathrm{id}_{\mathbb{Q}} : \mathrm{KH}_n(Y) \otimes \mathbb{Q} \longrightarrow \mathrm{KH}_n(X) \otimes \mathbb{Q}. \tag{3.50}$$

To simplify notation, we will denote $(\mathrm{KH}(-) \otimes \mathbb{Q})(f)$ *by* $\mathrm{KH}(f)_{\mathbb{Q}}$.

Remark 3.87. Formally, $\mathrm{K}(X) \otimes \mathbb{Q}$ and $\mathrm{KH}(X) \otimes \mathbb{Q}$ are obtained by taking a Bousfield localization at the Eilenberg-Maclane spectrum $H\mathbb{Q}$. However, as we are only interested in the fact that

$$\pi_n \, \mathrm{K}(X) \otimes \mathbb{Q} = \mathrm{K}_n(X) \otimes \mathbb{Q} \tag{3.51}$$

and that

$$\pi_n \, \mathrm{KH}(X) \otimes \mathbb{Q} = \mathrm{KH}_n(X) \otimes \mathbb{Q}, \tag{3.52}$$

we skip the construction.

As was mentioned in Section 1, $\mathrm{K}(X)$ and $\mathrm{KH}(X)$ agree with each other when X is smooth (see [Wei1, Proposition 1.5] and [Wei1, Example 4.7]). However, when X is not smooth, $\mathrm{K}(X)$ and $\mathrm{KH}(X)$ still share a relationship. The difference between $\mathrm{K}(X)$ and $\mathrm{KH}(X)$ is what we call $\mathcal{F}_{\mathrm{K}}(X)$. Our next goal will be to present the construction of $\mathcal{F}_{\mathrm{K}}(X)$. To do so, we begin with a definition (see [CHWW3, Section 3]).

Definition 3.88. *If E is a presheaf of complexes on* $\mathsf{Schemes/k}$*, then we denote by* $\mathbb{H}_{cdh}(-, E)$ *the cdh-fibrant replacement of E (see [CHWW3, Section 2]). We define $\mathcal{F}_E$ to be the shifted mapping cone of the map $E \longrightarrow \mathbb{H}_{cdh}(-, E)$. In other words, we have*

$$\mathbb{H}_{cdh}(X, E)[-1] \longrightarrow \mathcal{F}_E(X) \longrightarrow E(X) \longrightarrow \mathbb{H}_{cdh}(X, E). \tag{3.53}$$

In particular, if HC *is the presheaf of complexes on* $\mathsf{Schemes/k}$ *that maps to the cyclic homology complex, then we define $\mathcal{F}_{\mathrm{HC}}$ to be the shifted mapping cone of the map* $\mathrm{HC} \longrightarrow \mathbb{H}_{cdh}(-, \mathrm{HC})$.

Remark 3.89. While we will not treat the subject of triangulated categories explicitly in this paper, one should note that $\mathcal{F}_E$ is the choice of object that makes the sequence in Equation (3.53) into a distinguished triangle.

Definition 3.90. *If $\mathcal{E}$ is a presheaf of spectra on* $\mathsf{Schemes/k}$*, then we denote by* $\mathbb{H}_{cdh}(-,\mathcal{E})$ *the cdh-fibrant replacement of* $\mathcal{E}$*. We define* $\mathcal{F}_{\mathcal{E}}$ *to be the homotopy fiber of the map* $\mathcal{E} \longrightarrow \mathbb{H}_{cdh}(-,\mathcal{E})$ *(see [CHW, Definition* 1.4*]). In particular, if* K *is the presheaf of spectra on* $\mathsf{Schemes/k}$ *giving us the* K*-theory spectrum of Definition 3.77, then we define* $\mathcal{F}_{\mathrm{K}}$ *to be the homotopy fiber of the map* $\mathrm{K} \longrightarrow \mathbb{H}_{cdh}(-,\mathrm{K})$*.*

Remark 3.91. By [Hae, Theorem 6.4], $\mathbb{H}_{cdh}(-,\mathrm{K}) \simeq \mathrm{KH}$, so an equivalent formulation of Definition 3.90 is to say that $\mathcal{F}_{\mathrm{K}}$ is the homotopy fiber of the map $\mathrm{K} \longrightarrow \mathrm{KH}$.

It turns out that $\mathcal{F}_{\mathrm{K}}$ and $\mathcal{F}_{\mathrm{HC}}$ are related. We present that relationship in the following theorem.

Theorem 3.92. *The presheaves* $\mathcal{F}_{\mathrm{K}}$ *and* $\Omega^{-1}\mathcal{F}_{\mathrm{HC}}$ *are weakly equivalent as presheaves of spectra. In the more modern language, that means that, for every* X *in* $\mathsf{Schemes/k}$*, there is a long exact sequence*

$$\cdots \longrightarrow \mathrm{KH}_{n+1}(X) \longrightarrow H^{-n}_{Zar}(X, \mathcal{F}_{HC}[1]) \longrightarrow \mathrm{K}_n(X) \longrightarrow \mathrm{KH}_n(X) \longrightarrow \cdots \tag{3.54}$$

and $(\mathcal{F}_{\mathrm{K}})_n(X) = H^{-n}_{Zar}(X, \mathcal{F}_{HC}[1])$*.*

Proof. See [CHW, Theorem 1.6] and [CHWW, Theorem 5.5].

□

Remark 3.93. Strictly speaking, $\mathcal{F}_{\mathrm{HC}}$ is only a presheaf of complexes, not a presheaf of spectra. We will not focus on this detail too closely, but it is resolved by applying the Eilenberg-Mac Lane functor to $\mathcal{F}_{\mathrm{HC}}$ to yield an equivalent presheaf of spectra. So the $\mathcal{F}_{\mathrm{HC}}$ appearing in Theorem 3.92 is really the presheaf of spectra we get after applying the Eilenberg-Mac Lane functor to $\mathcal{F}_{\mathrm{HC}}$. The second result of Theorem 3.92 then follows in light of the work in [CHSW, Section 3] and [Tho, Scholium of Great Enlightenment 5.32], as well as the proofs given in [CHW, Theorem 1.6] and [CHWW, Theorem 5.5].

Since we study toric varieties, the results of [CHWW] are important for our purposes. In this paper, they prove the following theorem:

Theorem 3.94. *For every toric variety X over a field k of characteristic 0, the map $\mathrm{K}_*(X) \longrightarrow \mathrm{KH}_*(X)$ of Definition 3.91 is a split surjection. Hence*

$$\mathrm{K}_n(X) \cong \mathrm{KH}_n(X) \oplus (\mathcal{F}_{\mathrm{K}})_n(X) \tag{3.55}$$

where $(\mathcal{F}_{\mathrm{K}})_n(X) = H^{-n}_{Zar}(X, \mathcal{F}_{HC}[1])$.

Proof. See [CHWW, Proposition 5.6].

□

So for toric varieties, understanding of the K-theory can be accomplished by understanding the KH-theory and the $\mathcal{F}_{\mathrm{K}}$ groups. Given that this is the case, and given that our primary focus is on complete simplicial toric varieties, much of our work in this paper will focus on the calculation of $\mathrm{KH}(X)$ and $\mathcal{F}_{\mathrm{K}}(X)$.

3.7 Transfer Arguments in Algebraic K-Theory

In Section 3.6, we saw that K, KH, $\mathrm{K}(-) \otimes \mathbb{Q}$, and $\mathrm{KH}(-) \otimes \mathbb{Q}$ are all contravariant functors from schemes to spectra. Recalling that any ring homomorphism $f : A \longrightarrow B$ gives us an associated morphism $f^a : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$, we see that this is equivalent to saying that they are all covariant when defined on the category of rings, so that if $f : A \longrightarrow B$, then

$$f^* : \mathrm{K}(A) \longrightarrow \mathrm{K}(B) \tag{3.56}$$

and similarly for the other three. The goal of this section is to construct a morphism

$$f_* : \mathrm{K}(B) \longrightarrow \mathrm{K}(A) \tag{3.57}$$

which we call the *transfer morphism*, and to give conditions under which a transfer morphism exists. Throughout this section we will restrict our focus to regular rings, so that K and KH will be the same, as this will be the only situation in which we will use transfer arguments in this paper. To begin, we need the following lemmas.

Lemma 3.95. *Suppose $f : A \longrightarrow B$ is a flat, finite morphism of Noetherian rings. The functor $t_f : B - \mathrm{Mod} \longrightarrow A - \mathrm{Mod}$ sending a B-module M to the A-module M (via the A-module structure given in part (a) of Definition 3.5) has the following properties:*

(a) The functor t_f is always exact.

(b) The functor t_f sends finitely generated modules to finitely generated modules.

(c) The functor t_f sends finitely generated projective modules to finitely generated projective modules.

Proof. For (a), this is clear since any short exact sequence remains short exact when you use the A-module structure induced by f. For (b), since f is finite, B is finitely generated as an A-module via the action of f. Then any module that is finitely generated as a B-module is also automatically finitely generated as an A-module; indeed, if $x_1, ..., x_m$ generate B as an A-module, and $y_1, ..., y_n$ generate M as a B-module, then the mn products $x_i y_j$ generate M as an A-module. For (c), if M is a finitely generated projective B-module, then it is also flat. By Proposition 3.17, M is also flat as an A-module via the A-module structure given in part (a) of Definition 3.5. So t_f sends a finitely generated projective B-module to a finitely generated flat A-module. Since A is Noetherian, Corollary 3.19 implies that M is projective as an A-module; therefore, t_f sends finitely generated projective B-modules to a finitely generated projective A-modules as desired.

□

Lemma 3.96. *Suppose $f : A \longrightarrow B$ is a flat, finite morphism of Noetherian rings. Then there is an induced map $f_* : \mathrm{K}(B) \longrightarrow \mathrm{K}(A)$.*

Proof. By Lemma 3.95 part (c), t_f sends finitely generated projective modules to finitely generated projective modules. That means that t_f sends any strict perfect complex over B to a strict perfect complex over A. By Proposition 3.80, this induces a morphism

$\mathrm{K}^{naive}(B) \longrightarrow \mathrm{K}^{naive}(A)$; by Proposition 3.79, this induces a morphism $\mathrm{K}(B) \longrightarrow \mathrm{K}(A)$. We call this induced map f_*.

$\square$

One of the basic properties of transfer maps is that it satisfies the following version of additivity.

Proposition 3.97. *Suppose $f : A \longrightarrow B$ is a flat, finite morphism of Noetherian rings. If we have that $B = B' \times B''$, and that f decomposes as $f' \times f''$, then we have $f_* = f'_* + f''_*$.*

Proof. As usual, we induce transfer maps by restriction of scalars on finitely generated projective modules. Let P be any finitely generated projective B-module; then $t_f(P)$ is just P viewed as a projective A-module. Since $B = B' \times B''$, any module (and hence any finitely generated projective module) decomposes as $M = M' \times M''$; hence our module $P = P' \times P''$, which can be rewritten as $P = P' \oplus P''$. Here P' is a B'-module and P'' is a B''-module, and both are still projective. So $t_f(P) = t_f(P' \oplus P'') = t_f(P') \oplus t_f(P'')$. As f'' takes only the value 0 in P' and f' takes only the value 0 in P'', we have that $t_f(P') = t_{f'}(P')$ and $t_f(P'') = t_{f''}(P'')$. Taking the induced map on strict perfect complexes then yields the result.

$\square$

Lemma 3.96 shows that under certain conditions, a transfer map exists. One can now ask how the ordinary induced map f^* and the transfer map f_* are related. One immediate answer comes from the following lemma.

Lemma 3.98 (Projection Formula)**.** *Suppose $f : A \longrightarrow B$ is an injective, flat, finite morphism of Noetherian rings. Then if $x \in \mathrm{K}_n(B)$ and $y \in \mathrm{K}_m(A)$, we have*

$$f_*(x \cdot f^*(y)) = f_*(x) \cdot y \tag{3.58}$$

where the multiplications $x \cdot f^*(y)$ and $f_*(x) \cdot y$ are both cup product multiplications.

Proof. We sketch the proof given in [Wei3]. As mentioned in Section 3.6, and proven in [TT], the tensor product induces a pairing

$$\mathrm{K}(B) \wedge \mathrm{K}(A) \longrightarrow \mathrm{K}(A) \tag{3.59}$$

which represents the right hand side of the projection formula. If P is a projective A-module and Q is a projective B-module, the isomorphism

$$Q \otimes_A P \cong Q \otimes_B (B \otimes_A P) \tag{3.60}$$

induces a natural homotopy to a pairing that represents the left hand side of the projection formula, completing the proof. See [Wei3] for the full details.

□

Remark 3.99. The Projection Formula given in Lemma 3.98 is a special case of the more general Projection Formula: If $f : X \longrightarrow Y$ is a quasi-compact, quasi-separated morphism of schemes with Y quasi-compact, such that Rf_* preserves perfection (and therefore induces a transfer morphism $f_* : \mathrm{K}(X) \longrightarrow \mathrm{K}(Y)$), then f_* is a map of module spectra over the ring spectrum $\mathrm{K}(Y)$. In other words, for $x \in \mathrm{K}_n(X)$ and $y \in \mathrm{K}_m(Y)$, we have

$$f_*(x \cdot f^*(y)) = f_*(x) \cdot y. \tag{3.61}$$

For the proof of this version of the projection formula, see [TT, Proposition 3.17], [Qui1, Section 7.2.10], and [GroSGA6, IV, 2.12].

In what follows, we will begin to examine what happens when we tensor with $\mathbb{Q}$. To that end, we state the following lemma for convenience.

Lemma 3.100. *Suppose that A is an abelian group, and that $f : A \longrightarrow A$ is multiplication by n for some $n > 0$. Then the map*

$$f \otimes \mathrm{id}_{\mathbb{Q}} : A \otimes \mathbb{Q} \longrightarrow A \otimes \mathbb{Q} \tag{3.62}$$

is an isomorphism.

We are now ready to apply the Projection Formula of Lemma 3.98.

Theorem 3.101. *Suppose $f : A \longrightarrow B$ is an injective, flat, finite morphism of Noetherian rings. Then the map $f_* \circ f^* : \mathrm{K}_n(A) \longrightarrow \mathrm{K}_n(A)$ is multiplication by $[B]$, where $[B] \in \mathrm{K}_0(A)$ denotes the class of B. If B is a free A-module of rank d, then $f_* \circ f^*$ is multiplication by d.*

Proof. As we saw in Lemma 3.96, our morphism f as constructed induces a transfer map. By Lemma 3.98, for any $x \in K_0(B)$ and $y \in \mathrm{K}_n(A)$, we get

$$f_*(x \cdot f^*(y)) = f_*(x) \cdot y. \tag{3.63}$$

If $x \in K_0(B)$, then by construction $f_*(x) \in K_0(A)$. Letting $x = 1$ (that is, letting $x = [B] \in K_0(B)$), we see that our formula becomes

$$f_*(f^*(y)) = f_*(1) \cdot y. \tag{3.64}$$

So we need only determine what $f_*(1)$ is. As we saw in Lemma 3.95 and Lemma 3.96, f_* sends any finitely generated projective B-module to itself, except viewed now as a finitely generated projective A-module. So

$$f_*(1) = f_*([B]) = [B] \in \mathrm{K}_0(A) \tag{3.65}$$

making our formula

$$f_*(f^*(y)) = [B] \cdot y \tag{3.66}$$

as claimed. If B is a free A-module of rank d, then as A-modules, $B \cong A^d$. Therefore, $[B] = [A^d] = d \in K_0(A)$. In this case, our formula would then become

$$f_*(f^*(y)) = d \cdot y \tag{3.67}$$

as claimed.

$\square$

Corollary 3.102. *Suppose $f : A \longrightarrow B$ is an isomorphism of Noetherian rings. Then $f_* \circ f^*$ is the identity map, and $f_* = (f^*)^{-1}$.*

Proof. If f is an isomorphism, then B is a free A-module of rank 1; by Theorem 3.101, $f_* \circ f^*$ is multiplication by 1, which is the identity map as claimed. Since f being an isomorphism implies f^* is an isomorphism, taking $(f^*)^{-1}$ of both sides yields the second result.

$\square$

Corollary 3.103. *Suppose $f : A \longrightarrow B$ is an injective, flat, finite morphism of Noetherian rings, and B is a free A-module of rank d. Then for all n, the map*

$$(f_*)_{\mathbb{Q}} \circ (f^*)_{\mathbb{Q}} : \mathrm{K}_n(A) \otimes \mathbb{Q} \longrightarrow \mathrm{K}_n(A) \otimes \mathbb{Q} \tag{3.68}$$

is an isomorphism, and

$$(f^*)_{\mathbb{Q}} : \mathrm{K}_n(A) \otimes \mathbb{Q} \longrightarrow \mathrm{K}_n(B) \otimes \mathbb{Q} \tag{3.69}$$

is injective.

Proof. As usual, $(f^*)_{\mathbb{Q}} = f^* \otimes \mathrm{id}_{\mathbb{Q}}$; similarly, we define $(f_*)_{\mathbb{Q}} = f_* \otimes \mathrm{id}_{\mathbb{Q}}$. By Theorem 3.101, $f_* \circ f^* : \mathrm{K}_n(A) \longrightarrow \mathrm{K}_n(A)$ is multiplication by d. Therefore, by Lemma 3.100, the map

$$(f_* \circ f^*) \otimes \mathrm{id}_{\mathbb{Q}} = (f_*)_{\mathbb{Q}} \circ (f^*)_{\mathbb{Q}} : \mathrm{K}_n(A) \otimes \mathbb{Q} \longrightarrow \mathrm{K}_n(A) \otimes \mathbb{Q} \tag{3.70}$$

is an isomorphism. Since this composite is injective, the first map must also be injective, so $(f^*)_{\mathbb{Q}}$ is injective as claimed.

$\square$

One very nice application of transfer maps, and of Corollary 3.103 in particular, is in what is known as a *transfer argument*.

Theorem 3.104 (Transfer Argument)**.** *Suppose we have the following commutative square of rings*

$$\begin{tikzcd} A \arrow[r, "h_1"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "h_2"'] & D \end{tikzcd}$$

Suppose further that $f : A \longrightarrow C$ and $g : B \longrightarrow D$ satisfy the conditions of Lemma 3.96 (so that transfer maps exist), that C is a free module of finite rank over A via the structure induced by f, and that D is a free module of finite rank over B via the structure induced by g. Furthermore, suppose that $(g_*)_{\mathbb{Q}} \circ (h_2^*)_{\mathbb{Q}} = (h_1^*)_{\mathbb{Q}} \circ (f_*)_{\mathbb{Q}}$ for all n. Then if there is an n such that $(h_2^*)_{\mathbb{Q}}$ is an isomorphism, $(h_1^*)_{\mathbb{Q}}$ is an isomorphism as well for that same n. Consequently, if $(h_2^*)_{\mathbb{Q}}$ is an isomorphism for all n, then so is $(h_1^*)_{\mathbb{Q}}$. In particular, if $D \cong B \otimes_A C$, then the conclusion holds.

Proof. Applying $\mathrm{K}_n(-) \otimes \mathbb{Q}$ to the entire diagram, we get a diagram

$$\begin{tikzcd} \mathrm{K}_n(A) \otimes \mathbb{Q} \arrow[r, "(h_1^*)_{\mathbb{Q}}"] \arrow[d, bend right, "(f^*)_{\mathbb{Q}}"'] & \mathrm{K}_n(B) \otimes \mathbb{Q} \arrow[d, bend right, "(g^*)_{\mathbb{Q}}"'] \\ \mathrm{K}_n(C) \otimes \mathbb{Q} \arrow[r, "(h_2^*)_{\mathbb{Q}}"'] \arrow[u, bend right, "(f_*)_{\mathbb{Q}}"'] & \mathrm{K}_n(D) \otimes \mathbb{Q} \arrow[u, bend right, "(g_*)_{\mathbb{Q}}"'] \end{tikzcd}$$

By Corollary 3.103, both $(f_*)_{\mathbb{Q}} \circ (f^*)_{\mathbb{Q}}$ and $(g_*)_{\mathbb{Q}} \circ (g^*)_{\mathbb{Q}}$ are isomorphisms for all n. This gives us the following diagram:

$$\begin{tikzcd} \mathrm{K}_n(A) \otimes \mathbb{Q} \arrow[r, "(f^*)_{\mathbb{Q}}"] \arrow[d, "(h_1^*)_{\mathbb{Q}}"'] \arrow[rr, bend left, "\cong"] & \mathrm{K}_n(C) \otimes \mathbb{Q} \arrow[r, "(f_*)_{\mathbb{Q}}"] \arrow[d, "(h_2^*)_{\mathbb{Q}}"'] & \mathrm{K}_n(A) \otimes \mathbb{Q} \arrow[d, "(h_1^*)_{\mathbb{Q}}"'] \\ \mathrm{K}_n(B) \otimes \mathbb{Q} \arrow[r, "(g^*)_{\mathbb{Q}}"'] \arrow[rr, bend right, "\cong"'] & \mathrm{K}_n(D) \otimes \mathbb{Q} \arrow[r, "(g_*)_{\mathbb{Q}}"'] & \mathrm{K}_n(B) \otimes \mathbb{Q} \end{tikzcd}$$

Notice that the left square obviously commutes for all n, and the right square commutes for all n since by our assumption we have $(g_*)_{\mathbb{Q}} \circ (h_2^*)_{\mathbb{Q}} = (h_1^*)_{\mathbb{Q}} \circ (f_*)_{\mathbb{Q}}$ for all n. This means that for all n, $(h_1^*)_{\mathbb{Q}}$ is a direct summand of $(h_2^*)_{\mathbb{Q}}$. So if there is an n such that $(h_2^*)_{\mathbb{Q}}$ is an isomorphism, then $(h_1^*)_{\mathbb{Q}}$ (for that same n) is a direct summand of an isomorphism, and therefore is an isomorphism as well.

For the final comment concerning the case $D \cong B \otimes_A C$, note that in this case, the map $g_* \circ h_2^*$ is induced by taking a projective module P in the category of C-modules, mapping it to $P \otimes_C (C \otimes_A B)$, and then restricting scalars to the ring B. Similarly, the map $h_1^* \circ f_*$ is induced by restricting P to the ring A, and then mapping it to $P \otimes_A B$. Since

$$P \otimes_C (C \otimes_A B) \cong P \otimes_A B \tag{3.71}$$

as B-modules, the maps $g_* \circ h_2^*$ and $h_1^* \circ f_*$ are the same on the level of projective modules. Thus, they are the same on the level of strict perfect complexes, and therefore they are the same on the level of K-theory. Now tensoring with $\mathbb{Q}$ gives us the desired conditions, and the first part of the proof applies.

□

Theorem 3.101 tells us more or less all we need to know about the morphism $f_* \circ f^*$. But we would also like to examine the morphism $f^* \circ f_*$. This requires more effort. We begin with a proposition.

Proposition 3.105. *Let the following be a pullback diagram of quasi-compact schemes, with f a quasi-separated map.*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Suppose that f and g are Tor*-independent over Y. Suppose that f has finite* Tor*-dimension and that f and f' are such that Rf_* and Rf'_* preserve perfection, so that transfer maps*

$$f_* : \mathrm{K}(X) \longrightarrow \mathrm{K}(Y) \tag{3.72}$$

and

$$f'_* : \mathrm{K}(X') \longrightarrow \mathrm{K}(Y') \tag{3.73}$$

both exist. Then there is a canonical homotopy

$$g^* \circ f_* \simeq f'_* \circ g'^* : \mathrm{K}(X) \longrightarrow \mathrm{K}(Y'). \tag{3.74}$$

Proof. See [TT, Proposition 3.18], [Qui1, Section 7.2.11], and [GroSGA6, IV, 3.1.1]. ∎

Remark 3.106. While the statement of Proposition 3.105 initially only applies to ordinary K-theory, Thomason later proves that this homotopy extends to the non-connective K-theory case as well; see [TT, 6.5].

In order to understand $f^* \circ f_*$ properly, we need to be more restrictive in our choice of f. Making these extra restrictions allows us to explicitly write down the map $f^* \circ f_*$.

Theorem 3.107. *Suppose we have that $f : A \longrightarrow B$ is an injective, finite, étale morphism of Noetherian rings, and that B is a free A-module. Suppose further that B/A is a Galois extension of rings; in other words, suppose there is a finite group G acting on B such that $B^G = A$, and that the rank of B as an A-module is $|G|$, so that Theorem 3.101 tells us $f_* \circ f^*$ is multiplication by $|G|$. We call this G the* Galois group *of B over A. Then*

$$f^* \circ f_* = \sum_{g \in G} g^* : \mathrm{K}_n(B) \longrightarrow \mathrm{K}_n(B) \tag{3.75}$$

for all n, and

$$(f^*)_{\mathbb{Q}} \circ (f_*)_{\mathbb{Q}} = \sum_{g \in G} (g^*)_{\mathbb{Q}} : \mathrm{K}_n(B) \otimes \mathbb{Q} \longrightarrow \mathrm{K}_n(B) \otimes \mathbb{Q} \tag{3.76}$$

for all n.

Proof. The second claim clearly follows from the first by tensoring with $\mathbb{Q}$, so we restrict ourselves to proving the first claim. We model our proof after [Tho, Lemma 2.13] and [TT, Proposition 11.10].

Consider the following pushout square of rings

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "1_B \otimes f"] \\ B \arrow[r, "f \otimes 1_B"'] & B \otimes_A B \end{tikzcd}$$

Applying Spec will give us a pullback square of schemes, so Proposition 3.105 can be applied. Notice also that all morphisms in this scheme are injective, flat, and finite so the conditions for Tor-independence and finite Tor-dimension are trivially satisfied. So by Proposition 3.105, we have a canonical homotopy

$$f^* \circ f_* \simeq (f \otimes 1_B)_* \circ (1_B \otimes f)^* : \mathrm{K}(B) \longrightarrow \mathrm{K}(B). \tag{3.77}$$

In other words, for every n, we have

$$f^* \circ f_* = (f \otimes 1_B)_* \circ (1_B \otimes f)^* : \mathrm{K}_n(B) \longrightarrow \mathrm{K}_n(B). \tag{3.78}$$

Another way to express the canonical homotopy is to say that the square

$$\begin{tikzcd} \mathrm{K}(B \otimes_A B) \arrow[r, "(f\otimes 1_B)_*"] & \mathrm{K}(B) \\ \mathrm{K}(B) \arrow[u, "(1_B \otimes f)^*"] \arrow[r, "f_*"'] & \mathrm{K}(A) \arrow[u, "f^*"'] \end{tikzcd}$$

commutes up to canonical homotopy. So it is enough to understand the morphism $(f \otimes 1_B)_* \circ (1_B \otimes f)^*$.

From this perspective, we see that $f^* \circ f_*$ is induced by the functor

$$P \mapsto P \otimes_B (B \otimes_A B). \tag{3.79}$$

Galois Theory tells us that there is an isomorphism

$$\kappa : B \otimes_A B \xrightarrow{\cong} \prod_{g\in G} B$$

where $\kappa(x \otimes y) = (x \cdot g(y))_{g\in G}$. Since κ is an isomorphism, Corollary 3.102 says that $\kappa_* \circ \kappa^*$ is the identity map, and that $\kappa_* = (\kappa^*)^{-1}$. This allows us to construct the following commutative diagram:

$$\begin{tikzcd}
\prod_{g\in G} \mathrm{K}(B) \arrow[drr, bend left, "(f\otimes 1_B)_*\circ\kappa_*"] & & \\
& \mathrm{K}(B \otimes_A B) \arrow[ul, "\kappa^*", "\cong"'] \arrow[r, "(f\otimes 1_B)_*"] & \mathrm{K}(B) \\
& \mathrm{K}(B) \arrow[u, "(1_B\otimes f)^*"] \arrow[r, "f_*"'] \arrow[uul, bend left, "\Delta"] & \mathrm{K}(A) \arrow[u, "f^*"']
\end{tikzcd}$$

where Δ denotes the diagonal map. Notice that the map $(f \otimes 1_B)_* \circ \kappa_* = \sum_{g\in G} g_*$ by Proposition 3.97 applied to the map $\kappa \circ (f \otimes 1_B)$; indeed, notice that

$$\kappa \circ (f \otimes 1_B) : B \longrightarrow \prod_{g\in G} B \tag{3.80}$$

is given by sending $x = 1 \otimes x \mapsto (g(x))_{g\in G}$ so on the level of transfer maps we get that $(f \otimes 1_B)_* \circ \kappa_* = \sum_{g\in G} g_*$ as claimed. Composing with Δ gives us that

$$f^* \circ f_* \simeq \sum_{g\in G} g_* \circ \Delta \simeq \sum_{g\in G} g_* : \mathrm{K}(B) \longrightarrow \mathrm{K}(B) \tag{3.81}$$

and thus we have that

$$f^* \circ f_* = \sum_{g\in G} g^* : \mathrm{K}_n(B) \longrightarrow \mathrm{K}_n(B) \tag{3.82}$$

for all n as claimed.

$\square$

3.8 Descent Properties of Algebraic K-theory

Now that we have established the definitions and basic properties of K-theory and KH-theory, we can now examine what happens to these functors in the various Grothendieck topologies introduced in Section 3.2. For our purposes, the Nisnevich Topology is really only useful in introducing the cdh-Topology, so we will exclude descent properties with respect to the Nisnevich topology and instead only focus on the Zariski, Étale, and cdh topologies.

Definition 3.108. *Let F be a presheaf of* $\mathsf{Spectra}$ *on the Zariski site of X. We define the* Zariski Hypercohomology *of F (with respect to a Zariski cover $\mathcal{U}$), denoted $\mathbb{H}^{\cdot}(\mathcal{U}, F)$, to be* $\mathrm{holim}(F(\mathcal{U}))$. *In other words,* $\mathbb{H}^{\cdot}(\mathcal{U}, F) = \mathrm{holim}(F(\mathcal{U}))$; *that is, the homotopy limit of the diagram*

$$\prod_{i_0 \in I} F(U_{i_0}) \substack{\longleftarrow \\ \longrightarrow \\ \longrightarrow} \prod_{i_0, i_1 \in I} F(U_{i_0} \times_X U_{i_1}) \substack{\longleftarrow \\ \longleftarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} \cdots$$

Similarly, if F is a presheaf of $\mathsf{Spectra}$ *on the étale site of X, and we denote by* $\mathrm{Et}(\mathcal{U})$ *an étale cover of X, we define* $\mathbb{H}^{\cdot}_{\mathrm{Et}}(\mathrm{Et}(\mathcal{U}), F)$ *(the* Étale Hypercohomology *of F) to be* $\mathrm{holim}(F(\mathrm{Et}(\mathcal{U})))$. *Finally, if F is a presheaf of* $\mathsf{Spectra}$ *on the cdh site of X, and we denote by $(\mathcal{U})_{cdh}$ a cdh cover of X, we define $\mathbb{H}^{\cdot}_{cdh}((\mathcal{U})_{cdh}, F)$ (the* cdh Hypercohomology *of F) to be* $\mathrm{holim}(F((\mathcal{U})_{cdh}))$. *In the latter two cases, the only difference in our diagram is that the U_i's come from the appropriate cover (*$\mathrm{Et}(\mathcal{U})$ *or $(\mathcal{U})_{cdh}$) and F is replaced by the sheafication of F in the appropriate topology. These latter two are mentioned for completeness; our primary focus will be the Zariski topology case.*

Remark 3.109. We adopt this notation, given by Thomason in [Tho], to make the notation of this section a bit cleaner. We will continue to use the holim notation in other sections of this paper.

Definition 3.110. *Let F be a presheaf of* $\mathsf{Spectra}$ *on the Zariski site of X. We say that F satisfies the* Mayer-Vietoris Property *for the Zariski topology if for all Zariski open sub-*

schemes $U, V \subset X$, the square

$$\begin{array}{ccc} F(U\cup V) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U\cap V) \end{array}$$

is homotopy cartesian. In other words, if we apply F to a Zariski square, the resulting square is homotopy cartesian. Similarly, if F is a presheaf of $\mathsf{Spectra}$ *on the étale site of X (respectively, cdh site of X), we say that F satisfies the* Mayer-Vietoris Property *for the étale topology (respectively, the cdh-topology) if whenever we apply F to an étale square (respectively, cdh square), the resulting square is homotopy cartesian.*

Example 3.111. *[TT, Theorem 8.1] shows that, for X a quasi-separated scheme, the presheaf K^B satisfies the Mayer-Vietoris Property for the Zariski topology. We will use this fact without proof in this paper.*

Definition 3.112. *Let F be a presheaf of* $\mathsf{Spectra}$ *on the Zariski site of X. We say that F satisfies Zariski descent if the natural map $F(X) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{U}, F)$ is a weak equivalence for all Zariski covers $\mathcal{U}$ of X. Similarly, if F be a presheaf of* $\mathsf{Spectra}$ *on the étale site of X (respectively, cdh site of X), we say that F satisfies étale descent (respectively, the cdh descent) if the natrual map $F(X) \longrightarrow \mathbb{H}^{\cdot}_{\mathrm{Et}}(\mathrm{Et}(\mathcal{U}), F)$ (respectively, the natrual map $F(X) \longrightarrow \mathbb{H}^{\cdot}_{cdh}((\mathcal{U})_{cdh}, F)$) is a weak equivalence for all étale covers $\mathrm{Et}(\mathcal{U})$ (respectively, all cdh covers $(\mathcal{U})_{cdh}$).*

It turns out that satisfying Zariski descent is the same as satisfying the Mayer Vietoris property for a Zariski cover. For much of the remainder of this section, we will show that the functor KH satisfies Zariski descent in a special case by proving parts (b) and (c) of [TT, Exercise 9.11], and as such any mention of sheaves and the Mayer Vietoris property are assumed to be with respect to the Zariski topology unless otherwise stated. The general case is implied by a result of Brown and Gersten ([BG, Theorem 4]). We will then discuss other forms of descent that will be important to us.

We will be seeking to generalize the following proposition about K^B to KH.

Proposition 3.113. *Let X be a quasi-compact, quasi-separated scheme. Let*

$$\mathcal{U} = \{U_1, ..., U_n\} \tag{3.83}$$

be a cover of X by finitely many Zariski open subschemes, each of which is quasi-compact. The the augmentation map

$$\mathrm{K}^B(X) \longrightarrow \mathbb{H}^{\cdot}\left(\mathcal{U}, \mathrm{K}^B\right) \tag{3.84}$$

is a weak equivalence.

Proof. See [TT, Proposition 8.3].

□

To proceed further, we need to first make a definition. The following definition can be found in [Tho].

Definition 3.114. *Let $\mathcal{U} = \{U_i \longrightarrow X | i \in I\}$ and $\mathcal{V} = \{V_j \longrightarrow X | j \in J\}$ be two Zariski covers (or more generally, two covers in any fixed topology) of X. A map of covers $\mathcal{U} \longrightarrow \mathcal{V}$ consists of a function $\varphi : J \longrightarrow I$ and, for each $j \in J$, a morphism $f_j : V_j \longrightarrow U_{\varphi(j)}$ compatible with the projection to X. $\mathcal{V}$ is called a* refinement *of $\mathcal{U}$ if there is a map of covers $\mathcal{U} \longrightarrow \mathcal{V}$.*

We now state a couple of lemmas that we will use in the proof of Theorem 3.118 below.

Lemma 3.115. *Let $\mathcal{U}$ and $\mathcal{V}$ be two covers of X, and suppose there is a map $\mathcal{U} \longrightarrow \mathcal{V}$, so that $\mathcal{V}$ is a refinement of $\mathcal{U}$. Suppose that for every finite set I of $U_i \longrightarrow X$ drawn from $\mathcal{U}$, and for the fibre product (over X)*

$$U_I = U_{i_0} \times_X U_{i_1} \times_X \cdots \times_X U_{i_n} \tag{3.85}$$

of the elements of I, and for the induced cover $\mathcal{V} \times_X U_I$ of U_I that the augmentation map

$$F(U_I) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{V} \times_X U_I, F) \tag{3.86}$$

is a weak equivalence. In particular, for $I = \emptyset$, we suppose that

$$F(X) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{V}, F) \tag{3.87}$$

is a weak equivalence. Then the augmentation map for $\mathcal{U}$ is also a weak equivalence; namely,

$$F(X) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{U}, F) \tag{3.88}$$

is a weak equivalence.

Proof. See [TT, Lemma 8.2.5].

□

Lemma 3.116. *Let F be a presheaf of* $\mathsf{Spectra}$ *satisfying the Mayer-Vietoris property. Then* $\mathrm{hocolim}_{\Delta^{op}} F(- \times \Delta^{\cdot})$ *also satisfies the Mayer-Vietoris property. Here $\Delta^{\cdot}$ is the standard simplicial object, where $\Delta^n = \mathrm{Spec}\,(\mathbb{Z}[T_0, ..., T_n]/(\sum_{i=0}^n T_i = 1))$ and where the fiber product is taken over* $\mathrm{Spec}\,\mathbb{Z}$.

Proof. Let U and V be two open subschemes. Then we get a square of schemes

$$\begin{tikzcd}
(U \cap V) \times \Delta^{\cdot} \arrow[r] \arrow[d] & U \times \Delta^{\cdot} \arrow[d] \\
V \times \Delta^{\cdot} \arrow[r] & (U \cup V) \times \Delta^{\cdot}
\end{tikzcd}$$

which, by the Mayer-Vietoris property for F, gives us that the square

$$\begin{tikzcd}
F((U \cup V) \times \Delta^{\cdot}) \arrow[r] \arrow[d] & F(U \times \Delta^{\cdot}) \arrow[d] \\
F(V \times \Delta^{\cdot}) \arrow[r] & F((U \cap V) \times \Delta^{\cdot})
\end{tikzcd}$$

is homotopy cartesian. Since in Spectra we have that a square is homotopy cartesian if and only if it is homotopy cocartesian, we have that

$$\begin{tikzcd}
F((U \cup V) \times \Delta^{\cdot}) \arrow[r] \arrow[d] & F(U \times \Delta^{\cdot}) \arrow[d] \\
F(V \times \Delta^{\cdot}) \arrow[r] & F((U \cap V) \times \Delta^{\cdot})
\end{tikzcd}$$

is homotopy cocartesian as well. This means that the homotopy colimit of the diagram

$$\begin{tikzcd}
F((U \cup V) \times \Delta^{\cdot}) \arrow[r] \arrow[d] & F(U \times \Delta^{\cdot}) \\
F(V \times \Delta^{\cdot}) &
\end{tikzcd}$$

(which we denote by $\mathrm{hocolim}_i(F(\mathcal{U} \times \Delta^{\cdot}))$) has

$$\mathrm{hocolim}_i(F(\mathcal{U} \times \Delta^{\cdot})) \longrightarrow F((U \cap V) \times \Delta^{\cdot}) \tag{3.89}$$

and this map is a weak equivalence. Applying $\mathrm{hocolim}_{\Delta^{op}}$ to both sides and using [Tho, Lemma 5.16], we get that

$$\mathrm{hocolim}_i(\mathrm{hocolim}_{\Delta^{op}} F(\mathcal{U} \times \Delta^{\cdot})) \longrightarrow \mathrm{hocolim}_{\Delta^{op}} F((U \cap V) \times \Delta^{\cdot}) \tag{3.90}$$

is also a weak equivalence. This means that the square

$$\begin{tikzcd}
\mathrm{hocolim}_{\Delta^{op}} F((U \cup V) \times \Delta^{\cdot}) \arrow[r] \arrow[d] & \mathrm{hocolim}_{\Delta^{op}} F(U \times \Delta^{\cdot}) \arrow[d] \\
\mathrm{hocolim}_{\Delta^{op}} F(V \times \Delta^{\cdot}) \arrow[r] & \mathrm{hocolim}_{\Delta^{op}} F((U \cap V) \times \Delta^{\cdot})
\end{tikzcd}$$

is homotopy cocartesian, and therefore also homotopy cartesian. Therefore, the functor $\mathrm{hocolim}_{\Delta^{op}} F(- \times \Delta^{\cdot})$ satisfies the Mayer-Vietoris property as desired.

□

Corollary 3.117. *The presheaf of* Spectra KH *satisfies the Mayer-Vietoris property.*

Proof. Since $\mathrm{KH}(-) = \mathrm{hocolim}_{\Delta^{op}} \mathrm{K}^B(- \times \Delta^{\cdot})$, Example 3.111 and Lemma 3.116 give the result.

□

We are now ready to state the following important theorem.

Theorem 3.118. *Let F be a presheaf of* $\mathsf{Spectra}$ *satisfying the Mayer-Vietoris Property. Let X be a quasi-compact and quasi-separated scheme. Suppose that for any cover $\mathcal{C}$ of X, the presheaf F satisfies the property that the natural map*

$$F(X) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{C}, F) \tag{3.91}$$

is a weak equivalence. Let $\mathcal{U} = \{U_1, ..., U_n\}$ be a finite cover of X by open sets. Then the following map is a weak equivalence:

$$\mathrm{hocolim}_{\Delta^{op}}(\mathrm{holim}_{\Delta}(F(\mathcal{U} \times \Delta^{\cdot}))) \longrightarrow \mathrm{holim}_{\Delta}(\mathrm{hocolim}_{\Delta^{op}}(F(\mathcal{U} \times \Delta^{\cdot}))).$$

Since $\mathbb{H}^{\cdot}(\mathcal{U}, F) = \mathrm{holim}_{\Delta}(F(\mathcal{U}))$, we can rewrite this weak equivalence as

$$\mathrm{hocolim}_{\Delta^{op}}(\mathbb{H}^{\cdot}(\mathcal{U}, F(- \times \Delta^{\cdot}))) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{U}, \mathrm{hocolim}_{\Delta^{op}}(F(- \times \Delta^{\cdot}))).$$

Before we prove Theorem 3.118, we observe the following corollary, which is a special case of [BG, Theorem 4].

Corollary 3.119. *Let X be a quasi-compact, quasi-separated scheme. Let*

$$\mathcal{U} = \{U_1, ..., U_n\} \tag{3.92}$$

be a cover of X by finitely many Zariski open subschemes, each of which is quasi-compact. The the augmentation map

$$\mathrm{KH}(X) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{U}, \mathrm{KH}) \tag{3.93}$$

is a weak equivalence.

Proof. Under these conditions, Example 3.111 shows that K^B is a presheaf of $\mathsf{Spectra}$ satisfying the Mayer-Vietoris property. Proposition 3.113 tells us that

$$\mathrm{K}^B(X \times \Delta^{\cdot}) \longrightarrow \mathbb{H}^{\cdot}\left(\mathcal{U}, \mathrm{K}^B(- \times \Delta^{\cdot})\right) \tag{3.94}$$

is a weak equivalence. Applying $\mathrm{hocolim}_{\Delta^{op}}$ to both sides gives us that

$$\mathrm{KH}(X) \longrightarrow \mathrm{hocolim}_{\Delta^{op}}\left(\mathbb{H}^{\cdot}\left(\mathcal{U}, \mathrm{K}^B(- \times \Delta^{\cdot})\right)\right) \tag{3.95}$$

is also a weak equivalence. By Theorem 3.118, the map

$$\mathrm{hocolim}_{\Delta^{op}}\left(\mathbb{H}^{\cdot}\left(\mathcal{U}, \mathrm{K}^B(- \times \Delta^{\cdot})\right)\right) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{U}, \mathrm{KH}(-)) \tag{3.96}$$

is a weak equivalence. Composing these two maps, and applying the 2-out-of-3 axiom of Definition 3.44 gives us the result.

□

We are now ready to prove Theorem 3.118.

Proof of Theorem 3.118. From Lemma 3.116, we know that the functor

$$\mathrm{hocolim}_{\Delta^{op}} F(- \times \Delta^{\cdot}) \tag{3.97}$$

satisfies the Mayer-Vietoris property. We proceed by induction of the number of elements in $\mathcal{U}$. If $n = 1$ this result is trivially true as $\mathcal{U}$ is just the trivial cover. The case $n = 2$ is implied by Lemma 3.116, since the square

$$\begin{tikzcd}
\mathrm{hocolim}_{\Delta^{op}} F((U_1 \cup U_2) \times \Delta^{\cdot}) \arrow[r] \arrow[d] & \mathrm{hocolim}_{\Delta^{op}} F(U_1 \times \Delta^{\cdot}) \arrow[d] \\
\mathrm{hocolim}_{\Delta^{op}} F(U_2 \times \Delta^{\cdot}) \arrow[r] & \mathrm{hocolim}_{\Delta^{op}} F((U_1 \cap U_2) \times \Delta^{\cdot})
\end{tikzcd}$$

being homotopy cartesian means that, for the cover $\mathcal{U} = \{U_1, U_2\}$, the natural map

$$\operatorname{hocolim}_{\Delta^{op}} F((U_1 \cup U_2) \times \Delta^{\cdot}) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{U}, \operatorname{hocolim}_{\Delta^{op}} F(- \times \Delta^{\cdot})) \tag{3.98}$$

is a weak equivalence. Since the square

$$\begin{tikzcd} F((U_1 \cup U_2) \times \Delta^{\cdot}) \arrow[r] \arrow[d] & F(U_1 \times \Delta^{\cdot}) \arrow[d] \\ F(U_2 \times \Delta^{\cdot}) \arrow[r] & F((U_1 \cap U_2) \times \Delta^{\cdot}) \end{tikzcd}$$

is homotopy cartesian also, we get that the natural map

$$F((U_1 \cup U_2) \times \Delta^{\cdot}) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{U}, F(- \times \Delta^{\cdot})) \tag{3.99}$$

is also a weak equivalence. Applying $\operatorname{hocolim}_{\Delta^{op}}$ to both sides gives us that

$$\operatorname{hocolim}_{\Delta^{op}} F((U_1 \cup U_2) \times \Delta^{\cdot}) \longrightarrow \operatorname{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{U}, F(- \times \Delta^{\cdot})) \tag{3.100}$$

is a weak equivalence as well. Since

$$\operatorname{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{U}, F(- \times \Delta^{\cdot})) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{U}, \operatorname{hocolim}_{\Delta^{op}} F(- \times \Delta^{\cdot})) \tag{3.101}$$

is the natural augmentation map, the composite with the map

$$\operatorname{hocolim}_{\Delta^{op}} F((U_1 \cup U_2) \times \Delta^{\cdot}) \longrightarrow \operatorname{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{U}, F(- \times \Delta^{\cdot})) \tag{3.102}$$

gives us the natural augmentation map

$$\operatorname{hocolim}_{\Delta^{op}} F((U_1 \cup U_2) \times \Delta^{\cdot}) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{U}, \operatorname{hocolim}_{\Delta^{op}} F(- \times \Delta^{\cdot})) \tag{3.103}$$

which we already saw is a weak equivalence. Applying the 2-out-of-3 axiom of Definition

3.44 gives us that

$$\operatorname{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{U}, F(- \times \Delta^{\cdot})) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{U}, \operatorname{hocolim}_{\Delta^{op}} F(- \times \Delta^{\cdot})) \tag{3.104}$$

is a weak equivalence. This is the case $n = 2$.

Now suppose the statement is true for all covers of size $k < n$. Let

$$\mathcal{U} = \{U_1, ..., U_n\} \tag{3.105}$$

be any cover of X of size n. Set

$$V = U_1 \cup U_2 \cup \cdots \cup U_{n-1}, \tag{3.106}$$

and set $\mathcal{V} = \{U_1, ..., U_{n-1}\}$ the cover for V.

By the Mayer-Vietoris property and Lemma 3.116, we have that the square

$$\begin{tikzcd}
\operatorname{hocolim}_{\Delta^{op}} F(- \times \Delta^{\cdot}) \arrow[r] \arrow[d] & \operatorname{hocolim}_{\Delta^{op}} F((- \cap U_n) \times \Delta^{\cdot}) \arrow[d] \\
\operatorname{hocolim}_{\Delta^{op}} F((- \cap V) \times \Delta^{\cdot}) \arrow[r] & \operatorname{hocolim}_{\Delta^{op}} F((- \cap V \cap U_n) \times \Delta^{\cdot})
\end{tikzcd}$$

is homotopy cartesian. Applying $\mathbb{H}^{\cdot}(\mathcal{U}, -)$ to this square gives us, by Theorem 3.72, a homotopy cartesian square as well. So the natural maps give us a morphism of diagrams between the diagrams

$$\begin{tikzcd}
\operatorname{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{U}, F(- \times \Delta^{\cdot})) \arrow[r] \arrow[d] & \operatorname{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{U}, F((- \cap U_n) \times \Delta^{\cdot})) \arrow[d] \\
\operatorname{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{U}, F((- \cap V) \times \Delta^{\cdot})) \arrow[r] & \operatorname{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{U}, F((- \cap V \cap U_n) \times \Delta^{\cdot}))
\end{tikzcd}$$

and

$$\begin{array}{ccc} \mathbb{H}^{\cdot}(\mathcal{U}, \mathrm{hocolim}_{\Delta^{op}} F(- \times \Delta^{\cdot})) & \longrightarrow & \mathbb{H}^{\cdot}(\mathcal{U}, \mathrm{hocolim}_{\Delta^{op}} F((- \cap U_n) \times \Delta^{\cdot})) \\ \downarrow & & \downarrow \\ \mathbb{H}^{\cdot}(\mathcal{U}, \mathrm{hocolim}_{\Delta^{op}} F((- \cap V) \times \Delta^{\cdot})) & \longrightarrow & \mathbb{H}^{\cdot}(\mathcal{U}, \mathrm{hocolim}_{\Delta^{op}} F((- \cap V \cap U_n) \times \Delta^{\cdot})) \end{array}$$

and we want to explore the properties of this morphism of diagrams. I claim that this will induce our desired result. We will accomplish this by showing that the morphism between the diagrams

$$\begin{array}{ccc} & & \mathrm{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{U}, F((- \cap U_n) \times \Delta^{\cdot})) \\ & & \downarrow \\ \mathrm{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{U}, F((- \cap V) \times \Delta^{\cdot})) & \longrightarrow & \mathrm{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{U}, F((- \cap V \cap U_n) \times \Delta^{\cdot})) \end{array}$$

and

$$\begin{array}{ccc} & & \mathbb{H}^{\cdot}(\mathcal{U}, \mathrm{hocolim}_{\Delta^{op}} F((- \cap U_n) \times \Delta^{\cdot})) \\ & & \downarrow \\ \mathbb{H}^{\cdot}(\mathcal{U}, \mathrm{hocolim}_{\Delta^{op}} F((- \cap V) \times \Delta^{\cdot})) & \longrightarrow & \mathbb{H}^{\cdot}(\mathcal{U}, \mathrm{hocolim}_{\Delta^{op}} F((- \cap V \cap U_n) \times \Delta^{\cdot})) \end{array}$$

given by the natural maps is a weak equivalence of diagrams. Recall from Definition 3.57 that a weak equivalence of diagrams is a map of diagrams which is term-wise a weak equivalence. If we show this, then by applying Proposition 3.71, we will get that the homotopy limit of the first diagram maps to the homotopy limit of the second diagram, and this map is a weak equivalence. For ease of notation, call these homotopy limits holim(1) and holim(2). Then we get a commuting square

$$\begin{array}{ccc} \mathrm{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{U}, F(- \times \Delta^{\cdot})) & \longrightarrow & \mathbb{H}^{\cdot}(\mathcal{U}, \mathrm{hocolim}_{\Delta^{op}} F(- \times \Delta^{\cdot})) \\ \downarrow & & \downarrow \\ \mathrm{holim}(1) & \longrightarrow & \mathrm{holim}(2) \end{array}$$

where all but the top map are known weak equivalences. By applying the 2-out-of-3 axiom of Definition 3.44 twice, we get that the map

$$\operatorname{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{U}, F(- \times \Delta^{\cdot})) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{U}, \operatorname{hocolim}_{\Delta^{op}} F(- \times \Delta^{\cdot})) \tag{3.107}$$

is a weak equivalence also, which will complete the proof. So it remains only to show that the natural maps give us a weak equivalence of diagrams.

In each of the three cases we seek to make use of Lemma 3.115 and our induction hypothesis. Observe that, by its very construction,

$$\mathbb{H}^{\cdot}(\mathcal{U}, \operatorname{hocolim}_{\Delta^{op}} F((- \cap W) \times \Delta^{\cdot})) \tag{3.108}$$

is naturally isomorphic to

$$\mathbb{H}^{\cdot}(\mathcal{U} \cap W, \operatorname{hocolim}_{\Delta^{op}} F(- \times \Delta^{\cdot})) \tag{3.109}$$

for any open set W. Suppose $W = U_n$. Then $\mathcal{U} \cap U_n$ has a refinement $\mathcal{C}$ consisting of just the trivial cover $\{U_n \longrightarrow U_n\}$. Consider U_I as in Lemma 3.115; then $\mathcal{C} \times_X U_I$ is a cover of U_I of size 1, so by our induction hypothesis, we get

$$\operatorname{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{C} \times_X U_I, F(- \times \Delta^{\cdot})) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{C} \times_X U_I, \operatorname{hocolim}_{\Delta^{op}} F(- \times \Delta^{\cdot}))$$

is a weak equivalence for every I. Composing with the natural weak equivalence

$$\operatorname{hocolim}_{\Delta^{op}} F(U_I \times \Delta^{\cdot}) \longrightarrow \operatorname{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{C} \times_X U_I, F(- \times \Delta^{\cdot})) \tag{3.110}$$

we get that

$$\operatorname{hocolim}_{\Delta^{op}} F(U_I \times \Delta^{\cdot}) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{C} \times_X U_I, \operatorname{hocolim}_{\Delta^{op}} F(- \times \Delta^{\cdot})) \tag{3.111}$$

is a weak equivalence for all I. By Lemma 3.115, we get the following commuting triangle:

$$\begin{tikzcd}
\operatorname{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{U}\cap U_n, F(-\times\Delta^{\cdot})) \arrow[r] & \mathbb{H}^{\cdot}(\mathcal{U}\cap U_n, \operatorname{hocolim}_{\Delta^{op}} F(-\times\Delta^{\cdot})) \\
\operatorname{hocolim}_{\Delta^{op}} F(U_n\times\Delta^{\cdot}) \arrow[u, "\sim"] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

which, by the 2-out-of-3 axiom of Definition 3.44, implies that the morphism

$$\operatorname{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{U}\cap U_n, F(-\times\Delta^{\cdot})) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{U}\cap U_n, \operatorname{hocolim}_{\Delta^{op}} F(-\times\Delta^{\cdot}))$$

is a weak equivalence as desired.

The other two cases will work similarly. Indeed, notice that $\mathcal{V}$ is a refinement of $\mathcal{U}\cap V$ and $\mathcal{V}\cap U_n$ is a refinement of $\mathcal{U}\cap V\cap U_n$; both of these refinements are of size $n-1$ and will remain of size $n-1$ when taking the fibre product with U_I. So by our induction hypothesis, we have

$$\operatorname{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{V}\times_X U_I, F(-\times\Delta^{\cdot})) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{V}\times_X U_I, \operatorname{hocolim}_{\Delta^{op}} F(-\times\Delta^{\cdot}))$$

and

$$\operatorname{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}((\mathcal{V}\cap U_n)\times_X U_I, F(-\times\Delta^{\cdot})) \longrightarrow \mathbb{H}^{\cdot}((\mathcal{V}\cap U_n)\times_X U_I, \operatorname{hocolim}_{\Delta^{op}} F(-\times\Delta^{\cdot}))$$

are weak equivalences for every I. Composing, respectively, with the natural weak equivalences

$$\operatorname{hocolim}_{\Delta^{op}} F(U_I\times\Delta^{\cdot}) \longrightarrow \operatorname{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}(\mathcal{V}\times_X U_I, F(-\times\Delta^{\cdot}))$$

and

$$\operatorname{hocolim}_{\Delta^{op}} F(U_I\times\Delta^{\cdot}) \longrightarrow \operatorname{hocolim}_{\Delta^{op}} \mathbb{H}^{\cdot}((\mathcal{V}\cap U_n)\times_X U_I, F(-\times\Delta^{\cdot}))$$

we get that both

$$\mathrm{hocolim}_{\Delta^{op}} F(U_I \times \Delta^\cdot) \longrightarrow \mathbb{H}^\cdot(\mathcal{V} \times_X U_I, \mathrm{hocolim}_{\Delta^{op}} F(- \times \Delta^\cdot))$$

and

$$\mathrm{hocolim}_{\Delta^{op}} F(U_I \times \Delta^\cdot) \longrightarrow \mathbb{H}^\cdot((\mathcal{V} \cap U_n) \times_X U_I, \mathrm{hocolim}_{\Delta^{op}} F(- \times \Delta^\cdot))$$

are weak equivalences for all I. By Lemma 3.115, we get the following commuting triangles:

$$\begin{tikzcd}
\mathrm{hocolim}_{\Delta^{op}} \mathbb{H}^\cdot(\mathcal{U} \cap V, F(- \times \Delta^\cdot)) \arrow[r] & \mathbb{H}^\cdot(\mathcal{U} \cap V, \mathrm{hocolim}_{\Delta^{op}} F(- \times \Delta^\cdot)) \\
\mathrm{hocolim}_{\Delta^{op}} F(V \times \Delta^\cdot) \arrow[u, "\sim"] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{hocolim}_{\Delta^{op}} \mathbb{H}^\cdot(\mathcal{U} \cap V \cap U_n, F(- \times \Delta^\cdot)) \arrow[r] & \mathbb{H}^\cdot(\mathcal{U} \cap V \cap U_n, \mathrm{hocolim}_{\Delta^{op}} F(- \times \Delta^\cdot)) \\
\mathrm{hocolim}_{\Delta^{op}} F(V \cap U_n \times \Delta^\cdot) \arrow[u, "\sim"] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

which, by the 2-out-of-3 axiom of Definition 3.44, imply that the morphisms

$$\mathrm{hocolim}_{\Delta^{op}} \mathbb{H}^\cdot(\mathcal{U} \cap V, F(- \times \Delta^\cdot)) \longrightarrow \mathbb{H}^\cdot(\mathcal{U} \cap V, \mathrm{hocolim}_{\Delta^{op}} F(- \times \Delta^\cdot))$$

and

$$\mathrm{hocolim}_{\Delta^{op}} \mathbb{H}^\cdot(\mathcal{U} \cap V \cap U_n, F(- \times \Delta^\cdot)) \longrightarrow \mathbb{H}^\cdot(\mathcal{U} \cap V \cap U_n, \mathrm{hocolim}_{\Delta^{op}} F(- \times \Delta^\cdot))$$

are weak equivalences as desired. This completes the proof.

$\square$

Remark 3.120. Corollary 3.119 is not the true form of Brown and Gersten's Theorem. The true form asserts that if X is Noetherian and of finite dimension, and F is a presheaf of spectra with the Mayer-Vietoris property, then the augmentation map

$$F(X) \longrightarrow \mathbb{H}^{\cdot}(\mathcal{U}, F) \tag{3.112}$$

is a weak equivalence. This version is given in [Tho, Exercise 2.5]. It is proven in [TT] in the case that $F = \mathrm{K}^B$ (giving a stronger version of Proposition 3.113), and is proven completely in [Mit] using Jardine's model structure on the category of presheaves of spectra. The interested reader seeking this more general version is encouraged to read these papers.

So we have shown that the functor KH satisfies Zariski descent, assuming that K does. As an immediate corollary, if our scheme is a toric variety over a field of characteristic 0, then the functor $\mathcal{F}_{\mathrm{K}}$ (recall Definition 3.90) will also satisfy Zariski descent as a consequence of Theorem 3.94. The functors $\mathrm{K}(-) \otimes \mathbb{Q}$ and $\mathrm{KH}(-) \otimes \mathbb{Q}$ both satisfy étale descent, and thus Zariski descent since any Zariski cover is automatically an étale cover. The proof for $\mathrm{K}(-)\otimes\mathbb{Q}$ can be found in [TT]; we omit the details, but the proof uses very similar techniques to those presented in the proof of Theorem 3.107. The proof for $\mathrm{KH}(-) \otimes \mathbb{Q}$ can then be done using similar techniques to the ones presented in this section, using $\mathrm{K}(-) \otimes \mathbb{Q}$ in place of K. Finally, the functor KH also satisfies cdh descent; the proof of this fact can be found in [Hae]. If a functor F satisfies descent with respect to a topology then F also satisfies the Mayer-Vietoris property with respect to that topology, and as mentioned earlier in this section, satisfying Zariski descent is equivalent to satisfying the Mayer Vietoris property with respect to the Zariski topology. For the remainder of this paper we will, in the Zariski case, use these two ideas interchangeably.

4 KH-Theory for Complete Simplicial Toric Varieties

As we saw in Theorem 3.94, the algebraic K-theory of any toric variety X is determined completely by its KH-theory and the group $\mathcal{F}_{\mathrm{K}}$. In this section, we calculate as much of the KH-theory of complete simplicial toric varieties as we can.

The initial impulse the reader might have is to use the fact that toric varieties have a very nice resolution of singularities, and then combine that idea with the fact that KH satisfies cdh-descent, as mentioned in Section 3.8. For very simple examples, this is actually a reasonable approach, as we see in the following section.

4.1 KH-Theory of $\mathbb{P}(1,1,a)$

Recall Example 2.7, which examined the weighted projective space $\mathbb{P}(1,1,2)$. We showed in that example that if we resolve the singularity we get the Hirzebruch surface $\mathcal{H}_2$ and the exceptional variety is $\mathbb{P}^1$. Recall this gives us the blow-up square:

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[r, "i"] \arrow[d] & \mathcal{H}_2 \arrow[d] \\
\{*\} \arrow[r] & \mathbb{P}(1,1,2)
\end{tikzcd}$$

Let i denote the inclusion morphism $i : \mathbb{P}^1 \longrightarrow \mathcal{H}_2$ including the exceptional variety into the blow-up. Similarly, since $\mathcal{H}_2$ is the $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ associated to the sheaf $\mathcal{O}(0) + \mathcal{O}(-2)$, we get a structure morphism $\pi : \mathcal{H}_2 \longrightarrow \mathbb{P}^1$, which is induced by the lattice map $\widetilde{\pi} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ where $\widetilde{\pi}(x,y) = x$. Let $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be the composition of these maps; that is, $f = \pi \circ i$. On the level of lattices, notice that $\widetilde{N_\tau} = \mathbb{Z}$ in this case (remember from Example 2.7 that the exceptional variety is the toric variety associated to $\mathrm{Star}(\tau)$, which lives in the lattice $\widetilde{N_\tau}$) so begin by picking an element $z \in \mathbb{Z}$. This corresponds to a "line" in $\mathbb{Z}^2$, given by (z,t) for $t \in \mathbb{Z}$. Then under $\widetilde{\pi}$, this line again maps to z. Applying the appropriate functors, we see that f is an isomorphism. Now since KH satisfies cdh descent, we get the following long exact sequence:

$$\cdots \longrightarrow \mathrm{KH}_n(\mathbb{P}(1,1,2)) \longrightarrow \mathrm{KH}_n(\mathcal{H}_2) \oplus \mathrm{KH}_n(k) \xrightarrow{\alpha_n}$$

$$\mathrm{KH}_n(\mathbb{P}^1) \longrightarrow \mathrm{KH}_{n-1}(\mathbb{P}(1,1,2)) \longrightarrow \cdots$$

Now we want to analyze the morphism α_n. Recalling the construction of the Mayer-Vietoris long exact sequence, notice that α_n is the difference of the morphism

$i_n^* : \mathrm{KH}_n(\mathcal{H}_2) \longrightarrow \mathrm{KH}_n(\mathbb{P}^1)$ and the morphism $j_n^* : \mathrm{KH}_n(k) \longrightarrow \mathrm{KH}_n(\mathbb{P}^1)$. Our goal is to show that α_n is surjective; obviously the difference by j_n^* would not affect this provided that i_n^* is surjective. So it is enough to show that i_n^* is surjective. But $f = \pi \circ i$ was shown to be an isomorphism, so for every n, the composition $i_n^* \circ \pi_n^*$ is an isomorphism, and i_n^* is indeed surjective. By exactness, this means that the group $\mathrm{KH}_n(\mathbb{P}(1,1,2))$ is a subgroup of $\mathrm{KH}_n(\mathcal{H}_2) \oplus \mathrm{KH}_n(k)$ for every n; since the latter of these groups is 0 for $n \leq -1$, $\mathrm{KH}_n(\mathbb{P}(1,1,2)) = 0$ for $n \leq -1$ as well. Finally, in the case $n = 0$, we get the short exact sequence

$$0 \longrightarrow \mathrm{KH}_0(\mathbb{P}(1,1,2)) \longrightarrow \mathrm{KH}_0(\mathcal{H}_2) \oplus \mathrm{KH}_0(k) \xrightarrow{\alpha_0} \mathrm{KH}_0(\mathbb{P}^1) \longrightarrow 0$$

Now $\mathrm{KH}_0(k) = \mathbb{Z}$, $\mathrm{KH}_0(\mathbb{P}^1) = \mathbb{Z}^2$ and, by the projective bundle theorem (see [TT]), $\mathrm{KH}_0(\mathcal{H}_2) = \mathbb{Z}^4$. Therefore this short exact sequence reduces to

$$0 \longrightarrow \mathrm{KH}_0(\mathbb{P}(1,1,2)) \longrightarrow \mathbb{Z}^5 \longrightarrow \mathbb{Z}^2 \longrightarrow 0$$

which obviously splits. Therefore, we get $\mathrm{KH}_0(\mathbb{P}(1,1,2)) = \mathbb{Z}^3$.

Note that the choice of weight 2 did not really determine the answer. Had we looked at the weighted projective space $\mathbb{P}(1,1,a)$ for any $a > 1$, the above steps will still work (although will of course yield a different Hirzebruch surface), and still yield the same answer. We state this fact as the following theorem.

Theorem 4.1. *Consider the weighted projective space $\mathbb{P}(1,1,a)$, with $a \geq 2$. Then*

$$\mathrm{KH}_n(\mathbb{P}(1,1,a)) = 0 \tag{4.1}$$

for $n \leq -1$ and

$$\mathrm{KH}_0(\mathbb{P}(1,1,a)) = \mathbb{Z}^3. \tag{4.2}$$

Proof. The steps are almost word-for-word the same as the $\mathbb{P}(1,1,2)$ case. The only singular cone is $\langle(1,0),(-1,-a)\rangle$, and after refining our fan by adding the cone generated by $(0,-1)$, we get the Hirzebruch surface $\mathcal{H}_a$, which is the $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ associated to the sheaf $\mathcal{O}(0)+\mathcal{O}(-a)$. This gives us a blow-up square

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[r, "i"] \arrow[d] & \mathcal{H}_a \arrow[d] \\
\{*\} \arrow[r] & \mathbb{P}(1,1,a)
\end{tikzcd}$$

and because KH satisfies cdh descent, it gives rise to a long exact sequence

$$\cdots \longrightarrow \mathrm{KH}_n(\mathbb{P}(1,1,a)) \longrightarrow \mathrm{KH}_n(\mathcal{H}_a)\oplus\mathrm{KH}_n(k) \xrightarrow{\alpha_n}$$

$$\mathrm{KH}_n(\mathbb{P}^1) \longrightarrow \mathrm{KH}_{n-1}(\mathbb{P}(1,1,a)) \longrightarrow \cdots$$

By the exact same argument as in the $\mathbb{P}(1,1,2)$ case, the morphism α_n is surjective for every n and our long exact sequence splits into short exact sequences of the form

$$0 \longrightarrow \mathrm{KH}_n(\mathbb{P}(1,1,a)) \longrightarrow \mathrm{KH}_n(\mathcal{H}_a)\oplus\mathrm{KH}_n(k) \xrightarrow{\alpha_n} \mathrm{KH}_n(\mathbb{P}^1) \longrightarrow 0$$

Since $\mathrm{KH}_n(\mathcal{H}_a)\oplus\mathrm{KH}_n(k)=0$ for $n\leq -1$, $\mathrm{KH}_n(\mathbb{P}(1,1,a))=0$ for $n\leq -1$ as well. For the case $n=0$, we have

$$0 \longrightarrow \mathrm{KH}_0(\mathbb{P}(1,1,a)) \longrightarrow \mathrm{KH}_0(\mathcal{H}_a)\oplus\mathrm{KH}_0(k) \xrightarrow{\alpha_0} \mathrm{KH}_0(\mathbb{P}^1) \longrightarrow 0$$

Now $\mathrm{KH}_0(k)=\mathbb{Z}$, $\mathrm{KH}_0(\mathbb{P}^1)=\mathbb{Z}^2$ and, by the projective bundle theorem (see [TT, Theorem 4.1]), $\mathrm{KH}_0(\mathcal{H}_2)=\mathbb{Z}^4$. Therefore this short exact sequence reduces to

$$0 \longrightarrow \mathrm{KH}_0(\mathbb{P}(1,1,a)) \longrightarrow \mathbb{Z}^5 \longrightarrow \mathbb{Z}^2 \longrightarrow 0$$

which obviously splits. Therefore, we get $\mathrm{KH}_0(\mathbb{P}(1,1,a))=\mathbb{Z}^3$, as desired.

$\square$

However, this method is not effective as a general approach to calculating the KH-theory of even just weighted projective spaces (as opposed to all complete simplicial toric varieties). The problem is that even in dimension 2, the number of steps needed to resolve the singularities can be quite large, and therefore we can be confronted with uncontrollable exceptional varieties. Consider, for instance, the weighted projective space $\mathbb{P}(1,5,7)$. To completely resolve all singularities, we must add in five additional one-dimensional cones. The exceptional variety will, in this case, be a disjoint union of a chain of three copies of $\mathbb{P}^1$ and a chain of two copies of $\mathbb{P}^1$ (see [Ful], page 47).

However, this approach does suggest that if two complete simplicial toric varieties have the same simplicial structure, such as $\mathbb{P}^2$ and $\mathbb{P}(1,1,2)$ in this case, then there may be a relationship between their respective KH-theories. This motivates the approach we use, beginning in the next section.

4.2 The Simplicial Structure and Simplicial Scheme Structure Associated to a Complete Simplicial Toric Variety

We begin with calculating the KH-theory of U_σ for any cone σ. The intuition we use for doing this calculation goes all the way back to [Wei1]; the properties of KH that provide this intuition are given as Theorem 3.84. We begin by examining the KH-theory of U_σ in the case that σ is a maximal cone; that is, in the case that the dimension of the subspace generated by σ equals the dimension of $N_{\mathbb{R}}$.

Proposition 4.2. *If σ is a maximal cone, then*

$$\mathrm{KH}_n(U_\sigma) \cong \mathrm{KH}_n(k) \cong \mathrm{K}_n(k). \tag{4.3}$$

Proof. Note that if σ is maximal then the dual $\check{\sigma}$ is also strongly convex. But then $\check{\sigma} \cap M$ does not contain any lattice points along any linear subspace of M (it only contains points "on one side" of a linear subspace, but not both). That means that the ring $k[\check{\sigma} \cap M]$ has no non-trivial units, and is therefore an $\mathbb{N}$-graded polynomial ring. Then using either (a) or (d) of Theorem 3.84 gives the result.

$\square$

Now we recall from Proposition 2.2 that, given any p-dimensional cone σ in N, where $\dim N_{\mathbb{R}} = m$, we get $U_\sigma \cong U_{\sigma'} \times T$, where T is a torus of rank $m-p$. This brings us to our next proposition.

Proposition 4.3. *Let σ be any p-dimensional cone. Then*

$$\mathrm{KH}_n(U_\sigma) \cong \mathrm{KH}_n(\mathbb{G}_m^{m-p}) \cong \mathrm{K}_n(\mathbb{G}_m^{m-p}). \tag{4.4}$$

In other words, the KH *groups of an open set corresponding to a cone are just the* K *groups of its associated torus part.*

Proof. Given the splitting $N \cong N_\sigma \oplus N''$ of Proposition 2.2, taking duals gives us $M \cong M' \oplus M''$. Then $\sigma^\vee \cap M = (\sigma'^\vee \cap M') \oplus M''$, and

$$k[\sigma^\vee \cap M] = k[(\sigma'^\vee \cap M') \oplus M'']. \tag{4.5}$$

At this point in Proposition 2.2 we applied Spec. However, this time we instead apply KH to both sides to get:

$$\mathrm{KH}_n(k[\sigma^\vee \cap M]) \cong \mathrm{KH}_n(k[(\sigma'^\vee \cap M') \oplus M'']). \tag{4.6}$$

Now since σ' is maximal in N_σ, $\sigma'^\vee \cap M'$ is $\mathbb{N}$-graded. By a similar argument to the one given in Proposition 4.2, $k[(\sigma'^\vee \cap M') \oplus M'']$ is $\mathbb{N}$-graded in the variables given by $\sigma'^\vee \cap M'$. So by part (a) of Theorem 3.84,

$$\mathrm{KH}_n(k[(\sigma'^\vee \cap M') \oplus M'']) \cong \mathrm{KH}_n(k[M'']) = \mathrm{KH}_n(\mathbb{G}_m^{m-p}) \cong \mathrm{K}_n(\mathbb{G}_m^{m-p}) \tag{4.7}$$

where the last isomorphism is because $\mathbb{G}_m^{m-p}$ is smooth. This is what we wanted to show.

$\square$

Remark 4.4. One can also show that, as spectra,

$$\mathrm{KH}(U_\sigma) \cong \mathrm{KH}(\mathbb{G}_m^{m-p}) \cong \mathrm{K}(\mathbb{G}_m^{m-p}) \tag{4.8}$$

where the second isomorphism is because $\mathbb{G}_m^{m-p}$ is smooth. This is done in the proof of Proposition 5.6 of [CHWW], which appears as Theorem 3.94 in this paper.

It is the result of Proposition 4.3 that provides the intuition for the approach we use. Our goal will be to build some relationship between a complete simplicial toric variety and a scheme that is built from all the torus pieces of open sets of that toric variety, and then use this scheme of torus pieces to develop a relationship between the KH-theories of two complete simplicial toric varieties with the same simplicial structure.

As we saw above, the KH-theory of an open set associated to a cone depended only on the torus piece. This leads us to consider ways in which we might use the simplicial structure of a toric variety to determine its KH-theory. We want to see, in particular, the relationship between two complete simplicial toric varieties with the same simplicial structure. In order to do this, we must first make our definition of simplicial structure clear.

Definition 4.5. *Let X be a complete simplicial toric variety, and $\Delta_X(1)$ be the set of 1-dimensional rays in the fan of X. Let $\mathrm{S}(\Delta_X)$ denote the set of all sets of rays in $\Delta_X(1)$ that form a cone in the fan Δ_X. Since X is assumed simplicial, any subset of a set of rays forming a cone also forms a cone; therefore, $\mathrm{S}(\Delta_X)$ forms a simplicial complex. We define the "simplicial structure" of X to be the simplicial complex $\mathrm{S}(\Delta_X)$.*

By itself, this definition isn't very helpful. However, it now allows us to discuss what it means for two complete simplicial toric varieties to have the same simplicial structure.

Definition 4.6. *Consider two simplicial fans Δ and Δ', and their corresponding simplicial complexes $\mathrm{S}(\Delta)$ and $\mathrm{S}(\Delta')$. We say that the set map $\varphi : \mathrm{S}(\Delta) \longrightarrow \mathrm{S}(\Delta')$ is an isomorphism of simplicial complexes if it is a bijection as a set map and if $A \subset B$ in $\mathrm{S}(\Delta)$ then $\varphi(A) \subset \varphi(B)$ in $\mathrm{S}(\Delta')$. Let X and Y be two complete simplicial toric varieties; then their respective fans*

Δ_X and Δ_Y are simplicial fans. We say that the toric varieties X and Y have the same simplicial structure if their simplicial complexes $\mathrm{S}(\Delta_X)$ *and* $\mathrm{S}(\Delta_Y)$ *are isomorphic.*

Remark 4.7. Notice that, in particular, every element of $\Delta_X(1)$ is a cone in Δ_X and similarly for elements in $\Delta_Y(1)$. So an isomorphism of simplicial complexes $\varphi : \mathrm{S}(\Delta_X) \longrightarrow \mathrm{S}(\Delta_Y)$ induces a bijection $\phi : \Delta_X(1) \longrightarrow \Delta_Y(1)$. This statement has a partial converse that we can use. If X and Y are complete simplicial toric varieties, and if we have a bijection $\phi : \Delta_X(1) \longrightarrow \Delta_Y(1)$ that preserves adjacency relations (i.e. if x_i and x_j are adjacent rays, then $\phi(x_i)$ and $\phi(x_j)$ are adjacent rays as well), we can use this to build an isomorphism of simplicial complexes. To see this, note that every element of $\mathrm{S}(\Delta_X)$ and $\mathrm{S}(\Delta_Y)$ is just a set of elements in $\Delta_X(1)$ and $\Delta_Y(1)$, respectively. So we can map that set of elements in $\Delta_X(1)$ to the corresponding set of elements in $\Delta_Y(1)$ just by applying ϕ to every element in that set. The adjacency preserving condition will then ensure that the set of elements in $\Delta_Y(1)$ obtained by applying ϕ to every element in the set coming from $\Delta_X(1)$ actually still generates a cone. Then the two conditions for φ to be an isomorphism of simplicial complexes are trivially satisfied. So to define an isomorphism of simplicial complexes for two complete simplicial toric varieties, it is enough to define a bijection that preserves adjacency relations on the respective sets of 1-dimensional rays.

Caution 4.8. Both the completeness condition and the adjacency preserving condition of Remark 4.7 are necessary. Indeed, if one of the toric varieties is not complete, then a bijection $\phi : \Delta_X(1) \longrightarrow \Delta_Y(1)$ is not enough to construct an isomorphism of simplicial complexes $\varphi : \mathrm{S}(\Delta_X) \longrightarrow \mathrm{S}(\Delta_Y)$. As a counterexample, let $X = \mathbb{P}^2$ and let $Y = \mathbb{P}^2 \setminus \{[1:0:0]\}$. We saw in Section 2.2 that the fan Δ_X is given by all proper subsets of $\{x_0, x_1, x_2\}$ where x_i is the image of the basis element e_i under the surjection

$$N = \mathbb{Z}^3 / \langle e_0 + e_1 + e_2 \rangle. \tag{4.9}$$

This gives us the simplicial complex

$$\mathrm{S}(\Delta_X) = \{\emptyset, \{x_0\}, \{x_1\}, \{x_2\}, \{x_0, x_1\}, \{x_0, x_2\}, \{x_1, x_2\}\}. \tag{4.10}$$

Similarly, one can show (through the orbit-cone correspondence for toric varieties; see [Ful] and [Cox]) that the fan Δ_Y is the same as Δ_X except that the cone generated by x_1 and x_2 is not present; indeed, the distinguished point of the cone generated by x_1 and x_2 is $\{[1:0:0]\}$ so to delete the point is the same as to delete the corresponding cone. This gives us the simplicial complex

$$\mathrm{S}(\Delta_Y) = \{\emptyset, \{x_0\}, \{x_1\}, \{x_2\}, \{x_0, x_1\}, \{x_0, x_2\}\} \tag{4.11}$$

where again x_i is the image of the basis element e_i under the surjection

$$N = \mathbb{Z}^3 / \langle e_0 + e_1 + e_2 \rangle. \tag{4.12}$$

Observe that there is an obvious bijection between $\Delta_X(1)$ and $\Delta_Y(1)$, but that $\mathrm{S}(\Delta_X)$ and $\mathrm{S}(\Delta_Y)$ cannot possibly be isomorphic as simplicial complexes. So the completeness condition is essential to the partial converse.

Similarly, suppose we have the toric variety $\mathbb{P}^1 \times \mathbb{P}^1$ which is given by the fan in $\mathbb{Z}^2$ generated by the rays e_1, e_2, $-e_1$, and $-e_2$. Suppose we defined the bijection

$$\begin{aligned} e_1 &\mapsto e_2 \\ e_2 &\mapsto e_1 \\ -e_1 &\mapsto -e_1 \\ -e_2 &\mapsto -e_2 \end{aligned} \tag{4.13}$$

This will obviously not give rise to an automorphism of simplicial complexes as we described in Remark 4.7 because the cone $\langle -e_1, e_2 \rangle$ would not be mapped to a cone. So the adjaceny preserving condition is also essential to the partial converse.

Now that we have established our basic definitions, the remainder of Section 4 will be dedicated to proving and applying the following theorem, which is our main technical result.

Theorem 4.9. *Let X and Y be two complete simplicial toric varieties over k with the same* *simplicial structure; i.e.* $\varphi : \mathrm{S}(\Delta_X) \longrightarrow \mathrm{S}(\Delta_Y)$ *is an isomorphism of simplicial complexes, where* Δ_X *is the fan for* X *and* Δ_Y *is the fan for* Y*. Let* Δ_X *live in the lattice* N^X *and* Δ_Y *live in the lattice* N^Y*. Suppose we have a lattice morphism* $F : N^X \longrightarrow N^Y$ *which is injective with finite cokernel such that the restriction maps* $F|_{N^X_\sigma} : N^X_\sigma \longrightarrow N^Y_{\varphi(\sigma)}$ *are also injective with finite cokernel for any cone* $\sigma \in \Delta_X$*. Suppose further that the characteristic of* k *does not divide* $|\operatorname{coker}(F)|$*. Then* $\mathrm{KH}(X) \otimes \mathbb{Q}$ *and* $\mathrm{KH}(Y) \otimes \mathbb{Q}$ *are weakly equivalent as spectra; in particular,* $\mathrm{KH}_n(X) \otimes \mathbb{Q} \cong \mathrm{KH}_n(Y) \otimes \mathbb{Q}$ *for all* n*.*

There are several stages that go into the proof of Theorem 4.9. We begin by focusing on a single complete simplicial toric variety and, in the spirit of Proposition 4.3, seek to relate its KH theory to the KH theory of certain torus pieces that are related to X. Then, once such a relationship is established, we look to use the lattice conditions to build a map between these tori that let us (after considering the above relationship with the complete simplicial toric variety X) derive the relationship between the KH theories of X and Y.

To accomplish this, we first need to construct simplicial scheme structures associated to X and Y. Then we need to find a way to relate these associated simplicial scheme structures (as opposed to the structures of X and Y as complete simplicial toric varieties). For a discussion of general simplicial and cosimplicial objects over a given category, we refer the reader to Section 3.3. We can now prove our first important theorem, which allows us to apply the material of Section 3.3 to complete simplicial toric varieties.

Construction 4.10. *Let* X *be a complete simplicial toric variety. Then* X *gives rise to a simplicial scheme, which we call* $\mathcal{U}_X$*. This is a standard construction, but we make it explicit for use in Section 4.3. To construct the simplicial scheme structure, we need to give schemes* $(\mathcal{U}_X)_n$ *(for every* n*) and we need to give the face and degeneracy maps.*

Define $\mathcal{U}_X$ *to be the open cover of* X *by open sets corresponding to maximal cones (the construction using this cover is why we denote our simplicial scheme by the same notation). We define* $(\mathcal{U}_X)_n$ *to be the following:*

$$(\mathcal{U}_X)_n = \coprod (U_{\sigma_0} \times \cdots \times U_{\sigma_n}) \tag{4.14}$$

where this coproduct is taken over all elements of $\mathcal{U}_X$ *(equivalently, over all maximal cones). Notice that, in particular, the order of elements matters (so we can repeat elements in* $(\mathcal{U}_X)_n$*), and that the* U_{σ_j}*'s need not be distinct. Since* $U_{\sigma_0} \times \cdots \times U_{\sigma_n}$ *is the fiber product of known schemes, it is a scheme also; however, for practical use it's easier to view this as an intersection and in the remainder of the paper we switch seamlessly between these two viewpoints.*

Now that we have our $(\mathcal{U}_X)_n$*'s, we need to determine our face and degeneracy maps. Define* $d_j : (\mathcal{U}_X)_n \longrightarrow (\mathcal{U}_X)_{n-1}$ *to be the map where, in each term of the coproduct, you delete the* j^{th} *term. In other words, we have*

$$d_j : U_{\sigma_0} \times \cdots \times U_{\sigma_{j-1}} \times U_{\sigma_j} \times U_{\sigma_{j+1}} \times \cdots \times U_{\sigma_n} \mapsto U_{\sigma_0} \times \cdots \times U_{\sigma_{j-1}} \times U_{\sigma_{j+1}} \times \cdots \times U_{\sigma_n}$$

for each term in the coproduct. Note that we begin our "counting of terms" with 0*, not* 1 *(so the* 0^{th} *term is* U_{σ_0} *and so on). Observe that in each term of the coproduct, this is just an inclusion map. Similarly, we define our degeneracy map* $s_j : (\mathcal{U}_X)_n \longrightarrow (\mathcal{U}_X)_{n+1}$ *by repeating the* j^{th} *term in each term of the coproduct. In other words, we have*

$$s_j : U_{\sigma_0} \times \cdots \times U_{\sigma_j} \times U_{\sigma_{j+1}} \times \cdots \times U_{\sigma_n} \mapsto U_{\sigma_0} \times \cdots \times U_{\sigma_j} \times U_{\sigma_j} \times U_{\sigma_{j+1}} \times \cdots \times U_{\sigma_n}$$

for each term in the coproduct. Again, we begin our counting from 0*.*

From here, one can easily verify the usual simplicial identities by examining them on each term. In the first case, we examine $d_k d_j$ *for* $k < j$*. This deletes the term* U_{σ_j} *and then deletes the term* U_{σ_k}*. Looking at* $d_{j-1}d_k$*, we see that this first deletes* U_{σ_k}*. Since* $k < j$*, this now makes* U_{σ_j} *the* $(j-1)^{th}$ *term as the count for every term past the* k^{th} *is decreased by* 1*. But then* d_{j-1} *will delete* U_{σ_j} *since it is now the* $(j-1)^{th}$ *term. So* $d_k d_j$ *and* $d_{j-1}d_k$ *must be equal (for* $k < j$*).*

Next, we examine the behavior of $d_k s_j$*. If* $k < j$*, this repeats the* j^{th} *term* U_{σ_j} *and then deletes the* k^{th} *term* U_{σ_k}*, whose count is unaffected by adding a term ahead of it. If instead we look at* $s_{j-1}d_k$*, we first delete* U_{σ_k}*, which now makes* U_{σ_j} *the* $(j-1)^{th}$ *term. Applying* s_{j-1} *then repeats* U_{σ_j} *and we get the same thing. So* $d_k s_j = s_{j-1}d_k$ *for* $k < j$*. If* $k = j$ *or*

$k = j+1$, then this map is the identity map since applying s_j now makes both the j^{th} and $(j+1)^{th}$ terms U_{σ_j}; deleting either of these will give us back the term we started with. If $k > j+1$, then s_j repeats U_{σ_j} and d_k deletes what was originally the $(k-1)^{th}$ term, which is $U_{\sigma_{k-1}}$. This is the same as if we first applied d_{k-1} and then applied s_j since the j^{th} term didn't change by applying d_{k-1} $(k-1 > j)$. So $d_k s_j = s_j d_{k-1}$.

Finally, we look at $s_k s_j$ for $k > j$. After applying s_j, the k^{th} term in our result is what was the $(k-1)^{th}$ term from our original expression, which is $U_{\sigma_{k-1}}$. So this map repeats $U_{\sigma_{k-1}}$ and U_{σ_j}. This is the same as if we first applied s_{k-1}, repeating $U_{\sigma_{k-1}}$, and then since $k-1 \geq j$, this doesn't affect the count for j and applying s_j repeats U_{σ_j}. So $s_k s_j = s_j s_{k-1}$, verifying the usual simplicial identities.

For the remainder of this paper, when we say "simplicial scheme associated to X", we are referring to the simplicial scheme $\mathcal{U}_X$.

Remark 4.11. In Construction 4.10, we described the face and degeneracy maps geometrically. However, as our cover is given by open sets associated to cones, we can just as easily construct our face and degeneracy maps on the level of lattices, and then apply the appropriate functors to arrive at the ordinary face and degeneracy maps. For the face maps, recall that

$$d_j : U_{\sigma_0} \times \cdots \times U_{\sigma_{j-1}} \times U_{\sigma_j} \times U_{\sigma_{j+1}} \times \cdots \times U_{\sigma_n} \mapsto U_{\sigma_0} \times \cdots \times U_{\sigma_{j-1}} \times U_{\sigma_{j+1}} \times \cdots \times U_{\sigma_n}.$$

Letting $\tau = \sigma_0 \cap \cdots \cap \sigma_n$ and letting $\tau_j = \sigma_0 \cap \cdots \cap \widehat{\sigma_j} \cap \cdots \cap \sigma_n$, this map becomes

$$d_j : U_\tau \longrightarrow U_{\tau_j} \tag{4.15}$$

by inclusion. Notice that, since $\tau \prec \tau_j$, this face map can be obtained by taking $N^X \longrightarrow N^X$ to be the identity and taking $\tau \longrightarrow \tau_j$ by inclusion. For the degeneracy maps, if we again let $\tau = \sigma_0 \cap \cdots \cap \sigma_n$, then s_j becomes

$$s_j : U_\tau \longrightarrow U_\tau \tag{4.16}$$

by the identity, which can easily be obtained by taking $N^X \longrightarrow N^X$ to be the identity and taking $\tau \longrightarrow \tau$ to be the identity as well. Note that we can see very easily that the maps d_j and s_j above satisfy the usual simplicial relations in this form as well; we simply repeat the logic of the corresponding proofs from Construction 4.10. While the presentation in Construction 4.10 is more enlightening geometrically, the presentation from the perspective of lattices will prove to be more useful as we progress.

Corollary 4.12. *If X is a complete simplicial toric variety, then $\mathrm{KH}(\mathcal{U}_X)$ and $\mathrm{KH}(\mathcal{U}_X) \otimes \mathbb{Q}$ are cosimplicial objects over the category of spectra.*

Proof. If X is a complete simplicial toric variety then by Construction 4.10, $\mathcal{U}_X$ is a simplicial scheme. Since the functors KH and $\mathrm{KH}(-) \otimes \mathbb{Q}$ are contravariant functors from the category of schemes to the category of spectra, we then can immediately conclude that $\mathrm{KH}(\mathcal{U}_X)$ and $\mathrm{KH}(\mathcal{U}_X) \otimes \mathbb{Q}$ are cosimplicial objects over the category of spectra.

□

We conclude this section by making an observation about the simplicial scheme structures associated to two complete simplicial toric varieties X and Y whose simplicial structures are isomorphic in the sense of Definition 4.6.

Theorem 4.13. *Let X and Y be two complete simplicial toric varieties with isomorphic simplicial structures. Let $\mathcal{U}_X$ and $\mathcal{U}_Y$ be the simplicial scheme structures associated to X and Y, respectively. Then we have the following two facts:*

(a) For all $n \geq 0$, $(\mathcal{U}_X)_n$ and $(\mathcal{U}_Y)_n$ have the same number of terms in the coproduct, where we view $(\mathcal{U}_X)_n$ and $(\mathcal{U}_Y)_n$ as given in Theorem 4.10.

(b) For any $U_{\sigma_0} \cap \cdots \cap U_{\sigma_n}$ in $(\mathcal{U}_X)_n$ and $U_{\tau_0} \cap \cdots \cap U_{\tau_n}$ in $(\mathcal{U}_Y)_n$ corresponding under the isomorphism of simplicial structures, the torus parts for each of these intersections have the same rank.

Proof. For the first part, recall that $(\mathcal{U}_X)_n = \coprod(U_{\sigma_0} \times \cdots \times U_{\sigma_n})$ where each of the σ_i's are maximal cones in Δ_X. Similarly, we have $(\mathcal{U}_Y)_n = \coprod(U_{\tau_0} \times \cdots \times U_{\tau_n})$ where the τ_i's are maximal cones in Δ_Y. Since there is an isomorphism

$$\varphi : \mathrm{S}(\Delta_X) \longrightarrow \mathrm{S}(\Delta_Y), \tag{4.17}$$

and since $\mathrm{S}(\Delta_X)$ has as a subset all rays generating any maximal cone (and similarly for $\mathrm{S}(\Delta_Y)$), then Δ_X and Δ_Y each have the same number of maximal cones; otherwise, there would be rays in one that form a maximal cone but whose corresponding elements in the other does not. If this happens, then the rays (in the case they don't form a maximal cone) would generate a non-maximal cone, which would be the face of some larger dimensional cone (just add more rays). But then the set of rays generating this larger dimensional cone could not be in the image of φ (or possibly φ^{-1}, depending on whether non-maximality occurs in Δ_X or Δ_Y). This contradicts our assumption that X and Y have the same simplicial structure, so Δ_X and Δ_Y each have the same number of maximal cones. But since Δ_X and Δ_Y each have the same number of maximal cones, they must generate the same number of fiber products of the same length. This gives us (a).

For the second part, it's easier to use the intersection viewpoint

$$(\mathcal{U}_X)_n = \coprod (U_{\sigma_0} \cap \cdots \cap U_{\sigma_n}), \tag{4.18}$$

and similar for $(\mathcal{U}_Y)_n$. Recall from [Ful] that $U_{\sigma_1} \cap U_{\sigma_2} = U_\tau$, where $\tau = \sigma_1 \cap \sigma_2$.

Recall that, in a simplicial fan, if a cone σ is generated by k distinct rays (in a lattice of dimension n), then the torus part will have rank $n - k$. Let $\tau_i = \varphi(\sigma_i)$, where by $\varphi(\sigma_i)$ is the maximal cone in Δ_Y generated by the cones $\varphi(\rho_j)$, where the ρ_j's span σ_i. Since φ induces a bijection between the respective sets of rays (see Remark 4.7), τ_i and σ_i must be generated by the same number of rays. So U_{σ_i} and U_{τ_i} have torus parts of the same rank.

Now notice that $\varphi(\sigma_i \cap \sigma_j) = \tau_i \cap \tau_j$; therefore, they are both generated by the same number of rays and the open sets $U_{\sigma_i} \cap U_{\sigma_j} = U_{\sigma_i \cap \sigma_j}$ and $U_{\tau_i} \cap U_{\tau_j} = U_{\tau_i \cap \tau_j}$ have torus parts of the same rank. Repeating inductively gives us part (b).

$\square$

4.3 The Construction of the Simplicial Scheme BOT_X

In Section 4.2, we constructed the simplicial structure of a complete simplicial toric variety X (Definition 4.5), a simplicial scheme $\mathcal{U}_X$ associated to a complete simplicial toric variety X (Construction 4.10), and used it to construct cosimplicial spectra $\mathrm{KH}(\mathcal{U}_X)$ and $\mathrm{KH}(\mathcal{U}_X) \otimes \mathbb{Q}$ (Corollary 4.12). Now our goal is to relate the KH theory of a complete simplicial toric variety to the KH theory of its torus pieces, as mentioned in the outline following Theorem 4.9, by attempting to extend the viewpoint of Section 4.2. To do this, we must find some way of relating torus pieces (which are direct factors of open sets associated to cones) and the whole variety. Fortunately, our work has provided us a way to do this. Indeed, notice that our definition of the simplicial scheme structure associated to a complete simplicial toric variety is given completely in terms of open sets; thus, it will allow us to relate these two concepts. In what follows, we can present the material in one of two ways; we can either present everything purely in terms of the underlying lattices and then just apply the appropriate functors, or we can use a more geometric approach to describe the maps in question. Each presentation has its advantages and disadvantages: the lattice construction is very concrete and is therefore easy to understand and adapt to other situations, but its definition loses the intuition behind its construction; on the other hand, the geometric approach makes very clear the intuition behind its construction, but it is far more difficult to prove that this construction is well-defined and it is not very easy to adapt to the other situations that we encounter later in this paper. Given that each perspective offers a worthwhile opportunity for understanding, both constructions are presented below. We begin with some new definitions.

Definition 4.14. *Let X be a complete simplicial toric variety with associated simplicial scheme $\mathcal{U}_X$ as shown in Construction 4.10. We define two lattice maps, which we call $\widetilde{d}_j$ and $\widetilde{s}_j$, by the following construction. Let $\tau = \sigma_0 \cap \cdots \cap \sigma_n$ and $\tau_j = \sigma_0 \cap \cdots \cap \widehat{\sigma_j} \cap \cdots \cap \sigma_n$. Then construct*

$$\widetilde{d}_j : \widetilde{N}^X_\tau \longrightarrow \widetilde{N}^X_{\tau_j} \tag{4.19}$$

by first lifting $\widetilde{N}^X_\tau$ up to N^X, then mapping N^X to itself via the identity (with $\tau \longrightarrow \tau_j$ via *inclusion), and finally taking canonical projection onto $\widetilde{N}^X_{\tau_j}$. Similarly, construct*

$$\widetilde{s_j} : \widetilde{N}^X_\tau \longrightarrow \widetilde{N}^X_\tau \tag{4.20}$$

by first lifting $\widetilde{N}^X_\tau$ up to N^X, then mapping N^X to itself via the identity (with $\tau \longrightarrow \tau$ via the identity), and finally taking canonical projection onto $\widetilde{N}^X_\tau$. Observe that, with this construction, $\widetilde{s_j}$ is just the identity map.

Definition 4.15. *Let X be a complete simplicial toric variety with associated simplicial scheme $\mathcal{U}_X$ as shown in Construction 4.10. We define a new simplicial scheme, which we call BOT_X, by the following properties:*

- *(a) We define $(\mathrm{BOT}_X)_n = \coprod T_{\alpha(\sigma_0,\cdots,\sigma_n)}$, where $T_{\alpha(\sigma_0,\cdots,\sigma_n)}$ is the associated torus piece for the open set $U_{\sigma_0} \cap \cdots \cap U_{\sigma_n}$ and $\alpha(\sigma_0, \cdots, \sigma_n)$ is its rank. As before, the σ_i's are all maximal cones.*
- *(b) We define the face and degeneracy maps, denoted $d_j^{\mathrm{BOT}_X}$ and $s_j^{\mathrm{BOT}_X}$ respectively, to component-wise be the morphisms of toric varieties that are induced by the lattice maps $\widetilde{d_j}$ and $\widetilde{s_j}$ of Definition 4.14.*

Remark 4.16. We observe that, in the definition of $d_j^{\mathrm{BOT}_X}$ and $s_j^{\mathrm{BOT}_X}$, what we are actually doing geometrically is (component-wise) lifting a torus piece back to its open set, applying the appropriate face or degeneracy map under the simplicial structure of $\mathcal{U}_X$ to obtain some new open set, and then projecting back down onto the torus part of that new open set. So in other words, we could define our face and degeneracy maps purely geometrically in the following way: we map a given torus T into the associated open set $U_\tau \cong U_{\tau'} \times T$ by mapping $T \mapsto x^* \times T$ (for some point x^* in $U_{\tau'}$), apply the face or degeneracy map that comes from the simplicial scheme structure of X, and then project this new open set back onto its associated torus part. In symbols, component-wise we have

$$d_j^{\mathrm{BOT}_X} = (proj_2) \circ d_j \circ (x^* \times T) \tag{4.21}$$

and

$$s_j^{\mathrm{BOT}_X} = (proj_2) \circ s_j \circ (x^* \times T) \tag{4.22}$$

where $proj_2$ denotes projection onto the second component (which is the torus part).

We want to show that BOT_X, as defined above, is actually a simplicial scheme, as this is not clear from its definition. One possible point of concern is that in the definitions of $d_j^{\mathrm{BOT}_X}$ and $s_j^{\mathrm{BOT}_X}$, we make a choice (namely in lifting $\widetilde{N}_\tau^X$ up to N^X); therefore, they might not be well-defined. So before anything else, we need to make sure that the maps $\widetilde{d_j}$ and $\widetilde{s_j}$ of Definition 4.14 are well-defined.

Lemma 4.17. *The maps $\widetilde{d_j}$ and $\widetilde{s_j}$ of Definition 4.14 are well-defined maps of lattices; consequently, the maps $d_j^{\mathrm{BOT}_X}$ and $s_j^{\mathrm{BOT}_X}$ are well-defined.*

Proof. Suppose that $[x] = [y]$ in $\widetilde{N}_\tau^X$. We want to show that $\widetilde{d_j}([x]) = \widetilde{d_j}([y])$ and that $\widetilde{s_j}([x]) = \widetilde{s_j}([y])$. We begin with $\widetilde{d_j}$. Recall that d_j is given by mapping $i : N^X \longrightarrow N^X$ via the identity and sending $\tau \longrightarrow \tau_j$ via inclusion. This second inclusion induces an inclusion $i : N_\tau^X \longrightarrow N_{\tau_j}^X$. Since $[x] = [y]$ in $\widetilde{N}_\tau^X$, $x - y \in N_\tau^X$. By the inclusion $i : N_\tau^X \longrightarrow N_{\tau_j}^X$, we have $i(x) - i(y) \in N_{\tau_j}^X$; applying canonical surjection gives us $\widetilde{d_j}([x]) = \widetilde{d_j}([y])$.

Similarly, recall that s_j is given by mapping $i : N^X \longrightarrow N^X$ via the identity and sending $\tau \longrightarrow \tau$ via identity also. This implies that $i : N_\tau^X \longrightarrow N_\tau^X$ is the identity map as well. Since $[x] = [y]$ in $\widetilde{N}_\tau^X$, $x - y \in N_\tau^X$. By the inclusion $i : N_\tau^X \longrightarrow N_\tau^X$, we have $i(x) - i(y) \in N_\tau^X$; applying canonical surjection gives us $\widetilde{s_j}([x]) = \widetilde{s_j}([y])$. So $\widetilde{d_j}$ and $\widetilde{s_j}$ are well-defined; applying all the necessary functors, we see that $d_j^{\mathrm{BOT}_X}$ and $s_j^{\mathrm{BOT}_X}$ are well-defined as well.

□

There are two main advantages to using lattice techniques for the proofs in this section. The first is that later in the paper, these techniques will be more easily adapted to our proofs. The second is highlighted in the above proof of Lemma 4.17. From the lattice construction, it is reasonably easy to show that $d_j^{\mathrm{BOT}_X}$ and $s_j^{\mathrm{BOT}_X}$ are well-defined. If we instead use the geometric construction given in Remark 4.16, we need to show directly that they are independent of the choice of lift x^*. We prove this in the following lemma.

Lemma 4.18. *The maps $d_j^{\mathrm{BOT}_X}$ and $s_j^{\mathrm{BOT}_X}$ as defined in Remark 4.16 are, in each component, independent of our choice of lift; that is, they are independent of our choice of x^*. Therefore, the maps $d_j^{\mathrm{BOT}_X}$ and $s_j^{\mathrm{BOT}_X}$ are well-defined.*

Proof. Let T be the torus part of $U_{\sigma_0} \cap \cdots \cap U_{\sigma_n}$, let T_j be the torus part of $U_{\sigma_0} \cap \cdots \cap \widehat{U_{\sigma_j}} \cap \cdots \cap U_{\sigma_n}$, let $z \in T$ be any point, and let x_1^* and x_2^* be two choices of lift. Then we have the following two face maps

$$d_{j,1}^{\mathrm{BOT}_X} = (proj_2) \circ d_j \circ (x_1^* \times T) \tag{4.23}$$

and

$$d_{j,2}^{\mathrm{BOT}_X} = (proj_2) \circ d_j \circ (x_2^* \times T), \tag{4.24}$$

and we have the following two degeneracy maps

$$s_{j,1}^{\mathrm{BOT}_X} = (proj_2) \circ s_j \circ (x_1^* \times T) \tag{4.25}$$

and

$$s_{j,2}^{\mathrm{BOT}_X} = (proj_2) \circ s_j \circ (x_2^* \times T). \tag{4.26}$$

We begin with the two degeneracy maps. We want to show that $s_{j,1}^{\mathrm{BOT}_X}(z) = s_{j,2}^{\mathrm{BOT}_X}(z)$ for any $z \in T$. Recall that s_j is just the identity map on the respective components of $(\mathcal{U}_X)_n$ since all it does is repeat a term in the intersection (which does not change the actual set) and then includes into this intersection (which is just the identity as claimed). So we have

$$s_{j,1}^{\mathrm{BOT}_X}(z) = (proj_2) \circ s_j(x_1^*, z) = (proj_2)(x_1^*, z) = z \tag{4.27}$$

and

$$s_{j,2}^{\mathrm{BOT}_X}(z) = (proj_2) \circ s_j(x_2^*, z) = (proj_2)(x_2^*, z) = z. \tag{4.28}$$

This shows that $s_{j,1}^{\mathrm{BOT}_X}(z) = s_{j,2}^{\mathrm{BOT}_X}(z)$, establishing the claim of the lemma for the degeneracies.

Now we look at the face maps. We want to show that $d_{j,1}^{\mathrm{BOT}_X}(z) = d_{j,2}^{\mathrm{BOT}_X}(z)$. The proof is similar to the degeneracy case, with one slight modification. While $s_j^{\mathrm{BOT}_X}$ was mapping $T \longrightarrow T$, $d_j^{\mathrm{BOT}_X}$ is mapping $T \longrightarrow T_j$, which is in general a torus of smaller rank. So we have

$$d_{j,1}^{\mathrm{BOT}_X}(z) = (proj_2) \circ d_j(x_1^*, z) \tag{4.29}$$

and

$$d_{j,2}^{\mathrm{BOT}_X}(z) = (proj_2) \circ d_j(x_2^*, z). \tag{4.30}$$

As we saw before, d_j is just an inclusion map, so $d_j(x_1^*, z) = (x_1^*, z)$ and $d_j(x_2^*, z) = (x_2^*, z)$. If T_j is the same rank as T (which would mean that $T = T_j$), then the argument now proceeds exactly the same as the degeneracy case and we are done. However, if T_j is smaller rank, then when included into $U_{\sigma_0} \cap \cdots \cap \widehat{U_{\sigma_j}} \cap \cdots \cap U_{\sigma_n}$, there is some of z that misses T_j. Call that part y_j and denote by z_j the part of z that lands in T_j. Then $(x_1^*, z) = ((x_1^*, y_j), z_j)$ and $(x_2^*, z) = ((x_2^*, y_j), z_j)$; then the projection onto the torus part of each of these is just z_j. This gives us

$$d_{j,1}^{\mathrm{BOT}_X}(z) = (proj_2) \circ d_j(x_1^*, z) = (proj_2)((x_1^*, y_j), z_j) = z_j \tag{4.31}$$

and

$$d_{j,2}^{\mathrm{BOT}_X}(z) = (proj_2) \circ d_j(x_2^*, z) = (proj_2)((x_2^*, y_j), z_j) = z_j. \tag{4.32}$$

This shows that $d_{j,1}^{\mathrm{BOT}_X}(z) = d_{j,2}^{\mathrm{BOT}_X}(z)$ and completes the proof.

□

Using either the methods of Lemma 4.17 or Lemma 4.18, we see that the maps $d_j^{\mathrm{BOT}_X}$ and $s_j^{\mathrm{BOT}_X}$ are well-defined. Our next goal is to show that these maps indeed make BOT_X into a simplicial scheme. We begin with a lemma.

Lemma 4.19. *The maps $\widetilde{d}_j$ and $\widetilde{s}_j$ of Definition 4.14 satisfy the usual simplicial identities of Definition 3.39.*

Proof. We recall that the five usual simplicial identities are:

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i \text{ for } i < j \\
d_i s_j &= s_{j-1} d_i \text{ for } i < j \\
d_i s_j &= \mathrm{id} \quad\quad \text{for } i = j, j+1 \\
d_i s_j &= s_j d_{i-1} \text{ for } i > j+1 \\
s_i s_j &= s_j s_{i-1} \text{ for } i > j.
\end{aligned} \tag{4.33}$$

We show each of these separately for $\widetilde{d}_j$ and $\widetilde{s}_j$. Let

$$\begin{aligned}
\tau &= \sigma_0 \cap \cdots \cap \sigma_n \\
\tau_j &= \sigma_0 \cap \cdots \cap \widehat{\sigma_j} \cap \cdots \cap \sigma_n \\
\tau_{i,j} &= \sigma_0 \cap \cdots \cap \widehat{\sigma_i} \cap \cdots \cap \widehat{\sigma_j} \cap \cdots \cap \sigma_n
\end{aligned} \tag{4.34}$$

and so on. When including multiple indices, we write the indices in increasing order (so for $i < j$, we write $\tau_{i,j}$; if $i > j$, we write $\tau_{j,i}$). Similarly, we write

$$\begin{aligned}
\tau^j &= \sigma_0 \cap \cdots \cap \sigma_j \cap \sigma_j \cap \cdots \cap \sigma_n \\
\tau^{i,j} &= \sigma_0 \cap \cdots \cap \sigma_i \cap \sigma_i \cap \cdots \cap \sigma_j \cap \sigma_j \cap \cdots \cap \sigma_n
\end{aligned} \tag{4.35}$$

and so on. Again, when including multiple indices, we write the indices in increasing order.

Observe that both τ^j_j and τ^j_{j+1} are just τ itself.

By Lemma 4.17, we know that $\widetilde{d}_j$ and $\widetilde{s}_j$ are independent of the choice of lifting from $\widetilde{N}^X_\tau$ up to N^X; therefore, throughout this proof, we assume that the lifting takes $[x]$ to x. Call this lifting map α (by abuse of notation, we will use α to refer to all such lifting maps, instead of referring to the specific lifting map by index, as all these liftings have the same action). Then we can rewrite $\widetilde{d}_j$ and $\widetilde{s}_j$ as

$$\widetilde{d}_j = \pi \circ \mathrm{id}_{N^X} \circ \alpha \tag{4.36}$$

and

$$\widetilde{s}_j = \pi \circ \mathrm{id}_{N^X} \circ \alpha \tag{4.37}$$

where in each case π denotes the appropriate canonical surjection map. We now prove the identities.

For $i < j$, we have

$$\begin{aligned}
\widetilde{d}_i\widetilde{d}_j &= \pi_{N^X_{\tau_{i,j}}} \circ \mathrm{id}_{N^X} \circ \alpha \circ \pi_{N^X_{\tau_j}} \circ \mathrm{id}_{N^X} \circ \alpha \\
\widetilde{d}_i\widetilde{d}_j &= \pi_{N^X_{\tau_{i,j}}} \circ \mathrm{id}_{N^X} \circ \mathrm{id}_{N^X} \circ \alpha \\
\widetilde{d}_i\widetilde{d}_j &= \pi_{N^X_{\tau_{i,j}}} \circ \mathrm{id}_{N^X} \circ \alpha \circ \pi_{N^X_{\tau_i}} \circ \mathrm{id}_{N^X} \circ \alpha \\
\widetilde{d}_i\widetilde{d}_j &= \widetilde{d}_{j-1}\widetilde{d}_i
\end{aligned} \tag{4.38}$$

which is our first simplicial identity. Notice that the last equality comes from the fact that mapping τ to $\tau_{i,j}$ can be accomplished first by deleting the j^{th} term (which is σ_j) followed by deleting the i^{th} term (which is σ_i), or by deleting the i^{th} term followed by deleting the $(j-1)^{\text{th}}$ term (which has become σ_j as deleting σ_i shifts all indices bigger than i down by

1). Next we look at the identities for $\widetilde{d}_i\widetilde{s}_j$. For $i < j$, we have

$$\begin{aligned}
\widetilde{d}_i\widetilde{s}_j &= \pi_{N^X_{\tau^j_i}} \circ \mathrm{id}_{N^X_{\tau^j}} \circ\alpha \circ \pi_{N^X_{\tau^j}} \circ \mathrm{id}_{N^X} \circ\alpha \\
\widetilde{d}_i\widetilde{s}_j &= \pi_{N^X_{\tau^j_i}} \circ \mathrm{id}_{N^X} \circ \mathrm{id}_{N^X} \circ\alpha \\
\widetilde{d}_i\widetilde{s}_j &= \pi_{N^X_{\tau^j_i}} \circ \mathrm{id}_{N^X} \circ\alpha \circ \pi_{N^X_{\tau_i}} \circ \mathrm{id}_{N^X} \circ\alpha \\
\widetilde{d}_i\widetilde{s}_j &= \widetilde{s}_{j-1}\widetilde{d}_i && (4.39)
\end{aligned}$$

which is our second simplicial identity. Notice we can obtain the last equality by realizing that repeating the j^{th} term σ_j and then deleting the the i^{th} term σ_i is the same as first deleting the i^{th} term σ_i and then repeating the now $(j-1)^{\text{th}}$ term σ_j. We could also see this immediately since, as we saw in Definition 4.14 that $\widetilde{s}_j$ turns out to be the identity map. For $i = j, j+1$, we have

$$\begin{aligned}
\widetilde{d}_i\widetilde{s}_j &= \pi_{N^X_{\tau^j_i}} \circ \mathrm{id}_{N^X} \circ\alpha \circ \pi_{N^X_{\tau^j}} \circ \mathrm{id}_{N^X} \circ\alpha \\
\widetilde{d}_i\widetilde{s}_j &= \pi_{N^X_{\tau^j_i}} \circ \mathrm{id}_{N^X} \circ \mathrm{id}_{N^X} \circ\alpha \\
\widetilde{d}_i\widetilde{s}_j &= \pi_{N^X_{\tau^j_i}} \circ \alpha \\
\widetilde{d}_i\widetilde{s}_j &= \pi_{N^X_{\tau}} \circ \alpha \\
\widetilde{d}_i\widetilde{s}_j &= \mathrm{id} && (4.40)
\end{aligned}$$

which is our third simplicial identity. Here we use that both τ^j_j and τ^j_{j+1} are just τ. For $i > j+1$, we have

$$\begin{aligned}
\widetilde{d}_i\widetilde{s}_j &= \pi_{N^X_{\tau^j_i}} \circ \mathrm{id}_{N^X_{\tau^j}} \circ\alpha \circ \pi_{N^X_{\tau^j}} \circ \mathrm{id}_{N^X} \circ\alpha \\
\widetilde{d}_i\widetilde{s}_j &= \pi_{N^X_{\tau^j_i}} \circ \mathrm{id}_{N^X} \circ \mathrm{id}_{N^X} \circ\alpha \\
\widetilde{d}_i\widetilde{s}_j &= \pi_{N^X_{\tau^j_i}} \circ \mathrm{id}_{N^X} \circ\alpha \circ \pi_{N^X_{\tau_i}} \circ \mathrm{id}_{N^X} \circ\alpha \\
\widetilde{d}_i\widetilde{s}_j &= \widetilde{s}_j\widetilde{d}_{i-1} && (4.41)
\end{aligned}$$

which is our fourth simplicial identity. Notice we can obtain the last equality by realizing that repeating the j^{th} term σ_j and then deleting the new i^{th} term σ_{i-1} (after repeating σ_j, the i^{th} term becomes σ_{i-1}) is the same as first deleting the $(i-1)^{\text{th}}$ term σ_{i-1} and then repeating the j^{th} term σ_j (the j^{th} term is unaffected by deleting σ_{i-1} as $i-1>j$). We could also see this immediately since, as we saw in Definition 4.14 that $\widetilde{s}_j$ turns out to be the identity map. Finally, we look at the identities for $\widetilde{s}_i\widetilde{s}_j$. For $i>j$, we have

$$\begin{aligned}
\widetilde{s}_i\widetilde{s}_j &= \pi_{N^X_{\tau^{i-1,j}}} \circ \mathrm{id}_{N^X} \circ\alpha \circ \pi_{N^X_{\tau^j}} \circ \mathrm{id}_{N^X} \circ\alpha \\
\widetilde{s}_i\widetilde{s}_j &= \pi_{N^X_{\tau^{i-1,j}}} \circ \mathrm{id}_{N^X} \circ \mathrm{id}_{N^X} \circ\alpha \\
\widetilde{s}_i\widetilde{s}_j &= \pi_{N^X_{\tau^{i-1,j}}} \circ \mathrm{id}_{N^X} \circ\alpha \circ \pi_{N^X_{\tau^{i-1}}} \circ \mathrm{id}_{N^X} \circ\alpha \\
\widetilde{s}_i\widetilde{s}_j &= \widetilde{s}_j\widetilde{s}_{i-1}
\end{aligned} \tag{4.42}$$

which is our final simplicial identity. Notice we can obtain the last equality by realizing that repeating the j^{th} term σ_j and then repeating σ_{i-1} (the new i^{th} term after repeating σ_j) is the same as first repeating the $(i-1)^{\text{th}}$ term σ_{i-1} and then repeating the j^{th} term σ_j (the j^{th} term is unaffected by repeating σ_{i-1} as $i>j$). We could also see this immediately since, as we saw in Definition 4.14 that $\widetilde{s}_j$ turns out to be the identity map. So the maps $\widetilde{d}_j$ and $\widetilde{s}_j$ of Definition 4.14 satisfy the usual simplicial identities as claimed.

□

With Lemmas 4.17 and 4.19, we can now prove that BOT_X is a simplicial scheme.

Theorem 4.20. BOT_X*, as defined in Definition 4.15, is indeed a simplicial scheme.*

Proof. Definition 4.15 has already given us our objects $(\mathrm{BOT}_X)_n$ and our face and degeneracy maps. So to complete the proof, we need only show that the face and degeneracy maps given satisfy the usual simplicial identities. Recall that the maps $d_j^{\mathrm{BOT}_X}$ and $s_j^{\mathrm{BOT}_X}$ are defined to be given component-wise by applying the appropriate functors to $\widetilde{d}_j$ and $\widetilde{s}_j$ of Definition 4.14. By Lemma 4.19, the maps $\widetilde{d}_j$ and $\widetilde{s}_j$ satisfy the usual simplicial identities; applying the appropriate functors, we see that component-wise the maps $d_j^{\mathrm{BOT}_X}$ and $s_j^{\mathrm{BOT}_X}$ satisfy the usual simplicial identities. Since $d_j^{\mathrm{BOT}_X}$ and $s_j^{\mathrm{BOT}_X}$ are determined by what they do component-wise, we conclude that $d_j^{\mathrm{BOT}_X}$ and $s_j^{\mathrm{BOT}_X}$ satisfy the usual simplicial identities, making BOT_X a simplicial scheme, as desired.

$\square$

The above proof quotes Lemma 4.19, which uses only lattice techniques, in order to prove BOT_X is a simplicial scheme. However, we could use the geometric definitions of $d_j^{\mathrm{BOT}_X}$ and $s_j^{\mathrm{BOT}_X}$ and directly show that they satisfy the usual simplicial identities. This version of the proof (given below) demonstrates the intuition behind our choice of construction for $d_j^{\mathrm{BOT}_X}$ and $s_j^{\mathrm{BOT}_X}$; namely, that they are induced by (and therefore satisfy the usual simplicial relations because of) the face and degeneracy maps d_j and s_j of $\mathcal{U}_X$.

Alternate proof of Theorem 4.20. Definition 4.15 has already given us our objects $(\mathrm{BOT}_X)_n$ and our face and degeneracy maps. So to complete the proof, we need only show that the face and degeneracy maps given satisfy the usual simplicial identities. Since the maps $d_j^{\mathrm{BOT}_X}$ and $s_j^{\mathrm{BOT}_X}$ are defined by what they do on each component of $(\mathrm{BOT}_X)_n$, it is enough to show that the usual simplicial identities are satisfied component-wise. Each of these follow from the corresponding identities for the face and degeneracy maps for X as a simplicial scheme, since the face and degeneracy maps of $\mathcal{U}_X$ are also defined component-wise.

Now in each of these compositions, we get a term in the middle of the form $(x^* \times T) \circ (proj_2)$ for some choice of lift x^*. Since Lemma 4.18 showed that these maps are all independent of this choice of lift, we can simply choose x^* to be the point that was lost after applying $(proj_2)$. In other words, if $(proj_2)(x, z) = z$, then choose $x^* = x$. Making these choices, we can delete the term $(x^* \times T) \circ (proj_2)$ from any composite, and add it to any composite, as it is just an identity mapping when making the correct choice.

Now that we have made this observation, each of the usual simplicial identities are immediate from the corresponding identities on $\mathcal{U}_X$. First, we look at the identities for

$d_i^{\mathrm{BOT}_X} d_j^{\mathrm{BOT}_X}$. For $i < j$, we have

$$\begin{aligned}
d_i^{\mathrm{BOT}_X} d_j^{\mathrm{BOT}_X} &= (proj_2) \circ d_i \circ (x_i^* \times T) \circ (proj_2) \circ d_j \circ (x_j^* \times T) \\
d_i^{\mathrm{BOT}_X} d_j^{\mathrm{BOT}_X} &= (proj_2) \circ d_i \circ d_j \circ (x_j^* \times T) \\
d_i^{\mathrm{BOT}_X} d_j^{\mathrm{BOT}_X} &= (proj_2) \circ d_{j-1} \circ d_i \circ (x_j^* \times T) \\
d_i^{\mathrm{BOT}_X} d_j^{\mathrm{BOT}_X} &= (proj_2) \circ d_{j-1} \circ (x_{j-1}^* \times T) \circ (proj_2) \circ d_i \circ (x_j^* \times T) \\
d_i^{\mathrm{BOT}_X} d_j^{\mathrm{BOT}_X} &= d_{j-1}^{\mathrm{BOT}_X} d_i^{\mathrm{BOT}_X} \qquad (4.43)
\end{aligned}$$

which is our first simplicial identity. Next we look at the identities for $d_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X}$. For $i < j$, we have

$$\begin{aligned}
d_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= (proj_2) \circ d_i \circ (x_i^* \times T) \circ (proj_2) \circ s_j \circ (x_j^* \times T) \\
d_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= (proj_2) \circ d_i \circ s_j \circ (x_j^* \times T) \\
d_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= (proj_2) \circ s_{j-1} \circ d_i \circ (x_j^* \times T) \\
d_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= (proj_2) \circ s_{j-1} \circ (x_{j-1}^* \times T) \circ (proj_2) \circ d_i \circ (x_j^* \times T) \\
d_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= s_{j-1}^{\mathrm{BOT}_X} d_i^{\mathrm{BOT}_X} \qquad (4.44)
\end{aligned}$$

which is our second simplicial identity. For $i = j, j+1$, we have

$$\begin{aligned}
d_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= (proj_2) \circ d_i \circ (x_i^* \times T) \circ (proj_2) \circ s_j \circ (x_j^* \times T) \\
d_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= (proj_2) \circ d_i \circ s_j \circ (x_j^* \times T) \\
d_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= (proj_2) \circ \mathrm{id} \circ (x_j^* \times T) \\
d_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= (proj_2) \circ (x_j^* \times T) \\
d_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= \mathrm{id} \qquad (4.45)
\end{aligned}$$

which is our third simplicial identity. For $i > j+1$, we have

$$\begin{aligned}
d_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= (proj_2) \circ d_i \circ (x_i^* \times T) \circ (proj_2) \circ s_j \circ (x_j^* \times T) \\
d_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= (proj_2) \circ d_i \circ s_j \circ (x_j^* \times T) \\
d_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= (proj_2) \circ s_j \circ d_{i-1} \circ (x_j^* \times T) \\
d_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= (proj_2) \circ s_j \circ (x_j^* \times T) \circ (proj_2) \circ d_{i-1} \circ (x_j^* \times T) \\
d_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= s_j^{\mathrm{BOT}_X} d_{i-1}^{\mathrm{BOT}_X} \qquad (4.46)
\end{aligned}$$

which is our fourth simplicial identity. Finally, we look at the identities for $s_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X}$. For $i > j$, we have

$$\begin{aligned}
s_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= (proj_2) \circ s_i \circ (x_i^* \times T) \circ (proj_2) \circ s_j \circ (x_j^* \times T) \\
s_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= (proj_2) \circ s_i \circ s_j \circ (x_j^* \times T) \\
s_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= (proj_2) \circ s_j \circ s_{i-1} \circ (x_j^* \times T) \\
s_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= (proj_2) \circ s_j \circ (x_j^* \times T) \circ (proj_2) \circ s_{i-1} \circ (x_j^* \times T) \\
s_i^{\mathrm{BOT}_X} s_j^{\mathrm{BOT}_X} &= s_j^{\mathrm{BOT}_X} s_{i-1}^{\mathrm{BOT}_X} \qquad (4.47)
\end{aligned}$$

which is our final simplicial identity. So the face and degeneracy maps satisfy the usual simplicial identities and BOT_X is a simplicial scheme, as desired.

□

Now that we have shown BOT_X to be a simplicial scheme, we want to relate BOT_X to the simplicial scheme $\mathcal{U}_X$. Let $q_n^{\mathcal{U}_X}$ be the morphism of schemes obtained by projecting each component of $(\mathcal{U}_X)_n$ onto its torus part. By it's very definition, this gives us a morphism

$$q_n^{\mathcal{U}_X} : (\mathcal{U}_X)_n \longrightarrow (\mathrm{BOT}_X)_n \qquad (4.48)$$

which we now seek to show gives rise to a morphism of simplicial schemes.

Theorem 4.21. *The morphism* $q^{\mathcal{U}_X} : \mathcal{U}_X \longrightarrow \mathrm{BOT}_X$ *given in degree* $n \geq 0$ *by*

$$q_n^{\mathcal{U}_X} : (\mathcal{U}_X)_n \longrightarrow (\mathrm{BOT}_X)_n \tag{4.49}$$

is a morphism of simplicial schemes.

As we should expect by now, Theorem 4.21 can be proven two ways. We can either use the lattice techniques, or prove it directly using the geometric descriptions of $d_k^{\mathrm{BOT}_X}$ and $s_k^{\mathrm{BOT}_X}$. We present both proofs below.

Lattice proof of Theorem 4.21. We need to show that

$$d_k^{\mathrm{BOT}_X} \circ q_n^{\mathcal{U}_X} = q_{n-1}^{\mathcal{U}_X} \circ d_k \tag{4.50}$$

and that

$$s_k^{\mathrm{BOT}_X} \circ q_n^{\mathcal{U}_X} = q_{n+1}^{\mathcal{U}_X} \circ s_k. \tag{4.51}$$

As usual, we check this component-wise. Component-wise these maps are induced by the lattice maps $\widetilde{d}_k$ and $\widetilde{s}_k$ of Definition 4.14, and the canonical surjections $N^X \longrightarrow \widetilde{N}_\tau^X$ for the cones $\tau \in \Delta_X$. Recall that we can write $\widetilde{d}_k$ and $\widetilde{s}_k$ as

$$\widetilde{d}_k = \pi \circ \mathrm{id}_{N^X} \circ \alpha \tag{4.52}$$

and

$$\widetilde{s}_k = \pi \circ \mathrm{id}_{N^X} \circ \alpha \tag{4.53}$$

where in each case π denotes the appropriate canonical surjection map. On lattices, d_k is given by mapping $N^X \longrightarrow N^X$ via the identity, and $\tau \longrightarrow \tau_k$ via inclusion. So for the face maps we need

$$\pi_{\widetilde{N}^X_{\tau_k}} \circ \mathrm{id}_{N^X} \circ \alpha \circ \pi_{\widetilde{N}^X_{\tau}} = \pi_{\widetilde{N}^X_{\tau_k}} \circ \mathrm{id}_{N^X} \tag{4.54}$$

which is clearly true by our choice of α (and Lemma 4.17). The left hand side component-wise induces $d_k^{\mathrm{BOT}_X} \circ q_n^{\mathcal{U}_X}$, while the right-hand side component-wise induces $q_{n-1}^{\mathcal{U}_X} \circ d_k$. This establishes the first equality.

Similarly, on lattices, s_k is given by mapping $N^X \longrightarrow N^X$ via the identity, and $\tau \longrightarrow \tau$ via identity. So for the degeneracy maps we need

$$\pi_{\widetilde{N}^X_{\tau}} \circ \mathrm{id}_{N^X} \circ \alpha \circ \pi_{\widetilde{N}^X_{\tau}} = \pi_{\widetilde{N}^X_{\tau}} \circ \mathrm{id}_{N^X} \tag{4.55}$$

which is again clearly true by our choice of α (and Lemma 4.17). The left hand side component-wise induces $s_k^{\mathrm{BOT}_X} \circ q_n^{\mathcal{U}_X}$, while the right-hand side component-wise induces $q_{n+1}^{\mathcal{U}_X} \circ s_k$. This establishes the second equality. So $q^{\mathcal{U}_X}$ is indeed a morphism of simplicial schemes.

□

Geometric proof of Theorem 4.21. We need to show that

$$d_k^{\mathrm{BOT}_X} \circ q_n^{\mathcal{U}_X} = q_{n-1}^{\mathcal{U}_X} \circ d_k \tag{4.56}$$

and that

$$s_k^{\mathrm{BOT}_X} \circ q_n^{\mathcal{U}_X} = q_{n+1}^{\mathcal{U}_X} \circ s_k. \tag{4.57}$$

As usual, we check this component-wise. On each component of $(\mathcal{U}_X)_n$, $q_n^{\mathcal{U}_X}$ is just a projection map onto the torus. In other words,

$$q_n^{\mathcal{U}_X} : U_{\sigma_0} \cap \cdots \cap U_{\sigma_n} = U_\tau \cong U_{\tau'} \times T \mapsto T \tag{4.58}$$

is a projection (here T is our torus, and $\tau = \sigma_0 \cap \cdots \cap \sigma_n$). So pick any element x in $U_{\sigma_0} \cap \cdots \cap U_{\sigma_n}$; then that element is of the form $x = (y, z)$, where $z \in T$. $q_n^{\mathcal{U}_X}(x) = z$ so we have

$$d_k^{\mathrm{BOT}_X} \circ q_n^{\mathcal{U}_X}(x) = d_k^{\mathrm{BOT}_X}(z). \tag{4.59}$$

For the other direction, $d_k(x)$ is just an element of $U_{\sigma_0} \cap \cdots \cap \widehat{U_{\sigma_k}} \cap \cdots \cap U_{\sigma_n}$; in other words, $d_k(x) \in U_{\tau_k} \cong U_{\tau'_k} \times T'$ where $\tau_k = \sigma_0 \cap \cdots \cap \widehat{\sigma_k} \cap \cdots \cap \sigma_n$. So $d_k(x) = (y_k, z_k)$ and $q_{n-1}^{\mathcal{U}_X} \circ d_k(x) = q_{n-1}^{\mathcal{U}_X}(y_k, z_k) = z_k$. But notice that, by its very construction $d_k^{\mathrm{BOT}_X}(z) = z_k$. The reason is that the image of x under $q_{n-1}^{\mathcal{U}_X} \circ d_k$ depends only on where the torus part of x is sent under d_k; that is precisely what $d_k^{\mathrm{BOT}_X}(z)$ is since z is the torus part of x. So we establish the first condition.

Now we check the same condition on degeneracy maps. However, this condition is significantly easier. The reason is that s_k is just the identity on any given component, since all it does is send

$$U_{\sigma_0} \cap \cdots \cap U_{\sigma_k} \cap U_{\sigma_{k+1}} \cap \cdots \cap U_{\sigma_n} \mapsto U_{\sigma_0} \cap \cdots \cap U_{\sigma_k} \cap U_{\sigma_k} \cap U_{\sigma_{k+1}} \cap \cdots \cap U_{\sigma_n}$$

and this intersection is just the same set. Since it just includes a set into the same set, it is the identity on that set. As a consequence, $s_k^{\mathrm{BOT}_X}$ is also the identity on each component. Now pick any $x = (y, z)$. Then

$$s_k^{\mathrm{BOT}_X} \circ q_n^{\mathcal{U}_X}(y, z) = s_k^{\mathrm{BOT}_X}(z) = z \tag{4.60}$$

and

$$q_{n+1}^{\mathcal{U}_X} \circ s_k(y, z) = q_{n+1}^{\mathcal{U}_X}(y, z) = z. \tag{4.61}$$

So these commute as well and $q^{\mathcal{U}_X}$ is a morphism of simplicial schemes.

□

Corollary 4.22. *We get a morphism*

$$\mathrm{KH}(q^{\mathcal{U}_X}) : \mathrm{KH}(\mathrm{BOT}_X) \longrightarrow \mathrm{KH}(\mathcal{U}_X) \tag{4.62}$$

of cosimplicial objects over the category of spectra. Similarly, we get a morphism

$$\mathrm{KH}(q^{\mathcal{U}_X}) \otimes \mathrm{id}_{\mathbb{Q}} : \mathrm{KH}(\mathrm{BOT}_X) \otimes \mathbb{Q} \longrightarrow \mathrm{KH}(\mathcal{U}_X) \otimes \mathbb{Q} \tag{4.63}$$

of cosimplicial objects over the category of spectra.

Proof. Apply KH (or $\mathrm{KH}(-) \otimes \mathbb{Q}$) everywhere, and recall Corollary 4.12 and the fact that KH (or $\mathrm{KH}(-) \otimes \mathbb{Q}$) is a contravariant functor.

□

We can actually do even better. Recall from Theorem 3.54 that the category of spectra has a model category structure in which the class of weak equivalences are just quasi-isomorphisms; similarly, recall from Theorem 3.56 that the category of cosimplicial spectra has a model category structure in which the class of weak equivalences are just morphisms that are quasi-isomorphisms in each degree.

Theorem 4.23. *The morphism of cosimplicial spectra*

$$\mathrm{KH}(q^{\mathcal{U}_X}) : \mathrm{KH}(\mathrm{BOT}_X) \longrightarrow \mathrm{KH}(\mathcal{U}_X) \tag{4.64}$$

is a weak equivalence. The morphism of cosimplicial spectra

$$\mathrm{KH}(q^{\mathcal{U}_X}) \otimes \mathrm{id}_{\mathbb{Q}} : \mathrm{KH}(\mathrm{BOT}_X) \otimes \mathbb{Q} \longrightarrow \mathrm{KH}(\mathcal{U}_X) \otimes \mathbb{Q} \tag{4.65}$$

is also a weak equivalence.

Proof. Suppose we have a situation in which

$$\mathbb{G}_m^\alpha \xrightarrow{\ i\ } C \times \mathbb{G}_m^\alpha \xrightarrow{\ q\ } \mathbb{G}_m^\alpha \xrightarrow{\ i\ } C \times \mathbb{G}_m^\alpha$$

where i is an inclusion map and q is a projection map. For simplicity, we adopt the notation $f^* = \mathrm{KH}(f)$. Define a homotopy map $F : (C \times \mathbb{G}_m^\alpha) \times \mathbb{A}^1 \longrightarrow C \times \mathbb{G}_m^\alpha$ by $F(-,0) = i \circ q$ and $F(-,1) = \mathrm{id}_{C \times \mathbb{G}_m^\alpha}$.

The projection map $X \times \mathbb{A}^1 \longrightarrow X$ induces a homotopy equivalence

$$\mathrm{KH}(X) \xrightarrow{\simeq} \mathrm{KH}(X \times \mathbb{A}^1)$$

for any X with two homotopy inverses j_0^* and j_1^*, where $j_i : X \longrightarrow X \times \mathbb{A}^1$ by $x \mapsto (x,i)$ for $i = 0,1$. Now since j_0^* and j_1^* are both homotopy inverses to the same map, then in the homotopy category we must have $[j_0^*] = [j_1^*]$.

Applying this to $X = C \times \mathbb{G}_m^\alpha$, then we have $F \circ j_0 = F(-,0) = i \circ q$ and $F \circ j_1 = F(-,1) = \mathrm{id}_{C \times \mathbb{G}_m^\alpha}$. Then in the homotopy category, since $[j_0^*] = [j_1^*]$ and since $[F^*] = [F^*]$, we have that $[(F(-,0))^*] = [(F(-,1))^*]$. This means that $[q^*][i^*] = [\mathrm{id}^*]$ and $[q^*]$ is an isomorphism in the homotopy category. That means that q^* is a weak equivalence in the category of spectra.

But notice that the map $q^{\mathcal{U}_X}$ is, in each degree, made up of maps like the map q above. So by the above we have that in each degree, $\mathrm{KH}((q^{\mathcal{U}_X})_n)$ is a weak equivalence; therefore $\mathrm{KH}(q^{\mathcal{U}_X})$ is also a weak equivalence.

To see that $\mathrm{KH}(q^{\mathcal{U}_X}) \otimes \mathrm{id}_{\mathbb{Q}}$ is a weak equivalence, we need only note that $\mathrm{KH}(q^{\mathcal{U}_X})$ is a weak equivalence, and therefore induces an isomorphism on all KH groups of each component; since tensoring with $\mathbb{Q}$ preserves isomorphisms, the result is immediate.

$\square$

Corollary 4.24. *The morphism of cosimplicial spectra*

$$\mathrm{KH}(q^{\mathcal{U}_X}) : \mathrm{KH}(\mathrm{BOT}_X) \longrightarrow \mathrm{KH}(\mathcal{U}_X) \tag{4.66}$$

induces a weak equivalence

$$R(\mathrm{KH}(\mathrm{BOT}_X)) \longrightarrow R(\mathrm{KH}(\mathcal{U}_X)). \tag{4.67}$$

Similarly, the morphism of cosimplicial spectra

$$\mathrm{KH}(q^{\mathcal{U}_X}) \otimes \mathrm{id}_{\mathbb{Q}} : \mathrm{KH}(\mathrm{BOT}_X) \otimes \mathbb{Q} \longrightarrow \mathrm{KH}(\mathcal{U}_X) \otimes \mathbb{Q} \tag{4.68}$$

induces a weak equivalence

$$R(\mathrm{KH}(\mathrm{BOT}_X) \otimes \mathbb{Q}) \longrightarrow R(\mathrm{KH}(\mathcal{U}_X) \otimes \mathbb{Q}). \tag{4.69}$$

Proof. By Theorem 4.23, the morphisms

$$\mathrm{KH}(q^{\mathcal{U}_X}) : \mathrm{KH}(\mathrm{BOT}_X) \longrightarrow \mathrm{KH}(\mathcal{U}_X) \tag{4.70}$$

and

$$\mathrm{KH}(q^{\mathcal{U}_X}) \otimes \mathrm{id}_{\mathbb{Q}} : \mathrm{KH}(\mathrm{BOT}_X) \otimes \mathbb{Q} \longrightarrow \mathrm{KH}(\mathcal{U}_X) \otimes \mathbb{Q} \tag{4.71}$$

are weak equivalences, so the result is immediate by Proposition 3.51.

□

Corollary 4.25. *The morphism of cosimplicial spectra*

$$\mathrm{KH}(q^{\mathcal{U}_X}) : \mathrm{KH}(\mathrm{BOT}_X) \longrightarrow \mathrm{KH}(\mathcal{U}_X) \tag{4.72}$$

induces a morphism

$$\mathrm{holim}(\mathrm{KH}(\mathrm{BOT}_X)) \longrightarrow \mathrm{holim}(\mathrm{KH}(\mathcal{U}_X)), \tag{4.73}$$

and this induced morphism is a weak equivalence. Similarly, the morphism of cosimplicial spectra

$$\mathrm{KH}(q^{\mathcal{U}_X}) \otimes \mathrm{id}_{\mathbb{Q}} : \mathrm{KH}(\mathrm{BOT}_X) \otimes \mathbb{Q} \longrightarrow \mathrm{KH}(\mathcal{U}_X) \otimes \mathbb{Q} \tag{4.74}$$

induces a morphism

$$\operatorname{holim}(\mathrm{KH}(\mathrm{BOT}_X)\otimes\mathbb{Q}) \longrightarrow \operatorname{holim}(\mathrm{KH}(\mathcal{U}_X)\otimes\mathbb{Q}), \tag{4.75}$$

and this induced morphism is a weak equivalence.

Proof. This is immediate from the construction of the holim functor, Proposition 3.71, and Corollary 4.24.

□

This allows us to conclude the following important consequence for BOT_X.

Theorem 4.26. *Let X be a complete simplicial toric variety. Then the two spectra* $\operatorname{holim}(\mathrm{KH}(\mathrm{BOT}_X))$ *and* $\mathrm{KH}(X)$ *are weakly equivalent.*

Proof. Recall that two objects in a model category are said to be *weakly equivalent* if there is a "zig-zag" of weak equivalences between them; see [Hir, Definitions 7.9.1 and 7.9.2]. By Corollaries 3.119 and 4.25, we get a zig-zag

$$\operatorname{holim}(\mathrm{KH}(\mathrm{BOT}_X)) \longrightarrow \operatorname{holim}(\mathrm{KH}(\mathcal{U}_X)) \longleftarrow \mathrm{KH}(X) \tag{4.76}$$

where each map is a weak equivalence. Thus the spectra $\operatorname{holim}(\mathrm{KH}(\mathrm{BOT}_X))$ and $\mathrm{KH}(X)$ are weakly equivalent as claimed.

□

Theorem 4.26 completes our generalization of the intuition we presented in the introduction to Section 4. Since $\operatorname{holim}(\mathrm{KH}(\mathrm{BOT}_X))$ and $\mathrm{KH}(X)$ are weakly equivalent, the nth stable homotopy group of $\operatorname{holim}(\mathrm{KH}(\mathrm{BOT}_X))$ is isomorphic to the group $\mathrm{KH}_n(X)$, showing that $\mathrm{KH}_n(X)$ is indeed determined only by the torus pieces.

4.4 The Proof of Theorem 4.9

With the work of Sections 4.2 and 4.3, we are now ready to prove Theorem 4.9. Section 4.3 focused on the construction of BOT_X for a given complete simplicial toric variety X. Similarly, we could construct BOT_Y for a different complete simplicial toric variety Y. The question is: when are these two simplicial schemes related? In general this is not known; however, if we impose the conditions that X and Y have the same simplicial structure and that we have a lattice morphism

$$F : N^X \longrightarrow N^Y \tag{4.77}$$

which is injective with finite cokernel such that the restriction maps

$$F|_{N^X_\sigma} : N^X_\sigma \longrightarrow N^Y_{\varphi(\sigma)} \tag{4.78}$$

are also injective with finite cokernel for any cone $\sigma \in \Delta_X$, then we get a very useful relationship. It is the need to build this relationship that led to such strong conditions on the lattices associated to X and Y.

Recall that, for any cone $\sigma \in \Delta_X$, we write $\widetilde{N}^X_\sigma = N^X/N^X_\sigma$, where $\widetilde{N}^X_\sigma$ is the lattice that gives rise to the torus part of U_σ. See Remark 2.3.

Lemma 4.27. *Let X and Y be two complete simplicial toric varieties with the same simplicial structure, and suppose we have a lattice morphism $F : N^X \longrightarrow N^Y$ which is injective with finite cokernel such that the restriction maps $F|_{N^X_\sigma} : N^X_\sigma \longrightarrow N^Y_{\varphi(\sigma)}$ are also injective with finite cokernel for any cone $\sigma \in \Delta_X$. Then F induces a map*

$$\widetilde{F}_\sigma : \widetilde{N}^X_\sigma \longrightarrow \widetilde{N}^Y_{\varphi(\sigma)} \tag{4.79}$$

that is also injective with finite cokernel.

Proof. The first thing we need to show is that F induces a map

$$\widetilde{F}_\sigma : \widetilde{N}^X_\sigma \longrightarrow \widetilde{N}^Y_{\varphi(\sigma)}. \tag{4.80}$$

Once we have this map, we will show that it is injective with finite cokernel. Composing F with the canonical surjection morphism

$$\pi_{N^Y_{\varphi(\sigma)}} : N^Y \longrightarrow \widetilde{N}^Y_{\varphi(\sigma)} \tag{4.81}$$

we get a morphism

$$\pi_{N^Y_{\varphi(\sigma)}} \circ F : N^X \longrightarrow \widetilde{N}^Y_{\varphi(\sigma)}. \tag{4.82}$$

Since $F|_{N^X_\sigma} : N^X_\sigma \longrightarrow N^Y_{\varphi(\sigma)}$ is also injective, we see that the kernel of $\pi_{N^Y_{\varphi(\sigma)}} \circ F$ contains N^X_σ. Therefore, $\pi_{N^Y_{\varphi(\sigma)}} \circ F$ factors through $\widetilde{N}^X_\sigma$. Thus, we get a map

$$\widetilde{F}_\sigma : \widetilde{N}^X_\sigma \longrightarrow \widetilde{N}^Y_{\varphi(\sigma)} \tag{4.83}$$

as desired. Note that $\widetilde{F}_\sigma([x]) = \pi_{N^Y_{\varphi(\sigma)}} \circ F(x)$, and that this map is well-defined since if $[x] = [y]$, then $x - y \in N^X_\sigma$, which means that $F(x) - F(y) \in N^Y_{\varphi(\sigma)}$, and therefore that

$$\pi_{N^Y_{\varphi(\sigma)}} \circ F(x) = \pi_{N^Y_{\varphi(\sigma)}} \circ F(y). \tag{4.84}$$

So $\widetilde{F}_\sigma([x]) = \widetilde{F}_\sigma([y])$, showing that $\widetilde{F}_\sigma$ is well-defined and independent of the choice of lift.

Now by Lemma 2.1, we have that both $\widetilde{N}^X_\sigma$ and $\widetilde{N}^Y_{\varphi(\sigma)}$ are themselves lattices (in particular, they are free abelian groups). So we have the following diagram of lattices

$$\begin{tikzcd}
0 \arrow[r] & N^X_\sigma \arrow[r, hook] \arrow[d, "F|_{N^X_\sigma}"] & N^X \arrow[r, "\pi_{N^X_\sigma}"] \arrow[d, "F"] & \widetilde{N}^X_\sigma \arrow[r] \arrow[d, "\widetilde{F}_\sigma"] & 0 \\
0 \arrow[r] & N^Y_{\varphi(\sigma)} \arrow[r, hook] & N^Y \arrow[r, "\pi_{N^Y_{\varphi(\sigma)}}"'] & \widetilde{N}^Y_{\varphi(\sigma)} \arrow[r] & 0
\end{tikzcd}$$

where the rows are short exact sequences. Then by the Snake Lemma, we get the exact sequence

$$\ker\left(F|_{N_\sigma^X}\right) \longrightarrow \ker(F) \longrightarrow \ker\left(\widetilde{F}_\sigma\right) \longrightarrow \cdots$$

$$\cdots \operatorname{coker}\left(F|_{N_\sigma^X}\right) \longrightarrow \operatorname{coker}(F) \longrightarrow \operatorname{coker}\left(\widetilde{F}_\sigma\right)$$

Now since F and $F|_{N_\sigma^X}$ are assumed to be injective, we have

$$\ker\left(F|_{N_\sigma^X}\right) = \ker(F) = 0. \tag{4.85}$$

So our exact sequence reduces to

$$0 \longrightarrow \ker\left(\widetilde{F}_\sigma\right) \longrightarrow \operatorname{coker}\left(F|_{N_\sigma^X}\right) \longrightarrow \operatorname{coker}(F) \longrightarrow \operatorname{coker}\left(\widetilde{F}_\sigma\right)$$

Now we recall that $\operatorname{coker}\left(F|_{N_\sigma^X}\right)$ is assumed to be finite; therefore, as the above exact sequence forces $\ker\left(\widetilde{F}_\sigma\right)$ to inject into $\operatorname{coker}\left(F|_{N_\sigma^X}\right)$, we conclude that $\ker\left(\widetilde{F}_\sigma\right)$ is finite as well. Since $\ker\left(\widetilde{F}_\sigma\right)$ is a subgroup of the free abelian group $\widetilde{N}_\sigma^X$, we conclude that $\ker\left(\widetilde{F}_\sigma\right) = 0$ and the map $\widetilde{F}_\sigma$ is injective.

From here, the remainder of the proof is immediate. F injective with finite cokernel implies that $\operatorname{rank} N^X = \operatorname{rank} N^Y$ while $F|_{N_\sigma^X}$ injective with finite cokernel implies $\operatorname{rank} N_\sigma^X = \operatorname{rank} N_{\varphi(\sigma)}^Y$. Since $\operatorname{rank} \widetilde{N}_\sigma^X = \operatorname{rank} N^X - \operatorname{rank} N_\sigma^X$ and $\operatorname{rank} \widetilde{N}_{\varphi(\sigma)}^Y = \operatorname{rank} N^Y - \operatorname{rank} N_{\varphi(\sigma)}^Y$, we conclude that $\operatorname{rank} \widetilde{N}_\sigma^X = \operatorname{rank} \widetilde{N}_{\varphi(\sigma)}^Y$ and $\widetilde{F}_\sigma$ has finite cokernel as desired.

□

Corollary 4.28. *For $\widetilde{F}_\sigma$ as in Lemma 4.27, we have that $|\operatorname{coker}(\widetilde{F}_\sigma)|$ divides $|\operatorname{coker}(F)|$ for any cone σ.*

Proof. Choose splittings of N^X and N^Y to give us

$$N^X \cong N_\sigma^X \oplus M_\sigma^X \tag{4.86}$$

and

$$N^Y \cong N^Y_{\varphi(\sigma)} \oplus M^Y_{\varphi(\sigma)}. \tag{4.87}$$

Notice that $M^X_\sigma \cong \widetilde{N}^X_\sigma$ and that $M^Y_{\varphi(\sigma)} \cong \widetilde{N}^Y_{\varphi(\sigma)}$. One should also note that these splittings are non-canonical; however, as we are only interested in a relationship determined by orders of quotients of these groups (as opposed to the groups themselves), this will not be an issue. We showed in Lemma 4.27 that $\widetilde{F}_\sigma$ is given by lifting an element of $\widetilde{N}^X_\sigma$ back to N^X, applying F, and then applying the canonical surjection map $\pi_{N^Y_{\varphi(\sigma)}}$, and we also showed that this operation is independent of the choice of lift back to N^X. Therefore, if we assume that the lift of x is just $(0, x) \in N^X$, then the map $\widetilde{F}_\sigma$ is isomorphic to the map

$$F^M_\sigma : M^X_\sigma \longrightarrow M^Y_{\varphi(\sigma)} \tag{4.88}$$

given by

$$x \mapsto (0, x) \mapsto (0, F(x)) \mapsto F(x) \tag{4.89}$$

where the last map is just projection onto the second coordinate. In particular, $|\operatorname{coker}(\widetilde{F}_\sigma)| = |\operatorname{coker}(F^M_\sigma)|$ so we are done if we can show the result for $|\operatorname{coker}(F^M_\sigma)|$.

Now notice that $F = F|_{N^X_\sigma} \oplus F^M_\sigma$ so by elementary group theory, we have

$$N^Y/F(N^X) \cong \left(N^Y_{\varphi(\sigma)}/F|_{N^X_\sigma}(N^X_\sigma)\right) \oplus \left(M^Y_{\varphi(\sigma)}/F^M_\sigma(M^X_\sigma)\right). \tag{4.90}$$

In other words, we have

$$|\operatorname{coker}(F)| = |\operatorname{coker}(F|_{N^X_\sigma})| \cdot |\operatorname{coker}(F^M_\sigma)| \tag{4.91}$$

and conclude that $|\operatorname{coker}(F^M_\sigma)|$ divides $|\operatorname{coker}(F)|$. Therefore, $|\operatorname{coker}(\widetilde{F}_\sigma)|$ divides $|\operatorname{coker}(F)|$ as well and we are done.

$\square$

The importance of Lemma 4.27 is that $\widetilde{F}_\sigma : \widetilde{N}^X_\sigma \longrightarrow \widetilde{N}^Y_{\varphi(\sigma)}$ gives rise to a morphism between the torus pieces of U^X_σ and $U^Y_{\varphi(\sigma)}$, which will in turn allow us to construct a morphism $\mathrm{BOT}_X \longrightarrow \mathrm{BOT}_Y$ of simplicial schemes.

Theorem 4.29. *Let X and Y be two complete simplicial toric varieties with the same simplicial structure, and suppose we have a lattice morphism*

$$F : N^X \longrightarrow N^Y \tag{4.92}$$

which is injective with finite cokernel such that the restriction maps

$$F|_{N^X_\sigma} : N^X_\sigma \longrightarrow N^Y_{\varphi(\sigma)} \tag{4.93}$$

are also injective with finite cokernel for any cone $\sigma \in \Delta_X$. Then F induces a morphism of simplicial schemes

$$\mathrm{BOT}_X \longrightarrow \mathrm{BOT}_Y\,. \tag{4.94}$$

Furthermore, for every n, the morphism

$$(\mathrm{BOT}_X)_n \longrightarrow (\mathrm{BOT}_Y)_n \tag{4.95}$$

is, in each component, induced by a finite injection of rings (in fact, a finite injection of Hopf algebras).

Proof. As we've already seen, $\widetilde{N}^X_\sigma$ and $\widetilde{N}^Y_{\varphi(\sigma)}$ determine the torus parts of U^X_σ and $U^Y_{\varphi(\sigma)}$, and so

$$\widetilde{F}_\sigma : \widetilde{N}^X_\sigma \longrightarrow \widetilde{N}^Y_{\varphi(\sigma)} \tag{4.96}$$

determines a morphism between the tori T^X_σ and $T^Y_{\varphi(\sigma)}$. We will examine closely how this induced morphism is constructed. Because $\widetilde{F}_\sigma$ is injective with finite cokernel (Lemma 4.27), it gives us the following exact sequence:

$$0 \longrightarrow \widetilde{N}^X_\sigma \xrightarrow{\widetilde{F}_\sigma} \widetilde{N}^Y_{\varphi(\sigma)} \longrightarrow \widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma \longrightarrow 0$$

where $\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma$ is finite. Now we take the dual of these lattices, so we apply $\mathrm{Hom}(-,\mathbb{Z})$ to this sequence. This gives us the exact sequence

$$0 \longrightarrow \mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z}) \longrightarrow \cdots$$

$$\cdots \mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z}) \longrightarrow \mathrm{Ext}^1(\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma, \mathbb{Z}) \longrightarrow \mathrm{Ext}^1(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})$$

However, $\widetilde{N}^Y_{\varphi(\sigma)}$ is free, giving us

$$\mathrm{Ext}^1(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z}) = 0 \tag{4.97}$$

and since $\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma$ is finite,

$$\mathrm{Ext}^1(\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma, \mathbb{Z}) \cong \widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma \tag{4.98}$$

and

$$\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma, \mathbb{Z}) = 0. \tag{4.99}$$

So we reduce to the exact sequence

$$0 \longrightarrow \mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z}) \longrightarrow \widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma \longrightarrow 0\,.$$

This means that the induced morphism

$$\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z}) \tag{4.100}$$

is also injective with finite cokernel. Taking group rings, we get

$$k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})] \longrightarrow k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})] \longrightarrow k[\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma] \; .$$

From here, we take Spec of everything to get the sequence

$$\mathrm{Spec}\left(k[\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma]\right) \longrightarrow \mathrm{Spec}\left(k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]\right) \longrightarrow \mathrm{Spec}\left(k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]\right) \; .$$

This induces a morphism

$$(\mathrm{BOT}_X)_n \longrightarrow (\mathrm{BOT}_Y)_n \tag{4.101}$$

since in each component of $(\mathrm{BOT}_X)_n$, we get a map of the form

$$\mathrm{Spec}\left(k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]\right) \longrightarrow \mathrm{Spec}\left(k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]\right) . \tag{4.102}$$

Thus, we have the first piece of a morphism of simplicial schemes. To show this is a true morphism of simplicial schemes, we also need to show that it commutes with the face and degeneracy maps. However, before doing so, we show that, in each component of

$$(\mathrm{BOT}_X)_n \longrightarrow (\mathrm{BOT}_Y)_n, \tag{4.103}$$

this map is induced by a finite injection of rings.

To see this, we take a closer look at how the map is constructed in each component. As we saw above, in each component the map is just

$$\mathrm{Spec}\left(k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]\right) \longrightarrow \mathrm{Spec}\left(k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]\right) . \tag{4.104}$$

As we saw in Section 3.1, we can understand properties of this morphism by understanding the ring map

$$k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})] \longrightarrow k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]. \tag{4.105}$$

For ease of notation, we let $A = k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]$ and $B = k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]$. I claim that this ring map is injective and finite.

Injectivity is obvious. To see that this is finite, realize that, as an A-module, B is generated by $\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})$. Recall we have a surjection

$$\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z}) \longrightarrow \widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma \tag{4.106}$$

(where $\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma$ is finite). Lift each element of $\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma$ back to $\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})$; call these elements $\{\varphi_1, ..., \varphi_n\}$. Then every element $\psi \in \mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})$ is of the form $\psi = \phi + \varphi_i$ for some i and some $\phi \in \mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})$, as $\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma$ is (isomorphic to) the cokernel of the injection

$$\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z}). \tag{4.107}$$

When constructing the group ring, we need to view these groups as being multiplicative; therefore, we have $\chi^\psi = \chi^{\phi+\varphi_i} = \chi^\phi \cdot \chi^{\varphi_i}$. Now any element of B can be written as

$$\sum_{\psi \in \mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})} r_\psi \chi^\psi \tag{4.108}$$

where $r_\psi \in k$. Since $\chi^\psi = \chi^\phi \cdot \chi^{\varphi_i}$, we can rewrite this sum as

$$\sum_{i=1}^{n} \left(\sum_{\phi \in \mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})} r_\phi \chi^\phi \right) \chi^{\varphi_i}, \tag{4.109}$$

and $\sum_{\phi \in \mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})} r_\phi \chi^\phi \in A$. So every element of B is an A-linear combination of the χ^{φ_i}'s;

since there are finitely many χ^{φ_i}'s, B is finitely generated as an A-module. So the ring map

$$k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})] \longrightarrow k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})] \tag{4.110}$$

is finite as claimed.

Finally, we need to show that the maps

$$(\mathrm{BOT}_X)_n \longrightarrow (\mathrm{BOT}_Y)_n \tag{4.111}$$

commute with the face and degeneracy maps. Recall from Definitions 4.14 and 4.15 that the face and degeneracy maps for BOT_X are constructed by considering maps on the appropriate lattices. Recall that we called these maps $\widetilde{d}_j$ and $\widetilde{s}_j$; here, to differentiate between the case when they're constructed from N^X or N^Y, we denote these maps by $\widetilde{d}_j^{N^X}$ and $\widetilde{s}_j^{N^X}$ for the complete simplicial toric variety X and by $\widetilde{d}_j^{N^Y}$ and $\widetilde{s}_j^{N^Y}$ for the complete simplicial toric variety Y. Then, in the face map case, we get the following diagram:

$$\begin{tikzcd}
N^X \arrow[rrr, bend left=15, "\mathrm{id}_{N^X}"] \arrow[ddd, bend right=40, "F"'] & & & N^X \arrow[dl, bend right, "\pi_{N^X_{\tau_j}}"'] \arrow[ddd, bend left=40, "F"] \\
& \widetilde{N}^X_\tau \arrow[ul, "\alpha"'] \arrow[r, "\widetilde{d}_j^{N^X}"] \arrow[d, "\widetilde{F}_\tau"'] & \widetilde{N}^X_{\tau_j} \arrow[ur, bend right, "\alpha"'] \arrow[d, "\widetilde{F}_{\tau_j}"] & \\
& \widetilde{N}^Y_{\varphi(\tau)} \arrow[r, "\widetilde{d}_j^{N^Y}"] \arrow[dl, bend left, "\alpha"] & \widetilde{N}^Y_{\varphi(\tau_j)} & \\
N^Y \arrow[ur, bend left, "\pi_{N^Y_{\varphi(\tau)}}"] \arrow[rrr, bend right=15, "\mathrm{id}_{N^Y}"'] & & & N^Y \arrow[ul, "\pi_{N^Y_{\varphi(\tau_j)}}"']
\end{tikzcd}$$

Here F is injective by assumption, while $\widetilde{F}_\tau$ and $\widetilde{F}_{\tau_j}$ are injective by Lemma 4.27. Observe that, obviously, the outer square commutes. We want to show that the inner square commutes. In other words, we want to show that

$$\widetilde{F}_{\tau_j} \circ \widetilde{d}_j^{N^X} = \widetilde{d}_j^{N^Y} \circ \widetilde{F}_\tau; \tag{4.112}$$

then after applying all the necessary functors, we get that these induced maps on the tori commute with the face maps of BOT_X component-wise, and therefore commute overall.

Notice that, from the construction of the map $\widetilde{F}_\tau$ given in Lemma 4.27 we have that, for an element $[x] \in \widetilde{N}^X_\tau$

$$\widetilde{F}_\tau([x]) = \pi_{N^Y_{\varphi(\tau)}} \circ F(x). \tag{4.113}$$

In other words, we lift $[x]$ back to N^X, apply F, and then apply $\pi_{N^Y_{\varphi(\tau)}}$. As $\widetilde{F}_\tau$ is independent of our choice of lift, we can just take the lift α sending $[x]$ to x, as we did in the construction of $\widetilde{d}^{N^X}_j$ and $\widetilde{s}^{N^X}_j$ in Lemma 4.19. As before, we will abuse notation and use α to denote all such lifts of this type. This means that we can write

$$\widetilde{F}_\tau = \pi_{N^Y_{\varphi(\tau)}} \circ F \circ \alpha \tag{4.114}$$

and

$$\widetilde{F}_{\tau_j} = \pi_{N^Y_{\varphi(\tau_j)}} \circ F \circ \alpha. \tag{4.115}$$

Also recall that, from Lemma 4.19, we can write

$$\widetilde{d}^{N^X}_j = \pi_{N^X_{\tau_j}} \circ \mathrm{id}_{N^X} \circ \alpha \tag{4.116}$$

and

$$\widetilde{d}^{N^Y}_j = \pi_{N^Y_{\varphi(\tau_j)}} \circ \mathrm{id}_{N^Y} \circ \alpha. \tag{4.117}$$

With this, we now simply compute the composition:

$$
\begin{aligned}
\widetilde{d}_j^{N^Y} \circ \widetilde{F}_\tau &= \pi_{N^Y_{\varphi(\tau_j)}} \circ \mathrm{id}_{N^Y} \circ \alpha \circ \pi_{N^Y_{\varphi(\tau)}} \circ F \circ \alpha \\
\widetilde{d}_j^{N^Y} \circ \widetilde{F}_\tau &= \pi_{N^Y_{\varphi(\tau_j)}} \circ \mathrm{id}_{N^Y} \circ F \circ \alpha \\
\widetilde{d}_j^{N^Y} \circ \widetilde{F}_\tau &= \pi_{N^Y_{\varphi(\tau_j)}} \circ F \circ \mathrm{id}_{N^X} \circ \alpha \\
\widetilde{d}_j^{N^Y} \circ \widetilde{F}_\tau &= \pi_{N^Y_{\varphi(\tau_j)}} \circ F \circ \alpha \circ \pi_{N^X_{\tau_j}} \circ \mathrm{id}_{N^X} \circ \alpha \\
\widetilde{d}_j^{N^Y} \circ \widetilde{F}_\tau &= F|_{\widetilde{N}_j^X} \circ d_j^{N^X}. \qquad (4.118)
\end{aligned}
$$

So these maps commute with face maps. To see that they commute with degeneracy maps is even easier. This time our diagram is

$$
\begin{tikzcd}
N^X \arrow[rrr, bend left, "\mathrm{id}_{N^X}"] \arrow[ddd, bend right, "F"'] & & & N^X \arrow[dl, bend right, "\pi_{N^X_\tau}"'] \arrow[ddd, bend left, "F"] \\
& \widetilde{N}^X_\tau \arrow[ul, "\alpha"'] \arrow[r, "\widetilde{s}_j^{N^X}"] \arrow[d, "\widetilde{F}_\tau"'] & \widetilde{N}^X_\tau \arrow[ur, bend right, "\alpha"'] \arrow[d, "\widetilde{F}_\tau"] & \\
& \widetilde{N}^Y_{\varphi(\tau)} \arrow[r, "\widetilde{s}_j^{N^Y}"] \arrow[dl, bend left, "\alpha"] & \widetilde{N}^Y_{\varphi(\tau)} & \\
N^Y \arrow[ur, bend left, "\pi_{N^Y_{\varphi(\tau)}}"] \arrow[rrr, bend right, "\mathrm{id}_{N^Y}"'] & & & N^Y \arrow[ul, "\pi_{N^Y_{\varphi(\tau)}}"']
\end{tikzcd}
$$

and recalling from Definition 4.14 that $\widetilde{s}_j^{N^X}$ and $\widetilde{s}_j^{N^Y}$ turn out to just be the respective identity maps, the commutativity of the inner square is obvious. Thus, we have a morphism of simplicial schemes as desired.

□

Corollary 4.30. *For every n, the morphism $(\mathrm{BOT}_X)_n \longrightarrow (\mathrm{BOT}_Y)_n$ is, in each component, an isogeny; that is, in each component, this map is surjective with finite kernel.*

Proof. As before, let $A = k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]$ and let $B = k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]$. Recall from Theorem 4.29 that the ring homomorphism $A \longrightarrow B$ is injective and finite. Applying Corollary 3.13, we get that $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ is surjective. Thus, in each component, the map

$$(\mathrm{BOT}_X)_n \longrightarrow (\mathrm{BOT}_Y)_n \tag{4.119}$$

is surjective as claimed.

To show that, in each component, this map has finite kernel, we recall that each component is just an algebraic torus, which is just a diagonalizable group scheme. By Proposition 3.29, the category of diagonalizable group schemes is anti-equivalent to the category of abelian groups, which is an abelian category. That means that, under this equivalence, the short exact sequence

$$0 \longrightarrow \mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z}) \longrightarrow \widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma \longrightarrow 0$$

gets mapped to the short exact sequence

$$0 \longrightarrow \mathrm{Spec}\left(k[\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma]\right) \longrightarrow \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A) \longrightarrow 0\ .$$

So the kernel of any given component is of the form $\mathrm{Spec}\left(k[\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma]\right)$. Since $\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma$ was assumed finite, we have by Proposition 3.32 that the scheme $\mathrm{Spec}\left(k[\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma]\right)$ is a finite group scheme. So in each component, the map

$$(\mathrm{BOT}_X)_n \longrightarrow (\mathrm{BOT}_Y)_n \tag{4.120}$$

has finite kernel as desired.

$\square$

Remark 4.31. Notice that the kernel of the morphism $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ of Corollary 4.30 is the group scheme $\mathrm{Spec}\left(k[\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma]\right)$, which we denote by μ_σ. We will see later that, after writing $\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma$ in invariant factor form, if k contains all ${m_i}^{\text{th}}$ roots of unity (where $m_1, ..., m_n$ are the invariant factors of $\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma$) then

$$|\mu_\sigma| = m_1 \cdot m_2 \cdots m_n. \tag{4.121}$$

This will allow us to conclude that

$$|\mu_\sigma| = |\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma|. \tag{4.122}$$

In light of Corollary 4.28, once established we can conclude that the order of every kernel arising from an isogeny in Theorem 4.29 and Corollary 4.30 divides $|\operatorname{coker}(F)|$, where $F : N^X \longrightarrow N^Y$ is our lattice map from Theorem 4.29.

Corollary 4.32. *The morphism of Theorem 4.29 induces a morphism of cosimplicial spectra*

$$\mathrm{KH}(\mathrm{BOT}_Y) \longrightarrow \mathrm{KH}(\mathrm{BOT}_X) \tag{4.123}$$

which is, component-wise in each degree, given by induced morphisms f^* as defined in Definition 3.77. Similarly, the morphism of Theorem 4.29 induces a morphism of cosimplicial spectra

$$\mathrm{KH}(\mathrm{BOT}_Y) \otimes \mathbb{Q} \longrightarrow \mathrm{KH}(\mathrm{BOT}_X) \otimes \mathbb{Q} \tag{4.124}$$

which is, component-wise in each degree, given by induced morphisms $(f^*)_{\mathbb{Q}}$, with f^* as defined in Definition 3.77.

Proof. Recall that every component of $(\mathrm{BOT}_X)_n$ and $(\mathrm{BOT}_Y)_n$ is an algebraic torus. In particular, every component is smooth so the KH-theory of any component is just the K-theory of that same component. Similarly, $\mathrm{KH}(-) \otimes \mathbb{Q}$ of any component is just $\mathrm{K}(-) \otimes \mathbb{Q}$ of that same component.

Since KH is a contravariant functor from schemes to spectra, it sends any simplicial scheme to a cosimplicial spectrum, and sends any morphism of simplicial schemes to a morphism of cosimplicial spectra. KH of a simplicial scheme is given by applying KH to each component in each degree. Similarly, the morphism of simplicial schemes is mapped (via KH) to a morphism of cosimplicial spectra by applying the functor KH to every morphism. By the initial remark that the KH-theory of any component is just the K-theory of that same component, we see that these induced maps are just the maps f^* of Definition 3.77.

Similarly, since $\mathrm{KH}(-) \otimes \mathbb{Q}$ is a contravariant functor from schemes to spectra, it sends any simplicial scheme to a cosimplicial spectrum, and sends any morphism of simplicial schemes to a morphism of cosimplicial spectra. $\mathrm{KH}(-) \otimes \mathbb{Q}$ of a simplicial scheme is given by applying $\mathrm{KH}(-) \otimes \mathbb{Q}$ to each component in each degree. Similarly, the morphism of simplicial schemes is mapped (via $\mathrm{KH}(-) \otimes \mathbb{Q}$) to a morphism of cosimplicial spectra by applying the functor $\mathrm{KH}(-) \otimes \mathbb{Q}$ to every morphism. By the initial remark that $\mathrm{KH}(-) \otimes \mathbb{Q}$ of any component is just $\mathrm{K}(-) \otimes \mathbb{Q}$ of that same component, we see that these induced maps are just the maps $(f^*)_{\mathbb{Q}}$, with f^* as defined in Definition 3.77.

□

Theorem 4.29 has shown that $\mathrm{BOT}_X \longrightarrow \mathrm{BOT}_Y$ is in each degree component-wise induced by a finite, injective map of rings; however, these rings are actually Hopf algebras, and the Spec of these rings are actually diagonalizable group schemes. We can conclude from the following proposition that this injection of rings is also flat.

Proposition 4.33. *Let $f : A \longrightarrow B$ be an injection of Hopf algebras over k (where k is a field). Then B is faithfully flat as an A-module.*

Proof. See [KMRT, Proposition 22.1] and [Wat, Section 14.1].

□

Corollary 4.34. *The injection of rings $k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})] \longrightarrow k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]$ is faithfully flat.*

Proof. This is immediate by Proposition 4.33, since we've already seen that $k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})] \longrightarrow k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]$ is an injection of Hopf algebras.

□

Corollary 4.35. *The morphism*

$$\operatorname{Spec}\Big(k[\operatorname{Hom}(\widetilde{N}_\sigma^X,\mathbb{Z})]\Big) \longrightarrow \operatorname{Spec}\Big(k[\operatorname{Hom}(\widetilde{N}_{\varphi(\sigma)}^Y,\mathbb{Z})]\Big) \tag{4.125}$$

is flat.

Proof. This is immediate from Corollary 4.34, [Hart, Chapter III, Proposition 9.2], and the definition of a flat morphism of schemes. See [Hart, page 268].

□

We can actually say even more about the maps

$$\operatorname{Spec}\Big(k[\operatorname{Hom}(\widetilde{N}_\sigma^X,\mathbb{Z})]\Big) \longrightarrow \operatorname{Spec}\Big(k[\operatorname{Hom}(\widetilde{N}_{\varphi(\sigma)}^Y,\mathbb{Z})]\Big) \tag{4.126}$$

from Theorem 4.29. The first observation we can make is that because these maps are isogenies induced by a lattice morphism, they can be diagonalized.

Proposition 4.36. *Suppose that $f : T_1 \longrightarrow T_2$ is an isogeny between two split algebraic tori T_1 and T_2. Then f can be diagonalized; more specifically, we can choose coordinates $\{x_1, x_2, ..., x_n\}$ of T_1 and $\{y_1, y_2, ..., y_n\}$ of T_2 such that $y_i = x_i^{m_i}$ and the morphism f is just the powering map*

$$(x_1, x_2, ..., x_n) \mapsto (x_1^{m_1}, x_2^{m_2}, ..., x_n^{m_n}). \tag{4.127}$$

Proof. As T_1 and T_2 are algebraic tori, they are given by lattices and f is constructed by a lattice homomorphism. As f is assumed to be an isogeny, we have that the map of lattices inducing f, which we call $\widetilde{f}$, is injective with finite cokernel. In short, we have that $f : T_1 \longrightarrow T_2$ is given by the morphism

$$\widetilde{f} : N_1 \longrightarrow N_2 \tag{4.128}$$

where N_1 and N_2 are finite rank lattices. Let $\{e_1, ..., e_n\}$ denote the basis of N_1. Then the set $\{\widetilde{f}(e_1), ..., \widetilde{f}(e_n)\}$ is a linearly independent set in N_2. Let G denote the subgroup of N_2 generated by $\{\widetilde{f}(e_1), ..., \widetilde{f}(e_n)\}$; since $\widetilde{f}$ is injective with finite cokernel, G and N_2 have the same rank. Therefore, there exists a basis $\{\beta_1, ..., \beta_n\}$ of N_2, and integers $m_1, ..., m_n$ (with $m_1|m_2|\cdots|m_n$) such that the set $\{m_1\beta_1, ..., m_n\beta_n\}$ is a basis of G (this can be realized when giving an elementary proof of the invariant factor form of a finitely generated abelian group). As $\{m_1\beta_1, ..., m_n\beta_n\}$ is linearly independent and $\widetilde{f}$ is injective, $\{\alpha_1, ..., \alpha_n\}$ is linearly independent, where $\alpha_i = \widetilde{f}^{-1}(m_i\beta_i)$. Similarly, as $G = \operatorname{im}(\widetilde{f})$ is generated by $\{m_1\beta_1, ..., m_n\beta_n\}$ and $\widetilde{f}$ is injective, N_1 is generated by $\{\alpha_1, ..., \alpha_n\}$ as well (N_1 is isomorphic to G via $\widetilde{f}$). Therefore, $\{\alpha_1, ..., \alpha_n\}$ is a basis of N_1. Viewing N_1 in the basis $\{\alpha_1, ..., \alpha_n\}$ and N_2 in the basis $\{\beta_1, ..., \beta_n\}$ gives us that $\widetilde{f}$ is (in these bases) diagonal, with diagonal elements $m_1, m_2, ..., m_n$. Taking Hom and the appropriate group rings give rise to a morphism of rings

$$k[t_1, t_1^{-1}, ..., t_n, t_n^{-1}] \longrightarrow k[s_1, s_1^{-1}, ..., s_n, s_n^{-1}] \tag{4.129}$$

given by mapping k identically to itself and mapping $t_i \mapsto s_i^{m_i}$. Taking Spec everywhere, or equivalently taking $\operatorname{Hom}(-, k)$ everywhere, then gives the result.

□

Corollary 4.37. *If k contains all $m_i{}^{th}$ roots of unity, for $i = 1, ..., n$, then the group μ_σ of Remark 4.31 satisfies*

$$|\mu_\sigma| = |\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma| \tag{4.130}$$

as claimed.

Proof. Recall that the morphism $\widetilde{F}_\sigma : \widetilde{N}^X_\sigma \longrightarrow \widetilde{N}^Y_{\varphi(\sigma)}$ induces an isogeny of tori $T^X_\sigma \longrightarrow T^Y_{\varphi(\sigma)}$ by Theorem 4.29 and Corollary 4.30. By Proposition 4.36, we can diagonalize $T^X_\sigma \longrightarrow T^Y_{\varphi(\sigma)}$ using the invariant factors of the group $\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma$. If $m_1, ..., m_n$ are the invariant factors of

$\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma$, then

$$|\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma| = m_1 \cdot m_2 \cdots m_n \tag{4.131}$$

so we are done if we can show that $|\mu_\sigma| = m_1 \cdot m_2 \cdots m_n$ also. Using the bases of Proposition 4.36, we have that

$$\mu_\sigma = \{(x_1, ..., x_n) | x_i^{m_i} = 1\} \tag{4.132}$$

for all i. Since k contains all ${m_i}^{\text{th}}$ roots of unity by assumption, the ith coordinate of any element of μ_σ has m_i distinct possible values. This means that $|\mu_\sigma| = m_1 \cdot m_2 \cdots m_n$ and we are done. As mentioned in Remark 4.31, we can conclude that $|\mu_\sigma|$ divides $|\operatorname{coker}(F)|$ for every cone σ.

$\square$

As we just saw in Corollary 4.37, if $f : T_1 \longrightarrow T_2$ is an isogeny, Proposition 4.36 gives us a very nice way to view both the map and its kernel. Indeed, letting $\ker f = \mu$, we see that in these bases

$$\mu = \{(x_1, ..., x_n) | x_i^{m_i} = 1\} \tag{4.133}$$

for all i. The result of Corollary 4.37 leads us to want our underlying field k to have all of the ${m_i}^{\text{th}}$ roots of unity, which a priori need not be true. However, using a Transfer Argument as in Theorem 3.104 allows us to reduce to this case. Before doing so, we need a lemma.

Lemma 4.38. *Suppose $f : A \longrightarrow B$ is an injective homomorphism of rings and that B is a free A-module of rank d via the structure induced by f.*

(a) If s and t are indeterminants, we have an induced homomorphism

$$F : A[s] \longrightarrow B[t] \tag{4.134}$$

with A mapping to B via f and $s \mapsto t$. Then $B[t]$ is a free $A[s]$-module of rank d via the structure induced by F.

(b) If s and t are indeterminants, we have an induced homomorphism

$$\widetilde{F} : A[s, s^{-1}] \longrightarrow B[t, t^{-1}] \tag{4.135}$$

with A mapping to B via f and $s \mapsto t$. Then $B[t, t^{-1}]$ is a free $A[s, s^{-1}]$-module of rank d via the structure induced by $\widetilde{F}$.

Proof. To prove (a), first notice that since f is injective, F is injective as well. Since $A[s]$ is isomorphic to its image inside of $B[t]$, we can replace A by $f(A)$ and s by t and assume that F is just inclusion. So we have reduced to the case that $A[t] \longrightarrow B[t]$ via inclusion. Next, we recall that, for any indeterminant x and any A-module M, we have that

$$M[x] \cong A[x] \otimes_A M \tag{4.136}$$

as $A[x]$-modules; see [AM, Chapter 2, Exercise 6]. Since $B \cong A^d$ and since tensor products commute with direct sums, letting $M = B$ and $x = t$ gives us

$$B[t] \cong A[t] \otimes_A B \cong A[t] \otimes_A A^d \cong (A[t] \otimes_A A)^d \cong A[t]^d. \tag{4.137}$$

This establishes part (a).

For part (b), we use a similar method to reduce to the case that

$$A[t, t^{-1}] \longrightarrow B[t, t^{-1}] \tag{4.138}$$

via inclusion. By part (a), $B[t]$ is a free $A[t]$-module of rank d. Letting $S = \{1, t, t^2, t^3, ...\}$, observe that $A[t, t^{-1}] = S^{-1}A[t]$ and $B[t, t^{-1}] = S^{-1}B[t]$. Next, we recall that that, for any R-module M, we have

$$S^{-1}M \cong S^{-1}R \otimes_R M \tag{4.139}$$

as $S^{-1}R$-modules; see [AM, Chapter 3, Proposition 3.5]. Since $B[t] \cong A[t]^d$, letting $R = A[t]$, $M = B[t]$, and $S = \{1, t, t^2, t^3, ...\}$ gives us

$$\begin{aligned} B[t,t^{-1}] \cong S^{-1}B[t] &\cong S^{-1}A[t] \otimes_{A[t]} B[t] \cong S^{-1}A[t] \otimes_{A[t]} A[t]^d \cong \\ (S^{-1}A[t] \otimes_{A[t]} A[t])^d &\cong (S^{-1}A[t])^d \cong A[t,t^{-1}]^d. \end{aligned} \tag{4.140}$$

This establishes part (b).

$\square$

We are now ready to make our transfer argument. For convenience, we again use the notation of Proposition 4.36 and Corollary 4.37.

Proposition 4.39. *Suppose that the characteristic of k does not divide $|\mu_\sigma|$. Letting $k(\mu_\sigma)$ denote the field extension of k given by adjoining all the ${m_i}^{th}$ roots of unity, and letting $(T_j)_{k(\mu_\sigma)}$ (for $j = 1, 2$) denote the base extension of T_j to the field $k(\mu_\sigma)$, we have that if $((f_{k(\mu_\sigma)})^*)_{\mathbb{Q}}$ is an isomorphism for every n, then $(f^*)_{\mathbb{Q}}$ is an isomorphism for every n as well.*

Proof. Here we seek to use Theorem 3.104. Consider the square of rings

$$\begin{array}{ccc} k[t_1, t_1^{-1}, ..., t_n, t_n^{-1}] & \xrightarrow{f} & k[s_1, s_1^{-1}, ..., s_n, s_n^{-1}] \\ \Big\downarrow{\scriptstyle i_1} & & \Big\downarrow{\scriptstyle i_2} \\ k(\mu_\sigma)[t_1, t_1^{-1}, ..., t_n, t_n^{-1}] & \xrightarrow[f_{k(\mu_\sigma)}]{} & k(\mu_\sigma)[s_1, s_1^{-1}, ..., s_n, s_n^{-1}] \end{array}$$

where the two vertical maps i_1 and i_2 are given by base extension; in other words, they map the variables to themselves but extend the coefficients. This square obviously commutes. The morphism $k \longrightarrow k(\mu_\sigma)$ of rings is injective (as k is a field), and $k(\mu_\sigma)$ is a free k-module of (finite) rank $[k(\mu_\sigma) : k]$ (as $|\mu_\sigma|$ is finite, the degree $[k(\mu_\sigma) : k]$ must be finite as well). Therefore, by Lemma 4.38, we have that $k(\mu_\sigma)[t_1, t_1^{-1}, ..., t_n, t_n^{-1}]$ is a free module of rank $[k(\mu_\sigma) : k]$ over $k[t_1, t_1^{-1}, ..., t_n, t_n^{-1}]$, and similarly we have that $k(\mu_\sigma)[s_1, s_1^{-1}, ..., s_n, s_n^{-1}]$ is a free module of rank $[k(\mu_\sigma) : k]$ over $k[s_1, s_1^{-1}, ..., s_n, s_n^{-1}]$. Therefore, both morphisms i_1 and

i_2 are injective, flat and finite. Finally, notice that

$$k(\mu_\sigma)[s_1, s_1^{-1}, ..., s_n, s_n^{-1}] \cong k(\mu_\sigma)[t_1, t_1^{-1}, ..., t_n, t_n^{-1}] \otimes_{k[t_1,t_1^{-1},...,t_n,t_n^{-1}]} k[s_1, s_1^{-1}, ..., s_n, s_n^{-1}].$$

So the conditions of Theorem 3.104 are satisfied. Thus, if $((f_{k(\mu_\sigma)})^*)_\mathbb{Q}$ is an isomorphism for every n, then by Theorem 3.104, $(f^*)_\mathbb{Q}$ is an isomorphism for every n as well. This completes the proof.

□

So by Proposition 4.39, we may assume that our field k contains all ${m_i}^{\text{th}}$ roots of unity, where the numbers $m_1, ..., m_n$ are the diagonal elements of f determined in Proposition 4.36; we will do so from here on out.

Viewing the morphisms of Theorem 4.29 using Proposition 4.36, we can show that the morphism

$$\operatorname{Spec}\left(k[\operatorname{Hom}(\widetilde{N}_\sigma^X, \mathbb{Z})]\right) \longrightarrow \operatorname{Spec}\left(k[\operatorname{Hom}(\widetilde{N}_{\varphi(\sigma)}^Y, \mathbb{Z})]\right) \tag{4.141}$$

is actually even étale. This will be our next result.

Proposition 4.40. *If the characteristic of k does not divide $|\mu_\sigma|$, then the morphism*

$$\operatorname{Spec}\left(k[\operatorname{Hom}(\widetilde{N}_\sigma^X, \mathbb{Z})]\right) \longrightarrow \operatorname{Spec}\left(k[\operatorname{Hom}(\widetilde{N}_{\varphi(\sigma)}^Y, \mathbb{Z})]\right) \tag{4.142}$$

is étale.

Proof. Recall from Proposition 3.25 part (d) that if two morphisms are étale then their product is étale also. By inductively applying this, we can conclude that if we have a finite collection of étale morphisms then their product is étale also.

Proposition 4.36 shows that we can choose coordinates with

$$f(x_1, x_2, ..., x_n) = (x_1^{m_1}, x_2^{m_2}, ..., x_n^{m_n}). \tag{4.143}$$

This means that f is, in these bases, a product of powering maps. So if we show that the morphism

$$\mathbb{G}_m \longrightarrow \mathbb{G}_m \tag{4.144}$$

given by $x \mapsto x^n$ is étale for any n, where k contains all nth roots of unity, then we are done. Notice that the condition that the characteristic of k does not divide $|\mu_\sigma|$ reduces, in this case, to the characteristic of k not dividing n, since the kernel μ_σ of the nth powering map is $\{x|x^n = 1\}$, which has order n since k contains all nth roots of unity.

On the level of rings, observe that this is given by the ring homomorphism

$$k[t, t^{-1}] \longrightarrow k[s, s^{-1}] \tag{4.145}$$

where $t \mapsto s^n$. Then $k[s, s^{-1}] \cong k[t, t^{-1}][X]/(X^n - t)$, so it's enough to show that the morphism

$$k[t, t^{-1}] \longrightarrow k[t, t^{-1}][X]/(X^n - t) \tag{4.146}$$

is étale. By Corollary 3.27, we are done if we can show that $nt \in (k[t, t^{-1}])^*$. We know $t \in (k[t, t^{-1}])^*$ so this reduces to showing that $n \in (k[t, t^{-1}])^*$, which is the same as saying that the characteristic of k does not divide n. But that is true by assumption. So the morphism

$$k[t, t^{-1}] \longrightarrow k[t, t^{-1}][X]/(X^n - t) \tag{4.147}$$

is étale and we are done.

$\square$

Corollary 4.41. *If the characteristic of k does not divide $|\operatorname{coker}(F)|$, then the morphism*

$$\operatorname{Spec}\left(k[\operatorname{Hom}(\widetilde{N}_\sigma^X, \mathbb{Z})]\right) \longrightarrow \operatorname{Spec}\left(k[\operatorname{Hom}(\widetilde{N}_{\varphi(\sigma)}^Y, \mathbb{Z})]\right) \tag{4.148}$$

is étale for every cone $\sigma \in \Delta_X$.

Proof. By Corollary 4.37, $|\mu_\sigma|$ divides $|\operatorname{coker}(F)|$ for every cone σ. Therefore, if k does not divide $|\operatorname{coker}(F)|$, then k does not divide $|\mu_\sigma|$ for any cone σ. Thus, by applying Proposition 4.40 in every case, we see that the morphism

$$\operatorname{Spec}\left(k[\operatorname{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]\right) \longrightarrow \operatorname{Spec}\left(k[\operatorname{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]\right) \tag{4.149}$$

is étale for every cone $\sigma \in \Delta_X$, as desired.

□

We are now ready to begin the next stage of our argument. We know that $\mathrm{BOT}_X \longrightarrow \mathrm{BOT}_Y$ is in each degree component-wise given by the faithfully flat, finite injection of rings

$$k[\operatorname{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})] \longrightarrow k[\operatorname{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]. \tag{4.150}$$

By Corollary 3.19, we know that $k[\operatorname{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]$ is a projective module over $k[\operatorname{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]$. We now use the following theorem.

Theorem 4.42. *Let G be a free abelian group, k a field, and $k[G]$ the group algebra for G over k. Then any projective $k[G]$-module is free.*

Proof. The proof is essentially the same as the proofs of [Qui2, Theorems 3 and 4]; see [Swan] for the necessary modifications to make Quillen's solution to Serre's problem extend to Laurent polynomial rings.

□

Corollary 4.43. *The ring $k[\operatorname{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]$ is a free module over $k[\operatorname{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]$.*

Proof. By Corollary 3.19, we know that $k[\operatorname{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]$ is a projective module over $k[\operatorname{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]$. Since $\operatorname{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})$ is a free abelian group, the ring $k[\operatorname{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]$ is the group algebra of a free abelian group over a field. By Theorem 4.42, all projective modules over $k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]$ must be free. Therefore, $k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]$ is a free module over $k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]$ as desired.

□

Lemma 4.44. *The ring $k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]$ is Galois over $k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]$, with Galois group*

$$\mu_\sigma = \mathrm{Spec}\left(k[\widetilde{N}^Y_{\varphi(\sigma)}/\widetilde{N}^X_\sigma]\right) \tag{4.151}$$

the kernel of the associated morphism on schemes. As a consequence, the isogenies of Theorem 4.29 satisfy all the conditions of Theorem 3.107.

Proof. We need to show that the automorphism group of $k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]$ over $k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]$ is μ_σ and that the rank of $k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]$ over $k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]$ is $|\mu_\sigma|$. By Proposition 4.36, we can choose bases so that our map looks like

$$k[t_1, t_1^{-1}, ..., t_n, t_n^{-1}] \longrightarrow k[s_1, s_1^{-1}, ..., s_n, s_n^{-1}] \tag{4.152}$$

given by mapping k identically to itself and mapping $t_i \mapsto s_i^{m_i}$, and by Proposition 4.39, we may assume that all the ${m_i}^{\text{th}}$ roots of unity are in k. Then the group of automorphisms of $k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]$ over $k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]$ is actually the group of automorphisms of $k[s_1, s_1^{-1}, ..., s_n, s_n^{-1}]$ over $k[s_1^{m_1}, s_1^{-m_1}, ..., s_n^{m_n}, s_n^{-m_n}]$. Take any such automorphism g. Then $g(x) = x$ for all $x \in k$ and $g(s_i) = \alpha_i s_i$ for some α_i (otherwise, if g sent s_i to a non-monomial in s_i, the condition that $s_i^{m_i} \mapsto s_i^{m_i}$ would be violated). But since $g(s_i^{m_i}) = s_i^{m_i}$, we have that $\alpha_i^{m_i} s_i^{m_i} = s_i^{m_i}$, or that $\alpha_i^{m_i} = 1$. Notice that g is determined by the α_i's, and therefore can be represented by the tuple

$$g = (\alpha_1, \alpha_2, ..., \alpha_n) \tag{4.153}$$

and notice that the tuple $(\alpha_1, \alpha_2, ..., \alpha_n) \in \mu_\sigma$. So given any element of μ_σ we can build an automorphism of $k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]$ over $k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]$, and conversely given an automorphism of $k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]$ over $k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]$, we construct an element of μ_σ. So μ_σ is the automorphism group of $k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]$ over $k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]$ as desired.

For the second part, notice that if our field k contains all $m_i{}^{\text{th}}$ roots of unity, then $|\mu_\sigma| = m_1 \cdot m_2 \cdots m_n$. So we need only show the rank is $m_1 \cdot m_2 \cdots m_n$. But this is immediate, as the set of all products

$$\prod_{i=1}^{n} s_i^{\beta_i} \tag{4.154}$$

where $\beta_i \leq m_i$ will form a basis for $k[s_1, s_1^{-1}, ..., s_n, s_n^{-1}]$ over $k[t_1, t_1^{-1}, ..., t_n, t_n^{-1}]$. Thus, the ring $k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]$ is Galois over $k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]$ with Galois group μ_σ as claimed.

□

Corollary 4.45. *Given the morphism*

$$f : k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})] \longrightarrow k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})] \tag{4.155}$$

of Lemma 4.44, $f_* \circ f^*$ is multiplication by $|\mu_\sigma|$.

Proof. This is immediate from Corollary 4.43, Lemma 4.44, and Theorem 3.101.

□

Theorem 4.46. *The maps*

$$f : k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})] \longrightarrow k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})] \tag{4.156}$$

arising in each component of

$$(\mathrm{BOT}_X)_n \longrightarrow (\mathrm{BOT}_Y)_n \tag{4.157}$$

as constructed in Theorem 4.29 all have the property that

$$(f_*)_{\mathbb{Q}} \circ (f^*)_{\mathbb{Q}} : \mathrm{K}_n(k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]) \otimes \mathbb{Q} \longrightarrow \mathrm{K}_n(k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]) \otimes \mathbb{Q}$$

is an isomorphism for all n, and

$$(f^*)_{\mathbb{Q}} : \mathrm{K}_n(k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]) \otimes \mathbb{Q} \longrightarrow \mathrm{K}_n(k[\mathrm{Hom}(\widetilde{N}^X_{\sigma}, \mathbb{Z})]) \otimes \mathbb{Q}$$

is injective for all n.

Proof. This is an immediate application of Corollary 4.45 and Corollary 3.103.

□

Caution 4.47. The reader might be tempted at this point to, following the logic of Corollary 4.32, attempt to construct a morphism of cosimplicial spectra

$$\mathrm{KH}(\mathrm{BOT}_X) \longrightarrow \mathrm{KH}(\mathrm{BOT}_Y) \tag{4.158}$$

which is, in each degree, component-wise given by transfer maps f_* as constructed in Lemma 3.96. Unfortunately, this is not possible. The reason is that the transfer maps do not commute with the coface maps of $\mathrm{KH}(\mathrm{BOT}_X)$ and $\mathrm{KH}(\mathrm{BOT}_Y)$, and thus cannot give rise to a morphism of cosimplicial spectra.

Even though the transfer maps do not directly give rise to a morphism of cosimplicial spectra, we can still use them to gain information about f^* and, more importantly, $(f^*)_{\mathbb{Q}}$. Specifically, we can use them to show that if f is an isogeny between two algebraic tori, then $(f^*)_{\mathbb{Q}}$ is an isomorphism for every n. This will be our next goal.

To accomplish this goal, given any isogeny $f : T_1 \longrightarrow T_2$ with kernel μ, we need to see how the group μ acts on the group $\mathrm{K}_n(T_1) \otimes \mathbb{Q}$. Since k contains all $m_i{}^{\mathrm{th}}$ roots of unity, then any $g \in \mu$ is of the form

$$g = (\zeta_{m_1}^{\alpha_1}, \zeta_{m_2}^{\alpha_2}, ..., \zeta_{m_n}^{\alpha_n}) \tag{4.159}$$

where ζ_{m_j} is a primitive m_jth root of unity. Then g defines a morphism $T_1 \longrightarrow T_1$ given by

$$g(x_1, x_2, ..., x_n) = (\zeta_{m_1}^{\alpha_1} x_1, \zeta_{m_2}^{\alpha_2} x_2, ..., \zeta_{m_n}^{\alpha_n} x_n). \tag{4.160}$$

This corresponds to the morphism of rings

$$g : k[s_1, s_1^{-1}, ..., s_n, s_n^{-1}] \longrightarrow k[s_1, s_1^{-1}, ..., s_n, s_n^{-1}] \tag{4.161}$$

given by $g(x) = x$ for $x \in k$ and $g(s_i) = \zeta_{m_i}^{\alpha_i} s_i$ for each i (since μ is the Galois group of $k[s_1, s_1^{-1}, ..., s_n, s_n^{-1}]$ over $k[s_1^{m_1}, s_1^{-m_1}, ..., s_n^{m_n}, s_n^{-m_n}]$, then in view of Lemma 4.44, this should not be surprising). One can again easily see that these are equivalent by taking $\mathrm{Hom}(-, k)$ everywhere. This gives rise to a group action of μ on $\mathrm{K}_n(T_1) \otimes \mathbb{Q}$; if $g \in \mu$ and $x \in \mathrm{K}_n(T_1) \otimes \mathbb{Q}$, then define $g \cdot x = (g^*)_{\mathbb{Q}}(x)$, where

$$g^* : \mathrm{K}_n(T_1) \longrightarrow \mathrm{K}_n(T_1) \tag{4.162}$$

is the map induced by taking K-theory, and where $(g^*)_{\mathbb{Q}} = g^* \otimes \mathrm{id}_{\mathbb{Q}}$ as usual.

Theorem 4.48. *Let $f : T_1 \longrightarrow T_2$ be an isogeny with kernel μ. Then*

$$\mathrm{im}((f^*)_{\mathbb{Q}}) = [\mathrm{K}_n(T_1) \otimes \mathbb{Q}]^{\mu}, \tag{4.163}$$

the points of $\mathrm{K}_n(T_1) \otimes \mathbb{Q}$ fixed by the action of μ, for every n. As a consequence, we have that

$$(f^*)_{\mathbb{Q}} : \mathrm{K}_n(T_2) \otimes \mathbb{Q} \xrightarrow{\cong} [\mathrm{K}_n(T_1) \otimes \mathbb{Q}]^{\mu}$$

is an isomorphism for every n.

Proof. To see that $\mathrm{im}((f^*)_{\mathbb{Q}}) \subset [\mathrm{K}_n(T_1) \otimes \mathbb{Q}]^{\mu}$, observe that for any $g \in \mu$, we have that $f = f \circ g$. This can easily be seen by direct calculation:

$$\begin{aligned}
(f \circ g)(x_1, x_2, ..., x_n) &= f(\zeta_{m_1}^{\alpha_1} x_1, \zeta_{m_2}^{\alpha_2} x_2, ..., \zeta_{m_n}^{\alpha_n} x_n) \\
(f \circ g)(x_1, x_2, ..., x_n) &= ((\zeta_{m_1}^{\alpha_1} x_1)^{m_1}, (\zeta_{m_2}^{\alpha_2} x_2)^{m_2}, ..., (\zeta_{m_n}^{\alpha_n} x_n)^{m_n}) \\
(f \circ g)(x_1, x_2, ..., x_n) &= (x_1^{m_1}, x_2^{m_2}, ..., x_n^{m_n}) \\
(f \circ g)(x_1, x_2, ..., x_n) &= f(x_1, x_2, ..., x_n)
\end{aligned} \tag{4.164}$$

which shows that $f = f \circ g$ as claimed. Then taking the induced maps on $\mathrm{K}_n(-) \otimes \mathbb{Q}$, we get that $(f^*)_{\mathbb{Q}} = (g^*)_{\mathbb{Q}} \circ (f^*)_{\mathbb{Q}}$. Therefore, g fixes every element of $\mathrm{im}((f^*)_{\mathbb{Q}})$; since this can be done for any element $g \in \mu$, we get that

$$\mathrm{im}((f^*)_{\mathbb{Q}}) \subset [\mathrm{K}_n(T_1) \otimes \mathbb{Q}]^{\mu} \tag{4.165}$$

as claimed. One could also do this from the perspective of rings by appealing to the fact that μ is the Galois group of the associated morphism of rings.

For the other containment, we need to use a trace argument. Suppose that V is any $\mathbb{Q}$-vector space, and G is any finite group. Then we can define an operator $t : V \longrightarrow V$, where for $v \in V$ we have

$$t(v) = \sum_{g \in G} (g \cdot v). \tag{4.166}$$

Then $V^G = t(V)$. The containment $V^G \supset t(V)$ is obvious, since if we act on $t(V)$ by any element of G, we simply permute the terms in the sum. On the other hand, if $v \in V^G$, then $g \cdot v = v$ for all $g \in G$ and so $|G|v = \sum_{g\in G} v = \sum_{g \in G}(g \cdot v) = t(v)$ (remember that $|G|$ is finite). Thus we can write $v = \frac{1}{|G|} \cdot t(v) = t(\frac{1}{|G|} \cdot v) \in t(V)$, giving the other inclusion.

Letting $V = \mathrm{K}_n(T_1) \otimes \mathbb{Q}$ and letting $G = \mu$ above, we see that

$$[\mathrm{K}_n(T_1) \otimes \mathbb{Q}]^{\mu} = t(\mathrm{K}_n(T_1) \otimes \mathbb{Q}). \tag{4.167}$$

By Theorem 3.107, we have that $(f^*)_{\mathbb{Q}} \circ (f_*)_{\mathbb{Q}} = \sum_{g \in \mu} (g^*)_{\mathbb{Q}}$. Realizing that $g \cdot v = (g^*)_{\mathbb{Q}}(v)$ under this action, we have that $(f^*)_{\mathbb{Q}} \circ (f_*)_{\mathbb{Q}} = t$, where t is the trace operator from above, and that

$$(f^*)_{\mathbb{Q}} \circ (f_*)_{\mathbb{Q}}(\mathrm{K}_n(T_1) \otimes \mathbb{Q}) = t(\mathrm{K}_n(T_1) \otimes \mathbb{Q}) = [\mathrm{K}_n(T_1) \otimes \mathbb{Q}]^{\mu} \tag{4.168}$$

which shows that

$$[\mathrm{K}_n(T_1) \otimes \mathbb{Q}]^\mu \subset \mathrm{im}((f^*)_\mathbb{Q}), \tag{4.169}$$

giving the other inclusion.

The second part follows immediately from the first part, Theorem 4.46, and the fact that an injective morphism is an isomorphism onto its image.

□

By Theorem 4.48, we can show that $(f^*)_\mathbb{Q}$ is an isomorphism if we can show that $[\mathrm{K}_n(T_1) \otimes \mathbb{Q}]^\mu = \mathrm{K}_n(T_1) \otimes \mathbb{Q}$. This will be our next major goal. To that end, we need to understand better the action of μ on $\mathrm{K}_n(T_1) \otimes \mathbb{Q}$. Since any $g \in \mu$ acts on $\mathrm{K}_n(T_1) \otimes \mathbb{Q}$ by applying the map $(g^*)_\mathbb{Q}$, which is the identity on the $\mathbb{Q}$ part, it is enough to examine what g^* does to $\mathrm{K}_n(T_1)$. Since

$$T_1 = \mathrm{Spec}\left(k[t_1, t_1^{-1}, ..., t_s, t_s^{-1}]\right) \tag{4.170}$$

we can write

$$\mathrm{K}_n(T_1) = \mathrm{K}_n\left(k[t_1, t_1^{-1}, ..., t_s, t_s^{-1}]\right). \tag{4.171}$$

Here we can apply the Bass Fundamental Theorem (Theorem 3.82) inductively to see that

$$\mathrm{K}_n\left(k[t_1, t_1^{-1}, ..., t_s, t_s^{-1}]\right) = \bigoplus_{r \leq s} \{t_{j_1}, t_{j_2}, ..., t_{j_r}\} \cdot \mathrm{K}_{n-r}(k) \tag{4.172}$$

where $\{t_a, t_b\}$ denotes the cup product of $[t_a]$ and $[t_b]$ (viewed as elements of $\mathrm{K}_1\left(k[t_1, t_1^{-1}, ..., t_s, t_s^{-1}]\right)$ as usual).

Given any $g \in \mu$, recall that

$$g = (\zeta_{m_1}^{\alpha_1}, \zeta_{m_2}^{\alpha_2}, ..., \zeta_{m_n}^{\alpha_n}) \tag{4.173}$$

where ζ_{m_j} is the primitive m_jth root of unity. This g induces a map

$$k[t_1, t_1^{-1}, ..., t_s, t_s^{-1}] \longrightarrow k[t_1, t_1^{-1}, ..., t_s, t_s^{-1}] \tag{4.174}$$

given by mapping $t_i \mapsto \zeta_{m_i}^{\alpha_i} t_i$. This means that the element $t_i \in \mathrm{K}_1\left(k[t_1, t_1^{-1}, ..., t_s, t_s^{-1}]\right)$ is mapped to the element $\zeta_{m_i}^{\alpha_i} t_i \in \mathrm{K}_1\left(k[t_1, t_1^{-1}, ..., t_s, t_s^{-1}]\right)$. Therefore, we see that g acts on

$$\mathrm{K}_n\left(k[t_1, t_1^{-1}, ..., t_s, t_s^{-1}]\right) = \bigoplus_{r \leq s} \{t_{j_1}, t_{j_2}, ..., t_{j_r}\} \cdot \mathrm{K}_{n-r}(k) \tag{4.175}$$

by multiplying t_{j_i} by $\zeta_{m_{j_i}}^{\alpha_{j_i}}$, and by leaving any term from $\mathrm{K}_{n-r}(k)$ unchanged.

Lemma 4.49. *Let $x \in \mathrm{K}_n(T_1)$. Then $\zeta_{m_i}^{\alpha_i} \cdot (m_i \cdot x) = m_i \cdot x$.*

Proof. Write x as a tuple $(x_{j_1, j_2, ..., j_r})$, where

$$x_{j_1, j_2, ..., j_r} = \{t_{j_1}, t_{j_2}, ..., t_{j_r}\} \cdot y \tag{4.176}$$

for some $y \in \mathrm{K}_{n-r}(k)$. If $m \in \mathbb{Z}$ is any integer, then the action on $\mathrm{K}_n(T_1)$ is given by

$$m \cdot (x_{j_1, j_2, ..., j_r}) = \{t_{j_1}^m, t_{j_2}^m, ..., t_{j_r}^m\} \cdot (m \cdot y). \tag{4.177}$$

Letting $g = (1, ..., 1, \zeta_{m_i}^{\alpha_i}, 1, ..., 1) \in \mu$, we realize this acts by applying the map g^* as always. Since g^* is a homomorphism of abelian groups, we must have that

$$g \cdot (m_i \cdot x) = m_i \cdot (g \cdot x). \tag{4.178}$$

We show the latter of these is just $m_i \cdot x$. We do this componentwise. For every component of x, we have

$$m_i \cdot (x_{j_1, j_2, ..., j_r}) = \{t_{j_1}^{m_i}, t_{j_2}^{m_i}, ..., t_{j_r}^{m_i}\} \cdot (m_i \cdot y) \tag{4.179}$$

from the above $\mathbb{Z}$-action. On the other hand, applying g to x, we see that g acts trivially on any component where $i \notin \{j_1, j_2, ..., j_r\}$ and so obviously on those components $g \cdot (m_i \cdot x) = m_i \cdot x$. If $i \in \{j_1, j_2, ..., j_r\}$, then

$$g \cdot (x_{j_1, j_2, ..., j_r}) = \{t_{j_1}, t_{j_2}, ..., \zeta_{m_i}^{\alpha_i} t_i, ..., t_{j_r}\} \cdot y \tag{4.180}$$

and if we apply m_i to this component we get

$$\begin{aligned}
m_i \cdot (g \cdot (x_{j_1, j_2, ..., j_r})) &= m_i \cdot \left[\{t_{j_1}, t_{j_2}, ..., \zeta_{m_i}^{\alpha_i} t_i, ..., t_{j_r}\} \cdot y\right] \\
m_i \cdot (g \cdot (x_{j_1, j_2, ..., j_r})) &= \{t_{j_1}^{m_i}, t_{j_2}^{m_i}, ..., (\zeta_{m_i}^{\alpha_i} t_i)^{m_i}, ..., t_{j_r}^{m_i}\} \cdot (m_i \cdot y) \\
m_i \cdot (g \cdot (x_{j_1, j_2, ..., j_r})) &= \{t_{j_1}^{m_i}, t_{j_2}^{m_i}, ..., t_i^{m_i}, ..., t_{j_r}^{m_i}\} \cdot (m_i \cdot y) \\
m_i \cdot (g \cdot (x_{j_1, j_2, ..., j_r})) &= m_i \cdot (x_{j_1, j_2, ..., j_r})
\end{aligned} \tag{4.181}$$

as desired. So our statement is true on every component of x, and is therefore true on x. Since x was arbitrary, it is true on all of $\mathrm{K}_n(T_1)$.

□

Corollary 4.50. *Let $x \in \mathrm{K}_n(T_1)$. Then for any $g \in \mu$, $g \cdot (m \cdot x) = m \cdot x$, where $m = m_1 \cdot m_2 \cdots m_s$.*

Proof. As before, we have

$$g = (\zeta_{m_1}^{\alpha_1}, \zeta_{m_2}^{\alpha_2}, ..., \zeta_{m_n}^{\alpha_n}) \tag{4.182}$$

where ζ_{m_j} is the primitive m_jth root of unity. We can write this as

$$g = g_1 g_2 \cdots g_s \tag{4.183}$$

where

$$g_i = (1, ..., 1, \zeta_{m_i}^{\alpha_i}, 1, ..., 1). \tag{4.184}$$

Inductively applying g_1 through g_s and using Lemma 4.49 at each step then yields the result.

□

Theorem 4.51. *Let $f : T_1 \longrightarrow T_2$ be an isogeny with kernel μ. Then we have that*

$$(f^*)_{\mathbb{Q}} : \mathrm{K}_n(T_2) \otimes \mathbb{Q} \xrightarrow{\cong} \mathrm{K}_n(T_1) \otimes \mathbb{Q}$$

is an isomorphism for every n.

Proof. By Theorem 4.48, we need only show that $[\mathrm{K}_n(T_1) \otimes \mathbb{Q}]^{\mu} = \mathrm{K}_n(T_1) \otimes \mathbb{Q}$. The fact that

$$[\mathrm{K}_n(T_1) \otimes \mathbb{Q}]^{\mu} \subset \mathrm{K}_n(T_1) \otimes \mathbb{Q} \tag{4.185}$$

is obvious. For the other direction, let $x \in \mathrm{K}_n(T_1) \otimes \mathbb{Q}$. Then we can write $x = y \otimes \frac{p}{q}$, and we see that $qx = py \otimes 1$. In other words, $qx \in \mathrm{K}_n(T_1)$. Letting $m = m_1 \cdot m_2 \cdots m_s$ as in Corollary 4.50, and realizing that the action of μ on $\mathrm{K}_n(T_1) \otimes \mathbb{Q}$ only occurs in the $\mathrm{K}_n(T_1)$ coordinate, we see that $mqx = mpy \otimes 1$ is invariant under the action of μ by Corollary 4.50. However, as the action only occurs in the $\mathrm{K}_n(T_1)$ coordinate, we see that $mpy \otimes \frac{1}{mq}$ is invariant under the action of μ also, again by Corollary 4.50. But

$$mpy \otimes \frac{1}{mq} = y \otimes \frac{mp}{mq} = y \otimes \frac{p}{q} = x. \tag{4.186}$$

Thus, we see that x is invariant under the action of μ, which gives us that

$$\mathrm{K}_n(T_1) \otimes \mathbb{Q} \subset [\mathrm{K}_n(T_1) \otimes \mathbb{Q}]^{\mu}, \tag{4.187}$$

completing the proof.

□

With our above work, we are now ready to prove Theorem 4.9.

Proof of Theorem 4.9. Suppose we have a lattice morphism

$$F : N^X \longrightarrow N^Y \tag{4.188}$$

which is injective with finite cokernel such that the restriction maps

$$F|_{N^X_\sigma} : N^X_\sigma \longrightarrow N^Y_{\varphi(\sigma)} \tag{4.189}$$

are also injective with finite cokernel for any cone $\sigma \in \Delta_X$. We know from Theorem 4.29 that F induces a morphism

$$\mathrm{BOT}_X \longrightarrow \mathrm{BOT}_Y \tag{4.190}$$

which is component-wise in each degree an isogeny. By Theorem 4.51, the induced map

$$(f^*)_{\mathbb{Q}} : \mathrm{K}_n(k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]) \otimes \mathbb{Q} \longrightarrow \mathrm{K}_n(k[\mathrm{Hom}(\widetilde{N}^X_\sigma, \mathbb{Z})]) \otimes \mathbb{Q}$$

on each such component is an isomorphism for every n. Observe that it is this step which requires that the characteristic of k does not divide $|\operatorname{coker}(F)|$, since Theorem 4.51 depends on Theorem 4.48, which in turn depends on Theorem 3.107. One of the conditions of Theorem 3.107 is that the morphism in question is étale, and since we wish to use it for every map, we need all our isogenies to be étale. As we saw in Corollary 4.41, this is accomplished by assuming that the characteristic of k does not divide $|\operatorname{coker}(F)|$.

By Corollary 4.32, the induced morphism of cosimplicial spectra

$$\mathrm{KH}(\mathrm{BOT}_Y) \otimes \mathbb{Q} \longrightarrow \mathrm{KH}(\mathrm{BOT}_X) \otimes \mathbb{Q} \tag{4.191}$$

is in each degree component-wise given by morphisms of the form $(f^*)_{\mathbb{Q}}$. Recall from Theorem 3.54 that the model category structure we use for the category of spectra defines the class of weak equivalences to be quasi-isomorphisms. Since the morphism of spectra

$$(f^*)_{\mathbb{Q}} : \mathrm{K}(k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]) \otimes \mathbb{Q} \longrightarrow \mathrm{K}(k[\mathrm{Hom}(\widetilde{N}^X_{\sigma}, \mathbb{Z})]) \otimes \mathbb{Q}$$

induces the isomorphisms

$$(f^*)_{\mathbb{Q}} : \mathrm{K}_n(k[\mathrm{Hom}(\widetilde{N}^Y_{\varphi(\sigma)}, \mathbb{Z})]) \otimes \mathbb{Q} \longrightarrow \mathrm{K}_n(k[\mathrm{Hom}(\widetilde{N}^X_{\sigma}, \mathbb{Z})]) \otimes \mathbb{Q}$$

for every n, we see that $(f^*)_{\mathbb{Q}}$ is a weak equivalence of spectra. Observe from Theorem 3.56 that the model category structure on the category of cosimplicial spectra defines the class of weak equivalences to be the morphisms that are quasi-isomorphisms component-wise in each degree. Since the map

$$\mathrm{KH}(\mathrm{BOT}_Y) \otimes \mathbb{Q} \longrightarrow \mathrm{KH}(\mathrm{BOT}_X) \otimes \mathbb{Q} \tag{4.192}$$

is in each degree component-wise given by morphisms of the form $(f^*)_{\mathbb{Q}}$, and since we've shown these morphisms to be weak equivalences of spectra, we conclude that the morphism

$$\mathrm{KH}(\mathrm{BOT}_Y) \otimes \mathbb{Q} \longrightarrow \mathrm{KH}(\mathrm{BOT}_X) \otimes \mathbb{Q} \tag{4.193}$$

is a weak equivalence of cosimplicial spectra. By Proposition 3.71, we have that the morphism

$$\mathrm{holim}(\mathrm{KH}(\mathrm{BOT}_Y) \otimes \mathbb{Q}) \longrightarrow \mathrm{holim}(\mathrm{KH}(\mathrm{BOT}_X) \otimes \mathbb{Q}) \tag{4.194}$$

is a weak equivalence of spectra. This gives us the following diagram:

$$\begin{tikzcd}
\mathrm{holim}(\mathrm{KH}(\mathrm{BOT}_Y) \otimes \mathbb{Q}) \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathrm{holim}(\mathrm{KH}(\mathrm{BOT}_X) \otimes \mathbb{Q}) \arrow[d, "\sim"] \\
\mathrm{holim}(\mathrm{KH}(\mathcal{U}_Y) \otimes \mathbb{Q}) & \mathrm{holim}(\mathrm{KH}(\mathcal{U}_X) \otimes \mathbb{Q}) \\
\mathrm{KH}(Y) \otimes \mathbb{Q} \arrow[u, "\sim"] & \mathrm{KH}(X) \otimes \mathbb{Q} \arrow[u, "\sim"']
\end{tikzcd}$$

where the morphisms

$$\mathrm{holim}(\mathrm{KH}(\mathrm{BOT}_Y) \otimes \mathbb{Q}) \longrightarrow \mathrm{holim}(\mathrm{KH}(\mathcal{U}_Y) \otimes \mathbb{Q}) \tag{4.195}$$

and

$$\mathrm{holim}(\mathrm{KH}(\mathrm{BOT}_X) \otimes \mathbb{Q}) \longrightarrow \mathrm{holim}(\mathrm{KH}(\mathcal{U}_X) \otimes \mathbb{Q}) \tag{4.196}$$

are weak equivalences by Corollary 4.25, and the morphisms

$$\mathrm{KH}(Y) \otimes \mathbb{Q} \longrightarrow \mathrm{holim}(\mathrm{KH}(\mathcal{U}_Y) \otimes \mathbb{Q}) \tag{4.197}$$

and

$$\mathrm{KH}(X) \otimes \mathbb{Q} \longrightarrow \mathrm{holim}(\mathrm{KH}(\mathcal{U}_X) \otimes \mathbb{Q}) \tag{4.198}$$

are weak equivalences by the fact that $\mathrm{KH}(-) \otimes \mathbb{Q}$ satisfies étale (and therefore Zariski) descent (see our work at the end of Section 3.8). So the spectra $\mathrm{KH}(X) \otimes \mathbb{Q}$ and $\mathrm{KH}(Y) \otimes \mathbb{Q}$ are connected by a sequence of weak equivalences, making them weakly equivalent as spectra, and establishing $\mathrm{KH}_n(X) \otimes \mathbb{Q} \cong \mathrm{KH}_n(Y) \otimes \mathbb{Q}$ for all n. This completes the proof.

□

4.5 Applications of Theorem 4.9

With Theorem 4.9 proven, we can now apply it to calculate $\mathrm{KH}_n(\mathbb{P}(q_0, ..., q_d)) \otimes \mathbb{Q}$ for any n. To accomplish this, we first prove the following important lemma.

Lemma 4.52. *Any weighted projective space $\mathbb{P}(q_0, ..., q_d)$ has the same simplicial structure, in the sense of Definition 4.6, as ordinary projective space $\mathbb{P}^d$.*

Proof. Recall from Section 2.2 that both $\mathbb{P}^d$ and $\mathbb{P}(q_0, ..., q_d)$ are complete toric varieties. Recall that the fan of $\mathbb{P}^d$ is calculated by taking $\mathbb{Z}^{d+1}$, with basis $\{e_0, ..., e_d\}$, and building the lattice

$$N^{\mathbb{P}^d} = \mathbb{Z}^{d+1}/\langle e_0 + e_1 + \cdots + e_d \rangle. \tag{4.199}$$

Letting $x_i = \overline{e_i}$, the fan $\Delta_{\mathbb{P}^d}$ consists of all the cones generated by proper subsets of $\{x_0, x_1, ..., x_d\}$. Similarly, recall that the fan of $\mathbb{P}(q_0, ..., q_d)$ is calculated by taking $\mathbb{Z}^{d+1}$, with basis $\{f_0, ..., f_d\}$, and building the lattice

$$N^{\mathbb{P}(q_0,...,q_d)} = \mathbb{Z}^{d+1}/\langle q_0 f_0 + q_1 f_1 + \cdots + q_d f_d \rangle \tag{4.200}$$

Letting $y_i = \overline{f_i}$, the fan $\Delta_{\mathbb{P}(q_0,...,q_d)}$ consists of all the cones generated by proper subsets of $\{y_0, y_1, ..., y_d\}$. See Section 2.2 for the details of this. Define a map $\varphi : \mathrm{S}(\Delta_{\mathbb{P}^d}) \longrightarrow \mathrm{S}(\Delta_{\mathbb{P}(q_0,...,q_d)})$ given by mapping any cone $\langle x_{i_1}, x_{i_2}, ..., x_{i_r} \rangle$ in $\Delta_{\mathbb{P}^d}$ to the cone $\langle y_{i_1}, y_{i_2}, ..., y_{i_r} \rangle$ in $\Delta_{\mathbb{P}(q_0,...,q_d)}$. This map is clearly an isomorphism of simplicial complexes in the sense of Definition 4.6; therefore, weighted projective space $\mathbb{P}(q_0, ..., q_d)$ has the same simplicial structure as ordinary projective space $\mathbb{P}^d$.

□

We are now ready to calculate the groups $\mathrm{KH}_n(\mathbb{P}(q_0, ..., q_d)) \otimes \mathbb{Q}$, by comparing them to $\mathrm{KH}_n(\mathbb{P}^d) \otimes \mathbb{Q}$.

Theorem 4.53. *If $\mathbb{P}(q_0, ..., q_d)$ is any d-dimensional weighted projective space defined over a field k such that the characteristic of k does not divide the product $q_0 \cdot q_1 \cdots q_d$, then*

$$\mathrm{KH}_n(\mathbb{P}(q_0, ..., q_d)) \otimes \mathbb{Q} \cong \mathrm{KH}_n(\mathbb{P}^d) \otimes \mathbb{Q} \tag{4.201}$$

for all n.

Proof. We prove this by using Theorem 4.9. Let $N^{\mathbb{P}^d}$ and $N^{\mathbb{P}(q_0,...,q_d)}$ be as in the proof of Lemma 4.52. By Lemma 4.52, $\mathbb{P}^d$ and $\mathbb{P}(q_0, ..., q_d)$ have the same simplicial structure, so to apply Theorem 4.9, we need to find a lattice morphism $F : N^{\mathbb{P}^d} \longrightarrow N^{\mathbb{P}(q_0,...,q_d)}$ that satisfies the necessary conditions.

To see how to properly define F, we consider the map $\widetilde{F} : \mathbb{Z}^{d+1} \longrightarrow \mathbb{Z}^{d+1}$ defined by $\widetilde{F}(e_i) = q_i f_i$. We get the following diagram:

$$\begin{tikzcd}
\mathbb{Z}^{d+1} \arrow[r, "\widetilde{F}"] \arrow[d, "\pi_{N^{\mathbb{P}^d}}"'] & \mathbb{Z}^{d+1} \arrow[d, "\pi_{N^{\mathbb{P}(q_0,\ldots,q_d)}}"] \\
N^{\mathbb{P}^d} \arrow[r, dashed] & N^{\mathbb{P}(q_0,\ldots,q_d)}
\end{tikzcd}$$

where the two vertical maps are the canonical surjection maps and the dotted bottom horizontal line is the obvious map that makes this diagram commute. Define

$$F : N^{\mathbb{P}^d} \longrightarrow N^{\mathbb{P}(q_0,\ldots,q_d)} \tag{4.202}$$

to be this map; by construction, $F(x_i) = q_i y_i$. To see this map is well-defined, suppose that $\overline{a_1} = \overline{a_2}$ in $N^{\mathbb{P}^d}$ (that is, that $\pi_{N^{\mathbb{P}^d}}(a_1) = \pi_{N^{\mathbb{P}^d}}(a_2)$); then

$$a_1 = a_2 + z(e_0 + e_1 + \cdots + e_d) \tag{4.203}$$

for some $z \in \mathbb{Z}$. Applying $\widetilde{F}$ to both sides gives us

$$\widetilde{F}(a_1) = \widetilde{F}(a_2) + z(q_0 f_0 + q_1 f_1 + \cdots q_d f_d) \tag{4.204}$$

which, after applying canonical surjection, gives us that

$$\pi_{N^{\mathbb{P}(q_0,\ldots,q_d)}}\left(\widetilde{F}(a_1)\right) = \pi_{N^{\mathbb{P}(q_0,\ldots,q_d)}}\left(\widetilde{F}(a_2)\right). \tag{4.205}$$

By construction,

$$F\left(\overline{a_i}\right) = F\left(\pi_{N^{\mathbb{P}^d}}(a_i)\right) = \pi_{N^{\mathbb{P}(q_0,\ldots,q_d)}}\left(\widetilde{F}(a_i)\right); \tag{4.206}$$

therefore, $F\left(\overline{a_1}\right) = F\left(\overline{a_2}\right)$, showing that F is a well-defined morphism of lattices.

The next thing we need to show is that F is injective with finite cokernel. Suppose there is an α such that $F(\alpha) = 0$; since $\{x_1, ..., x_d\}$ is a basis for $N^{\mathbb{P}^d}$, we can write

$$\alpha = a_1 x_1 + \cdots + a_d x_d. \tag{4.207}$$

Applying F to this expression gives

$$F(\alpha) = a_1 q_1 y_1 + \cdots + a_d q_d y_d = 0$$

in $N^{\mathbb{P}(q_0,...,q_d)}$. Lifting back to $\mathbb{Z}^{d+1}$, there is a $z \in \mathbb{Z}$ such that

$$a_1 q_1 f_1 + \cdots + a_d q_d f_d = z q_0 f_0 + z q_1 f_1 + \cdots + z q_d f_d.$$

Subtracting the right side from the left side, we get

$$-(q_0 z) f_0 + q_1(a_1 - z) f_1 + \cdots q_d(a_d - z) f_d = 0.$$

Since $\{f_0, ..., f_d\}$ is a basis for $\mathbb{Z}^{d+1}$, and since $q_0 \neq 0$ by assumption, $z = 0$. Therefore, this reduces the above expression to

$$a_1 q_1 f_1 + \cdots + a_d q_d f_d = 0$$

in $\mathbb{Z}^{d+1}$. Again, since $\{f_0, ..., f_d\}$ is a basis for $\mathbb{Z}^{d+1}$, and since none of the q_i's are 0, we get that $a_i = 0$ for $i = 1, 2, ..., d$. This proves that F is injective. Since rank $\left(N^{\mathbb{P}^d}\right)$ and rank $\left(N^{\mathbb{P}(q_0,...,q_d)}\right)$ both equal d and F is injective, F automatically has finite cokernel as well. Notice that $|\operatorname{coker}(F)| = q_0 \cdot q_1 \cdots q_d$; since we want to apply Theorem 4.9, we see the assumption that the characteristic of k does not divide the product $q_0 \cdot q_1 \cdots q_d$ is a necessary one.

Now we need to show that, for any cone $\sigma \in \Delta_{\mathbb{P}^d}$, the restriction map

$$F|_{N_\sigma^{\mathbb{P}^d}} : N_\sigma^{\mathbb{P}^d} \longrightarrow N_{\varphi(\sigma)}^{\mathbb{P}(q_0,...,q_d)} \tag{4.208}$$

is injective with finite cokernel, where

$$\varphi : \mathrm{S}(\Delta_{\mathbb{P}^d}) \longrightarrow \mathrm{S}(\Delta_{\mathbb{P}(q_0,\dots,q_d)}) \tag{4.209}$$

is the isomorphism constructed in Lemma 4.52. To see that $F|_{N_\sigma^{\mathbb{P}^d}}$ takes values in $N_{\varphi(\sigma)}^{\mathbb{P}(q_0,\dots,q_d)}$ note that if there is a set of indices $\{i_1, \dots, i_k\}$ such that

$$\sigma = \langle x_{i_1}, \dots, x_{i_k} \rangle, \tag{4.210}$$

then

$$\varphi(\sigma) = \langle y_{i_1}, \dots, y_{i_k} \rangle. \tag{4.211}$$

Since $F(x_{i_j}) = q_{i_j} y_{i_j}$, the image of $F|_{N_\sigma^{\mathbb{P}^d}}$ is a sublattice generated by $\{q_{i_1} y_{i_1}, \dots, q_{i_k} y_{i_k}\}$, which is a sublattice of $N_{\varphi(\sigma)}^{\mathbb{P}(q_0,\dots,q_d)}$. Now, since $F|_{N_\sigma^{\mathbb{P}^d}}$ is the restriction of an injective map, it must also be injective, and since σ and $\varphi(\sigma)$ have the same dimension, rank $\left(N_\sigma^{\mathbb{P}^d}\right)$ and rank $\left(N_{\varphi(\sigma)}^{\mathbb{P}(q_0,\dots,q_d)}\right)$ must be equal, forcing the cokernel to be finite. Then by Theorem 4.9,

$$\mathrm{KH}_n(\mathbb{P}(q_0, \dots, q_d)) \otimes \mathbb{Q} \cong \mathrm{KH}_n(\mathbb{P}^d) \otimes \mathbb{Q} \tag{4.212}$$

for all n.

$\square$

Remark 4.54. Theorem 4.53 establishes part (a) of Theorem 1.1.

Corollary 4.55. *If $\mathbb{P}(q_0, \dots, q_d)$ is any d-dimensional weighted projective space defined over a field k such that the characteristic of k does not divide the product $q_0 \cdot q_1 \cdots q_d$, then for $n < 0$, $\mathrm{KH}_n(\mathbb{P}(q_0, \dots, q_d))$ has rank 0, and $\mathrm{KH}_0(\mathbb{P}(q_0, \dots, q_d))$ has rank $d + 1$.*

Proof. This is immediate from Theorem 4.53. Theorem 4.53 showed that

$$\mathrm{KH}_n(\mathbb{P}(q_0, \dots, q_d)) \otimes \mathbb{Q} \cong \mathrm{KH}_n(\mathbb{P}^d) \otimes \mathbb{Q} \tag{4.213}$$

for all n. Then for $n < 0$, $\mathrm{KH}_n(\mathbb{P}^d) = \mathrm{K}_n(\mathbb{P}^d) = 0$. Therefore, $\mathrm{KH}_n(\mathbb{P}^d) \otimes \mathbb{Q} = 0$, forcing $\mathrm{KH}_n(\mathbb{P}(q_0, ..., q_d)) \otimes \mathbb{Q} = 0$. So the rank of $\mathrm{KH}_n(\mathbb{P}(q_0, ..., q_d))$ (for $n < 0$) is 0 as claimed.

Now recalling that $\mathrm{KH}_0(\mathbb{P}^d) = \mathrm{K}_0(\mathbb{P}^d) = \mathbb{Z}^{d+1}$, $\mathrm{KH}_0(\mathbb{P}^d) \otimes \mathbb{Q} = \mathbb{Q}^{d+1}$. This forces $\mathrm{KH}_0(\mathbb{P}(q_0, ..., q_d)) \otimes \mathbb{Q} = \mathbb{Q}^{d+1}$. So the rank of $\mathrm{KH}_0(\mathbb{P}(q_0, ..., q_d))$ is $d + 1$ as claimed.

□

We can use Corollary 4.55 and the methods of Section 4.1 to calculate the KH-theory (up to degree 0) of weighted projective spaces of the form $\mathbb{P}(1, 1, 1, 1, ..., 1, a)$. We do so in the following corollary.

Corollary 4.56. *Consider the d-dimensional weighted projective space $\mathbb{P}(1, 1, 1, 1, ..., 1, a)$, with $a \geq 2$. Then*

$$\mathrm{KH}_n(\mathbb{P}(1, 1, 1, 1, ..., 1, a)) = 0 \tag{4.214}$$

for $n \leq -1$ and

$$\mathrm{KH}_0(\mathbb{P}(1, 1, 1, 1, ..., 1, a)) = \mathbb{Z}^{d+1}. \tag{4.215}$$

Proof. The fan for $\mathbb{P}(1, 1, 1, 1, ..., 1, a)$ is generated by the 1-dimensional cones

$$\{e_1, e_2, ..., e_d, -e_1 - e_2 - \cdots - e_{d-1} - ae_d\}. \tag{4.216}$$

The steps are almost word-for-word the same as the $\mathbb{P}(1, 1, a)$ case. The only singular cone is $\langle e_1, e_2, ..., e_{d-1}, -e_1 - e_2 - \cdots - e_{d-1} - ae_d\rangle$, and after refining our fan by adding the cone generated by $-e_d$, we get a smooth toric variety; call this smooth variety $\widetilde{X}$. Now notice that the star of the cone $-e_d$ is just the fan for $\mathbb{P}^{d-1}$, so we get the blow-up square

$$\begin{tikzcd}
\mathbb{P}^{d-1} \arrow[r, "i"] \arrow[d] & \widetilde{X} \arrow[d] \\
\{*\} \arrow[r] & \mathbb{P}(1,1,1,1,...,1,a)
\end{tikzcd}$$

and because KH satisfies cdh descent, it gives rise to a long exact sequence

$$\cdots \longrightarrow \mathrm{KH}_n(\mathbb{P}(1,1,1,1,...,1,a)) \longrightarrow \mathrm{KH}_n(\widetilde{X}) \oplus \mathrm{KH}_n(k) \xrightarrow{\alpha_n}$$

$$\mathrm{KH}_n(\mathbb{P}^{d-1}) \longrightarrow \mathrm{KH}_{n-1}(\mathbb{P}(1,1,1,1,...,1,a)) \longrightarrow \cdots$$

Now we get a morphism $\pi : \widetilde{X} \longrightarrow \mathbb{P}^{d-1}$ induced by the lattice morphism $(x_1,...,x_{d-1},x_d) \mapsto (x_1,...,x_{d-1})$. Making the obvious analogous argument to the one presented in the $\mathbb{P}(1,1,2)$ case, we see that i_n^* is surjective for all n; therefore, the morphism α_n is surjective for every n as well (see Section 4.1). As before, our long exact sequence splits into short exact sequences of the form

$$0 \longrightarrow \mathrm{KH}_n(\mathbb{P}(1,1,1,1,...,1,a)) \longrightarrow \mathrm{KH}_n(\widetilde{X}) \oplus \mathrm{KH}_n(k) \xrightarrow{\alpha_d} \mathrm{KH}_n(\mathbb{P}^{d-1}) \longrightarrow 0$$

Since $\mathrm{KH}_n(\widetilde{X}) \oplus \mathrm{KH}_n(k) = 0$ for $n \leq -1$, $\mathrm{KH}_n(\mathbb{P}(1,1,1,1,...,1,a)) = 0$ for $n \leq -1$ as well. For the case $n = 0$, we have

$$0 \longrightarrow \mathrm{KH}_0(\mathbb{P}(1,1,1,1,...,1,a)) \longrightarrow \mathrm{KH}_0(\widetilde{X}) \oplus \mathrm{KH}_0(k) \xrightarrow{\alpha_0} \mathrm{KH}_0(\mathbb{P}^{d-1}) \longrightarrow 0$$

We could at this point attempt to calculate $\mathrm{KH}_0(\widetilde{X})$ and then use the fact that this above sequence splits. However, in [MP, Corollary 7.8], the authors prove that if Y is any smooth projective toric variety, then $\mathrm{K}_0(Y)$ is a free abelian group of finite rank; since $\widetilde{X}$ is a smooth, projective toric variety, we have that $\mathrm{KH}_0(\widetilde{X})$ is a free abelian group of finite rank. Since $\mathrm{KH}_0(k)$ is also free, $\mathrm{KH}_0(\mathbb{P}(1,1,1,1,...,1,a))$ is a subgroup of a free abelian group, and is therefore itself free abelian. By Corollary 4.55, the rank of $\mathrm{KH}_0(\mathbb{P}(1,1,1,1,...,1,a))$ is $d+1$;

therefore, we get $\mathrm{KH}_0(\mathbb{P}(1,1,1,1,...,1,a)) = \mathbb{Z}^{d+1}$, as desired.

$\square$

5 The $\mathcal{F}_{\mathrm{K}}$ groups for Weighted Projective Spaces of Dimension 2

Having calculated the $\mathrm{KH}(-) \otimes \mathbb{Q}$ groups for weighted projective spaces in Theorem 4.53, we are now ready to examine the $\mathcal{F}_{\mathrm{K}}$ groups. The calculation of these groups is in general much more difficult, so we will restrict ourselves to the case when the characteristic of the underlying field is 0 and when the weighted projective space has dimension 2. We begin with a definition.

Definition 5.1. *Let F be a field of characteristic 0 and X an F-scheme essentially of finite type over F. We say that X is K_n-regular if $\mathrm{K}_i(X) \cong \mathrm{KH}_i(X)$ for all $i \leq n$. Equivalently, we say X is K_n-regular if $(\mathcal{F}_{\mathrm{K}})_i(X) = 0$ for $i \leq n$.*

In [CHSW], the authors prove that if k is a field of characteristic 0 and X an k-scheme essentially of finite type and of dimension d, then X is K_{-d}-regular and for $n < -d$, we have $\mathrm{K}_n(X) = 0$. Our goal will be to derive stronger K-regularity results for complete toric surfaces, and in particular weighted projective spaces of dimension 2. We will begin by returning to the example of $\mathbb{P}(1,1,2)$ in Section 5.1 before progressing to weighted projective spaces of the form $\mathbb{P}(1,1,a)$, for $a > 1$, in Section 5.2. We will then proceed to the general case of complete toric surfaces, with a particular emphasis on $\mathbb{P}(a,b,c)$ (where $a,b,c > 1$ and all are pairwise relatively prime) in Section 5.3. We then conclude this paper with a discussion of what the $\mathcal{F}_{\mathrm{K}}$ groups will look like for certain classes of higher dimensional complete toric varieties and weighted projective spaces in Section 5.4.

5.1 The $\mathcal{F}_{\mathrm{K}}$ groups for $\mathbb{P}(1,1,2)$

We begin just as we did in Section 4, by returning to the example of $\mathbb{P}(1,1,2)$. Recall that in Section 4.1, we concluded that $\mathrm{KH}_n(\mathbb{P}(1,1,2))$ was 0 when $n \leq -1$ and was $\mathbb{Z}^3$ when $n = 0$. We proved that result using cdh-descent. While $\mathcal{F}_{\mathrm{K}}$ does not satisfy cdh-descent, it does satisfy Zariski descent as discussed in Section 3.8. To that end, we want to construct the correct Zariski cover for $\mathbb{P}(1,1,2)$ to allow us to use Zariski descent. The obvious starting point is to use the open cover given by the fan. Recall that the fan is generated by the one-dimensional cones $(1,0)$, $(0,1)$, and $(-1,-2)$ in the lattice $\mathbb{Z}^2$. The affine open subsets that form the cover we want are the affine open sets associated to each of the maximal cones. Following the outline given in Section 2.1 for constructing an affine scheme associated to a cone, we see that the cone $\sigma_1 = \langle (1,0),(0,1)\rangle$ gives us $U_{\sigma_1} = \mathbb{A}^2$, that the cone $\sigma_2 = \langle (0,1),(-1,-2)\rangle$ also gives us $U_{\sigma_2} = \mathbb{A}^2$ (with different coordinates of course), and that the cone $\sigma_3 = \langle (1,0),(-1,-2)\rangle$ gives us

$$U_{\sigma_3} = \operatorname{Spec}\left(k[u,v,w]/(uw - v^2)\right). \tag{5.1}$$

Notice that σ_3 is the only non-smooth cone in the fan of $\mathbb{P}(1,1,2)$; indeed, the other dimension 2 cones are smooth, and all 1-dimensional cones are smooth since any toric variety is normal (and therefore smooth in codimension 1). Then we get the following theorem.

Theorem 5.2. *For U_{σ_3} as above, we have that $(\mathcal{F}_{\mathrm{K}})_n(\mathbb{P}(1,1,2)) \cong (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_3})$ for all n.*

Proof. Let $Y = U_{\sigma_1} \cup U_{\sigma_2}$ and $Z = Y \cap U_{\sigma_3}$. We begin by showing that

$$(\mathcal{F}_{\mathrm{K}})_n(Y) = (\mathcal{F}_{\mathrm{K}})_n(Z) = 0 \tag{5.2}$$

for all n. Covering Y by U_{σ_1} and U_{σ_2} and using Zariski descent, we have the long exact sequence:

$$\begin{array}{rcl} \cdots \longrightarrow & (\mathcal{F}_{\mathrm{K}})_n(Y) & \longrightarrow (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2}) \longrightarrow \cdots \\ \cdots \longrightarrow & (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1} \cap U_{\sigma_2}) & \longrightarrow (\mathcal{F}_{\mathrm{K}})_{n-1}(Y) \longrightarrow \cdots \end{array} \tag{5.3}$$

Noting that U_{σ_1} and U_{σ_2} are smooth (and therefore that $U_{\sigma_1} \cap U_{\sigma_2}$ is smooth), we have that

$$(\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1}) = (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2}) = (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1} \cap U_{\sigma_2}) = 0 \tag{5.4}$$

for all n. Therefore, by exactness, we have that $(\mathcal{F}_{\mathrm{K}})_n(Y) = 0$ for all n as well.

Now $Z = Y \cap U_{\sigma_3} = U_{\sigma_1 \cap \sigma_3} \cup U_{\sigma_2 \cap \sigma_3}$. Covering Z by the open sets $U_{\sigma_1 \cap \sigma_3}$ and $U_{\sigma_2 \cap \sigma_3}$, and noticing that

$$U_{\sigma_1 \cap \sigma_3} \cap U_{\sigma_2 \cap \sigma_3} = U_{\sigma_1 \cap \sigma_2 \cap \sigma_3} = U_0 = \mathbb{G}_m^2 \tag{5.5}$$

we get the long exact sequence (again using Zariski descent):

$$\begin{aligned} \cdots \longrightarrow \quad (\mathcal{F}_{\mathrm{K}})_n(Z) \quad &\longrightarrow (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1 \cap \sigma_3}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2 \cap \sigma_3}) \longrightarrow \cdots \\ \cdots \longrightarrow \quad (\mathcal{F}_{\mathrm{K}})_n(\mathbb{G}_m^2) \quad &\longrightarrow (\mathcal{F}_{\mathrm{K}})_{n-1}(Z) \longrightarrow \cdots \end{aligned} \tag{5.6}$$

Noting that $U_{\sigma_1 \cap \sigma_3}$, $U_{\sigma_2 \cap \sigma_3}$, and $\mathbb{G}_m^2$ are smooth, we have that

$$(\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1 \cap \sigma_3}) = (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2 \cap \sigma_3}) = (\mathcal{F}_{\mathrm{K}})_n(\mathbb{G}_m^2) = 0 \tag{5.7}$$

for all n. Therefore, by exactness, we have that $(\mathcal{F}_{\mathrm{K}})_n(Z) = 0$ for all n as well.

Finally, covering $\mathbb{P}(1,1,2)$ by the open sets Y and U_{σ_3} and using Zariski descent, we get the following long exact sequence:

$$\begin{aligned} \cdots \longrightarrow \quad (\mathcal{F}_{\mathrm{K}})_n(\mathbb{P}(1,1,2)) \quad &\longrightarrow (\mathcal{F}_{\mathrm{K}})_n(Y) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_3}) \longrightarrow \cdots \\ \cdots \longrightarrow \quad (\mathcal{F}_{\mathrm{K}})_n(Z) \quad &\longrightarrow (\mathcal{F}_{\mathrm{K}})_{n-1}(\mathbb{P}(1,1,2)) \longrightarrow \cdots \end{aligned} \tag{5.8}$$

However, $(\mathcal{F}_{\mathrm{K}})_n(Y) = (\mathcal{F}_{\mathrm{K}})_n(Z) = 0$ for all n by our above work; therefore, by exactness, we get that $(\mathcal{F}_{\mathrm{K}})_n(\mathbb{P}(1,1,2)) \cong (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_3})$ for all n.

□

So now we have reduced the problem to determining $(\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_3})$. Here we use the techniques presented in [CHWW2]. In that paper the authors show that if R is a graded ring with $R_0 = k$, and R is the homogeneous coordinate ring of a smooth projective variety, then we can calculate $\widetilde{\mathrm{K}}_n(R) = \mathrm{K}_n(R)/\mathrm{K}_n(k)$. In our case, since $\mathrm{KH}_n(U_{\sigma_3}) = \mathrm{K}_n(k)$, if

$U_{\sigma_3} = \mathrm{Spec}(R)$, where R is the homogeneous coordinate ring of a smooth projective variety (and its weight 0 component is k), then $\widetilde{\mathrm{K}}_n(R) \cong (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_3})$.

Proposition 5.3. *The ring $k[u,v,w]/(uw - v^2)$ is the homogeneous coordinate ring of the smooth projective curve $uw - v^2 = 0$.*

Proof. First notice that this is the affine coordinate ring of the cone $uw - v^2 = 0$. Following [Hart, Chapter I, Exercise 2.10], we immediately see that this ring is the homogeneous coordinate ring of the projective curve $uw - v^2 = 0$ since it is the affine coordinate ring of the cone over that projective curve. Now we simply need to show the projective curve $uw - v^2 = 0$ is indeed smooth. The singularities are the points where the Jacobian is singular; in this case, that means where all the partial derivatives are 0. But one can quickly see that this only occurs when $(u, v, w) = (0, 0, 0)$, which is not a projective point. So for all projective points this curve is non-singular, making it a smooth projective curve as desired.

□

Now letting $R = k[u,v,w]/(uw - v^2)$, we seek to calculate $\widetilde{\mathrm{K}}_n(R)$. Fortunately, this calculation was already done in [CHWW2]. We state the result below.

Theorem 5.4. *For $R = k[u,v,w]/(uw - v^2)$ and for all n, we have*

$$\widetilde{\mathrm{K}}_n(R) \cong \Omega_k^{n-1} \oplus \Omega_k^{n-3} \oplus \Omega_k^{n-5} \oplus \cdots \tag{5.9}$$

In particular, $\mathrm{K}_1(R) = \mathrm{K}_1(k) \oplus k$ and $\mathrm{K}_2(R) = \mathrm{K}_2(k) \oplus \Omega_k^1$.

Proof. See [CHWW2, Theorem 4.3]. Note that by Ω_k^i we mean $\Omega_{k/\mathbb{Q}}^i$.

□

Corollary 5.5. *If k is algebraic over $\mathbb{Q}$, the formula in Theorem 5.4 reduces to $\widetilde{\mathrm{K}}_n(R) = k$ when $n \geq 1$ and odd, and $\widetilde{\mathrm{K}}_n(R) = 0$ otherwise.*

Proof. By [Hart, Chapter II, Theorem 8.6A], since every extension of $\mathbb{Q}$ is separable, $\Omega_{k/\mathbb{Q}} = 0$ if $k/\mathbb{Q}$ is algebraic (because if $k/\mathbb{Q}$ is algebraic, its transcendence degree is 0). From here the result is immediate from Theorem 5.4.

□

Corollary 5.6. *We have that*

$$(\mathcal{F}_{\mathrm{K}})_n(\mathbb{P}(1,1,2)) \cong \Omega_k^{n-1} \oplus \Omega_k^{n-3} \oplus \Omega_k^{n-5} \oplus \cdots \tag{5.10}$$

As a consequence, we have that $\mathrm{K}_n(\mathbb{P}(1,1,2)) = 0$ *if* $n \leq -1$ *and* $\mathrm{K}_0(\mathbb{P}(1,1,2)) = \mathbb{Z}^3$*.*

Proof. The first assertion is immediate from Theorem 5.4 and the fact that

$$(\mathcal{F}_{\mathrm{K}})_n(\mathbb{P}(1,1,2)) \cong (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_3}) \cong \widetilde{\mathrm{K}}_n(R). \tag{5.11}$$

The second assertion follows from the first, the fact that

$$\mathrm{K}_n(\mathbb{P}(1,1,2)) \cong \mathrm{KH}_n(\mathbb{P}(1,1,2)) \oplus (\mathcal{F}_{\mathrm{K}})_n(\mathbb{P}(1,1,2)), \tag{5.12}$$

the work in Section 4.1 showing that $\mathrm{KH}_n(\mathbb{P}(1,1,2)) = 0$ if $n \leq -1$ and $\mathrm{KH}_0(\mathbb{P}(1,1,2)) = \mathbb{Z}^3$, and that fact that $\Omega_k^i = 0$ if $i < 0$.

□

5.2 The $\mathcal{F}_{\mathrm{K}}$ groups for $\mathbb{P}(1,1,a)$

Building on our work from Section 5.1, we now seek to generalize this situation for the case $\mathbb{P}(1,1,a)$, just as we did in the KH case. This time, $\sigma_3 = \langle (1,0), (-1,-a) \rangle$ and this gives us the ring

$$R = k[y^{-1}, xy^{-1}, x^2y^{-1}, ..., x^ay^{-1}]. \tag{5.13}$$

Then just as before we get the following theorem.

Theorem 5.7. *For* $\sigma_3 = \langle (1,0), (-1,-a) \rangle$*, and* $U_{\sigma_3} = \mathrm{Spec}(R)$ *the associated open affine subset, we have that* $(\mathcal{F}_{\mathrm{K}})_n(\mathbb{P}(1,1,a)) \cong (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_3}) \cong (\mathcal{F}_{\mathrm{K}})_n(R)$ *for all* n*.*

Proof. The proof is analogous to the proof of Theorem 5.2. Note that this theorem also will follow from Theorem 5.28.

□

Before proceeding, we make an important definition that we will use throughout this section.

Definition 5.8. *Consider the map $\mathbb{P}^1 \longrightarrow \mathbb{P}^d$ given by*

$$[S : T] \mapsto [S^d : S^{d-1}T : \cdots : ST^{d-1} : T^d]; \tag{5.14}$$

in other words, the Veronese Embedding of degree d on $\mathbb{P}^1$. We define the Veronese Curve of degree *d to be the image of the Veronese Embedding of degree d on $\mathbb{P}^1$. We will just say* Veronese Curve *when the degree of the embedding is unimportant or when the degree is understood from the context. Observe that the Veronese Curve of degree d is the projective variety given by the common zero locus of the homogeneous equations $u_iu_j - u_{i+1}u_{j-1}$ where $0 \leq i \leq d-1$ and $1 \leq j \leq d$. Notice that we do not require i and j to be different.*

The Veronese Curve has very nice properties that we plan to use. However, in order to examine these properties, we first need to recall the following proposition, due to Gubeladze.

Proposition 5.9. *Let R be a domain and M a monoid. Then the monoid algebra $R[M]$ is normal if and only if R is normal and M is saturated.*

Proof. See [Gub2, Theorem 1.5.2]. Note that what he calls a normal monoid is what we call a monoid that is saturated.

□

With Proposition 5.9, we are ready to prove the following useful properties of Veronese Curves.

Proposition 5.10. *Any Veronese Curve is rational, nonsingular, and projective.*

Proof. By its very definition, any Veronese Curve of degree a is a closed subvariety of the projective space $\mathbb{P}^a$; therefore it must be projective (see Definition 5.8). To check that any such curve is nonsingular, we show that the curve is normal. Since any normal variety is smooth in codimension 1, our curve being normal is equivalent to our curve being nonsingular. So to see normality, use Proposition 5.9 with the domain k and the monoid $\sigma^\vee \cap M$ where M is the dual of our given lattice N, as always, and $\sigma^\vee$ is the dual of the maximal cone $\sigma = \langle (1,0), (-1,-a) \rangle$. Since σ is maximal, $\sigma^\vee$ is also a rational, strongly convex polyhedral cone; therefore $\sigma^\vee \cap M$ is saturated by Lemma 2.1. Since any field is obviously normal, $k[\sigma^\vee \cap M]$ is normal. Since $k[\sigma^\vee \cap M]$ is isomorphic to the homogeneous coordinate ring for the Veronese Curve, the Veronese Curve is projectively normal and therefore normal (see [Hart, Chapter II, Exercise 5.4]). Finally, to show that the Veronese Curve is rational, we recall [Hart, Chapter I, Corollary 4.5], which says that two varieties X and Y are birationally equivalent if and only if there is an open set $U \subset X$ and $V \subset Y$ such that U and V are isomorphic. For convenience, let C denote the Veronese Curve of degree a inside $\mathbb{P}^a$. Suppose that $\mathbb{P}^1$ is given by the homogeneous coordinates $[S : T]$ and $\mathbb{P}^a$ is given by the homogeneous coordinates $[X_0 : X_1 : \cdots : X_a]$. Consider the affine open set $\widetilde{V}$ in $\mathbb{P}^a$ defined by $X_0 \neq 0$, and let $V = \widetilde{V} \cap C$ be the induced open set on C. Then V is given by the coordinates $\left[1 : \frac{T}{S} : \left(\frac{T}{S}\right)^2 : \cdots : \left(\frac{T}{S}\right)^a\right]$. Let U be the affine open set of $\mathbb{P}^1$ defined by $S \neq 0$; then U is given by coordinates $\left[1 : \frac{T}{S}\right]$. Then, when restricted to U, the Veronese Embedding is just the morphism between the affine varieties U and V given by

$$x \mapsto (x, x^2, x^3, ..., x^a) \tag{5.15}$$

which is a clear isomorphism, making the Veronese Curve rational.

□

Proposition 5.10 showed that any Veronese Curve is rational, nonsingular, and projective. However, any curve that is rational, nonsingular, and projective is in fact isomorphic to $\mathbb{P}^1$, which we can conclude from the following proposition.

Proposition 5.11. *Let X be a rational, nonsingular, projective curve. Then X is isomorphic to $\mathbb{P}^1$.*

Proof. This follows from [Hart, Chapter II, Propositions 6.7 and 6.8] as well as the discussion in [Hart, Chapter II, Example 6.10.1].

$\square$

Corollary 5.12. *If X is a Veronese Curve of degree a, then $X \cong \mathbb{P}^1$.*

Proof. This is immediate from Propositions 5.10 and 5.11.

$\square$

Veronese Curves will play a very important role in our understanding of $\mathbb{P}(1,1,a)$. Recall from Section 5.1 that $\mathcal{F}_{\mathrm{K}}(\mathbb{P}(1,1,2)) = \mathcal{F}_{\mathrm{K}}(R)$ where

$$R = k[y^{-1}, xy^{-1}, x^2y^{-1}]. \tag{5.16}$$

Here R is the affine coordinate ring for the cone over the Veronese Curve of degree 2. We claim that this statement is true for all $\mathbb{P}(1,1,a)$, with $a > 2$; namely, $\mathcal{F}_{\mathrm{K}}(\mathbb{P}(1,1,a)) = \mathcal{F}_{\mathrm{K}}(R)$, where R is the affine coordinate ring for the cone over the Veronese Curve of degree a. We prove this claim in the following theorem.

Theorem 5.13. *The ring*

$$R = k[y^{-1}, xy^{-1}, x^2y^{-1}, ..., x^ay^{-1}] \tag{5.17}$$

is the affine coordinate ring of the cone over the Veronese Curve of degree a.

Proof. Letting R be generated by the algebraic variables $u_i = x^iy^{-1}$, we notice right away that, given any i and j, we have $u_iu_j - u_{i+1}u_{j-1} = 0$, and in fact, all relationships between the u_i's can be derived from these ones. Notice that we do not require i and j to be different. Then R will be the affine coordinate ring of the cone over the projective variety given by the common zero locus of the homogeneous equations $u_i u_j - u_{i+1}u_{j-1}$. But this common zero locus is precisely Veronese Curve of degree a, by Definition 5.8. This is what we wanted.

□

Unfortunately, there is a Lemma that is used in [CHWW2] that is crucial for the proof of Theorem 5.4; namely, that the curve we are taking the cone over must be a complete intersection. Since the Veronese Curve of degree d is not a complete intersection for $d > 2$, there is no simple analog of Theorem 5.4. So our goal is to calculate as much of the K-theory as we can in spite of this.

Let X be a smooth projective variety of dimension d, and R be the affine coordinate ring of the cone over X. We define the $\mathbb{Q}$-vector space $\mathrm{K}_n^{(i)}(R)$ to be the i^{th} weight eigenspace of the Adams operation. Then for any n, we have

$$\mathrm{K}_n(R) \otimes \mathbb{Q} = \bigoplus_{i=0}^{d+1} \mathrm{K}_n^{(i)}(R) \tag{5.18}$$

as $\mathbb{Q}$-vector spaces. We can derive a similar result for $\widetilde{\mathrm{K}}_n(R)$. Indeed, since R is a graded ring, notice that $\widetilde{\mathrm{K}}_n(R)$ is an R_0-module for every n. In our case, $R_0 = k$, a field of characteristic 0. Therefore, k contains $\mathbb{Q}$, making $\widetilde{\mathrm{K}}_n(R)$ a $\mathbb{Q}$-vector space for every n. We define the space $\widetilde{\mathrm{K}}_n^{(i)}(R)$ to be the i^{th} weight space induced by the Adams operation on $\mathrm{K}_n(R) \otimes \mathbb{Q}$. Then for any n, we have

$$\widetilde{\mathrm{K}}_n(R) = \bigoplus_{i=0}^{d+1} \widetilde{\mathrm{K}}_n^{(i)}(R). \tag{5.19}$$

This is immediate from the corresponding decomposition for $\mathrm{K}_n(R) \otimes \mathbb{Q}$ and the fact that $\widetilde{\mathrm{K}}_n(R)$ is already a $\mathbb{Q}$-vector space. See [CHWW2] for a full discussion of this.

Theorem 5.14. *Let X be a smooth projective variety in $\mathbb{P}_k^N$ with homogeneous coordinate ring R. Then*

(a) $\mathrm{K}_{-m}^{(0)}(R) = 0$ *for all* $m > 0$ *and* $\widetilde{\mathrm{K}}_n^{(0)}(R) = 0$ *for* $n \geq 0$*;*

(b) $\mathrm{K}_0^{(1)}(R) \cong R^+/R$, *where* R^+ *denotes the seminormalization of* R*;*

(c) $\mathrm{K}_0^{(i+1)}(R) \cong \bigoplus_{t=1}^{\infty} H^i(X, \Omega_X^i(t))$ *for* $i \geq 1$*;*

(d) $\mathrm{K}_{-m}^{(i+1)}(R) \cong \bigoplus_{t=1}^{\infty} H^{m+i}(X, \Omega_X^i(t))$ *for any* $m > 0$ *and all* $i \geq 0$*.*

If k *has finite transcendence degree over* $\mathbb{Q}$*, then each* $\mathbb{Q}$*-vector space* $\mathrm{K}_0(R)/\mathbb{Z}$ *and* $\mathrm{K}_{-m}(R)$ *is finite dimensional.*

Proof. See [CHWW2, Proposition 1.5 and Theorem 2.1]. For the final remark, recall that we already showed that $\widetilde{\mathrm{K}}_n(R)$ a $\mathbb{Q}$-vector space for every n. Since $\widetilde{\mathrm{K}}_{-m}(R) = \mathrm{K}_{-m}(R)$ and $\widetilde{\mathrm{K}}_0(R) = \mathrm{K}_0(R)/\mathbb{Z}$, $\mathrm{K}_0(R)/\mathbb{Z}$ and $\mathrm{K}_{-m}(R)$ are indeed $\mathbb{Q}$-vector spaces. To see that they are finite dimensional, observe that if k has finite transcendence degree over $\mathbb{Q}$, then Ω_X^i is a coherent sheaf. For each $q \geq 0$, the $H^q(X, \Omega_X^i(t))$ are finite dimensional, and only finitely many are non-zero by Serre's Theorem B; for more details, see [CHWW2, Theorem 2.1] and [Hart, Chapter III, Theorem 5.2].

□

Theorem 5.15. *Let* X *be a smooth projective variety in* $\mathbb{P}_k^N$ *with homogeneous coordinate ring* R*. Then for all* $n \geq 1$ *we have graded isomorphisms:*

$$\begin{aligned} \mathrm{K}_n^{(n+1)}(R) &\cong \operatorname{coker}\left(\Omega_R^n/d\Omega_R^{n-1} \longrightarrow \bigoplus_{t=1}^{\infty} H^0(X, \Omega_X^n(t))\right) \\ \mathrm{K}_n^{(i)}(R) &\cong \oplus_{t=1}^{\infty} H^{i-n-1}(X, \Omega_X^{i-1}(t)) \text{ for } i \geq n+2. \end{aligned} \tag{5.20}$$

The graded decomposition of $\mathrm{K}_n^{(n+1)}(R) = \oplus_{t=1}^{\infty} \mathrm{K}_n^{(n+1)}(R)_t$ *is:*

$$\mathrm{K}_n^{(n+1)}(R)_t \cong \operatorname{coker}\left(\left(\Omega_R^n/d\Omega_R^{n-1}\right)_t \longrightarrow H^0(X, \Omega_X^n(t))\right). \tag{5.21}$$

Proof. See [CHWW2, Proposition 2.12].

□

Theorem 5.16. *Let X be a curve of genus g, embedded in $\mathbb{P}^N_k$ be a complete linear system of degree $d > 1$. Assume the twisted Gauss-Manin connection*

$$\bigtriangledown : H^0(X, \Omega^1_{X/k}(1)) \longrightarrow \Omega^1_k \otimes H^1(X, \mathcal{O}_X(1)) \tag{5.22}$$

is zero. Then $\mathrm{K}_1^{(2)}(R)_1 \cong k^{d+g-1} \neq 0$ and

$$\mathrm{K}_n^{(n+1)}(R)_1 \cong \Omega^1_k \otimes k^{d+g-1} \tag{5.23}$$

for $n \geq 1$. In particular, $\mathrm{K}_n^{(n+1)}(R)_1 \neq 0$ whenever n is between 1 and the transcendence degree of $k/\mathbb{Q}$. Here

Proof. See [CHWW2, Theorem 3.8].

□

Remark 5.17. The group $\mathrm{K}_n^{(n+1)}(R)_1$ denotes the weight 1 part of $\mathrm{K}_n^{(n+1)}(R)$ under the graded isomorphism of Theorem 5.15.

Theorem 5.18. *Let X be a smooth projective variety in $\mathbb{P}^N_k$ with homogeneous coordinate ring R. Suppose that X is a curve, so that $\dim(R) = 2$, and suppose further that R is reduced. The we have*

(a) $\mathrm{K}_1(R) \cong k^\times \oplus \mathrm{K}_1^{(2)}(R) \oplus \mathrm{K}_1^{(3)}(R)$ *and* $\mathrm{K}_1^{(i)}(R) = 0$ *for* $i \geq 4$*;*

(b) $\mathrm{K}_2(R) \cong \mathrm{K}_2(k) \oplus \operatorname{tors} \Omega^1_R \oplus \mathrm{K}_2^{(3)}(R) \oplus \mathrm{K}_2^{(4)}(R)$*;*

(c) $\mathrm{K}_n(R) \cong \mathrm{K}_n(k) \oplus \bigoplus_{i=2}^{n+2} \widetilde{\mathrm{K}}_n^{(i)}(R)$.

Proof. This is a special case of [CHWW2, Theorem 1.15].

□

At this point, we restrict ourselves to fields which are algebraic over $\mathbb{Q}$; this will avoid Ω^1_k being non-zero, and will force $\mathrm{K}_0(R)/\mathbb{Z}$ and $\mathrm{K}_{-m}(R)$ to be finite dimensional. This assumption will make Theorem 5.16 trivially true in our case. It also reduces Theorem 5.18 to the following Corollary.

Corollary 5.19. *If $R = k \oplus R_1 \oplus R_2 \oplus \cdots$ is seminormal of dimension 2 and k is algebraic over $\mathbb{Q}$, then*

(a) $\mathrm{K}_1(R) \cong k^\times \oplus \Omega^1_{cdh}(R)/\Omega^1(R)$ *where* $\Omega^1_{cdh}(R) = H^0_{cdh}(R, \Omega^1)$*;*

(b) $\mathrm{K}_2(R) \cong \mathrm{K}_2(k) \oplus \operatorname{tors} \Omega^1_R$*;*

(c) $\mathrm{K}_n(R) \cong \mathrm{K}_n(k) \oplus \widetilde{\mathrm{HC}}_{n-1}(R)$.

Proof. See [CHWW2, Proposition 1.17]. Note that $\widetilde{\mathrm{HC}}_n(R) = \mathrm{HC}_n(R)/\mathrm{HC}_n(k)$, where HC denotes the cyclic homology functor.

□

Using Corollary 5.19 above, along with [CHWW2, Proposition 2.12], one can prove the following theorem.

Theorem 5.20. *Suppose k is algebraic over $\mathbb{Q}$ and that R is the homogeneous coordinate ring of a smooth curve X over k. Then*

$$\widetilde{\mathrm{K}}_1(R) \cong \left(\oplus_{t=1}^{\infty} H^0_{Zar}(X, \Omega^1_{X/k}(t))\right)/\Omega^1_{R/k}. \tag{5.24}$$

The calculation of $\widetilde{\mathrm{HC}}_{n-1}(R)$ is beyond the scope of this paper, but Theorems 5.14, 5.16, 5.18, and 5.20, along with Corollary 5.19 give us the ability to calculate the K-theory of $\mathbb{P}(1,1,a)$ for all negative degrees as well as degree 0. We also give a description of the rational K-theory of $\mathbb{P}(1,1,a)$ in all positive degrees.

We begin with calculating the negative K-theory. By Theorem 5.14 part (d), we have

$$\mathrm{K}^{(i+1)}_{-m}(R) \cong \bigoplus_{t=1}^{\infty} H^{m+i}(X, \Omega^i_X(t)) \tag{5.25}$$

for any $m > 0$ and all $i \geq 0$. For our case, X is the Veronese Curve of degree a, and R is the affine coordinate ring of the cone over X. By Corollary 5.12, $X \cong \mathbb{P}^1$; therefore we also know that $\Omega^i_X(t) = \Omega^i_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(a \cdot t)$ by [CHWW2, Remark 2.13]. So our groups become

$$\mathrm{K}^{(i+1)}_{-m}(R) \cong \bigoplus_{t=1}^{\infty} H^{m+i}(\mathbb{P}^1, \Omega^i_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(a \cdot t)) \text{ for } m > 0,\ i \geq 0. \tag{5.26}$$

By Grothendieck's Vanishing Theorem (see [Hart, Chapter III, Theorem 2.7]), if $m \geq 2$ or $i \geq 2$ or $m = i = 1$, then these groups are 0. So we are left with the case $m = 1$ and $i = 0$ as the only possible non-zero case. So we have

$$\mathrm{K}^{(1)}_{-1}(R) \cong \bigoplus_{t=1}^{\infty} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a \cdot t)). \tag{5.27}$$

By [Hart, Chapter III, Theorem 5.1 part (d)], we have that

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a \cdot t)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2 - a \cdot t)) \tag{5.28}$$

as k-vector spaces. Since the latter of these is 0 for $t \geq 1$, the former is also 0 for all $t \geq 1$. Thus, we have just proven the following theorem.

Theorem 5.21. *Let X be the Veronese Curve of degree a, and let R be the affine coordinate ring of the cone over X. Then $\mathrm{K}_{-m}(R) = 0$ for all $m > 0$.*

Corollary 5.22. *For all $m \geq 1$ and all $a \geq 2$, we have that*

$$\mathrm{K}_{-m}(\mathbb{P}(1, 1, a)) = 0. \tag{5.29}$$

Proof. Recall Theorem 3.94, which says that

$$\mathrm{K}_{-m}(\mathbb{P}(1, 1, a)) = \mathrm{KH}_{-m}(\mathbb{P}(1, 1, a)) \oplus (\mathcal{F}_{\mathrm{K}})_{-m}(\mathbb{P}(1, 1, a)). \tag{5.30}$$

We saw in Theorem 4.1 that $\mathrm{KH}_{-m}(\mathbb{P}(1, 1, a)) = 0$ for $m \geq 1$. To calculate

$(\mathcal{F}_{\mathrm{K}})_{-m}(\mathbb{P}(1,1,a))$, recall that Theorem 5.7 says that

$$(\mathcal{F}_{\mathrm{K}})_{-m}(\mathbb{P}(1,1,a)) \cong (\mathcal{F}_{\mathrm{K}})_{-m}(R) \tag{5.31}$$

where R is the affine coordinate ring for the cone over the Veronese Curve of degree a. By Theorem 5.21, $\mathrm{K}_{-m}(R) = 0$ for all $m \geq 1$; consequently, $(\mathcal{F}_{\mathrm{K}})_{-m}(R) = 0$ for all $m \geq 1$, and we are done.

□

We next turn our attention to calculating $\mathrm{K}_0(R)$. By Theorem 5.14 parts (a), (b) and (c), we have that

$$\mathrm{K}_0^{(1)}(R) \cong R^+/R \tag{5.32}$$

where R^+ denotes the seminormalization of R,

$$\mathrm{K}_0^{(i+1)}(R) \cong \bigoplus_{t=1}^{\infty} H^i(X, \Omega_X^i(t)) \tag{5.33}$$

for $i \geq 1$, and that $\mathrm{K}_0^{(0)}(R) \cong \mathbb{Z}$. By Proposition 5.9, we know that R is in fact normal, and so $R^+ = R$. Therefore, $\mathrm{K}_0^{(1)}(R) = 0$. For the calculation of $\mathrm{K}_0^{(i+1)}(R)$, we again know by Grothendieck's Vanishing Theorem ([Hart, Chapter III, Theorem 2.7]) that these groups are all 0 if $i \geq 2$. So the only remaining case is $i = 1$:

$$\mathrm{K}_0^{(2)}(R) \cong \bigoplus_{t=1}^{\infty} H^1(X, \Omega_X^1(t)). \tag{5.34}$$

Since $\Omega_X^1(t) = \Omega_{\mathbb{P}^1}^1 \otimes \mathcal{O}_{\mathbb{P}^1}(a \cdot t)$ and since $\Omega_{\mathbb{P}^1}^1 \cong \mathcal{O}_{\mathbb{P}^1}(-2)$, we have that

$$\Omega_X^1(t) \cong \mathcal{O}_{\mathbb{P}^1}(a \cdot t - 2). \tag{5.35}$$

So this gives us

$$\mathrm{K}_0^{(2)}(R) \cong \bigoplus_{t=1}^{\infty} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a \cdot t - 2)). \tag{5.36}$$

By [Hart, Chapter III, Theorem 5.1 part (d)], we have that

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a \cdot t - 2)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-a \cdot t)). \tag{5.37}$$

Since the latter of these is 0 for all $t \geq 1$, the former must be also. Therefore, $\mathrm{K}_0^{(2)}(R) = 0$. Thus, we have just proven the following theorem.

Theorem 5.23. *Let X be the Veronese Curve of degree a, and let R be the affine coordinate ring of the cone over X. Then $\mathrm{K}_0(R) = \mathbb{Z}$.*

Corollary 5.24. *For all $a \geq 2$, we have that*

$$\mathrm{K}_0(\mathbb{P}(1, 1, a)) = \mathbb{Z}^3. \tag{5.38}$$

Proof. Recall Theorem 3.94, which says that

$$\mathrm{K}_0(\mathbb{P}(1, 1, a)) = \mathrm{KH}_0(\mathbb{P}(1, 1, a)) \oplus (\mathcal{F}_{\mathrm{K}})_0(\mathbb{P}(1, 1, a)). \tag{5.39}$$

We saw in Theorem 4.1 that $\mathrm{KH}_0(\mathbb{P}(1, 1, a)) = \mathbb{Z}^3$. To calculate $(\mathcal{F}_{\mathrm{K}})_0(\mathbb{P}(1, 1, a))$, recall that Theorem 5.7 says that

$$(\mathcal{F}_{\mathrm{K}})_0(\mathbb{P}(1, 1, a)) \cong (\mathcal{F}_{\mathrm{K}})_0(R) \tag{5.40}$$

where R is the affine coordinate ring for the cone over the Veronese Curve of degree a. By Theorem 5.23, $\mathrm{K}_0(R) = \mathbb{Z}$. Since $\mathrm{KH}_0(R) = \mathbb{Z}$ by Proposition 4.2, $(\mathcal{F}_{\mathrm{K}})_0(R) = 0$, and we are done.

□

For the higher K-theory, we can no longer derive a nice formula for the K-theory of

$\mathbb{P}(1,1,a)$; however, we can still use the description of $\mathrm{K}_n(R)$ to give a general expression.

Theorem 5.25. *Let X be the Veronese Curve of degree a, let R be the affine coordinate ring of the cone over X, and suppose k is algebraic over $\mathbb{Q}$. Then we have the following:*

$$\mathrm{K}_n(\mathbb{P}(1,1,a)) \otimes \mathbb{Q} \cong \begin{cases} 0 & \text{for } n < 0 \\ \mathbb{Q}^3 & \text{for } n = 0 \\ ((\mathrm{K}_1(k)) \otimes \mathbb{Q})^3 \oplus \left(\left(\Omega^1_{cdh}(R)/\Omega^1_{R/k} \right) \otimes \mathbb{Q} \right) & \text{for } n = 1 \\ ((\mathrm{K}_2(k)) \otimes \mathbb{Q})^3 \oplus ((\operatorname{tors} \Omega^1_R) \otimes \mathbb{Q}) & \text{for } n = 2 \\ ((\mathrm{K}_n(k)) \otimes \mathbb{Q})^3 \oplus \left(\left(\widetilde{\mathrm{HC}}_{n-1}(R) \right) \otimes \mathbb{Q} \right) & \text{for } n \geq 3 \end{cases} \tag{5.41}$$

where

$$\Omega^1_{cdh}(R)/\Omega^1_{R/k} \cong \left(\oplus_{t=1}^{\infty} H^0_{Zar}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a \cdot t - 2)) \right) / \Omega^1_{R/k} \tag{5.42}$$

and

$$\operatorname{tors} \Omega^1_R = \ker \left(\Omega^1_{R/k} \longrightarrow \Omega^1_{cdh}(R) \right). \tag{5.43}$$

Proof. The case when $n < 0$ is immediate from Corollary 5.22, after tensoring with $\mathbb{Q}$. Similarly, the case $n = 0$ is immediate from Corollary 5.24, after tensoring with $\mathbb{Q}$. The remaining cases follow from applying Theorem 4.53 (and the fact that $\mathrm{K}_n(\mathbb{P}^2) \cong \mathrm{K}_n(k)^3$), Corollary 5.19, Theorem 5.20, the fact that $\mathrm{KH}_n(R) \cong \mathrm{K}_n(k)$ (by Proposition 4.2), the result of Theorem 5.7 that says that $(\mathcal{F}_{\mathrm{K}})_n(\mathbb{P}(1,1,a)) \cong (\mathcal{F}_{\mathrm{K}})_n(R)$, and tensoring with $\mathbb{Q}$.

□

While this doesn't give us a complete calculation like in the $\mathbb{P}(1,1,2)$ case, it still gives us a fairly good description of the rational K-theory of $\mathbb{P}(1,1,a)$. Notice that Corollaries 5.22 and 5.24 are proven, in part, by showing that $(\mathcal{F}_{\mathrm{K}})_n(\mathbb{P}(1,1,a)) = 0$ for $n \leq 0$; in other words, by showing that $\mathbb{P}(1,1,a)$ is at least K_0-regular. One final question from our work in this section remains: is $\mathbb{P}(1,1,a)$ K_d-regular, for some $d > 0$? While we can't fully calculate

$(\mathcal{F}_{\mathrm{K}})_1(\mathbb{P}(1,1,a))$, we can say with certainty that it is non-zero, and thus that K_0-regularity is the best we can hope for (and which agrees with our earlier calculations in the $\mathbb{P}(1,1,2)$ case). We prove this as the following theorem.

Theorem 5.26. *The group* $(\mathcal{F}_{\mathrm{K}})_1(\mathbb{P}(1,1,a)) \neq 0$*. Therefore,* $\mathbb{P}(1,1,a)$ *is* K_0*-regular and no better.*

Proof. By Theorem 5.7, we have that $(\mathcal{F}_{\mathrm{K}})_1(\mathbb{P}(1,1,a)) \cong (\mathcal{F}_{\mathrm{K}})_1(R)$. As we saw in Theorem 5.16, $\mathrm{K}_1^{(2)}(R) \neq 0$ since $\mathrm{K}_1^{(2)}(R)_1 \cong k^{a-1} \neq 0$. As we saw in Theorem 5.18, we have that $\mathrm{K}_1^{(2)}(R) \subset (\mathcal{F}_{\mathrm{K}})_1(R)$; therefore, $(\mathcal{F}_{\mathrm{K}})_1(R) \neq 0$, and so $(\mathcal{F}_{\mathrm{K}})_1(\mathbb{P}(1,1,a)) \neq 0$ also, as desired. □

Remark 5.27. Combining Corollary 5.22, Corollary 5.24, and Theorem 5.26 establishes part (c) of Theorem 1.1.

5.3 The $\mathcal{F}_{\mathrm{K}}$ groups for Complete Simplicial Toric Surfaces and for Weighted Projective Spaces $\mathbb{P}(a,b,c)$

Our goal is to now proceed to the general case $\mathbb{P}(a,b,c)$. We begin by showing that there is an analog of Theorem 5.2 that extends to all complete toric surfaces.

Theorem 5.28 ($\mathcal{F}_{\mathrm{K}}$ Decomposition Theorem)**.** *Let X be any complete toric surface, and let U_{σ_1}, U_{σ_2},...,U_{σ_m} be all the open sets associated to a maximal cone in the fan Δ_X. Then we have*

$$(\mathcal{F}_{\mathrm{K}})_n(X) \cong (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2}) \oplus \cdots \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_m}) \tag{5.44}$$

for all n.

Proof. We proceed by induction on the number of open sets associated to maximal cones. We begin with the base case $m = 2$.

So let $X = U_{\sigma_1} \cup U_{\sigma_2}$. We want to show that

$$(\mathcal{F}_{\mathrm{K}})_n(X) \cong (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2}) \tag{5.45}$$

for all n. Covering X by U_{σ_1} and U_{σ_2} and using Zariski descent, we have the long exact sequence:

$$\begin{aligned} \cdots \longrightarrow \quad & (\mathcal{F}_{\mathrm{K}})_n(X) && \longrightarrow (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2}) \longrightarrow \cdots \\ \cdots \longrightarrow \quad & (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1} \cap U_{\sigma_2}) && \longrightarrow (\mathcal{F}_{\mathrm{K}})_{n-1}(X) \longrightarrow \cdots \end{aligned} \tag{5.46}$$

Now recalling that $U_{\sigma_1} \cap U_{\sigma_2} = U_{\sigma_1 \cap \sigma_2}$, we know immediately that $U_{\sigma_1} \cap U_{\sigma_2}$ is smooth. The reason is that $\sigma_1 \cap \sigma_2$ is either the 0 cone (in which case $U_{\sigma_1} \cap U_{\sigma_2} \cong \mathbb{G}_m^2$ and is obviously smooth) or is a 1-dimensional cone; for any toric variety a 1-dimensional cone must be smooth (since any toric variety is normal and therefore smooth in codimension 1). So we have that

$$(\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1} \cap U_{\sigma_2}) = 0 \tag{5.47}$$

for all n. Therefore, by exactness, we have that

$$(\mathcal{F}_{\mathrm{K}})_n(X) \cong (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2}) \tag{5.48}$$

for all n as desired. This establishes the case $m = 2$.

Now suppose the result is true for all $k < m$. Then we have

$$X = \bigcup_{i=1}^{m} U_{\sigma_i} \tag{5.49}$$

so we can cover X by the open sets Y and U_{σ_m}, where

$$Y = \bigcup_{i=1}^{m-1} U_{\sigma_i}. \tag{5.50}$$

Let $Z = Y \cap U_{\sigma_m}$. Then $(\mathcal{F}_{\mathrm{K}})_n(Z) = 0$ for all n. To see this, again proceed by induction on the number of open sets, where we notice that

$$Z = Y \cap U_{\sigma_m} = \bigcup_{i=1}^{m-1} U_{\sigma_i \cap \sigma_m} \tag{5.51}$$

is our cover. The case $m = 2$ is trivial. The case $m = 3$ is done by considering the cover $U_{\sigma_1 \cap \sigma_3}$ and $U_{\sigma_2 \cap \sigma_3}$. By the same reasoning as above, $U_{\sigma_i \cap \sigma_3}$ is smooth for $i = 1, 2$, and noticing that we have

$$U_{\sigma_1 \cap \sigma_3} \cap U_{\sigma_2 \cap \sigma_3} = U_{\sigma_1 \cap \sigma_2 \cap \sigma_3} = U_0 = \mathbb{G}_m^2 \tag{5.52}$$

we get the long exact sequence (again using Zariski descent):

$$\begin{array}{rcl} \cdots \longrightarrow & (\mathcal{F}_{\mathrm{K}})_n(Z) & \longrightarrow (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1 \cap \sigma_3}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2 \cap \sigma_3}) \longrightarrow \cdots \\ \cdots \longrightarrow & (\mathcal{F}_{\mathrm{K}})_n(\mathbb{G}_m^2) & \longrightarrow (\mathcal{F}_{\mathrm{K}})_{n-1}(Z) \longrightarrow \cdots \end{array} \tag{5.53}$$

Since $U_{\sigma_1 \cap \sigma_3}$, $U_{\sigma_2 \cap \sigma_3}$, and $\mathbb{G}_m^2$ are smooth, we have that

$$(\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1 \cap \sigma_3}) = (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2 \cap \sigma_3}) = (\mathcal{F}_{\mathrm{K}})_n(\mathbb{G}_m^2) = 0 \tag{5.54}$$

for all n. Therefore, by exactness, we have that $(\mathcal{F}_{\mathrm{K}})_n(Z) = 0$ for all n as well. This gives the case $m = 3$.

Now for the inductive step for $(\mathcal{F}_{\mathrm{K}})_n(Z)$, cover Z by the open sets $\widetilde{Z}$ and $U_{\sigma_{m-1} \cap \sigma_m}$, where

$$\widetilde{Z} = \bigcup_{i=1}^{m-2} U_{\sigma_i \cap \sigma_m}. \tag{5.55}$$

By the same reasoning as above, $U_{\sigma_i \cap \sigma_m}$ is smooth for all i, and noticing that, for any $i \neq j$, we have

$$U_{\sigma_i \cap \sigma_m} \cap U_{\sigma_j \cap \sigma_m} = U_{\sigma_i \cap \sigma_j \cap \sigma_m} = U_0 = \mathbb{G}_m^2 \tag{5.56}$$

which implies that

$$\widetilde{Z} \cap U_{\sigma_{m-1}\cap\sigma_m} = \bigcup_{i=1}^{m-2} U_{\sigma_i\cap\sigma_m} \cap U_{\sigma_{m-1}\cap\sigma_m} = \bigcup_{i=1}^{m-2} \mathbb{G}_m^2 = \mathbb{G}_m^2. \tag{5.57}$$

So we get the long exact sequence (again using Zariski descent):

$$\begin{aligned} \cdots \longrightarrow \quad (\mathcal{F}_{\mathrm{K}})_n(Z) \quad &\longrightarrow (\mathcal{F}_{\mathrm{K}})_n(\widetilde{Z}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_{m-1}\cap\sigma_m}) \longrightarrow \cdots \\ \cdots \longrightarrow \quad (\mathcal{F}_{\mathrm{K}})_n(\mathbb{G}_m^2) \quad &\longrightarrow (\mathcal{F}_{\mathrm{K}})_{n-1}(Z) \longrightarrow \cdots \end{aligned} \tag{5.58}$$

But $(\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_{m-1}\cap\sigma_m}) = 0$ because $U_{\sigma_{m-1}\cap\sigma_m}$ is smooth, and $(\mathcal{F}_{\mathrm{K}})_n(\widetilde{Z}) = 0$ by our inductive hypothesis. Since $(\mathcal{F}_{\mathrm{K}})_n(\mathbb{G}_m^2) = 0$ also, exactness gives us that $(\mathcal{F}_{\mathrm{K}})_n(Z) = 0$, giving the inductive step for Z.

Now we return to the inductive step for X, with the covering by Y and U_{σ_m}. Using Zariski descent, we get the following long exact sequence:

$$\begin{aligned} \cdots \longrightarrow \quad (\mathcal{F}_{\mathrm{K}})_n(X) \quad &\longrightarrow (\mathcal{F}_{\mathrm{K}})_n(Y) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_m}) \longrightarrow \cdots \\ \cdots \longrightarrow \quad (\mathcal{F}_{\mathrm{K}})_n(Z) \quad &\longrightarrow (\mathcal{F}_{\mathrm{K}})_{n-1}(X) \longrightarrow \cdots \end{aligned} \tag{5.59}$$

However, $(\mathcal{F}_{\mathrm{K}})_n(Z) = 0$ for all n by our above work; therefore, by exactness, we get that

$$(\mathcal{F}_{\mathrm{K}})_n(X) \cong (\mathcal{F}_{\mathrm{K}})_n(Y) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_m}) \tag{5.60}$$

for all n. By our induction hypothesis, we have that

$$(\mathcal{F}_{\mathrm{K}})_n(Y) \cong (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2}) \oplus \cdots \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_{m-1}}) \tag{5.61}$$

for all n. Substituting this in gives us

$$(\mathcal{F}_{\mathrm{K}})_n(X) \cong (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2}) \oplus \cdots \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_m}) \tag{5.62}$$

for all n, completing the inductive step.

$\square$

Corollary 5.29. *Let $\mathbb{P}(a,b,c)$ be any 2-dimensional weighted projective space, and let U_{σ_1}, U_{σ_2}, and U_{σ_3} be the open sets associated to the three maximal cones in the fan of $\mathbb{P}(a,b,c)$. Then we have that*

$$(\mathcal{F}_{\mathrm{K}})_n(\mathbb{P}(a,b,c)) \cong (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_3}) \tag{5.63}$$

for all n.

Proof. This is immediate from Theorem 5.28

$\square$

So just as we saw in Section 5.2, our problem reduces to calculation $(\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_i})$, for all i. The problem is that this time, while U_{σ_i} will still be the prime spectrum of the affine coordinate ring of a cone over a projective variety, that projective variety will rarely be smooth. It will most often be isomorphic to a chain of copies of $\mathbb{P}^1$ intersecting at a collection of points, and these intersection points will be singular (see [Ful, page 47]). So the results of [CHWW2] cannot be applied here.

Nevertheless, the results of Section 5.2 do still suggest that $(\mathcal{F}_{\mathrm{K}})_n(\mathbb{P}(a,b,c))$ should be 0 for $n \leq 0$. So even though we cannot calculate all of the $\mathcal{F}_{\mathrm{K}}$ groups, we seek to determine if $\mathbb{P}(a,b,c)$ is K_0-regular. To do so, we use the following results, originally due to Gubeladze.

Lemma 5.30 (Gubeladze)**.** *For any regular ring R and any monoid M, we have*

$$\mathrm{K}_n(R) = \mathrm{K}_n(R[M]) = 0 \tag{5.64}$$

for $n \leq -1$.

Proof. See [Gub3, Theorem 1.3], which in turn uses elements of [Gub] and [Gub2].

$\square$

Theorem 5.31. *If $X = U_\sigma$ is an affine toric variety, then* $\mathrm{K}_0(X) = \mathbb{Z}$ *and* $\mathrm{K}_n(X) = 0$ *for* $n \leq -1$. *Consequently,* $(\mathcal{F}_{\mathrm{K}})_n(X) = 0$ *for* $n \leq 0$.

Proof. For the K_0 part, see [CHWW, Proposition 5.7]. Note that while I write this as a consequence in the statement above, the proof in the $n = 0$ case centers around using that $\mathrm{KH}_0(X) = \mathbb{Z}$ (see Proposition 4.3 of this paper or the proof of [CHWW, Proposition 5.6]) and then showing by direct calculation that $(\mathcal{F}_{\mathrm{K}})_0(X) = 0$. A stronger version of this statement can be found in [Gub3].

For the case of K_n with $n \leq -1$, we use Lemma 5.30, the fact that $U_\sigma = \mathrm{Spec}\,(k[\sigma^\vee \cap M])$, the fact that all fields are regular, and the fact that $\sigma^\vee \cap M$ is a submonoid of M.

□

Corollary 5.32. *Let X be any complete toric surface. For all $n \leq 0$, we have*

$$(\mathcal{F}_{\mathrm{K}})_n(X) = 0. \tag{5.65}$$

Proof. This is immediate from Theorem 5.28 and Theorem 5.31.

□

Corollary 5.33. *For all $n \leq 0$, we have*

$$(\mathcal{F}_{\mathrm{K}})_n(\mathbb{P}(a,b,c)) = 0. \tag{5.66}$$

Proof. This is immediate from Corollary 5.32.

□

Corollary 5.34. *For any two complete toric surfaces X and Y satisfying the conditions of Theorem 4.9, we have that*

$$\mathrm{K}_n(X) \otimes \mathbb{Q} \cong \mathrm{K}_n(Y) \otimes \mathbb{Q} \tag{5.67}$$

for $n \leq 0$.

Proof. By Corollary 5.32, we have that

$$\mathrm{K}_n(X) \cong \mathrm{KH}_n(X) \tag{5.68}$$

and

$$\mathrm{K}_n(Y) \cong \mathrm{KH}_n(Y) \tag{5.69}$$

for $n \leq 0$. Tensoring each of these with $\mathbb{Q}$ gives us

$$\mathrm{K}_n(X) \otimes \mathbb{Q} \cong \mathrm{KH}_n(X) \otimes \mathbb{Q} \tag{5.70}$$

and

$$\mathrm{K}_n(Y) \otimes \mathbb{Q} \cong \mathrm{KH}_n(Y) \otimes \mathbb{Q} \tag{5.71}$$

for $n \leq 0$. By Theorem 4.9, we have that

$$\mathrm{KH}_n(X) \otimes \mathbb{Q} \cong \mathrm{KH}_n(Y) \otimes \mathbb{Q} \tag{5.72}$$

for all n. Composing all the isomorphisms gives the result.

□

Corollary 5.35. *If $\mathbb{P}(a, b, c)$ is any weighted projective space, we have that*

$$\mathrm{K}_n(\mathbb{P}(a, b, c)) \otimes \mathbb{Q} = 0 \tag{5.73}$$

for all $n < 0$, and that

$$\mathrm{K}_0(\mathbb{P}(a, b, c)) \otimes \mathbb{Q} \cong \mathbb{Q}^3. \tag{5.74}$$

Proof. By Corollary 5.34 applied to the toric varieties $\mathbb{P}(a, b, c)$ and $\mathbb{P}^2$, we have that

$$\mathrm{K}_n(\mathbb{P}(a, b, c)) \otimes \mathbb{Q} \cong \mathrm{K}_n(\mathbb{P}^2) \otimes \mathbb{Q} \tag{5.75}$$

for $n \leq 0$. But $\mathrm{K}_n(\mathbb{P}^2) = 0$ for $n < 0$ and $\mathrm{K}_0(\mathbb{P}^2) \cong \mathbb{Z}^3$. Therefore we have

$$\mathrm{K}_n(\mathbb{P}(a, b, c)) \otimes \mathbb{Q} \cong 0 \otimes \mathbb{Q} = 0 \tag{5.76}$$

for $n < 0$ and

$$\mathrm{K}_0(\mathbb{P}(a, b, c)) \otimes \mathbb{Q} \cong \mathbb{Z}^3 \otimes \mathbb{Q} \cong \mathbb{Q}^3 \tag{5.77}$$

which completes the proof.

□

Remark 5.36. Combining Corollaries 5.33 and 5.35 establishes part (b) of Theorem 1.1.

5.4 The $\mathcal{F}_{\mathrm{K}}$ groups for Weighted Projective Spaces of Higher Dimensions

In Section 5.3, we proved Theorem 5.28 and then used it, along with Theorem 4.9 to determine the rational K-theory (in degree $n \leq 0$) for complete toric surfaces, and in particular for 2-dimensional weighted projective spaces. Unfortunately, Theorem 5.28 does not, in general, extend to higher dimensions. The problem that arises is that, while we could always conclude that $U_{\sigma_i} \cap U_{\sigma_j}$ was smooth when our variety was dimension 2, it is not true in general that $U_{\sigma_i} \cap U_{\sigma_j}$ is smooth if the dimension of our variety is bigger than 2.

Example 5.37. *Consider the* 3*-dimensional weighted projective space* $\mathbb{P}(1, 1, 2, 4)$*. The fan is generated by the* 1*-dimensional cones*

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -2, -4)\}. \tag{5.78}$$

Let

$$\sigma_1 = \langle (1,0,0), (0,1,0), (-1,-2,-4) \rangle \tag{5.79}$$

and let

$$\sigma_2 = \langle (1,0,0), (0,0,1), (-1,-2,-4) \rangle. \tag{5.80}$$

Then we have that

$$\sigma_1 \cap \sigma_2 = \langle (1,0,0), (-1,-2,-4) \rangle. \tag{5.81}$$

I claim that this cone is singular, and therefore that $U_{\sigma_1} \cap U_{\sigma_2}$ is not smooth. Indeed, for this cone to be smooth, we would need to be able to find a vector $(a,b,c) \in \mathbb{Z}^3$ such that the matrix

$$\begin{pmatrix} 1 & -1 & a \\ 0 & -2 & b \\ 0 & -4 & c \end{pmatrix} \tag{5.82}$$

has determinant ± 1. But this is impossible because the determinant of this matrix is $4b - 2c$ and there are no integers b and c that make this equation equal to ± 1 (the $\gcd(2,4) = 2 \neq 1$). So this cone is indeed singular.

As Example 5.37 shows, $U_{\sigma_i} \cap U_{\sigma_j}$ does not need to be smooth even in dimension 3, and so we cannot express $(\mathcal{F}_{\mathrm{K}})_n(X)$ as a direct sum of the $(\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_i})$'s. However, if we impose additional conditions on X, we can still recover an analog of Theorem 5.28 in dimensions $d > 2$.

Theorem 5.38. *Let X be a complete toric variety of dimension $d > 2$, and suppose that the dimension of the singular set of X is 0 (that is, X is smooth in all codimensions $\leq d-1$). Let U_{σ_1}, U_{σ_2},...,U_{σ_m} be all the open sets associated to a maximal cone in the fan Δ_X. Then* *we have*

$$(\mathcal{F}_{\mathrm{K}})_n(X) \cong (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2}) \oplus \cdots \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_m}) \tag{5.83}$$

for all n.

Proof. First notice that the dimension of a singular cone is precisely the codimension of the singularities created by that cone. So the statement that X is smooth in all codimensions $\leq d-1$ is equivalent to saying that the only possible singular cones of Δ_X are maximal cones.

We now proceed by induction on the number of open sets associated to maximal cones, as we did before. We begin with the base case $m = 2$.

So let $X = U_{\sigma_1} \cup U_{\sigma_2}$. We want to show that

$$(\mathcal{F}_{\mathrm{K}})_n(X) \cong (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2}) \tag{5.84}$$

for all n. Covering X by U_{σ_1} and U_{σ_2} and using Zariski descent, we have the long exact sequence:

$$\begin{aligned} \cdots \longrightarrow \quad & (\mathcal{F}_{\mathrm{K}})_n(X) & \longrightarrow (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2}) \longrightarrow \cdots \\ \cdots \longrightarrow \quad & (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1} \cap U_{\sigma_2}) & \longrightarrow (\mathcal{F}_{\mathrm{K}})_{n-1}(X) \longrightarrow \cdots \end{aligned} \tag{5.85}$$

Now recalling that $U_{\sigma_1} \cap U_{\sigma_2} = U_{\sigma_1 \cap \sigma_2}$, we know that $U_{\sigma_1} \cap U_{\sigma_2}$ is smooth because $\sigma_1 \cap \sigma_2$ is not a maximal cone (it is a cone of smaller dimension) and is therefore nonsingular by assumption. So we have that

$$(\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1} \cap U_{\sigma_2}) = 0 \tag{5.86}$$

for all n. Therefore, by exactness, we have that

$$(\mathcal{F}_{\mathrm{K}})_n(X) \cong (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2}) \tag{5.87}$$

for all n as desired. This establishes the case $m = 2$.

Now suppose the result is true for all $k < m$. Then we have

$$X = \bigcup_{i=1}^{m} U_{\sigma_i} \tag{5.88}$$

so we can cover X by the open sets Y and U_{σ_m}, where

$$Y = \bigcup_{i=1}^{m-1} U_{\sigma_i}. \tag{5.89}$$

Let $Z = Y \cap U_{\sigma_m}$. Then $(\mathcal{F}_{\mathrm{K}})_n(Z) = 0$ for all n. To see this, again proceed by induction on the number of open sets, where we notice that

$$Z = Y \cap U_{\sigma_m} = \bigcup_{i=1}^{m-1} U_{\sigma_i \cap \sigma_m} \tag{5.90}$$

is our cover. The case $m = 2$ was done above; $U_{\sigma_1} \cap U_{\sigma_2}$ is smooth (by assumption) because $\sigma_1 \cap \sigma_2$ is not a maximal cone. The case $m = 3$ is done by considering the cover $U_{\sigma_1 \cap \sigma_3}$ and $U_{\sigma_2 \cap \sigma_3}$. By the same reasoning as above, $U_{\sigma_i \cap \sigma_3}$ is smooth for $i = 1, 2$.

Noticing also that the intersection of these two open sets is

$$U_{\sigma_1 \cap \sigma_3} \cap U_{\sigma_2 \cap \sigma_3} = U_{\sigma_1 \cap \sigma_2 \cap \sigma_3} \tag{5.91}$$

we get that $U_{\sigma_1 \cap \sigma_3} \cap U_{\sigma_2 \cap \sigma_3}$ is also smooth because $\sigma_1 \cap \sigma_2 \cap \sigma_3$ is not a maximal cone; hence

$$(\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1 \cap \sigma_3} \cap U_{\sigma_2 \cap \sigma_3}) = 0 \tag{5.92}$$

for all n as well.

Using Zariski descent, we get the long exact sequence:

$$\begin{array}{lll} \cdots \longrightarrow & (\mathcal{F}_{\mathrm{K}})_n(Z) & \longrightarrow (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1 \cap \sigma_3}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2 \cap \sigma_3}) \longrightarrow \cdots \\ \cdots \longrightarrow & (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1 \cap \sigma_3} \cap U_{\sigma_2 \cap \sigma_3}) & \longrightarrow (\mathcal{F}_{\mathrm{K}})_{n-1}(Z) \longrightarrow \cdots \end{array} \tag{5.93}$$

Since $U_{\sigma_1\cap\sigma_3}$, $U_{\sigma_2\cap\sigma_3}$, and $U_{\sigma_1\cap\sigma_3}\cap U_{\sigma_2\cap\sigma_3}$ are smooth, we have that

$$(\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1\cap\sigma_3}) = (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2\cap\sigma_3}) = (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1\cap\sigma_3}\cap U_{\sigma_2\cap\sigma_3}) = 0 \tag{5.94}$$

for all n. Therefore, by exactness, we have that $(\mathcal{F}_{\mathrm{K}})_n(Z) = 0$ for all n as well. This gives the case $m = 3$.

Now for the inductive step for $(\mathcal{F}_{\mathrm{K}})_n(Z)$, cover Z by the open sets $\widetilde{Z}$ and $U_{\sigma_{m-1}\cap\sigma_m}$, where

$$\widetilde{Z} = \bigcup_{i=1}^{m-2} U_{\sigma_i\cap\sigma_m}. \tag{5.95}$$

By the same reasoning as above, $U_{\sigma_i\cap\sigma_m}$ is smooth for all i, and noticing that, for any $i \neq j$, we have

$$U_{\sigma_i\cap\sigma_m}\cap U_{\sigma_j\cap\sigma_m} = U_{\sigma_i\cap\sigma_j\cap\sigma_m} \tag{5.96}$$

is smooth (again, $\sigma_i\cap\sigma_j\cap\sigma_m$ is not a maximal cone, and therefore smooth by assumption) which implies that

$$\widetilde{Z}\cap U_{\sigma_{m-1}\cap\sigma_m} = \bigcup_{i=1}^{m-2} U_{\sigma_i\cap\sigma_m}\cap U_{\sigma_{m-1}\cap\sigma_m} \tag{5.97}$$

is also smooth, since it is the union of smooth open subschemes. As a consequence,

$$(\mathcal{F}_{\mathrm{K}})_n(\widetilde{Z}\cap U_{\sigma_{m-1}\cap\sigma_m}) = 0. \tag{5.98}$$

So we get the long exact sequence (again using Zariski descent):

$$\begin{array}{llll} \cdots \longrightarrow & (\mathcal{F}_{\mathrm{K}})_n(Z) & \longrightarrow (\mathcal{F}_{\mathrm{K}})_n(\widetilde{Z}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_{m-1}\cap\sigma_m}) \longrightarrow \cdots \\ \cdots \longrightarrow & (\mathcal{F}_{\mathrm{K}})_n(\widetilde{Z}\cap U_{\sigma_{m-1}\cap\sigma_m}) & \longrightarrow (\mathcal{F}_{\mathrm{K}})_{n-1}(Z) \longrightarrow \cdots \end{array} \tag{5.99}$$

But $(\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_{m-1}\cap\sigma_m}) = 0$ because $U_{\sigma_{m-1}\cap\sigma_m}$ is smooth, and $(\mathcal{F}_{\mathrm{K}})_n(\widetilde{Z}) = 0$ by our inductive hypothesis. Since $(\mathcal{F}_{\mathrm{K}})_n(\widetilde{Z} \cap U_{\sigma_{m-1} \cap \sigma_m}) = 0$ also, exactness gives us that $(\mathcal{F}_{\mathrm{K}})_n(Z) = 0$, giving the inductive step for Z.

Now we return to the inductive step for X, with the covering by Y and U_{σ_m}. Using Zariski descent, we get the following long exact sequence:

$$\begin{aligned} \cdots \longrightarrow \quad (\mathcal{F}_{\mathrm{K}})_n(X) \quad &\longrightarrow (\mathcal{F}_{\mathrm{K}})_n(Y) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_m}) \longrightarrow \cdots \\ \cdots \longrightarrow \quad (\mathcal{F}_{\mathrm{K}})_n(Z) \quad &\longrightarrow (\mathcal{F}_{\mathrm{K}})_{n-1}(X) \longrightarrow \cdots \end{aligned} \tag{5.100}$$

However, $(\mathcal{F}_{\mathrm{K}})_n(Z) = 0$ for all n by our above work; therefore, by exactness, we get that

$$(\mathcal{F}_{\mathrm{K}})_n(X) \cong (\mathcal{F}_{\mathrm{K}})_n(Y) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_m}) \tag{5.101}$$

for all n. By our induction hypothesis, we have that

$$(\mathcal{F}_{\mathrm{K}})_n(Y) \cong (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2}) \oplus \cdots \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_{m-1}}) \tag{5.102}$$

for all n. Substituting this in gives us

$$(\mathcal{F}_{\mathrm{K}})_n(X) \cong (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_1}) \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_2}) \oplus \cdots \oplus (\mathcal{F}_{\mathrm{K}})_n(U_{\sigma_m}) \tag{5.103}$$

for all n, completing the inductive step.

□

Using Theorem 5.38, we can now derive results that are analogous to those proven at the end of Section 5.3.

Corollary 5.39. *Let X be a complete toric variety of dimension $d > 2$, and suppose that the dimension of the singular set of X is 0. For all $n \leq 0$, we have*

$$(\mathcal{F}_{\mathrm{K}})_n(X) = 0. \tag{5.104}$$

Proof. This is immediate from Theorems 5.38 and 5.31.

□

Corollary 5.40. *If X and Y are any two complete simplicial toric varieties satisfying the conditions of Theorem 4.9, and satisfying the extra condition that the singular sets of X and Y both have dimension 0, then for $n \leq 0$, we have*

$$\mathrm{K}_n(X) \otimes \mathbb{Q} \cong \mathrm{K}_n(Y) \otimes \mathbb{Q}. \tag{5.105}$$

Proof. By Corollary 5.39, we have that

$$\mathrm{K}_n(X) \cong \mathrm{KH}_n(X) \tag{5.106}$$

and

$$\mathrm{K}_n(Y) \cong \mathrm{KH}_n(Y) \tag{5.107}$$

for $n \leq 0$. Tensoring each of these with $\mathbb{Q}$ gives us

$$\mathrm{K}_n(X) \otimes \mathbb{Q} \cong \mathrm{KH}_n(X) \otimes \mathbb{Q} \tag{5.108}$$

and

$$\mathrm{K}_n(Y) \otimes \mathbb{Q} \cong \mathrm{KH}_n(Y) \otimes \mathbb{Q} \tag{5.109}$$

for $n \leq 0$. By Theorem 4.9, we have that

$$\mathrm{KH}_n(X) \otimes \mathbb{Q} \cong \mathrm{KH}_n(Y) \otimes \mathbb{Q} \tag{5.110}$$

for all n. Composing all the isomorphisms gives the result.

□

Corollary 5.41. *If $\mathbb{P}(q_0, q_1, q_2, ..., q_d)$ is any d-dimensional weighted projective space such* *that the singular set has dimension* 0*, we have that*

$$\mathrm{K}_n(\mathbb{P}(q_0, q_1, q_2, ..., q_d)) \otimes \mathbb{Q} = 0 \tag{5.111}$$

for all $n < 0$*, and that*

$$\mathrm{K}_0(\mathbb{P}(q_0, q_1, q_2, ..., q_d)) \otimes \mathbb{Q} \cong \mathbb{Q}^{d+1}. \tag{5.112}$$

Proof. By Corollary 5.40 applied to the toric varieties $\mathbb{P}(q_0, q_1, q_2, ..., q_d)$ and $\mathbb{P}^d$, we have that

$$\mathrm{K}_n(\mathbb{P}(q_0, q_1, q_2, ..., q_d)) \otimes \mathbb{Q} \cong \mathrm{K}_n(\mathbb{P}^d) \otimes \mathbb{Q} \tag{5.113}$$

for $n \leq 0$. But $\mathrm{K}_n(\mathbb{P}^d) = 0$ for $n < 0$ and $\mathrm{K}_0(\mathbb{P}^d) \cong \mathbb{Z}^{d+1}$. Therefore we have

$$\mathrm{K}_n(\mathbb{P}(q_0, q_1, q_2, ..., q_d)) \otimes \mathbb{Q} \cong 0 \otimes \mathbb{Q} = 0 \tag{5.114}$$

for $n < 0$ and

$$\mathrm{K}_0(\mathbb{P}(q_0, q_1, q_2, ..., q_d)) \otimes \mathbb{Q} \cong \mathbb{Z}^{d+1} \otimes \mathbb{Q} \cong \mathbb{Q}^{d+1} \tag{5.115}$$

which completes the proof.

□

Remark 5.42. Combining Corollaries 5.39 and 5.41 establishes part (d) of Theorem 1.1.

Corollary 5.41 gives us a way to examining non-trivial classes of higher dimensional weighted projective spaces, as the following example demonstrates.

Example 5.43. *Consider the* 3*-dimensional weighted projective space* $\mathbb{P}(1, a, b, c)$ *where*

$$\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1. \tag{5.116}$$

The fan is generated by the 1*-dimensional cones*

$$\{(1,0,0),(0,1,0),(0,0,1),(-a,-b,-c)\}. \tag{5.117}$$

Every 1*-dimensional cone is smooth as before, so to apply Corollary 5.41, we need only check that all* 2*-dimensional cones are smooth. Obviously all* 2*-dimensional cones involving only the cones* $(1,0,0)$*,* $(0,1,0)$*, and* $(0,0,1)$ *will be smooth, so we need to only consider the three* 2*-dimensional cones that involve* $(-a,-b,-c)$*. That gives us the following cones:*

$$\begin{aligned}\tau_1 &= \langle (1,0,0),(-a,-b,-c)\rangle\\ \tau_2 &= \langle (0,1,0),(-a,-b,-c)\rangle\\ \tau_3 &= \langle (0,0,1),(-a,-b,-c)\rangle\end{aligned} \tag{5.118}$$

For τ_1 *to be smooth, we need to be able to find a vector* $(x,y,z)\in\mathbb{Z}^3$ *such that the matrix*

$$\begin{pmatrix} 1 & -a & x\\ 0 & -b & y\\ 0 & -c & z\end{pmatrix} \tag{5.119}$$

has determinant ± 1*. The determinant is* $cy-bz$ *so if we can find integers* y *and* z *such that* $cy-bz=1$ *then we have extended to a* $\mathbb{Z}$*-basis of* $\mathbb{Z}^3$ *as desired. But since* $\gcd(b,c)=1$*, such a* y *and* z *can indeed be found; taking those choices for* y *and* z *and letting* $x=0$ *gives us the desired extension. The argument is analogous for* τ_2 *and* τ_3*. Therefore, provided that*

$$\gcd(a,b)=\gcd(a,c)=\gcd(b,c)=1 \tag{5.120}$$

we see that $\mathbb{P}(1,a,b,c)$ *satisfies the conditions for Corollary 5.41, and we conclude that*

$$\mathrm{K}_n(\mathbb{P}(1,a,b,c))\otimes\mathbb{Q}=0 \tag{5.121}$$

for $n < 0$ and

$$\mathrm{K}_0(\mathbb{P}(1,a,b,c)) \otimes \mathbb{Q} \cong \mathbb{Q}^4. \tag{5.122}$$

Example 5.44. *Following from Example 5.43, consider the d-dimensional weighted projective space $\mathbb{P}(1, q_1, q_2, ..., q_d)$ where $\gcd(q_i, q_j) = 1$ for $i \neq j$. The fan is generated by the 1-dimensional cones*

$$\{e_1, e_2, ..., e_d, -q_1e_1 - q_2e_2 - \cdots - q_de_d\}. \tag{5.123}$$

As before, every 1-dimensional cone is smooth, and obviously every cone involving only the e_i's are smooth also. So the only possibly non-smooth cones are those involving the cone $-q_1e_1 - q_2e_2 - \cdots - q_de_d$. Just as in Example 5.43, we need to consider non-maximal cones involving $-q_1e_1 - q_2e_2 - \cdots - q_de_d$ and see that they are still smooth. The idea is analogous. Let us consider the cone

$$\sigma = \langle e_{i_1}, e_{i_2}, ..., e_{i_k}, -q_1e_1 - q_2e_2 - \cdots - q_de_d\rangle. \tag{5.124}$$

Notice that $k \leq d-2$ since if $k = d-1$ then σ would be a maximal cone. Also notice that if σ is shown to be smooth whenever $k = d-2$, then it is smooth for all choices of k since we can just extend by the $(d-2)-k$ vectors that are ommitted. So we can assume $k = d-2$. Without loss of generality, suppose that $e_{i_j} = e_j$, so that we have

$$\sigma = \langle e_1, e_2, ..., e_{d-2}, -q_1e_1 - q_2e_2 - \cdots - q_de_d\rangle. \tag{5.125}$$

For σ to be smooth, we need to be able to find a vector $(\alpha_1, \alpha_2, ..., \alpha_d) \in \mathbb{Z}^d$ such that the *matrix*

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -q_1 & \alpha_1 \\ 0 & 1 & 0 & \cdots & 0 & -q_2 & \alpha_2 \\ 0 & 0 & 1 & \cdots & 0 & -q_3 & \alpha_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -q_{d-2} & \alpha_{d-2} \\ 0 & 0 & 0 & \cdots & 0 & -q_{d-1} & \alpha_{d-1} \\ 0 & 0 & 0 & \cdots & 0 & -q_d & \alpha_d \end{pmatrix} \tag{5.126}$$

has determinant ± 1. *The determinant is* $\alpha_{d-1}q_d - \alpha_d q_{d-1}$ *so if we can find integers* α_{d-1} *and* α_d *such that* $\alpha_{d-1}q_d - \alpha_d q_{d-1} = 1$ *then we have extended to a* $\mathbb{Z}$*-basis of* $\mathbb{Z}^d$ *as desired. But since* $\gcd(q_{d-1}, q_d) = 1$, *such an* α_{d-1} *and* α_d *can indeed be found; taking those choices for* α_{d-1} *and* α_d *and letting* $\alpha_i = 0$ *for* $i \leq d-2$ *gives us the desired extension. The argument is analogous for all other possible choices for* σ. *Therefore, provided that*

$$\gcd(q_i, q_j) = 1 \tag{5.127}$$

for $i \neq j$, *we see that* $\mathbb{P}(1, q_1, q_2, ..., q_d)$ *satisfies the conditions for Corollary 5.41, and we conclude that*

$$\mathrm{K}_n(\mathbb{P}(1, q_1, q_2, ..., q_d)) \otimes \mathbb{Q} = 0 \tag{5.128}$$

for $n < 0$ *and*

$$\mathrm{K}_0(\mathbb{P}(1, q_1, q_2, ..., q_d)) \otimes \mathbb{Q} \cong \mathbb{Q}^{d+1}. \tag{5.129}$$

Example 5.44 shows that weighted projective spaces of the form $\mathbb{P}(1, q_1, q_2, ..., q_d)$, with $\gcd(q_i, q_j) = 1$ for all $i \neq j$, satisfy the conditions for Corollary 5.41 by showing that they satisfy the conditions for Corollary 5.39. In particular, weighted projective spaces of the form $\mathbb{P}(1, 1, 1, ..., 1, a)$, where $a \geq 2$, satisfy the conditions for Corollary 5.39. This gives rise to our final theorem.

Theorem 5.45. *Consider the d-dimensional weighted projective space* $\mathbb{P}(1,1,1,...,1,a)$, *where* $a \geq 2$. *Then*

$$\mathrm{K}_n(\mathbb{P}(1,1,1,...,1,a)) = 0 \tag{5.130}$$

for $n \leq -1$ *and*

$$\mathrm{K}_0(\mathbb{P}(1,1,1,...,1,a)) = \mathbb{Z}^{d+1}. \tag{5.131}$$

Proof. Recall from Theorem 3.94 that

$$\mathrm{K}_n(\mathbb{P}(1,1,1,...,1,a)) = \mathrm{KH}_n(\mathbb{P}(1,1,1,...,1,a)) \oplus (\mathcal{F}_{\mathrm{K}})_n(\mathbb{P}(1,1,1,...,1,a)). \tag{5.132}$$

Corollary 5.39 and our work in Example 5.44 shows that $(\mathcal{F}_{\mathrm{K}})_n(\mathbb{P}(1,1,1,...,1,a)) = 0$ for $n \leq 0$, and gives us that

$$\mathrm{K}_n(\mathbb{P}(1,1,1,...,1,a)) = \mathrm{KH}_n(\mathbb{P}(1,1,1,...,1,a)). \tag{5.133}$$

Applying Corollary 4.56 then gives us the result.

□

Remark 5.46. Theorem 5.45 establishes part (e) of Theorem 1.1, and therefore completes the proof of Theorem 1.1.

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