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The unfolded Seiberg-Witten Floer spectrum: Definition, properties and applications

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The unfolded Seiberg-Witten Floer spectrum: Definition, properties and applications

A dissertation submitted in partial satisfaction
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Jianfeng Lin

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Abstract of the Dissertation

The unfolded Seiberg-Witten Floer spectrum: Definition, properties and applications

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Jianfeng Lin

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2016

Professor Ciprian Manolescu, Chair

In this thesis, we define different versions of unfolded Seiberg-Witten Floer spectra for general 3-manifolds. They generalize Manolescu's and Kronheimer-Manolescu's construction of Floer stable homotopy type. We prove some properties of these new invariants and give some topological applications (Joint works with collaborators.) Along the way, as an application of the Seiberg-Witten Floer spectrum, we study the Pin(2)-equivariant Seiberg-Witten Floer KO-theory and prove new Furuta-type inequalities on the intersection forms of spin cobordisms between homology 3-spheres.

The dissertation of Jianfeng Lin is approved.

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Robert F. Brown

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2016

To my parents, Dongxiang Tan and Min Lin

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Part of this manuscript (Chapter 2, 3, 5, 6, 7, 8) is a reorganized version of a series of articles [26], [27] and [28] joint with Tirasan Khandhawit and Hirofumi Sasahira. Chapter 4 is a slightly expanded version of the paper [39] first published in Algebraic & Geometric Topology in [15(2), 2015], published by Mathematical Sciences Publishers.

Vita

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CHAPTER 1

Introduction

1.1 The Seiberg-Witten equations and the monopole Floer homology

In 1994, following up the work of Seiberg and Witten [62], Witten [67] introduced a set of nonlinear partial differential equations over spin^c 4-manifolds. These equations (later called the Seiberg-Witten equations) turn out to be extremely powerful in the study of four dimensional topology. In particular, by counting the number of solutions (called the Seiberg-Witten invariant), topologists are able to distinguish many smooth 4-manifolds that are homeomorphic to each other. For example, Fitushel and Stern proved the following theorem:

Theorem 1.1.1 (Fintushel-Stern [14])**.** *Let X be a closed, oriented, simply connected smooth 4-manifold with $b_2^+(X) > 1$. Suppose that X has nonzero Seiberg-Witten invariant (e.g. X admits a symplectic structure) and contains a homologically essential torus T of self-intersection 0 and $\pi_1(X \setminus T) = 1$. Then there exist infinitely many smooth structures on X.*

In dimension three, following the idea of Floer [16] in the setting of the anti-self-dual Yang-Mills equation, Kronheimer and Mrowka [33] developed the corresponding theory for 3-manifolds, i.e., the theory of monopole Floer homology. Roughly speaking, the monopole Floer homology of a 3-manifold Y is defined as the homology of a certain chain complex, whose generators are the critical points of the Chern-Simons-Dirac functional (corresponding to the solutions of the Seiberg-Witten equations on Y) and differential given by counting the number of negative gradient flow lines connecting these critical points (corresponding to the finite-energy solutions of the Seiberg-Witten equations on $\mathbb{R} \times Y$). The monopole Floer homology, together with its counter parts (instanton Floer homology and Heegaard Floer homology), is now an important tool in 3-dimensional topology and has many remarkable applications.

1.2 The Seiberg-Witten Floer spectrum for rational homology 3-spheres

In the contexts of symplectic Floer theory and instanton Floer theory, Cohen, Jones and Segal [9] posed a question of constructing a "Floer spectrum," an object whose homology recovers the Floer homology. (Recall that a spectrum is a generalization of a topological space in the setting of stable homotopy theory.) In 2003, Manolescu [40] first constructed the Seiberg-Witten Floer spectrum for rational homology 3-spheres by incorporating Furuta's technique of finite dimensional approximation in Seiberg-Witten theory [21] and Conley index theory [10]. It has been just recently shown by Lidman and Manolescu [36] that the homology of this spectrum is isomorphic to the monopole Floer homology.

Working with the Seiberg-Witten Floer spectrum has several advantages. First, suppose that the Seiberg-Witten equations have a certain "additional symmetry" (e.g. the underlying 3-manifold has a spin structure or has a finite group action). Then defining the "equivariant Seiberg-Witten Floer theory" is usually easier in the context of a spectrum invariant. A major reason is that the construction of a spectrum invariant requires very weak (or no) transversality conditions, while obtaining equivariant transversality in the setting of Morse homology is usually more difficult. A remarkable application in this direction is the following theorem (we stick to the case of integer homology spheres):

Theorem 1.2.1 (Manolescu [43]). *To each integer homology sphere Y, by studying the Pin(2)-equivariant Seiberg-Witten Floer spectrum of Y, we can associate an invariant $\beta(Y) \in \mathbb{Z}$ with the following properties:*

- *If $-Y$ denotes the orientation reversal of Y, then $\beta(-Y) = -\beta(Y)$;*
- *The mod-2 reduction of $\beta(Y)$ equals the Rohlin invariant $\mu(Y)$;*
- *Suppose Y_0, Y_1 are homology cobordent to each other. Then $\beta(Y_0) = \beta(Y_1)$.*

Combining the earlier work of Galewski-Stern [25] and Matumoto [45], Theorem 1.2.1 disproves the Triangulation Conjecture in high dimensions:

Corollary 1.2.2 (Manolescu [43]). *For every $n \geq 5$, there exists a closed n-dimensional topological manifold that does not admits a simplicial triangulation.*

Remark. A different construction of Pin(2)-equivariant Seiberg-Witten Floer homology was given by Lin in [37]. Instead of doing finite dimensional approximations, Lin extends Kronheimer-Mrowka's construction to the Morse-Bott setting. This construction gives an alternative disproof of the triangulation conjecture.

The second advantage of the Seiberg-Witten Floer spectrum is that: in principle, this invariant contains more information than the monopole Floer homology. For example, by applying the K-theory or KO-theory functor to the spectrum invariant, one can define "Seiberg-Witten Floer K-theory" or "Seiberg-Witten Floer KO-theory" and obtain interesting topological applications. This leads us to the topic of the next section.

1.3 Pin(2)-equivariant Seiberg-Witten Floer KO-theory

In this section, we discuss the following natural question in 4-dimensional topology:

Question 1.3.1. *Which nontrivial symmetric bilinear form can be realized as the intersection form of a smooth, spin 4-manifold X (closed or with boundary)?*

We first assume that X is closed. In this case, the intersection form should be even and unimodular. Therefore, it is indefinite by Donaldson's diagonalizability theorem [12, 13]. After changing the orientation of X if necessary, we can assume that the signature $\sigma(X)$ is non-positive. Then the intersection form can be decomposed as

$p(-E_8) \oplus q\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with $p \geq 0, q > 0$. Matsumoto's 11/8 conjecture [44] states that $b_2(X) \geq \frac{11}{8}|\sigma(X)|$, which can be rephrased as $q \geq \frac{3p}{2}$. An important result is the following 10/8 theorem of Furuta.

Theorem 1.3.2 (Furuta [21])**.** *Suppose X is an oriented closed spin 4-manifold with intersection form $p(-E_8) \oplus q\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $p \geq 0, q > 0$. Then we have $q \geq p + 1$.*

Furuta's proof made use of the finite dimensional approximation of the Seiberg-Witten equations on closed 4-manifolds and Pin(2)-equivariant K-theory. By doing destabilization and appealing to a result by Stolz [64], Minami [47] and Schmidt [60] independently proved the following improvement:

Theorem 1.3.3 (Minami [47], Schmidt [60])**.** *Let X be a smooth, oriented, closed spin 4-manifold with intersection form $p(-E_8) \oplus q\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $p \geq 0, q > 0$. Then we have:*

$$q \geq \begin{cases} p+1, & p \equiv 0, 2 \ mod\ 8 \\ p+2, & p \equiv 4 \ mod\ 8 \\ p+3, & p \equiv 6 \ mod\ 8. \end{cases} \tag{1.1}$$

Remark. p is always an even integer by Rokhlin's theorem [55].

An interesting observation is that Schmidt's calculation in [60] about the Adams operations actually implies an alternative proof of the following further improvement, which was first proved by Furuta-Kametani [22]. As a natural by-product of defining the Seiberg-Witten Floer KO-theoretic invariants, we will give this alternative proof in Section 4.2.

Theorem 1.3.4 (Furuta-Kametani [22])**.** *Let X be a smooth, oriented, closed spin 4-manifold with intersection form $p(-E_8) \oplus q\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $p, q > 0$. Then $q \geq p + 3$ when $p \equiv 0 \ mod\ 8$.*

Now we turn to the case that X is not closed but has boundary components, which are homology 3-spheres. The intersection form of X is still even and unimodular but can be definite now. For the definite case, various constrains are found in [17, 18, 19, 51, 31, 40].

For the indefinite case, Furuta-Li [24] and Manolescu [42] proved the following theorem independently[1].

Theorem 1.3.5 (Furuta-Li [24], Manolescu [40]). *To each oriented homology 3-sphere Y, we can associate an invariant $\kappa(Y) \in \mathbb{Z}$ with the following properties:*

(i) Suppose W is a smooth, spin cobordism from Y_0 to Y_1, with intersection form $p(-E_8) \oplus q \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then:

$$\kappa(Y_1) + q \geq \kappa(Y_0) + p - 1.$$

(ii) Suppose W is a smooth, oriented spin manifold with connected boundary Y, with intersection form $p(-E_8) \oplus q \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $q > 0$. Then:

$$\kappa(Y) + q \geq p + 1.$$

Both Furuta-Li and Manolescu proved this theorem by considering the Pin(2)-equivariant K-theory on the Seiberg-Witten Floer spectrum. Some new bounds can be obtained from this theorem. For example, the Brieskorn sphere $+\Sigma(2, 3, 12n + 1)$ does not bound a spin 4-manifold with intersection form $p(-E_8) \oplus p \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $p > 0$.

The main purpose of Chapter 4 will be extending Theorem 1.3.3 to the case of spin cobordisms tor get more constraints on the intersection form of a spin 4-manifold with boundary. The results in this chapter have also appeared in the paper [39].

Here is the first result:

Theorem 1.3.6. *For any $k \in \mathbb{Z}/8$, we can associate an invariant $\kappa o_k(Y)$ to each oriented homology sphere Y, with the following properties:*

- *(1) $2\kappa o_k(Y)$ is an integer whose mod 2 reduction is the Rokhlin invariant $\mu(Y)$.*
- *(2) Suppose W is an oriented smooth spin cobordism from Y_0 to Y_1, with intersection form $p(-E_8) \oplus q \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $p, q \geq 0$. Let $p = 4l + m$ for $l \in \mathbb{Z}$ and $m = 0, 1, 2, 3$. Then for any $k \in \mathbb{Z}/8$, we have the following inequalities:*

[1] We give Manolescu's statement here. Furuta-Li's statement is slightly different.

(i) If $(\mu(Y_0), m) = (0,0), (0,3), (1,0), (1,1)$*, then:*

$$\kappa o_k(Y_0) + 2l + h(\mu(Y_0), m) \leq \kappa o_{k+q}(Y_1) + \beta^q_{k+q}. \tag{1.2}$$

(ii) If $(\mu(Y_0), m) = (0,1), (0,2), (1,2), (1,3)$*, then:*

$$\kappa o_{k+4}(Y_0) + 2l + h(\mu(Y_0), m) \leq \kappa o_{k+q}(Y_1) + \beta^{4+q}_{k+q}. \tag{1.3}$$

Here $\beta^j_k = \sum_{i=0}^{j-1} \alpha_{k-i}$ *where* $\alpha_i = 1$ *for* $i \equiv 1,2,3,5$ *mod* 8 *and* $\alpha_i = 0$ *for* $i \equiv 0,4,6,7$ *mod* 8 *(*β^0_k *is defined to be* 0*). The constants* $h(\mu(Y_0), m)$ *are listed below:*

	$m=0$	$m=1$	$m=2$	$m=3$
$\mu(Y_0)=0$	0	5/2	3	3/2
$\mu(Y_0)=1$	0	1/2	3	7/2

Remark. When m is even, $\mu(Y_0) = \mu(Y_1)$ and $h(\mu(Y_0), m)$ is an integer. When m is odd, $\mu(Y_0) \neq \mu(Y_1)$ and $h(\mu(Y_0), m)$ is a half-integer.

Setting $p = q = 0$ in (2) of Theorem 1.3.6, we get:

Corollary 1.3.7. *If two homology spheres* Y_0, Y_1 *are homology cobordant to each other, then* $\kappa o_k(Y_0) = \kappa o_k(Y_1)$ *for any* $k \in \mathbb{Z}/8$.

The definition of κo_k is similar to that of κ [24, 42]. Roughly, $\kappa o_k(Y)$ is defined as follows. Pick a metric g on Y. By doing finite dimensional approximation to the Seiberg-Witten equations on (Y, g), we get a topological space I_ν with an action by $G = Pin(2)$. After changing I_ν by suitable suspension or desuspension, we consider the following construction: The inclusion of the S^1-fixed point set $I_\nu^{S^1}$ induces a map between the equivariant KO-groups $i^* : \widetilde{KO}_G(I_\nu) \to \widetilde{KO}_G(I_\nu^{S^1})$. We choose a specific reduction $\varphi : \widetilde{KO}_G(I_\nu^{S^1}) \to \mathbb{Z}$. It can be proved that the image of $\varphi \circ i^*$ is an ideal generated by $2^a \in \mathbb{Z}$. We define a as $\kappa o_k(Y)$. Different $k \in \mathbb{Z}/8$ correspond to different suspensions.

In Section 8, we calculate some examples using the results in [42] about the Seiberg-Witten Floer spectrum of $\pm\Sigma(2,3,r)$.

Theorem 1.3.8. *(a) We have $\kappa o_i(S^3) = 0$ for any $i \in \mathbb{Z}/8$.*

(b) For a positive integer r with $gcd(r, 6) = 1$, let $\Sigma(2, 3, r)$ be the Brieskorn spheres oriented as boundaries of negative plumbings and let $-\Sigma(2, 3, r)$ be the same Brieskorn spheres with the orientations reversed. Then $\kappa o_i(\pm\Sigma(2, 3, r))$ are listed below:

	κo_0	κo_1	κo_2	κo_3	κo_4	κo_5	κo_6	κo_7
$\Sigma(2,3,12n-1)$	1	1	1	0	0	0	0	0
$-\Sigma(2,3,12n-1)$	0	0	-1	-1	0	0	0	0
$\Sigma(2,3,12n-5)$	$1/2$	$1/2$	$1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
$-\Sigma(2,3,12n-5)$	$3/2$	$3/2$	$1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$1/2$
$\Sigma(2,3,12n+1)$	0	0	0	0	0	0	0	0
$-\Sigma(2,3,12n+1)$	0	0	0	0	0	0	0	0
$\Sigma(2,3,12n+5)$	$3/2$	$3/2$	$1/2$	$-1/2$	$-1/2$	$-1/2$	$1/2$	$3/2$
$-\Sigma(2,3,12n+5)$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$

Remark. We see that $\kappa o_k(-Y) \neq -\kappa o_k(Y)$ in general, while $\kappa o_k(Y\#(-Y))$ is always 0 by Corollary 1.3.7. Therefore, κo_k is not additive under connected sum.

If we apply (2) of Theorem 1.3.6 to the case $Y_0 = Y_1 = S^3$, the result is weaker than Theorem 1.3.3. As is the case in K-theory (See [42]), we can remedy this by considering the special property of $Y_0 \cong S^3$, which is called the Floer KO_G-split condition.

Theorem 1.3.9. *Let W be an oriented, smooth spin cobordism from Y_0 to Y_1, with intersection form $p(-E_8) \oplus q \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $p \geq 0, q > 0$. Suppose Y_0 is Floer KO_G-split. Let $p = 4l + m$ for $l \in \mathbb{Z}$ and $m = 0, 1, 2, 3$. Then we have the following inequalities:*

(1)If $(\mu(Y_0), m) = (0, 0), (0, 3), (1, 0), (1, 1)$, then:

$$\kappa o_4(Y_0) + 2l + h(\mu(Y_0), m) + 1 \leq \kappa o_{4+q}(Y_1) + \beta^q_{4+q}. \tag{1.4}$$

(2)If $(\mu(Y_0), m) = (0, 1), (0, 2), (1, 2), (1, 3)$, then:

$$\kappa o_4(Y_0) + 2l + h(\mu(Y_0), m) + 1 \leq \kappa o_q(Y_1) + \beta^{4+q}_q. \tag{1.5}$$

Here β^*_* and $h(\mu(Y_0), m)$ are the constants defined in Theorem 1.3.6.

In particular, S^3 is Floer KO_G-split. Applying $Y_0 = S^3$ to the previous theorem, we get the following useful corollary:

Corollary 1.3.10. *Let W be an oriented smooth spin 4-manifold whose boundary is a homology sphere Y. Suppose the intersection form of W is $p(-E_8) \oplus q \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with $p \geq 0, q > 0$. Then we have the following inequalities:*

- *If $p = 4l$, then $2l < \kappa o_{4+q}(Y) + \beta^q_{4+q}$.*
- *If $p = 4l + 1$, then $2l + \frac{5}{2} < \kappa o_q(Y) + \beta^{4+q}_q$.*
- *If $p = 4l + 2$, then $2l + 3 < \kappa o_q(Y) + \beta^{4+q}_q$.*
- *If $p = 4l + 3$, then $2l + \frac{3}{2} < \kappa o_{4+q}(Y) + \beta^q_{4+q}$.*

Remark. If we set $Y = S^3$ in Corollary 1.3.10, we will recover Theorem 1.3.3. However, Corollary 1.3.10 is not enough to prove Theorem 1.3.4. In order to get the relative version of Theorem 1.3.4, we have to apply similar constructions on the fixed point set of the Adams operation. This will not be done in the present paper.

Combining the results in Theorem 1.3.8 with Corollary 1.3.10, we get some new explicit bounds on the intersection forms of spin 4-manifolds bounded by $\pm\Sigma(2, 3, r)$. We give two of them here and refer to Section 4.7 for a complete list.

Example 1.3.11. We have the following conclusions:

- $-\Sigma(2, 3, 12n-1)$ does not bound a spin 4-manifold with intersection form $p(-E_8)\oplus (p + 1) \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $p > 0$.
- $-\Sigma(2, 3, 12n-5)$ does not bound a spin 4-manifold with intersection form $p(-E_8)\oplus p \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $p > 1$.

1.4 The unfolded Seiberg-Witten Floer spectrum for general 3-manifolds

From the discussion in previous sections, we see that it is natural and desirable to extend Manolescu's construction to any 3-manifold Y with $b_1(Y) > 0$, as monopole

Floer homology is defined for general 3-manifolds. In [29], Khandhawit gave an approach to constructing Seiberg-Witten Floer spectrum for a general case. The main goal of Chapter 5 and Chapter 6 is to rigorously construct the "unfolded" version of Seiberg-Witten Floer spectrum for general 3-manifolds. These results are joint with Khandhawit and Sasahira, and have also appeared [28].

Our invariants come with two variations: type-A invariant and type-R invariant. The letters "A" and "R" stand for attractor and repeller, which are notions in dynamical system and play a role in our construction.

Theorem 1.4.1. *Let Y be a closed, oriented 3-manifold and let $\mathfrak{s}$ be a spin^c structure on Y. Given a Riemannian metric g on Y and a spin^c connection A_0 which induces a connection on the determinant bundle of the spinor bundle with harmonic curvature, we can define*

$$\underline{\mathrm{swf}}^A(Y,\mathfrak{s},A_0,g;S^1) \text{ and } \underline{\mathrm{swf}}^R(Y,\mathfrak{s},A_0,g;S^1)$$

as a direct system and an inverse system in the S^1-equivariant stable category. These objects are well-defined up to canonical isomorphisms in the corresponding categories.

In the case that $c_1(\mathfrak{s})$ is nontorsion and $l = \gcd\{(h \cup [c_1(\mathfrak{s})])[Y] \mid h \in H^1(Y;\mathbb{Z})\}$, the objects $\underline{\mathrm{swf}}^A(Y,\mathfrak{s},A_0,g;S^1)$ and $\underline{\mathrm{swf}}^R(Y,\mathfrak{s},A_0,g;S^1)$ are l-periodic in the sense that

$$\Sigma^{\frac{l}{2}\mathbb{C}}\underline{\mathrm{swf}}^A(Y,\mathfrak{s},A_0,g;S^1) \cong \underline{\mathrm{swf}}^A(Y,\mathfrak{s},A_0,g;S^1),$$
$$\Sigma^{\frac{l}{2}\mathbb{C}}\underline{\mathrm{swf}}^R(Y,\mathfrak{s},A_0,g;S^1) \cong \underline{\mathrm{swf}}^R(Y,\mathfrak{s},A_0,g;S^1).$$

When the metric g or the connection A_0 changes, the objects $\underline{\mathrm{swf}}^A(Y,\mathfrak{s},A_0,g;S^1)$ and $\underline{\mathrm{swf}}^R(Y,\mathfrak{s},A_0,g;S^1)$ can change only by suspending or desuspending by copies of the complex representation $\mathbb{C}$ of S^1.

In the case that $c_1(\mathfrak{s})$ is torsion, we can normalize the above objects to obtain invariants

$$\underline{\mathrm{SWF}}^A(Y,\mathfrak{s};S^1) \text{ and } \underline{\mathrm{SWF}}^R(Y,\mathfrak{s};S^1)$$

of the spin-c manifold $(Y,\mathfrak{s})$.

For further reference, we denote the S^1-equivariant stable category by $\mathfrak{C}$ and denote the category of inductive systems (resp. projective systems) in $\mathfrak{C}$ by $\mathfrak{S}$ and (resp. $\mathfrak{S}^*$). (See Section 3.1 for the precise definition.)

Note that, for rational homology 3-spheres, the invariants $\underline{\mathrm{SWF}}^A$ and $\underline{\mathrm{SWF}}^R$ are the same and they agree with Manolescu's spectrum. In the case $b_1(Y) = 1$ and $\mathfrak{s}$ is nontorsion, $\underline{\mathrm{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1)$ is equivalent to $\mathrm{SWF}_0(Y, \mathfrak{s}, g, A_0)$ constructed by Kronheimer and Manolescu.

Remark. According to Furuta [23], one could set up a periodically graded category so that it is possible to define $\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}; S^1)$ and $\underline{\mathrm{SWF}}^R(Y, \mathfrak{s}; S^1)$ as invariants of the manifold in the nontorsion case.

When $\mathfrak{s}$ is a spin structure, there is an additional Pin(2)-symmetry on the Seiberg-Witten equations. The Pin(2)-equivariant Seiberg-Witten Floer spectrum for a rational homology sphere is instrumental in Manolescu's solution [43] of the Triangulation Conjecture. For a general spin 3-manifold, we have the following generalization:

Theorem 1.4.2. *Let Y be a closed, oriented 3-manifold and let $\mathfrak{s}$ be a spin structure on Y. We can obtain*

$$\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}; Pin(2)) \text{ and } \underline{\mathrm{SWF}}^R(Y, \mathfrak{s}; Pin(2))$$

as Pin(2)-equivariant analogs of $\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}; S^1)$ and $\underline{\mathrm{SWF}}^R(Y, \mathfrak{s}; S^1)$.

Let us try to explain the motivation of our "unfolded" construction. Intuitively, the monopole Floer homology is a Morse-Floer homology of a quotient configuration space $Coul(Y)/H^1(Y;\mathbb{Z})$, where $Coul(Y)$ is a Hilbert space of configurations with gauge fixing. We see that this is a Hilbert bundle when $b_1(Y) > 0$ but we cannot simply use vector spaces for finite dimensional approximation. Actually, there is a topological obstruction to finding a good sequence of subbundles for finite dimensional approximation (cf. [32, Proposition 6]). Thus, we instead do finite dimensional approximation on $Coul(Y)$. Since the Seiberg-Witten solutions and trajectories are no longer compact on $Coul(Y)$, we will be required to consider spectra obtained from an increasing sequence of bounded sets with nice properties on $Coul(Y)$. Our unfolded spectrum is then obtained as a direct (or inverse) system from these spectra.

From the construction, we expect the homology of our unfolded invariants to agree with monopole Floer homology with fully twisted coefficients, i.e. homology with a local system on the blown up configuration space whose fiber is the group ring $\mathbb{Z}[H^1(Y;\mathbb{Z})]$. By equivalence of monopole Floer homology and Heegaard Floer homology, the corresponding Heegaard Floer group with totally twisted coefficient $\underline{HF}(Y,\mathfrak{s})$ is constructed by Ozsváth and Szabó [53, Section 8]. This inspires us to use underline notation $\underline{\mathrm{SWF}}$ for the unfolded spectrum. Moreover, it should be possible to give a rigorous proof of this speculation with techniques developed by Lidman and Manolescu [36]. However, this is not the aim of this thesis.

In another direction, Sasahira [58] defined a folded version of Seiberg-Witten Floer spectra in the case that the topological obstruction, as mentioned above, vanishes. Khandhawit [30, Chapter 6] also gave an approach to defining a folded invariant, called the twisted Floer spectrum, for general 3-manifolds as a twisted parametrized spectrum. These theories will not be discussed here either.

In general, our invariants are quite difficult to compute. However, by using Mrowka-Ozváth-Yu's explicit description of the Seiberg-Witten moduli space for Seifert manifolds [48] and a refinement of the rescaling technique developed by Khandhawit [29], we are able to give explicit computation of the invariants in torsion cases of the following manifolds:

1. The manifold $S^2 \times S^1$;
2. Large degree circle bundles over surfaces;
3. All nil manifolds;
4. All flat manifolds except T^3.

Example 1.4.3. Let Y be the circle bundle over T^2 with degree $d > 0$. Then Y has a canonical spin^c structure $\mathfrak{s}_0$. The spectrum invariants for $\mathfrak{s}_0$ are given by

$$\underline{\mathrm{SWF}}^A(Y,\mathfrak{s}_0;S^1) \cong \left(\mathbb{C}^+ \to \bigvee_{S^0}^{2} \mathbb{C}^+ \to \bigvee_{S^0}^{3} \mathbb{C}^+ \to \cdots, 0, \frac{d-17}{8}\right)$$

and

$$\underline{\mathrm{SWF}}^R(Y,\mathfrak{s}_0;S^1) \cong \left((\mathbb{C}^2)^+ \setminus (S^1) \leftarrow (\mathbb{C}^2)^+ \setminus (\coprod^{2} S^1) \leftarrow (\mathbb{C}^2)^+ \setminus (\coprod^{3} S^1) \leftarrow \cdots, 0, \frac{d-1}{8}\right).$$

Here the connecting morphisms are given by natural inclusions. The notation $(*, 0, x)$ indicates that we formally desuspend the inductive/projective system by x-copies of $\mathbb{C}$.

We refer to Section 8.1 for other examples and refer to [28] for the detailed proofs.

1.5 Invariance of the spectrum

Chapter 6 is devoted to proving that our construction is well-defined, i.e. it does not depend on choices involved in the construction up to canonical isomorphisms. Let us explain this more carefully. We focus on $\underline{\mathrm{SWF}}^A(Y,\mathfrak{s};S^1)$ and the other versions are similar.

Choices of auxiliary parameters are involved in our construction of the spectrum invariant. For example, we need to fix a Riemannian metric on Y and we need to choose an specific index pair for each isolated invariant set. See the beginning of Chapter 6 for a complete list of parameters involved. For now, let us just denote by A_1 the set of all possible combinations of parameters. Instead of a single object in $\mathfrak{S}$, we obtain a family $\{\underline{\mathrm{SWF}}^A(Y,\mathfrak{s};S^1)_a\}$ of objects in $\mathfrak{S}$, parameterized by $a \in A_1$. Moreover, for any $a, b \in A_1$, there exists an isomorphism

$$\phi_{a,b} : \underline{\mathrm{SWF}}^A(Y,\mathfrak{s};S^1)_a \to \underline{\mathrm{SWF}}^A(Y,\mathfrak{s};S^1)_b$$

with the following properties

- $\phi_{a,a}$ is the identity morphism for any a;

- $\phi_{b,c} \circ \phi_{a,b} = \phi_{a,c}$ for any $a, b, c \in A_1$.

The spectrum invariant $\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}; S^1)$ actually consists of the set A_1, the family of objects $\{\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}; S^1)_a\}$ and a collection of isomorphisms $\phi = \{\phi_{a,b}\}$.

An alternative view (following [33]) is that $\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}; S^1)$ is an object of a new category $\mathfrak{S}/\mathrm{CAN}$. An object x in $\mathfrak{S}/\mathrm{CAN}$ consists of a set A, a family $\{x_a\}$ of objects in $\mathfrak{S}$ parameterized by $a \in A$ and a collection ϕ of isomorphisms $\phi_{a,b}$ for *all* a, b satisfying the above two properties. A morphism m from $\{A, \{x_a\}, \phi\}$ to $\{B, \{y_b\}, \psi\}$ is a collection of morphisms $m_{a,b} : x_a \to y_b$ for all $(a, b) \in A \times B$, satisfying the relation

$$\psi_{b',b} \circ m_{a',b'} \circ \phi_{a,a'} = m_{a,b}.$$

Note however that to specify m, it suffices to give a single morphism $m_{a,b}$ for some a, b.

To make the current thesis more readable, we will not use the language of CAT/CAN. Instead, we will just talk about *objects that are well-defined up to canonical isomorphisms*. Note that the main applications of our theory will be to apply different kinds of generalized cohomology functors on the spectrum invariant to obtain a well-defined object in the category GROUP/CAN, while there exists a functor from GROUP/CON to GROUP (the category of groups). This functor assigns an object $(A, \{G_a\}, \phi)$ to the group of "cross sections": the subgroup of ΠG_a consisting of collections $\{g_a\}$ satisfying $\phi(g_a) = g_b$ for any $a, b \in A$. By composing with this functor, we can obtain an actual group instead of "a group well-defined up to canonical isomorphisms".

We end this section by mentioning that one of the main complication in proving the well-definedness of our invariants is that we need to perturb the Chern-Simons-Dirac functional in the construction. First, we perturb the functional by a nonexact 2-form so that the functional is balanced (see Section 2.1). Second, we require that the set of critical points is discrete modulo gauge otherwise we cannot construct a good sequence of bounded subsets to apply finite dimensional approximation. As a result, the space of such perturbations may not be path connected and we cannot use standard homotopy argument here. (Note that this difficulty was avoided in Manolescu's original construction because perturbations are not necessary in the case of homology spheres.) We sketch the idea as follows and refer to Chapter 6 for more detailed discussion.

As before, we focus on the case of $\underline{\mathrm{SWF}}^A(Y,\mathfrak{s};S^1)$. This invariant is an inductive system in the S^1-equivariant stable category. Let f and $\tilde{f}$ be two perturbations on the Chern-Simons Dirac functional and let

$$S_1 : I_1^+ \to I_2^+ \to I_3^+ \to \cdots$$

and

$$S_2 : \tilde{I}_1^+ \to \tilde{I}_2^+ \to \tilde{I}_3^+ \to \cdots$$

be the corresponding inductive systems. In order to obtain an isomorphism between S_1 and S_2, we just need to construct a "mixed system"

$$S_{1,2} : I_{m_1}^+ \xrightarrow{i_1^+} \tilde{I}_{m_2}^+ \xrightarrow{i_2^+} I_{m_3}^+ \xrightarrow{i_3^+} \tilde{I}_{n_4}^+ \cdots \quad (m_1 < m_2 < m_3 < \cdots)$$

that contains both a subsystem of S_1 and a subsystem of S_2. The connecting morphism i_k^+ is obtained by considering a "mixed perturbation" f_k: a perturbation that equals f on some subset of $Coul(Y)$ and equals f' on another subset of $Coul(Y)$. We note that a technical difficulty in this argument is to prove a *uniform boundedness* result for the *noncompact* family $\{f_k\}_{k\in\mathbb{Z}}$ of perturbations. This will be done in Lemma 6.3.9 and Lemma 6.3.11.

1.6 The relative Bauer-Furuta invariants for 4-manifolds with boundary

In [8], Bauer and Furuta used the finite dimensional approximation technique to define an invariant Ψ of closed 4-manifolds. This invariant takes values in the equivariant stable cohomotopy groups of spheres. As shown by Bauer [7], the Bauer-Furuta invariant is strictly stronger than the Seiberg-Witten invariant. For example, by doing Fintushel-Stern knot surgeries on the manifold $X = K3\#K3$, one can obtain an infinite family of smooth manifolds $\{X_p\}_{p\in\mathbb{Z}}$ such that each X_p is homeomorphic to X and has vanishing Seiberg-Witten invariant (just like X). However, these manifolds are not diffeomorphic to each other because they have different Bauer-Furuta invariants.

In [40], Manolescu extended the Bauer-Furuta invariant to 4-manifolds whose boundary components are all rational homology 3-spheres. This invariant is called the relative Bauer-Furuta invariant. In Chapter 7, we will extend the definition of the relative Bauer-Furuta invariants to all compact, oriented 4-manifolds with boundary. Just like our spectrum invariants, the relative Bauer-Furuta invariants also have different versions: namely the type-A invariant and the type-R invariant. To define them, we need the following topological inputs:

- A spin^c 4-manifold $(X, \hat{\mathfrak{s}})$ with boundary $(Y, \mathfrak{s})$;
- A base point $\hat{o} \in X$;
- A *based path data* $[\vec{\gamma}]$: that is an equivalent class of pathes $(\gamma_1, \gamma_2, \cdots \gamma_{b_0(Y)})$ from $\hat{o}$ to each component of Y (see Definition 7.1.1).

Theorem 1.6.1. *Let $(X, \hat{\mathfrak{s}}, \hat{o}, [\gamma])$ be defined as above. Given a Riemannian metric g on the boundary Y and a spin^c connection A_0 with $F_{A_0^t}$ harmonic, we have an object $T(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}; S^1)$ of the equivariant stable homotopy category, well-defined up to canonical isomorphisms. (We call this object the S^1-equivariant Thom spectrum associated to the virtual index bundle of the Dirac operator.)*

By doing finite dimensional approximation on the Seiberg-Witten equations over X, we can define two versions of relative Bauer-Furuta invariants

$$\underline{\mathrm{bf}}^A(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}, [\vec{\eta}]; S^1) \in \mathrm{mor}_{\mathfrak{S}}(\Sigma^{-I_+(X)} T(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}; S^1), \underline{\mathrm{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1))$$

and

$$\underline{\mathrm{bf}}^R(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}, [\vec{\eta}]; S^1)$$
$$\in \mathrm{mor}_{\mathfrak{S}^*}(\Sigma^{-(I_+(X) \oplus \mathrm{coker}(H^1(X;\mathbb{R}) \to H^1(Y;\mathbb{R})))} T(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}; S^1), \underline{\mathrm{swf}}^R(Y, \mathfrak{s}, A_0, g; S^1))$$

($I_+(X)$ denotes a maximal non-negative subspace of $\mathrm{im}(H^2(X, Y; \mathbb{R}) \to H^2(X; \mathbb{R}))$ *under the cup product pairing.) When (A_0, g) changes, these invariants can only change by suspension or desuspension by copies of the complex S^1-representation $\mathbb{C}$.*

In the special case that $\mathfrak{s} = \hat{\mathfrak{s}}|_Y$ *is torsion, we can normalize these invariants to define*

$$\underline{\mathrm{BF}}^A(X, \hat{\mathfrak{s}}, \hat{o}, [\vec{\eta}]; S^1) \in \mathrm{mor}_{\mathfrak{S}}(\Sigma^{-I_+(X)}\tilde{T}(X, \hat{\mathfrak{s}}, \hat{o}; S^1), \underline{\mathrm{SWF}}^A(Y, \mathfrak{s}; S^1))$$

and

$$\begin{aligned}&\underline{\mathrm{BF}}^R(X, \hat{\mathfrak{s}}, \hat{o}, [\vec{\eta}]; S^1) \\ &\qquad \in \mathrm{mor}_{\mathfrak{S}^*}(\Sigma^{-(I_+(X)\oplus \mathrm{coker}(H^1(X;\mathbb{R})\to H^1(Y;\mathbb{R})))}\tilde{T}(X, \hat{\mathfrak{s}}, \hat{o}; S^1), \underline{\mathrm{SWF}}^R(Y, \mathfrak{s}; S^1)),\end{aligned}$$

where $\tilde{T}(X, \hat{\mathfrak{s}}, \hat{o}; S^1)$ *is the normalized Thom spectrum. These normalized invariants are topological invariants of* $(X, \hat{\mathfrak{s}}, \hat{o}, [\vec{\gamma}])$.

There are several remarks that we want to make here. First, both the source and the target of the relative Bauer-Furuta invariants depend on the choice of some parameters (e.g. the Riemannian metric $\hat{g}$ on X). Therefore, instead of a single morphism (in the category $\mathfrak{S}$ or $\mathfrak{S}^*$), we get a collection of morphisms that are compatible to each other under the natural isomorphisms on both the source and the target. In other words, our type-A (resp. type-R) relative Bauer-Furuta invariant is actually a morphism in the category $\mathfrak{S}/\mathrm{CAN}$ (resp. $\mathfrak{S}^*/\mathrm{CAN}$).

Second, we do not remove the base point $\hat{o}$ and the based path data $[\vec{\gamma}]$ from our notations of the invariants (even when $\mathfrak{s}$ is torsion). The reason is that our invariants actually depend on the choice of $(\hat{o}, [\vec{\gamma}])$. (In fact, the author is even unaware of *a canonical way* to identify the Thom spectra for different base points.) To relate the relative Bauer-Furuta invariants associated to different $(\hat{o}, [\vec{\gamma}])$, we have to define the spectrum invariant of Y as inductive/projective system in the $(S^1)^{b_0(Y)}$-equivariant stable category and to introduce the notion of twisting a morphism by a map from the Picard torus to $(S^1)^{b_0(Y)}$. Since this is beyond the scope of the current thesis, we simply include $(\hat{o}, [\vec{\gamma}])$ as the topological input.

Note that in either one of the following two cases, the dependence on $(\hat{o}, [\gamma])$ can be avoided and we remove $(\hat{o}, [\vec{\gamma}])$ from our notations.

- If Y is connected, then we can choose any $\hat{o} \in Y$ and set γ to be the constant path. It can be proved that any choice of such $(\hat{o}, [\gamma])$ gives the *same* result;

- If the map $H^1(X;\mathbb{R}) \to H^1(Y;\mathbb{R})$ is injective, then the relative Picard torus $Pic^0(X,Y)$ (see (7.4)) is a single point and we can construct a *natural isomorphism* between the relative Bauer-Furuta invariants given by two different choices of $(\hat{o},[\gamma])$.

Third, suppose the type-A spectrum invariant of Y is defined as an inductive system

$$I_1^+ \xrightarrow{i_1^+} I_2^+ \xrightarrow{i_2^+} I_3^+ \to \cdots$$

To define the type-A relative Bauer-Furuta invariant, we will define a morphism ψ_m^+ from a suspension of the Thom spectrum to I_m^+ for all m greater or equal to a certain constant m_0. In order to show that ψ_m^+ represents the same morphism in the category $\mathfrak{S}$, we have to show that these morphisms are compatible with the connecting morphisms in the following sense:

$$i_m^+ \circ \psi_m^+ = \psi_{m+1} \text{ for any } m \geq m_0. \tag{1.6}$$

Similarly, suppose the type-R spectrum invariant of Y is defined as a projective system

$$I_1^- \xleftarrow{i_1^-} I_2^- \xleftarrow{i_2^-} I_3^- \leftarrow \cdots .$$

Then the type-R Bauer-Furuta invariant will be given by a collection of morphisms ψ_m satisfying

$$i_m^- \circ \psi_{m+1}^- = \psi_m \text{ for any } m \geq 1. \tag{1.7}$$

Proving (1.6) and (1.7) is one of the main difficulties in the definition of relative Bauer-Furuta invariants. This turns out to be a problem in Conley index theory. The corresponding results are Proposition 3.2.18 and Proposition 3.2.19. In order to prove these results, we prove Theorem 3.2.14, an quantitative refinement of Theorem 3.2.7 by Manolescu [40], which states the existence of index pairs containing a pre-pair. We note that Theorem 3.2.14 also plays an important role in the proof of the gluing theorem [27] because we need to use it to get a uniform control on the length of the neck as we do finite dimensional approximations. We also note that results in subsection 3.2.2 may be of independent interest for some readers.

Last, we mention that the relative Bauer-Furuta invariants fit into the general framework of the *TQFT-property of Floer theories*: these invariants assign morphisms (from the Thom spectra to the spectrum invariants of Y) to the cobordism X (from the empty set to the boundary Y). However, we are still not able to define unfolded Seiberg-Witten Floer spectra as functors from a cobordism category to the category of spectra. A major difficulty is the loss of compactness because of our unfolding operation. It would be an interesting question to set up a suitable cobordism category and define the functors corresponding to our invariants.

We end this section by the following theorem about the Pin(2)-equivariant relative Bauer-Furuta invariants for spin 4-manifolds with boundary.

Theorem 1.6.2. *When $\hat{\mathfrak{s}}$ is a spin structure, we can define the normalized Thom spectrum as an object $\tilde{T}(X, \hat{\mathfrak{s}}, \hat{o}; Pin(2))$ of the Pin(2)-equivariant stable category. In this case, we have two versions of the Pin(2)-equivariant relative Bauer-Furuta invariant*

$$\underline{\mathrm{BF}}^A(X, \hat{\mathfrak{s}}, \hat{o}, [\vec{\eta}]; Pin(2)) \in \mathrm{mor}_{\mathfrak{S}_{Pin(2)}}(\Sigma^{-I_+(X)}\tilde{T}(X, \hat{\mathfrak{s}}, \hat{o}; Pin(2)), \underline{\mathrm{SWF}}^A(Y, \mathfrak{s}; Pin(2)))$$

and

$$\underline{\mathrm{BF}}^R(X, \hat{\mathfrak{s}}, \hat{o}, [\vec{\eta}]; Pin(2))$$
$$\in \mathrm{mor}_{\mathfrak{S}^*_{Pin(2)}}(\Sigma^{-(I_+(X)\oplus \mathrm{coker}(H^1(X;\mathbb{R})\to H^1(Y;\mathbb{R})))}\tilde{T}(X, \hat{\mathfrak{s}}, \hat{o}; Pin(2)), \underline{\mathrm{SWF}}^R(Y, \mathfrak{s}; Pin(2))).$$

Both of them are topological invariants of $(X, \hat{\mathfrak{s}}, \hat{o}, [\vec{\gamma}])$.

1.7 Further developments

Finally, we close this introductory chapter by listing the further developments on the theory of the unfolded Seiberg-Witten Floer spectrum. We refer to later chapters (mostly Chapter 8) for the precise statements of the results and refer to our papers [28], [26] and [27] for detailed proofs.

- By applying different equivariant generalized cohomology functors to the spectrum invariants, we will define all kinds of Frøyshov-type numerical invariants for a general 3-manifold

- S^1-equivariant homology: d^A and d^R;
- Pin(2)-equivariant homology: $\alpha^A, \beta^A, \gamma^A$ and $\alpha^R, \beta^R, \gamma^R$;
- Pin(2)-equivariant K-theory: κ^A, κ^R.

All of them are invariant under homology cobordisms. Using these invariants, we will give new constraints on the intersection form of 4-manifolds with boundary.

- We will define the smash product as a bifunctor $\mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$. This makes $\mathfrak{C}$ a symmetric monoidal category. Similar result holds for the Pin(2)-equivariant stable category $\mathfrak{C}_{Pin(2)}$;

- We will define the Spanier-Whitehead duality between the categories $\mathfrak{S}$ and $\mathfrak{S}^*$ (and also between $\mathfrak{S}_{Pin(2)}$ and $\mathfrak{S}^*_{Pin(2)}$). Under this definition, the type-A invariants of Y and the corresponding type-R invariants of $-Y$ are Spanier-Whitehead dual to each other;

- The statement of a gluing theorem for the relative Bauer-Furuta invariants will be given. Under certain technical assumptions, this will allow us to compute the Bauer-Furuta invariant of a closed 4-manifold $X = X_1 \cup_Y X_2$ in terms of the type-A relative Bauer-Furuta invariant of X_1 and the type-R relative Bauer-Furuta invariant of X_2. This is a generalization of a gluing theorem by Manolescu [41], which corresponds to the case that Y is a rational homology sphere;

- We will give various applications of the generalized gluing theorem: behavior of the fiberwise Bauer-Furuta invariant under surgery along loops, generalization of Bauer's connected sum theorem [7]; nonexistence of essential spheres with trivial normal bundle in a 4-manifold with nontrivial Bauer-Furuta invariant.

- Using a variation of the above mentioned gluing theorem, we will prove a connected sum formula for Manolescu's spectrum, i.e.

$$\mathrm{SWF}(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2; S^1) \cong \mathrm{SWF}(Y_1, \mathfrak{s}_1; S^1) \wedge \mathrm{SWF}(Y_2, \mathfrak{s}_2; S^1)$$

  when Y_1, Y_2 are rational homology spheres. Similar result holds for the Pin(2)-spectrum invariants.

1.8 Outline

The outline of this thesis is as follows. In Chapter 2, we first cover some of the basics of the Seiberg-Witten equations and then prove some analytical results which are needed in our later constructions. In the first half of Chapter 3, after defining the stable categories and the S^1-fixed point functor(s), we set up the Spanier-Whitehead duality between these categories. In the second half of Chapter 3, we review some elementary facts about the Conley index theory. Some further results on Conley index theory are also given here. These results are needed in Chapter 6. In Chapter 4, we construct the unfolded Seiberg-Witten Floer spectrua, as objects of the stable categories we defined in Chapter 3. The invariance of these spectra is proved in Chapter 5. In Chapter 6, we define different versions of relative Bauer-Furuta invariant for a general 4-manifold with boundary. In Chapter 7, we discuss some further developments in this theory. In particular, the statements of the results mentioned in Section 1.7 will be given and some of the proofs will be sketched.

CHAPTER 2

The approximated Seiberg-Witten flow

2.1 The Chern-Simons-Dirac functional and Seiberg-Witten trajectories

Let Y be a closed, oriented (but not necessarily connected) 3-manifold endowed with a spin^c structure $\mathfrak{s}$ and a Riemannian metric g. We denote its connected components by $Y_1, \ldots, Y_{b_0}$ and denote by $b_1 = b_1(Y)$ its first Betti number. Let S_Y be the associated spinor bundle and $\rho\colon TY \to \operatorname{End}(S_Y)$ be the Clifford multiplication. After fixing a base spin^c connection A_0, the space of spin^c connections on S_Y can be identified with $i\Omega^1(Y)$ via the correspondence $A \mapsto A - A_0$.

Let A_0^t be the connection on $\det(S_Y)$ induced by A_0. We choose A_0 such that the curvature $F_{A_0^t}$ equals $2\pi i\nu_0$, where ν_0 is the harmonic 2-form representing $-c_1(\mathfrak{s})$. For a 1-form $a \in i\Omega^1(Y)$, we let $\not{D}_{A_0+a}$ be the Dirac operator associated to the connection $A_0 + a$. We also denote by $\not{D} := \not{D}_{A_0}$ the Dirac operator corresponding to the base connection, so we have $\not{D}_{A_0+a} = \not{D} + \rho(a)$.

The gauge group $\operatorname{Map}(Y, S^1)$ acts on the space $i\Omega^1(Y) \oplus \Gamma(S_Y)$ by

$$u \cdot (a, \phi) = (a - u^{-1}du, u\phi),$$

where $u \in \operatorname{Map}(Y, S^1)$ and $(a, \phi) \in i\Omega^1(Y) \oplus \Gamma(S_Y)$. In practice, we will work with the Sobolev completion of the spaces $i\Omega^1(Y) \oplus \Gamma(S_Y)$ and $\operatorname{Map}(Y, S^1)$ by the L^2_k and L^2_{k+1} norms respectively. We fix an integer $k > 4$ throughout the paper and denote the completed spaces by $\mathcal{C}_Y$ and $\mathcal{G}_Y$ respectively. We will also consider the following subgroups of $\mathcal{G}_Y$:

- $\mathcal{G}^e_Y := \{u \in \mathcal{G}_Y \mid u = e^{\xi} \text{ for some } \xi\colon Y \to i\mathbb{R}\}$;

- $\mathcal{G}_Y^{e,0} := \{u \in \mathcal{G}_Y^e \mid u = e^\xi \text{ with } \int_{Y_j} \xi d\text{vol} = 0 \text{ for } j = 1, \ldots, b_0\}$;
- $\mathcal{G}_Y^h := \{u \in \mathcal{G}_Y \mid \Delta(\log u) = 0\}$ the harmonic gauge group, where $\Delta = d^*d$;
- $\mathcal{G}_Y^{h,o} := \{u \in \mathcal{G}_Y^h \mid u(o_j) = 1 \text{ for } j = 1, \ldots, b_0\}$ the based harmonic gauge group, where o_j is a chosen base point on Y_j.

Note that $\mathcal{G}_Y^e \cong \mathcal{G}_Y^{e,0} \times (S^1)^{b_0}$ and $\mathcal{G}_Y^h \cong \mathcal{G}_Y^{h,0} \times (S^1)^{b_0}$.

The balanced Chern-Simons-Dirac functional $CSD_{\nu_0} \colon \mathcal{C}_Y \to \mathbb{R}$ is defined as

$$CSD_{\nu_0}(a, \phi) := -\frac{1}{2}\left(\int_Y a \wedge da - \int_Y \langle \phi, \not{D}_{A_0+a}(\phi)\rangle d\text{vol}\right).$$

Note that this is a perturbation of the standard Chern-Simons-Dirac functional by the closed but nonexact 2-form ν_0 so that CSD_{ν_0} becomes invariant under the full gauge group (cf.[33, Definition 29.1.1]). The formal L^2-gradient is given by

$$\operatorname{grad} CSD_{\nu_0}(a, \phi) = (*da + \rho^{-1}(\phi\phi^*)_0, \not{D}_a\phi), \tag{2.1}$$

where $(\phi\phi^*)_0$ is the traceless part the endomorphism $\phi\phi^*$ on S_Y.

If we slightly perturb CSD_{ν_0}, the critical points of CSD_{ν_0} are discrete modulo gauge transformations. To ensure this property, we will need to pick a function $f \colon \mathcal{C}_Y \to \mathbb{R}$ which is invariant under $\mathcal{G}_Y$ and consider a twice perturbed functional $CSD_{\nu_0,f} := CSD_{\nu_0} + f$. We will make use of a large Banach space of perturbations constructed by Kronheimer and Mrowka [33, Section 11].

Definition 2.1.1. Let $\{\hat{f}_j\}_{j=1}^\infty$ be a countable collection of cylinder functions as in [33, Page 193]. Given a sequence $\{C_j\}_{j=1}^\infty$ of positive real numbers, we consider a separable Banach space

$$\mathcal{P} = \left\{ \sum_{j=1}^\infty \eta_j \hat{f}_j \,\middle|\, \eta_j \in \mathbb{R}, \ \sum_{j=1}^\infty C_j|\eta_j| < \infty \right\}, \tag{2.2}$$

where the norm is defined by $\left\|\sum_{j=1}^\infty \eta_j \hat{f}_j\right\| = \sum_{j=1}^\infty |\eta_j| C_j$. An element of $\mathcal{P}$ will be called an *extended cylinder function.*

The Banach space $\mathcal{P}$ will be fixed throughout the paper. In particular, we will choose a real sequence $\{C_j\}_j$ satisfying our requirements as in the following result.

Proposition 2.1.2. *The sequence $\{C_j\}_j$ can be chosen so that any extended cylinder function $\bar{f}$ in $\mathcal{P}$ has the following properties:*

(i) *$\bar{f}$ is a bounded function;*

(ii) *The formal L^2-gradient $\operatorname{grad}\bar{f}$ is a tame perturbation (see [33, Definition 10.5.1]);*

(iii) *For any positive integer m, the gradient $\operatorname{grad}\bar{f}$ defines a smooth vector field on the Hilbert space $L^2_m(i\Omega^1(Y)\oplus\Gamma(S_Y))$. Moreover, for each nonnegative integer n, we have*

$$\|\mathcal{D}^n_{(a,\phi)}\operatorname{grad}\bar{f}\| \leq C\, p_{m,n}(\|(a,\phi)\|_{L^2_m}),$$

where $p_{m,n}$ is a polynomial depending only on m,n and C is a constant depending on m,n and $\bar{f}$. The norm of $\mathcal{D}^n_{(a,\phi)}\operatorname{grad}\bar{f}$ is taken considering $\mathcal{D}^n_{(a,\phi)}\operatorname{grad}\bar{f}$ as an element of

$$\operatorname{Mult}^n(\times_n L^2_m(i\Omega^1(Y)\oplus\Gamma(S_Y)), L^2_m(i\Omega^1(Y)\oplus\Gamma(S_Y))).$$

(iv) *$\{C_j\}_j$ is taken so that the statement of Lemma 6.3.7 and 6.3.10 holds.*

Proof. By the definition of cylinder functions, each $\hat{f}_j$ is bounded. Therefore, property (i) can be ensured by taking $\{C_j\}_j$ increasing fast enough. Property (ii) is a consequence of [33, Theorem 11.6.1]. For property (iii), let $\hat{f}_j$ be a cylinder function from the collection. By [33, Proposition 11.3.3], the gradient $\operatorname{grad}\hat{f}_j$ defines a smooth vector field over $L^2_m(i\Omega^1(Y)\oplus\Gamma(S_Y))$ with the property that

$$\|\mathcal{D}^n_{(a,\phi)}\operatorname{grad}\hat{f}_j\| \leq C'_{j,m,n}(1+\|\phi\|_{L^2})^n(1+\|a\|_{L^2_{m-1}})^m(1+\|\phi\|_{L^2_{m,A_0+a}}),$$

where $C'_{j,m,n}$ is a constant and $\|\cdot\|_{L^2_{m,A_0+a}}$ denotes the L^2_m-norm defined using the connection A_0+a. Therefore, we only need to estimate $\|\phi\|_{L^2_{m,A_0+a}}$ by a polynomial of $\|(a,\phi)\|_{L^2_m}$.

Notice that the expansion of $\nabla^{(m)}_{A_0+a}\phi$ consists of terms of the form $\nabla^{(n_1)}a\cdot\nabla^{(n_2)}a\cdots\nabla^{(n_i)}a\cdot\nabla^{(n_{i+1})}_{A_0}\phi$ where ∇ denotes the Levi-Civita connection and $i,n_1,\ldots,n_{i+1}$ are nonnegative integers satisfying $n_1+n_2+\cdots+n_{i+1}+i=m$. As we want to control the L^2-norm of this term using $\|(a,\phi)\|_{L^2_m}$, there are three cases:

- $i = 0$: This is trivial since $\|\phi\|_{L^2_m} \leq \|(a,\phi)\|_{L^2_m}$;
- $i = 1$ and $n_1 = m-1$: We apply Sobolev multiplication $L^2_1 \times L^2_m \to L^2$ and obtain $\left\|\nabla^{(m-1)} a \cdot \phi\right\|_{L^2} \leq C \left\|\nabla^{(m-1)} a\right\|_{L^2_1} \|\phi\|_{L^2_m} \leq C \|(a,\phi)\|^2_{L^2_m}$. The case $i = 1$ and $n_2 = m-1$ can be done in the same manner;
- Otherwise, we will have $i \geq 1$ and $n_1, \ldots, n_{i+1} < m-1$. Similarly, we consider $n_{\max} = \max\{n_1, \ldots, n_{i+1}\}$ and apply Sobolev multiplication

$$L^2_{m-n_1} \times \cdots \times L^2_{m-n_i} \times L^2_{m-n_{i+1}} \to L^2_{m-n_{\max}} \hookrightarrow L^2.$$

Putting these together, we can find a polynomial $p_{m,n}$ (independent of j) such that

$$\|\mathcal{D}^n_{(a,\phi)} \operatorname{grad} \hat{f}_j\| \leq C'_{j,m,n}\, p_{m,n}(\|(a,\phi)\|_{L^2_m}).$$

For each j, take a constant C_j with

$$C_j \geq \max\{\ C'_{l_1,l_2,l_3} \mid 0 \leq l_1, l_2, l_3 \leq j\ \}.$$

We will prove that condition (iii) is satisfied. Take any element $\bar{f} = \sum_j \eta_j \hat{f}_j$ of $\mathcal{P}$. Then we have

$$\begin{aligned}
\|\mathcal{D}^n_{(a,\phi)} \operatorname{grad} \bar{f}\| &\leq \sum_j |\eta_j| \|\mathcal{D}^n_{(a,\phi)} \operatorname{grad} \hat{f}_j\| \\
&\leq \sum_j |\eta_j| C'_{j,m,n} p_{m,n}(\|(a,\phi)\|_{L^2_m}) \\
&\leq \left(\sum_{1 \leq j \leq N} |\eta_j| C'_{j,m,n} + \sum_{j \geq N} |\eta_j| C_j \right) p_{m,n}(\|(a,\phi)\|_{L^2_m}).
\end{aligned}$$

Here $N = \max\{m, n\}$. Putting $C := \left(\sum_{1\leq j\leq N} |\eta_j| C'_{j,m,n} + \sum_{j\geq N} |\eta_j| C_j\right)$, we obtain

$$\|\mathcal{D}^n_{(a,\phi)} \operatorname{grad} \bar{f}\| \leq C p_{m,n}(\|(a,\phi)\|_{L^2_m}).$$

Thus $\mathcal{P}$ satisfies (iii).

By further shrinking C_j, we may suppose that C_j satisfies Lemma 6.3.7 and Lemma 6.3.10 (2). That is, condition (iv) is satisfied.

□

The perturbation we consider in the current paper will be of the form

$$f(a,\phi) = \bar{f}(a,\phi) + \frac{\delta}{2}\|\phi\|^2_{L^2},$$

where $\bar{f}$ is an extended cylinder function and δ is a real number. We sometimes write the above perturbation as a pair $(\bar{f},\delta)$.

Definition 2.1.3. A perturbation $f = (\bar{f},\delta)$ is called *good* if the critical points of $CSD_{\nu_0,f}$ are discrete modulo gauge transformations.

When $\delta = 0$, we know that good perturbations are generic in $\mathcal{P}$ by virtue of [33, Theorem 12.1.2]. It is immediate to extend the result to a general case and we only give a statement here.

Lemma 2.1.4. *For any real δ, a subset of extended cylinder functions $\bar{f}$ in $\mathcal{P}$ such that $(\delta,\bar{f})$ is a good perturbation is residual.*

Remark. To define our invariants, it is sufficient to take $\delta = 0$. We include the term $\frac{\delta}{2}\|\phi\|^2$ as it will facilitate computations of many examples.

Our main object of interest is the negative gradient flow of the functional $CSD_{\nu_0,f}$ on the space $\mathcal{C}_Y$ modulo the gauge group. Let $I \subset \mathbb{R}$ be an interval. A trajectory $\gamma\colon I \to \mathcal{C}_Y$ of the negative gradient flow is described by the equation

$$-\frac{\partial}{\partial t}\gamma(t) = \operatorname{grad} CSD_{\nu_0,f}(\gamma(t)).$$

As in [40] and [29], it is more convenient to study the flow on the subspace called the Coulomb slice

$$Coul(Y) = \{(a,\phi) \mid d^*a = 0\} \subset \mathcal{C}_Y.$$

Since any configuration $(a,\phi) \in \mathcal{C}_Y$ can be gauge transformed into $Coul(Y)$ by a unique element of $\mathcal{G}^{e,0}_Y$, the Coulomb slice is isomorphic to the quotient $\mathcal{C}_Y/\mathcal{G}^{e,o}_Y$ with residual action by the harmonic gauge group $\mathcal{G}^h_Y$.

Let $\Pi : \mathcal{C}_Y \to \mathcal{C}_Y/\mathcal{G}^{e,o}_Y \cong Coul(Y)$ be the nonlinear Coulomb projection. The formula for Π is given by

$$\Pi(a,\phi) = \left(a - d\bar{\xi}(a), e^{\bar{\xi}(a)}\phi\right), \tag{2.3}$$

where $\bar{\xi}(a)\colon Y \to i\mathbb{R}$ is a unique function which solves

$$\Delta\bar{\xi}(a) = d^*a \text{ and } \int_{Y_j} \bar{\xi}(a) = 0 \text{ for each } j = 1, \dots, b_0. \tag{2.4}$$

To describe the Seiberg-Witten vector field on $Coul(Y)$, we first consider a trivial bundle $\mathcal{T}_{k-1}$ over $\mathcal{C}_Y$ with fiber $L^2_{k-1}(i\Omega^1(Y) \oplus \Gamma(S_Y))$. Note that the vector field $\operatorname{grad} CSD_{\nu_0,f}$ is a section of $\mathcal{T}_{k-1}$. Similarly, we have a trivial bundle $Coul_{k-1}$ over $Coul(Y)$ whose fiber is the L^2_{k-1}-completion of $\ker d^* \oplus \Gamma(S_Y)$. At a point $(a, \phi) \in Coul(Y)$, the pushforward $\Pi_*\colon \mathcal{T}_{k-1} \to Coul_{k-1}$ of the Coulomb projection Π is given by

$$\Pi_{*(a,\phi)}(b, \psi) = \left(b - d\bar{\xi}(b), \psi + \bar{\xi}(b)\phi\right). \tag{2.5}$$

We now project the negative gradient flow lines from $\mathcal{C}_Y$ to $Coul(Y)$ using Π. Such projected trajectories $\gamma\colon I \to Coul(Y)$ are described by an equation

$$-\frac{\partial}{\partial t}\gamma(t) = \Pi_* \operatorname{grad} CSD_{\nu_0,f}(\gamma(t)). \tag{2.6}$$

From (2.1) and (2.5), we can write down an explicit formula for the induced vector field on $Coul(Y)$ as a section of $Coul_{k-1}$

$$\Pi_* \operatorname{grad} CSD_{\nu_0,f}(a, \phi) = l(a, \phi) + c(a, \phi), \tag{2.7}$$

where $l = (*d, \not{D})$ is a first order elliptic operator and $c = (c^1, c^2)$ is given by

$$c^1(a, \phi) = \rho^{-1}(\phi\phi^*)_0 + \operatorname{grad}^1 f(a, \phi) - d\bar{\xi}(\rho^{-1}(\phi\phi^*)_0 + \operatorname{grad}^1 f(a, \phi)), \tag{2.8}$$

$$c^2(a, \phi) = \rho(a)\phi + \operatorname{grad}^2 f(a, \phi) + \bar{\xi}(\rho^{-1}(\phi\phi^*)_0 + \operatorname{grad}^1 f(a, \phi))\phi. \tag{2.9}$$

Note that l is linear and the nonlinear term c has nice compactness properties which will be explored in Section 2.2. We will call those trajectories γ satisfying (2.6) the *Seiberg-Witten trajectories*. By the standard elliptic bootstrapping argument, γ is actually a smooth path in $Coul(Y)$ when restricted to interior of I.

We would also like to interpret the vector field $\Pi_* \operatorname{grad} CSD_{\nu_0,f}$ from (2.6) as a gradient vector field on $Coul(Y)$. However, $\Pi_* \operatorname{grad} CSD_{\nu_0,f}$ is not the gradient of the restriction $CSD_{\nu_0,f}|_{Coul(Y)}$ with respect to the standard L^2-metric and we need to introduce another metric on $Coul(Y)$. Roughly speaking, we have to measure only the component of a vector on $Coul(Y)$ which is orthogonal to the linearized gauge group action. More specifically, consider a bundle decomposition over $\mathcal{C}_Y$

$$\mathcal{T}_{k-1} = \mathcal{J}_{k-1} \oplus \mathcal{K}_{k-1},$$

where the fiber of $\mathcal{J}_{k-1}$ at (a, ϕ) consists of a vector of the form $(-d\xi, \xi\phi)$ where $\xi \in L^2_k(Y; i\mathbb{R})$ with $\int_{Y_j} \xi = 0$ and the fiber of $\mathcal{K}_{k-1}$ is the L^2-orthogonal complement. Note that this decomposition is slightly different from the decomposition which appeared in [33, Section 9.3] as we use the derivative of the action of $\mathcal{G}^{e,0}_Y$ rather than $\mathcal{G}^e_Y$. Let $\widetilde{\Pi}$ be the L^2-orthogonal projection onto $\mathcal{K}_{k-1}$. Explicitly, the projection $\widetilde{\Pi}$ at (a, ϕ) is given by

$$\widetilde{\Pi}_{(a,\phi)}(b, \psi) = \left(b - d\tilde{\xi}(b, \psi, \phi), \psi + \tilde{\xi}(b, \psi, \phi)\phi\right),$$

where $\tilde{\xi}(b, \psi, \phi) : Y \to i\mathbb{R}$ is a unique function such that $-d^*(b - d\tilde{\xi}(b, \psi, \phi)) + i\mathrm{Re}\langle i\phi, \psi + \tilde{\xi}(b, \psi, \phi)\phi\rangle$ is a locally constant function and $\int_{Y_j} \tilde{\xi}(b, \psi, \phi) = 0$. It is not hard to see that we have a bundle isomorphism

$$\begin{array}{ccc} Coul_{k-1} & \underset{\Pi_*}{\overset{\widetilde{\Pi}}{\rightleftarrows}} & \mathcal{K}_{k-1} \\ & \searrow \quad \swarrow & \\ & Coul(Y) & \end{array}$$

since both are complementary to the derivative of the action of $\mathcal{G}^{e,0}_Y$.

We now define a metric $\tilde{g}$ for the bundle $Coul_{k-1}$ by setting

$$\langle (b_1, \psi_1), (b_2, \psi_2)\rangle_{\tilde{g}} := \langle \widetilde{\Pi}(b_1, \psi_1), \widetilde{\Pi}(b_2, \psi_2)\rangle_{L^2}.$$

Since $\widetilde{\Pi}$ and Π_* are inverse of each other and $\widetilde{\Pi}$ is an orthogonal projection, we have the following identity

$$\langle \Pi_* v, w\rangle_{\tilde{g}} = \langle v, w\rangle_{L^2} \quad \text{whenever } v \in \mathcal{K}_{k-1}.$$

Since $CSD_{\nu_0,f}$ is gauge invariant, $\operatorname{grad} CSD_{\nu_0,f}$ lies in $\mathcal{K}_{k-1}$. From this point on, we will denote by $\widetilde{\operatorname{grad}}$ the gradient on $Coul(Y)$ with respect to the metric $\tilde{g}$ and put

$$\mathcal{L} := CSD_{\nu_0,f}|_{Coul(Y)}.$$

We then have

$$\widetilde{\operatorname{grad}}\,\mathcal{L} = \Pi_* \operatorname{grad} CSD_{\nu_0,f} = l + c \text{ and } \|\widetilde{\operatorname{grad}}\,\mathcal{L}\|_{\tilde{g}} = \|\operatorname{grad} CSD_{\nu_0,f}\|_{L^2}. \tag{2.10}$$

Note that analogous results hold for any functional on $\mathcal{C}_Y$ which is $\mathcal{G}_Y^{e,0}$-invariant.

2.2 Analysis of approximated Seiberg-Witten trajectories

In this section, we review some boundedness and convergence results relevant to finite dimensional approximation which will be used in the main construction.

Definition 2.2.1. A smooth path in $Coul(Y)$ is called *finite type* if it is contained in a fixed bounded set (in the L_k^2-norm).

It can be proved that a Seiberg-Witten trajectory $\gamma(t) = (\alpha(t), \phi(t))$ is of finite type if and only if both $CSD_{\nu_0,f}(\gamma(t))$ and $\|\phi(t)\|_{C^0}$ are bounded (cf. [40, Definition 1]).

Recall that the set of the Seiberg-Witten solutions is compact modulo the full gauge group. However, there is a residual action by the group $\mathcal{G}_Y^{h,o} \cong H^1(Y;\mathbb{Z})$ on $Coul(Y)$. This motivates us to consider a strip of balls

$$Str(R) = \{x \in Coul(Y) \mid \exists h \in \mathcal{G}_Y^{h,o} \text{ s.t. } \|h \cdot x\|_{L_k^2} \leq R\},$$

where R is a positive real number.

Since $CSD_{\nu_0,f}$ is invariant under the full gauge group $\mathcal{G}_Y$, we have a uniform bound for the topological energy of all finite type trajectories (see [29, Proposition 10]). As a result, we have the following boundedness result.

Theorem 2.2.2 ([29])**.** *There exists a constant R_0 such that all finite type Seiberg-Witten trajectories are contained in the interior of $Str(R_0)$. In particular, the set $Str(R_0)$ contains all the critical points of $\mathcal{L}$ and trajectories between them.*

We now discuss finite dimensional approximation of Seiberg-Witten trajectories following [40] and [30]. To describe various projections, we first specify the L_m^2-inner product ($m \geq 1$) on $i\Omega^1(Y) \oplus \Gamma(S_Y)$. From the Hodge decomposition $\Omega^1(Y) = \ker d^* \oplus$

$\operatorname{im} d$, we will just define an inner product on each summand. On $i \ker d^* \oplus \Gamma(S_Y)$, we use the elliptic operator $l = (*d, \not{D})$

$$\langle (a_1, \phi_1), (a_2, \phi_2) \rangle_{L^2_m} := \langle (a_1, \phi_1), (a_2, \phi_2) \rangle_{L^2} + \langle l^m(a_1, \phi_1), l^m(a_2, \phi_2) \rangle_{L^2}.$$

For $\beta_1, \beta_2 \in i \operatorname{im} d$, we define

$$\langle \beta_1, \beta_2 \rangle_{L^2_m} := \langle \beta_1, \beta_2 \rangle_{L^2} + \langle \Delta^m \beta_1, \beta_2 \rangle_{L^2}.$$

Definition 2.2.3. With the Sobolev inner product defined above, a projection π will be called a *nice* projection if it satisfies the following properties:

(i) π is an L^2_m-orthogonal projection for any $m \geq 0$;

(ii) π extends to a map on a cylinder $I \times Y$ with $\|\pi\|_{L^2_m(I \times Y)} \leq 1$ for any $m \geq 0$.

Consider the spectral decomposition of $Coul(Y)$ with respect to the eigenspaces of $l = (*d, \not{D})$. For any real numbers $\lambda < 0 \leq \mu$, let V^μ_λ be the span of the eigenspaces of l with eigenvalues in the interval $(\lambda, \mu]$ and let p^μ_λ be the L^2-orthogonal projection onto V^μ_λ. It is not hard to see that p^μ_λ is a nice projection.

Recall that a Seiberg-Witten trajectory is an integral curve of the vector field $l + c$ on $Coul(Y)$. This leads us to consider a trajectory on a finite-dimensional subspace $\gamma \colon I \to V^\mu_\lambda$ satisfying an equation

$$-\frac{d\gamma(t)}{dt} = (l + p^\mu_\lambda \circ c)(\gamma(t)).$$

Such a trajectory will be loosely called an approximated Seiberg-Witten trajectory. We will also call a sequence of approximated Seiberg-Witten trajectories $\left\{\gamma_n \colon I \to V^{\mu_n}_{\lambda_n}\right\}_{n \in \mathbb{N}}$ an exhausting sequence when $-\lambda_n, \mu_n \to \infty$. The next proposition is the main convergence result of this section.

Proposition 2.2.4. *Let $\{\gamma_n \colon [a,b] \to V^{\mu_n}_{\lambda_n}\}$ be an exhausting sequence of approximated Seiberg-Witten trajectories whose L^2_k-norms are uniformly bounded. Then there exists a Seiberg-Witten trajectory $\gamma_\infty \colon (a,b) \to Coul(Y)$, such that, after passing to a subsequence, $\gamma_n(t) \to \gamma_\infty(t)$ uniformly in any Sobolev norm on any compact subset of (a,b).*

The proof of this proposition will be at end of this section. We basically follow the same strategy as in the proof of [40, Proposition 3] and [29, Proposition 11]. Since our vector field $l + c$ has an extra term coming from $\operatorname{grad} f$, we need to assure that the nonlinear part c still has nice compactness properties similar to those of the quadratic term in the Seiberg-Witten equation. For this purpose, we recall the notion of "quadratic-like" map and related results in [29, Section 4.2]. Since our setting here is slightly different, we give out some details for completeness.

Definition 2.2.5. Let E be a vector bundle over Y. A smooth map $Q\colon Coul(Y) \to L^2_k(\Gamma(E))$ is called *quadratic-like* if it has the following properties:

(i) The map Q sends a bounded subset in L^2_k to a bounded subset in L^2_k;

(ii) Let m be a nonnegative integer not greater than $k-1$. If there is a convergence of paths over a compact interval $(\frac{d}{dt})^s\gamma_n(t) \to (\frac{d}{dt})^s\gamma_\infty(t)$ uniformly in L^2_{k-1-s} for each $s = 0, 1, \cdots, m$, then we have $(\frac{d}{dt})^m Q(\gamma_n(t)) \to (\frac{d}{dt})^m Q(\gamma_\infty(t))$ uniformly in L^2_{k-2-m};

(iii) The map Q extends to a continuous map from $L^2_m(I \times Y)$ to $L^2_m(I \times Y)$ (with suitable bundles understood) for each integer $m \geq k-1$. Here I is a compact interval.

The sum of two quadratic-like maps is obviously quadratic-like. Furthermore, it can be shown that the pointwise tensor product of two quadratic-like maps is also quadratic-like (cf. [29, Lemma 10]).

Lemma 2.2.6 (cf. Lemma 9 of [29]). *Let f be a perturbation given by a pair $(\delta, \bar{f})$ with $\delta \in \mathbb{R}$ and $\bar{f} \in \mathcal{P}$. Then the map $\operatorname{grad} f\colon Coul(Y) \to L^2_k(i\Omega^1(Y) \oplus \Gamma(S_Y))$ is quadratic-like.*

Proof. We see that $\operatorname{grad} f(a, \phi) = (0, \delta\phi) + \operatorname{grad} \bar{f}(a, \phi)$ and the first term is obviously quadratic-like. We just need to show that $\operatorname{grad} \bar{f}$ is quadratic-like. First, we will check properties (i) and (ii) when $m = 0$ of Definition 2.2.5 .

For two configurations (a_0, ϕ_0) and (a_1, ϕ_1), we consider a straight segment $(a_t, \phi_t) = (1-t)(a_0, \phi_0) + t(a_1, \phi_1)$ joining them and apply the fundamental theorem of calculus

$$\left\| \operatorname{grad} \bar{f}(a_1, \phi_1) - \operatorname{grad} \bar{f}(a_0, \phi_0) \right\|_{L^2_j} = \left\| \int_{[0,1]} \mathcal{D}_{(a_t, \phi_t)} \operatorname{grad} \bar{f}(a_1 - a_0, \phi_1 - \phi_0) dt \right\|_{L^2_j}$$
$$\leq C \int_{[0,1]} p_{j,1}(\|a_t, \phi_t\|_{L^2_j}) \, \|(a_1, \phi_1) - (a_0, \phi_0)\|_{L^2_j} \, dt,$$

where the last inequality follows from Proposition 2.1.2 (iii). When $j = k$ and $(a_0, \phi_0) = (0, 0)$, this implies property (i) of Definition 2.2.5. Property (ii) when $m = 0$ also follows from the above inequality when $j = k - 1$.

We now check property (ii) when $1 \leq m \leq k - 2$. Suppose that $(\frac{d}{dt})^s \gamma_n(t) \to (\frac{d}{dt})^s \gamma_\infty(t)$ uniformly in L^2_{k-1-s} for each $s = 0, 1, \ldots, m$. We observe that an expansion of $(\frac{d}{dt})^m \operatorname{grad} \bar{f}(\gamma(t))$ consists of terms of the form

$$\mathcal{D}^s_{\gamma(t)} \operatorname{grad} \bar{f} \left(\left(\frac{d}{dt} \right)^{\alpha_1} \gamma(t), \ldots, \left(\frac{d}{dt} \right)^{\alpha_s} \gamma(t) \right) \text{ with } \alpha_i \geq 1 \text{ and } \alpha_1 + \cdots + \alpha_s = m.$$

From Proposition 2.1.2 (iii), $\|\mathcal{D}^s_{\gamma(t)} \operatorname{grad} \bar{f}\| \leq C \, p_{k-1-m,s}(\|\gamma(t)\|_{L^2_{k-1-m}})$ as an element of $\operatorname{Mult}^s(\times_s L^2_{k-1-m}, L^2_{k-1-m})$. We see that γ_n is uniformly bounded in L^2_{k-1-m} and that the convergence $(\frac{d}{dt})^{\alpha_i} \gamma_n(t) \to (\frac{d}{dt})^{\alpha_i} \gamma_\infty(t)$ is uniform in L^2_{k-1-m} as $\alpha_i \leq m$. These imply property (ii).

Properties (iii) easily follows from the fact that grad $\bar{f}$ is a tame perturbation.

□

As a result, we can deduce compactness property of the induced vector field on $Coul(Y)$.

Corollary 2.2.7. *The nonlinear part c of the induced Seiberg-Witten vector field in (2.7) is quadratic-like.*

Proof. It is clear that the composition of a quadratic-like map with a linear operator of nonpositive order is quadratic-like. Since the operator $\bar{\xi}$ in (2.3) is of order -1, Lemma 2.2.6 and closure under pointwise multiplication imply that the map c is quadratic-like.

□

We are now ready to prove Proposition 2.2.4. Although, we will only give outline of the proof as the reader can find more details in [40] and [29].

Proof of Proposition 2.2.4. Let $\{\gamma_n\}$ be an exhausting sequence of approximated trajectories which are all contained in a ball $B(R)$ in L^2_k. The norm $\|\frac{d}{dt}\gamma_n(t)\|_{L^2_{k-1}}$ is uniformly bounded by boundedness of the map $l+c$. By the Rellich lemma and the Arzela-Ascoli theorem, we can pass to a subsequence of $\{\gamma_n\}$ which converges to a path γ_∞ uniformly in L^2_{k-1}. Moreover, it can be shown that γ_∞ is a Seiberg-Witten trajectory. By property (ii) of Definition 2.2.5 of c, we can inductively prove uniform convergence $(\frac{d}{dt})^m(\gamma_n(t)) \to (\frac{d}{dt})^m(\gamma_\infty(t))$ in L^2_{k-1-m} for $m = 1, \ldots, k-1$. This implies that $\hat{\gamma}_n \to \hat{\gamma}_\infty$ in $L^2_{k-1}([a,b] \times Y)$. (Here we treat $\gamma_n(t)$ and $\gamma_\infty(t)$ as sections over $I \times Y$ and denote them respectively by $\hat{\gamma}_n$ and $\hat{\gamma}_\infty$.) Property (iii) of Definition 2.2.5 allows us to do the bootstrapping argument over any shorter cylinder $I \times Y$. This finishes the proof of the proposition. □

Proposition 2.2.4 has the following consequence.

Corollary 2.2.8. *For a closed and bounded subset S of $Coul(Y)$ in L^2_k, there exist large numbers $-\bar{\lambda}, \bar{\mu}, -\bar{T} \gg 0$ such that if $\lambda < \bar{\lambda}$, $\mu > \bar{\mu}$ and $T > \bar{T}$ then for any approximated Seiberg-Witten trajectory $\gamma\colon [-T,T] \to V^\mu_\lambda$ contained in S, we have $\gamma(0) \in Str(R_0)$. Here R_0 is the universal constant from Theorem 2.2.2.*

Proof. Suppose the contrary: we can find an exhausting sequence of approximated trajectories $\gamma_n\colon [-T_n,T_n] \to V^{\mu_n}_{\lambda_n} \cap S$, with $T_n \to \infty$, with $\gamma_n(0) \notin Str(R_0)$. Since S is bounded, we can apply Proposition 2.2.4 and the diagonalization argument to find a Seiberg-Witten trajectory $\gamma_\infty\colon \mathbb{R} \to S$ of finite type such that, after passing to a subsequence, $\gamma_n(0) \to \gamma_\infty(0)$ in L^2_k. However, $\gamma_\infty(0)$ is in the interior of $Str(R_0)$ by Theorem 2.2.2. This is a contradiction. □

Remark. In Corollary 2.2.8, we can also consider more generalized approximated trajectories. For example, we can use interpolation between two projections for approximation, i.e. a trajectory satisfying

$$-\frac{d\gamma(t)}{dt} = \left(l + ((1-s)p_{\lambda}^{\mu} + sp_{\lambda'}^{\mu'}) \circ c\right)(\gamma(t)),$$

where $0 \leq s \leq 1$ and $\lambda' < \lambda < \bar{\lambda}$ and $\mu' > \mu > \bar{\mu}$.

CHAPTER 3

Categorical and topological preliminaries

3.1 The stable categories.

3.1.1 Definition of the stable categories

In this subsection, we briefly review algebraic-topological constructions which will be needed later. In particular, we will define three S^1-equivariant stable categories $\mathfrak{C}, \mathfrak{S}$ and $\mathfrak{S}^*$ in which our invariants live as objects. The categories $\mathfrak{S}$ and $\mathfrak{S}^*$ are defined as direct systems and inverse systems of $\mathfrak{C}$ respectively. In the end, we will also define $\mathfrak{C}_{Pin(2)}, \mathfrak{S}_{Pin(2)}$ and $\mathfrak{S}^*_{Pin(2)}$. These are Pin(2)-analogue of $\mathfrak{C}, \mathfrak{S}$ and $\mathfrak{S}^*$ respectively. Our treatment follows closely with [40] and [42]. See [1] and [46] for more systematic and detailed discussions regarding equivariant stable homotopy theory.

The category $\mathfrak{C}$, which was defined in [40], is the S^1-equivariant analog of the classical Spanier-Whitehead category with $\mathbb{R}^\infty \oplus \mathbb{C}^\infty$ as the universe. In other words, we will only consider suspensions involving the following two representations:

1. $\mathbb{R}$ the one-dimensional trivial representation;
2. $\mathbb{C}$ the two-dimensional representation where $S^1 = \{e^{i\theta} | \theta \in [0, 2\pi)\}$ acts by complex multiplication.

For a representation V, we will denote by V^+ its one-point compactification and by V^{S^1} its S^1-fixed point set. Note that the transposition $(\mathbb{R}^{u_1})^+ \wedge (\mathbb{R}^{u_2})^+ \to (\mathbb{R}^{u_2})^+ \wedge (\mathbb{R}^{u_1})^+$ is homotopic to identity only when u_1 or u_2 is even.

The objects of $\mathfrak{C}$ are triples (A, m, n) consisting of a pointed topological space A with an S^1-action, an even integer m and a rational number n. We require that A is

S^1-homotopy equivalent to a finite S^1-CW complex. The set of morphisms between two objects is given by

$$\mathrm{mor}_{\mathfrak{C}}((A,m,n),(A',m',n')) := \operatorname*{colim}_{u,v\to\infty}[(\mathbb{R}^u \oplus \mathbb{C}^v)^+ \wedge A, (\mathbb{R}^{u+m-m'} \oplus \mathbb{C}^{v+n-n'})^+ \wedge A']_{S^1},$$

if $n - n' \in \mathbb{Z}$, where $[\cdot,\cdot]_{S^1}$ denotes the set of pointed S^1-equivariant homotopy classes. We define $\mathrm{mor}_{\mathfrak{C}}((A,m,n),(A',m',n'))$ to be the empty set if $n - n' \notin \mathbb{Z}$. As in [40], there is a full subcategory $\mathfrak{C}_0$ inside of $\mathfrak{C}$ consisting of objects of the form $(A,0,0)$, which we also denote by A. For an object $Z = (A,m,n) \in \mathrm{ob}\,\mathfrak{C}$, an even integer m' and a rational number n', we also write (Z,m',n') for $(A,m+m',n+n')$.

We now turn to the description of the category $\mathfrak{S}$. An object of $\mathfrak{S}$ consists of a collection $Z = (\{Z_p\},\{i_p\})_{p\in\mathbb{N}})$ of objects $\{Z_p\}_{p\in\mathbb{N}}$ of $\mathfrak{C}$ and a collection of morphisms $\{i_p \in \mathrm{mor}_{\mathfrak{C}}(Z_p, Z_{p+1})\}_{p\in\mathbb{N}}$. In other word, an object Z of $\mathfrak{S}$ is a direct system

$$Z_1 \xrightarrow{i_1} Z_2 \xrightarrow{i_2} \cdots .$$

For two objects $Z = (\{Z_p\}_p, \{i_p\}_p)$ and $Z' = (\{Z'_p\}_p, \{i'_p\}_p)$ of $\mathfrak{S}$, we define the set of morphisms as

$$\mathrm{mor}_{\mathfrak{S}}(Z,Z') := \lim_{\infty\leftarrow p}\lim_{q\to\infty} \mathrm{mor}_{\mathfrak{C}}(Z_p, Z'_q). \tag{3.1}$$

The identity morphism and the composition law are defined in the obvious way. Notice that here we first take the direct limit and then take the inverse limit. This order should not be changed.

As for the category $\mathfrak{S}^*$, its objects are the inverse systems

$$\bar{Z}_1 \xleftarrow{j_1} \bar{Z}_2 \xleftarrow{j_2} \cdots ,$$

where $\bar{Z}_p \in \mathrm{ob}\,\mathfrak{C}$ and $j_p \in \mathrm{mor}_{\mathfrak{C}}(\bar{Z}_{p+1}, \bar{Z}_p)$. For two objects $\bar{Z} = (\{\bar{Z}_p\}_p, \{j_p\}_p)$ and $\bar{Z}' = (\{\bar{Z}'_p\}_p, \{j'_p\}_p)$ of $\mathfrak{S}^*$, we define the set of morphisms as

$$\mathrm{mor}_{\mathfrak{S}^*}(\bar{Z},\bar{Z}') := \lim_{\infty\leftarrow q}\lim_{p\to\infty} \mathrm{mor}_{\mathfrak{C}}(\bar{Z}_p, \bar{Z}'_q). \tag{3.2}$$

Again, we first take the direct limit and then take the inverse limit.

Remark. The full subcategory of $\mathfrak{C}$ consisting of objects $\{(A,m,n) \mid m \in 2\mathbb{Z}, n \in \mathbb{Z}\}$ can be naturally embedded into the homotopy category of the S^1-equivariant spectra modeled on the standard universe $\mathbb{R}^\infty \oplus \mathbb{C}^\infty$. Therefore, an object $(\{(A_p, m_p, n_p)\}_p, \{i_p\}_p)$ of $\mathfrak{S}$ (resp. $\mathfrak{S}^*$) with $m_p \in 2\mathbb{Z}$ and $n_p \in \mathbb{Z}$ corresponds to an inductive system (resp. projective system) of S^1-equivariant spectra. For this reason, we call an object of $\mathfrak{S}$ an ind-spectum and an object of $\mathfrak{S}^*$ a pro-spectrum. However, this is not so accurate because, in the usual sense, an ind-spectrum (resp. pro-spectrum) refers to an inductive system (resp. projective system) in the category of spectra, not the homotopy category of spectra. Also, with a slightly abuse of language, we call all our invariants spectrum invariants.

Now we turn to the Pin(2) case. Recall that the group Pin(2) is defined as the the subgroup $S^1 \cup jS^1 \subset \mathbb{H}$ of the algebra of quaternions, containing S^1 as the set of unit complex numbers. We are interested in the following real representations of Pin(2):

1. $\mathbb{R}$ the trivial one-dimensional representation;
2. $\tilde{\mathbb{R}}$ the nontrivial one-dimensional representation where S^1 acts trivially and j acts as multiplication by -1;
3. $\mathbb{H}$ the 4-dimensional representation where Pin(2) acts by left quaternionic multiplication.

We introduce the category $\mathfrak{C}_{Pin(2)}$, $\mathfrak{S}_{Pin(2)}$ and $\mathfrak{S}^*_{Pin(2)}$ which are the Pin(2)-version of the categories $\mathfrak{C}, \mathfrak{S}$ and $\mathfrak{S}^*$. The objects of $\mathfrak{C}_{Pin(2)}$ are triples (A, m, n) consisting of an even integer m, a rational number n and a pointed Pin(2)-space A which is Pin(2)-homotopy equivalent to a finite Pin(2)-CW complex. The set $\mathrm{mor}_{\mathfrak{C}_{Pin(2)}}((A, m, n), (A', m', n'))$ is given by

$$\operatorname*{colim}_{u,v,w\to\infty} [(\mathbb{R}^u \oplus \tilde{\mathbb{R}}^v \oplus \mathbb{H}^w)^+ \wedge A, (\mathbb{R}^u \oplus \tilde{\mathbb{R}}^{v+m-m'} \oplus \mathbb{H}^{w+n-n'})^+ \wedge A']_{Pin(2)}$$

when $n - n' \in \mathbb{Z}$ and is empty otherwise. The objects of $\mathfrak{S}_{Pin(2)}$ (resp. $\mathfrak{S}^*_{Pin(2)}$) are the sequential direct systems (resp. sequential inverse systems) in $\mathfrak{C}_{Pin(2)}$. We call an object of $\mathfrak{S}_{Pin(2)}$ a Pin(2)-equivariant ind-spectrum and call an object of $\mathfrak{S}^*_{Pin(2)}$ a Pin(2)-equivariant pro-spectrum. The sets of morphisms are defined in the same way as (3.1) and (3.2). For an object W of $\mathfrak{C}_{Pin(2)}$, $\mathfrak{S}_{Pin(2)}$ and $\mathfrak{S}^*_{Pin(2)}$, the natation (W, m, n) will be used as in the S^1-case.

We end this subsection with the following useful lemma, which is directly implied by our definition.

Lemma 3.1.1. *Let $Z = (\{Z_p\}_{p\in\mathbb{N}}, \{i_p\}_{p\in\mathbb{N}})$ be an object of $\mathfrak{S}$. For any infinite sequence of positive integers $0 < p_1 < p_2 < \cdots$, the subsystem*

$$Z_{p_1} \xrightarrow{i_{p_2-1}\circ\cdots\circ i_{p_1}} Z_{p_2} \xrightarrow{i_{p_3-1}\circ\cdots\circ i_{p_2}} Z_{p_3} \to \cdots$$

of Z is canonically isomorphic to Z as an object of $\mathfrak{S}$. Similarly, let $\bar{Z} = (\{\bar{Z}_p\}_{p\in\mathbb{N}}, \{j_p\}_{p\in\mathbb{N}})$ be an object $\mathfrak{S}^*$, then the subsystem

$$\bar{Z}_{p_1} \xleftarrow{j_{p_1}\circ\cdots\circ j_{p_2-1}} \bar{Z}_{p_2} \xleftarrow{j_{p_2}\circ\cdots\circ j_{p_3-1}} \bar{Z}_{p_3} \leftarrow \cdots$$

of $\bar{Z}$ is canonically isomorphic to $\bar{Z}$ as an object of $\mathfrak{S}^*$. Similar result holds for $\mathfrak{S}_{Pin(2)}$ and $\mathfrak{S}^*_{Pin(2)}$.

3.1.2 The S^1-fixed point functor

The S^1-fixed point functor will play an important rule when we define the Froyshøv type numerical invariants in Chapter 8. Before discussing these functors, we briefly introduce some more categories

$$\bar{\mathfrak{C}},\ \bar{\mathfrak{S}},\ \bar{\mathfrak{S}}^*,\ \bar{\mathfrak{C}}_{\mathbb{Z}_2},\ \bar{\mathfrak{S}}_{\mathbb{Z}_2} \text{ and } \bar{\mathfrak{S}}^*_{\mathbb{Z}_2}.$$

They correspond to the S^1-fixed point sets of $\mathfrak{C}$, $\mathfrak{S}$, $\mathfrak{S}^*$, $\mathfrak{C}_{Pin(2)}$, $\mathfrak{S}_{Pin(2)}$ and $\mathfrak{S}^*_{Pin(2)}$ respectively.

An object of $\bar{\mathfrak{C}}$ consists of an integer m and a pointed topological space $\bar{H}$, which is homotopy equivalent to a finite CW complex. An object of $\bar{\mathfrak{C}}_{\mathbb{Z}_2}$ consists of an integer m and a pointed $\mathbb{Z}_2$-space $\tilde{H}$, which is $\mathbb{Z}_2$-homotopic equivalent to a finite $\mathbb{Z}_2$-CW complex. We define

$$\text{mor}_{\bar{\mathfrak{C}}}((\bar{H}, m), (\bar{H}', m')) := \operatorname*{colim}_{k\in\mathbb{Z}}[(\mathbb{R}^k)^+ \wedge \bar{H}, (\mathbb{R}^{k+m-m'})^+ \wedge \bar{H}'],$$

$$\text{mor}_{\bar{\mathfrak{C}}_{\mathbb{Z}_2}}((\tilde{H}, m), (\tilde{H}', m')) := \operatorname*{colim}_{k,l\in\mathbb{Z}}[(\mathbb{R}^k \oplus \tilde{\mathbb{R}}^l)^+ \wedge \tilde{H}, (\mathbb{R}^k \oplus \tilde{\mathbb{R}}^{l+m-m'})^+ \wedge \tilde{H}']_{\mathbb{Z}_2},$$

where we treat $\tilde{\mathbb{R}}$ as a nontrivial $\mathbb{Z}_2$-representation.

Having defined $\bar{\mathfrak{C}}$ and $\bar{\mathfrak{C}}_{\mathbb{Z}_2}$, we can define the other four categories as follows:

- $\bar{\mathfrak{S}}$: the category of sequential direct systems in $\bar{\mathfrak{C}}$;
- $\bar{\mathfrak{S}}^*$: the category of sequential projective systems in $\bar{\mathfrak{C}}$;
- $\bar{\mathfrak{S}}_{\mathbb{Z}_2}$: the category of sequential direct systems in $\bar{\mathfrak{C}}_{\mathbb{Z}_2}$;
- $\bar{\mathfrak{S}}^*_{\mathbb{Z}_2}$: the category of sequential projective systems in $\bar{\mathfrak{C}}_{\mathbb{Z}_2}$.

Now we discuss the fixed point functors. First, we can define the functor $\Phi^{S^1}_{\mathfrak{C}}$: $\mathfrak{C} \to \bar{\mathfrak{C}}$ by

$$\Phi^{S^1}_{\mathfrak{C}}(A, m, n) := (A^{S^1}, m) \text{ and } \Phi^{S^1}_{\mathfrak{C}}([f]) := [f^{S^1}].$$

The functor

$$\Phi^{S^1}_{\mathfrak{C}_{Pin(2)}} : \mathfrak{C}_{Pin(2)} \to \bar{\mathfrak{C}}_{\mathbb{Z}_2}$$

can be defined by the same formula. By extending these two functors in an obvious way, we can define the other four functors:

$$\Phi^{S^1}_{\mathfrak{S}} : \mathfrak{S} \to \bar{\mathfrak{S}};$$

$$\Phi^{S^1}_{\mathfrak{S}^*} : \mathfrak{S}^* \to \bar{\mathfrak{S}}^*;$$

$$\Phi^{S^1}_{\mathfrak{S}_{Pin(2)}} : \mathfrak{S}_{Pin(2)} \to \bar{\mathfrak{S}}_{\mathbb{Z}_2};$$

$$\Phi^{S^1}_{\mathfrak{S}^*_{Pin(2)}} : \mathfrak{S}^*_{Pin(2)} \to \bar{\mathfrak{S}}^*_{\mathbb{Z}_2}.$$

We call all these functors the S^1-fixed point functors.

Remark. The functors $\Phi^{S^1}_*$ correspond to the "geometric fixed-point functor" (see [46, Page 166]), which should be distinguished with the usual fixed-point functor in the equivariant stable homotopy theory (see, for example, [35, Page 21]). A further discussion about this difference can be found in [1, Section 7].

3.1.3 Smash product and suspension/desuspension functors

In this subsection, we establish the symmetric monoidal structure on the category $\mathfrak{C}$. To do this, we will define the smash product as a bifunctor $\wedge : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$. First, we define the smash product of two objects as

$$(A_1, m_1, n_2) \wedge (A_2, m_2, n_2) := (A_1 \wedge A_2, m_1 + m_2, n_1 + n_2)$$

where $A_1 \wedge A_2$ denotes the classical smash product on pointed topological spaces.

Next, we define the smash product of morphisms. Suppose for $i = 1, 2$, the map

$$f_i : (\mathbb{R}^{j_i} \oplus \mathbb{C}^{l_i})^+ \wedge A_i \to (\mathbb{R}^{(j_i+m_i-m_i')} \oplus \mathbb{C}^{(l_i+n_i-n_i')})^+ \wedge A_i'$$

represents a morphism $[f_i] \in \mathrm{mor}_{\mathfrak{C}}((A_i, m_i, n_i), (A_i', m_i', n_i'))$. We define the map

$$f_1 \wedge f_2 : (\mathbb{R}^{j_1} \oplus \mathbb{R}^{j_2} \oplus \mathbb{C}^{l_1} \oplus \mathbb{C}^{l_2})^+ \wedge X_1 \wedge X_2 \to$$
$$(\mathbb{R}^{(j_1+m_1-m_1')} \oplus \mathbb{R}^{(j_2+m_2-m_2')} \oplus \mathbb{C}^{(l_1+n_1-n_1')} \oplus \mathbb{C}^{(l_2+n_2-n_2')})^+ \wedge X_1' \wedge X_2'$$

by putting the suspension indices for f_1 on the left and the indices for f_2 on the right. We let $[f_1] \wedge [f_2]$ be the morphism represented by $f_1 \wedge f_2$. To prove that this operation is well defined, we need to check that for $a, b \in \mathbb{N}$, we have

$$\mathrm{id}_{(\mathbb{R}^a \oplus \mathbb{C}^b)^+} \wedge (f_1 \wedge f_2) \cong (\mathrm{id}_{(\mathbb{R}^a \oplus \mathbb{C}^b)^+} \wedge f_1) \wedge f_2 \cong f_1 \wedge (\mathrm{id}_{(\mathbb{R}^a \oplus \mathbb{C}^b)^+} \wedge f_2),$$

where $\cong$ means S^1-equivariant stably homotopic. This a consequence of equivariant Hopf theorem (cf. [66, Section 2.4]). Note that to prove the second isomorphism, we make use of the assumption that m_i and m_i' are even. There is an isomorphism

$$\tau_{(A_1,m_1,n_1),(A_2,m_2,n_2)} : (A_1, m_1, n_1) \wedge (A_2, m_2, n_2) \to (A_2, m_2, n_2) \wedge (A_1, m_1, n_1)$$

represented by the obvious homeomorphism $A_1 \wedge A_2 \to A_2 \wedge A_1$. It is not difficult to prove that τ is actually an natural isomorphism (again here we need the assumption that m_1, m_2 are even). Once the well definedness of $\wedge$ and the naturality of τ are established, we can prove the following lemma easily by checking the axioms at the level of topological spaces.

Lemma 3.1.2. *Under the operations $\wedge$ and τ, the category $\mathfrak{C}$ is a symmetric monoidal category, with the unit given by $S = (S^0, 0, 0)$.*

An S^1-representation E is called *admissible* if it is isomorphic to $\mathbb{R}^a \oplus \mathbb{C}^b$ for some nonnegative integers a, b. For such representation E, we define S^E as $(E^+, 0, 0)$ and define S^{-E} as $(E^{S^1}, 2a, 0)$, where E^{S^1} denotes the S^1-fixed point set of E. Now we choose an identification

$$I : E \to \mathbb{R}^a \oplus \mathbb{C}^b$$

and consider the composition maps

$$E \oplus E^{S^1} \xrightarrow{I \oplus I^{S^1}} \mathbb{R}^a \oplus \mathbb{C}^b \oplus \mathbb{R}^a \xrightarrow{\tau_{2,3}} \mathbb{R}^a \oplus \mathbb{R}^a \oplus \mathbb{C}^b$$

and

$$E^{S^1} \oplus E \xrightarrow{I^{S^1} \oplus I} \mathbb{R}^a \oplus \mathbb{C}^b \oplus \mathbb{R}^a \xrightarrow{\tau_{2,3}} \mathbb{R}^a \oplus \mathbb{R}^a \oplus \mathbb{C}^b.$$

Here I^{S^1} denotes the restriction of I to the S^1-fixed point set and $\tau_{2,3}$ denotes the map interchanging the second and the third factor. These maps induce isomorphisms

$$\iota_+ : S^E \wedge S^{-E} \xrightarrow{\cong} S \text{ and } \iota_- : S^{-E} \wedge S^E \xrightarrow{\cong} S. \tag{3.3}$$

By equivariant Hopf theorem, one can check that $\iota_\pm$ do not depend on the choice of I.

For an admissible representation E, we define the suspension functor $\Sigma^E \colon \mathfrak{C} \to \mathfrak{C}$ as the left smash product by S^E. We also define desuspension functor $\Sigma^{-E} \colon \mathfrak{C} \to \mathfrak{C}$ as the left smash product by S^{-E}. We have the following lemma, whose proof is straightforward.

Lemma 3.1.3. *For any admissible S^1-representations E, we have canonical isomorphisms*

$$\iota_+^* : \Sigma^E \circ \Sigma^{-E} \to \mathrm{id} \ \textit{and}\ \iota_-^* : \Sigma^{-E} \circ \Sigma^E \to \mathrm{id}$$

induced by $\iota_\pm$ in (3.3). Here id *denotes the identity morphism on $\mathfrak{C}$.*

Note that the suspension and desuspension functors can be extended to the category $\mathfrak{S}$ and $\mathfrak{S}^*$ in an obvious way and Lemma 3.1.3 still holds.

Now we briefly discuss the Pin(2)-case. The smash product $\wedge$ and the commuting operation τ can be defined on the category $\mathfrak{C}_{Pin(2)}$ in exactly the same way as before. As a result, the category $\mathfrak{C}_{Pin(2)}$ is also an symmetric monoidal category. An Pin(2)-representation E is called admissible if it is isomorphic to $\tilde{\mathbb{R}}^a \oplus \mathbb{H}^b$ for some nonnegative a, b. For such representation E, we define the objects $S^E := (E^+, 0, 0)$ and $S^{-E} := (E^{S^1}, 2a, b)$. As before, the functors Σ^E and Σ^{-E} are defined as the left smash product with S^E and S^{-E} respectively. These two functors can be extended to the categories $\mathfrak{S}_{Pin(2)}$ and $\mathfrak{S}^*_{Pin(2)}$. A similar result as Lemma 3.1.3 holds in the current case.

3.1.4 Equivariant Spanier-Whitehead duality

In this subsection, we will set up the equivariant Spanier-Whitehead duality between the categories $\mathfrak{S}$ and $\mathfrak{S}^*$. Although we will mostly focus on the S^1-case for simplicity, all definitions and proofs can be easily adapted to the Pin(2)-case. As a result, a duality between $\mathfrak{S}_{Pin(2)}$ and $\mathfrak{S}^*_{Pin(2)}$ can also be set up in a similar way.

The following definition is motivated by [46, Chapterr XVI Theorem 7.6].

Definition 3.1.4. Let U, W be objects of $\mathfrak{C}$. Suppose there exists morphisms

$$\epsilon : W \wedge U \to S \text{ and } \eta : S \to U \wedge W$$

such that the compositions

$$U \cong S \wedge U \xrightarrow{\eta \wedge \mathrm{id}} U \wedge W \wedge U \xrightarrow{\mathrm{id} \wedge \epsilon} U \wedge S \cong S$$

and

$$W \cong W \wedge S \xrightarrow{\mathrm{id} \wedge \eta} W \wedge U \wedge W \xrightarrow{\epsilon \wedge \mathrm{id}} S \wedge W \cong W$$

are the respective identity morphisms. Then we say U, W are Spanier-Whitehead dual to each other and call ϵ, η the duality maps.

Now we generalize Definition 3.1.4 to define the duality between $\mathfrak{S}$ and $\mathfrak{S}^*$. Let

$$Z : Z_1 \to Z_2 \to Z_3 \to \cdots$$

be an object of $\mathfrak{S}$ and

$$\bar{Z} : \bar{Z}_1 \leftarrow \bar{Z}_2 \leftarrow \bar{Z}_3 \leftarrow \cdots$$

be an object of $\mathfrak{S}^*$. We consider an element

$$\epsilon \in \lim_{\infty \leftarrow m} \lim_{n \to \infty} \mathrm{Mor}_{\mathfrak{C}}(\bar{Z}_n \wedge Z_m, S)$$

represented by a collection of morphisms $\{\epsilon_{m,n} : \bar{Z}_n \wedge Z_m \to S\}_{m>0, n \gg m}$ and an element

$$\eta \in \lim_{\infty \leftarrow n} \lim_{m \to \infty} \mathrm{Mor}_{\mathfrak{C}}(S, Z_m \wedge \bar{Z}_n)$$

represented by the collection of morphisms $\{\eta_{m,n} : S \to Z_m \wedge \bar{Z}_n\}_{n>0, m \gg n}$.

Definition 3.1.5. For $Z \in \mathrm{ob}\, \mathfrak{S}$ and $\bar{Z} \in \mathrm{ob}\, \mathfrak{S}^*$, we say that Z and $\bar{Z}$ are Spanier-Whitehead dual to each other if there exist ϵ and η as above satisfying the following conditions:

- For any $m > 0$, there exists n large enough relative to m and m' large enough relative to n such that the composition

$$Z_m \cong S \wedge Z_m \xrightarrow{\eta_{m',n} \wedge \mathrm{id}} Z_{m'} \wedge \bar{Z}_n \wedge Z_m \xrightarrow{\mathrm{id} \wedge \epsilon_{m,n}} Z_{m'} \wedge S \cong Z_{m'}$$

  equals the connecting morphism $Z_m \to Z_{m'}$ in the inductive system Z.

- For any $n > 0$, there exists m large enough relative to n and n' large enough relative to m such that the composition

$$\bar{Z}_{n'} \cong \bar{Z}_{n'} \wedge S \xrightarrow{\mathrm{id} \wedge \eta_{m,n}} \bar{Z}_{n'} \wedge Z_m \wedge \bar{Z}_n \xrightarrow{\epsilon_{m,n'} \wedge \mathrm{id}} S \wedge \bar{Z}_n \cong \bar{Z}_n$$

  equals the connecting morphism $\bar{Z}_{n'} \to \bar{Z}_n$ in the projective system $\bar{Z}$.

In this case, we call ϵ, η the duality morphisms.

We end this subsection by introducing the smashing operation $\tilde{\epsilon}(\cdot, \cdot)$, which will be used to give the statement of gluing theorem for the Bauer-Furuta invariant.

Definition 3.1.6. Let $Z \in \operatorname{ob}\mathfrak{S}$ and $\bar{Z} \in \operatorname{ob}\mathfrak{S}^*$ be objects that are S^1-equivariant Spanier-Whitehead dual to each other, with duality morphisms given by ϵ, η. Given two morphisms

$$\rho = [\rho_m] \in \operatorname{mor}_{\mathfrak{S}}(W, Z) \text{ and } \bar{\rho} = \{\bar{\rho}_n\}_{n>0} \in \operatorname{mor}_{\mathfrak{S}^*}(\bar{W}, \bar{Z}),$$

where $W, \bar{W}$ belong to the subcategory $\mathfrak{C}$, we define the morphism $\tilde{\epsilon}(\rho, \bar{\rho}) \in \operatorname{mor}_{\mathfrak{C}}(S, W \wedge \bar{W})$ as the composition

$$S \xrightarrow{\epsilon_{m,n}} Z_m \wedge \bar{Z}_n \xrightarrow{\rho_m \wedge \bar{\rho}_n} W \wedge \bar{W}$$

for any n large enough relative to m. It can be proved that $\tilde{\epsilon}(\rho, \bar{\rho})$ does not depend on the choice of m, n and ρ_m. (Note that $\bar{\rho}_n$ is determined by n and $\bar{\rho}$.)

3.2 The Conley index

3.2.1 Definition and basic properties

In this section, we recall basic facts regarding the Conley index theory. See [10] ,[40] and [57] for more details.

Let V be a finite dimensional manifold and φ be a smooth flow on V, i.e. a C^∞-map $\varphi\colon V \times \mathbb{R} \to V$ such that $\varphi(x, 0) = x$ and $\varphi(x, s+t) = \varphi(\varphi(x, s), t)$ for any $x \in V$ and $s, t \in \mathbb{R}$. We denote by $\operatorname{inv}(\varphi, A) := \{x \in A \mid \varphi(x, \mathbb{R}) \subset A\}$ the maximal invariant set of A. We sometimes write $\operatorname{inv}(A)$ when the flow φ is obvious from the context.

A compact set $A \subset V$ is called an *isolating neighborhood* if $\operatorname{inv}(A)$ lies in the interior of A. A compact set $S \subset V$ is called an *isolated invariant set* if there exists an isolating neighborhood A such that $\operatorname{inv}(A) = S$. In this situation, we also say that A is an isolating neighborhood of S. For an isolated invariant set S, a pair (N, L) of compact sets $L \subset N$ is called an *index pair* of S if the following conditions hold:

(i) $\operatorname{inv}(N \setminus L) = S \subset \operatorname{int}(N \setminus L)$, where $\operatorname{int}(N \setminus L)$ denotes the interior of $N \setminus L$;

(ii) L is an exit set for N, i.e. for any $x \in N$ and $t > 0$ such that $\varphi(x, t) \notin N$, there exists $\tau \in [0, t)$ with $\varphi(x, \tau) \in L$;

(iii) L is positively invariant in N, i.e. for $x \in L$ and $t > 0$, if we know $\varphi(x, [0, t]) \subset N$, then we have $\varphi(x, [0, t]) \subset L$.

We list two fundamental facts regarding index pairs:

- For an isolated invariant set S with an isolating neighborhood A, we can always find an index pair (N, L) of S such that $L \subset N \subset A$.
- The pointed homotopy type of N/L with $[L]$ as a base point only depends on S and φ. More precisely, for any two index pairs (N, L) and (N', L') of S, there is a natural pointed homotopy equivalence $N/L \to N'/L'$ induced by the flow.

These lead to us the definition of the Conley index.

Definition 3.2.1. Given an isolated invariant set S of a flow φ, we denote by $I(\varphi, S, N, L)$ the pointed space of $(N/L, [L])$, where (N, L) is an index pair of S. This is called the *Conley index* of S. We will always suppress (N, L) from our notation and write $I(\varphi, S)$ instead. We may also write $I(S)$ when the flow is clear from the context.

Remark. In [57], the Conley was defined as a connected simple system of pointed spaces. I.e., a collection of pointed spaces (given by different index pairs) together with natural homotopy equivalences between them (given by the flow map). In Definition 3.2.1, we actually pick a representative of this connected simple system by making a choice of the index pair (N, L). As we will see in next section, we need to make choices of all kinds of index pairs in our construction of spectrum invariants. Just like the Riemannian metric g and the perturbation on f, these choices will be treated as auxiliary data involved in the construction and we will prove that our spectrum invariant is independent of this data upto canonical isomorphism.

We further provide relevant properties of the Conley index.

1. (Product flow) If φ_j is a flow on V_j for $j = 1, 2$ and S_j is an isolated invariant set for φ_j, then we have a canonical homotopy equivalence $I(\varphi_1 \times \varphi_2, S_1 \times S_2) \cong I(\varphi_1, S_1) \wedge I(\varphi_2, S_2)$, where "$\wedge$" is the smash product.

2. (Continuation) Let φ_t is a continuous family of flows parametrized by $t \in [0,1]$. Suppose that A is an isolating neighborhood of φ_t for any $t \in [0,1]$, and let S_t be $\mathrm{inv}(\varphi_t, A)$. Then we have a canonical homotopy equivalence $I(\varphi_0, S_0) \cong I(\varphi_1, S_1)$.

The following concept will be useful for explicitly computing the Conley index.

Definition 3.2.2 ([56]). For a compact subset A, we consider the following subsets of its boundary

$$n^+(A) := \{x \in \partial A \mid \exists \epsilon > 0 \text{ s.t } \varphi(x, (-\epsilon, 0)) \cap A = \emptyset\},$$
$$n^-(A) := \{x \in \partial A \mid \exists \epsilon > 0 \text{ s.t } \varphi(x, (0, \epsilon)) \cap A = \emptyset\}.$$

A compact subset N is called an *isolating block* if $\partial N = n^+(N) \cup n^-(N)$.

It is easy to verify that an isolating block is an isolating neighborhood. When N is an isolating block, its index pair can be given by $(N, n^-(N))$.

Next, we consider a situation when an isolated invariant set can be decomposed to smaller isolated invariant sets.

Definition 3.2.3.

(i) For a subset $A \subset V$, we define its α-limit and ω-limit set as

$$\alpha(A) = \bigcap_{t<0} \overline{\varphi(A, (-\infty, t])} \quad \text{and} \quad \omega(A) = \bigcap_{t>0} \overline{\varphi(A, [t, +\infty))}.$$

(ii) Let S be an isolated invariant set. A subset $T \subset S$ is called an *attractor* (resp. *repeller*) if there exists a neighborhood U of T in S such that $\omega(U) = T$ (resp. $\alpha(U) = T$).

(iii) When T is an attractor in S, we define the set $T^* := \{x \in S \mid \omega(x) \cap T = \emptyset\}$, which is a repeller in S. We call (T, T^*) an *attractor-repeller pair* in S.

Note that an attractor and a repeller are always an isolated invariant sets. We give an important result relating Conley indices of an attractor-repeller pair.

Proposition 3.2.4 (Salamon [57]). *Let S be an isolated invariant set with an isolating neighborhood A and (T, T^*) be an attractor-repeller pair in S. Then there exist compact sets $\tilde{N}_3 \subset \tilde{N}_2 \subset \tilde{N}_1 \subset A$ such that the pairs $(\tilde{N}_2, \tilde{N}_3), (\tilde{N}_1, \tilde{N}_3), (\tilde{N}_1, \tilde{N}_2)$ are index pairs for T, S and T^* respectively. The maps induced by inclusions give a natural coexact sequence of Conley indices*

$$I(\varphi, T) \xrightarrow{i_1} I(\varphi, S) \xrightarrow{i_2} I(\varphi, T^*) \to \Sigma I(\varphi, T) \to \Sigma I(\varphi, S) \to \cdots .$$

We call the triple $(\tilde{N}_3, \tilde{N}_2, \tilde{N}_1)$ an index triple for the pair (T, T^*) and call the maps i_1 and i_2 the attractor map and the repeller map respectively.

By Corollary 4.4 of [57], the attractor maps are transitive in the following sense. Suppose that S_1 is an attractor in S_2 and S_2 is an attractor in S_3. Then S_1 is also an attractor in S_3. Moreover, the corresponding attractor maps

$$i_1 : I(\varphi, S_1) \to I(\varphi, S_2),\ i_1' : I(\varphi, S_2) \to I(\varphi, S_3) \text{ and } i_1'' : I(\varphi, S_1) \to I(\varphi, S_3)$$

satisfy the relation $i_1'' = i_1' \circ i_1$. Similar statements hold for the repeller maps.

Lastly, we briefly discuss the equivariant Conley index theory, which has been developed in [15] and [54]. Let G be a compact Lie group acting on V while preserving the flow φ. For a G-invariant isolated invariant set S, we can find a G-invariant isolating neighborhood as well as a G-invariant index pair (N, L). As in the non-equivariant case, with the choice of (N, L), we denote by $I_G(\varphi, S)$ the pointed G-space $(N/L, [L])$, whose G-equivariant homotopy type only depends on S and φ. In particular, $I_G(\varphi, S)$ is the *G-equivariant Conley index* of S. All the non-equivariant results stated above can be adapted to the G-equivariant setting. From now on, we will work on this equivariant setting with $G = S^1$ or $Pin(2)$.

3.2.2 Further properties of Conley index

In this subsection, we collect some further properties and prove some results regarding Conley index theory that will be needed in Chapter 7. These results may be of independent interest for some readers. Although we focus on the non-equivariant case for simplicity, all the results can be adapted to the equivariant setting. Throughout this subsection, we fix a smooth flow $\varphi : V \times \mathbb{R} \to V$, an isolated invariant set S and an isolating neighborhood A for S. We use the notation $f \cong g$ to indicate that two maps f, g are pointed homotopic to each other.

Notation. For $B \subset A$ and $I \subset \mathbb{R}$, we define

$$B^I := \{x \in A \mid \varphi(x, I) \subset B\}.$$

We also write $B^{[0,+\infty)}$ and $B^{(-\infty,0]}$ as B^+ and B^- respectively.

3.2.2.1 The flow map

As mentioned in the remark after Definition 3.2.1, different Conley indices for S form a simple connected system. In particular, they are connected to each other by natural homotopy equivalences, whose precise formula is given in the following theorem.

Theorem 3.2.5 (Kurland [34], Salamon [57], Conley-Zender [11])**.** *If (N, L) and (N', L') are two index pairs for the same isolated invariant set S, then there exists $\bar{T} > 0$ such that*

- $\varphi(x, [-\bar{T}, \bar{T}]) \subset N' \setminus L'$ *implies* $x \in N \setminus L$*;*
- $\varphi(x, [-\bar{T}, \bar{T}]) \subset N \setminus L$ *implies* $x \in N' \setminus L'$*.*

Moreover, for any $T \geq \bar{T}$, the map $s_T : N/L \to N'/L'$ given by

$$s_T([x]) := \begin{cases} [\varphi_{3T}(x)] & \text{if } \varphi(x, [0, 2T]) \subset N \setminus L \text{ and } \varphi(x, [T, 3T]) \subset N' \setminus L' \\ [L] & \text{otherwise} \end{cases}$$

is well defined and continuous. For different $T \geq \bar{T}$, the maps s_T are all homotopic to each other and they give natural homotopy equivalences between N/L and N'/L'. We call s_T the flow map at time T (also called the Salamon map at time T).

3.2.2.2 T-tame pre-index pair and T-tame index pair

Definition 3.2.6. A pair (K_1, K_2) of compact subsets of A is called *a pre-index pair* if

- For any $x \in K_1 \cap A^+$, we have $\varphi(x, [0, +\infty)) \subset \mathrm{int}(A)$;
- $K_2 \cap A^+ = \emptyset$.

Theorem 3.2.7 (Manolescu [40]). *For any pre-index pair* (K_1, K_2)*, there exists an index pair* (N, L) *satisfying*

$$K_1 \subset N \subset A, \;\; K_2 \subset L. \tag{3.4}$$

We call such (N, L) *an* index pair containing (K_1, K_2).

Theorem 3.2.8 (Khandhawit [30]). *Let* (K_1, K_2) *be a pre-index pair and* (N_j, L_j) $(j = 1, 2)$ *be two index pairs containing* (K_1, K_2)*. Denote by* $l_j : K_1/K_2 \to N_j/L_j$ *the map induced by the natural inclusion. Then we have*

$$s_t \circ l_1 \cong l_2 \text{ for } t \gg 0,$$

where $s_t : N_1/L_1 \to N_2/L_2$ *is the flow map at time* t.

For an pre-index pair (K_1, K_2), by Theorem 3.2.7, we can find index pair (N, L) containing it and define $l_{(K_1,K_2,N,L)} : K_1/K_2 \to I(\varphi, S, N, L)$. Moreover, by Theorem 3.2.8, for different choices of (N, L), the maps $l_{(K_1,K_2,N,L)}$ are compatible with each other. This allows us to suppress N, L (and also φ) from our notations and simply write the map as

$$l : K_1/K_2 \to I(S). \tag{3.5}$$

We call this map *the canonical map for* (K_1, K_2).

Next, we discuss the quantitative refinement of Theorem 3.2.7, which will be especially useful when defining the relative Bauer-Furuta invariant and proving the gluing theorem. We first introduce some definitions.

Definition 3.2.9. For $T \in \mathbb{R}^{>0}$, an isolating neighborhood A is called a *T-tame isolating neighborhood* if it satisfies the following condition:

$$A^{[-T,T]} \subset \mathrm{int}(A).$$

Definition 3.2.10. Let (K_1, K_2) be a pre-index pair in a T-tame isolating neighborhood A. We call (K_1, K_2) a *T-tame pre-index pair* if it satisfies the following conditions:

(1) There exists a compact set A' satisfying

- $A^{[-T,T]} \subset A' \subset \mathrm{int}(A)$;
- If $x \in K_1 \cap A^{[0,T']}$ for some $T' \geq T$, then $\varphi(x, [0, T' - T]) \subset A'$.

(2) $K_2 \cap A^{[0,T]} = \emptyset$.

Lemma 3.2.11. *Let (K_1, K_2) be a pre-index pair in an isolating neighborhood A. Then there exists $T > 0$ such that A and (K_1, K_2) are both T-tame. This also implies that A and (K_1, K_2) are $\bar{T}$-tame for any $\bar{T} \geq T$.*

Proof. We know that $A^{(-\infty,+\infty)} \cap \partial A = \emptyset$ and $K_2 \cap A^{[0,+\infty)} = \emptyset$. Since both ∂A and K_2 are compact, we have $A^{[T,T]} \cap \partial A = \emptyset$ and $K_2 \cap A^{[0,T]} = \emptyset$ for $T \gg 0$. Therefore, we are left with checking condition (1) of Definition 3.2.10. Suppose it does not for any T. Then we can find sequences $\{x_j\} \subset K_1$, $\{T_j\}$ and $\{T'_j\}$ such that $T_j \to +\infty$, $T'_j \geq T_j$, $\varphi(x_j, [0, T'_j]) \subset A$ and $\varphi(x_j, T'_j - T_j) \to y \in \partial A$. If $T'_j - T_j$ does not goes to infinity, then after passing a subsequence, we can assume that $(x_j, T'_j - T_j) \to (x_\infty, T'') \in K_1 \times \mathbb{R}^{\geq 0}$. We have $\varphi(x_\infty, [0, +\infty)) \subset A$ and $\varphi(x_\infty, T'') = y \in \partial A$. This is a contraction with Definition 3.2.6. Now suppose $T'_j - T_j$ goes to infinity. For any $t \in \mathbb{R}$, we have $T'_j - T_j + t \in [0, T'_j]$ for $j \gg 0$, which implies $\varphi(x_j, T'_j - T_j + t) \in A$. Notice that $\varphi(y, t) = \lim_{j \to \infty} \varphi(x_j, T'_j - T_j + t)$. Therefore, we have $y \in A^{(-\infty,+\infty)}$. This is a contradiction because $A^{(-\infty,+\infty)} \cap \partial A = \emptyset$. □

Definition 3.2.12. For $T \in \mathbb{R}^{>0}$, an index pair (N, L) in an isolating neighborhood A is called a *T-tame index pair* if it satisfies the following conditions:

(i) Both N, L are positively invariant in A;

(ii) $A^{[-T,T]} \subset N$;

(iii) $A^{[0,T]} \cap L = \emptyset$.

One important reason that we are interested tame index pair is the following lemma.

Lemma 3.2.13. *Let (N, L) and (N', L') be two index pairs in A. For $T > 0$, suppose (N, L) is T-tame and the flow map $s_T : N/L \to N/L'$ is defined. Then we have the following equivalent definition of s_T*

$$s_T([x]) = \begin{cases} [\varphi_{3T}(x)] & \text{if } \varphi(x, [0, 3T]) \subset A \text{ and } \varphi(x, [T, 3T]) \subset N' \setminus L' \\ [L] & \text{otherwise} \end{cases}.$$

Proof. We need to show that for any $x \in N$ the following two conditions are equivalent.

(1) $\varphi(x, [0, 3T]) \subset A$ and $\varphi(x, [T, 3T]) \subset N' \setminus L'$;

(2) $\varphi(x, [0, 2T]) \subset N \setminus L$ and $\varphi(x, [T, 3T]) \subset N' \setminus L'$.

It is easy to see that (2) implies (1). To see the converse, we pick any x satisfying (1). Then since N is positively invariant in A, we have $\varphi(x, [0, 3T]) \subset N$. By condition (iii) of 3.2.12, we have $\varphi(x, [0, 2T]) \cap L = \emptyset$. Therefore, x satisfies (2). $\square$

Theorem 3.2.14. *For any $T > 1$, let A be a $(T-1)$-tame insolating neighborhood and (K_1, K_2) be a $(T-1)$-tame pre-index pair contained in A. Then there exists a T-tame index pair (N, L) that contains (K_1, K_2).*

To simplify the proof, we introduce the following notation:

Notation. For subsets $B, B' \subset V$, we define the set

$$P_{B'}(B) := \{\varphi(x, t) \mid x \in B,\ t \geq 0 \text{ and } \varphi(x, [0, t]) \subset B'\}.$$

Proof of Theorem 3.2.14. The proof is an adaption of the argument in [40, Appendix A]. Let $\tilde{K}_1 = K_1 \cup A^{[-T+1, T-1]}$. We claim that $(\tilde{K}_1, K_2)$ is also an pre-index pair. To see this, we need to show that $\varphi(y, [0, +\infty)) \subset \operatorname{int}(A)$ for any $y \in \tilde{K}_1 \cap A^+$. This is clear when $y \in K_1 \cap A^+$ because (K_1, K_2) is a pre-index pair. For those y belonging to $A^{[-T+1, T-1]} \cap A^+ = A^{[-T+1, +\infty)}$, we have

$$\varphi(y, [0, +\infty)) \subset A^{[-T+1, T-1]} \subset \operatorname{int}(A),$$

where the second inclusion make use of the condition that A is $(T-1)$-tame. Therefore, the claim is verified.

Now since $(\tilde{K}_1, K_2)$ is a pre-index pair, by Theorem 3.2.7, there exists an index pair (N, L) containing $(\tilde{K}_1, K_2)$. We have

$$A^{[-T,T]} \subset A^{[-T+1,T-1]} \subset \tilde{K}_1 \subset N.$$

Condition (ii) in Definition 3.2.12 is verified.

Recall that in the proof of Theorem 3.2.7 [40], the index pair (N, L) is constructed as

$$(P_A(B) \cup P_A(A \setminus \tilde{V}), P_A(A \setminus \tilde{V}))$$

where $A, \tilde{V}$ are certain specific subsets of A. (The notation V is used in [40] as $\tilde{V}$ here.) Since $P_A(*)$ is always positively invariant in A and the union of two positively invariant sets in A is again positively invariant in A. We see that both N, L are positively invariant.

We are left with checking condition (iii). To do this, let us recall the construction of the set $\tilde{V}$ in [40]. Let C be a compact subset of A, with the following properties:

1. C is a neighborhood of $A^+ \cap \partial A$ in A;
2. $C \cap P_A(\tilde{K}_1) = \emptyset$;
3. $C \cap A^- = \emptyset$.

Then $\tilde{V}$ is any open neighborhood of A^+ satisfying the following conditions:

4. $\overline{\tilde{V} \setminus C} \subset \text{int}(A)$;
5. $K_2 \cap \tilde{V} = \emptyset$.

Now we construct specific $C, \tilde{V}$ that meet all these requirements: Let A'' be a compact set such that

$$A'' \subset \text{int}(A), \;\; A' \cup A^{[-T+1,T-1]} \subset \text{int}(A''),$$

where the set $A' \subset \text{int}(A)$ has the following property:

- For any $T' \geq T-1$ and $x \in A^{[0,T']} \cap K_1$, we have $\varphi(x, [0, T'-T+1]) \subset A'$.

Such A' exists because (K_1, K_2) is a $(T-1)$-tame pre-index pair (see Definition 3.2.10).

Now we fix two numbers $T-1 < T_2 < T_1 < T$ and set

$$C_0 = (A \setminus \mathrm{int}(A'')) \cap A^{[0,T_2]} \text{ and } \tilde{V}_0 = A^{[0,T_1]}.$$

We have the following observations:

- $C_0 \cap A^- = \emptyset$: This is because

  $$C_0 \cap A^- = (A \setminus \mathrm{int}(A'')) \cap A^{[-\infty,T_2]} \subset (A \setminus \mathrm{int}(A'')) \cap A^{[-T+1,T-1]} = \emptyset.$$

  We their distance by d_1. This is positive since both sets are compact.

- The distance between C_0 and $P_A(\tilde{K}_1)$, denoted by d_2, is positive: Assume the contrary. Notice that C_0 is compact. We can find sequences $\{x_j\} \subset \tilde{K}_1$ and $\{t_j\} \subset \mathbb{R}^{\geq 0}$ that satisfy $\varphi(x_j, [0, t_j]) \subset A$ and $\varphi(x_j, t_j) \to y \in C_0$. There are two cases:

  - Suppose t_j does not goes to infinity. Then after passing to a subsequence, we can assume $(x_j, t_j) \to (x_\infty, t_\infty) \in \tilde{K}_1 \times \mathbb{R}^{\geq 0}$. We have $\varphi(x_\infty, [0, t_\infty]) \subset A$ and $\varphi(x_\infty, t_\infty) = y$. If $x_\infty \in K_1$, then since $\varphi(y, T-1) \subset A$, we have $y \in A'$ by our choice of A'. This is impossible because $A' \cap C_0 = \emptyset$. If $x_\infty \in A^{[-T,T]}$, then $y \in A^{[-T-t_\infty, T_2]} \subset A^{[-T+1,T-1]}$. This is also impossible because $A^{[-T+1,T-1]} \cap C_0 = \emptyset$.
  - Suppose $t_j \to +\infty$. Then for any $t \geq 0$, we have $\varphi(y, -t) = \lim_{j\to\infty} \varphi(x_j, t_j - t) \in A$. This is a contradiction because $C_0 \cap A^- = \emptyset$.

- $\overline{\tilde{V}_0 \setminus C_0} \subset A''$: This is because $T_1 > T_2$.

- $K_2 \cap \tilde{V}_0 = \emptyset$: This is because (K_1, K_2) is $(T-1)$-tame and $T_2 > T-1$. Denote the distance between K_2 and $\tilde{V}_0$ by d_2. We have $d_2 > 0$ because K_2 and $\tilde{V}_0$ are both compact.

Let $d_3 > 0$ be the distance between ∂A and A''. We choose any positive number d_4 strictly smaller than $\min(d_0, d_1, d_2, d_3)$ and set our specific $C, \tilde{V}$ to be

$$C = \{x \in A \mid d(x, C_0) \leq d_4\}; \ \tilde{V} = \{x \in A \mid d(x, \tilde{V}_0) < d_4\}.$$

Now we show that C, $\tilde{V}$ satisfy the requirements: Conditions 1, 2, 3, 5 are directly implied by our choice of d_4. As for condition 4, for any $y \in \partial A \cap \bar{\tilde{V}}$, there exists $y' \in \tilde{V}_0$ such that $d(y, y') \leq d_4 < d_3$. This implies that $y' \notin A''$. Since $\overline{\tilde{V}_0 \setminus C_0} \subset A''$, we have $y' \in C_0$, which implies $y \in C$. Therefore, condition 4 is verbified.

Now we have $L = P_A(A \setminus \tilde{V})$ and we want to verify condition (iii) in Definition 3.2.12 (i.e., $L \cap A^{[0,T]} = \emptyset$). Suppose $x \in P_A(A \setminus \tilde{V}) \cap A^{[0,T]}$. Then we can have $x = \varphi_t(y)$ where $y \notin \tilde{V}$ and $\varphi_{[0,T+t]}(y) \subset A$. This is a contradiction because $A^{[0,T+t]} \subset \tilde{V}$. □

3.2.2.3 The attractor-repeller pair arising from a strong Morse decomposition

In our later construction of spectrum invariants, our isolating neighborhood will have a nice decomposition, which gives an attractor-repeller pair with good properties. To clarify the situation, we give the following definition.

Definition 3.2.15. Let (A_1, A_2) be pair of compact sets. We call (A_1, A_2) a *strong Morse decomposition of* A if

- $A = A_1 \cup A_2$;
- For any $x \in A_1 \cap A_2$, there exists $\epsilon > 0$ such that

$$\varphi(x, (0, \epsilon)) \cap A_1 = \emptyset \text{ and } \varphi(x, (-\epsilon, 0)) \cap A_2 = \emptyset. \tag{3.6}$$

We summarize the basic properties of a strong decomposition in the following lemma. We omit the proof since it is straightforward.

Lemma 3.2.16. *Suppose (A_1, A_2) is a strong Morse decomposition of an isolating invariant set A. Then we have the following results.*

(1) A_1 *(resp.* A_2*) is negatively (resp. positively) invariant in* A*;*

(2) $A_1 \cap A_2 = \partial A_1 \cap \partial A_2$ *and* $\partial A_i \subset \partial A \cup (A_1 \cap A_2)$ *for* $i = 1, 2$*;*

(3) Both A_1 *and* A_2 *are isolating invariant sets;*

(4) $(\mathrm{inv}(A_2), \mathrm{inv}(A_1))$ *is an attractor-repeller pair in* $\mathrm{inv}(A)$*.*

The attractor-repeller pair arising from a strong Morse decomposition has the following special property.

Lemma 3.2.17. *Let* (A_1, A_2) *be a strong Morse decomposition of* A *and* $(\tilde{N}_3, \tilde{N}_2, \tilde{N}_1)$ *be an index triple for* $(\mathrm{inv}(A_2), \mathrm{inv}(A_1))$ *(see Proposition 3.2.4). Then* $(\tilde{N}_3, \tilde{N}_2', \tilde{N}_1)$ *is again an index triple, where* $\tilde{N}_2' = \tilde{N}_2 \cup (\tilde{N}_1 \cap A_2)$*. In particular, we can always assume* $(\tilde{N}_1 \cap A_2) \subset \tilde{N}_2$ *by replacing* $\tilde{N}_2$ *with* $\tilde{N}_2'$*.*

Proof. First consider the pair $(\tilde{N}_1, \tilde{N}_2')$. We have the following observations, which implies that $(\tilde{N}_1, \tilde{N}_2')$ is an index pair for $\mathrm{inv}(A_1)$.

- $\tilde{N}_2'$ is positively invariant in $\tilde{N}_1$: Since A_2 is positively invariant in A, the set $A_2 \cap \tilde{N}_1$ is positively invariant in $\tilde{N}_1$. Because $\tilde{N}_2$ is also positively invariant in $\tilde{N}_1$ (by the definition of an index pair) and the union of two positively invariant set is again positively invariant. We see that $\tilde{N}_2'$ is positively invariant in $\tilde{N}_1$.

- $\tilde{N}_2'$ is an exit set for $\tilde{N}_1$: This is because $\tilde{N}_2'$ contains $\tilde{N}_2$, which is an exit set for $\tilde{N}_1$.

- $\mathrm{inv}(A_1) = \mathrm{inv}(\tilde{N}_1 \setminus \tilde{N}_2') \subset \mathrm{int}(\tilde{N}_1 \setminus \tilde{N}_2')$: We have $\mathrm{inv}(\tilde{N}_1 \setminus \tilde{N}_2') \subset \mathrm{inv}(\tilde{N}_1 \setminus \tilde{N}_2)$ because $\tilde{N}_1 \setminus \tilde{N}_2' \subset \tilde{N}_1 \setminus \tilde{N}_2$. To see the converse, we take any $x \in \mathrm{inv}(\tilde{N}_1 \setminus \tilde{N}_2) = \mathrm{inv}(A_1)$. Then $\varphi(x, (-\infty, +\infty))$ is contained in $(\tilde{N}_1 \setminus \tilde{N}_2) \cap \mathrm{int}(A_1)$. Notice that $\mathrm{int}(A_1)$ does not intersect $\tilde{N}_1 \cap A_2$. We have $\varphi(x, (-\infty, +\infty)) \subset \tilde{N}_1 \setminus (\tilde{N}_2 \cup (\tilde{N}_1 \cap A_2))$. This implies that $x \in \mathrm{inv}(\tilde{N}_1 \setminus \tilde{N}_2')$. We have shown that $\mathrm{inv}(\tilde{N}_1 \setminus \tilde{N}_2') = \mathrm{inv}(\tilde{N}_1 \setminus \tilde{N}_2) = \mathrm{inv}(A)$. Moreover, we have $\mathrm{inv}(\tilde{N}_1 \setminus \tilde{N}_2') \subset \mathrm{int}(\tilde{N}_1 \setminus \tilde{N}_2) \cap \mathrm{int}(A_1) \subset \mathrm{int}(\tilde{N}_1 \setminus \tilde{N}_2')$.

Now consider the pair $(\tilde{N}_2', N_3)$. We have the following observations, which implies that $(\tilde{N}_1', \tilde{N}_3)$ is an index pair for $\mathrm{inv}(A_2)$.

- $\tilde{N}_3$ is positively invariant in $\tilde{N}_2'$: This is because $\tilde{N}_3$ is positively invariant in $\tilde{N}_1$, which contains $\tilde{N}_2'$.
- $\tilde{N}_3$ is an exit set for $\tilde{N}_2'$: This is because $\tilde{N}_3$ is an exit set for $\tilde{N}_1$ and $\tilde{N}_2'$ is positively invariant in $\tilde{N}_1$.
- $\mathrm{inv}(A_2) = \mathrm{inv}(\tilde{N}_2' \setminus \tilde{N}_3) \subset \mathrm{int}(\tilde{N}_2' \setminus \tilde{N}_3)$: We have $\mathrm{inv}(\tilde{N}_2 \setminus \tilde{N}_3) \subset \mathrm{inv}(\tilde{N}_2' \setminus \tilde{N}_3)$ because $\tilde{N}_2 \setminus \tilde{N}_3 \subset \tilde{N}_2' \setminus \tilde{N}_3$. To see the converse, take any $x \in \mathrm{inv}(\tilde{N}_2' \setminus \tilde{N}_3)$. Then we have $\varphi(x, (-\infty, \infty)) \subset \tilde{N}_2' \setminus \tilde{N}_3$. Suppose $x \notin \mathrm{inv}(\tilde{N}_2 \setminus \tilde{N}_3) = \mathrm{inv}(A_2)$. Then $\varphi_{x,t_0} \notin \tilde{N}_2$ for some t_0. Since $\varphi(x, (-\infty, +\infty))$ does not intersect $\tilde{N}_3$, which is an exit set for $\tilde{N}_2$, we see that $\varphi(x, (-\infty, t_0])$ does not intersect $\tilde{N}_2$. Since $\varphi(x, (-\infty, t_0])$ is contained in $\tilde{N}_2'$, it has to be contained in $\tilde{N}_1 \cap A_2$. Notice that $\tilde{N}_3$ is also an exit set for $\tilde{N}_1 \cap A_2$ while $\varphi(x, (-\infty, +\infty)) \cap \tilde{N}_3 = \emptyset$. We see that $\varphi(x, (-\infty, +\infty)) \subset \tilde{N}_1 \cap A_2$, which implies $x \in \mathrm{inv}(A_2)$. This is a contradiction and we have shown that $\mathrm{inv}(\tilde{N}_2' \setminus \tilde{N}_3)$ equals $\mathrm{inv}(A_2)$, which is contained in $\mathrm{int}(\tilde{N}_2 \setminus \tilde{N}_3)$ and hence also in $\mathrm{int}(\tilde{N}_2' \setminus \tilde{N}_3)$.

□

Now we study the relation between the strong Morse decomposition and the pre-index pair. More precisely, we will show that the canonical map (3.5) is compatible with the attractor and repeller maps given by strong Morse decomposition. These results will play a crucial role when we define the relative Bauer-Furuta invariant in Chapter 7.

Proposition 3.2.18. *Let (A_1, A_2) be a strong Morse decomposition of A and let (K_1, K_2) be a pre-index pair in A_2. Then we have*

(1) (K_1, K_2) is also a pre-index pair in A;

(2) Consider the corresponding canonical maps (see (3.5)):

$$l_1 : K_1/K_2 \to I(\mathrm{inv}(A)) \text{ and } l_2 : K_1/K_2 \to I(\mathrm{inv}(A_2)).$$

Then we have $i_1 \circ l_2 \cong l_1$, where $i_1 : I(\mathrm{inv}(A_2)) \to I(\mathrm{inv}(A))$ is the attractor map (see Proposition 3.2.4).

Proof. (1) We check those two conditions in Definition 3.2.6.

- Suppose $x \in K_1$ satisfies $\varphi(x, [0, +\infty)) \subset A$. Since A_2 is positively invariant in A and $x \in A_2$, we have $\varphi(x, [0, +\infty)) \subset A_2$. This implies $\varphi(x, [0, +\infty)) \subset \mathrm{int}(A_2) \subset \mathrm{int}(A)$ because (K_1, K_2) is an index pair for A_2.
- Suppose $x \in K_2$ satisfies $\varphi(x, [0, +\infty)) \subset A$. Again since A_2 is positively invariant in A, we have $\varphi(x, [0, +\infty)) \subset A_2$. This is impossible because $K_2 \cap A_2^+ = \emptyset$.

(2) Let $\tilde{N}_3 \subset \tilde{N}_2 \subset \tilde{N}_1 \subset A$ be an index triple for $(\mathrm{inv}(A_2), \mathrm{inv}(A_1))$ and let $L \subset N \subset A$ (resp. $L_2 \subset N_2 \subset A_2$) be an index pair for $\mathrm{inv}(A)$ (resp. $\mathrm{inv}(A_2)$) that contains (K_1, K_2). The map $i_1 \circ l_2$ is given by the decomposition

$$K_1/K_2 \xrightarrow{l_{(N_2,L_2)}} N_2/L_2 \xrightarrow{s_T} \tilde{N}_2/\tilde{N}_3 \xrightarrow{i_1} \tilde{N}_1/\tilde{N}_3 \tag{3.7}$$

while the map l_1 is given by the composition

$$K_1/K_2 \xrightarrow{l_{(N,L)}} N/L \xrightarrow{s'_T} \tilde{N}_1/\tilde{N}_3. \tag{3.8}$$

Here l_* are canonical maps (induced by the corresponding inclusions) and s_T, s'_T are flow maps. (We choose $T \gg 0$ such that they are defined.) The attractor map i_1 is given by the inclusion. By Lemma 3.2.17, we may assume $\tilde{N}_1 \cap A_2 \subset \tilde{N}_2$. By Lemma 3.2.11 and Theorem 3.2.14, we may also assume that both (N, L) and (N_2, L_2) are T-tame. Under these assumptions, by Lemma 3.2.13, one can check that the composition (3.7) sends $[x]$ to $[\varphi(x, 3T)]$ if

$$\varphi(x, [0, 3T]) \subset A_2,\ \varphi(x, [T, 3T]) \subset \tilde{N}_2 \setminus \tilde{N}_3 \tag{3.9}$$

and sends $[x]$ to the base point otherwise. One the other hand, the composition (3.8) sends $[x]$ to $[\varphi(x,3T)]$ is

$$\varphi(x,[0,3T])\subset A,\ \ \varphi(x,[T,3T])\subset \tilde{N}_1\setminus\tilde{N}_3 \tag{3.10}$$

and sends $[x]$ to the base point otherwise. Therefore, we just need to show that for $x\in K_1\subset A_2$, (3.9) is equivalent to (3.10). This is simply because that A_2 is positively invariant in A and $\tilde{N}_1\cap A_2\subset\tilde{N}_2$. □

Proposition 3.2.19. *Let (A_1,A_2) be a strong Morse decomposition of A and let (K_3,K_4) be a pre-index pair in A. Then we have*

(1) The pair $(K_3\cap A_1,(K_4\cap A_1)\cup(K_3\cap A_1\cap A_2))$, denoted by (K_3',K_4'), is a pre-index pair in A_1;

(2) Consider the corresponding canonical maps

$$l_3:K_3/K_4\to I(\mathrm{inv}(A))\ \textit{and}\ l_4:K_3'/K_4'\to I(\mathrm{inv}(A_1)).$$

Then we have $i_2\circ l_3\cong l_4\circ q$, where $i_2:I(\mathrm{inv}(A))\to I(\mathrm{inv}(A_1))$ is the repeller map (see Proposition 3.2.4) and $q:K_3/K_4\to K_3'/K_4'$ is given by

$$q([x])=\begin{cases}[x] & \textit{if } x\in K_3'\\ [K_4'] & \textit{otherwise}\end{cases}.$$

(Note that q is continuous because $(\overline{K_3\setminus K_3'}\cap K_3')\subset K_4'$.)

Proof. (1) We check those two conditions in Definition 3.2.6.

- Suppose $x\in K_3'$ satisfies $\varphi(x,[0,+\infty)\subset K_3'$. Then we have $\varphi(x,[0,+\infty))\cap\partial A=\emptyset$ because (K_3,K_4) is a pre-index pair. By (3.6), we also have $\varphi(x,[0,+\infty))\cap A_1\cap A_2=\emptyset$. Note that ∂A_1 is contained in $(\partial A\cup(A_1\cap A_2))$. We see that $\varphi(x,[0,+\infty))\cap\partial A_1=\emptyset$.
- Since $K_4\cap A^+=\emptyset$, we have $(K_4\cap A_1)\cap A_1^+=\emptyset$. By (3.6), we have $(K_3\cap A_1\cap A_2)\cap A_1^+=\emptyset$. Therefore, K_4' does not intersect A_1^+.

(2) As in the proof of Proposition 3.2.18, let $\tilde{N}_3 \subset \tilde{N}_2 \subset \tilde{N}_1 \subset A$ be an index triple for $(\mathrm{inv}(A_2), \mathrm{inv}(A_1))$ and let $L \subset N \subset A$ (resp. $L_1 \subset N_1 \subset A_1$) be an index pair $\mathrm{inv}(A)$ (resp. for $\mathrm{inv}(A_1)$) that contains (K_3, K_4) (resp. (K_3', K_4')). Then the map $q \circ l_4$ is given by the composition

$$K_3/K_4 \xrightarrow{q} K_3'/K_4' \xrightarrow{l_{(N_1,L_1)}} N_1/L_1 \xrightarrow{s_T} \tilde{N}_1/\tilde{N}_2, \tag{3.11}$$

and the map $i_2 \circ l_3$ is given by the composition

$$K_3/K_4 \xrightarrow{l_{(N,L)}} N/L \xrightarrow{s_T'} \tilde{N}_1/\tilde{N}_3 \xrightarrow{i_2} \tilde{N}_1/\tilde{N}_2. \tag{3.12}$$

Here s_T, s_T' are flow maps, l_* are canonical maps and i_2 is the quotient map. As before, we can assume that (N, L) and (N_1, L_1) are both T-tame and $\tilde{N}_1 \cap A_2 \subset \tilde{N}_2$. Under these assumptions, one can check that the composition (3.11) sends $[x]$ to $[\varphi(x, 3T)]$ if

$$\varphi(x, [0, 3T]) \subset A_1 \text{ and } \varphi(x, [T, 3T]) \subset \tilde{N}_1 \setminus \tilde{N}_2 \tag{3.13}$$

and send $[x]$ to the base point otherwise. On the other hand, the composition (3.12) sends $[x]$ to $\varphi(x, 3T)$ if

$$\varphi(x, [0, 3T]) \subset A,\ \ \varphi(x, [T, 3T]) \subset \tilde{N}_1 \setminus \tilde{N}_3 \text{ and } \varphi(x, 3T) \notin \tilde{N}_2 \tag{3.14}$$

and send $[x]$ to the base point otherwise. We just need to show that for $x \in K_3$, condition (3.13) is equivalent to (3.14). Clearly, (3.13) implies (3.14). To see the converse, we take any $x \in K_3$ satisfying (3.14). We have $\varphi(x, 3T) \in \tilde{N}_1 \setminus \tilde{N}_2 \subset A_1$. Since A_1 is negatively invariant in A, we have $\varphi(x, [0, 3T]) \subset A_1$. Moreover, because $\varphi(x, 3T) \notin \tilde{N}_2$ and $\varphi(x, [T, 3T]) \cap \tilde{N}_3 = \emptyset$, we have $\varphi(x, [T, 3T]) \cap \tilde{N}_2 = \emptyset$ since $\tilde{N}_3$ is an exit set for $\tilde{N}_2$. We have proved that x satisfies (3.13). $\square$

CHAPTER 4

Pin(2)-equivariant Seiberg-Witten Floer KO-theory

Through out this chapter, we will make the following assumptions:

- Y is a connected, 3-manifold with $b_1 = 0$;
- The spin structure $\mathfrak{s}$ is induced by a spin structure. With a slightly abuse of notations, we also denote this spin structure by $\mathfrak{s}$.
- Except in subsection 4.1.1, G denotes the group Pin(2).

4.1 Equivariant KO-theory

4.1.1 General Theory

In this subsection, we review some general facts about equivariant KO-theory, mostly from [61] and [4]. See [3], [2] for basic facts about ordinary K-theory and KO-theory.

Let G be a compact topological group and X be a compact G-space. We denote the Grothendieck group of real G-bundles over X by $KO_G(X)$.

Fact 4.1.1. *$KO_G(pt) = RO(G)$. Here $RO(G)$ denotes the real representation ring of G. For a general X, $KO_G(X)$ is a $RO(G)$-algebra (with unit).*

Remark. In this paper, we will not distinguish a representation of G with its representation space.

Fact 4.1.2. *A continuous G-map $f : X \to Y$ induces a map $f^* : KO_G(Y) \to KO_G(X)$.*

Fact 4.1.3. *For each subgroup $H \subseteq G$, by restricting the G action to H, which makes a G-bundle into an H-bundle, we get a functorial restriction map $r : KO_G(X) \to KO_H(X)$.*

Fact 4.1.4. *If G acts freely on X, then the pull back map $KO(X/G) \to KO_G(X)$ is a ring isomorphism.*

Fact 4.1.5. *For a real irreducible representation space V of G, $End_G(V)$ is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $\mathbb{Z}Ir_{\mathbb{R}}$, $\mathbb{Z}Ir_{\mathbb{C}}$ and $\mathbb{Z}Ir_{\mathbb{H}}$ denote the free abelian groups generated by irreducible representations of respective types and let $KSp(X)$ be the the Grothendieck group of quaternionic vector bundles over X. Then if G acts trivially on X, we have:*

$$KO_G(X) = (KO(X) \otimes \mathbb{Z}Ir_{\mathbb{R}}) \oplus (K(X) \otimes \mathbb{Z}Ir_{\mathbb{C}}) \oplus (KSp(X) \otimes \mathbb{Z}Ir_{\mathbb{H}}). \tag{4.1}$$

Now suppose X has a distinguished base point p which is fixed by G. Then we define $\widetilde{KO}_G(X)$ (the reduced KO-group) to be the kernel of the map $KO_G(X) \to KO_G(p)$. For based space X with trivial action, we also have:

$$\widetilde{KO}_G(X) = (\widetilde{KO}(X) \otimes \mathbb{Z}\mathrm{Ir}_{\mathbb{R}}) \oplus (\widetilde{K}(X) \otimes \mathbb{Z}\mathrm{Ir}_{\mathbb{C}}) \oplus (\widetilde{KSp}(X) \otimes \mathbb{Z}\mathrm{Ir}_{\mathbb{H}}). \tag{4.2}$$

The following fact is proved as Corollary 3.1.6 in [3]. ([3] only proved the complex K-theory case but the proof works without modification in the real case.)

Fact 4.1.6. *Suppose X is a finite, based G-CW complex and the G-action is free away from the base point. Then any element in $\widetilde{KO}_G(X) \cong \widetilde{KO}(X/G)$ is nilpotent.*

Recall that the augmentation ideal $\mathfrak{a} \subset RO(G)$ is the kernel of the forgetful map $RO(G) \cong KO_G(pt) \to KO(pt) \cong \mathbb{Z}$. Any element in $\mathfrak{a}$ defines an element in $\widetilde{KO}_G(X)$. By the above fact, we get:

Fact 4.1.7. *Suppose X is a finite, based G-CW complex and the G-action is free away from the base point. Then any element in the augmentation ideal acts on $\widetilde{KO}^*_G(X)$ nilpotently.*

Fact 4.1.8. *For pointed spaces X, Y, there is a natural product map $\widetilde{KO}_G(X) \otimes \widetilde{KO}_G(Y) \to \widetilde{KO}_G(X \wedge Y)$.*

Fact 4.1.9. *For pointed spaces X, Y, we have $\widetilde{KO}_G(X \vee Y) \cong \widetilde{KO}_G(X) \oplus \widetilde{KO}_G(Y)$*

Let V be a real representation space of G. Denote the reduced suspension $V^+ \wedge X$ by $\Sigma^V X$. The following equivariant version of real Bott periodicity theorem was proved in [4].

Fact 4.1.10. *Suppose the dimension n of V is divisible by 8 and V is a spin representation (which means the group action $G \to SO(n) \subset End(V)$ factors through $Spin(n)$). Then we have the Bott isomorphism $\varphi_V : \widetilde{KO}_G(X) \cong \widetilde{KO}_G(\Sigma^V X)$, given by the multiplication of the Bott Class $b_V \in \widetilde{KO}_G(V^+)$ under the natural map $\widetilde{KO}_G(V^+) \otimes \widetilde{KO}_G(X) \to \widetilde{KO}_G(\Sigma^V X)$. Bott isomorphism is funtorial under the pointed map $X \to X'$.*

Fact 4.1.11. *Bott classes behave well under the restriction map, which means that $i^* b_V = b_{i^*(V)}$. Here i^* is the restriction map (see Fact 4.1.3) and $i^*(V)$ is the the restriction of the representation to the subgroup.*

4.1.2 Pin(2)-equivariant KO-theory

In this section, we will review some important facts about Pin(2)-equivariant KO-theory. The detailed discussions can be found in [60]. From now on to the end of this chapter, we assume $G \cong Pin(2)$ unless otherwise noted. We have:

$$RO(G) \cong \mathbb{Z}[D, K, H]/(D^2 - 1, DK - K, DH - H, H^2 - 4(1 + D + K)).$$

Here $D = [\tilde{\mathbb{R}}]$ and $H = [\tilde{\mathbb{H}}]$. As for K, the representation space is $\mathbb{C} \cong \mathbb{R} \oplus i\mathbb{R}$ where $z \in S^1 \subset Pin(2)$ acts as multiplication by z^2 (in $\mathbb{C}$) and j acts as reflection along the diagonal. We will also write $\mathbb{R}$ as the trivial one dimensional representation of G.

Notation. For $l, m, n \in \mathbb{Z}_{\geq 0}$ and a pointed G-space X, we denote $(\mathbb{R}^l \oplus \tilde{\mathbb{R}}^m \oplus \mathbb{H}^n)^+ \wedge X$ by $\Sigma^{m\mathbb{R}+lD+nH} X$. We also write $(\mathbb{R}^l \oplus \tilde{\mathbb{R}}^m \oplus \mathbb{H}^n)^+$ as $(l\mathbb{R} + mD + nH)^+$.

Following the notation of [60], we denote $\widetilde{KO}_G((kD + lH)^+)$ by $KO_G(kD + lH)$ (we choose ∞ as the base point). Then for $k, l, m, n \in \mathbb{Z}_{\geq 0}$ we have the multiplication map:

$$KO_G(kD + lH) \otimes KO_G(mD + nH) \to KO_G((k + m)D + (l + n)H). \qquad (4.3)$$

In order to define this map, we need to fix the identification between $(kD \oplus lH) \oplus (mD \oplus nH)$ and $(k+m)D \oplus (l+n)H$ by sending $(x_1 \oplus y_1) \oplus (x_2 \oplus y_2)$ to $(x_1, x_2) \oplus (y_1, y_2)$. By considering the G-equivariant homotopy, it is not hard to see that the multiplication map is commutative when K or l is even. (We will prove that the multiplication map is actually commutative for any k, l, after we give the structure of $KO_G(kD + lH)$ in Theorem 4.1.12.)

It is easy to prove (see [60]) that $8D$, $H + 4D$ and $2H$ are spin representations. Therefore, we can choose Bott classes $b_{8D} \in KO_G(8D)$, $b_{2H} \in KO_G(2H)$ and $b_{H+4D} \in KO_G(H+4D)$. Multiplication by these classes induces isomorphism $KO_G(kD+lH) \cong KO_G((k + 8)D + lH) \cong KO_G((k + 4)D + (l + 1)H) \cong KO_G(kD + (l + 2)H)$. Since the Bott classes are in the center, it doesn't matter whether we multiply on the left or on the right. Moreover, we can choose the Bott classes to be compatible with each other, which means that $b_{8D}b_{2H} = b_{H+4D}^2$. We will fix the choice of these Bott classes throughout this paper.

For $k, l \in \mathbb{Z}$, the $RO(G)$-module $KO_G(kD+lH)$ is defined to be $KO_G((k+8a)D+(l + 2b)H)$ for any $a, b \in \mathbb{Z}$ which make $k + 8a \geq 0$ and $l + 2b \geq 0$. Since the Bott Classes are chosen to be compatible, the groups defined by different choices of a, b are canonically identified to each other. Again because the Bott classes are in the center, the multiplication map (4.3) can now be extended to all $k, l, m, n \in \mathbb{Z}$.

Consider the inclusion $i : 7D^+ \to 8D^+$. There is a unique element $\gamma(D) \in KO_G(-D)$ which satisfies $\gamma(D)b_{8D} = i^*(b_{8D})$. The map $KO_G((k + 1)D + lH) \xrightarrow{\cdot\gamma(D)} KO_G(kD+lH)$ is just the map induced by the inclusion $kD \oplus lH \to (k+1)D \oplus lH$ for $k, l \geq 0$. Similarly, we can define $\gamma(H) \in KO_G(-H)$ and $\gamma(H + 4D) = \gamma(H)\gamma(D)^4$. Since left multiplication and right multiplication by $\gamma(D)$ or $\gamma(H)$ just correspond to different inclusions of subspaces, which are homotopic to each other, we see that $\gamma(D)$ and $\gamma(H)$ are both in the center.

By Bott periodicity, we only have to compute $KO_G(lD)$ for $l = -2, -1, 0, \cdots, 5$. This was done in [60] and we list the result here:

Theorem 4.1.12 (Schmidt [60]). *As $\mathbb{Z}$-modules we have the following isomorphisms:*

- *1) $KO_G(pt) \cong RO(Pin(2)) \cong \mathbb{Z}[D, A, B]/(D^2 - 1, DA - A, DB - B, B^2 - 4(A - 2B))$, where $A = K - (1 + D)$ and $B = H - 2(1 + D)$.* [1]
- *2) $KO_G(-lD) \cong \mathbb{Z} \oplus \oplus_{n \geq 1} \mathbb{Z}/2$ for $l = 1, 2$ generated by $\gamma(D)^{|l|}$ and $\gamma(D)^{|l|}A^n$.*
- *3) $KO_G(D) \cong \mathbb{Z}$, generated by $\eta(D)$.*
- *4) $KO_G(lD) \cong \mathbb{Z} \oplus \oplus_{m \geq 0} \mathbb{Z}/2$ for $l = 2, 3$. The generators are $\eta(D)^2$ and $\gamma(D)^2 A^m c$ for $l = 2$; $\gamma(D)\lambda(D)$ and $\gamma(D)A^m c$ for $l = 3$.*
- *5) $KO_G(4D)$ is freely generated by $\lambda(D), D\lambda(D), A^n\lambda(D)$ and $A^m c$ for $m \geq 0$ and $n \geq 1$.*
- *6) $KO_G(5D) \cong \mathbb{Z}$, generated by $\eta(D)\lambda(D)$.*

Corollary 4.1.13. *The multiplication map (4.3) is commutative.*

Proof. We just need to check $\gamma(D)$, $\eta(D)$, $\lambda(D)$, c commute with each other. This is easy since $\lambda(D)$ and c are in $KO_G(kD)$ for even K, while $\gamma(D)$ is in the center by our discussion before. □

For our purpose, we don't need to know the explicit constructions of $\eta(D), \lambda(D)$ and c. We just need to know the following properties of them.

$\eta(D)$ is the Hurewicz image of an element $\tilde{\eta}(D) \in \pi^0_G(D)$ (G-equivariant stable cohomotopy group of D^+). If we forget about the G-action, $\tilde{\eta}(D)$ is just the Hopf map in $\pi^{\mathrm{st}}_1(\mathrm{pt})$.

For $\lambda(D)$ and $c \in KO_G(4D)$, by Bott periodicity and formula (4.2), we have isomorphisms:

$$KO_G(4D) \cong KO_G(8D + 4) \cong KO_G(4)$$

[1] *There is a typo in [60], where the relation between A and B is $B^2 - 2(A - 2B)$.*

$$\cong (\widetilde{KO}(S^4) \otimes \mathbb{Z}\mathrm{Ir}_{\mathbb{R}}) \oplus (\widetilde{K}(S^4) \otimes \mathbb{Z}\mathrm{Ir}_{\mathbb{C}}) \oplus (\widetilde{KSp}(S^4) \otimes \mathbb{Z}\mathrm{Ir}_{\mathbb{H}}).$$

(Here $4 \in RO(G)$ denotes the trivial 4-dimensional real representation.)

We can choose suitable Bott classes such that under these isomorphisms, $\lambda(D)$ corresponds to $([V_H]-4\mathbb{R})\otimes 1 \in \widetilde{KO}(S^4)\otimes\mathbb{Z}\mathrm{Ir}_{\mathbb{R}}$ and c corresponds to $([V_{\mathbb{H}}]-\mathbb{H})\otimes H \in \widetilde{KSp}(S^4) \otimes \mathbb{Z}\mathrm{Ir}_{\mathbb{H}}$. Here $V_{\mathbb{H}}$ is the quaternion Hopf bundle over $S^4 \cong \mathbb{H}P^2$. $\mathbb{H}$ and $\mathbb{R}$ denote the trivial bundles and $1, H$ are elements in $RO(G)$.

Let $\lambda(H)$ and $c(H)$ be the image of $\lambda(D)$ and c under the Bott isomorphism $KO_G(4D) \cong KO_G(8D+H) \cong KO_G(H)$. Then $KO_G(H)$ is generated by $\lambda(H)$ and $c(H)$ as $RO(G)$-algebra.

Remark. Notice that the element $[V_H] \otimes H \in KSpS^4 \otimes \mathbb{Z}Ir_{\mathbb{H}}$ is represented by the bundle $V_H \otimes_{\mathbb{H}} H$. Hence it is a real bundle of dimension 4 (not 16).

For further discussions, we need to know the multiplicative structures of $KO_G(lD)$, which are also given in [60]. We list some of them that are useful for us:

Theorem 4.1.14 (Schmidt [60])**.** *The following relations hold:*

- *1)* $H\lambda(D) = 4c,\ Hc = (A+2+2D)\lambda(D),\ Dc = c.$
- *2)* $(D+1)\gamma(D) = 2A\gamma(D) = B\gamma(D) = 0.$
- *3)* $(D+1)\eta(D) = A\eta(D) = B\eta(D) = 0.$
- *4)* $\gamma(D)\eta(D) = 1-D,\ \gamma(D)\lambda(D) = \eta(D)^3.$
- *5)* $\gamma(D)^8 b_{8D} = 8(1-D),\ \ \gamma(H)^2 b_{2H} = K - 2H + D + 5.$
- *6)* $\gamma(H+4D)b_{H+4D} = 4(1-D).$
- *7)* $\eta(D)\lambda(D) = \gamma(D)^3 b_{8D},\ \eta(D)c = 0.$
- *8)* $\gamma(H)\lambda(H) = 4 - H$ *and* $\gamma(H)c(H) = H - 1 - D - K.$

4.2 The Adams operations

4.2.1 Basic properties

In this subsection, we give a quick review about the basic properties of the Adams operations. See [3] and [65] for more detailed discussions. Some of the calculations can be found in [60] but we give them here for completeness. For simplicity and concreteness, we only deal with $\psi^k : KO_G(X) \to KO_G(X)$ for an actual G-space X and we don't do localizations (like [60]).

Let $KO_G(X)[[t]]$ be the formal power series with coefficients in $KO_G(X)$. For a bundle E over X, we define $\lambda_t(E) \in KO_G(X)[[t]]$ to be $\sum\limits_{i=0} t^i[\lambda^i(E)]$. Here $\lambda^i(E)$ is the i-th exterior power of E. We let $\psi^0(E) = \text{rank}(E)$ and define $\psi_t(E) = \sum\limits_{i=0} t^i\psi^i(E) \in KO_G(X)[[t]]$ by

$$\psi_t(E) = \psi^0(E) - t\frac{d}{dt}(\log\lambda_{-t}(x)). \tag{4.4}$$

It turns out that for any $k \in \mathbb{Z}_{\geq 0}$, ψ^k extends to a well defined operation on $KO_G(X)$, which satisfies the following nice properties:

- (1) ψ^k is functorial with respect to continuous maps $f : X \to X'$.
- (2) ψ^k maps $\widetilde{KO}_G(X)$ to $\widetilde{KO}_G(X)$.
- (3) For all $x, y \in KO_G(X)$, $\psi^k(x+y) = \psi^k(x)+\psi^k(y)$ and $\psi^k(xy) = \psi^k(x)\psi^k(y)$.
- (4) If x is a line bundle, then $\psi^k(x) = x^k$.

The effect of the Adams operations on the Bott classes can be described by the Bott cannibalistic class. Given a spin G-bundle E over X with rank $n \equiv 0$ mod 8, the Bott cannibalistic class $\theta_k^{\text{or}}(E) \in RO(G)$ is defined by the equation:

$$\psi^k(b_E) = \theta_k^{\text{or}}(E) \cdot b_E \text{ for } k > 1. \tag{4.5}$$

When K is odd, this can be explicitly written as (see [65]):[2]

$$\theta_k^{\text{or}}(E) = k^{n/2} \prod_{u \in J} \lambda_{-u}(E)(1-u)^{-n}. \tag{4.6}$$

[2]There is a typo in 3.10.4 [65].

Here J is a set of K-th unit roots $u \neq 1$ such that J contains exactly one element from each pair $\{u, u^{-1}\}$. Notice that we can define $\theta_k^{\mathrm{or}}(E)$ for any real bundle E of even dimension using formula (4.6). It can be shown that:

$$\theta_k^{\mathrm{or}}(E+F) = \theta_k^{\mathrm{or}}(E)\theta_k^{\mathrm{or}}(F).$$

Now let's specialize to the case $k = 3$. By formula (4.4), it is easy to check that $\psi^3(x) = x^3 - 3\lambda^2(x)x + 3\lambda^3(x)$. We want to calculate the action of ψ^3 on $RO(G)$. Since the G-action on H preserves the orientation, we have $\lambda^3(H) = \lambda^1(H) = H$. Using complexification, it is easy to show $\lambda^2(H) = K + D + 3$. Also, we have $\lambda^2(K) = D$. Therefore, we get[3]:

$$\psi^3(D) = D,\ \psi^3(H) = HK - H,\ \psi^3(K) = K^3 - 3K,$$

$$\psi^3(A) = A^3 + 6A^2 + 9A, \quad \psi^3(B) = AB + B + 4A.$$

Also, applying formula (4.6), we get:

$$\theta_3^{\mathrm{or}}(2) = 3,\ \theta_3^{\mathrm{or}}(2D) = 1 + 2D,\ \theta_3^{\mathrm{or}}(H) = A + B + 4D + 5.$$

4.2.2 Proof of Theorem 1.3.4

The central part of the proof is the following proposition:

Proposition 4.2.1. *For any integers $r, a, b \geq 0$ and $l > 0$, there does not exist G-equivariant map*

$$f : (r\mathbb{R} + aD + (4l+b)H)^+ \to (r\mathbb{R} + (a + 8l + 2)D + bH)^+$$

which induces homotopy equivalence on the G-fixed point set.

Proof. Suppose there exists such a map f. After suspension by copies of $\mathbb{R}, D$ and H, we can assume $a = 8l' + 6$, $r = 8d$ and $b = 2k$. Let $V_1 = 8d\mathbb{R} + 2kH + 8(l + l' + 1)D$ and $V_2 = 8d\mathbb{R} + (4l + 2k)H + (8l' + 8)D$. Let b_{V_1} and b_{V_2} be the Bott classes of V_1

[3]There is a typo in [60], where $\psi^3(H) = HK - K$.

and V_2, respectively. Consider the element $x = f^*(b_{V_1})$. By the Bott isomorphism and (2) of Theorem 4.1.12, we can write x as $b_{V_2}\gamma(D)^2\alpha$ for some $\alpha \in RO(G)$. Moreover, we can assume $\alpha = p + Ah(A)$ for some integer p and some polynomial $h(A)$ whose coefficients are either 0 or 1.

Claim: p is even and $h = 0$.

This is essentially a special case of Proposition 5.21 in [60] for $\mathcal{KO}(4l, 8l+2)$.[4]

By formula (4.5), we have: $\psi^3(b_{V_1}) = \theta_3^{\text{or}}(V_1)\cdot b_{V_1}$, which implies:

$$\psi^3(x) = f^*(\psi^3(b_{V_1})) = \theta_3^{\text{or}}(V_1)\cdot x. \tag{4.7}$$

Notice that $x = i^*(b_{V_2}\cdot\alpha)$ where $i : (8d\mathbb{R} + (4l+2k)H + (8l'+6)D)^+ \to V_2^+$ is the standard inclusion. By formula (4.5), we have:

$$\psi^3(x) = i^*(\psi^3(b_{V_2}\cdot\alpha)) = \theta_3^{\text{or}}(V_2)b_{V_2}\psi^3(\alpha)\cdot\gamma(D)^2. \tag{4.8}$$

Comparing equation (4.7) and equation (4.8), we get:

$$(\theta_3^{\text{or}}(V_2)\psi^3(\alpha) - \theta_3^{\text{or}}(V_1)\alpha)\gamma(D)^2 = 0 \tag{4.9}$$

We can calculate:

$$\theta_3^{\text{or}}(V_1) = 3^{4d}(1+2D)^{4l+4l'+4}(A+B+4D+5)^{2k},$$

$$\theta_3^{\text{or}}(V_2) = 3^{4d}(1+2D)^{4l'+4}(A+B+4D+5)^{2k+4l}.$$

Notice that $2A\gamma(D) = B\gamma(D) = (1+D)\gamma(D) = 0$, we can simplify equation (4.9) as:

$$3^{4d}((A+1)^{2k}\alpha - (A+1)^{4l+2k}\psi^3(\alpha))\cdot\gamma(D)^2 = 0. \tag{4.10}$$

Since $\alpha = p + Ah(A)$, we have $\psi^3(\alpha) = p + (A^3+6A^2+9A)h(A^3+6A^2+9A)$. Using the relation $2A\gamma(D) = 0$, we can further simplify equation (4.10) and get:

$$3^{4d}\cdot g(A)\cdot\gamma(D)^2 = 0 \tag{4.11}$$

[4] There is an error in [60] for $\mathcal{KO}(c,d)$ when $4c - d \equiv -3 \bmod 8$, but we will not consider this case here.

Here $g(A) = (A+1)^{2k}(p + Ah(A)) - (A+1)^{2k+4l}(p + (A^3 + A)h(A^3 + A))$.

By (2) of Theorem 4.1.12, we see that if we expand $g(A)$ as a polynomial in A, the degree-0 coefficient should be 0 and all other coefficients should be even. By our assumption, the coefficients of h are either 0 or 1. Checking the leading coefficient of $g(A)$, it is easy to see that $h = 0$ and $g(A) = p((A+1)^{2k} - (A+1)^{2k+4l})$. This implies that p is even. The claim is proved.

Now consider the commutative diagram:

$$\begin{array}{ccc} \widetilde{KO}_G(V_1^+) & \xrightarrow{f^*} & \widetilde{KO}_G((8d\mathbb{R} + (8l'+6)D + (4l+2k)H)^+) \\ \downarrow \cdot\gamma(H)^{2k}\gamma(D)^{8l+8l'+8} & & \downarrow \cdot\gamma(H)^{4l+2k}\gamma(D)^{8l'+6} \\ \widetilde{KO}_G((8d\mathbb{R})^+) & \xrightarrow{\cong} & \widetilde{KO}_G((8d\mathbb{R})^+). \end{array} \tag{4.12}$$

The vertical maps are given by the inclusions of subspaces. The bottom map is an isomorphism because f induces a homotopy equivalence on the G-fixed point set. Any automorphism on $\widetilde{KO}_G((8d\mathbb{R})^+)$ is given by the multiplication of a unit $\tilde{u} \in RO(G)$. Therefore, we obtain :

$$\tilde{u} \cdot b_{V_1} \cdot \gamma(H)^{2k}\gamma(D)^{8l+8l'+8} = x \cdot \gamma(H)^{4l+2k}\gamma(D)^{8l'+6} = b_{V_2} \cdot \gamma(D)^{8l'+8}\gamma(H)^{4l+2k} \cdot p \tag{4.13}$$

Applying the relations in Theorem 4.1.14, we simplify this as :

$$(K - 2H + D + 5)^{2l+k}(8(1-D))^{l'+1} \cdot p = (K - 2H + D + 5)^k(8(1-D))^{l+l'+1} \cdot \tilde{u}. \tag{4.14}$$

Now consider the ring homomorphism $\varphi_0 : RO(G) \to \mathbb{Z}$ defined by $\varphi_0(D) = -1, \varphi_0(A) = \varphi_0(B) = 0$. Notice that $\varphi_0(\tilde{u}) = \pm 1$ since $\tilde{u}$ is a unit. We get $p = \pm 1$, which is a contradiction. This finishes the proof of Proposition 4.2.1. □

Now suppose W is a closed, oriented, smooth spin 4-manifold with intersection form $p(-E_8) \oplus q\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $p = 8l > 0$ and $q < p + 3$. After doing surgery on loops and connect sum copies of $S^2 \times S^2$, we can assume $b_1(W) = 0$ and $q = 8l + 2$. As shown in [21], by doing finite dimensional approximation of the Seiberg-Witten equations on W, we get an G-equivariant map:

$$f : (aD + (4l + b)H)^+ \to ((a + 8l + 2)D + bH)^+ \text{ for some } a, b > 0.$$

Moreover, f induces a homotopy equivalence on the G-fixed point set. This is a contradiction to Proposition 4.2.1. Therefore, Theorem 1.3.4 is proved.

4.3 Seiberg-Witten Floer spectrum for spin 3-manifolds with $b_1 = 0$

In [40], [42] and [43], Manolescu constructed a Pin(2)-equivariant spectrum class $S(Y, \mathfrak{s})$. Let us briefly review the constructions and collect some useful properties here. See [40], [42] and [43] for details.

As in Chapter 2, we let $Coul(Y)$ be the (global) Coulomb slice and V_λ^μ be its finite dimensional subspace spanned by the eigenvectors of $l = (*d, \not{D}_{A_0})$. We choose the base connection A_0 to be the spin connection take the perturbation f to be zero. Since $\mathfrak{s}$ is torsion, the harmonic form ν_0 is also zero.

By our assumption, we have $\mathcal{G}_Y^{h,o} = H^1(Y; \mathbb{Z}) = 0$. By Theorem 2.2.2, there exists R_0 such that all finite type Seiberg-Witten trajectories are contained in the interior of the ball

$$B(R_0) = \{x \in Coul(Y) | \|x\|_{L_k^2} \leq R_0\}.$$

As a consequence of consequence of Proposition 2.2.4, we have the following corollary:

Corollary 4.3.1 (Manolescu, [40])**.** *There exists $\mu_0 \gg 0$ and $\lambda_0 \ll 0$ such that for any $\mu > \mu_0$ and $\lambda < \lambda_0$. The set $B(2R_0) \cap V_\lambda^\mu$ is an isolating neighborhood under the approximated Seiberg-Witten flow. We denote the corresponding Conley index by I_λ^μ.*

There is an G-action on $Coul(Y)$ preserving the subspace V_λ^μ and the Seiberg-Witten flow (see [43] for details). As a result, the Conley index I_λ^μ is a G-space. We consider the object

$$\mathrm{SWF}(Y, \mathfrak{s}; Pin(2)) = \Sigma^{-V_\lambda^0}(I_\lambda^\mu, 0, \tfrac{n(Y,\mathfrak{s},A_0,g)}{2}) \in \mathrm{ob}\, \mathfrak{C}_{Pin(2)},$$

where $n(Y, \mathfrak{s}, A_0, g) \in \mathbb{Q}$ is the correction term defined in [40] (see Section 5.2 for exact definition).

Theorem 4.3.2 (Manolescu, [42]). *The object* $\mathrm{SWF}(Y, \mathfrak{s}; Pin(2))$ *is a topological invariant of the pair* $(Y, \mathfrak{s})$. *More precisely, under canonical isomorphisms in the category* $\mathfrak{C}_{Pin(2)}$, *the object* $\mathrm{SWF}(Y, \mathfrak{s}; Pin(2))$ *is independent of the metric* g, *the numbers* λ, μ *and other auxiliary data involved in the construction.*

In order to define the numerical invariant from the Seiberg-Witten-Floer KO theory in later sections, we make a digression into the following definitions.

Definition 4.3.3. Let $s \in \mathbb{Z}_{\geq 0}$. A space of type SWF (at level s) is a pointed, finite G-CW complex X with the following properties:

- (a) The S^1-fixed point set X^{S^1} is G-homotopy equivalent to the sphere $(s\tilde{\mathbb{R}})^+$. We define $\mathrm{lev}(X)$ to be s.
- (b) The action of G is free on the complement $X - X^{S^1}$.

An object (X, m, n) in the category $\mathfrak{C}_{Pin(2)}$ is of type SWF if X is of type SWF.

Definition 4.3.4. A spectrum class is an equivalent class objects of type SWF in $\mathfrak{C}_{Pin(2)}$ under equivalence relation of the isomorphisms in $\mathfrak{C}_{Pin(2)}$. We denote by $\mathfrak{B}$ the set of spectrum classes.

Definition 4.3.5. For a spectrum class $S = [(X, a, b)] \in \mathfrak{B}$, we let

$$\mathrm{lev}(S) = \mathrm{lev}(X) - a.$$

Remark. By considering the S^1-fixed point set, we see that two spaces of type SWF at different levels are not G-homotopic to each other. Using this fact, it is easy to prove that $lev(S)$ is a well defined quality.

For $r \in \mathbb{Z}$ and $s \in \mathbb{Q}$, we can define the formal desuspension $\Sigma^{rD+sH} : \mathfrak{B} \to \mathfrak{B}$ by sending $[(X, a, b)]$ to $[(\Sigma^{rD} X, a - 2r, b - s)]$. It's easy to check that this is a well defined operation on the set $\mathfrak{B}$.

Definition 4.3.6. Let X, X' be two spaces of type SWF at level K and k' respectively. A pointed G-map $f : X \to X'$ is called admissible if f preserves the base point and satisfies one of the following two conditions:

- (1) $k < k'$ and the induced map on the G-fixed point set $f^G : X^G \to X'^G$ is a homotopy equivalence.

- (2) $k = k'$ and the induced map on the S^1-fixed point set $f^{S^1} : X^{S^1} \to X'^{S^1}$ is a homotopy equivalence.

Definition 4.3.7. Let S, S' be two spectrum classes. We say S dominates S' if there exist representatives (X, m, n) and (X', m, n), respectively for S, S', such that the following conditions are satisfied:

- X, X' are pointed G-spaces of type SWF;

- There exists an admissible map $f : X' \to X'$.

Now we return to the spectrum invariant $\mathrm{SWF}(Y, \mathfrak{s}; Pin(2))$. It is proved in [42] that this object is of type SWF. As a result of Theorem 4.3.2, the equivalent class $[\mathrm{SWF}(Y, \mathfrak{s}; Pin(2))] \in \mathfrak{B}$ is a well defined invariant for the pair $(Y, \mathfrak{s})$. We denote it by $S(Y, \mathfrak{s})$. We have the following theorem, which will be important for our further constructions.

Theorem 4.3.8 (Manolescu [40], [42]). *Suppose $(Y, \mathfrak{s})$ and $(Y', \mathfrak{s}')$ are two spin 3-manifolds with $b_1 = 0$ and $(W, \hat{\mathfrak{s}})$ is a spin cobordism from $(Y, \mathfrak{s})$ to $(Y', \mathfrak{s}')$. Then for any $n \in \mathbb{Z}$, the spectrum class $\Sigma^{\frac{-\sigma(W)}{16}H}\Sigma^{nD} S(Y, \mathfrak{s})$ dominates the spectrum class $\Sigma^{b_2^+(W)D}\Sigma^{nD} S(Y', \mathfrak{s}')$.*

4.4 Numerical Invariants

Let Y be a rational homology sphere and $\mathfrak{s}$ be a spin structure on Y. In the previous section, we defined an invariant $S(Y, \mathfrak{s}) \in \mathfrak{B}$. In this section, we will extract a set of numerical invariants $\kappa o_i(Y, \mathfrak{s})$ from $S(Y, \mathfrak{s})$, for $i \in \mathbb{Z}/8$.

Definition 4.4.1. For $l = -2, -1, 0, \cdots, 5$, we define the group homomorphisms $\varphi_l : KO(lD) \to \mathbb{Z}$ as following (see Theorem 4.1.12):

- 1) For $l = 0$, $\varphi_l(D) = -1$ and $\varphi_l(A) = \varphi_l(B) = 0$, then extend φ_l by the multiplicative structure on $RO(G)$.
- 2) For $l = -1, -2$, $\varphi_l(\gamma(D)^{|l|}) = 1$ and $\varphi_l(\gamma(D)^{|l|}A^n) = 0$ for $n \geq 1$.
- 3) For $l = 1$, $\varphi_l(\eta(D)) = 1$.
- 4) For $l = 2$, $\varphi_l(\eta(D)^2) = 1$ and $\varphi_l(\gamma(D)^2 A^m c) = 0$.
- 5) For $l = 3$, $\varphi_l(\gamma(D)\lambda(D)) = 1$ and $\varphi_l(\gamma(D)A^m c) = 0$.
- 6) For $l = 4$, $\varphi_l(\lambda(D)) = 1$, $\varphi_l(D\lambda(D)) = -1$, and $\varphi_l(A^n\lambda(D)) = \varphi_l(A^m c) = 0$.
- 7) For $l = 5$, $\varphi_l(\eta(D)\lambda(D)) = 1$.

For the other $l \in \mathbb{Z}$, we use the Bott isomorphism to identify $KO(lD)$ with $KO((l - 8k)D)$ for $-2 \leq l - 8k \leq 5$ and apply the above definition.

Lemma 4.4.2. *For any $a \in KO_G(pt)$ and $b \in KO_G(kD)$, we have $\varphi_0(a)\varphi_k(b) = \varphi_k(a \cdot b)$.*

Proof. This is a straightforward calculation using Theorem 4.1.12 and Theorem 4.1.14. □

Remark. φ_0 is just taking the trace of $j \in Pin(2)$. While the other φ_l are defined such that the torsion elements are killed and Lemma 4.4.2 holds.

We consider the map $\tau : D^+ \to D^+$ which maps x to $-x$. By suspension with copies of D, we get an admissible involution $\tau : (kD)^+ \to (kD)^+$ for $k > 0$.

The following lemma is a straightforward corollary of the equivariant Hopf theorem.

Lemma 4.4.3. *When $0 \leq k < l$, any admissible map $f : (kD)^+ \to (lD)^+$ is G-homotopic to the standard inclusion. For $0 \leq k = l$, any admissible map $f : (kD)^+ \to (kD)^+$ is either homotopic to τ or to the identity map, depending on $deg(f)$.*

τ induces the involution $\tau^* : KO_G(kD) \to KO_G(kD)$. For $k, l > 0$ and any $a \in KO_G(kD), b \in KO_G(lD)$, the following equality is easy to check by Lemma 4.4.3:

$$\tau^*(a) \cdot b = a \cdot \tau^*(b) = \tau^*(a \cdot b) \text{ and } \tau^*(a) \cdot \tau^*(b) = a \cdot b. \tag{4.15}$$

Using this fact, we can define $\tau^* : KO_G(kD) \to KO_G(kD)$ for any $k \in \mathbb{Z}$ by identifying $KO_G(kD)$ with $KO_G(k'D)$ for any $0 < k' \equiv k \bmod 8$ using Bott periodicity. Moreover, formula (4.15) now holds for all $k, l \in \mathbb{Z}$.

Now consider the element $u \in RO(G)$ defined by $\tau^*(b_{8D}) = u \cdot b_{8D}$. Then for $l \in \mathbb{Z}$ and any element $\alpha \in KO_G(lD)$, we have $\tau^*(\alpha) \cdot b_{8D} = \alpha \cdot \tau^*(b_{8D}) = (u\alpha) \cdot b_{8D}$, which implies $\tau^*(\alpha) = u\alpha$.

Lemma 4.4.4. *We have the following properties about τ^* and u:*

- *(1) τ^* acts as identity on $KO_G(lD)$ for $l \neq 0, 4 \bmod 8$.*
- *(2) u is a unit with $\varphi_0(u) = 1$.*
- *(3) $\varphi_l \circ \tau^* = \varphi_l$ for any $l \in \mathbb{Z}$.*

Proof. (1) We have $\gamma(D)b_{8D} = i^*(b_{8D})$ where i^* is the inclusion $(7D)^+ \to (8D)^+$. Therefore, we get $\tau^*(\gamma(D)b_{8D}) = (\tau \circ i)^*(b_{8D})$. By Lemma 4.4.3, $\tau \circ i$ is G-homotopic to i, thus $\tau^*(\gamma(D)b_{8D}) = i^*(b_{8D}) = \gamma(D)b_{8D}$, which implies that $\tau^*(\gamma(D)) = \gamma(D)$.

Since τ^* induces an involution on $KO_G(D) \cong \mathbb{Z}$, we have $\tau^*(\eta(D)) = \pm\eta(D)$. But since $\tau^*(\eta(D)) \cdot \gamma(D) = \eta(D) \cdot \tau^*(\gamma(D)) = \eta(D)\gamma(D) = 1 - D \neq -\eta(D)\gamma(D)$, we get $\tau^*(\eta(D)) = \eta(D)$.

By formula (4.15), $\tau^*(a) = a$ implies $\tau^*(ab) = ab$ for any a, b. Therefore we see that τ^* acts as the identity map on $KO_G(kD)$ for $k \neq 0, 4 \bmod 8$.

(2) $u^2 = 1$ because $\tau^2 = \mathrm{id}$. Since $u \cdot (1 - D) = \tau^*(1 - D) = \tau^*(\gamma(D) \cdot \eta(D)) = \gamma(D) \cdot \eta(D) = 1 - D$, we see that $(u - 1)(1 - D) = 0$. We get $\varphi_0(u) = 1$ by Lemma 4.4.2.

(3) is straightforward from (2) and Lemma 4.4.2. $\square$

Now suppose X is a space of type SWF at level l. A choice of G-homotopy equivalence $X^{S^1} \cong (lD)^+$ gives us an inclusion map $i : (lD)^+ \to X$, which we call a trivialization. A trivialization induces the map $i^* : \widetilde{KO}_G(X) \to KO_G(lD)$. Consider the map $\varphi_l \circ i^* : \widetilde{KO}_G(X) \to \mathbb{Z}$.

Proposition 4.4.5. *The submodule* $Im(i^*)$ *and the map* $\varphi_l \circ i^*$ *are both independent of the choice of the trivialization. Moreover, we have* $Im(\varphi_l \circ i^*) = (2^k)$ *for some* $k \in \mathbb{Z}_{\geq 0}$.

Proof. By Lemma 4.4.3, there are two possible trivializations i and $i \circ \tau$. We have $\mathrm{Im}(i \circ \tau)^* = \tau^*(\mathrm{Im} i^*) = u \cdot \mathrm{Im}(i^*)$. Since u is a unit, the multiplication by u does not change the submodule $\mathrm{Im}(i^*)$. Moreover, we have $\varphi_l \circ (i \circ \tau)^* = \varphi_l \circ \tau^* \circ i^* = \varphi_l \circ i^*$ by (3) of Lemma 4.4.4.

For the second statement, we consider the exact sequence:

$$\cdot \to \widetilde{KO}_G(X) \xrightarrow{i^*} KO_G(lD) \xrightarrow{\delta} \widetilde{KO}^1_G(X/X^{S^1}) \to \cdot$$

Since the G action is free away from the basepoint and $(1 - D) \in RO(G)$ is in the augmentation ideal, $(1-D)$ acts on $\widetilde{KO}^1_G(X/X^{S^1})$ nilpotently by Fact 4.1.7. Therefore, we can find $m \gg 0$ such that $(1 - D)^m KO_G(lD) \subset \ker(\delta) = \mathrm{Im}(i^*)$. It follows that $2^m \in \mathrm{Im}(\varphi_l \circ i^*)$ and $\mathrm{Im}(\varphi_l \circ i^*) = (2^k)$ for some $0 \leq k \leq m$. □

Proposition 4.4.5 justify the following definition:

Definition 4.4.6. For a G-space X of type SWF at level l, we define $\mathcal{J}(X)$ to be the image of i^* for any trivialization i and let $\kappa o(X)$ be the integer K such that $\varphi_l(\mathcal{J}(X)) = (2^k)$.

Let's study the property of $\mathcal{J}(X)$ and $\kappa o(X)$. First recall that we defined the constants $\beta^0_k = 0$ and $\beta^j_k = \sum_{i=0}^{j-1} \alpha_{k-i}$ for $j \geq 1$, where $\alpha_i = 1$ for $i \equiv 1, 2, 3, 5 \bmod 8$ and $\alpha_i = 0$ for $i \equiv 0, 4, 6, 7 \bmod 8$. It's easy to see that $\beta^k_j = \beta^k_{j'}$ for $j \equiv j' (\mathrm{mod} 8)$. The integers β^k_j are important because of the following proposition:

Proposition 4.4.7. *For integers* $0 \leq j \leq k$ *and an admissible map* $i : ((k-j)D)^+ \to (kD)^+$*, we have the following commutative diagram, where the map* $m^j_k : \mathbb{Z} \to \mathbb{Z}$ *is the multiplication of* $2^{\beta^j_k}$.

$$\begin{array}{ccc} KO_G(kD) & \xrightarrow{i^*} & KO_G((k-j)D) \\ \downarrow \varphi_k & & \downarrow \varphi_{k-j} \\ \mathbb{Z} & \xrightarrow{m^j_k} & \mathbb{Z} \end{array} \tag{4.16}$$

Proof. The case $j = 0$ follows from Lemma 4.4.4. When $j > 0$, by Lemma 4.4.3, the map i is G-homotopic to the standard inclusion. Because of the associativity of i^* and m_l^k, we only need to prove the case $j = 1$. In this case, the map i^* is just the multiplication by $\gamma(D)$ and m_k^1 is the multiplication by 2^{α_k}. Since both φ_k and i^* are compatible with Bott isomorphism, we only need to check the case $k = 1, 2, \cdots, 8$. This can be proved by straightforward calculations using Definition 4.4.1, Theorem 4.1.14 and Theorem 4.1.12. □

The following proposition studies the behavior of $\mathcal{J}(X)$ and $\kappa o(X)$ under the Bott isomorphism:

Proposition 4.4.8. *Let X be a space of type SWF at level K. We have the following:*

- *(1) $\mathcal{J}(X) \cdot b_{8D} = \mathcal{J}(\Sigma^{8D} X)$ and $\kappa o(\Sigma^{8D} X) = \kappa o(X)$.*
- *(2) $\mathcal{J}(X) \cdot (K - 2H + D + 5) = \mathcal{J}(\Sigma^{2H} X)$ and $\kappa o(\Sigma^{2H} X) = \kappa o(X) + 2$.*
- *(3) $\kappa o(\Sigma^{H+4D} X) = \kappa o(X) + 3 - \beta_{k+4}^4$.*

Proof. (1) Since $(\Sigma^{8D} X)^{S^1} = \Sigma^{8D}(X^{S^1})$, statement (1) follows from the functoriality of the Bott isomorphism.

(2) We have the commutative diagram induced by the inclusions of subspaces:

$$\begin{array}{ccc} \widetilde{KO}_G(\Sigma^{2H} X) & \longrightarrow & \widetilde{KO}_G(X) \\ \downarrow & & \downarrow \\ \widetilde{KO}_G((\Sigma^{2H} X)^{S^1}) & \xrightarrow{\cong} & \widetilde{KO}_G(X^{S^1}). \end{array} \tag{4.17}$$

Since $(\Sigma^{2H} X)^{S^1} = \Sigma^{2H}(X^{S^1})$, the map in the bottom row is the identity. If we identify $\widetilde{KO}_G(\Sigma^{2H} X)$ with $\widetilde{KO}_G(X)$ using the Bott isomorphism, then the top horizontal map is the multiplication by $\gamma(H)^2 b_{2H} = K - 2H + D + 5$ (by Theorem 4.1.14). This implies $\mathcal{J}(\Sigma^{2H} X) = (K - 2H + D + 5)\mathcal{J}(X)$. We also have $\kappa o(\Sigma^{2H} X) = \kappa o(X) + 2$ since $\varphi_0(K - 2H + D + 5) = 4$.

(3) Again, by inclusions of subspaces, we have:

$$\begin{array}{ccc} \widetilde{KO}_G(\Sigma^{H+4D}X) & \longrightarrow & \widetilde{KO}_G(X) \\ \downarrow & & \downarrow \\ KO_G((\Sigma^{H+4D}X)^{S^1}) & \xrightarrow{\cdot\gamma(D)^4} & KO_G(X^{S^1}). \end{array}$$

Since $(\Sigma^{H+4D}X)^{S^1} \cong \Sigma^{4D}(X^{S^1})$, the bottom horizontal map is the multiplication by $\gamma(D)^4$. If we identify $\widetilde{KO}_G(\Sigma^{H+4D}X)$ with $\widetilde{KO}_G(X)$ using the Bott isomorphism, the top horizontal map is the multiplication by $\gamma(H+4D)b_{H+4D} = 4(1-D)$ (by Theorem 4.1.14). Therefore, under appropriate trivializations, we see that the maps $i_1^* : \widetilde{KO}_G(X) \cong \widetilde{KO}_G(\Sigma^{H+4D}X) \to KO_G((k+4)D)$ and $i_2^* : \widetilde{KO}_G(X) \to KO_G(kD)$ are related by $\gamma(D)^4 \cdot i_1^*(x) = 4(1-D) \cdot i_2^*(x)$. Since $\varphi_0(4(1-D)) = 8$, statement (3) follows from Proposition 4.4.7 (for $j = 4$) and Lemma 4.4.2. □

We have the following proposition, which is the analogue of Lemma 3.8 in [42].

Proposition 4.4.9. *Let X_1 and X_2 be spaces of type SWF. Suppose there is a based G-equivariant homotopy equivalence f from $\Sigma^{r\mathbb{R}}X_1$ to $\Sigma^{r\mathbb{R}}X_2$, for some $r \geq 0$. Then we have $\mathcal{J}(X_1) = \mathcal{J}(X_2)$ and $\kappa o(X_1) = \kappa o(X_2)$.*

Proof. The proof in [42] works with some modifications. Suppose X_1, X_2 are both at level K. By (1) of Proposition 4.4.8, we can replace X_i by $\Sigma^{8D}X_i$ and assume $k > 1$. Also, we can suspend some more copies of $\mathbb{R}$ and assume that $8|r$. Choose trivilizations i_1, i_2 of X_1 and X_2, respectively. They give homotopy equivalences $(r\mathbb{R}+kD)^+ \cong (\Sigma^{r\mathbb{R}}X_1)^{S^1}$ and $(r\mathbb{R}+kD)^+ \cong (\Sigma^{r\mathbb{R}}X_2)^{S^1}$. Composing them with $f^{S^1} : (\Sigma^{r\mathbb{R}}X_1)^{S^1} \to (\Sigma^{r\mathbb{R}}X_2)^{S^1}$, we get the equivariant homotopy equivalence $h : (r\mathbb{R}+kD)^+ \to (r\mathbb{R}+kD)^+$. Since $k > 1$, by equivariant Hopf theorem, h is based homotopic to $\tau_1 \wedge \tau_2$. The map $\tau_1 : (r\mathbb{R})^+ \to (r\mathbb{R})^+$ is either identity or a map with degree -1. Therefore, $\tau_1^*(b_{r\mathbb{R}}) = a \cdot b_{r\mathbb{R}}$ where $b_{r\mathbb{R}}$ is the Bott class and $a \in RO(G)$ is a unit. Also, $\tau_2 : (kD)^+ \to (kD)^+$ is either identity or the map τ we defined before. Therefore, $\tau_2^*(x)$ is either x or ux (see Lemma 4.4.4). We have shown that the map $h^* : \widetilde{KO}_G((r\mathbb{R}+kD)^+) \to \widetilde{KO}_G((r\mathbb{R}+kD)^+)$ is just multiplication by some unit in $RO(G)$, which does not change any submodule.

Now consider the following commutative diagram:

$$\begin{array}{ccccccc}
\widetilde{KO}_G(X_2) & \xrightarrow{\cong} & \widetilde{KO}_G(\Sigma^{r\mathbb{R}}X_2) & \xrightarrow{f^*} & \widetilde{KO}_G(\Sigma^{r\mathbb{R}}X_1) & \xrightarrow{\cong} & \widetilde{KO}_G(X_1) \\
\downarrow i_2^* & & \downarrow (\Sigma^{r\mathbb{R}}i_2)^* & & \downarrow (\Sigma^{r\mathbb{R}}i_1)^* & & \downarrow i_1^* \\
KO_G(kD) & \xrightarrow{\cong} & \widetilde{KO}_G((r\mathbb{R}+kD)^+) & \xrightarrow{h^*} & \widetilde{KO}_G((r\mathbb{R}+kD)^+) & \xrightarrow{\cong} & KO_G(kD).
\end{array}$$

In each row, the first map is a Bott isomorphism and the third map is the inverse to a Bott isomorphism. We see that $b_{r\mathbb{R}} \cdot \mathrm{Im}(i_2^*) = h^*(b_{r\mathbb{R}} \cdot \mathrm{Im}(i_2^*)) = b_{r\mathbb{R}} \cdot \mathrm{Im}(i_1^*)$. Therefore, we have $\mathrm{Im}(i_1^*) = \mathrm{Im}(i_2^*)$, which implies $\kappa o(X_1) = \kappa o(X_2)$. □

Definition 4.4.10. For a spectrum class $S = [(X, a, b)] \in \mathfrak{B}$, we let

$$\kappa o(S) = \kappa o(\Sigma^{(8M-a)D}\Sigma^{(2N-b')H}X) - 2N - s \tag{4.18}$$

for any $M, N, b' \in \mathbb{Z}$ and $s \in [0, 1)$ making $8M - a \geq 0, 2N - b' \geq 0$ and $b = b' + s$.

Proposition 4.4.11. *$\kappa o(S)$ is well defined.*

Proof. By (1) and (2) of Proposition 4.4.8, it's easy to prove that the righthand side of formula (4.4.10) is independent of the choice of M, N. By choosing $M, N \gg 0$, we see that changing the representative of S from (X, a, b) to $(\Sigma^{2D}X, a+2, b)$ or $(\Sigma^{H}X, a, b+1)$ does not change the value of $\kappa o(S)$. By Proposition 4.4.9, we proved that $\kappa o(S)$ does not change when we change the representative of the spectrum class. □

By definition of the suspension of a spectrum class and Proposition 4.4.8, it is easy to prove:

Proposition 4.4.12. *For any spectrum class $S \in \mathfrak{B}$ at level K, we have:*

- $\kappa o(\Sigma^{8D}S) = \kappa o(S)$.
- $\kappa o(\Sigma^{2H}S) = \kappa o(S) + 2$.
- $\kappa o(\Sigma^{H+4D}S) = \kappa o(S) + 3 - \beta^4_{k+4}$.

With these discussions, we can now define the invariants for three manifolds.

Definition 4.4.13. For an oriented rational homology sphere Y and a spin structure $\mathfrak{s}$ on Y, we define $\kappa o_i(Y, \mathfrak{s}) = \kappa o(\Sigma^{iD}S(Y, \mathfrak{s}))$ for any $i \in \mathbb{Z}_{\geq 0}$. Then $\kappa o_i(Y, \mathfrak{s}) = \kappa o_{i+8}(Y, \mathfrak{s})$, which allow us to define $\kappa o_i(Y, \mathfrak{s})$ for $i \in \mathbb{Z}/8$.

4.5 Proof of Theorem 1.3.6

In this section, we will prove Theorem 1.3.6.

Let X_0, X_1 be be two spaces of type SWF at level k_0 and k_1, respectively. Suppose there is an admissible map $f : X_0 \to X_1$ (which implies $k_0 \leq k_1$). By Lemma 4.4.7, we can choose suitable trivializations such that the following diagram commutes.

$$\begin{array}{ccc}
\widetilde{KO}_G(X_1) & \xrightarrow{f^*} & \widetilde{KO}_G(X_0) \\
\downarrow{\scriptstyle i_1^*} & & \downarrow{\scriptstyle i_0^*} \\
KO_G(k_1 D) & \xrightarrow{(f^{S^1})^*} & KO_G(k_0 D) \\
\downarrow{\scriptstyle \varphi_{k_1}} & & \downarrow{\scriptstyle \varphi_{k_0}} \\
\mathbb{Z} & \xrightarrow{m_{k_1}^{k_1-k_0}} & \mathbb{Z}
\end{array}$$

Therefore, we get $m_{k_1}^{k_1-k_0}(\mathrm{Im}(\varphi_{k_1} \circ i_1^*)) \subset \mathrm{Im}(\varphi_{k_0} \circ i_0^*)$. This implies that $(2^{\kappa o(X_1)+\beta_{k_1}^{k_1-k_0}}) \subset (2^{\kappa o(X_0)}) \subset \mathbb{Z}$. Therefore, we get the following proposition:

Proposition 4.5.1. *Let X_0, X_1 be two spaces of type SWF at level k_0 and k_1, respectively. Suppose there is an admissible map $f : X_0 \to X_1$. Then we have:*

$$\kappa o(X_0) \leq \kappa o(X_1) + \beta_{k_1}^{k_1-k_0}. \tag{4.19}$$

Next we generalize the above inequality to the spectrum classes:

Definition 4.5.2. Let $S_0, S_1 \in \mathfrak{B}$ be two spectrum classes. We call S_0 dominates S_1 if we can find representatives $S_i = [(X_i, a, b)]$ for $i = 1, 2$ and an admissible map f from X_0 to X_1.

Proposition 4.5.3. *Let $S_0, S_1 \in \mathfrak{B}$ be two spectrum classes at level k_0 and k_1 respectively. Suppose S_0 dominates S_1, then we have:*

$$\kappa o(S_0) \leq \kappa o(S_1) + \beta_{k_1}^{k_1-k_0}. \tag{4.20}$$

Proof. Since an admissible map $f : X_0 \to X_1$ gives an admissible map $\Sigma^{aH+bD} f : \Sigma^{aH+bD} X_0 \to \Sigma^{aH+bD} X_1$ for any $a, b \in \mathbb{Z}_{\geq 0}$. This proposition is a straightforward corollary of Proposition 4.5.1 and Definition 4.4.10. □

By considering the natural inclusion $X \to \Sigma^D X$, it is easy to see that S always dominates $\Sigma^D S$. Therefore ,we get the following corollary, which will be useful in Section 8.

Corollary 4.5.4. *For any spectrum class $S \in \mathfrak{B}$ at level K. We have:*

$$\kappa o(S) \leq \kappa o(\Sigma^D S) + \alpha_{k+1}.$$

Now let Y_0, Y_1 be two rational homology 3-spheres and $\mathfrak{s}_i$ be spin structures on them respectively. Suppose $(W, \mathfrak{s})$ is a smooth oriented spin cobordism from $(Y_0, \mathfrak{s}_0)$ to $(Y_1, \mathfrak{s}_1)$. By Theorem 4.3.8, we see that $\Sigma^{-\frac{\sigma(W)}{16}\mathbb{H}}(\Sigma^{kD} S(Y_0, \mathfrak{s}_0))$ dominates $\Sigma^{(b_2^+(W)+k)D} S(Y_1, \mathfrak{s}_1)$ for any $k \in \mathbb{Z}$. Applying Proposition 4.5.3, we get:

Theorem 4.5.5. *Suppose $(W, \mathfrak{s})$ is a smooth, oriented spin cobordism from $(Y_0, \mathfrak{s}_0)$ to $(Y_1, \mathfrak{s}_1)$. Then for any $k \in \mathbb{Z}$, we have the inequality:*

$$\kappa o_{k+b_2^+(W)}(Y_1, \mathfrak{s}_1) + \beta^{b_2^+(W)}_{k+b_2^+(W)} \geq \kappa o(\Sigma^{-\frac{\sigma(W)}{16}\mathbb{H}}(\Sigma^{kD} S(Y_0, \mathfrak{s}_0))). \tag{4.21}$$

In general, $\kappa o(\Sigma^{-\frac{\sigma(W)}{16}\mathbb{H}}(\Sigma^{kD} S(Y_0, \mathfrak{s}_0)))$ can be expressed by $\kappa o_k(Y_0, \mathfrak{s}_0)$ or $\kappa o_{k+4}(Y_0, \mathfrak{s}_0)$, but the explicit formula is messy. For simplicity, we now focus on the integral homology sphere case.

Remark. Suppose Y is an oriented integral homology 3-sphere. There is a unique spin structure $\mathfrak{s}$ on Y and we simply write $S(Y, \mathfrak{s})$ and $\kappa o_i(Y, \mathfrak{s})$ as $S(Y)$ and $\kappa o_i(Y)$, respectively.

Suppose both Y_i are integral homology spheres, then the intersection form of W is a unimodular, even form. Let's assume that the intersection from can be decomposed as:

$$p(-E_8) \oplus q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ for } p, q \geq 0.$$

In this case, we have $\frac{\sigma(W)}{16} = -\frac{p}{2}$ and $b_2^+(W) = q$. Recall that the spectrum class invariant $S(Y_0)$ is defined as

$$[\Sigma^{-V^0_\lambda}(I^\mu_\lambda, 0, \tfrac{n(Y,\mathfrak{s},A_0,g)}{2})] = [(\Sigma^{(\dim_{\tilde{\mathbb{R}}} V^0_\lambda)\tilde{\mathbb{R}}} I^\mu_\lambda, -2\dim_{\tilde{\mathbb{R}}} V^0_\lambda, \tfrac{n(Y,\mathfrak{s},A_0,g)}{2} + \dim_{\tilde{\mathbb{H}}} V^0_\lambda)].$$

The third component of this triple may be an integer or a half integer, depending on the Rokhlin invariant $\mu(Y_0)$.

Proposition 4.5.6. *Let Y_0 be an integral homology three sphere and $p \in \mathbb{Z}_{\geq 0}$. Then we have the following relations.*

(1) Suppose $\mu(Y_0) = 0 \in \mathbb{Z}_2$.

- *For $p = 4l$, we have $\kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0))) = \kappa o_k(Y_0) + 2l$.*
- *For $p = 4l + 1$, we have $\kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0))) = \kappa o_{k+4}(Y_0) + \frac{5}{2} + 2l - \beta_k^4$.*
- *For $p = 4l + 2$, we have $\kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0))) = \kappa o_{k+4}(Y_0) + 3 + 2l - \beta_k^4$.*
- *For $p = 4l + 3$, we have $\kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0))) = \kappa o_k(Y_0) + 2l + \frac{3}{2}$.*

(2) Suppose $\mu(Y_0) = 1 \in \mathbb{Z}_2$.

- *For $p = 4l$, we have $\kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0))) = \kappa o_k(Y_0) + 2l$.*
- *For $p = 4l + 1$, we have $\kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0))) = \kappa o_k(Y_0) + 2l + \frac{1}{2}$.*
- *For $p = 4l + 2$, we have $\kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0))) = \kappa o_{k+4}(Y_0) + 3 + 2l - \beta_k^4$.*
- *For $p = 4l + 3$, we have $\kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0))) = \kappa o_{k+4}(Y_0) + \frac{7}{2} + 2l - \beta_k^4$.*

Proof. We denote $(\Sigma^{(\dim_{\tilde{\mathbb{R}}} V_\lambda^0)\tilde{\mathbb{R}}} I_\lambda^\mu, -2\dim_{\tilde{\mathbb{R}}} V_\lambda^0, \frac{n(Y,\mathfrak{s},A_0,g)}{2} + \dim_{\tilde{\mathbb{H}}} V_\lambda^0)$ by (X, a, b).

For $\mu(Y_0) = 0$ and $p = 4l$, we have $b \in \mathbb{Z}$. Take $M, N \gg 0$ and let $N' = N + l$. Then by Definition 4.4.10, we have:

$$\begin{aligned} \kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0)) &= \kappa o(\Sigma^{(8M+k-a)D}\Sigma^{(2N+2l-b)H} X) - 2N \\ &= \kappa o(\Sigma^{(8M+k-a)D}\Sigma^{(2N'-b)H} X) - 2N' + 2l = \kappa o_k(Y) + 2l. \end{aligned} \tag{4.22}$$

For $p = 4l + 1$, take $M, N \gg 0$ and let $N' = N + l$. Then we have:

$$\begin{aligned} \kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0)) &= \kappa o(\Sigma^{(8M+k-a)D}\Sigma^{(2N+2l+1-b)H} X) - 2N - \frac{1}{2} \\ &= \kappa o(\Sigma^H(\Sigma^{kD}(X, a, b))) + 2l - \frac{1}{2} = \kappa o_{k+4}(Y_0) + \frac{5}{2} + 2l - \beta_k^4. \end{aligned} \tag{4.23}$$

The other cases can be proved similarly. ◻

Now combining the above proposition and Theorem 4.5.5, we proved Theorem 1.3.6.

4.6 KO_G-Split condition

Now consider the space $X = (8kD + (2l+1)H)^+$ for $k, l \in \mathbb{Z}_{\geq 0}$. We have the map induced by the inclusion:

$$i^* : \widetilde{KO}_G(X) \to KO_G(8kD).$$

By Theorem 4.1.12, we see that $KO_G(8kD+(2l+1)H)$ is generated by $(b_{2H})^l(b_{8D})^k\lambda(H)$ and $(b_{2H})^l(b_{8D})^k c(H)$ as $RO(G)$-module and the map i^* is multiplication by $\gamma(H)^{2l+1}$. Using Proposition 4.1.14, we get:

$$\begin{aligned} i^*((b_{2H})^l(b_{8D})^k\lambda(H)) &= (2 + A - 2D - 2B)^l(2 - 2D - B) \cdot (b_{8D})^k, \\ i^*((b_{2H})^l(b_{8D})^k c(H)) &= (A - 2B)^l(B - A) \cdot (b_{8D})^k. \end{aligned} \tag{4.24}$$

The above discussion motivates the following definition:

Definition 4.6.1. Let X be a space of type SWF at level $8k$. X is called even KO_G-split if $\mathcal{J}(X)$ is the submodule generated by $(2 + A - 2D - 2B)^l(2 - 2D - B) \cdot (b_{8D})^k$ and $(A - 2B)^l(B - A) \cdot (b_{8D})^k$ for some $l \in \mathbb{Z}_{\geq 0}$.

Next, we consider the space $X = ((8k+4)D + 2lH)^+$. The map:

$$i^* : \widetilde{KO}_G(X) \to KO_G((8k+4)D)$$

is just multiplication of $\gamma(H)^{2l}$. We know $\widetilde{KO}_G(X) = KO_G((8k+4)D) \cdot (b_{2H})^l$ by the Bott isomorphism. Since $\gamma(H)^{2l}(b_{2H})^l = (K - 2H + D + 5)^l = (A + 2D + 6 - 2H)^l$ (see Theorem 4.1.14), we have $\mathrm{Im}(i^*) = (A + 2D + 6 - 2H)^l \cdot KO_G((8k+4)D) \subset KO_G((8k+4)D)$. This motivates the following definition:

Definition 4.6.2. Let X be a space of type SWF at level $8k+4$. X is called odd KO_G-split if $\mathcal{J}(X) = (A + 2D + 6 - 2H)^l \cdot KO_G((8k+4)D)$ for some $l \in \mathbb{Z}_{\geq 0}$.

KO_G-split spaces are special because of the following proposition (compare Proposition 4.5.1).

Proposition 4.6.3. *Let X_0, X_1 be two spaces of type SWF at level k_0, k_1 respectively and f be an admissible map from X_0 to X_1. Suppose $k_0 < k_1$ and X_0 is odd or even KO_G-split (which implies that $k_0 \equiv 0$ or 4 mod 8). Then we have:*

$$\kappa o(X_0) < \kappa o(X_1) + \beta_{k_1}^{k_1 - k_0}. \tag{4.25}$$

Before proving this proposition, we need to make a digression into the general properties of $KO_G(4D)$ and $RO(G)$.

Lemma 4.6.4. *The following properties holds:*

- *(1) Any element in $RO(G)$ can be uniquely written as $bD + f(A) + Bg(A)$ for some polynomials f, g and integer b.*
- *(2) Any element in $RO(G)$ can be uniquely written as $bD + f(A) + Hg(A)$ for some polynomials f, g and integer b.*
- *(3) Any element in $KO_G(4D)$ can be uniquely written as $bD\lambda(D) + f(A)\lambda(D) + g(A)c$ for some polynomials f, g and integer b.*
- *(4) The map $RO(G) \to KO_G(4D)$ defined by multiplication of $\lambda(D)$ is injective.*
- *(5) An element $\omega = bD\lambda(D) + f(A)\lambda(D) + g(A)c$ belongs to $RO(G)\lambda(D)$ if and only if $4|g(A)$. Moreover, if $(A + 2D + 6 - 2H)^l\omega \in RO(G) \cdot \lambda(D)$ for some l, then $\omega \in RO(G) \cdot \lambda(D)$.*
- *(6) Suppose $(A - 2B)^l h(A, B) = 0 \in RO(G)$ for some two-variable polynomial h in A, B. Then we have $h(A, B) = 0$ in $RO(G)$.*
- *(7) Suppose $f(D) = h(A, B)$ for some 2-variable polynomial h without degree-0 term and some polynomial f. Then $h(A, B) = 0$.*

Proof. (1),(2),(3),(4) can be proved by straightforward calculation using Theorem 4.1.12. The first statement of (5) is the corollary of (2),(3) and the relation $H\lambda(D) =$

4c. Let's prove the second statement of (5). We have $Hc = (1 + D + K)\lambda(D)$ and $(2D + 6)c = 8c = 2H\lambda(D)$. Therefore, $(A + 2D + 6 - 2H)^l\omega \in RO(G)\lambda(D)$ implies $A^l\omega \in RO(G)\lambda(D)$. It follows that $4|A^l g(A)$, which implies $4|g(A)$ and $\omega \in RO(G)\lambda(D)$.

For (6), we can assume that $h(A, B) = f(A) + Bg(A)$ for some polynomials f, g. Consider the map $\psi : RO(G) \to \mathbb{Q}[x]$ defined by $\psi(D) = 1, \psi(B) = x$ and $\psi(A) = \frac{x^2}{4} + 2x$. Then $0 = \psi((A - 2B)^l(f(A) + Bg(A))) = (\frac{x^2}{4})^l(f(\frac{x^2}{4} + 2x) + xg(\frac{x^2}{4} + 2x))$, which implies $0 = f(\frac{x^2}{4} + 2x) + xg(\frac{x^2}{4} + 2x)$. Considering the leading term in x, we see that $f(x) = g(x) = 0$.

For (7), we can simplify $h(A, B)$ as $Ag_1(A) + Bg_2(A)$ for some polynomials g_1, g_2 by the relation $B^2 - 4(A - 2B) = 0$. Then the conclusion follows from (1). □

Lemma 4.6.5. *Suppose* $a(1-D)\lambda(D) \in (A+2D+6-2H)^l KO_G(4D)$ *for some* $a \in \mathbb{Z}$ *and* $l \in \mathbb{Z}_{\geq 0}$. *Then we have* $2^{2l+1}|\varphi_4(a(1 - D)\lambda(D))$.

Proof. Since $\varphi_4(a(1-D)\lambda(D)) = 2a$, the conclusion is trivial when $l = 0$. Now suppose $l > 0$. Let $a(1 - D)\lambda(D) = (A + 2D + 6 - 2H)^l \cdot \omega$ for some $\omega \in KO_G(4D)$. By (5) of Lemma 4.6.4, we see that $\omega \in RO(G)\lambda(D)$. Write ω as $(bD + f(A) + Bg(A))\lambda(D)$. By (4) of Lemma 4.6.4, we get $a(1 - D) = (A - 2B - 2D + 2)^l(bD + f(A) + Bg(A))$. Using the relation $(1 - D)A = (1 - D)B = 0$, we can simplify this equality as $a(1 - D) - (f(0) + bD)(2 - 2D)^l = (A - 2B)^l(b + f(A) + Bg(A))$. By (7) of Lemma 4.6.4, we get that $(A - 2B)^l(b + f(A) + Bg(A)) = 0 \in RO(G)$. By (6) of Lemma 4.6.4, we have $b + f(A) + Bg(A) = 0$. This implies that $\omega = b(D - 1)\lambda(D)$ and $\varphi_4(a(1 - D)\lambda(D)) = -2^{2l+1}b$ for some $b \in \mathbb{Z}$. □

Lemma 4.6.6. *Suppose* $a(1 - D)$ *is in the ideal of* $RO(G)$ *generated by* $(2 + A - 2D - 2B)^l(2 - 2D - B)$ *and* $(A - 2B)^l(B - A)$ *for some* $l \in \mathbb{Z}_{\geq 0}$. *Then we have* $2^{2l+3}|\varphi_0(a(1 - D))$.

Proof. We assume $l > 0$ first. By (1) of Lemma 4.6.4 and the relation $A(1 - D) =$

$B(1-D)=0$, we have can express $a(1-D)$ as:

$$\begin{aligned}(2-2D-B)(2-2D+A-2B)^l(b(1-D)+f_1(A)+Bg_1(A))\\ +(A-2B)^l(B-A)(f_2(A)+Bg_2(A))\end{aligned} \tag{4.26}$$

for some integer b and polynomials f_1, f_2, g_1, g_2.

As in the proof of Lemma 4.6.5, we can simplify this formula and use (7) of Lemma 4.6.4 to get:

$$-B(A-2B)^l(f_1(A)+Bg_1(A))+(A-2B)^l(B-A)(f_2(A)+Bg_2(A))=0\in RO(G). \tag{4.27}$$

We have $-B(f_1(A)+Bg_1(A))+(B-A)(f_2(A)+Bg_2(A))=0$ by (6) of Lemma 4.6.4. Simplifying this, we obtain:

$$-4Ag_1(A)-Af_2(A)+4Ag_2(A)+B(-f_1(A)+f_2(A)+8g_1(A)-Ag_2(A)-8g_2(A))=0. \tag{4.28}$$

This implies $-4Ag_1(A)-Af_2(A)+4Ag_2(A)=0$ and $-f_1(A)+8g_1(A)+f_2(A)-Ag_2(A)-8g_2(A)=0$. Considering the degree-1 term of the first identity, we get $4|f_2(0)$. Also, we have $8|-f_1(0)+f_2(0)$ by checking the degree-0 term of the second identity. Therefore, we have $4|f_1(0)$, which implies $\varphi_0(a(1-D))=2^{2l+2}(2b+f_1(0))$ can be divided by 2^{2l+3}.

The case $l=0$ is similar. We also get the identity (4.28). □

Proof of Proposition 4.6.3: Consider the commutative diagram:

$$\begin{array}{ccc}\widetilde{KO}_G(X_1) & \xrightarrow{f^*} & \widetilde{KO}_G(X_0)\\ \downarrow{\scriptstyle i_1^*} & & \downarrow{\scriptstyle i_0^*}\\ KO_G(k_1D) & \xrightarrow{(f^{S^1})^*} & KO_G(k_0D)\\ \downarrow{\scriptstyle \varphi_{k_1}} & & \downarrow{\scriptstyle \varphi_{k_0}}\\ \mathbb{Z} & \xrightarrow{m_{k_1}^{k_1-k_0}} & \mathbb{Z}.\end{array}$$

(1) Suppose X_0 is odd KO_G-split. Then $k_0=8k+4$ for some integer K and $KO_G(k_0D)=KO_G(4D)\cdot(b_{8D})^k$ by the Bott isomorphism. $\mathrm{Im}(i_0^*)=(A+2D+6-2H)^l\cdot KO_G(4D)\cdot(b_{8D})^k$ for some $l\in\mathbb{Z}_{\geq 0}$. A simple calculation shows that

$\kappa o(X_0) = 2l$. Suppose $\kappa o(X_1) = r$. Then we can find an element $z \in \widetilde{KO}_G(X_1)$ such that $\varphi_{k_1} i_1^*(z) = 2^r$. Therefore, $\varphi_{k_0}(\omega) = 2^{r+\beta_{k_1}^{k_1-k_0}}$ where $\omega = (f^{S^1})^*(i_1^*(z))$. Since $k_1 > k_0$, the map $(f^{S^1})^*$ factors through $KO_G((k_0+1)D) \to KO_G(k_0 D)$. Therefore, we see that $\omega = \gamma(D) \cdot (a\eta(D)\lambda(D)) \cdot (b_{8D})^k = a(1-D)\lambda(D) \cdot (b_{8D})^k$ for some $a \in \mathbb{Z}$. Because of the commutative diagram, we have $\omega \in \mathrm{Im}(i_0^*)$. By Lemma 4.6.5, we get $2^{2l+1} | \varphi_{k_0}(\omega)$. This implies $2l + 1 \leq r + \beta_{k_1}^{k_1-k_0}$.

(2) Suppose X_0 is even KO_G-split with $k_0 = 8k$. Notice that $\kappa o(X) = 2l + 2$ if $\mathcal{J}(X)$ is the submodule generated by $(2 + A - 2D - 2B)^l(2 - 2D - B)(b_{8D})^k$ and $(A - 2B)^l(B - A)(b_{8D})^k$. Using Lemma 4.6.6, the proof is almost the same with the previous case. □

By Proposition 4.4.8, we see that $\Sigma^{2H}X$ and $\Sigma^{8D}X$ are even (odd) KO_G-split if X is even (odd) KO_G-split. Therefore, Proposition 4.4.9 justifies the following definition:

Definition 4.6.7. A spectrum class $S = [(X, a, b + r)]$ with $a, b \in \mathbb{Z}, r \in [0, 1)$ is called even (odd) KO_G-split if for integers $M, N \gg 0$, $\Sigma^{(8M-a)D}\Sigma^{(2N-b)H}X$ is even (odd) KO_G-split.

Example 4.6.8. For any $a, b \in \mathbb{Z}$ and $r \in [0, 1)$, $[(S^0, 8a, 2b+1+r)]$ is even KO_G-split and $[(S^0, 8a + 4, 2b + r)]$ is odd KO_G-split.

The following proposition is easy to prove using Proposition 4.6.3

Proposition 4.6.9. *Let $S_0, S_1 \in \mathfrak{C}$ be two spectrum classes at level k_0, k_1 respectively, with $k_0 < k_1$. Suppose S_0 is even or odd KO_G-split and S_0 dominates S_1, then we have:*

$$\kappa o(S_0) < \kappa o(S_1) + \beta_{k_1}^{k_1-k_0}. \tag{4.29}$$

Now let Y be a homology sphere. Recall that we have a spectrum class invariant $S(Y)$ at level 0.

Definition 4.6.10. Y is called Floer KO_G-split if $\Sigma^H S(Y)$ is even KO_G-split and $\Sigma^{4D}S(Y)$ is odd KO_G-split.

Remark. For simple examples like $Y = \pm\Sigma(2,3,12n+1)$ or $\pm\Sigma(2,3,12n+5)$, the two conditions in the above definition are either both true or both false. We expect that this fails in more complicated examples. If we only assume one of these two conditions, only half of the cases in Theorem 1.3.9 are still true.

Remark. We will see in Section 8 that $S^3, \pm\Sigma(2,3,12n+1)$ and $-\Sigma(2,3,12n+5)$ are Floer KO_G-split, while $+\Sigma(2,3,12n+5)$ is not Floer KO_G-split.

Proof of Theorem 1.3.9: (1) When $\mu(Y_0) = 0$, $S(Y_0) = [(X, a, b)]$ for some space X and some integers a, b. For large integers M, N, we have the following:

(i) The space $\Sigma^{(8M-a)D}\Sigma^{(2N-b+1)H}X$ is even KO_G-split.

(ii) The space $\Sigma^{(8M-a+4)D}\Sigma^{(2N-b)H}X$ is odd KO_G-split.

Now consider $p = 4l + m$ for $m = 0, 1, 2, 3$:

- For $p = 4l$, $\Sigma^{\frac{p}{2}H}\Sigma^{4D}S(Y_0) = [(\Sigma^{4D}X, a, b - 2l)]$ is odd KO_G-split by (ii).
- For $p = 4l + 1$, $\Sigma^{\frac{p}{2}H}S(Y_0) = [(\Sigma^{H}X, a, b - 2l + \frac{1}{2})]$ is even KO_G-split by (i).
- For $p = 4l + 2$, $\Sigma^{\frac{p}{2}H}S(Y_0) = [(\Sigma^{H}X, a, b - 2l)]$ is even KO_G-split by (i).
- For $p = 4l + 3$, $\Sigma^{\frac{p}{2}H}\Sigma^{4D}S(Y_0) = [(\Sigma^{4D}X, a, b - 2l - 2 + \frac{1}{2})]$ is odd KO_G-split by (ii).

Similarly, we can prove that when $\mu(Y_0) = 1$, $\Sigma^{\frac{p}{2}H}S(Y_0)$ is even KO_G-split for $p = 4l+2$ and $4l + 3$ while $\Sigma^{\frac{p}{2}H}\Sigma^{4D}S(Y_0)$ is odd KO_G-split for $p = 4l$ and $4l + 1$.

Now repeat the proof of Theorem 1.3.6 for $k = 0$ or 4, using Proposition 4.6.9 instead of Proposition 4.5.3. Notice that the two sides of the same inequalities are either both integers or both half-integers. The inequalities are proved. □

4.7 Examples and Explicit bounds

In this section, we will prove Theorem 1.3.8 about the values of $\kappa o_i(S^3)$ and $\kappa o_i(\pm\Sigma(2,3,r))$ with $\gcd(r, 6) = 1$. We will also use Corollary 1.3.10 to give some new bounds about the intersection forms of spin four manifolds with given boundaries.

4.7.1 Basic Examples

If Y is a rational homology sphere admitting metric g with a positive scaler curvature, then by the arguments in [40], we obtain:

$$S(Y,\mathfrak{s}) = [(S^0, 0, n(Y,\mathfrak{s})/2)].$$

In particular, S^3 is Floer KO_G-split and $\kappa o_i(S^3) = 0$ for any $i \in \mathbb{Z}/8$.

In [42], Manolescu gave two examples of spaces of type SWF that are related to the spectrum class invariants of the Brieskorn spheres $\pm\Sigma(2,3,r)$. We recall the construction here.

Suppose that G acts freely on a finite G-CW complex Z, with the quotient space $Q = Z/G$. Let

$$\widetilde{Z} = ([0,1]\times Z)/(0,z) \sim (0,z') \text{ and } (1,z) \sim (1,z') \text{ for all } z,z' \in Z$$

denote the unreduced suspension of Z, where G acts trivially on the $[0,1]$ factor. We can take one of the two cone points (say $(0,z) \in \widetilde{Z}$) as the base point and view $\widetilde{Z}$ as a pointed G-space. It's easy to see that $\widetilde{Z}$ is of type SWF at level 0.

We want to compute $\kappa o(\Sigma^{kD}\tilde{Z})$ for $k = 0, 1, \cdots, 7$. It turns out that the method in [42] also works here. Namely, the inclusion $i : (\Sigma^{kD}\tilde{Z})^{S^1} = \Sigma^{kD}S^0 \to \Sigma^{kD}\tilde{Z}$ gives the long exact sequence:

$$\cdot \to \widetilde{KO}_G(\Sigma^{kD}\tilde{Z}) \xrightarrow{i^*} KO_G(kD) \xrightarrow{p^*} KO^1_G(\Sigma^{kD}\tilde{Z},(kD)^+) \to \cdot. \tag{4.30}$$

By exactness of the sequence, we have $\mathrm{Im}(i^*) = \ker(p^*)$. By definition, we have:

$$KO^1_G(\Sigma^{kD}\tilde{Z},(kD)^+) \cong \widetilde{KO}^1_G(\Sigma^{kD}\Sigma Z_+) \cong \widetilde{KO}_G(\Sigma^{kD}Z_+).$$

By abuse of notation, we still use p^* to represent the map between $KO_G(kD)$ and $\widetilde{KO}_G(\Sigma^{kD}Z_+)$. Checking the maps in the exact sequence, one can see that the p^* is induced by the natural projection $p : \Sigma^{kD}Z_+ \to (kD)^+$. Since G acts freely on $\Sigma^{kD}Z_+$

away from the base point, we see that $\widetilde{KO}_G(\Sigma^{kD}Z_+) \cong \widetilde{KO}((\Sigma^{kD}Z_+)/G)$. Notice that $(Z \times kD)/G$ is a vector bundle over Q and $(\Sigma^{kD}Z_+)/G$ is the Thom space of this bundle. We are interested in two cases:

- $Z \cong G$, acting on itself via left multiplication.
- $Z \cong T \cong S^1 \times jS^1 \subset \mathbb{C} \times j\mathbb{C} \subset \mathbb{H}$ and G acts on T by left multiplication in $\mathbb{H}$.

The first case is easy since the isomorphism $\widetilde{KO}_G(\Sigma^{kD}Z_+) \cong \widetilde{KO}(S^k)$ is given by $i_1^* \circ r_0$, where $i_1 : S^k \to \Sigma^{k\mathbb{R}}Z_+$ is the standard inclusion and $r_0 : \widetilde{KO}_G(\Sigma^{kD}Z_+) \to \widetilde{KO}(\Sigma^{k\mathbb{R}}Z_+)$ is the restriction map (See Fact 4.1.3 in Section 2). It follows that $\mathrm{Im}(i^*) = \ker(p^*) = \ker(i_1^* \circ r_0 \circ p^*) = \ker(r)$, where $r : KO_G(kD) \to \widetilde{KO}(S^k)$ is the restriction map.

We know the structure of $\widetilde{KO}(S^k)$:

- $\widetilde{KO}(S^0) \cong KO(pt) \cong \mathbb{Z}$.
- $\widetilde{KO}(S^1) \cong \mathbb{Z}_2$, generated by the Hurewicz image of the Hopf map in $\pi_3(S^2)$.
- $\widetilde{KO}(S^2) \cong \mathbb{Z}_2$, generated by the Hurewicz image of the square of the Hopf map.
- $\widetilde{KO}(S^4) \cong \mathbb{Z}$, generated by $V_{\mathbb{H}} - 4$, where $V_{\mathbb{H}}$ is the quaternion Hopf bundle.
- $\widetilde{KO}(S^k) \cong 0$ for $k = 3, 5, 6, 7$.

Therefore, by the explicit description of $\eta(D), \lambda(D), c$ after Theorem 4.1.12. We get the following results about the kernel of $r : KO_G(kD) \to \widetilde{KO}(S^k)$.

- For $k = 0$, $\ker(r)$ is the submodule generated by $1 - D, A, B$.
- For $k = 1$, $\ker(r)$ is generated by $2\eta(D)$.
- For $k = 2$, $\ker(r)$ is generated by $2\eta(D)^2$ and $\gamma(D)^2c$.
- For $k = 4$, $\ker(r)$ is generated by $\lambda(D) - c, (1 - D)\lambda(D), A\lambda(D)$ and Ac.
- For $k = 3, 5, 6, 7$, $\ker(r) \cong KO_G(kD)$.

From this, we get:

Proposition 4.7.1. *$\kappa o(\Sigma^{kD}\tilde{G}) = 0$ for $k = 3, 4, 5, 6, 7$ and $\kappa o(\Sigma^{kD}\tilde{G}) = 1$ for $k = 0, 1, 2$.*

Now let's consider the case $Z \cong T$. We want to find $\ker(p^*)$ for $p^* : KO_G(kD) \to KO_G(\Sigma^{kD}T_+)$. Notice that $S^1 \subset G$ acts trivially on $(kD)^+$ and freely on T with $T/S^1 = S^1$. We have $\widetilde{KO}_G(\Sigma^{kD}T_+) = \widetilde{KO}((\Sigma^{kD}S^1_+)/\mathbb{Z}_2)$. The space $(\Sigma^{kD}S^1_+)/\mathbb{Z}_2$ can be identified with:

$[0,1]\times(kD)^+/(0,x) \sim (1,-x)$ and $(t_1,\infty) \sim (t_2,\infty)$ for any $x \in (kD)^+$ and $t_1, t_2 \in [0,1]$.

Consider the inclusion $i_2 : \{0\}\times(kD)^+ \to (\Sigma^{kD}S^1_+)/\mathbb{Z}_2$. Notice that $((\Sigma^{kD}S^1_+)/\mathbb{Z}_2)/(kD)^+ \cong S^{k+1}$. We get the long exact sequence:

$$\cdot \to \widetilde{KO}(S^{k+1}) \xrightarrow{\delta} \widetilde{KO}(S^{k+1}) \to \widetilde{KO}((\Sigma^{kD}S^1_+)/\mathbb{Z}_2) \xrightarrow{i_2^*} \widetilde{KO}(S^k) \to \cdot \tag{4.31}$$

By checking the iterated mapping cone construction, which gives us this long exact sequence, it is not hard to prove that δ is induced by the map $f : S^{k+1} \to S^{k+1}$ with $\deg(f) = 0$ for even K and $\deg(f) = 2$ for odd K.

When $k = 2, 4, 5, 6$, we have $\widetilde{KO}(S^{k+1}) = 0$. Therefore, i_2^* is injective, which implies $i_1^* \circ r_0 : \widetilde{KO}_G(\Sigma^{kD}T_+) \to \widetilde{KO}((kD)^+)$ is injective (i_1^* and r_0 are defined as in the case $Z \cong G$). We see that when $k = 2, 4, 5, 6$, just like the case $Z \cong G$, the kernel of p^* is the kernel of the restriction map $r : KO_G(kD) \to \widetilde{KO}(S^k)$. Thus, we get $\kappa o(\Sigma^{kD}\tilde{T}) = \kappa o(\Sigma^{kD}\tilde{G})$ for $k = 2, 4, 5, 6$.

For $k = 0$, consider $[0,1]$ as the subset $\{1 + je^{i\theta} | \theta \in [0,\pi]\} \subset T$. The left endpoint is mapped to the right endpoint under the action of $-j \in G$. This embedding of $[0,1]$ gives us the following explicit description of the map $p^* : RO(G) \cong \widetilde{KO}_G(S^0) \to \widetilde{KO}_G(T_+) \cong KO_G(T) \cong KO(T/G) = KO(S^1)$.

Starting from a representation space V of G, we get an trivial bundle $V \times [0,1]$ over $[0,1]$. Identifying $(x, 0)$ with $((-j) \circ x, 1)$ for any $x \in V$, we get a bundle E over S^1. $[E] \in KO(S^1)$ is the image of $[V] \in RO(G)$ under p^*.

We know that $KO(S^1)$ is generated by the one dimensional trivial bundle [1] and the one dimensional nontrivial bundle $[m]$, subject to the relation 2([1]-[m])=0. Using the explicit description of p^*, we see that $p^*(1) = [1]$, $p^*(D) = [m]$ and $p^*(A) = p^*(B) = 0$. Therefore, we get $\kappa o(\tilde{T}) = 2$.

Applying Corollary 4.5.4 for $S = \Sigma^{2D}\tilde{T}$, we get $\kappa o(\Sigma^{3D}\tilde{T}) + 1 \geq \kappa o(\Sigma^{2D}\tilde{T}) = 1$. Applying Corollary 4.5.4 for $S = \Sigma^{3D}\tilde{T}$, we get $0 = \kappa o(\Sigma^{4D}\tilde{T}) + 0 \geq \kappa o(\Sigma^{3D}\tilde{T})$. Therefore, we see that $\kappa o(\Sigma^{3D}\tilde{T}) = 0$.

Applying Corollary 4.5.4 for $S = \Sigma^{2D}\tilde{T}$ and $S = \Sigma^{D}\tilde{T}$, we get $\kappa o(\Sigma^{D}\tilde{T}) = 1$ or 2.

For $k = 7$, the map $\delta : \widetilde{KO}(S^8) \to \widetilde{KO}(S^8)$ is multiplication by 2. Since $\widetilde{KO}(S^7) = 0$, we get $\widetilde{KO}((\Sigma^{kD}S^1_+)/\mathbb{Z}_2) = \mathbb{Z}_2$. This implies $p^*(2b_{8D} \cdot \gamma(D)) = 2p^*(b_{8D} \cdot \gamma(D)) = 0$. Therefore, $2b_{8D} \cdot \gamma(D) \in \ker(p^*)$ and $\kappa o(\Sigma^{7D}\tilde{T}) = 0$ or 1.

Lemma 4.7.2. *$\kappa o(\Sigma^{D}\tilde{T}) = 2$ and $\kappa o(\Sigma^{7D}\tilde{T}) = 1$.*

Proof. This can be proved directly using Gysin sequence. But here we use a different approach. In [42] and [41], Manolescu proved that $S(-\Sigma(2,3,11)) = [(\tilde{T}, 0, 1)]$, where $-\Sigma(2,3,11)$ is a negative oriented Brieskorn sphere. Therefore, by Definition 4.4.10 and Proposition 4.4.12, we get:

$$\kappa o_i(-\Sigma(2,3,11)) = \kappa o(\Sigma^{(i+4)D}\tilde{T}) + 1 - \beta_i^4.$$

In particular, $\kappa o_3(-\Sigma(2,3,11)) = \kappa o(\Sigma^{7D}\tilde{T}) - 2$ and $\kappa o_5(-\Sigma(2,3,11)) = \kappa o(\Sigma^{D}\tilde{T}) - 2$. Since $-\Sigma(2,3,11)$ bounds a smooth spin four manifold with intersection from $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (see [42]). We can apply Corollary 1.3.10 for $p = 0, q = 1$ and get $\kappa o_5(-\Sigma(2,3,11)) \geq 0$, which implies $\kappa o(\Sigma^{D}\tilde{T}) \geq 2$. We get $\kappa o(\Sigma^{D}\tilde{T}) = 2$ by our discussion before the lemma.

We can also apply Theorem 1.3.6 for $Y_0 = S^3, Y_1 = -\Sigma(2,3,11), p = 0, q = 1$ and $k = 2$. We have $\kappa o_3(-\Sigma(2,3,11)) \geq -1$ and $\kappa o(\Sigma^{7D}\tilde{T}) \geq 1$. Therefore, $\kappa o(\Sigma^{7D}\tilde{T}) = 1$ by our discussions before. □

We summarise our results in the following proposition.

Proposition 4.7.3. *$\kappa o(\Sigma^{kD}\tilde{T}) = 2$ for $k = 0, 1$; $\kappa o(\Sigma^{kD}\tilde{T}) = 1$ for $k = 2, 7$ and $\kappa o(\Sigma^{kD}\tilde{T}) = 0$ for $k = 3, 4, 5, 6$.*

Now we calculate $\kappa o_i(\pm\Sigma(2,3,r))$ with $\gcd(6,r)=1$. Actually, the spectrum class invariants $S(\pm\Sigma(2,3,r))$ are given in [42].

Proposition 4.7.4 (Manolescu [42]). *We have the following results about $S(\pm\Sigma(2,3,r))$.*

- $S(\Sigma(2,3,12n-1)) = [(\tilde{G} \vee \underbrace{\Sigma G_+ \vee \cdot \vee \Sigma G_+}_{n-1}, 0, 0)]$.
- $S(-\Sigma(2,3,12n-1)) = [(\tilde{T} \vee \underbrace{\Sigma^2 G_+ \vee \cdot \vee \Sigma^2 G_+}_{n-1}, 0, 1)]$.
- $S(\Sigma(2,3,12n-5)) = [(\tilde{G} \vee \underbrace{\Sigma G_+ \vee \cdot \vee \Sigma G_+}_{n-1}, 0, 1/2)]$.
- $S(-\Sigma(2,3,12n-5)) = [(\tilde{T} \vee \underbrace{\Sigma^2 G_+ \vee \cdot \vee \Sigma^2 G_+}_{n-1}, 0, 1/2)]$.
- $S(\Sigma(2,3,12n+1)) = [(S^0 \vee \underbrace{\Sigma^{-1} G_+ \vee \cdot \vee \Sigma^{-1} G_+}_{n}, 0, 0)]$.[5]
- $S(-\Sigma(2,3,12n+1)) = [(S^0 \vee \underbrace{G_+ \vee \cdot \vee G_+}_{n}, 0, 0)]$.
- $S(\Sigma(2,3,12n+5)) = [(S^0 \vee \underbrace{\Sigma^{-1} G_+ \vee \cdot \vee \Sigma^{-1} G_+}_{n}, 0, -1/2)]$.
- $S(-\Sigma(2,3,12n+5)) = [(S^0 \vee \underbrace{G_+ \vee \cdot \vee G_+}_{n}, 0, 1/2)]$.

As we mentioned in Remark 4.6, $\pm\Sigma(2,3,12n+1)$ and $-\Sigma(2,3,12n+5)$ are KO_G-split because of Example 4.6.8. While using the relations in Theorem 4.1.12 and Theorem 4.1.14, it is not hard to prove that the space $(8MD \oplus (2N+2)H)^+$ is not even KO_G-split for integers $M, N \gg 0$. This implies that $+\Sigma(2,3,12n+5)$ is not KO_G-split.

Since it's easy to see that wedging with a free G-space does not change the κo invariants, we don't need to consider those $\Sigma^l G_+$ factors. By Definition 4.4.10 and Proposition 4.4.12, we can use Proposition 4.7.1 and Proposition 4.7.3 to prove the results in Theorem 1.3.8 easily.

[5] *Strictly speaking, by this we mean the spectrum class of* $(\mathbb{H}^+ \vee \underbrace{\Sigma^3 G_+ \vee \cdot \vee \Sigma^3 G_+}_{n}, 0, 1)$.

4.7.2 Explicit Bounds

Now we use Corollary 1.3.10 and Proposition 4.2.1 to get explicit bounds on the intersection forms of spin 4-manifolds with boundary $\pm\Sigma(2,3,r)$.

Theorem 4.7.5. *Let W be an oriented, smooth spin 4-manifold with $\partial W = \pm\Sigma(2,3,r)$. Assume that the intersection form of W is $p(-E_8) \oplus q\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $p > 1, q > 0$.*[6] *If the mod 8 reduction of p is m, then we have $q - p \geq c_m$, where c_m are constants listed below. (Recall that the mod 2 reduction of p is the Rohklin invariant of the boundary.)*

	$m=0$	$m=2$	$m=4$	$m=6$
$\Sigma(2,3,12n-1)$	2	0	1	2
$-\Sigma(2,3,12n-1)$	3	(2)	(3)	3
$\Sigma(2,3,12n+1)$	(3)	1	(2)	(3)
$-\Sigma(2,3,12n+1)$	3	1	2	3

	$m=1$	$m=3$	$m=5$	$m=7$
$\Sigma(2,3,12n-5)$	1	2	3	3
$-\Sigma(2,3,12n-5)$	2	(1)	(2)	2
$\Sigma(2,3,12n+5)$	(2)	0	(1)	(2)
$-\Sigma(2,3,12n+5)$	2	3	4	4

Remark. Some of the bounds in Theorem 4.7.5 can also be obtained by other methods. For example, the case $m = 2$ for $\Sigma(2,3,12n+1)$ can be obtained using κ-invariant (see [42]). Also, some bounds can be obtained by filling method for small n. For example, the case $m = 2, 4$ for $-\Sigma(2,3,11)$ can be deduced from Theorem 1.3.3, using the fact that $\Sigma(2,3,11)$ bounds a spin 4-manifold with intersection form $2(-E_8) \oplus 2\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. However, the bounds that we put in the brackets in Theorem 4.7.5 appear to be new for general n.

Proof. Since we can do surgeries on loops without changing intersection forms, we will always assume $b_1(W) = 0$.

[6] *It is easy to see that the conclusions are not true for $p = 0, 1$. For example, $\pm\Sigma(2,3,12n-1)$ bounds a spin manifold with intersection form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.*

(1) Suppose $\Sigma(2,3,12n+1)$ bounds a spin 4-manifold with intersection form $8l(-E_8)\oplus(8l+2)\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ for $l>0$. Then we get a spin cobordism from $-\Sigma(2,3,12n+1)$ to S^3 with the same intersection form. By Theorem 4.3.8, $\Sigma^{4lH}S(-\Sigma(2,3,12n+1))$ dominates $\Sigma^{8l+2}S(S^3)$. Since $S(-\Sigma(2,3,12n+1)) = [(S^0\vee G_+\vee\cdot\vee G_+,0,0)]$ and $S(S^3)=[(S^0,0,0)]$, we get a map:

$$f:\Sigma^{r\mathbb{R}+(4l+M)H+ND}(S^0\vee G_+\vee\cdot\vee G_+)\to\Sigma^{r\mathbb{R}+MH+(8l+2+N)D}S^0$$

for some $M,N\in\mathbb{Z}$. Restricting to the first factor of $S^0\vee G_+\vee\cdot\vee G_+$, we obtain:

$$g:\Sigma^{r\mathbb{R}+(4l+M)H+ND}S^0\to\Sigma^{r\mathbb{R}+MH+(8l+2+N)D}S^0,$$

which induces homotopy equivalence between the G-fixed point sets. This a contradiction with Proposition 4.2.1. The case $m=0$ for $\Sigma(2,3,12n+1)$ is proved.

(2) Suppose $\Sigma(2,3,12n+5)$ bounds a smooth spin manifold with intersection form $(8l+1)(-E_8)\oplus(8l+2)\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ for $l>0$. Then we get a spin cobordism from $-\Sigma(2,3,12n+5)$ to S^3. As the previous case, this implies $\Sigma^{(4l+1/2)H}S(-\Sigma(2,3,12n+5))$ dominates $\Sigma^{(8l+2)D}S(S^3)$. Since $\Sigma^{(4l+1/2)H}S(-\Sigma(2,3,12n+5)) = [(\Sigma^{4lH}S^0,0,0)]$, we get the contradiction as before. This proves the case $m=1$ for $\Sigma(2,3,12n+5)$.

(3) Suppose $-\Sigma(2,3,12n-1)$ bounds a spin 4-manifold with intersection form $(8l+2)(-E_8)\oplus(8l+3)\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ for $l\geq 0$. By Corollary 1.3.10, we get $4l+3<\kappa o_{3+8l}(-\Sigma(2,3,12n-1))+\beta^{8l+7}_{8l+3}=-1+4+4l$, which is a contradiction. This proves the case $m=2$ for $-\Sigma(2,3,12n-1)$.

Using similar method as (3), we can prove all the other cases except:

- $m=0$ for $\pm\Sigma(2,3,12n-1)$ and $-\Sigma(2,3,12n+1)$,
- $m=7$ for $\Sigma(2,3,12n-5)$ and $-\Sigma(2,3,12n+5)$,
- $m=1$ for $-\Sigma(2,3,12n-5)$.

(4) We need to introduce another approach in order to prove the rest of the cases. Consider the orbifold D^2-bundle over $S^2(2,3,r)$. This gives us an orbifold W' with boundary $+\Sigma(2,3,r)$. We have $b_2^+(W')=0, b_2^-(W)=1$ and W' has a unique spin

structure $\mathfrak{t}$. Now suppose $-\Sigma(2,3,r)$ bounds a spin manifold W with intersection form $p(-E_8)\oplus q\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Then we can glue W and W' together to get an oriented closed spin 4-orbifold. We have:

$$\text{ind}_{\mathbb{C}}\not{D}(W\cup W') = p + \omega(\Sigma(2,3,r), W', \mathfrak{t}).$$

Here $\omega(\Sigma(2,3,r), W', \mathfrak{t})$ is the Fukumoto-Furuta invariant defined in [20]. Saveliev [59] proved that $\omega(\Sigma(2,3,r), W', \mathfrak{t}) = -\overline{\mu}(\Sigma(2,3,r)) = \overline{\mu}(-\Sigma(2,3,r))$, where $\overline{\mu}$ is the Neumann-Siebenmann invariant [49, 50]. In [20], Fukumoto and Furuta considered the finite dimensional approximation of the Seiberg-Witten equations on the orbifold $W\cup W'$ and constructed a stable Pin(2)-equivariant map: $(\frac{\text{ind}_{\mathbb{C}}\not{D}(W\cup W')}{2}H)^+ \to (b_2^+(W\cup W')D)^+$ which induces homotopy equivalence on the Pin(2)-fixed point set. (Recall that H and D are Pin(2)-representations defined in Section 2). Since $b_2^+(W\cup W') = q$ and $\text{ind}_{\mathbb{C}}\not{D}(W\cup W') = p + \overline{\mu}(-\Sigma(2,3,r))$, we can apply Proposition 4.2.1 to get:

$$q - p \geq 3 + \overline{\mu}(-\Sigma(2,3,r)) \text{ if } 0 < p + \overline{\mu}(-\Sigma(2,3,r)) \text{ can be divided by } 8.$$

Similarly, suppose $\Sigma(2,3,r)$ bounds a spin 4-manifold W' with intersection form $p(-E_8)\oplus q\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. We can consider $W'\cup(-W)$ and repeat the argument above. We get:

$$q - p \geq 2 + \overline{\mu}(\Sigma(2,3,r)) \text{ if } 0 < p + \overline{\mu}(\Sigma(2,3,r)) \text{ can be divided by } 8.$$

The invariants $\overline{\mu}(\pm\Sigma(2,3,r))$ were computed in [49, 50]:

$$\overline{\mu}(\pm\Sigma(2,3,12n-1)) = \overline{\mu}(\pm\Sigma(2,3,12n+1)) = 0,$$

$$\overline{\mu}(\Sigma(2,3,12n-5)) = \overline{\mu}(-\Sigma(2,3,12n+5) = 1,$$

$$\overline{\mu}(-\Sigma(2,3,12n-5)) = \overline{\mu}(\Sigma(2,3,12n+5) = -1.$$

Therefore, simple calculations prove the rest of the cases. □

CHAPTER 5

Construction of the unfolded spectrum invariants

In this chapter, we define different versions of unfolded Seiberg-Witten-Floer spectra for the spin^c manifold $(Y, \mathfrak{s})$. First, we will define the spectrum $\underline{\mathrm{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1)$ and $\underline{\mathrm{swf}}^R(Y, \mathfrak{s}, A_0, g; S^1)$ for a general spin^c structure $\mathfrak{s}$. In Section 5.2, we consider the situation when $\mathfrak{s}$ is torsion and define normalized spectra $\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}; S^1)$ and $\underline{\mathrm{SWF}}^R(Y, \mathfrak{s}; S^1)$ which are independent of the choices of base connection A_0 and metric g. In Section 5.3, we deal with the Pin(2)-equivariant case for a spin structure $\mathfrak{s}$ and define $\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}; Pin(2))$, $\underline{\mathrm{SWF}}^R(Y, \mathfrak{s}; Pin(2))$.

5.1 The unfolded spectrum invariants for general spin^c structures.

The main idea of the construction follows [42] and [29]. In summary, we want to apply finite dimensional approximation of Conley indices to the set $Str(R)$ which contains all critical points and flow lines between them. However, the set $Str(R)$ is unbounded owing to the action of $\mathcal{G}^h_Y$. We then need to introduce transverse functions and use their level sets to obtain a collection of bounded subsets of $Str(R)$.

Notice that the space of imaginary-valued harmonic 1-forms, denoted by $i\Omega^1_h(Y)$, is a subspace of $Coul(Y)$. Let $p_{\mathcal{H}} \colon Coul(Y) \to i\Omega^1_h(Y)$ be the L^2-orthogonal projection. Here, we identify $i\Omega^1_h(Y)$ with $\mathbb{R}^{b_1}$ by choosing harmonic forms $\{h_1, h_2, \ldots h_{b_1}\} \subset i\Omega^1_h(Y)$ representing a set of free generators of the group

$$2\pi i \,\mathrm{im}(H^1(Y;\mathbb{Z}) \to H^1(Y;\mathbb{R})) \cong \mathbb{Z}^{b_1}.$$

With this identification, we can write the projection as

$$p_{\mathcal{H}} = (p_{\mathcal{H},1}, \ldots, p_{\mathcal{H},b_1}).$$

From now on, we assume that our perturbation f is good (see Definition 2.1.3). Together with the compactness result [33, Theorem 10.7.1], the critical points of $\mathcal{L}$ in $Coul(Y)$ is finite modulo the action of $\mathcal{G}^h_Y$. Consequently, we can find a small interval $[r,s] \subset (0,1)$ such that $\bigcup_{j=1}^{b_1} p_{\mathcal{H},j}^{-1}([-s,-r] \cup [r,s])$ contains no critical point of $\mathcal{L}$. Let us pick a positive number $\tilde{R}$ greater than the universal constant R_0 from Theorem 2.2.2.

Lemma 5.1.1. *There exists a positive number* $\tilde{\epsilon} > 0$ *such that we have* $\|\widetilde{\text{grad}}\,\mathcal{L}(x)\|_{\tilde{g}} > \tilde{\epsilon}$ *for any* $x \in \left(\bigcup_{j=1}^{b_1} p_{\mathcal{H},j}^{-1}([-s,-r] \cup [r,s])\right) \cap Str(\tilde{R})$.

Proof. Suppose that the result is not true. We can then find a sequence $\{x_n\}$ contained in $\left(\bigcup_{j=1}^{b_1} p_{\mathcal{H},j}^{-1}([-s,-r] \cup [r,s])\right) \cap Str(\tilde{R})$ with $\|\widetilde{\text{grad}}\,\mathcal{L}(x_n)\|_{\tilde{g}} \to 0$. Notice that the sequence $\{x_n\}$ is contained in $p_{\mathcal{H}}^{-1}([-1,1]^{b_1}) \cap Str(\tilde{R})$, which is bounded in L^2_k. Hence, after passing to a subsequence, x_n converges to some point x_∞ of $Coul(Y)$ weakly in L^2_k and strongly in L^2_{k-1} by Rellich lemma. Consequently, we have $p_{\mathcal{H}}(x_n) \to p_{\mathcal{H}}(x_\infty)$ and $\widetilde{\text{grad}}\,\mathcal{L}(x_\infty) = 0$ by continuity. This is a contradiction since x_∞ is a critical point of $\widetilde{\text{grad}}\,\mathcal{L}$ and lies in $\bigcup_{j=1}^{b_1} p_{\mathcal{H},j}^{-1}([-s,-r] \cup [r,s])$. □

Note that $\tilde{\epsilon}$ in the above lemma depends on the choice of r, s and $\tilde{R}$. With these data, we choose a smooth "staircase" function $\bar{g} \colon \mathbb{R} \to [0,\infty)$ satisfying the following properties:

(i) $\bar{g}$ is even, i.e. $\bar{g}(x) = \bar{g}(-x)$ for all $x \in \mathbb{R}$;

(ii) There is a positive constant $\bar{\tau}$ such that $\bar{g}(x+1) = \bar{g}(x) + \bar{\tau}$ for all $x \in [0,\infty)$;

(iii) $\bar{g}$ is increasing on the interval $[r,s]$ and $\bar{g}' = 0$ on $[0,r] \cup [s,1]$;

(iv) $|\bar{g}'(x)| < \tilde{\epsilon} \cdot \epsilon''$ for all $x \in \mathbb{R}$, where ϵ'' is a positive constant with the property that
$$\epsilon'' \cdot \|\sum_{j=1}^{b} a_j h_j\|_{L^2} \leq (\sum_{j=1}^{b} a_j^2)^{\frac{1}{2}}, \ \forall (a_1, a_2, \cdots, a_b) \in \mathbb{R}^b. \tag{5.1}$$

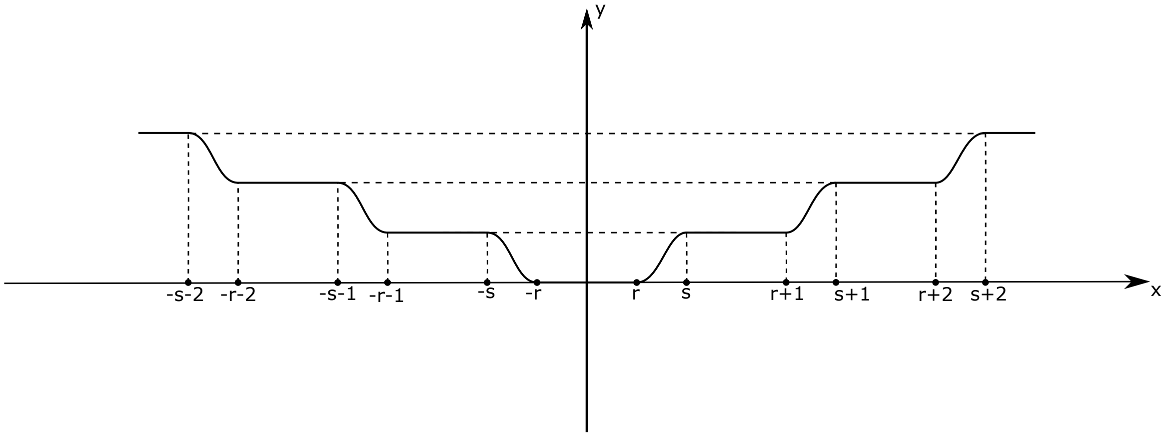


Figure 5.1: the function $\bar{g}$

Next we use the function $\bar{g}$ to define a small perturbation of $\mathcal{L}$ which is not invariant under $\mathcal{G}_Y^h$ but transverse to level sets of $\mathcal{L}$. For each $j = 1, \ldots, b_1$, we define

$$g_{j,+} = \bar{g} \circ p_{\mathcal{H},j} + \mathcal{L} \text{ and } g_{j,-} = \bar{g} \circ p_{\mathcal{H},j} - \mathcal{L}.$$

With our assumptions on $\bar{g}$, we have the following result.

Lemma 5.1.2. *For each $j = 1, \ldots, b_1$, we have*

$$\langle \widetilde{\text{grad}}\, \mathcal{L}(x), \widetilde{\text{grad}}\, g_{j,+}(x) \rangle_{\tilde{g}} \geq 0 \text{ and } \langle \widetilde{\text{grad}}\, \mathcal{L}(x), \widetilde{\text{grad}}\, g_{j,-}(x) \rangle_{\tilde{g}} \leq 0,$$

where the equalities hold only when x is a critical point of $\mathcal{L}$.

Proof. By (2.10) and a straightforward computation, we can prove that

$$\left\| \widetilde{\text{grad}}\, (\bar{g} \circ p_{\mathcal{H},j})(x) \right\|_{\tilde{g}} = \| \text{grad}\, (\bar{g} \circ p_{\mathcal{H},j})(x) \|_{L^2} \leq \frac{1}{\epsilon''} \cdot |\bar{g}'(p_{\mathcal{H},j}(x))| < \tilde{\epsilon}.$$

If $|p_{\mathcal{H},j}(x)| \in [n, n+r]$ or $\left|p_{\mathcal{H}_j}(x)\right| \in [n+s, n+1]$ for some integer n, then $\bar{g}'(p_{\mathcal{H},j}(x)) = 0$ and $\langle \widetilde{\text{grad}}\, \mathcal{L}(x), \widetilde{\text{grad}}\, g_{j,+}(x) \rangle_{\tilde{g}} = \| \widetilde{\text{grad}}\, \mathcal{L}(x) \|_{\tilde{g}}^2$ which is zero if and only if x is a critical point of $\mathcal{L}$. Otherwise, $|p_{\mathcal{H},j}(x)| \in [n+r, n+s]$ for some integer n and Lemma 5.1.1 implies

$$\begin{aligned} \langle \widetilde{\text{grad}}\, \mathcal{L}(x), \widetilde{\text{grad}}\, g_{j,+}(x) \rangle_{\tilde{g}} &= \| \widetilde{\text{grad}}\, \mathcal{L}(x) \|_{\tilde{g}}^2 + \left\langle \widetilde{\text{grad}}\, \mathcal{L}(x), \widetilde{\text{grad}}\, (\bar{g} \circ p_{\mathcal{H},j})(x) \right\rangle_{\tilde{g}} \\ &\geq \| \widetilde{\text{grad}}\, \mathcal{L}(x) \|_{\tilde{g}}^2 - \left\| \widetilde{\text{grad}}\, (\bar{g} \circ p_{\mathcal{H},j})(x) \right\|_{\tilde{g}} \cdot \left\| \widetilde{\text{grad}}\, \mathcal{L}(x) \right\|_{\tilde{g}} \\ &> \| \widetilde{\text{grad}}\, \mathcal{L}(x) \|_{\tilde{g}} \left(\| \widetilde{\text{grad}}\, \mathcal{L}(x) \|_{\tilde{g}} - \tilde{\epsilon} \right) > 0. \end{aligned}$$

The same argument applies to the inner product $\langle \widetilde{\operatorname{grad}}\, \mathcal{L}(x), \widetilde{\operatorname{grad}}\, g_{j,-}(x)\rangle_{\tilde{g}}$.

□

Since the number of critical points of $\mathcal{L}$ is finite modulo gauge, we can find a real number $\theta \in \mathbb{R}$ such that $g_{j,\pm}(x) \neq \theta$ for any critical point x of $\mathcal{L}$ and $j \in \{1, 2, \ldots, b_1\}$. For convenience, we also choose a decreasing sequence of negative real numbers $\{\lambda_n\}$ and an increasing sequence of positive real numbers $\{\mu_n\}$ such that $-\lambda_n, \mu_n \to \infty$. We are now ready to define a collection of bounded sets in $Str(\tilde{R})$.

Definition 5.1.3. With the choice of $\tilde{R}, \bar{g}$ and θ above, we define the sets

$$\begin{aligned} J_m^+ &:= Str(\tilde{R}) \cap \bigcap_{1 \leq j \leq b_1} g_{j,+}^{-1}(-\infty, \theta + m], \\ J_m^- &:= Str(\tilde{R}) \cap \bigcap_{1 \leq j \leq b_1} g_{j,-}^{-1}(-\infty, \theta + m], \end{aligned} \tag{5.2}$$

for each positive integer m. This collection of J_m^+ (resp. J_m^-) will be called a *positive (resp. negative) transverse system*. With the choice of $\{\lambda_n\}$ and $\{\mu_n\}$, we also define

$$J_m^{n,\pm} := J_m^\pm \cap V_{\lambda_n}^{\mu_n}.$$

Notice that the functional $\mathcal{L}$ is bounded on $Str(\tilde{R})$, and the perturbed functional $g_{j,\pm}$ is bounded below on $Str(\tilde{R})$. Since a subset $S \subset Str(\tilde{R})$ is bounded if and only if $p_{\mathcal{H}}(S)$ is bounded, we can see that the set $J_m^\pm$ is bounded in the L_k^2-norm.

We will start to derive some properties of the finite-dimensional bounded sets $J_m^{n,\pm}$. Although some of the following results are slightly stronger than what we need to define the 3-dimensional invariants, they will be useful when we develop the 4-dimensional theory and prove the gluing theorem in [26, 27].

Lemma 5.1.4. *For any positive integer m, there exist positive real numbers ϵ_m, θ_m and an integer $N_m \gg 0$ such that for any $n > N_m$ and $1 \leq j \leq b_1$ we have*

(i) $\langle (l + p_{\lambda_n}^{\mu_n} \circ c)(x), \widetilde{\operatorname{grad}}\, g_{j,+}(x)\rangle_{\tilde{g}} > \epsilon_m$ *for any* $x \in J_m^{n,+} \cap g_{j,+}^{-1}[\theta + m - \theta_m, \theta + m]$*;*

(ii) $\langle (l + p_{\lambda_n}^{\mu_n} \circ c)(x), \widetilde{\operatorname{grad}}\, g_{j,-}(x)\rangle_{\tilde{g}} < -\epsilon_m$ *for any* $x \in J_m^{n,-} \cap g_{j,-}^{-1}[\theta + m - \theta_m, \theta + m]$.

Proof. We only prove this lemma for $g_{1,+}$ and the other cases can be proved similarly. Suppose that the result is not true, then we can find sequences $n_i \to +\infty$, $\epsilon_{m,i}, \theta_{m,i} \to 0$ and $\{x_i\}$ with $x_i \in J_m^{n_i,+} \cap g_{1,+}^{-1}[\theta+m-\theta_{m,i}, \theta+m]$ and $\langle (l+p^{\mu_{n_i}}_{\lambda_{n_i}} \circ c)(x_i), \widetilde{\mathrm{grad}}\, g_{1,+}(x_i)\rangle_{\tilde{g}} \leq \epsilon_{m,i}$. Since $\{x_i\}$ is contained in the L^2_k-bounded set J^+_m, we can pass to a convergent subsequence $x_i \to x_\infty$ in L^2_{k-1} by the Rellich lemma. By continuity, we have $x_\infty \in g_{1,+}^{-1}(\theta+m)$ and $\widetilde{\mathrm{grad}}\, g_{1,+}(x_i) \to \widetilde{\mathrm{grad}}\, g_{1,+}(x_\infty)$ in L^2_{k-2}. Since $p^{\mu_n}_{\lambda_n}$ converges to the identity map pointwise, we also have $(l + p^{\mu_{n_i}}_{\lambda_{n_i}} \circ c)(x_i) \to (l+c)(x_\infty) = \widetilde{\mathrm{grad}}\, \mathcal{L}(x_\infty)$ in L^2_{k-2}. Therefore, we obtain

$$\langle (l + p^{\mu_n}_{\lambda_n} \circ c)(x_n), \widetilde{\mathrm{grad}}\, g_{j,+}(x_n)\rangle_{\tilde{g}} \to \langle \widetilde{\mathrm{grad}}\, \mathcal{L}(x_\infty), \widetilde{\mathrm{grad}}\, g_{j,+}(x_\infty)\rangle_{\tilde{g}},$$

which implies that $\langle \widetilde{\mathrm{grad}}\, \mathcal{L}(x_\infty), \widetilde{\mathrm{grad}}\, g_{j,+}(x_\infty)\rangle_{\tilde{g}} \leq 0$ and x_∞ is a critical point by Lemma 5.1.2. This is a contradiction with the choice of θ. □

Now we start applying the Conley index theory to the flow on $V^{\mu_n}_{\lambda_n}$ generated by the vector field $-(l + p^{\mu_n}_{\lambda_n} \circ c)$. There is a technical point here. Since $V^{\mu_n}_{\lambda_n}$ is non-compact, this flow may go to infinity within a finite time. As in [40], we can fix this by choosing a bump function $\iota_m \colon Coul(Y) \to [0,1]$ for each m such that ι_m is supported in a bounded subset of $Coul(Y)$ and $J^{\pm}_{m+1}$ is contained in the interior of $\iota_m^{-1}(1)$. We denote by φ^n_m the flow on $V^{\mu_n}_{\lambda_n}$ generated by $-\iota_m \cdot (l + p^{\mu_n}_{\lambda_n} \circ c)$. Note that the flow $\varphi^n_{m'}$ on $J^{n,\pm}_m$ does not depend on m' whenever $m' \geq m-1$ so that its invariant set and its Conley index remain unchanged.

Lemma 5.1.5. *For a positive integer M, there exist large numbers N, T such that, for any positive integers $m \leq M$ and $n \geq N$, we have the following statements.*

(a) If $\gamma \colon [-T,T] \to V^{\mu_n}_{\lambda_n}$ is an approximated Seiberg-Witten trajectory contained in $J^{n,+}_m$, then we have

$$\gamma(0) \in Str(R_0) \cap \bigcap_{1 \leq j \leq b_1} g_{j,+}^{-1}(-\infty, \theta + m - \theta_m]. \tag{5.3}$$

In particular, $J^{n,+}_m$ is an isolating neighborhood for the flow φ^n_m.

(b) The set $\mathrm{inv}(\varphi^n_{m-1}, J^{n,+}_{m-1})$ is an attractor in $\mathrm{inv}(\varphi^n_m, J^{n,+}_m)$ with respect to the flow φ^n_m.

Proof. Let $\bar{T}$, $\bar{\lambda}$ and $\bar{\mu}$ be the large constants from Corollary 2.2.8 with $S = J_M^+$. Let θ_m, ϵ_m and N_m be the constants obtained from Lemma 5.1.4 for $m = 1, \dots, M$. Put $T = \max\{\bar{T}, \frac{\theta_1}{\epsilon_1}, \dots, \frac{\theta_M}{\epsilon_M}\} + 1$. We choose a positive integer $N > \max\{N_1, \dots, N_M\}$ such that $\lambda_N < \bar{\lambda}$ and $\mu_N > \bar{\mu}$. Let $m \le M$ and $n > N$ be arbitrary positive integers.

(a) Let $\gamma\colon [-T, T] \to V_{\lambda_n}^{\mu_n}$ be an approximated Seiberg-Witten trajectory contained in $J_m^{n,+}$. Corollary 2.2.8 and the choice of N, T ensure that $\gamma(0) \in Str(R_0)$. For the sake of contradiction, let us suppose that $g_{j,+}(\gamma(0)) > \theta + m - \theta_m$ for some $j \in \{1, \dots, b\}$. By Lemma 5.1.4, the value of $g_{j,+}(\gamma(t))$ decreases along the trajectory γ on $[-T, 0]$ with

$$\frac{dg_{j,+}(\gamma(t))}{dt} = \langle -(l + p_{\lambda_n}^{\mu_n} \circ c)(\gamma(t)), \widetilde{\operatorname{grad}}\, g_{j,+}(\gamma(t))\rangle_{\tilde{g}} < -\epsilon_m.$$

Hence, we obtain $g_{j,+}(\gamma(-T)) > g_{j,+}(\gamma(0)) + T\epsilon_m > \theta + m$ from the fundamental theorem of calculus. This is a contradiction with our assumption that $\gamma(-T) \in J_m^+ \subset g_{j,+}^{-1}(-\infty, \theta + m]$.

(b) From Lemma 5.1.4 and the choice of N, we have $\langle -(l+p_{\lambda_n}^{\mu_n}\circ c)(x), \widetilde{\operatorname{grad}}\, g_{j,+}(x)\rangle_{\tilde{g}} < 0$ for any $x \in J_{m-1}^{n,+} \cap g_{j,+}^{-1}(\theta + m - 1)$. Consequently, the flow φ_m^n goes inside $J_{m-1}^{n,+}$ along $\partial J_{m-1}^{n,+} \setminus \partial Str(\tilde{R})$ and $\operatorname{inv}(\varphi_{m-1}^n, J_{m-1}^{n,+})$ is an attractor in $\operatorname{inv}(\varphi_m^n, J_m^{n,+})$ with respect to the flow φ_m^n. $\square$

Consequently, we can acquire the S^1-equivariant Conley index $I_{S^1}(\varphi_m^n, \operatorname{inv}(J_m^{n,+}))$ from a compact finite-dimensional subset $J_m^{n,+}$ when n is large enough relative to m as in Lemma 5.1.5. Using the orthogonal complement $\bar{V}_\lambda^0$ of $i\Omega_h^1(Y)$ in V_λ^0, we define

$$I_m^{n,+} := \Sigma^{-\bar{V}_{\lambda_n}^0} I_{S^1}(\varphi_m^n, \operatorname{inv}(J_m^{n,+}))$$

as an object of $\mathfrak{C}$. Note that here a choice of index pair for $\operatorname{inv}(J_m^{n,+})$ is made to get the Conley index (see the remark following Definition 3.2.1). Eventually, we will show that our invariants are independent of this choice up to canonical isomorphisms.

Let $i_m^{n,+}\colon I_{S^1}(\varphi_m^n, \operatorname{inv}(J_m^{n,+})) \to I_{S^1}(\varphi_m^n, \operatorname{inv}(J_{m+1}^{n,+}))$ be the attractor map and denote by $\tilde{i}_m^{n,+}$ the morphism

$$\Sigma^{-\bar{V}_{\lambda_n}^0} i_m^{n,+} : I_m^{n,+} \to I_{m+1}^{n,+}. \tag{5.4}$$

We will show that the object $I_m^{n,+}$ is stable in the following sense.

Proposition 5.1.6. *For any positive integer $M > 0$, there exists a positive integer N such that, for any positive integers $m \leq M$ and $n \geq N$, there is a canonical isomorphism $\tilde{\rho}^{n,+}_m \in \operatorname{mor}_{\mathfrak{C}}(I^{n,+}_m, I^{n+1,+}_m)$. Moreover, we have the following commutative diagram*

$$\begin{array}{ccc} I^{n,+}_{m-1} & \xrightarrow{\tilde{i}^{n,+}_{m-1}} & I^{n,+}_m \\ {\scriptstyle \tilde{\rho}^{n,+}_{m-1}} \downarrow & & \downarrow {\scriptstyle \tilde{\rho}^{n,+}_m} \\ I^{n+1,+}_{m-1} & \xrightarrow{\tilde{i}^{n+1,+}_{m-1}} & I^{n+1,+}_m . \end{array} \tag{5.5}$$

Proof. Following the remark after Corollary 2.2.8, we can extend the result of Lemma 5.1.5 to interpolated projections. With the integer N depending on M from Lemma 5.1.5, we can deduce that $J^{n+1,+}_m$ is an isolating neighborhood for the flow generated by

$$-\iota_m \cdot (l + (s p^{\mu_{n+1}}_{\lambda_{n+1}} + (1-s) p^{\mu_n}_{\lambda_n}) \circ c) \tag{5.6}$$

for any $n > N$ and $s \in [0,1]$.

The rest of proof follows from the arguments given in [40, p.910] and [29, Proposition 4.7]. By continuation property of the Conley index, we have a natural homotopy equivalence

$$\rho^{n,+}_m : \Sigma^{V^{\lambda_n}_{\lambda_{n+1}}} I_{S^1}(\varphi^n_m, \operatorname{inv}(J^{n,+}_m)) \to I_{S^1}(\varphi^{n+1}_m, \operatorname{inv}(J^{n+1,+}_m)).$$

The isomorphism $\tilde{\rho}^{n,+}_m$ is then given by the composition

$$\begin{aligned} &\Sigma^{-\bar{V}^0_{\lambda_n}} I_{S^1}(\varphi^n_m, \operatorname{inv}(J^{n,+}_m)) \to \Sigma^{-\bar{V}^0_{\lambda_n}} \Sigma^{-V^{\lambda_n}_{\lambda_{n+1}}} \Sigma^{V^{\lambda_n}_{\lambda_{n+1}}} I_{S^1}(\varphi^n_m, \operatorname{inv}(J^{n,+}_m)) \\ &\quad \to \Sigma^{-\bar{V}^0_{\lambda_n}} \Sigma^{-V^{\lambda_n}_{\lambda_{n+1}}} I_{S^1}(\varphi^{n+1}_m, \operatorname{inv}(J^{n+1,+}_m)) = \Sigma^{-\bar{V}^0_{\lambda_{n+1}}} I_{S^1}(\varphi^{n+1}_m, \operatorname{inv}(J^{n+1,+}_m)), \end{aligned}$$

where the first morphism is given by $\Sigma^{-\bar{V}^0_{\lambda_n}} \eta^{-1}$ and the second morphism equals $\Sigma^{-\bar{V}^0_{\lambda_n}} \rho^{n,+}_m$. The diagram (5.5) commutes because of the continuation property of attractor-repeller pairs [57, Theorem 6.10].

□

For each positive integer M, we pick a positive integer n_M larger than the constant N from Proposition 5.1.6 and we require that $\{n_M\}$ is an increasing sequence. We are now ready to define the spectrum invariant.

Definition 5.1.7. The S^1-equivariant ind-spectrum $\underline{\mathrm{swf}}^A(Y, \mathfrak{s}_Y, A_0, g; S^1)$ is defined to be an object of $\mathfrak{S}$ given by

$$I_1^{n_1,+} \to I_2^{n_2,+} \to I_3^{n_3,+} \to \cdots, \tag{5.7}$$

where the morphism from $I_m^{n_m,+}$ to $I_{m+1}^{n_{m+1},+}$ is a composition $\tilde{i}_m^{n_{m+1},+} \circ \tilde{\rho}_m^{n_{m+1}-1,+} \circ \cdots \circ \tilde{\rho}_m^{n_m,+}$ of the morphisms in Proposition 5.1.6.

We will prove in the next section that this gives a well-defined object of the category $\mathfrak{S}$ independent of the choices made in the construction up to canonical isomorphism.

To define another invariant $\underline{\mathrm{swf}}^R(Y, \mathfrak{s}_Y, A_0, g; S^1)$, we follow almost the same steps for the construction of $\underline{\mathrm{swf}}^A$ except that there are two main differences. First, the set $\mathrm{inv}(\varphi_m^n, J_m^{n,-})$ is a repeller in $\mathrm{inv}(\varphi_m^n, J_{m+1}^{n,-})$, so the arrows in the system will be reversed. Second, we use $V_{\lambda_n}^0$ for desuspension instead of $\bar{V}_{\lambda_n}^0$. We define

$$I_m^{n,-} := \Sigma^{-V_{\lambda_n}^0} I_{S^1}(\varphi_m^n, \mathrm{inv}(J_m^{n,-})) \in \mathrm{ob}\, \mathfrak{C},$$

where n is large enough relative to m, and we have a morphism

$$I_m^{n,-} \leftarrow I_{m+1}^{n,-}$$

induced by the repeller map. The following collection of results can be proved in the same way as the corresponding results for $J_m^{n,+}$.

Proposition 5.1.8. *For a positive integer M, there exist large numbers N, T such that, for any positive integers $m \le M$ and $n \ge N$, we have the following statements.*

(a) For any approximated Seiberg-Witten trajectory $\gamma : [-T, T] \to V_{\lambda_n}^{\mu_n}$ which is contained in $J_m^{n,-}$, we have

$$\gamma(0) \in Str(R_0) \cap \bigcap_{1 \le j \le b_1} g_{j,-}^{-1}(-\infty, \theta + m - \theta_m].$$

In particular, $J_m^{n,-}$ is an isolating neighborhood for the flow φ_n^m.

(b) The set $\mathrm{inv}(\varphi_n^m, J_{m-1}^{n,-})$ is a repeller in $\mathrm{inv}(\varphi_n^m, J_m^{n,-})$ with respect to the flow φ_n^m. Consequently, we have the repeller map

$$i_{m-1}^{n,-} \colon I_{S^1}(\varphi_m^n, \mathrm{inv}(J_m^{n,-})) \to I_{S^1}(\varphi_m^n, \mathrm{inv}(J_{m-1}^{n,-})).$$

(c) *There is a canonical isomorphism* $\tilde{\rho}^{n,-}_m \in \mathrm{mor}_{\mathfrak{C}}(I^{n,-}_m, I^{n+1,-}_m)$ *such that the following diagram commutes*

$$\begin{array}{ccc} I^{n,-}_{m-1} & \xleftarrow{\tilde{i}^{n,-}_{m-1}} & I^{n,-}_m \\ \Big\downarrow{\scriptstyle \tilde{\rho}^{n,-}_{m-1}} & & \Big\downarrow{\scriptstyle \tilde{\rho}^{n,-}_m} \\ I^{n+1,-}_{m-1} & \xleftarrow{\tilde{i}^{n+1,-}_{m-1}} & I^{n+1,-}_m, \end{array} \tag{5.8}$$

where $\tilde{i}^{n,-}_{m-1}$ *is given by* $\Sigma^{-V^0_{\lambda_n}} i^{n,-}_{m-1}$.

For each positive integer M, we also choose a positive integer n_M larger than the constant N from Proposition 5.1.8 so that $\{n_M\}$ is an increasing sequence.

Definition 5.1.9. The S^1-equivariant pro-spectrum $\underline{\mathrm{swf}}^R(Y, \mathfrak{s}_Y, A_0, g; S^1)$ is defined to be an object of $\mathfrak{S}^*$ given by

$$I^{n_1,-}_1 \leftarrow I^{n_2,-}_2 \leftarrow I^{n_3,-}_3 \leftarrow \cdots, \tag{5.9}$$

where the connecting morphisms are defined in the same manner as in Definition 5.1.7.

We will also prove well-definedness of $\underline{\mathrm{swf}}^R(Y, \mathfrak{s}_Y, A_0, g; S^1)$ in the next section.

5.2 The torsion case

When the spin^c structure $\mathfrak{s}$ is torsion, we will be able to further normalize the spectrum invariants $\underline{\mathrm{swf}}^A$ and $\underline{\mathrm{swf}}^R$ following the idea of [40]. The resulting objects will not depend on A_0 and g.

We will need to define a rational number $n(Y, \mathfrak{s}_Y, A_0, g)$. Choose a 4-manifold X with boundary Y with $H^3(X, Y; \mathbb{Z}) \cong H_1(X; \mathbb{Z}) = 0$. Such X always exists as we can construct X by attaching 2-handles on D^4 according the surgery diagram of Y. By the homology long exact sequence for the pair (X, Y), we see that $H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is surjective. Therefore, we can extend $\mathfrak{s}$ to a spin^c structure $\mathfrak{s}_X$ over X and extend A_0 to a connection $\hat{A}_0$ over X. Recall that we have a nondegenerate pairing

$$\cup : \mathrm{im}(H^2(X, Y; \mathbb{Q}) \to H^2(X; \mathbb{Q})) \otimes \mathrm{im}(H^2(X, Y; \mathbb{Q}) \to H^2(X; \mathbb{Q})) \to \mathbb{Q}.$$

Denote by $b^+(X)$ (resp. $b^-(X)$) the dimension of a maximal positive (resp. negative) subspace with respect to this pairing and denote by $\sigma(X)$ the signature of this pairing. Notice that we can define $c_1(\mathfrak{s}_X)^2 = c_1(\mathfrak{s}_X) \cup c_1(\mathfrak{s}_X) \in \mathbb{Q}$ because $c_1(\mathfrak{s}_X)|_Y = c_1(\mathfrak{s})$ is torsion. We define

$$n(Y, \mathfrak{s}, A_0, g) := \mathrm{Ind}_{\mathbb{C}}(\hat{\not{D}}^+_{\hat{A}_0}) - \frac{c_1(\mathfrak{s}_X)^2 - \sigma(X)}{8}, \tag{5.10}$$

where $\hat{\not{D}}^+_{\hat{A}_0}$ is the positive Dirac operator on X coupled with $\hat{A}_0$ and $\mathrm{Ind}_{\mathbb{C}}(\hat{\not{D}}^+_{\hat{A}_0})$ is its index defined by using spectral boundary condition as in [5]. It was proved in [40] that $n(Y, \mathfrak{s}, A_0, g)$ does not depend on the choices of $X, \mathfrak{s}_X$ and $\hat{A}_0$ ([40] only considered a rational homology sphere Y but the proof works for a general 3-manifold Y without any changes). In fact, we have

$$n(Y, \mathfrak{s}, A_0, g) = \frac{1}{2}\left(\eta(\not{D}) - \dim_{\mathbb{C}}(\ker \not{D}) + \frac{\eta_{\mathrm{sign}}}{4}\right), \tag{5.11}$$

where $\eta(\not{D})$ and η_{sign} denote the eta-invariant of the Dirac operator and the odd signature operator respectively (see [40] and [5]).

The normalized invariant $\underline{\mathrm{SWF}}^A$ and $\underline{\mathrm{SWF}}^R$ will be obtained by formally desuspending $\underline{\mathrm{swf}}^A$ and $\underline{\mathrm{swf}}^R$ with the rational number $n(Y, \mathfrak{s}, A_0, g)$ as follows.

Definition 5.2.1. We define the S^1-equivariant ind-spectrum and pro-spectrum by

$$\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}; S^1) := \left(\underline{\mathrm{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1), 0, n(Y, \mathfrak{s}, A_0, g)\right),$$
$$\underline{\mathrm{SWF}}^R(Y, \mathfrak{s}; S^1) := \left(\underline{\mathrm{swf}}^R(Y, \mathfrak{s}, A_0, g; S^1), 0, n(Y, \mathfrak{s}, A_0, g)\right).$$

as objects of $\mathfrak{S}$ and $\mathfrak{S}^*$ respectively.

The proof of invariance of $\underline{\mathrm{SWF}}^A$ and $\underline{\mathrm{SWF}}^R$ will also be in the next section.

5.3 The Pin(2)-spectrum invariants for spin structures

In this subsection, we will define Pin(2)-analogue of the spectrum invariants for a 3-manifold Y equipped with a spin structure $\mathfrak{s}$. Since all the constructions are similar to the S^1-case, some of the discussions will be brief. Recall that we defined the categories $\mathfrak{C}_{Pin(2)}$, $\mathfrak{S}_{Pin(2)}$ and $\mathfrak{S}^*_{Pin(2)}$ in Chapter 3.1, which are the Pin(2)-version of the categories $\mathfrak{C}, \mathfrak{S}$ and $\mathfrak{S}^*$. Our spectrum invariants will be objects of $\mathfrak{S}_{Pin(2)}$ and $\mathfrak{S}^*_{Pin(2)}$.

The spin structure $\mathfrak{s}$ induces a torsion spin^c structure on Y. With a slight abuse of notations, we also denote this spin^c structure by $\mathfrak{s}$. We will have the same setup from the spin^c structure $\mathfrak{s}$ with the following new features coming from a spin structure.

1. The structure group of S_Y can be reduced to $SU(2) \cong S(\mathbb{H})$. Therefore, S_Y is a quaternionic bundle. Here we follow the convention of [43] and let the structure group act by the right multiplication.

2. The bundle $\det(S_Y)$ has a canonical trivialization. The Levi-Civita connection on TY then induces a canonical spin connection A_0 on S_Y with $F_{A_0^t} = 0$. We will always choose A_0 for our base connection.

3. We have an additional action $\jmath : \mathcal{C}_Y \to \mathcal{C}_Y$ sending (a, ϕ) to $(-a, j\phi)$. This action, together with the constant gauge group S^1, gives a Pin(2)-action on $\mathcal{C}_Y$. All the objects in the setup are Pin(2)-invariant, e.g. the functional CSD_{ν_0}, the Coulomb slice $Coul(Y)$ and the L^2_k-inner product etc.

In order to respect the additional $\jmath$-symmetry, we have two new requirements in our construction.

1. The perturbation f should be invariant under $\jmath$. In other worlds, we should have $f(a, \phi) = f(-a, j\phi)$.

2. The sets $J_m^{n,\pm}$ should be invariant under $\jmath$.

A slight adaption of [37, Theorem 2.6] shows that for any real number δ, we can find a $\jmath$-invariant extended cylinder function $\bar{f}$ such that $(\delta, \bar{f})$ is a good perturbation. Since we required the staircase function $\bar{g}$ from Section 5.1 is even, it is not hard to see that $J_m^{n,\pm}$ is $\jmath$-invariant once the perturbation f is $\jmath$-invariant.

We can now follow the construction from Section 5.1 . In particular, the sets $J_m^{n,\pm}$ are isolating neighborhoods for the Pin(2)-invariant flow φ_m^n when n is sufficiently large relative to m and we define

$$I_m^{n,+}(Pin(2)) := \Sigma^{-\bar{V}^0_{\lambda_n}} I_{Pin(2)}(\varphi_n^m, \mathrm{inv}(J_m^{n,+})),$$
$$I_m^{n,-}(Pin(2)) := \Sigma^{-V^0_{\lambda_n}} I_{Pin(2)}(\varphi_n^m, \mathrm{inv}(J_m^{n,-}))$$

as objects of $\mathfrak{C}_{Pin(2)}$. As before, we obtain an object $\underline{\mathrm{swf}}^A(Y, \mathfrak{s}, A_0, g; Pin(2))$ of $\mathfrak{S}_{Pin(2)}$ given by

$$I_1^{n_1,+}(Pin(2)) \to I_2^{n_2,+}(Pin(2)) \to \cdots$$

and an object $\underline{\mathrm{swf}}^R(Y, \mathfrak{s}, A_0, g; Pin(2))$ of $\mathfrak{S}^*_{Pin(2)}$ given by

$$I_1^{n_1,-}(Pin(2)) \leftarrow I_2^{n_2,-}(Pin(2)) \leftarrow \cdots$$

for an increasing sequence of large positive integers $\{n_i\}$. We define spectrum invariants as in the torsion spin^c case.

Definition 5.3.1. With the above setup, the Pin(2)-equivariant ind-spectrum and pro-spectrum are defined by

$$\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}; Pin(2)) := \left(\underline{\mathrm{swf}}^A(Y, \mathfrak{s}, A_0, g; Pin(2)), 0, \frac{n(Y, \mathfrak{s}, A_0, g)}{2}\right),$$
$$\underline{\mathrm{SWF}}^R(Y, \mathfrak{s}; Pin(2)) := \left(\underline{\mathrm{swf}}^R(Y, \mathfrak{s}, A_0, g; Pin(2)), 0, \frac{n(Y, \mathfrak{s}, A_0, g)}{2}\right).$$

as objects of $\mathfrak{S}_{Pin(2)}$ and $\mathfrak{S}^*_{Pin(2)}$ respectively. Here $n(Y, \mathfrak{s}, A_0, g)$ is the rational number defined in (5.10). As before, these objects are independent of the choices made in the construction up to canonical isomorphism.

CHAPTER 6

The invariance for the unfolded spectrum

In this chapter we will prove the invariance of our ind-spectrum (pro-spectrum). In other words, we will show that the spectra given by different choices of parameters are canonically isomorphic to each other (as objects of the category in which they are defined). We focus on the S^1-equivariant case and the Pin(2)-case can be proved in the same way.

First, let us list the parameters in the order that the choices of a parameter can only depend on the parameters listed before it (for example, $\tilde{R}$ is any number greater R_0, where R_0 is the constant of Theorem 2.2.2 depending on g, A_0 and f):

(I) The Riemannian metric g and the base connection A_0;

(II) The good perturbation $f\colon Coul(Y) \to \mathbb{R}$;

(III) The sequences of real numbers $\{\lambda_n\}, \{\mu_n\}$;

(IV) The number $\tilde{R}$ (in the definition of $Str(\tilde{R})$);

(V) The harmonic forms $\{h_j\}$, the cutting function $\bar{g}$ and the cutting value θ;

(VI) The positive integers n_m in (5.7) and (5.9);

(VII) The Conley indices for the isolated invariant sets.

The invariance for (VII) is a direct consequence of the invariance of the Conley index (see Subsection 3.2 and [57]). The commutative diagrams (5.5) and (5.8) imply the invariance for (VI).

In subsection 6.1, we will make a digression into the discussion of the finite dimensional approximation for a family of flows. In subsection 6.2, we will prove the invariance for (III), (IV), (V). The invariance for (II) (which is the most interesting one) and (I) will be proved in subsection 6.3 and subsection 6.4 respectively. In subsection 6.5, we will discuss the restriction of our invariant to the S^1-fixed point sets.

6.1 The finite dimensional approximation for a family of flows

In this subsection, we extend finite dimensional approximation results in Section 2.2 for a continuous family of flows. This setup will be useful for proving the invariance and calculating examples.

Let S be a compact manifold (possibly with boundary) and consider a smooth family of Riemannian metrics $\{g_s\}_{s\in S}$ and a smooth family of base connections $\{A_{0,s}\}_{s\in S}$. As before, we require that $\frac{i}{2\pi}F_{A^t_{0,s}}$ equals the harmonic form representing $c_1(\mathfrak{s})$. We denote by $Coul(Y,s)$ the (L^2_k-completed) Coulomb slice for $(\tilde{g}_s, A_{0,s})$. For each s, we have an elliptic operator $l_s\colon Coul(Y,s)\to Coul(Y,s)$ given by $(*d, \not{D}_{A_{0,s}})$. Although $\{Coul(Y,s)|s\in S\}$ is a Hilbert bundle over S, by the Kuiper's theorem, this bundle is trivial and we can identify it with $S\times Coul(Y)$ by fixing a trivialization. We have the following generalization of Definition 2.2.5:

Definition 6.1.1. Let E be a vector bundle over Y. A family of smooth and bounded maps $\{Q_s\colon Coul(Y,s)\to L^2_k(\Gamma(E))\}_{s\in S}$ is called a *continuous family of quadratic-like maps* if Q_s is quadratic-like for each $s\in S$ and, for each nonnegative integer $m<k$, we have a uniform convergence $(\frac{d}{dt})^m Q_{s_n}(\gamma_n(t))\to(\frac{d}{dt})^m Q_{s_\infty}(\gamma_\infty(t))$ in L^2_{k-2-m} whenever there is a uniform convergent of compact paths $(\frac{d}{dt})^j\gamma_n(t)\to(\frac{d}{dt})^j\gamma_\infty(t)$ uniformly in L^2_{k-1-j} for each $j=0,1,\cdots,m$ with $\gamma_n\colon I\to Coul(Y,s_n)$ and $s_n\to s_\infty$.

We now let $\{Q_s\colon Coul(Y,s)\to L^2_k(\ker d^*\oplus\Gamma(S_Y))\}_{s\in S}$ be a continuous family of quadratic-like maps. As before, for real numbers $\lambda<0\le\mu$, we define $V^\mu_\lambda(s)\subset Coul(Y,s)$ to be the space spanned by the eigenvectors of l_s with eigenvalue in $(\lambda,\mu]$.

We also consider $\bar{V}_\lambda^0(s)$, which is the orthogonal complement of $i\Omega_h^1(Y)$ in $V_\lambda^0(s)$. Note that these spaces usually do not change continuously with s because the dimension can jump at eigenvalues of l_s.

Throughout this subsection, we say that, for an interval I, a path $\gamma\colon I \to Coul(Y,s)$ is an actual trajectory if it satisfies $\frac{d}{dt}\gamma(t) = -(l+Q_s)\gamma(t)$ and a path $\gamma\colon I \to V_\lambda^\mu(s)$ is an approximated trajectory if it satisfies $\frac{d}{dt}\gamma(t) = -(l+p_\lambda^\mu \circ Q_s)\gamma(t)$ for some μ, λ. We denote by $\varphi(\lambda,\mu,s)$ the flow generated by $-\iota\cdot(l+p_\lambda^\mu\circ Q_s)$, where ι is a bump function which equals 1 on any bounded subset involved in our discussion.

Theorem 6.1.2. *Let B be a closed and bounded subset of $Coul(Y)$ and suppose that there exists a closed subset $A \subset int(B)$ such that, for any $s \in S$ and any actual trajectory $\gamma\colon \mathbb{R} \to Coul(Y,s)$ contained in B, we have γ contained in* $\mathrm{int}(A)$. *Then there exist constants $T, -\bar{\lambda}, \bar{\mu} \gg 0$ such that the following statements hold:*

(i) For any $\lambda < \bar{\lambda}$, $\mu > \bar{\mu}$ and $s \in S$, if an approximated trajectory $\gamma\colon [-T,T] \to V_\lambda^\mu(s)$ is contained in B, then we have $\gamma(0) \in A$. In particular, $B \cap V_\lambda^\mu$ is an isolating neighborhood for the flow $\varphi(\lambda,\mu,s)$;

(ii) The spectra $\Sigma^{-V_\lambda^0(s)} I_{S^1}(\varphi(\lambda,\mu,s), \mathrm{Inv}(B\cap V_\lambda^\mu(s)))$ and $\Sigma^{-\bar{V}_\lambda^0(s)} I_{S^1}(\varphi(\lambda,\mu,s), \mathrm{Inv}(B\cap V_\lambda^\mu(s)))$ do not depend on the choice of $\lambda < \bar{\lambda}$ and $\mu > \bar{\mu}$ up to canonical isomorphisms in $\mathfrak{C}$. We denote these objects by $I(B,s)$ and $\bar{I}(B,s)$ respectively.

(iii) For any path $\alpha\colon [0,1] \to S$, we have well defined isomorphisms

$$\rho(B,\alpha)\colon I(B,\alpha(0)) \to \Sigma^{\mathrm{sf}(-\not{D},\alpha)\mathbb{C}} I(B,\alpha(1)),$$

$$\bar{\rho}(B,\alpha)\colon \bar{I}(B,\alpha(0)) \to \Sigma^{\mathrm{sf}(-\not{D},\alpha)\mathbb{C}} \bar{I}(B,\alpha(1)),$$

where $\mathrm{sf}(-\not{D},\alpha)$ denotes the spectral flow of $-\not{D}$ along the path α. Moreover, the isomorphisms ρ and $\bar{\rho}$ only depend on the homotopy class of α relative to its end points.

Proof. For the first part, the proof is similar to that of Corollary 2.2.8: we suppose there exists no such $\bar{\lambda}, \bar{\mu}, T$. Then we can find a sequence of approximated trajectories

$\gamma_n\colon [-T_n, T_n] \to Coul(Y, s_n)$ with $T_n, -\lambda_n, \mu_n \to +\infty$ such that γ_n is contained in B but $\gamma_n(0) \notin A$. Since S is compact, we can assume $s_n \to s_\infty$ after passing to a subsequence. The properties in Definition 6.1.1 allow us to repeat the argument in the proof of Proposition 2.2.4 and find an actual trajectory $\gamma_\infty\colon \mathbb{R} \to Coul(Y, s_\infty)$ as the limit of γ_n. Consequently, we have γ contained in B and $\gamma_\infty(0) \notin \mathrm{int}(A)$. This is a contradiction with our hypothesis. Thus, the proof of (i) is finished.

The proof of (ii) is a straight forward adaption of arguments from Proposition 5.1.6 and we omit it. For (iii), we will focus on the case $\rho(B, \alpha)$ as the other case can be proved similarly. For brevity, we will denote by $E^\mu_\lambda(s)$ the Conley index $I_{S^1}(\varphi(\lambda, \mu, s), \mathrm{Inv}(B \cap V^\mu_\lambda(s)))$. The isomorphism $\rho(B, \alpha)$ is constructed as follows: we consider the interval $[0, 1]$ as the union of subintervals $[t_j, t_{j+1}]$ with $j = 1, \cdots, m$ such that, for each j, we can find $\mu_j > \bar{\mu}$ and $\lambda_j < \bar{\lambda}$ which are not eigenvalues of $l_{\alpha(t)}$ for any $t \in [t_j, t_{j+1}]$. Then $V^{\mu_j}_{\lambda_j}(\alpha(t))$ from $t = t_j$ to $t = t_{j+1}$ is a continuous family of linear subspaces and $\varphi(\lambda_j, \mu_j, \alpha(t))$ is a continuous family of flows on them. By the homotopy invariance of the Conley index [57, Section 6], we get an isomorphism

$$\rho_j\colon E^{\mu_j}_{\lambda_j}(\alpha(t_j)) \xrightarrow{\cong} E^{\mu_j}_{\lambda_j}(\alpha(t_{j+1})). \tag{6.1}$$

Notice that

$$[V^0_{\lambda_j}(\alpha(t_j))] + [\mathrm{sf}(-\not{D}, \alpha([t_j, t_{j+1}]))\mathbb{C}] = [V^0_{\lambda_j}(\alpha(t_{j+1}))]$$

as elements of the representation ring of S^1. We can desuspend both sides of (6.1) and get an isomorphism

$$I(B, \alpha(t_j)) \to \Sigma^{\mathrm{sf}(-\not{D}, \alpha([t_j, t_{j+1}]))\mathbb{C}} I(B, \alpha(t_{j+1})).$$

The isomorphism $\rho(B, \alpha)$ is defined as the composition of the above isomorphisms for $j = 1, \ldots, m$.

We will see that $\rho(B, \alpha)$ is independent of the choices of t_j, λ_j and μ_j. First, fix a choice of $\{t_j\}$ and choose different choices of $\{\lambda'_j\}$ and $\{\mu'_j\}$. Without loss of generality, we may assume that $\lambda_j < \lambda'_j$, $\mu_j > \mu'_j$. As before, we have an isomorphism

$$\rho'_j\colon E^{\mu'_j}_{\lambda'_j}(\alpha(t_j)) \xrightarrow{\cong} E^{\mu'_j}_{\lambda'_j}(\alpha(t_{j+1})).$$

As in Proposition 5.1.6, we have isomorphisms for stability of conley indices

$$\sigma_j \colon E^{\mu_j}_{\lambda_j}(\alpha(t_j)) \xrightarrow{\cong} \Sigma^{V^{\lambda'_j}_{\lambda_j}} E^{\mu'_j}_{\lambda'_j}(\alpha(t_j)),$$
$$\sigma_{j+1} \colon E^{\mu_j}_{\lambda_j}(\alpha(t_{j+1})) \xrightarrow{\cong} \Sigma^{V^{\lambda'_j}_{\lambda_j}} E^{\mu'_j}_{\lambda'_j}(\alpha(t_{j+1})).$$

Using the formula in [57, Theorem 6.7], we can easily see that $\sigma_{j+1} \circ \rho_j$ is S^1-equivariantly homotopic to $\rho'_j \circ \sigma_j$. This implies that $\Sigma^{-V^0_{\lambda_j}} \rho_j$ and $\Sigma^{-V^0_{\lambda'_j}} \rho'_j$ are equal to each other as morphisms in $\mathfrak{C}$. Therefore $\rho(B, \alpha)$ does not depended on the choices of $\{\lambda_j\}$ and $\{\mu_j\}$. Next we prove the independence of the choice of $\{t_j\}$. Let us pick another sequence $\{t'_j\}_{j=1}^{m'}$. Without loss of generality, we will only work on the case $\{t'_j\} \subset \{t_j\}$, i.e. $\{t_j\}$ is a finer subdivision. Let us suppose that

$$t_j = t'_{j'} < t_{j+1} < t_{j+2} = t'_{j'+1}$$

for some $j' \in \{1, \dots, m'\}$. An equivariant version of [57, Corollary 6.8] implies that $\rho_{j+1} \circ \rho_j$ is S^1-equivariantly homotopic to $\rho'_{j'}$. This discussion implies that $\rho(B, \alpha)$ is independent of the choice of $\{t_j\}$.

Now suppose that we have two paths α_0, α_1 which are homotopic to each other relative to their end points by a homotopy α_u as $u \in [0, 1]$. For any $(t_0, u_0) \in [0, 1]^2$, one can also find $\mu > \bar{\mu}$ and $\lambda < \bar{\lambda}$ and a small neighborhood O of (t_0, u_0) such that μ, λ are not eigenvalues of $l_{\alpha_u(t)}$ for any (t, u) in O. By the definition of ρ and the homotopy invariance of the Conley index, we see that $\rho(B, \alpha_u)$ does not change as u varies inside O. By considering a finite cover of $[0, 1]^2$ by such neighborhoods, we see that $\rho(B, \alpha_0) = \rho(B, \alpha_1)$. This finishes the proof of the theorem. □

The following corollary is directly implied by the homotopy invariance of the attractor-repeller map.

Corollary 6.1.3. *Let $B_1 \subset B_2$ be two closed and bounded sets both satisfying the hypothesis of Theorem 6.1.2. Suppose that for any sufficiently large $-\lambda, \mu$ and any $s \in S$, the set $\mathrm{Inv}(\varphi(\lambda, \mu, s), B_1 \cap V^{\mu}_{\lambda}(s))$ is an attractor in $\mathrm{Inv}(\varphi(\lambda, \mu, s), B_2 \cap V^{\mu}_{\lambda}(s))$. Then the desuspensions of the corresponding attractor maps give well defined morphisms*

$i(s) : I(B_1, s) \to I(B_2, s)$ and $\bar{i}(s) : \bar{I}(B_1, s) \to \bar{I}(B_2, s)$. Moreover, for any path $\alpha : [0, 1] \to S$, we have

$$\rho(B_2, \alpha) \circ i(\alpha(0)) = (\Sigma^{\mathrm{sf}(-\not{D},\alpha)\mathbb{C}} i(\alpha(1))) \circ \rho(B_1, \alpha),$$

$$\bar{\rho}(B_2, \alpha) \circ \bar{i}(\alpha(0)) = (\Sigma^{\mathrm{sf}(-\not{D},\alpha)\mathbb{C}} \bar{i}(\alpha(1))) \circ \bar{\rho}(B_1, \alpha).$$

The repeller version of this result also holds given that $\mathrm{Inv}(\varphi(\lambda, \mu, s), B_1 \cap V_\lambda^\mu(s))$ *is a repeller in* $\mathrm{Inv}(\varphi(\lambda, \mu, s), B_2 \cap V_\lambda^\mu(s))$ *for any $s \in S$.*

6.2 The invariance for (III),(IV),(V)

Notice that the three parameters in (V) only affect our results through the definition of the bounded set $J_m^\pm$. Suppose that we choose two different triples of parameters $(\{h_j\}, \bar{g}, \theta)$ and $(\{\tilde{h}_j\}, \tilde{g}, \tilde{\theta})$ and use them to define the sets J_m^+ and $\tilde{J}_m^+$ respectively. From these subsets, we construct two direct systems, which we denote by (5.7) and (5.7') respectively. Notice that J_m^+ and $\tilde{J}_m^+$ are bounded subsets of $Str(\tilde{R})$. We can find $0 < m_1 < m_2 \cdot$ and $0 < \tilde{m}_1 < \tilde{m}_2 < \cdot$ such that:

$$J_{m_1}^+ \subset \tilde{J}_{\tilde{m}_1}^+ \subset J_{m_2}^+ \subset \tilde{J}_{\tilde{m}_2}^+ \subset \cdots , \tag{6.2}$$

which also implies the following inclusions for any positive integer n

$$J_{m_1}^{n,+} \subset \tilde{J}_{\tilde{m}_1}^{n,+} \subset J_{m_2}^{n,+} \subset \tilde{J}_{\tilde{m}_2}^{n,+} \subset \cdots$$

Notice that for any $j > 0$ and any n, m large enough relative to $m_j, \tilde{m}_j$. The flow φ_m^n goes inside $J_{m_j}^{n,+}$ and $\tilde{J}_{\tilde{m}_j}^{n,+}$ along $\partial J_{m_j}^{n,+} \setminus \partial Str(\tilde{R})$ and $\partial \tilde{J}_{\tilde{m}_j}^{n,+} \setminus \partial Str(\tilde{R})$ respectively. Therefore, the attractor maps, together with the isomorphisms $\tilde{\rho}_*^{*,+}$ (as defined in Proposition 5.1.6) give a direct system in the category $\mathfrak{C}$

$$I_{m_1}^{n_1,+} \to \tilde{I}_{\tilde{m}_1}^{\tilde{n}_1,+} \to I_{m_2}^{n_2,+} \to \tilde{I}_{\tilde{m}_2}^{\tilde{n}_2,+} \to \cdots \tag{6.3}$$

for suitable choices of $n_1 < \tilde{n}_1 < n_2 < \tilde{n}_2 < \cdots$, where the connecting maps are defined in a similar way as (5.7). Since the attractor maps are transitive as mentioned after Proposition 3.2.4, the composition of the connecting morphisms $I_{m_j}^{n_{m_j},+} \to \tilde{I}_{\tilde{m}_j}^{\tilde{n}_j,+} \to$

$I^{n_{j+1},+}_{m_{j+1}}$ is the same as the attractor map for $\mathrm{Inv}(J^{n,+}_{m_j}) \subset \mathrm{Inv}(J^{n,+}_{m_{j+1}})$. Therefore, we see that (6.3) contains both a subsystem of (5.7) and a subsystem of (5.7'). By Lemma 3.1.1, this implies that (5.7) and (5.7') are canonically isomorphic as objects of $\mathfrak{S}$. In other words, up to canonical isomorphisms, the spectrum invariants $\underline{\mathrm{swf}}^A$ and $\underline{\mathrm{SWF}}^A$ do not depend on the choice of $\{h_j\}, \bar{g}$ and θ. The case of $\underline{\mathrm{swf}}^R$ and $\underline{\mathrm{SWF}}^R$ can be shown similarly. We have proved the invariance for (V).

The proof of the invariance for (IV) is easy: Let $\tilde{R}_0 < \tilde{R}_1$ be two numbers which are both larger than the constant R_0 from Theorem 2.2.2. Notice that when we choose a suitable choice of parameters $(\{h_j\}, \bar{g}, \theta)$ for $\tilde{R}_1$, these parameters also work for $\tilde{R}_0$ since $\tilde{R}_0 < \tilde{R}_1$. Denote by $J^{n,\pm}_m(\tilde{R}_i)$ the corresponding bounded set corresponding for $i = 0, 1$. Then it is straightforward to see that, for any positive integer m and any sufficiently large integer n (relative to m), the sets $J^{n,\pm}_m(\tilde{R}_0)$ and $J^{n,\pm}_m(\tilde{R}_1)$ are both isolating neighborhoods of the same isolated invariant set. Therefore, their Conley indices are related to each other by canonical isomorphims which are compatible with attractor-repeller maps. This implies the invariance for (IV).

Remark. Actually, from the above argument, we can replace $Str(R)$ in our construction with any set $C \subset Coul(Y)$ satisfying the following conditions:

1. For any bounded subset $A \subset i\Omega^1_h(Y)$, the set $p^{-1}_{\mathcal{H}}(A) \cap C$ is also bounded;

2. Any finite type Seiberg-Witten trajectory is contained in the interior of C.

Also, we can define $\{J^{\pm}_m\}$ to be any sequence of bounded, closed subsets of C such that $J^{\pm}_m \subset J^{\pm}_{m+1}$, $\cup_{m=1}^{\infty} J^{\pm}_m = C$ and for any $m > 0$ and n large enough relative to m the flow φ^n_m goes inside (resp. outside) $J^{n,+}_m$ (resp. $J^{n,-}_m$) along $\partial J^{n,+}_m \setminus \partial C$ (resp. $\partial J^{n,-}_m \setminus \partial C$).

As for (III), we choose different sequences $\{\lambda_n\}, \{\mu_n\}$ and $\{\tilde{\lambda}_n\}, \{\tilde{\mu}_n\}$. By Lemma 3.1.1, we can pass to their subsequences and assume that $\lambda_{n+1} < \tilde{\lambda}_n < \lambda_n$ and $\mu_n < \tilde{\mu}_n < \mu_{n+1}$ for any n. Let $I^{n,+}_m$ and $\tilde{I}^{n,+}_m$ be the objects of $\mathfrak{C}$ obtained by desuspending the Conley indices corresponding to $\{\lambda_n\}, \{\mu_n\}$ and $\{\tilde{\lambda}_n\}, \{\tilde{\mu}_n\}$ respectively. We can repeat the proof of Proposition 5.1.6 and establish canonical isomorphisms $I^{n,+}_m \cong \tilde{I}^{n,+}_m$ and $I^{n+1,+}_m \cong \tilde{I}^{n+1,+}_m$ for any positive integer m and any sufficiently large integer n (relative to m). Moreover, they form commutative diagrams similar to (5.5). This implies that $\underline{\text{swf}}^A$ and $\underline{\text{SWF}}^A$ are independent of (III). The repeller case follows in the same manner.

6.3 The invariance for (II)

In this subsection, we will consider any two choices of good perturbation $f_j \colon \mathcal{C}_Y \to \mathbb{R}$ for $j = 1, 2$. Recall that $f_j(a, \phi) = \frac{\delta_j}{2}\|\phi\|^2_{L^2} + \bar{f}_j(a, \phi)$, where δ_j is a real constant and $\bar{f}_j$ is an extended cylinder function. We first assume that $\delta_1 = \delta_2 = \delta$. Since we do not know whether the space of good perturbation is path connected, the usual homotopy invariance argument does not work. Therefore, we follow a different approach here. Because the whole argument is relatively long and technical, we first sketch the the rough idea as follows.

Denote by $\mathcal{L}_j$ the restriction of CSD_{ν_0, f_j} to $Coul(Y)$. Recall that we identify $i\Omega^1_h(Y)$ with $\mathbb{R}^{b_1}$ by choosing independent harmonic forms $\{h_j\}$. For any real number $e \geq 1$, we will construct a family of "mixed" functionals $\pounds^s_e$ for $s \in [0, 1]$ such that $\pounds^1_e = \mathcal{L}_2$ and $\pounds^0_e$ equals $\mathcal{L}_1$ on $p^{-1}_{\mathcal{H}}([-e+1, e-1]^{b_1})$ and equals $\mathcal{L}_2$ on $p^{-1}_{\mathcal{H}}(\mathbb{R}^{b_1} \setminus (-e, e)^{b_1})$. Suppose that all finite type flow lines of $\pounds^s_e$ are contained in $Str(\tilde{R})$ and consider an increasing sequence of bounded subsets

$$J^+_{m_1} \subset \tilde{J}^+_{\tilde{m}_1} \subset J^+_{m_2} \subset \tilde{J}^+_{\tilde{m}_2} \subset \cdots$$

where $J^+_{m_j}$ and $\tilde{J}^+_{m_j}$ are the bounded subsets of $Str(\tilde{R})$ corresponding to $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. We will require that, for each positive integer j, there exists a real number $e_j \geq 1$ satisfying

$$J^+_{m_j} \subset p^{-1}_{\mathcal{H}}([-e_j + 1, e_j - 1]^b) \cap Str(\tilde{R}) \subset p^{-1}_{\mathcal{H}}([-e_j, e_j]^b) \cap Str(\tilde{R}) \subset J^+_{\tilde{m}_j}.$$

Let $\varphi^n(\mathcal{L})$ be the approximated gradient flow of $\mathcal{L}$ on the compact set $J^{n,+}_m$. Since $\mathcal{L}_1$ equals $\pounds^0_{e_j}$ when restricted to $J^+_{m_j}$ and the flow goes inside $J^{n,+}_{m_j}$, we have an attractor map

$$I_{S^1}(\varphi^n(\mathcal{L}_1), \mathrm{Inv}(J^{n,+}_{m_j})) = I_{S^1}(\varphi^n(\pounds^0_{e_j}), \mathrm{Inv}(J^{n,+}_{m_j})) \to I_{S^1}(\varphi^n(\pounds^0_{e_j}), \mathrm{Inv}(\tilde{J}^{n,+}_{\tilde{m}_j})).$$

On the other hand, we have $I_{S^1}(\varphi^n(\mathcal{L}^0_{e_j}), \mathrm{Inv}(\tilde{J}^{n,+}_{\tilde{m}_j})) \cong I_{S^1}(\varphi^n(\mathcal{L}_2), \mathrm{Inv}(\tilde{J}^{n,+}_{\tilde{m}_j}))$ by continuity of Conley indices. We combine these and obtain a map

$$I_{S^1}(\varphi^n(\mathcal{L}_1), \mathrm{Inv}(J^{n,+}_{m_j})) \to I_{S^1}(\varphi^n(\mathcal{L}_2), \mathrm{Inv}(\tilde{J}^{n,+}_{\tilde{m}_j})).$$

We also construct another family of functionals $\tilde{\mathcal{L}}^s_e$ to obtain a map $I_{S^1}(\varphi^n(\mathcal{L}_2), \mathrm{Inv}(\tilde{J}^{n,+}_{\tilde{m}_j})) \to I_{S^1}(\varphi^n(\mathcal{L}_1), \mathrm{Inv}(J^{n,+}_{m_{j+1}}))$. We will then prove that the composition

$$I_{S^1}(\varphi^n(\mathcal{L}_1), \mathrm{Inv}(J^{n,+}_{m_j})) \to I_{S^1}(\varphi^n(\mathcal{L}_2), \mathrm{Inv}(\tilde{J}^{n,+}_{\tilde{m}_j})) \to I_{S^1}(\varphi^n(\mathcal{L}_1), \mathrm{Inv}(J^{n,+}_{m_{j+1}}))$$

is just the attractor map corresponding to $\mathcal{L}_1$. A similar result holds for $\mathcal{L}_2$. Therefore, we have constructed a "mixed direct system" in the category $\mathfrak{C}$ and the spectra corresponding to f_1, f_2 are both subsequential colimit of it. Therefore, the invariance of $\underline{\mathrm{swf}}^A$ is implied by Lemma 3.1.1. The $\underline{\mathrm{swf}}^R$ case can be proved similarly.

There is one technical difficulty here. We need to find a uniform constant R_2 (independent of e, s) such that $Str(R_2)$ contains all the finite type trajectories of $\mathcal{L}^s_e$ and $\tilde{\mathcal{L}}^s_e$. This will be taken care by Lemma 6.3.9 and Lemma 6.3.11, which generalize Theorem 2.2.2.

Let us prepare some general results regarding the perturbations. Recall that we have a canonical isomorphism

$$\pi_0(\mathcal{G}_Y) \cong \pi_0(\mathcal{G}^h_Y) \cong H^1(Y; \mathbb{Z}).$$

For any positive integer m, we denote by $m\mathcal{G}_Y$ (resp. $m\mathcal{G}^h_Y$) the subgroup of $\mathcal{G}_Y$ (resp. $\mathcal{G}^h_Y$) consisting of the connected components corresponding to $m \cdot H^1(Y; \mathbb{Z})$.

Definition 6.3.1. For a positive integer m, a continuous function $f\colon Coul(Y) \to \mathbb{R}$ is called m-periodic if f is invariant under the action of $m\mathcal{G}^h_Y$, which implies that $f \circ \Pi$ is invariant under $m\mathcal{G}_Y$.

We will also need the following definition of tame functions.

Definition 6.3.2. A smooth function $f\colon Coul(Y) \to \mathbb{R}$ is called a tame function if the formal gradient $\mathrm{grad}(f \circ \Pi)$ satisfies all the conditions of the tame perturbations

[33, Definition 10.5.1] except that it needs not be invariant under the full gauge group $\mathcal{G}_Y$. where $\Pi : \mathcal{C}_Y \to Coul(Y)$ is the non-linear Coulomb projection.

Furthermore, a continuous family of functions $\{f_w\}$ parametrized by a compact manifold W (possibly with boundary) is called a continuous family of tame functions if each function is tame and $\operatorname{grad}(f_w \circ \Pi)$ extends to a continuous family of maps on the cylinder $I \times Y$. In addition, we require that the constant m_2 and the function μ_1 from [33, Definition 10.5.1] are uniform with respect to $w \in W$.

Now we describe a way to construct a continuous family of tame functions from any pair of extended cylinder function, given a family of smooth function.

Lemma 6.3.3. *Let W be a compact manifold and $\tilde{\tau}_w \colon i\Omega^1_h(Y) \cong \mathbb{R}^{b_1} \to \mathbb{R}$ be a smooth family of smooth functions parametrized by $w \in W$. Then, we can choose a sequence of constants $\{C_j\}$ in the definition of the space of perturbations $\mathcal{P}$ (c.f. Definition 2.1.1) so that, for any $\delta \in \mathbb{R}$ and any $\bar{f}_1, \bar{f}_2 \in \mathcal{P}$, a family of functions $\tilde{f}_w \colon Coul(Y) \to \mathbb{R}$ given by*

$$\tilde{f}_w(a, \phi) := \frac{\delta}{2}\|\phi\|^2_{L^2} + (\tilde{\tau}_w \circ \pi_{\mathcal{H}}(a)) \cdot \bar{f}_1(a, \phi) + (1 - \tilde{\tau}_w \circ \pi_{\mathcal{H}}(a)) \cdot \bar{f}_2(a, \phi). \tag{6.4}$$

is a continuous family of tame functions. Moreover, if $\tilde{\tau}_w$ is $m\mathbb{Z}^{b_1}$-periodic, then $\tilde{f}_w$ is m-periodic.

Proof. This is actually a parametrized version of [33, Theorem 11.6.1] and we will focus only on the term $(\tilde{\tau}_w \circ \pi_{\mathcal{H}}(a)) \cdot \bar{f}_1(a, \phi)$. To avoid repeating complicated analysis there, we introduce a trick turning a family of functions into a single function. Let Y' be another spin^c 3-manifold with $b_1(Y') > 2 \dim W$ so that we can embed W in the torus $i\Omega^1_h(Y')/\mathcal{G}^{h,o}_{Y'}$. We now consider the family $\{\tilde{\tau}_w\}_{w \in W}$ as a single function on $i\Omega^1_h(Y) \times W$ and extend it to $\tilde{\tau} \colon i\Omega^1_h(Y) \times i\Omega^1_h(Y') \to \mathbb{R}$. Recall that $\bar{f}_1 = \sum_{j=1}^{\infty} \eta_j \hat{f}_j$, where $\hat{f}_j$ is a cylinder function of Y with $\sum_{j=1}^{\infty} C_j |\eta_j| < \infty$. We define a function

$$\hat{f}'_j \colon \mathcal{C}_Y \times \mathcal{C}_{Y'} \to \mathbb{R}$$

given by

$$(a, \phi) \times (a', \phi') \mapsto \bar{\tau}(\pi'_{\mathcal{H}}(a'), \pi_{\mathcal{H}}(a)) \cdot \hat{f}_j(a, \phi),$$

where $\pi'_{\mathcal{H}}\colon i\Omega^1(Y') \to i\Omega^1_h(Y')$ denotes the projection onto harmonic forms on Y'. These functions $\hat{f}'_j$ almost fit into the definition of cylinder functions (cf. [33, Section 11]), on $\mathcal{C}(Y) \times \mathcal{C}(Y')$. We can still repeat the argument the proof of [33, Theorem 11.6.1] and show that, by setting $\{C_j\}$ to increase fast enough, the formal gradient $\mathrm{grad}(\sum_j \eta_j \hat{f}'_j)$ is a tame perturbation for the manifold $Y \cup Y'$ except that it is not invariant under the full gauge group. As a result, it is not hard to see that this actually implies that $(\tilde{\tau}_w \circ \pi_{\mathcal{H}}(a)) \cdot \bar{f}_1(a, \phi)$ is a continuous family of tame functions. □

For a general functional $\mathcal{L}\colon Coul(Y) \to \mathbb{R}$, we can consider its negative gradient flow line $\gamma\colon I \to Coul(Y)$, described by the equation $\frac{d\gamma(t)}{dt} = -\widetilde{\mathrm{grad}}\,\mathcal{L}(\gamma(t))$. Such a trajectory will be called an $\mathcal{L}$-trajectory. As before, we define the topological energy by

$$\mathcal{E}^{\mathrm{top}}(\gamma, \mathcal{L}) := 2(\sup_{t\in I} \mathcal{L}(\gamma(t)) - \inf_{t\in I} \mathcal{L}(\gamma(t))). \tag{6.5}$$

Recall that a trajectory is called finite type if it is contained in a bounded subset of $Coul(Y)$. We have the following uniform boundedness result for functionals perturbed by periodic functions.

Proposition 6.3.4. *Let $\{f_w\}$ be a continuous family of m-periodic tame functions parametrized by a compact manifold W and consider a family of functionals $\mathcal{L}_w = CSD_{\nu_0}|_{Coul(Y)} + f_w$. Then for any $C > 0$, there exist constants R, C' such that for any $w \in W$ and any $\mathcal{L}_w$-trajectory $\gamma\colon [-1,1] \to Coul(Y)$ with topological energy $\mathcal{E}^{top}(\gamma, \mathcal{L}_w) \le C$, we have $\gamma(0) \in Str(R)$ and $|\mathcal{L}_w(\gamma(0))| \le C'$.*

Proof. The proof is a slight adaption of [33, Theorem 10.7.1]. Suppose that the statement is not true. Then we can find a sequence $\{\gamma_j\}$ of $\mathcal{L}_{w_j}$-trajectory $\gamma_j\colon [-1,1] \to Coul(Y)$ with $w_j \in W$ such that at least one of the following two situations happens:

- $\limsup_{j\to\infty} \|u_j \cdot \gamma_j(0)\|_{L^2_k} = \infty$ for any sequence $\{u_j\} \subset m\mathcal{G}^h_Y$;
- $\limsup_{j\to\infty} |\mathcal{L}_{w_j}(\gamma_j(0))| = \infty$.

Since W is compact, after passing to a subsequence, we may assume that $w_j \to w_\infty$.

We lift γ_j to a path $\tilde{\gamma}_j\colon [-\frac{1}{2}, \frac{1}{2}] \to \mathcal{C}_Y$, which is the negative gradient flow line of $CSD_{\nu_0} + f_{w_j} \circ \Pi$. Note that we only consider an interior domain here to avoid a possible regularity issue. With $X = [-\frac{1}{2}, \frac{1}{2}] \times Y$, we treat $\tilde{\gamma}_j$ as a section over X and denote it by $(\hat{a}_j, \hat{\phi}_j)$. We can find a gauge transformation $\hat{u}_j$ over X whose restrictions to $\{0\} \times Y$ belong to $m\mathcal{G}_Y$ such that the following conditions hold:

1. $\hat{d}^*(\hat{a}_j - \hat{u}_j^{-1} d\hat{u}_j) = 0$ on X.
2. $(\hat{a}_j - \hat{u}_j^{-1} d\hat{u}_j)(\mathbf{n}) = 0$ on ∂X, where $\mathbf{n}$ is the outward normal vector ;
3. For each for $l = 1, \ldots, b_1$, we have $\int_{\tilde{X}} (\hat{a}_j - \hat{u}_j^{-1} d\hat{u}_j) \wedge (*_4 \hat{h}_l) \in [0, m)$ where $\hat{h}_l$ is the pull-back of h_l on X;

The conditions in Definition 6.3.2 allow us to repeat the bootstrapping argument in the proof of [33, Theorem 10.7.1] and obtain the following statement. After passing to a further subsequence, $(\hat{a}_j - \hat{u}_j^{-1} d\hat{u}_j, \hat{u}_j \cdot \hat{\phi}_j)$ is convergent in $L^2_{k+\frac{1}{2}}$ when restricted to any interior cylinder. In particular, this implies that $\Pi(\hat{u}_j|_{\{0\}\times Y} \cdot \tilde{\gamma}_j(0))$ is convergent in L^2_k. Notice that $\Pi(\hat{u}_j|_{\{0\}\times Y} \cdot \tilde{\gamma}_j(0))$ equals $u_j \cdot \gamma_j(0)$ for some $u_j \in m\mathcal{G}^h_Y$. Also $\mathcal{L}_{w_j}(\gamma_j(0)) = \mathcal{L}_{w_j}(u_j \cdot \gamma_j(0))$ is a convergent sequence since $\mathcal{L}_w(a, \phi)$ is continuous in w and (a, ϕ). Therefore, we arrive at a contradiction with the above two situations.

□

We also have the following lemma, whose proof is essentially the same as Lemma 2.2.7 and we omit it.

Lemma 6.3.5. *Let $\{f_w\}$ be a continuous family of tame functions. For each $w \in W$, we define a nonlinear term $c_w\colon Coul(Y) \to L^2_k(i \ker d^* \oplus \Gamma(S_Y))$ of the gradient of $CSD_{\nu_0}|_{Coul(Y)} + f_w$ as in (2.8) and (2.9). Then $\{c_w\}$ is a continuous family of quadratic-like maps.*

Now we construct explicit "the mixed perturbation" as follows. Choose a smooth function $\tau\colon \mathbb{R} \to [0, 1]$ satisfying $\tau|_{(-\infty, \frac{1}{4}]} \equiv 0$ and $\tau|_{[\frac{1}{2}, \infty)} \equiv 1$. For any real number

$e \geq 1$, we define a bump function $\tau_e \colon i\Omega^1_h(Y) \to [0,1]$ from τ by

$$\tau_e(x_1, x_2, \cdots, x_{b_1}) = \prod_{1 \leq j \leq b_1} \tau(e + x_j)\tau(e - x_j).$$

Each τ_e gives an induced tame function $\tilde{f}^0_e \colon Coul(Y) \to \mathbb{R}$ as in (6.4), i.e.

$$\tilde{f}^0_e(a,\phi) := \frac{\delta}{2}\|\phi\|^2_{L^2} + (\tau_e \circ p_{\mathcal{H}}(a,\phi))) \cdot \bar{f}_1(a,\phi) + (1 - \tau_e \circ p_{\mathcal{H}}(a,\phi)) \cdot \bar{f}_2(a,\phi),$$

where $\bar{f}_1, \bar{f}_2 \in \mathcal{P}$. With $f_j = \frac{\delta}{2}\|\phi\|^2_{L^2} + \bar{f}_j$, we note that the function $\tilde{f}^0_e$ equals f_1 on $p^{-1}_{\mathcal{H}}([-e+1, e-1]^b)$ and equals f_2 on $p^{-1}_{\mathcal{H}}(\mathbb{R}^b \setminus (-e,e)^b)$. For $s \in [0,1]$, we also consider an interpolation $\tau^s_e = (1-s)\tau_e$ and define

$$\tilde{f}^s_e = (1-s)\tilde{f}^0_e + sf_2 \text{ and } \mathcal{L}^s_e = CSD_{\nu_0}|_{Coul(Y)} + \tilde{f}^e_s. \tag{6.6}$$

Notice that $\tilde{f}^s_e$ is essentially a tame function induced from τ^s_e which is not m-periodic for any positive integer m. To utilize Proposition 6.3.4, we will introduce an explicit family of smooth periodic functions such that the induced periodic tame functions agree with $\tilde{f}^s_e$ on desirable regions.

For any positive integer M, we consider a family of $(6M+6)$-periodic smooth functions parametrized by compact manifold W_M described as follows. The manifold W_M is of the form $W_{M,1} \amalg W_{M,2} \amalg W_{M,3}$ where $W_{M,1} := [1, M+1] \times [0,1]$ and $W_{M,2} := \{(B,\sigma) \mid \emptyset \neq B \subset \{1,2,\ldots,b_1\} \text{ and } \sigma \colon B \to \{\pm 1\}\} \times (\mathbb{R}/(6M+6)\mathbb{Z}) \times [0,1]$ and $W_{M,3} = \{1,2\}$. We construct a family of functions $\{\tilde{\tau}_w\}$ as following:

- For each positive integer M and $(e,0) \in W_{M,1}$, we assign the unique $(6M+6)\mathbb{Z}^{b_1}$-periodic function $\tilde{\tau}_e \colon \mathbb{R}^{b_1} \to \mathbb{R}$ which extends $\tau_e|_{[-3M-3, 3M+3]^{b_1}}$.
- For each positive integer M, we pick a $(6M+6)$- periodic function $\bar{\tau}_M \colon \mathbb{R} \to [0,1]$ which extends $\tau|_{[-2M-2, 2M+2]}$. For each $(B,\sigma,\theta,0) \in W_{M,2}$, we assign a function $\tilde{\tau}_{B,\sigma,\theta} \colon \mathbb{R}^{b_1} \to [0,1]$ given by

$$\tilde{\tau}_{(B,\sigma,\theta)}(x_1, \cdots, x_{b_1}) := \prod_{j \in B} \bar{\tau}_M(\theta + \sigma(j)x_j).$$

- For general $w = (w', s) \in W_{M,1} \coprod W_{M,2}$, we simply define $\tilde{\tau}_{(w',s)} := (1-s)\tilde{\tau}_{w'}$.

- We set $\tilde{\tau}_j \equiv 2 - j$ for $j \in W_{M,3}$ so that $\tilde{f}_j = f_j$.

Lemma 6.3.6. *For each positive integer M, any $(s,e) \in [0,1]\times[1,\infty)$ and $(e_1, e_2, \cdots, e_{b_1}) \in \mathbb{R}^{b_1}$, there exists an element $w \in W_M$ such that the induced function $\tilde{f}_w$ equals $\tilde{f}_e^s$ on $p_{\mathcal{H}}^{-1}([e_1 - M, e_1 + M] \times \cdot \times [e_{b_1} - M, e_{b_1} + M])$.*

Proof. For convenience, we denote $E = [e_1 - M, e_1 + M] \times \cdot \times [e_{b_1} - M, e_{b_1} + M]$. We will consider two main cases with several subcases:

Case $e \in [1, M+1]$; If $E \cap [-M-1, M+1]^{b_1} \neq \emptyset$, then we have $E \subset [-3M-3, 3M+3]^{b_1}$. This implies $\tilde{\tau}_e|_A = \tau_e|_A$. Therefore, we can just choose $w = (e,s) \in W_{M,1}$. If $E \cap [-M-1, M+1]^{b_1} = \emptyset$, then we have $p_{\mathcal{H}}^{-1}(E) \subset p_{\mathcal{H}}^{-1}(\mathbb{R}^{b_1} \setminus (-e,e)^{b_1})$ and $\tilde{f}_e^s = f_2$ on $p_{\mathcal{H}}^{-1}(E)$. We just take $w = 2 \in W_{M,3}$ so that $\tilde{f}_w = f_2$ in this case.

Case $e > M+1$; We consider the following subsets of $[1, 2, \cdots, b]$:

$$B_1 = \{j \mid [e_j - M, e_j + M] \cap [e-1, e] \neq \emptyset\},$$

$$B_2 = \{j \mid [e_j - M, e_j + M] \cap [-e, -e+1] \neq \emptyset\},$$

$$B_3 = \{j \mid [e_j - M, e_j + M] \cap [-e, e] = \emptyset\}.$$

If $B_1 \cup B_2 = \emptyset$, then E is either contained in $[-e+1, e-1]^{b_1}$ or $\mathbb{R}^{b_1} \setminus (-e,e)^{b_1}$ and we can just take $w \in W_{M,3}$. If $B_3 \neq \emptyset$, then we have $\tau_e|_E \equiv 0$ and $\tilde{f}_e^s|_{p_{\mathcal{H}}^{-1}(E)} = f_2|_{p_{\mathcal{H}}^{-1}(E)}$. We can take $w = 2 \in W_{M,3}$ again in this subcase. We are now left with the case $B_1 \cup B_2 \neq \emptyset$ and $B_3 = \emptyset$. Notice that for any $(x_1, \cdots, x_{b_1}) \in E$, the following holds:

$$j \in B_1 \Rightarrow e + x_j \geq 2e - 1 - 2M \geq 1 \text{ and } e - x_j \in [-2M, 2M+1];$$

$$j \in B_2 \Rightarrow e - x_j \geq 2e - 1 - 2M \geq 1 \text{ and } e + x_j \in [-2M, 2M+1];$$

$$j \notin B_1 \cup B_2 \Rightarrow e - |x_j| \geq 1.$$

Therefore, for such $(x_1, \cdots, x_{b_1})$, we have

$$\tau_e(x_1, \cdots, x_b) = \prod_{j \in B_1} \tau(e - x_j) \cdot \prod_{j \in B_2} \tau(e + x_j) = \prod_{j \in B_1} \bar{\tau}_M(e - x_j) \cdot \prod_{j \in B_2} \bar{\tau}_M(e + x_j),$$

where we use the fact that $\bar{\tau}_M|_{[-2M-2, 2M+2]} = \tau|_{[-2M-2, 2M+2]}$. As a result, we see that $\tilde{f}_e^s = \tilde{f}_w$ on $p_{\mathcal{H}}^{-1}(E)$ when we set $w = (B_1 \cup B_2, \sigma, e, s) \in W_{M,2}$ with $\sigma \colon B_1 \cup B_2 \to \{\pm 1\}$ sending B_1 to -1 and B_2 to 1. Notice that $B_1 \cap B_2 = \emptyset$ because $e > M+1$.

□

We also have the following extension of Lemma 6.3.3 to a countable union of compact sets.

Lemma 6.3.7. *We can choose a sequence of constants $\{C_j\}$ in the definition of $\mathcal{P}$ (see Definition 2.1.1) such that for any positive integer M and any $\bar{f}_1, \bar{f}_2 \in \mathcal{P}$, the induced family $\{\tilde{f}_w\}_{w\in W_M}$ is a continuous family of $(6M+6)$-periodic tame functions.*

Proof. For each W_M, there exists a sequence $\{C_{M,j}\}_j$ such that, for any $f_1, f_2 \in \mathcal{P}(\{C_{M,j}\}_j)$, the family $\{\tilde{f}_w\}_{w\in W_M}$ is a continuous family of $(6M+6)$-periodic tame functions. It is straightforward to see that a sequence of positive real numbers $\{C_j\}$ such that

$$C_j \geq \max_{1\leq M\leq j} C_{M,j}$$

satisfies our requirement. □

Next is the boundedness result for functionals with mixed perturbations.

Lemma 6.3.8. *For any $C > 0$, there exist constants R, C' such that for any $(e,s) \in [1,\infty) \times [0,1]$ and any $\mathcal{L}^s_e$-trajectory $\gamma\colon [-2,2] \to Coul(Y)$ with topological energy $\mathcal{E}^{top}(\gamma; \mathcal{L}^s_e) \leq C$, we have $\gamma(0) \in Str(R)$ and $|\mathcal{L}^s_e(\gamma(0))| < C'$.*

Proof. We first write down $\widetilde{\operatorname{grad}}\, \tilde{f}^s_e$ as

$$\begin{aligned}&\widetilde{\operatorname{grad}}\, \tilde{f}^s_e(a,\phi) = \delta\phi + (1-s)(\bar{f}_1(a,\phi) - \bar{f}_2(a,\phi))\, \widetilde{\operatorname{grad}}(\tau_e \circ p_{\mathcal{H}})(a,\phi)\\ &+ (1-s)(\tau_e \circ p_{\mathcal{H}}(a,\phi))\, \widetilde{\operatorname{grad}}\, \bar{f}_1(a,\phi) + (1-(1-s)(\tau_e \circ p_{\mathcal{H}}(a,\phi)))\, \widetilde{\operatorname{grad}}\, \bar{f}_2(a,\phi).\end{aligned}$$

By boundedness and tameness conditions of $\bar{f}_j$, we see that

$$\|\operatorname{grad}(\xi^s_e \circ \Pi)(a,\phi)\|_{L^2} = \|\widetilde{\operatorname{grad}}\, \xi^s_e(a,\phi)\|_{\tilde{g}} \leq m(1 + \|\phi\|_{L^2}),$$

where m is a constant independent of (e,s). This implies

$$\|\operatorname{grad}(\xi^s_e \circ \Pi)(a,\phi)\|^2_{L^2} \leq 2m^2 + 2m^2\|\phi\|^2_{L^2} \tag{6.7}$$

We can lift $\gamma|_{[-1,1]}$ to $\tilde{\gamma} : [-1,1] \to \mathcal{C}_Y$, which is a negative gradient flow line for the functional $\mathcal{L}^s_e \circ \Pi$. Now we follow the argument on Page 161 of [33]. Since $\mathcal{L}^s_e \circ \Pi = CSD_{\nu_0} + \xi^s_e \circ \Pi$, we have

$$\|\operatorname{grad} CSD_{\nu_0}\|^2_{L^2} - 2\|\operatorname{grad}(\xi^s_e \circ \Pi)\|^2_{L^2} \leq 2\|\operatorname{grad}(\mathcal{L}^s_e \circ \Pi)\|^2_{L^2}.$$

By formula (6.7), this implies

$$\begin{aligned}\int_{-1}^{1}(\|\operatorname{grad} CSD_{\nu_0}(\tilde{\gamma}(t))\|_{L^2}^2+\|\tilde{\gamma}'(t)\|_{L^2}^2)dt-2m^2\int_{-1}^{1}\|\phi(t)\|_{L^2}^2dt-4m^2\\ \leq 2\int_{-1}^{1}(\|\operatorname{grad}(\mathcal{L}_e^s\circ\Pi)(\tilde{\gamma}(t))\|_{L^2}^2+\|\tilde{\gamma}'(t)\|_{L^2}^2)dt<2\mathcal{E}^{top}(\gamma,\mathcal{L}_s^e)\leq 2C.\end{aligned} \tag{6.8}$$

We can treat $\tilde{\gamma}$ as a section over the 4-manifold $[-1,1]\times Y$ and denote it by $(\hat{a},\hat{\phi})$. By Definition 4.5.4 and formula (4.19) of [33], the above estimate on the analytical energy actually implies

$$\frac{1}{4}\int_{[-1,1]\times Y}|d\hat{a}|^2+\int_{[-1,1]\times Y}|\nabla_{\hat{A}}\hat{\phi}|^2+\frac{1}{4}\int_{[-1,1]\times Y}(|\hat{\phi}|^2-C_2)^2\leq C_3$$

where $\hat{A}$ is the connection corresponding to $\hat{a}$ and C_2 is a constant independent of e,s. By Corollary 4.5.3, Lemma 5.1.2 and Lemma 5.1.3 of [33], we can find a gauge transformation $u:[-1,1]\times Y\to S^1$ such that $\|u\cdot\tilde{\gamma}\|_{L_1^2([-1,1]\times Y)}$ is bounded by a uniform constant C_4. Let u_t equals $u|_{\{t\}\times Y}$. Then there exists C_5 such that for any $t_1,t_2\in[-1,1]$, we have

$$\|\Pi_{\mathcal{H}}(u_{t_1}\cdot\tilde{\gamma}(t_1))-\Pi_{\mathcal{H}}(u_{t_2}\cdot\tilde{\gamma}(t_2))\|_{L^2}\leq\|u_{t_1}\cdot\tilde{\gamma}(t_1)\|_{L^2}+\|u_{t_2}\cdot\tilde{\gamma}(t_2)\|_{L^2}\leq C_5$$

Recall that $\Pi_{\mathcal{H}}:\mathcal{C}_Y\to i\Omega_h^1(Y)$ is just the orthogonal projection. Since u_{t_1} and u_{t_2} are in the same component of the gauge group $\mathcal{G}_Y$, we have

$$\|p_{\mathcal{H}}(\gamma(t_1))-p_{\mathcal{H}}(\gamma(t_2))\|_{L^2}=\|\Pi_{\mathcal{H}}(u_{t_1}\cdot\tilde{\gamma}(t_1))-\Pi_{\mathcal{H}}(u_{t_2}\cdot\tilde{\gamma}(t_2))\|_{L^2}\leq C_5.$$

This implies that $\gamma([-1,1])$ is contained in $p_{\mathcal{H}}^{-1}([e_1-M_0,e_1+M_0]\times\cdot\times[e_b-M_0,e_b+M_0])$ for some $(e_1,\cdots,e_b)\in\mathbb{R}^b$ and some uniform constant $M_0\in\mathbb{Z}_{\geq 1}$. By Lemma 6.3.6, we have $\xi_e^s|_{p_{\mathcal{H}}^{-1}([e_1-M_0,e_1+M_0]\times\cdot\times[e_b-M_0,e_b+M_0])}=f_w$ for some $w\in W_{M_0}$. This implies that $\gamma|_{[-1,1]}$ is also a trajectory for $CSD_{\nu_0}|_{Coul(Y)}+f_w$. Notice that $\mathcal{E}^{\text{top}}(\gamma|_{[-1,1]},CSD_{\nu_0}|_{Coul(Y)}+f_w)<C$. Our result is directly implied by Proposition 6.3.4.

□

The previous results implies uniform boundedness for finite type trajectories for the family $\{\mathcal{L}_e^s\}$. For convenience, we will say that functional $\mathcal{L}\colon Coul(Y)\to\mathbb{R}$ is called R-bounded if any finite type $\mathcal{L}$-trajectory is contained in $Str(R)$.

Corollary 6.3.9. *There exists a uniform constant $R_1 > 0$ such that for any $e \in \mathbb{R}_{\geq 1}$ and $s \in [0,1]$, the functionals $\mathcal{L}_e^s$ is R_1-bounded.*

Proof. Let $\gamma\colon \mathbb{R} \to Coul(Y)$ be a finite type $\mathcal{L}_e^s$-trajectory. Since $\mathcal{E}^{\text{top}}(\gamma, \mathcal{L}_e^s) < \infty$, we have $\mathcal{E}^{\text{top}}(\gamma|_{[t-1,t+1]}, \mathcal{L}_e^s) < 1$ for any t with $|t|$ sufficiently large. By Lemma 6.3.8 (with $C = 1$), we have $|\mathcal{L}_e^s(\gamma(t))| \leq C'$ for such t. Since $\mathcal{L}_e^s$ is decreasing along γ, we see that $\mathcal{L}_e^s(\gamma(t-1)) - \mathcal{L}_e^s(\gamma(t+1)) < 2C'$ for any $t \in \mathbb{R}$. We apply Lemma 6.3.8 again (now $C = 2C'$), so there is a uniform constant R_1 such that $\gamma(t) \in Str(R_1)$ for any $t \in \mathbb{R}$. □

For the reader's convenience, we summarize the functionals we will be dealing with. Two extended cylinder functions $\bar{f}_1, \bar{f}_2$ are now fixed, along with their corresponding functional $\mathcal{L}_1, \mathcal{L}_2$. We have the continuous family of functionals $\{\mathcal{L}_e^s\}$ (see (6.6)) such that, for each $(e,s) \in [1,\infty) \times [0,1]$, they satisfy

$$\mathcal{L}_e^1 = \mathcal{L}_2,$$
$$\mathcal{L}_e^0(x) = \begin{cases} \mathcal{L}_1(x) & \text{if } x \in p_{\mathcal{H}}^{-1}([-e+1, e-1]^{b_1}), \\ \mathcal{L}_2(x) & \text{if } x \in p_{\mathcal{H}}^{-1}(\mathbb{R}^{b_1} \setminus (-e,e)^{b_1}), \end{cases}$$
$$\mathcal{L}_e^s(x) = \mathcal{L}_2(x) \text{ if } x \in p_{\mathcal{H}}^{-1}(\mathbb{R}^{b_1} \setminus (-e,e)^{b_1}).$$

Since the above construction is asymmetrical in $\bar{f}_1$ and $\bar{f}_2$, we also consider another family of functionals $\{\tilde{\mathcal{L}}_e^s\}$ where the role of $\bar{f}_1$ and $\bar{f}_2$ are reversed. In other words, we have

$$\tilde{\mathcal{L}}_e^1 = \mathcal{L}_1,$$
$$\tilde{\mathcal{L}}_e^0(x) = \begin{cases} \mathcal{L}_2(x) & \text{if } x \in p_{\mathcal{H}}^{-1}([-e+1, e-1]^{b}), \\ \mathcal{L}_1(x) & \text{if } x \in p_{\mathcal{H}}^{-1}(\mathbb{R}^{b} \setminus (-e,e)^{b}), \end{cases}$$
$$\tilde{\mathcal{L}}_e^s(x) = \mathcal{L}_1(x) \text{ if } x \in p_{\mathcal{H}}^{-1}(\mathbb{R}^{b} \setminus (-e,e)^{b}).$$

Roughly speaking, the family $\{\mathcal{L}_e^s\}$ will give a morphism from Conley indices given by $\mathcal{L}_1$ to Conley indices given by $\mathcal{L}_2$ and vice versa.

To show equivalence, we need to introduce (final) two more families of functionals.

For two real numbers e, e' with $e-1 \geq e' \geq 1$ and $s \in [0,1]$, we define

$$\mathcal{L}^s_{e,e'}(x) = \begin{cases} \tilde{\mathcal{L}}^s_{e'}(x) & \text{if } x \in p_{\mathcal{H}}^{-1}([-e', e']^{b_1}) \\ \mathcal{L}^0_e(x) & \text{otherwise,} \end{cases}$$

$$\tilde{\mathcal{L}}^s_{e,e'}(x) = \begin{cases} \mathcal{L}^s_{e'}(x) & \text{if } x \in p_{\mathcal{H}}^{-1}([-e', e']^{b_1}) \\ \tilde{\mathcal{L}}^0_e(x) & \text{otherwise.} \end{cases}$$

These functionals have the following properties:

1. $\mathcal{L}^1_{e,e'} = \mathcal{L}^0_e$ and $\tilde{\mathcal{L}}^1_{e,e'} = \tilde{\mathcal{L}}^0_e$.
2. $\mathcal{L}^0_{e,e'}(x) = \begin{cases} \mathcal{L}_2(x) & \text{if } x \in p_{\mathcal{H}}^{-1}([-e'+1, e'-1]^{b_1} \cup (\mathbb{R}^{b_1} \setminus (-e,e)^{b_1})), \\ \mathcal{L}_1(x) & \text{if } x \in p_{\mathcal{H}}^{-1}([-e+1, e-1]^{b_1} \setminus (-e', e')^{b_1}). \end{cases}$
3. $\tilde{\mathcal{L}}^0_{e,e'}(x) = \begin{cases} \mathcal{L}_1(x) & \text{if } x \in p_{\mathcal{H}}^{-1}([-e'+1, e'-1]^{b_1} \cup (\mathbb{R}^{b_1} \setminus (-e,e)^{b_1})), \\ \mathcal{L}_2(x) & \text{if } x \in p_{\mathcal{H}}^{-1}([-e+1, e-1]^{b_1} \setminus (-e', e')^{b_1})). \end{cases}$

We have the following extension of Lemma 6.3.6 and 6.3.7. The proof is essentially the same and we omit it.

Lemma 6.3.10. *(1) For each positive integer M, we can find a smooth family of $(6M+6)\mathbb{Z}^{b_1}$-periodic functions $\tilde{\tau}_w : \mathbb{R}^{b_1} \to [0,1]$, parametrized by a compact manifold W'_M, with the following property: for any functional in the family $\{\mathcal{L}^s_{e,e'} \mid s \in [0,1], e-1 \geq e' \geq 1\}$ and any $(e_1, \cdots, e_{b_1}) \in \mathbb{R}^{b_1}$, we can find $w \in W'_M$ such that*

$$\mathcal{L}^s_{e,e'} = CSD_\nu|_{Coul(Y)} + f_w$$

when restricted to $p_{\mathcal{H}}^{-1}([e_1 - M, e_1 + M] \times \cdot [e_{b_1} - M, e_{b_1} + M])$. Here f_w is the function on $Coul(Y)$ induced by $\tilde{\tau}_w$ (see (6.4)).
(2) We can choose a sequence of constants $\{C_j\}$ in the definition of $\mathcal{P}$ (see Definition 2.1.1) such that for any positive integer M and any $\bar{f}_1, \bar{f}_2 \in \mathcal{P}$, the induced family $\{\tilde{f}_w\}_{w \in W'_M}$ is a continuous family of $(6M+6)$-periodic tame functions.
(3) Similar result holds if we consider any one of the following families instead

- $\{\tilde{\mathcal{L}}^s_{e,e'} \mid s \in [0,1], e-1 \geq e' \geq 1\}$;

- $\{(1-s)\mathcal{L}_2 + s\mathfrak{L}^{s'}_{e,e'} \mid s, s' \in [0,1], e-1 \geq e'+1, \}$;
- $\{(1-s)\mathcal{L}_1 + s\tilde{\mathfrak{L}}^{s'}_{e,e'} \mid s, s' \in [0,1], e \geq e'+1, \}$.

With Lemma 6.3.10 at hand, we can repeat the proof of Corollary 6.3.9 and get the following result.

Lemma 6.3.11. *There exists a uniform R_2 such that for $e-1 \geq e' \geq 1$ and $s, s' \in [0,1]$, the functionals $\mathfrak{L}^s_{e,e'}, \tilde{\mathfrak{L}}^s_{e,e'}, (1-s)\mathcal{L}_2 + s\mathfrak{L}^{s'}_{e,e'}$ and $(1-s)\mathcal{L}_1 + s\tilde{\mathfrak{L}}^{s'}_{e,e'}$ are all R_2-bounded.*

Now we start constructing a mixed direct system relating the spectra given by $\mathcal{L}_1$ and $\mathcal{L}_2$. As usual, we focus on the case of $\underline{\text{swf}}^A$. We first choose a constant $\tilde{R}$ greater than $\max(R_1, R_2)$, where R_1 is the constant in Corollary 6.3.9 and R_2 is a constant that we will specify later in Lemma 6.3.11. Let $J^+_1 \subset J^+_2 \subset \ldots$ and $\tilde{J}^+_1 \subset \tilde{J}^+_2 \subset \ldots$ be increasing sequences of bounded subsets corresponding to $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively (see (5.2)). Although these bounded sets come from $Str(\tilde{R})$, they are different as we use different cutting functions and different cutting values. Since the sequences of subsets are increasing, we can find increasing sequences of positive integers $\{m_j\}$, $\{\tilde{m}_j\}$, $\{e_j\}$ and $\{\tilde{e}_j\}$ such that

$$\begin{aligned} J^+_{m_j} &\subset p^{-1}_{\mathcal{H}}([-e_j+1, e_j-1]^b) \cap Str(\tilde{R}) \subset p^{-1}_{\mathcal{H}}([-e_j, e_j]^b) \cap Str(\tilde{R}) \subset \tilde{J}^+_{\tilde{m}_j} \\ &\subset p^{-1}_{\mathcal{H}}([-\tilde{e}_j+1, \tilde{e}_j-1]^b) \cap Str(\tilde{R}) \subset p^{-1}_{\mathcal{H}}([-\tilde{e}_j, \tilde{e}_j]^b) \cap Str(\tilde{R}) \subset J^+_{m_{j+1}}. \end{aligned} \tag{6.9}$$

Let $\{\mu_n\}$ and $\{\lambda_n\}$ be an increasing sequence and a decreasing sequence of real numbers with $-\lambda_n, \mu_n \to \infty$ and denote by $V^{\mu_n}_{\lambda_n}$ the corresponding eigenspace. For a functional $\mathcal{L}$ on $Coul(Y)$, we denote by $\varphi^n(\mathcal{L})$ the flow generated by $\iota \circ p^{\mu_n}_{\lambda_n} \widetilde{\text{grad}}\, \mathcal{L}$ on $V^{\mu_n}_{\lambda_n}$ where ι is a bump function with value 1 on a specific bounded set. Since we are only interested in the Conley index which will be independent of ι, we can drop ι from our notation.

Consider $J^{n,+}_{m_j} = J^+_{m_j} \cap V^{\mu_n}_{\lambda_n}$ and $\tilde{J}^{n,+}_{m_j} = \tilde{J}^+_{m_j} \cap V^{\mu_n}_{\lambda_n}$. By Theorem 6.1.2, we can fix a sufficiently large integer n so that $J^{n,+}_{m_j}, \tilde{J}^{n,+}_{m_j}$ are isolating neighborhoods for all of the above families of approximated flows. For the family $\{\mathcal{L}^s_{e_j}\}$, we get a homotopy equivalence from homotopy invariance of Conley indices

$$\rho_1 \colon I_{S^1}(\varphi^n(\mathcal{L}^0_{e_j}), \mathrm{Inv}(\tilde{J}^{n,+}_{\tilde{m}_j})) \xrightarrow{\cong} I_{S^1}(\varphi^n(\mathcal{L}_2), \mathrm{Inv}(\tilde{J}^{n,+}_{\tilde{m}_j})),$$

where we recall that $\mathcal{L}^1_{e_j} = \mathcal{L}_2$. Since $\mathcal{L}^0_{e_j}$ is equal to $\mathcal{L}_1$ on $p^{-1}_{\mathcal{H}}([-e_j + 1, e_j - 1]^{b_1})$, which contains $J^+_{m_j}$, we see that the flow $\varphi^n(\mathcal{L}^0_{e_j})$ goes inside $J^{n,+}_{m_j}$ along the boundary $\partial J^{n,+}_{m_j} \setminus \partial Str(\tilde{R})$. Consequently, the subset $J^{n,+}_{m_j} \subset \tilde{J}^{n,+}_{\tilde{m}_1}$ is an attractor with respect to $\varphi^n(\mathcal{L}^0_{e_j})$ and we obtain an attractor map

$$\rho_2 \colon I_{S^1}(\varphi^n(\mathcal{L}_1), \mathrm{Inv}(J^{n,+}_{m_j})) \to I_{S^1}(\varphi^n(\mathcal{L}^0_{e_j}), \mathrm{Inv}(\tilde{J}^{n,+}_{\tilde{m}_j})).$$

We combine the above two maps and obtain the following map

$$\bar{\imath}^{n,+}_{m_j} := \rho_1 \circ \rho_2 \colon I_{S^1}(\varphi^n(\mathcal{L}_1), \mathrm{Inv}(J^{n,+}_{m_j})) \to I_{S^1}(\varphi^n(\mathcal{L}_2), \mathrm{Inv}(\tilde{J}^{n,+}_{\tilde{m}_j})). \tag{6.10}$$

Similarly, we use the family $\{\tilde{\mathcal{L}}^s_{\tilde{e}_j}\}$ to get a homotopy equivalence

$$\tilde{\rho}_1 \colon I_{S^1}(\varphi^n(\tilde{\mathcal{L}}^0_{\tilde{e}_j}), \mathrm{Inv}(J^{n,+}_{m_{j+1}})) \xrightarrow{\cong} I_{S^1}(\varphi^n(\mathcal{L}_1), \mathrm{Inv}(J^{n,+}_{m_{j+1}})).$$

Since $\tilde{J}^{n,+}_{\tilde{m}_j} \subset J^{n,+}_{m_{j+1}}$ is an attractor with respect to $\tilde{\mathcal{L}}^0_{\tilde{e}_j}$, we also get an attractor map

$$\tilde{\rho}_2 \colon I_{S^1}(\varphi^n(\mathcal{L}_2), \mathrm{Inv}(\tilde{J}^{n,+}_{\tilde{m}_j})) \to I_{S^1}(\varphi^n(\tilde{\mathcal{L}}^0_{\tilde{e}_j}), \mathrm{Inv}(J^{n,+}_{m_{j+1}})).$$

We compose the above two maps and get the following map

$$\hat{\imath}^{n,+}_{\tilde{m}_j} := \tilde{\rho}_1 \circ \tilde{\rho}_2 \colon I_{S^1}(\varphi^n(\mathcal{L}_2), \mathrm{Inv}(\tilde{J}^{n,+}_{\tilde{m}_j})) \to I_{S^1}(\varphi^n(\mathcal{L}_1), \mathrm{Inv}(J^{n,+}_{m_{j+1}})). \tag{6.11}$$

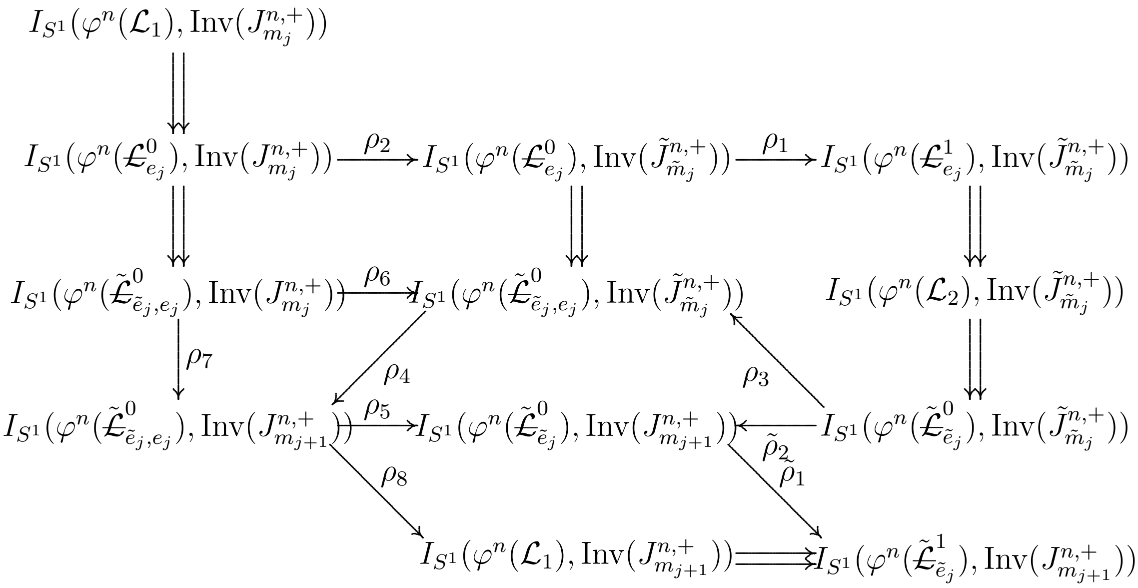
After appropriate desuspension, we obtain a direct system in the category $\mathfrak{C}$

$$I^{n_1,+}_{m_1} \to \tilde{I}^{\tilde{n}_1,+}_{\tilde{m}_1} \to I^{n_2,+}_{m_2} \to \tilde{I}^{\tilde{n}_2,+}_{\tilde{m}_2} \to \cdots, \tag{6.12}$$

where $I^{n,+}_m$ (resp. $\tilde{I}^{n,+}_m$) be the object of $\mathfrak{C}$ obtained from desuspending the Conlex indices of $J^{n,+}_m$ (resp. $\tilde{J}^{n,+}_m$) by $\bar{V}^0_{-\lambda_n}$ and we can pick a suitable sequence of integers $0 \ll n_1 < \tilde{n}_1 < n_2 < \tilde{n}_2 < \cdots$. The main result of this section follows from the following proposition.

Proposition 6.3.12. *The map $\hat{i}^{n,+}_{\tilde{m}_j} \circ \bar{i}^{n,+}_{m_j}$ is S^1-homotopic to attractor map for the attractor $Inv(\varphi^n(\mathcal{L}_1), J^{n,+}_{m_j}) \subset Inv(\varphi^n(\mathcal{L}_1), J^{n,+}_{m_{j+1}})$.*

Proof. We consider the following commutative (up to S^1-homotopy) diagram.



The maps are defined as follows.

1. Different flows are generated by the same vector field when restricted to some isolating neighborhood. This defines all the identifications "=" in the diagram.
2. The maps $\rho_1, \rho_2, \tilde{\rho}_1, \tilde{\rho}_2$ are defined as before.
3. The maps ρ_3, ρ_5 are the homotopy equivalences given by the deformation $\tilde{\mathcal{L}}^s_{\tilde{e}_j,e_j}$, $s \in [0,1]$.
4. The maps ρ_4, ρ_6, ρ_7 are the attractor maps for the flow $\varphi^n(\tilde{\mathcal{L}}^0_{\tilde{e}_j,e_j})$.
5. The map ρ_8 is homotopy equivalence given by the deformation

$$(1-s)\mathcal{L}_1 + s\tilde{\mathcal{L}}^0_{\tilde{e}_j,e_j}, \ s \in [0,1]. \tag{6.13}$$

Now we check that the above diagram commutes:

1. The maps ρ_2 and ρ_6 are the defined as attractor maps for the flows $\varphi^n(\mathcal{L}^0_{e_j})$ and $\varphi^n(\tilde{\mathcal{L}}^0_{\tilde{e}_j,e_j})$ respectively. Since these two flows are generated by the same vector

field when restricted to $\tilde{J}^{n,+}_{\tilde{m}_j}$, we see that ρ_2 is S^1-homotopic to ρ_6, written as $\rho_2 \cong \rho_6$.

2. Because the attractor maps for the same flow are transitive, we have $\rho_7 \cong \rho_4 \circ \rho_6$.

3. We deform $\tilde{\mathcal{L}}^0_{\tilde{e}_j} = \tilde{\mathcal{L}}^1_{\tilde{e}_j,e_j}$ to $\tilde{\mathcal{L}}^0_{\tilde{e}_j,e_j}$ through the family $\tilde{\mathcal{L}}^s_{\tilde{e}_j,e_j}$. In the process of this deformation, nothing is changed on the set $p^{-1}_{\mathcal{H}}(\mathbb{R}^b \setminus (-e_j, e_j)^b)$, which contains both $\partial J^{n,+}_{m_{j+1}} \setminus \partial Str(\tilde{R})$ and $\partial \tilde{J}^{n,+}_{\tilde{m}_j} \setminus \partial Str(\tilde{R})$. Therefore, we obtain a family of attractor maps: we get ρ_4 when $s = 0$ and get $\tilde{\rho}_2$ when $s = 1$. Notice that ρ_3 and ρ_5 are the homotopy equivalences induced by this deformation. The identity $\tilde{\rho}_2 \cong \rho_5 \circ \rho_4 \circ \rho_3$ can be proved using the homotopy invariance of the attractor map.

4. The map ρ_3 is induced by the deformation $\tilde{\mathcal{L}}^s_{\tilde{e}_j,e_j}$ with s going from 1 to 0. We just get $\mathcal{L}^s_{e_j}$ if we restrict this deformation to the set $\tilde{J}^{n,+}_{\tilde{m}_j}$. Therefore, we have $\rho_1 \cong \rho_3^{-1}$.

5. Notice that $\tilde{\rho}_1 \circ \rho_5$ is the homotopy equivalence induced by the following deformation:

$$\tilde{\mathcal{L}}^0_{\tilde{e}_j,e_j}|_{J^+_{m_{j+1}}} \to \tilde{\mathcal{L}}^1_{\tilde{e}_j,e_j}|_{J^+_{m_{j+1}}} = \tilde{\mathcal{L}}^0_{\tilde{e}_j}|_{J^+_{m_{j+1}}} \to \tilde{\mathcal{L}}^1_{\tilde{e}_j}|_{J^+_{m_{j+1}}} = \mathcal{L}_1|_{J^+_{m_{j+1}}}. \tag{6.14}$$

In order to prove the identity $\rho_8 = \tilde{\rho}_1 \circ \rho_5$, we just need to show that the homotopy equivalences $I_{S^1}(\varphi^n(\tilde{\mathcal{L}}^0_{\tilde{e}_j,e_j}), \mathrm{Inv}(J^{n,+}_{m_{j+1}})) \xrightarrow{\cong} I_{S^1}(\varphi^n(\mathcal{L}_1), \mathrm{Inv}(J^{n,+}_{m_{j+1}}))$ which are induced by deformations (6.13) and (6.14) are S^1-homotopic to each other. To see this, for any $r \in [0,1]$, we consider the following 2-step deformation.

   (a) First deform $\tilde{\mathcal{L}}^0_{\tilde{e}_j,e_j}$ to $r\mathcal{L}_1 + (1-r)\tilde{\mathcal{L}}^1_{\tilde{e}_j,e_j} = \tilde{\mathcal{L}}^r_{\tilde{e}_j}$ through the family $rs\mathcal{L}_1 + (1-rs)\tilde{\mathcal{L}}^s_{\tilde{e}_j,e_j}$, with s going from 0 to 1.

   (b) Then deform $\tilde{\mathcal{L}}^r_{\tilde{e}_j}$ to $\tilde{\mathcal{L}}^1_{\tilde{e}_j} = \mathcal{L}_1$ through the family $\tilde{\mathcal{L}}^s_{\tilde{e}_j}$, with s going from r to 1.

Setting r to be 0 and 1 in the above deformation, we will get (6.14) and (6.13) respectively. As before, the flow near $\partial J^+_{m_{j+1}} \setminus \partial Str(\tilde{R})$ is not changed. By Lemma

6.3.11, all the functionals involved in the above deformation are R_2-bounded. Since $\tilde{R} > R_2$, $J^{n,+}_{m_{j+1}}$ is an isolating neighborhood for all these functionals when n is large enough. Therefore, as r goes from 0 to 1, we get a S^1-homotopy between the homotopy equivalences induced by (6.13) and (6.14).

We have proved that the diagram is commutative up to S^1-homotopy. As a corollary, the map $\hat{i}^{n,+}_{\tilde{m}_j} \circ \bar{i}^{n,+}_{m_j} = \tilde{\rho}_1 \circ \tilde{\rho}_2 \circ \rho_1 \circ \rho_2$ is S^1-homotopic to $\rho_8 \circ \rho_7$. Now we consider the attractor map for the flow $\mathcal{L}_1$, which we denote by

$$i^+ : I_{S^1}(\varphi^n(\mathcal{L}_1), \mathrm{Inv}(J^{n,+}_{m_j})) \to I_{S^1}(\varphi^n(\mathcal{L}_1), \mathrm{Inv}(J^{n,+}_{m_{j+1}})).$$

We deform $\mathcal{L}_1$ to $\tilde{\mathcal{L}}^0_{\tilde{e}_j, e_j}$ through the family $(1-s)\mathcal{L}_1 + s\tilde{\mathcal{L}}^0_{\tilde{e}_j, e_j}$ $(s \in [0,1])$. Notice that for any s, $(1-s)\mathcal{L}_1 + s\tilde{\mathcal{L}}^0_{\tilde{e}_j, e_j}$ equals $\mathcal{L}_1$ on the set $p^{-1}_{\mathcal{H}}([-e_j+1, e_j-1]^b \cup (\mathbb{R}^b \setminus (-\tilde{e}_j, \tilde{e}_j)^b))$, which contains both $\partial J^+_{m_j} \setminus \partial Str(\tilde{R})$ and $\partial J^+_{m_{j+1}} \setminus \partial Str(\tilde{R})$. Therefore, we get a family of attractors:

$$\mathrm{Inv}(\varphi^n((1-s)\mathcal{L}_1 + s\tilde{\mathcal{L}}^0_{\tilde{e}_j, e_j}), J^{n,+}_{m_j}) \subset \mathrm{Inv}(\varphi^n((1-s)\mathcal{L}_1 + s\tilde{\mathcal{L}}^0_{\tilde{e}_j, e_j}), J^{n,+}_{m_{j+1}}).$$

By the homotopy invariance of the attractor maps, we see that i^+ also is homotopic to $\rho_8 \circ \rho_7$. This finish the proof of the proposition. □

Proposition 6.3.12 actually implies that the direct system (6.12) contains a subsystem whose colimit gives the ind-spectrum $\underline{\mathrm{swf}}^A$ for the perturbation f_1. Similarly, we can prove that the ind-spectrum for the perturbation f_2 is also a subsequential colimit of (6.12). Therefore, by Lemma 3.1.1, we see that f_1 and f_2 gives the same ind-spectrum up to canonical isomorphism.

Finally, we address the situation when $f_1(a,\phi) = \frac{\delta_1}{2}\|\phi\|^2_{L^2} + \bar{f}_1(a,\phi)$ and $f_2(a,\phi) = \frac{\delta_2}{2}\|\phi\|^2_{L^2} + \bar{f}_2(a,\phi)$ with $\delta_1 \neq \delta_2$. This can now be proved the standard homotopy invariance argument as follows. We set $\delta_t = (2-t)\delta_1 + (t-1)\delta_2$. For each $t_0 \in [1,2]$, we can find an extended cylinder function $\bar{f}$ such that the pair $(\delta_t, \bar{f})$ gives a perturbed Chern-Simons-Dirac functional whose critical points are all nondegenerate in the sense of [33, Definition 12.1.1] for any t near t_0. Here we essentially use the compactness result for critical points, which is a special case of [33, Proposition 11.6.4]. Hence, we can find a subdivision $1 = t_1 < \cdots < t_n = 2$ and $\bar{f}'_1, \ldots, \bar{f}'_{n-1} \in \mathcal{P}$ with $\bar{f}'_1 = \bar{f}_1$ and $\bar{f}'_{n-1} = \bar{f}_2$ such that the pair $(\delta_t, \bar{f}'_j)$ gives a good perturbation for any $t \in [t_j, t_{j+1}]$. By homotopy invariance of the Conley index, we see that $(\delta_{t_j}, \bar{f}'_j)$ and $(\delta_{t_{j+1}}, \bar{f}'_j)$ give the same ind-spectrum $\underline{\mathrm{swf}}^A$. Since we already showed that the ind-spectrum does not depend on the choice of the extended cylinder function when δ is fixed, we can conclude that f_1 and f_2 give the same $\underline{\mathrm{swf}}^A$ (up to canonical isomorphisms). This finishes the proof of the invariance for (II).

6.4 The invariance for (I)

Now we discuss what happens when we vary the metric g and the base connection A_0. Let (A_0, g_0) and (A_1, g_1) be two pairs of base connections and metrics. We can connect them by a smooth path $\alpha(s) = (A_s, g_s)$ with $s \in [0, 1]$. As in the proof of the invariance for δ, we can divide $[0, 1]$ into small subintervals $[s_j, s_{j+1}]$ such that, for each subinterval $[s_j, s_{j+1}]$, we can fix the choice of the auxiliary data $(f, \bar{g}, \theta, \tilde{R}, \{\lambda_n\}, \{\mu_n\})$.

As s varies between s_j and s_{j+1}, we get a continuous family of Coulomb slices $Coul(Y, s)$ and a family of sequences of bounded sets

$$J^+_{1,s} \subset J^+_{2,s} \subset \cdots .$$

For any positive integer n, we have a (usually not continuous) family of finite-dimensional spaces $V^{\mu_n}_{\lambda_n}(s)$. As before, we denote by $\bar{V}^0_{\mu_n}(s)$ the orthogonal complement of $i\Omega^1_h(Y)$ in $V^0_{\mu_n}(s)$. Let $J^{n,+}_{m,s} = J^+_{m,s} \cap V^{\mu_n}_{\lambda_n}(s)$ and $\varphi_{n,s}$ be the approximated Seiberg-Witten flow on $V^{\mu_n}_{\lambda_n}(s)$. The following lemma is a direct consequence of Theorem 6.1.2.

Lemma 6.4.1. *For any positive integer m and a sufficiently large integer n relative to m, we have*

$$\Sigma^{-\bar{V}^0_{\mu_n}(s_j)} I_{S^1}(\varphi_{n,s_j}, \mathrm{Inv}(J^{n,+}_{m,s_j})) \cong \Sigma^{\mathrm{sf}(-\not{D}, \alpha[s_j, s_{j+1}])\mathbb{C}} \Sigma^{-\bar{V}^0_{\mu_n}(s_{j+1})} I_{S^1}(\varphi_{n,s_{j+1}}, \mathrm{Inv}(J^{n,+}_{m,s_{j+1}}))$$

as objects of $\mathfrak{C}$.

Taking colimit of the above isomorphisms, we obtain

$$\underline{\mathrm{swf}}^A(Y, \mathfrak{s}, A_{s_j}, g_{s_j}; S^1) \cong \Sigma^{\mathrm{sf}(-\not{D}, \alpha([s_j, s_{j+1}]))\mathbb{C}} \underline{\mathrm{swf}}^A(Y, \mathfrak{s}, A_{s_{j+1}}, g_{s_{j+1}}; S^1).$$

By additivity of spectral flow, we can conclude that

$$\underline{\mathrm{swf}}^A(Y, \mathfrak{s}_Y, A_0, g_0; S^1) \cong \Sigma^{\mathrm{sf}(-\not{D},\alpha)\mathbb{C}} \underline{\mathrm{swf}}^A(Y, \mathfrak{s}_Y, A_1, g_1; S^1). \tag{6.15}$$

Therefore, $\underline{\mathrm{swf}}^A$ can only change by suspension or desuspension of copies of $\mathbb{C}$ when we vary the pair (A_0, g_0). Now we discuss the following two cases separately.

(1) $\mathfrak{s}$ is torsion: In this case, we recall that there is a well defined quantity $n(Y, \mathfrak{s}, A_0, g)$. By excision argument as in [40], we have

$$n(Y, \mathfrak{s}, A_0, g_0) + \mathrm{sf}(-\not{D}, \alpha) = n(Y, \mathfrak{s}, A_1, g_1).$$

This implies

$$(\underline{\mathrm{swf}}^A(Y, \mathfrak{s}, A_0, g_0; S^1), 0, n(Y, \mathfrak{s}, A_0, g_0)) \cong (\underline{\mathrm{swf}}^A(Y, \mathfrak{s}, A_1, g_1; S^1), 0, n(Y, \mathfrak{s}, A_1, g_1))$$

and the same result holds for $\underline{\mathrm{swf}}^R$. This finishes the proof of invariance of $\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}; S^1)$ and $\underline{\mathrm{SWF}}^R(Y, \mathfrak{s}; S^1)$ in the torsion case.

(2) $\mathfrak{s}$ is non-torsion: In this case, let $l = \mathrm{g.c.d}\{(c_1(\mathfrak{s}) \cup h)[Y] \mid h \in H^1(Y; \mathbb{Z})\}$. We pick a harmonic gauge transformation $u_0 \in \mathcal{G}_Y^{h,o} = H^1(Y; \mathbb{Z})$ such that $(c_1(\mathfrak{s}) \cup [u_0])[Y] = l$ and denote by $Coul(Y, A_0)$ and $Coul(Y, u_0(A_0))$ the Coulomb slices with the base connections A_0 and $u_0(A_0) = A_0 - u_0^{-1} du_0$ respectively. (Actually, these two slices correspond the same subspace of $\mathcal{C}_Y$. However, since the base connections are different, this subspace is identified with $L^2_k(i \ker d^* \oplus \Gamma(S))$ in different ways. For this reason, we distinguish them for clarity.) The gauge transformation $u_0 : Coul(Y, A_0) \to Coul(Y, u_0(A_0))$ preserves the functional $CSD_{\nu_0, f}$, its formal gradient, the subspace $i\Omega^1_h(Y)$, the finite dimensional subspaces $V^{\mu_n}_{\lambda_n}$ and both the L^2-metric and the non-linear metric $\| \cdot \|_{\tilde{g}}$. From this fact, we get a natural isomorphism

$$\underline{\mathrm{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1) \cong \underline{\mathrm{swf}}^A(Y, \mathfrak{s}, u_0(A_0), g; S^1). \tag{6.16}$$

Let α be any path going from A_0 to $u_0(A_0)$. As the spectral flow $\mathrm{sf}(-\not{D}_A, \alpha)$ can be calculated using excision and the Atiyah-Singer index theorem (see of [33, Lemma 14.4.6]), it is not hard to check that $\mathrm{sf}(-\not{D}_A, \alpha) = \frac{l}{2}$. Combining the above two equivalences with (6.15) and (6.16), we get

$$\underline{\mathrm{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1) \cong \Sigma^{\frac{l}{2}\mathbb{C}} \underline{\mathrm{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1)$$

and similar results hold for $\underline{\mathrm{swf}}^R$. This proves the periodicity result in the main theorem.

CHAPTER 7

The relative Bauer-Furuta invariants

7.1 The setup and notations

Let X be a compact, oriented 4-manifold with boundary Y. We give X a Riemannian metric $\hat{g}$ so that X is isomorphic to $[-3,0] \times Y$ near the boundary, with ∂X identified with $\{0\} \times Y$. (Note that we shall often put a hat over the 4-dimensional objects.) Assume that X has a spin^c structure $\hat{\mathfrak{s}}$ which extends $\mathfrak{s}$. Denote by S_X^+ and S_X^- the associated positive and negative spinor bundles respectively and let $\hat{\rho}$ be the Clifford multiplication. We fix a base spin^c connection $\hat{A}_0$ on X and require that $\nabla_{\hat{A}_0} = \frac{d}{dt} + \nabla_{A_0}$ when restricted to $[-3,0] \times Y$. As in the 3-dimensional case, the space of spin^c connections on X can be identified with $i\Omega^1(X)$ via $\hat{A} \to \hat{A} - \hat{A}_0$. For $\hat{a} \in i\Omega^1(X)$, we let $\not{D}_{\hat{a}}^+ = \not{D}^+ + \hat{\rho}(\hat{a})$ be the corresponding Dirac operator, acting on sections of S_X^+. In particular, $\not{D}^+$ corresponds to the base connection $\hat{A}_0$.

In order to define the relative Bauer-Furuta invariants, we need to make a choice of the following axillary data:

- A base point $\hat{o}$
- A set of loops $\{\alpha_1, \alpha_2, \cdots, \alpha_{b_1}\}$ in Y representing a basis of $H_1(Y;\mathbb{R})$ such that $\{\alpha_1, \alpha_2, \cdots, \alpha_e\}$ also represents a basis for $\operatorname{im}(H_1(Y;\mathbb{R}) \to H_1(X;\mathbb{R}))$;
- A set of loops $\{\beta_1, \cdots, \beta_l\}$ in X representing a basis of $\operatorname{coker}(H_1(Y;\mathbb{R}) \to H_1(X;\mathbb{R}))$;
- A *based path data* $[\vec{\eta}]$, whose definition is given below.

Definition 7.1.1. A based path data is an equivalent class of paths $(\eta_1, \eta_2, \cdots, \eta_{b_0})$, where η_j is a path from $\hat{o}$ to some point in Y_j. We define $(\eta_1, \cdots, \eta_{b_0})$ and $(\eta'_1, \cdots, \eta'_{b_0})$ to be equivalent if for any $1 \leq j \leq b_0$, the composed path $\eta'_j \cdot \eta_j^{-1}$ represents the zero class in $H_1(X, Y; \mathbb{R})$.

Now we start setting up the gauge theory on X. Similar to the three dimensional case, it will be convenient to restrict the discussion to a specific subspace of the whole configuration space called the double Coulomb slice. Let us briefly recall its definition following [30].

For a differential form $\hat{a}$ on X, we have a decompose $\hat{a}|_Y = \mathbf{t}\hat{a} + \mathbf{n}\hat{a}$, where $\mathbf{t}\hat{a}$ and $\mathbf{n}\hat{a}$ are the tangential part and the normal part respectively. When Y has several connected components, we denote by $\mathbf{t}_i\hat{a}$ and $\mathbf{n}_i\hat{a}$ the corresponding parts of $\hat{a}|_{Y_i}$. We say that a 1-form $\hat{a}$ satisfies the double Coulomb condition if:

1. $\hat{a}$ is coclosed, i.e. $d^*\hat{a} = 0$;
2. The restriction of $\hat{a}$ to the boundary is coclosed, i.e. $d^*(\mathbf{t}\hat{a}) = 0$;
3. For each j, we have $\int_{Y_j} \mathbf{t_j}(*\hat{a}) = 0$.

Denote by $\Omega^1_{CC}(X)$ the space of all 1-forms satisfying the double Coulomb condition. The double Coulomb slice $Coul^{CC}(X)$, which is a subspace of $L^2_{k+1/2}(i\Omega^1(X) \oplus S^+_X)$, is defined as:

$$Coul^{CC}(X) := L^2_{k+1/2}(i\Omega^1_{CC}(X) \oplus \Gamma(S^+_X)).$$

There is a natural restriction map

$$r : Coul^{CC}(X) \to Coul(Y) \tag{7.1}$$

given by

$$(\hat{a}, \hat{\phi}) \mapsto (\mathbf{t}\hat{a}, \hat{\phi}|_Y).$$

Now consider the space of harmonic 1-forms on X with the Dirichlet boundary condition

$$\Omega^1_D(X) := \{\hat{a} \in \Omega^1(X) \mid d\alpha = d^*\alpha = 0, \mathbf{t}(\hat{a}) = 0\}.$$

By classical Hodge theory on X, there is a natural isomorphism

$$\Omega^1_D(X) \cong \ker(H^1(X;\mathbb{R}) \to H^1(Y;\mathbb{R})). \tag{7.2}$$

We can treat $i\Omega^1_D(X)$ as a subspace of $Coul^{CC}(X)$ via the embedding $\hat{a} \to (\hat{a}, 0)$. There is a (nonorthogonal) projection

$$\hat{p}_\beta : Coul^{CC}(X) \to i\Omega^1_D(X)$$

sending $(\hat{a}, \hat{\phi})$ to $\hat{\pi}_{\mathcal{H}}(\hat{a})$, where $\hat{\pi}_{\mathcal{H}}(\hat{a})$ is the unique element in $i\Omega^1_D(X)$ satisfying

$$\int_{\beta_j} \hat{a} = \int_{\beta_j} \hat{\pi}_{\mathcal{H}}(\hat{a}) \text{ for } j = 1, 2, \cdots, l.$$

For later convenience, we also define the map

$$\hat{p}_\alpha : Coul^{CC}(X) \to \mathbb{R}^e$$

given by

$$(\hat{a}, \hat{\phi}) \mapsto (-i \int_{\alpha_1} \hat{a}, \cdots, -i \int_{\alpha_e} \hat{a}).$$

and let

$$Coul^{CC}_0(X) := \ker \hat{p}_\alpha.$$

Now we consider the group of gauge transformations. Let $\mathcal{G}_X$ be the $L^2_{k+3/2}$-completion of $\mathrm{Map}(X, S^1)$. The proof of the following lemma is a slight adaption of [30, Propositioon 1] and we omit it.

Lemma 7.1.2. *Inside each connected component of $\mathcal{G}_X$, there is a unique element $\hat{u} : X \to S^1$ satisfying*

$$\hat{u}(\hat{o}) = 1, \;\; u^{-1}du \in i\Omega^1_{CC}(X).$$

These elements form a subgroup of $\mathcal{G}_X$, which we denote by $\mathcal{G}^{h,\hat{o}}_X$.

By Lemma 7.1.2, we have a natural isomorphism

$$\mathcal{G}^{h,\hat{o}}_X \cong \pi_0(\mathcal{G}_X) \cong H^1(X;\mathbb{Z}). \tag{7.3}$$

We denote by $\mathcal{G}^{h,\hat{o}}_{X,Y}$ the subgroup of $\mathcal{G}^{h,\hat{o}}_X$ that corresponds to the subgroup $\ker(H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z}))$. Elements in $\mathcal{G}^{h,\hat{o}}_{X,Y}$ restrict to a constant function on each component of Y.

Now we define the relative Picard torus

$$\begin{aligned} \mathrm{Pic}^0(X,Y) :&= i\Omega^1_D(X)/\mathcal{G}^{h,\hat{o}}_{X,Y} \\ &\cong \ker(H^1(X;\mathbb{R}) \to H^1(Y;\mathbb{R}))/\ker(H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})). \end{aligned} \tag{7.4}$$

The double Coulomb slice $Coul^{CC}(Y)$ is preserved by $\mathcal{G}^{h,\hat{o}}_{X,Y}$ and $\mathcal{G}^{h,\hat{o}}_{X}$. Our main concern will be the quotient space $Coul^{CC}(Y)/\mathcal{G}^{h,\hat{o}}_{X,Y}$. We treat it as a Hilbert bundle over $\mathrm{Pic}^0(X,Y)$, where the bundle structure is induced by the projection $\hat{p}_\beta$.

Remark. A different Hilbert bundle structure of $Coul^{CC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y}$ can be induced by the orthogonal projection

$$\hat{p}_\perp : Coul^{CC}(X) \to i\Omega^1_D(X).$$

The main reason we are using $\hat{p}_\beta$ here is that $\hat{p}_\beta$ behaves better than $\hat{p}_\perp$ under the gluing of 4-manifolds, which simplifies the proof of gluing theorem for the relative Bauer-Furuta invariant.

Notation. For a pair $(\hat{a},\hat{\phi}) \in Coul^{CC}(X)$, we denote by $[\hat{a},\hat{\phi}]$ the corresponding point in the Hilbert bundle $Coul^{CC}(Y)/\mathcal{G}^{h,\hat{o}}_{X,Y}$. We write $\|\cdot\|_F$ for the fiber-direction norm on $Coul^{CC}(Y)/\mathcal{G}^{h,\hat{o}}_{X,Y}$.

Note that $\|\cdot\|_F$ is not directly given by the restriction of $L^2_{k+1/2}$-norm on $Coul^{CC}(Y)$, because the latter is not invariant under $\mathcal{G}^{h,\hat{o}}_{X,Y}$. However, we have the following lemma, which is a simple consequence of compactness of $\mathrm{Pic}^0(X,Y)$. Let $\mathbf{D} \subset Coul^{CC}(X)$ be a fundamental domain for $\mathrm{Pic}^0(X,Y)$, fixed through out this chapter.

Lemma 7.1.3. *There exists a constant C such that for any $(\hat{a},\hat{\phi}) \in Coul^{CC}(X)$ with $\hat{p}_\beta(\hat{a},\hat{\phi}) \in \mathbf{D}$, we have*

$$\frac{\|[\hat{a},\hat{\phi}]\|_F}{C} \le \|(\hat{a},\hat{\phi})\|_{L^2_{k+1/2}} \le C\cdot(\|[\hat{a},\hat{\phi}]\|_F + 1).$$

One can check that the map $\hat{p}_\alpha$ is invariant under the action of $\mathcal{G}^{h,\hat{o}}_{X,Y}$. As a result, it induce a well-defined map

$$\tilde{p}_\alpha : Coul^{CC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y} \to \mathbb{R}^e.$$

The primage $\tilde{p}_\alpha^{-1}(0)$ is just $Coul_0^{CC}(X)/\mathcal{G}_{X,Y}^{h,\hat{o}}$. This is subbundle of $Coul^{CC}/\mathcal{G}_{X,Y}^{h,\hat{o}}$ with codimension e.

The last task of this section is to define the restriction map from $Coul^{CC}(X)/\mathcal{G}_{X,Y}^{h,\hat{o}}$ to the $Coul(Y)$. This is where the based path data $[\vec{\eta}]$ comes into play. First notice that the restriction map r in formula (7.1) is not invariant under the action of $\mathcal{G}_{X,Y}^{h,\hat{o}}$. To remedy this, we consider the "twisted version"

$$\begin{aligned} r' : Coul^{CC}(X) &\to \prod_{j=1}^{b_0} Coul(Y_j) \cong Coul(Y) \\ (\hat{a}, \hat{\phi}) &\mapsto \prod_{j=1}^{b_0} (\mathbf{t}_j \hat{a}, e^{-\int_{\eta_j} \hat{p}_\beta(\hat{a},\hat{\phi})} \cdot \hat{\phi}|_{Y_j}). \end{aligned} \tag{7.5}$$

The following result can be verified by simple calculation.

Lemma 7.1.4. *r' does not depend on the choice of the representative $\vec{\eta}$ in its equivalent class (see Definition 7.1.1). Moreover, the map r' is invariant under the action of $\mathcal{G}_{X,Y}^{h,\hat{o}}$. As a result, we can define the induced map*

$$\tilde{r} : Coul^{CC}(X)/\mathcal{G}_{X,Y}^{h,\hat{o}} \to Coul(Y). \tag{7.6}$$

We denote by $\tilde{r}^0$ the restriction of $\tilde{r}$ on the subgundle $Coul_0^{CC}(X)/\mathcal{G}_{X,Y}^{h,\hat{o}}$.

7.2 The Seiberg-Witten map and its finite dimensional approximations

In this section, we will define the Seiberg-Witten maps $SW, \widetilde{SW}, \widetilde{SW^0}$ and their finite dimensional approximations. These maps will be important when we define the relative Bauer-Furuta invariant in Section 7.4.

7.2.1 Definition of the Seiberg-Witten map

The (perturbed) Seiberg-Witten map

$$SW : Coul^{CC}(X) \to L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S_X^-))$$

is given by

$$SW(\hat{a},\hat{\phi}) := (d^+\hat{a}, \not{D}^+\hat{\phi}) + (\frac{1}{2}F^+_{\hat{A}^t_0} - \hat{\rho}^{-1}(\hat{\phi}\hat{\phi}^*)_0 - \omega_0^+, \hat{\rho}(\hat{a})\hat{\phi}) + \hat{q}(\hat{a},\hat{\phi}), \tag{7.7}$$

explained as below:

- $(\hat{\phi}\hat{\phi}^*)_0$ denotes the trace-free part of $\hat{\phi}\hat{\phi}^* \in \Gamma(\mathrm{End}(S^+_X))$;
- ω_0^+ is the self-dual part of the closed perturbation 2-from $\omega_0 \in \Omega^2(X)$. We require that $\omega_0|_{[-3,0]\times Y} = \pi i \nu_0$. (Recall that ν_0 is the harmonic form representing $-c_1(\mathfrak{s})$.) Such ω_0 can be obtained from $-\frac{1}{2}F_{\hat{A}^t_0} + dh$ for some $h \in i\Omega^0(X)$.
- To define the perturbation term $\hat{q}(\hat{a},\hat{\phi})$, we take a bump-function $\iota : [-3,0] \to [0,1]$ satisfying $\iota([-3,-2]) = 0$, $\iota([-1,0]) = 1$ and $0 \le \iota'(x) \le 2$. For $t \in [-3,0]$, let a_t be the pull back of $\hat{a}$ by the inclusion $\{t\}\times Y \to X$ and let $\phi_t = \hat{\phi}|_{\{t\}\times Y}$. Then we define [1]

$$\hat{q}(\hat{a},\hat{\phi}) := \iota(t)((dt \wedge \mathrm{grad}^1 f(a_t,\phi_t) + *\,\mathrm{grad}^1 f(a_t,\phi_t)), \mathrm{grad}^2 f(a_t,\phi_t)). \tag{7.8}$$

  (Recall that a good perturbation f (see Definition 2.1.3) is involved in the construction of the spectrum invariants. We write its gradient as $(\mathrm{grad}^1 f, \mathrm{grad}^2 f)$.)

We have a decomposition

$$SW = L + Q \tag{7.9}$$

where

$$L(\hat{a},\hat{\phi}) = (d^+\hat{a}, \not{D}^+_{\hat{p}_\beta(\hat{a})}\hat{\phi}) \text{ and } Q = SW - L.$$

A fundamental computation, making use of the tameness condition on grad f (see [33, Definition 10.5.1]), gives the following lemma:

Lemma 7.2.1. *For any number $j \ge 2$ and any subset $U \subset Coul^{CC}(Y)$ which is bounded in L^2_j, the set $Q(U)$ is also bounded in L^2_j.*

---

[1] We write $*$ for the 3-dimensional Hodge star operator and wirte $*_4$ for the 4-dimensional Hodge star operator.

Now we start defining the map $\widetilde{SW}$, which is the folded version of SW. Notice that the map

$$(SW, \hat{p}_\beta) : Coul^{CC}(X) \to L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S^-_X)) \times (i\Omega^1_D(X))$$

is equivariant under the action of $\mathcal{G}^{h,\hat{o}}_{X,Y}$, where the action on the target space is given by

$$u \cdot ((\omega, \hat{\phi}), \hat{h}) := ((\omega, u\hat{\phi}), \hat{h} - u^{-1}du).$$

As a result, $(SW, \hat{p}_\beta)$ induces a map

$$\overline{SW} : Coul^{CC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y} \to (L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S^-_X)) \times (i\Omega^1_D(X)))/\mathcal{G}^{h,\hat{o}}_{X,Y}.$$

By Kuiper's theorem, the Hilbert bundle $(L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S^-_X)) \times (i\Omega^1_D(X)))/\mathcal{G}^{h,\hat{o}}_{X,Y}$ can be trivialized (unique to homotopy). We fix a trivialization and consider the induced projection from this bundle to its fiber $L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S^-_X))$. Composing the map $\overline{SW}$ with this projection. We get the map

$$\widetilde{SW} : Coul^{CC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y} \to L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S^-_X)).$$

The map

$$\widetilde{SW^0} : Coul^{CC}_0(X)/\mathcal{G}^{h,\hat{o}}_{X,Y} \to L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S^-_X))$$

is defined as the restriction of $\widetilde{SW}$.

7.2.2 The approximated Seiberg-Witten map

Just like the map $(SW, \hat{p}_\beta)$, the map $(L, \hat{p}_\beta)$ is also equivariant under the action of $\mathcal{G}^{h,\hat{o}}_{X,Y}$. As a result, the decomposition (7.9) induces a decomposition

$$\widetilde{SW} = \tilde{L} + \tilde{Q} : Coul^{CC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y} \to L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S^-_X))$$

where $\tilde{L}$ is a fiberwise linear map. By restricting the subbundle $Coul^{CC}_0(X)/\mathcal{G}^{h,\hat{o}}_{X,Y}$, we get a similar decomposition

$$\widetilde{SW^0} = \tilde{L}^0 + \tilde{Q}^0. \tag{7.10}$$

Now we define the finite dimensional approximations of $\widetilde{SW}$ and $\widetilde{SW}^0$ using these decompositions. Recall that we chose an increasing sequence $\{\mu_n\}$ and a decreasing sequence $\{\lambda_n\}$ when defining the spectrum invariants for Y (see Section 5.1). We have the following lemma:

Lemma 7.2.2. *For any n, the map*

$$(\tilde{L}, p^{\mu_n}_{-\infty} \circ \tilde{r}) : Coul^{CC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y} \to L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S^-_X)) \oplus V^{\mu_n}_{-\infty}$$

and the map

$$(\tilde{L}^0, p^{\mu_n}_{-\infty} \circ \tilde{r}^0) : Coul^{CC}_0(X)/\mathcal{G}^{h,\hat{o}}_{X,Y} \to L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S^-_X)) \oplus V^{\mu_n}_{-\infty}$$

are fiberwise Freedholm.

Proof. This is a straightforward application of the Atiyah-Patodi-Singer theory [5]. □

Now we choose an increasing sequence $\{U_n\}$ of finite dimensional subspaces of $L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S^-_X))$ with the following two properties:

(i) As $n \to \infty$, the orthogonal projection $P_{U_n} : L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S^-_X)) \to U_n$ converges to the identity map pointwisely.

(ii) For any point $p \in \text{Pic}^0(X, Y)$ and any n, the restriction of $(\tilde{L}^0, p^{\mu_n}_{-\infty} \circ \tilde{r}^0)$ to the fiber over p is transverse to U_n. This implies the similar result for the map $(\tilde{L}, p^{\mu_n}_{-\infty} \circ \tilde{r})$.

Let W_n denotes $(\tilde{L}, p^{\mu_n}_{-\infty} \circ \tilde{r})^{-1}(U_n)$ and W^0_n be the intersection $W_n \cap (Coul^{CC}_0(X)/\mathcal{G}^{h,\hat{o}}_{X,Y})$. By Lemma 7.2.2 and property (ii) above, both W_n and W^0_n are finite dimensional vector spaces over the Picard torus $\text{Pic}^0(X, Y)$. We let

$$\widetilde{SW}_n = \tilde{L} + P_{U_n} \circ \tilde{Q} : W_n \to U_n$$

and let $\widetilde{SW}^0_n : W^0_n \to U_n$ be its restriction. These are called *the approximated Seiberg-Witten maps.*

7.3 The boundedness results

In this section, we will prove some analytical results on the (approximated) Seiberg-Witten map. To state the result, we introduce the following definition.

Definition 7.3.1. A finite type X-trajectory is a pair (x, γ) with

- $\tilde{x} = [\hat{a}, \hat{\phi}] \in Coul^{CC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y}$ satisfies $\widetilde{SW}(\tilde{x}) = 0$;
- $\gamma : [0, +\infty) \to Coul(Y)$ is a finite type Seiberg-Witten trajectory;
- $\tilde{r}(\tilde{x}) = \gamma(0)$.

A finite type X-trajectory can be thought of a finite energy solution of the Seiberg-Witten equations on the manifold $X^* = X \cup ([0, +\infty) \times Y)$. We have the following theorem, which is a refinement of in [30, Corolary 2].

Theorem 7.3.2. *For any $M > 0$, there exists a constant $R > 0$ such that for any finite type X-trajectory $(\tilde{x}, \gamma)$ satisfying*

$$\tilde{p}_\alpha(\tilde{x}) \in [-M, M]^l \tag{7.11}$$

we have

$$\|\tilde{x}\|_F < R \text{ and } \gamma([0, \infty)) \subset \operatorname{int}(Str(R)).$$

We need a lemma before proving Theorem 7.3.2. First recall that the topological energy of a perturbed Seiberg-Witten trajectory $\gamma : [a, b] \to Coul(Y)$ is defined as

$$\mathcal{E}^{\text{top}}(\gamma, CSD_{\nu_0,f}) := 2(\sup_{t \in I} CSD_{\nu_0,f}(\gamma(t)) - \inf_{t \in I} CSD_{\nu_0,f}(\gamma(t))).$$

Lemma 7.3.3. *For any $C > 0$, there exist a constant R such that for any perturbed Seiberg-Witten trajectory $\gamma : [-1, 1] \to Coul(Y)$ with $\mathcal{E}^{top}(\gamma, CSD_{\nu_0,f}) < C$, we have $\gamma(0) \in \operatorname{int}(Str(R))$.*

Proof of Lemma 7.3.3. Suppose this is not true. Then we can find a sequence of trajectories $\{\gamma_n\}$ with $\mathcal{E}^{\text{top}}(\gamma_n, CSD_{\nu_0,f}) < C$ such that

$$\lim_{n \to +\infty} \|u_n \cdot \gamma_n(0)\|_{L^2_k} \to +\infty, \ \forall \{u_n\} \subset \mathcal{G}^h_Y. \tag{7.12}$$

To get the contradiction, we lift γ_n to $\tilde{\gamma}_n : [-\frac{1}{2}, -\frac{1}{2}] \to \mathcal{C}_Y$ satisfying:

$$\tilde{\gamma}_n(0) = \gamma_n(0), \ \frac{d}{dt}\tilde{\gamma}_n(t) = -\operatorname{grad} CSD_{\nu_0,f}(\tilde{\gamma}_n(t)).$$

Such $\tilde{\gamma}_n$ can be treated as solutions of the perturbed Seiberg-Witten equations over the manifold $[-\frac{1}{2}, \frac{1}{2}] \times Y$. By [33, Theorem 10.7.1] (adapted to the balanced perturbed case), after passing to a subsequence, there exists a sequence of gauge transformations $\hat{u}_n : [-\frac{1}{2}, \frac{1}{2}] \times Y \to S^1$ such that $\hat{u}_n \cdot \tilde{\gamma}_n$ converges in $L^2_{k+1/2}$ on any interior domain. In particular, this implies that $\Pi(\hat{u}_n|_{\{0\}\times Y} \cdot \tilde{\gamma}_n(0))$ converges in L^2_k. (Here Π denotes the nonlinear gauge projection defined in (2.3).) Notice that

$$\Pi(\hat{u}_n|_{\{0\}\times Y} \cdot \tilde{\gamma}_n(0)) = u_n \cdot \gamma_n(0)$$

for some $u_n \in \mathcal{G}^h_Y$. We get the contradiction with (7.12). □

Proof of Theorem 7.3.2. Suppose the theorem is not true for some constant M. Then we can find a sequence $\{([\hat{a}_n, \hat{\phi}_n], \gamma_n)\}_{n\in\mathbb{N}}$ of finite type X-trajectories satisfying (7.11) such that at least one of the following conditions is satisfied

(i) $\|[\hat{a}_n, \hat{\phi}_n]\|_F \to +\infty$;

(ii) There exists a sequence $\{t_n\} \subset \mathbb{R}^{\geq 0}$ such that

$$\|u_n \cdot \gamma_n(t_n)\|_{L^2_k} \to +\infty, \ \forall \{u_n\} \subset \mathcal{G}^h_Y.$$

Without loss of generality, we may also assume

$$\hat{p}_\beta(\hat{a}_n, \hat{\phi}_n) \in \mathbf{D} \tag{7.13}$$

where $\mathbf{D}$ is the fundamental domain fixed before Lemma 7.1.3.

Notice that for any fixed n, the energy $\mathcal{E}^{\text{top}}(\gamma_n|_{[t-1,t+1]}, CSD_{\nu_0,f})$ is less than 1 when t is large enough. By Lemma 7.3.3, we have $\gamma_n(t) \in \operatorname{int}(Str(R'))$ for any n and any t large enough compared to n. Here R' is a constant independent of n, j. Since $CSD_{\nu_0,f}$ is bounded on $\operatorname{int}(Str(R'))$ and $CSD_{\nu_0,f}$ is decreasing along γ_n, we get a uniform lower bound C_1 on $CSD_{\nu_0,f}(\gamma_n(t))$ for any $n \in \mathbb{N}, t \geq 0$. This in turn gives a uniform upper bound on the perturbed topological energy (see [33, formula (24.25)] for definition) of s_n, where s_n is the solution on the manifold

$$X' = X \cup ([0,1] \times Y)$$

obtained by gluing together $(\hat{a}_n, \hat{\phi}_n)$ and $\gamma_n|_{[0,1]}$. By [33] (adapted to the balanced perturbed situation), after passing to a subsequence and applying suitable gauge transformations, the solution s_n on X' converges in C^∞ on the interior domain X. In particular, this implies the following two results:

(1) We can find $\hat{u}_n \in \mathcal{G}_X^{h,\hat{o}}$ such that $\hat{u}_n \cdot (\hat{a}_n, \hat{\phi}_n)$ converges in $L^2_{k+1/2}$ to some $(\hat{a}_\infty, \hat{\phi}_\infty) \in Coul^{CC}(X)$;

(2) $CSD_{\nu_0,f}((\hat{a}_n, \hat{\phi}_n)|_{\{-1\}\times Y})$ is uniformly bounded below for any n.

We denote by $[\hat{u}] \in H^1(X;\mathbb{Z})$ the image of $\hat{u}$ under the isomorphism (7.3). By (1), we have

$$(\int_{\beta_j} \hat{a}_n) - 2\pi i \langle [\hat{u}_n], [\beta_j] \rangle = \int_{\beta_j} (\hat{a}_n - \hat{u}_n^{-1} d\hat{u}_n) \to \int_{\beta_j} \hat{a}_\infty, \; j = 1, 2, \cdots, l$$

and

$$(\int_{\alpha_j} \hat{a}_n) - 2\pi i \langle [\hat{u}_n], [\alpha_l] \rangle = \int_{\alpha_j} (\hat{a}_n - \hat{u}_n^{-1} d\hat{u}_n) \to \int_{\alpha_j} \hat{a}_\infty, \; j = 1, 2, \cdots, e.$$

By (7.11) and (7.13), we see that the set

$$\{\langle [\hat{u}_n], [\beta_j] \rangle \mid 0 \leq n, 1 \leq j \leq l\} \cup \{\langle [\hat{u}_n], [\alpha_j] \rangle \mid 0 \leq n, 1 \leq j \leq e\}$$

is a bounded subset of $\mathbb{Z}$. Since $\{\alpha_1, \cdots, \alpha_e, \beta_1, \cdots, \beta_l\}$ represents a set of basis for $H_1(X;\mathbb{R})$. We see that $\{\hat{u}_n\}$ only takes finitely many values in $\mathcal{G}_X^{h,\hat{o}}$. After passing to a subsequence, we can assume that $\hat{u}_n$ does not depend on n. Then (1) implies that after passing to a subsequence, $(\hat{a}_n, \hat{\phi}_n)$ converges in $L^2_{k+1/2}$, which rules out the possibility of (i) by Lemma 7.1.3.

As for (ii), since $(\hat{a}_n, \hat{\phi}_n)$ converges in $L^2_{k+1/2}$, we have a uniform upper bound C_2 for $CSD_{\nu_0,f}((\hat{a}_n, \hat{\phi}_n)|_{\{-1\}\times Y})$. By restricting $(\hat{a}_n, \hat{\phi}_n)$ to the slices $\{t\} \times Y$ $(-1 \leq t \leq$

0), projecting to $Coul(Y)$ using the nonlinear projection Π and gluing to the half trajectories γ_n, we obtain trajectories

$$\gamma_n' : [-1, \infty) \to Coul(Y)$$

whose topological energy is bounded above by $C_2 - C_1$. Applying Lemma 7.3.3 on $\gamma_n'|_{[t-1,t+1]}$ with $t \in [0, \infty)$, we see that

$$\gamma_n(t) \in \text{int}(Str(R'')),\ \forall n \in \mathbb{N}, 0 \leq t$$

for some uniform constant R''. This rules out the possibility of (ii). □

Corollary 7.3.4. *There exists a uniform constant R_1 such that for any finite type X-trajectory $(\tilde{x}, \gamma)$, we have $\gamma(t) \in Str(R_1)$ for any $t \in [0, +\infty)$.*

Proof. We can find a constant C with the following significance:

- For any $(y_1, \cdots, y_e) \in \mathbb{R}^e$, there exists a class $[\xi] \in H^1(X; \mathbb{Z})$ such that

$$(y_1, \cdots, y_e) - 2\pi(\langle[\alpha_1], [\xi]\rangle, \cdots, \langle[\alpha_l], [\xi]\rangle) \in [-C, C]^e$$

Let R_1 be the constant in Theorem 7.3.2 with $M = C$. Then by our choice of C, for any finite type X-trajectory $(\tilde{x}, \gamma)$, we can find a transformation $\hat{u} \in \mathcal{G}_X^{h,\hat{o}} : X \to S^1$ such that $(\hat{u} \cdot \tilde{x}, (\hat{u}|_Y) \cdot \gamma)$ is a finite type X-trajectory satisfying condition (7.11) with $M = C$. By Theorem 7.3.2, we have $(\hat{u}|_Y) \cdot \gamma(t) \in \text{int}(Str(R_1))$ for any $t \in [0, +\infty)$. This implies $\gamma(t) \in \text{int}(Str(R_1))$ for any $t \in [0, +\infty)$. □

Now we turn to the boundedness result for approximated X-trajectories.

Definition 7.3.5. For $n \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{\geq 0}$, a finite type (n, ϵ)-approximated X-trajectory is a pair $(\tilde{x}, \gamma)$, where

- $\tilde{x} \in W$ satisfies $\|\widetilde{SW}_n(\tilde{x}_n)\|_{L^2_{k-1/2}} \leq \epsilon$;
- $\gamma : [0, T) \to V_{\lambda_n}^{\mu_n}$ is a finite type approximated trajectory. Here $T \in \mathbb{R}^{>0} \cup \{+\infty\}$ is called the length of the approximated X-trajectory.

- $\gamma(0) = p^{\mu_n}_{-\infty} \circ \tilde{r}(\tilde{x}_n)$. (Note that $p^{\lambda_n}_{-\infty} \circ \tilde{r}(\tilde{x})$ always belongs to $V^{\mu_n}_{\lambda_n}$ by the definition of W_n.)

Definition 7.3.6. For each $j \in \mathbb{N}$, let $(\tilde{x}_j, \gamma_j)$ be a finite type (n_j, ϵ_j)-approximated X-trajectory of length $T_j < +\infty$. Then the sequence $\{(x_j, \gamma_j)\}_{j\in\mathbb{N}}$ is called an exhausting sequence if

$$n_j \to +\infty,\ \epsilon_j \to 0 \text{ and } T_j \to +\infty.$$

The proof of the following lemma is a slight adaption of [30, Lemma 2] and we omit it.

Lemma 7.3.7. *Let $\tilde{S}, S$ be bounded subsets of $Coul^{CC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y}$ and $Coul(Y)$ respectively. Let $\{(\tilde{x}_j, \gamma_j)\}_{j\in\mathbb{N}}$ be an exhausting sequence of finite type approximated X-trajectory with $\tilde{x}_j \in \tilde{S}, \gamma_j \subset S$ for any j. Then there exists a X-trajectory $(\tilde{x}_\infty, \gamma_\infty)$ such that, after passing to a subsequence, we have*

- *$\tilde{x}_j$ converges to $\tilde{x}_\infty$ in the topology of $Coul^{CC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y}$;*
- *γ_j converges to γ_∞ uniformly in L^2_k on any compact subset of $\mathbb{R}^{\geq 0}$.*

Proposition 7.3.8. *There exists a constant R_2 with the following significance: for any bounded subsets $\tilde{S} \subset Coul^{CC}_0(X)/\mathcal{G}^{h,\hat{o}}_{X,Y}$ and $S \subset Coul(Y)$, there exists $\epsilon_0, N, \bar{T} \in (0, \infty)$ such that: for any finite type (n, ϵ)-approximated X-trajectory $(\tilde{x}, \gamma)$ of length $T \geq \bar{T}$ satisfying*

$$n \geq N,\ \epsilon \leq \epsilon_0,\ \tilde{x} \in \tilde{S} \text{ and } \gamma \subset S,$$

we have $\|\tilde{x}\|_F < R_2$.

Proof. Let R_2 be the constant R given by Theorem 7.3.2, with $M = 0$. Suppose the result is not true for some $\tilde{S}, S$. Then we can find an exhausting sequence $\{(\tilde{x}_j, \gamma_j)\}_{j\in\mathbb{N}}$ of finite type approximated X-trajectory such that

$$\gamma_j \subset S,\ \tilde{x}_j \in \tilde{S} \text{ and } \|\tilde{x}_j\|_F \geq R_2.$$

By Lemma 7.3.7, after passing to a subsequence, we can find a finite type X-trajectory $(\tilde{x}_\infty, \gamma_\infty)$ such that $\tilde{x}_j \to \tilde{x}_\infty$ in the topology of $Coul^{CC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y}$. In particular, this implies

$$\|\tilde{x}_\infty\|_F = \lim_{j\to+\infty} \|\tilde{x}_j\|_F \geq R_2 \text{ and } \tilde{p}_\alpha(x_\infty) = \lim_{j\to+\infty} \tilde{p}_\alpha(\tilde{x}_j).$$

Since $\tilde{x}_j \in Coul_0^{CC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y}$, we have $\tilde{p}_\alpha(\tilde{x}_j) = 0$, which implies $\tilde{p}_\alpha(\tilde{x}_\infty) = 0$. By Theorem 7.3.2, we have $\|\tilde{x}_\infty\|_F < R_2$. This is a contradiction. □

Proposition 7.3.9. *For any bounded subset $S \subset Coul(Y)$, we can find a constant $R(S)$ with the following significance: for any bounded subset $\tilde{S} \subset Coul^{CC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y}$ there exist $\epsilon, N, \bar{T} \in (0,\infty)$ such that for any finite type (n,ϵ)-approximated X-trajectory $(\tilde{x},\gamma)$ of length $T \geq \bar{T}$ satisfying*

$$n \geq N,\ \epsilon \leq \epsilon_0,\ \tilde{x} \in \tilde{S} \text{ and } \gamma \subset S,$$

we have $\|\tilde{x}\|_F < R(S)$.

Proof. Consider the number

$$C(S) = \sup\{|i\int_{\alpha_j} a| \mid (a,\phi) \in S,\ 1 \leq j \leq e\}.$$

We let $R(S)$ be the constant R given by Theorem 7.3.2, with $M = C(S)$. The rest of the proof is identical with the previous one. □

Proposition 7.3.10. *There exists a constant R_3 with the following significance: for any bounded subsets $\tilde{S} \subset Coul(X)/\mathcal{G}^{h,\hat{o}}_{X,Y}$ and $S \subset Coul(Y)$, there exist $\epsilon_0, N, \bar{T} \in (0,+\infty)$ such that for any finite type (n,ϵ)-approximated X-trajectory $(\tilde{x},\gamma)$ of length $T \geq \bar{T}$ satisfying*

$$n \geq N,\ \epsilon \leq \epsilon_0,\ \tilde{x} \in \tilde{S} \text{ and } \gamma \subset S$$

We have $\gamma|_{[0,T-\bar{T}]} \subset Str(R_3)$.

Proof. Let $R_3 = \max\{R_0, R_1\}$ where R_0 and R_1 are the constants given by Theorem 2.2.2 and Proposition 7.3.4 respectively. Suppose the result is not true for some $\tilde{S}, S$. Then we can find an exhausting sequence $\{(\tilde{x}_j,\gamma_j)\}_{j\in\mathbb{N}}$ of finite type approximated X-trajectories with $\tilde{x}_j \in \tilde{S}, \gamma_j \subset S$, together with a sequence $\{t_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^{\geq 0}$ such that $\gamma_j(t_j) \notin Str(R_3)$ for any j. By Lemma 7.3.7, there exists an finite type X-trajectory $(\tilde{x}_\infty,\gamma_\infty)$ such that, after passing to a subsequence, γ_j converges to γ_∞ uniformly in

L^2_k on any compact subset of $\mathbb{R}^{\geq 0}$. Since $\gamma_\infty \subset \mathrm{int}(Str(R_3))$ by Proposition 7.3.4, we see that $\lim_{j\to+\infty} t_j = +\infty$. However, by Corollary 2.2.8, the sequence $\{t_j\}_{j\in\mathbb{N}}$ should be bounded above. This is a contradiction. $\square$

7.4 Construction of the relative Bauer-Furuta invariants

7.4.1 Thom spectrum of the virtual index bundle

In this subsection, we will define the Thom spectrum of the virtual index bundle $\mathrm{ind}(\not{D}^+)$ as an object of $\mathfrak{C}$. (When $\hat{\mathfrak{s}}$ is spin, we also have the corresponding object in $\mathfrak{C}_{Pin(2)}$.) The relative Bauer-Furuta invariant will be defined as a morphism from (a suitable suspension of) this object to the spectrum invariant of Y.

Let H^-_S be the closure in $L^2_k(\Gamma(S_Y))$ of the subspace spanned by the eigenvectors of $\not{D}_{A_0}$ with nonpositive eigenvalues and let

$$\Pi^-_S : L^2_k(\Gamma(S_Y)) \to H^-_S$$

be the orthogonal projection. We consider the map

$$D_S : i\Omega^1_D(X) \times L^2_{k+1/2}(\Gamma(S^+_X)) \to i\Omega^1_D(X) \times (L^2_{k-1/2}(\Gamma(S^-_X)) \oplus H^-_S)$$

given by

$$(\hat{h}, \hat{\phi}) \mapsto (\hat{h}, \not{D}^+_{\hat{h}}\hat{\phi}, \Pi^-_S\hat{\phi}|_Y).$$

We treat D_S as the bundle map over $i\Omega^1_D(X)$ that is equivariant under the action of $\mathcal{G}^{h,\hat{o}}_{X,Y}$. Here the actions on source and target bundles are respectively given by

$$\hat{u}\cdot(\hat{a}, \hat{\phi}) := (\hat{a} - \hat{u}^{-1}d\hat{u}, \hat{u}\hat{\phi})$$

and

$$\hat{u}\cdot(\hat{a}, \hat{\phi}, \psi) := (\hat{h} - \hat{u}^{-1}d\hat{u}, \hat{u}\hat{\phi}, (u|_Y)\psi).$$

Let $\bar{D}_S$ be the bundle map over $Pic^0(X, Y)$, obtained by taking the quotient of D_S under the action $\mathcal{G}^{h,\hat{o}}_{X,Y}$. We fix a trivialization of the target Hilbert bundle

$$\tau : Pic^0(X,Y) \times (L^2_{k-1/2}(\Gamma(S^-_X)) \oplus H^-_S) \xrightarrow{\cong} (i\Omega^1_D(X) \times (L^2_{k-1/2}(\Gamma(S^-_X)) \oplus H^-_S))/\mathcal{G}^{h,\hat{o}}_{X,Y}.$$

Composing $\bar{D}_S$ with the projection from the target bundle to its fiber (which is induced by τ), we obtain a map

$$\tilde{D}_S : (i\Omega^1_D(X) \times L^2_{k+1/2}(\Gamma(S^+_X)))/\mathcal{G}^{h,\hat{o}}_{X,Y} \to L^2_{k-1/2}(\Gamma(S^-_X)) \oplus H^-_S.$$

By the Atiyah-Patodi-Singer theory (c.f [5]), the fiberwise restrictions of $\tilde{D}_S$ are Freedholm. As a result, we can find a finite dimensional subspace $U \subset L^2_{k-1/2}(\Gamma(S^-_X)) \oplus H^-_S$ such that it is transversal to all these restrictions. In this case, $\tilde{D}_S^{-1}(U)$ is a finite dimensional vector space over $Pic^0(X,Y)$. We let $T(\tilde{D}_S^{-1}(U))$ be its Thom space and choose ∞ as the base point. Then $\Sigma^{-U}T(\tilde{D}_S^{-1}(U))$ is an object of the stable category $\mathfrak{C}$. We write this object as $T(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}; S^1)$. This notation is justified by the following lemma, whose proof follows from standard homotopy argument.

Lemma 7.4.1. *Let $\Sigma^{-U_j}T(\tilde{D}_S^{-1}(U_j))$ $(j = 1, 2)$ be two objects arising from different choices of auxiliary data $(\hat{g}_j, \hat{A}_{0,j}, \hat{o}_j)$. Suppose $\hat{g}_1|_Y = \hat{g}_2|_Y$, $\hat{A}_{0,1}|_Y = \hat{A}_{0,2}|_Y$ and $\hat{o}_1 = \hat{o}_2$. Then there is a natural isomorphism between $\Sigma^{-U_1}T(\tilde{D}_S^{-1}(U_1))$ and $\Sigma^{-U_2}T(\tilde{D}_S^{-1}(U_2))$.*

Remark. For different choices of base points $\hat{o}_1, \hat{o}_2$, one can construct an isomorphism between the resulting objects by choosing a path γ from $\hat{o}_1$ to $\hat{o}_2$. However, isomorphisms given by different pathes γ_1, γ_2 are usually different, unless the composed loop $\gamma_1^{-1}\gamma_2$ represents the zero class in $H_1(X, Y; \mathbb{R})$.

Now we relate the object $T(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}; S^1)$ with our earlier constructions. To state the result, we recall that there is an nondegenerate, symmetric quadratic form on $\mathrm{im}(H^2(X, Y; \mathbb{R}) \to H^2(X; \mathbb{R}))$, which is given by cup product. We make a choice of a maximal non-negative subspace for this quadratic form and denote it by $I_+(X)$. Note that for any two choices of $I_+(X)$, their orientation set can be naturally identified.

Lemma 7.4.2. *Let U_n, W_n, W^0_n be defined as in subsection 7.2.2. Then there are natural isomorphisms (in the category $\mathfrak{C}$)*

$$\Sigma^{-U_n}\Sigma^{-\bar{V}^0_{\lambda_n}}T(W^0_n) \cong \Sigma^{-I_+(X)}T(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}; S^1) \tag{7.14}$$

and

$$\Sigma^{-U_n}\Sigma^{-V^0_{\lambda_n}}T(W_n)\cong\Sigma^{-(\text{coker}(H^1(X;\mathbb{R})\to H^1(Y;\mathbb{R}))\oplus I_+(X))}T(X,\hat{\mathfrak{s}},A_0,g,\hat{o};S^1). \tag{7.15}$$

Here $T(W^0_n)$ *(resp.* $T(W_n)$*) denotes the Thom space of the bundle* W^0_n *(resp.* W_n*).*

Proof. This is a bundle version of the index computation in [30, Propositionn 2]. The detailed argument is omitted because it is straightforward but messy. We only mention that in the proof, the based point data $[\vec{\eta}]$ (which is involved in the construction of W_n and W^0_n) plays the role of specifying a bundle trivialization

$$(i\Omega^1_D(X)\times Coul(Y))/\mathcal{G}^{h,\hat{o}}_{X,Y}\cong Pic^0(X,Y)\times Coul(Y)$$

by sending $[(\hat{h},\prod_{j=1}^{b_0}(a_j,\phi_j))]$ to $([\hat{h}],\prod_{j=1}^{b_0}(a_j,e^{-\int_{\eta_j}\hat{h}}\cdot\phi_j))$ (c.f. (7.5)). □

Next, we turn to the special case that $\mathfrak{s}=\hat{\mathfrak{s}}|_Y$ is torsion. In this case, we consider the normalization $(T(X,\hat{\mathfrak{s}},A_0,g,\hat{o};S^1),0,n(Y,\mathfrak{s},A_0,g))\in\text{ob}\,\mathfrak{C}$. By simple excision argument, one can show that there is an natural isomorphism

$$(T(X,\hat{\mathfrak{s}},A_0,g,\hat{o};S^1),0,n(Y,\mathfrak{s},A_0,g))\cong(T(X,\hat{\mathfrak{s}},g',A'_0,\hat{o};S^1),0,n(Y,\mathfrak{s},A'_0,g'))$$

for any (A_0,g) and (A'_0,g'). As a result, we are safe to suppress (A_0,g) from our notation and denote this normalized object by $\widetilde{T}(X,\hat{\mathfrak{s}},\hat{o};S^1)$.

When $\mathfrak{s}$ is a spin structure, all our construction can be adapted to the Pin(2)-equivariant setting. As a result, $\Sigma^{-U}T(\tilde{D}^{-1}_S(U))$ can also be treated as an object of $\mathfrak{C}_{Pin(2)}$, which we denote by $T(X,\hat{\mathfrak{s}},A_0,g,\hat{o};Pin(2))$. Similar result as Lemma 7.4.2 still holds in this case. (Note that in the spin case, both $I_+(X)$ and $\text{coker}(H^1(X;\mathbb{R})\to H^1(Y;\mathbb{R})$ are isomorphic to copies of $\tilde{\mathbb{R}}$ (the nontrivial one dimensional representation of Pin(2)). Also, the element $j\in Pin(2)$ acts nontrivially on $Pic^0(X,Y)$.) We denote the normalized object

$$(T(X,\hat{\mathfrak{s}},A_0,g,\hat{o};Pin(2)),0,\frac{n(Y,\mathfrak{s},A_0,g)}{2})\in\text{ob}\,\mathfrak{C}_{Pin(2)}$$

by $\widetilde{T}(X,\hat{\mathfrak{s}},\hat{o};Pin(2))$.

7.4.2 Type-A relative Bauer-Furuta invariant

With all the results in place, we can finally start defining the relative Bauer-Furuta invariant for a spin^c 4-manifolds with boundary. We make the following conventions.

Notation. Let R be a positive number, V be a normed vector space and W be a normed vector bundle. We denote by $B(V, R)$ (resp. $B(W, R)$) the closed disk (resp. disk bundle) with radius R and denote by $S(V, R)$ (resp. $S(W, R)$) the sphere (resp. sphere bundle) with radius R.

Recall that the spectrum invariants of Y are obtained by cutting the unbounded set $Str(\tilde{R})$ into bounded subsets $J_m^{\pm}$ and doing finite dimensional approximations. In order the define the relative Bauer-Furuta invariants, we impose the following requirements on $\tilde{R}$:

(i) $\tilde{R} > R_3$, with contant R_3 given in Proposition 7.3.10;

(ii) $\tilde{r}(B(Coul_0^{CC}(X)/\mathcal{G}_{X,Y}^{h,\hat{o}}, R_2)) \subset Str(\tilde{R})$, with constant R_2 given in Proposition 7.3.8.

For each $n \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{>0}$, we define the following bounded subsets of $Str(\tilde{R})$:

$$K_1^+(n, \epsilon) := p_{-\infty}^{\mu_n} \circ \tilde{r}(\widetilde{SW_n^0}^{-1}(B(U_n, \epsilon)) \cap B(W_n^0, R_2)),$$

$$K_2^+(n, \epsilon) := p_{-\infty}^{\mu_n} \circ \tilde{r}(\widetilde{SW_n^0}^{-1}(B(U_n, \epsilon)) \cap S(W_n^0, R_2)).$$

We define the map

$$\upsilon^+(n, \epsilon) : B(W_n^0, R_2)/S(W_n^0, R_2) \to (B(U_n, \epsilon)/S(U_n, \epsilon)) \wedge (K_1^+(n, \epsilon)/K_2^+(n, \epsilon))$$

by the formla

$$\upsilon^+(n, \epsilon)(x) := \begin{cases} (\widetilde{SW_n^0}(x), p_{\infty}^{\mu_n} \circ \tilde{r}(x)) & \text{if } \|\widetilde{SW_n^0}(x)\|_{L^2_{k-1/2}} \leq \epsilon \\ \text{basepoint} & \text{otherwise} \end{cases}$$

It is easy to see that $\upsilon^+(n, \epsilon)$ is well-defined and continuous.

Since $\tilde{r}(B(Coul_0^{CC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y}, R_2))$ is a bounded subset of $Str(\tilde{R})$, there exists $m_0 \in \mathbb{N}$ such that

$$\tilde{r}(B(Coul_0^{CC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y}, R_2)) \subset J^+_{m_0}.$$

This also implies

$$K_2^+(n,\epsilon) \subset K_1^+(n,\epsilon) \subset J_m^{n,+} \text{ for any } n \in \mathbb{N}, \epsilon > 0 \text{ and } m \geq m_0.$$

Lemma 7.4.3. *For any $m \geq m_0$, there exist $N_m \in \mathbb{N}$ and $T_m, \epsilon_m \in \mathbb{R}^{>0}$ such that for any $n \geq N_m$ and $\epsilon \leq \epsilon_m$, we have*

(1) $J_m^{n,+}$ is an T_m-tame isolating neighborhood under the flow φ_m^n;

(2) $(K_1^+(n,\epsilon), K_2^+(n,\epsilon))$ is a T_m-tame pre-index pair in $J_m^{n,+}$.

Proof. (1) is a direct consequence of Lemma 5.1.5. Under the assumption (1), we let $A = J_m^{n,+}$ and

$$A' = A \cap Str(\max(R_0, R_3)) \cap \bigcap_{1 \leq j \leq b_1} g_{j,+}^{-1}(-\infty, \theta + m - \theta_m]$$

(c.f. (5.3)). Then by Lemma 5.1.4, Lemma 5.1.5 and Proposition 7.3.10, there exists a constant T_m independent of n, ϵ such that conditions in Definition 3.2.10 holds for $(K_1^+(n,\epsilon), K_2^+(n,\epsilon))$ when n is large and ϵ is small. □

For each $m \geq m_0, n \geq N_m$, we choose any $\epsilon \leq \epsilon_m$ and let

$$l^+_{n,m,\epsilon} : K_1^+(n,\epsilon)/K_2^+(n,\epsilon) \to I_{S^1}(\varphi_m^n, \mathrm{inv}(J_m^{n,+}))$$

be the canonical map (see (3.5)). We consider the following morphism in $\mathfrak{C}$

$$\psi_m^{n,+} := \Sigma^{-\bar{V}^0_{\lambda_n}} \Sigma^{-U_n}((\mathrm{id} \wedge l^+_{n,m,\epsilon}) \circ \upsilon^+(n,\epsilon)) : \Sigma^{-I_+(X)} T(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}; S^1) \to I_m^{n,+}.$$

(id denotes the identity map on $B(U_n,\epsilon)/S(U_n,\epsilon)$.) Here we made use of the natural isomorphism

$$\Sigma^{-U_n} \Sigma^{-\bar{V}^0_{\lambda_n}} B(W_n^0, R_2)/S(W_n^0, R_2) \cong \Sigma^{-I_+(X)} T(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}; S^1)$$

provided by Lemma 7.4.2. Note that for different choices of $\epsilon \in (0, \epsilon_m]$, the maps $(\mathrm{id} \wedge l^+_{n,m,\epsilon}) \circ \upsilon^+(n,\epsilon)$ are homotopic to each other. Therefore, as a morphism in $\mathfrak{C}$, $\psi_m^{n,+}$ does not depend on the choice of ϵ.

Lemma 7.4.4. *For any $m \geq m_0$ and any n large enough relative to m, we have*

$$\tilde{i}_m^{n,+} \circ \psi_m^{n,+} = \psi_{m+1}^{n,+} \tag{7.16}$$

where $\tilde{i}_m^{n,+} = \Sigma^{-\bar{V}^0_{\lambda_n}} i_m^{n,+} \in \mathrm{mor}_{\mathfrak{C}}(I_m^{n,+}, I_{m+1}^{n,+})$ is given in (5.4).

Proof. Notice that when n is large enough relative to m, $(\overline{J_{m+1}^{n,+} \setminus J_m^{n,+}}, J_m^{n,+})$ is a strong Morse decomposition for $J_{m+1}^{n,+}$ (see Definition 3.2.15). By Proposition 3.2.18, we have $l_{m+1,n} \cong i_m^{n,+} \circ l^+_{n,m,\epsilon}$, which implies $\tilde{i}_m^{n,+} \circ \psi_m^{n,+} = \psi_{m+1}^{n,+}$. □

Lemma 7.4.5. *For any $m \geq m_0$ and any n large enough relative to m, we have*

$$\tilde{\rho}_m^{n,+} \circ \psi_m^{n,+} = \psi_m^{n+1,+}. \tag{7.17}$$

where $\tilde{\rho}_m^{n,+}$ is given in Proposition 5.1.6.

Proof. The proof is very similar to Proposition 5.1.6. First, by replacing U_{n+1} with $U_n + U_{n+1}$, we can assume that $U_n \subset U_{n+1}$, which implies $W_n^0 \subset W_{n+1}^0$. Then for $s \in [0,1]$, we consider the following map from W_{n+1}^0 to $U_{n+1} \oplus V_{\lambda_{n+1}}^{\mu_{n+1}}$

$$\widetilde{SW}_{n+1}^{0,s} := \tilde{L}^0 + (sP_{U_{n+1}} + (1-s)P_{U_n}) \circ \tilde{Q}^0 \circ (sP_{W_{n+1}^0} + (1-s)P_{W_n^0}).$$

By repeating our construction in this subsection with $\widetilde{SW}_{n+1}^{0,s}$ in place of $\widetilde{SW}_{n+1}^{0}$ and $\varphi_m^{n+1,s}$ (the interpolated flow generated by the vector field (5.6)) in place of φ_m^{n+1}, we obtain a map

$$\begin{aligned}(B(W_{n+1}^0, R_2) \times [0,1])/(S(W_{n+1}^0, R_2) \times [0,1]) \to \\ B(U_{n+1}, \epsilon)/S(U_{n+1}, \epsilon) \wedge I_{S^1}(\varphi_m^{n+1,s}, \mathrm{inv}(J_m^{n+1,+})).\end{aligned} \tag{7.18}$$

(7.17) is directly implied by the homotopy induced by (7.18). □

Recall that the spectrum invariant $\underline{\mathrm{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1)$ is defined as the inductive system

$$I_1^{n_1,+} \xrightarrow{\eta_1^+} I_2^{n_2,+} \xrightarrow{\eta_2^+} I_3^{n_3,+} \xrightarrow{\eta_3^+} \cdot$$

with connecting morphism $\eta_m^+ = \tilde{i}_m^{n_{m+1},+} \circ \tilde{\rho}_m^{n_{m+1}-1,+} \circ \cdots \circ \tilde{\rho}_m^{n_m,+}$. We can take n_m large enough relative to m, such that (7.16) and (7.17) holds. Then we have

$$\eta_m^+ \circ \psi_m^{n_m,+} = \psi_{m+1}^{n_{m+1},+} \text{ for any } m \geq m_0.$$

As a result, the element

$$[\psi_m^{n_m}] \in \mathrm{mor}_{\mathfrak{S}}(\Sigma^{-I_+(X)}T(X,\hat{\mathfrak{s}},A_0,g,\hat{o};S^1),\underline{\mathrm{swf}}^A(Y,\mathfrak{s},A_0,g;S^1))$$

$$= \lim_{m\to\infty} \mathrm{mor}_{\mathfrak{C}}(\Sigma^{-I_+(X)}T(X,\hat{\mathfrak{s}},A_0,g,\hat{o};S^1),I_m^{n_m,+})$$

does not dependent on the choice of $m \in \mathbb{Z}^{\geq m_0}$.

Definition 7.4.6. (1) The S^1-equivariant relative Bauer-Furuta invariant

$$\underline{\mathrm{bf}}^A(X,\hat{\mathfrak{s}},A_0,g,\hat{o},[\vec{\eta}];S^1) \in \mathrm{mor}_{\mathfrak{S}}(\Sigma^{-I_+(X)}T(X,\hat{\mathfrak{s}},A_0,g,\hat{o};S^1),\underline{\mathrm{swf}}^A(Y,\mathfrak{s},A_0,g;S^1))$$

is defined as the morphism represented by $\psi_m^{n_m,+}$ for any $m \geq m_0$.

(2) When $\mathfrak{s} = \hat{\mathfrak{s}}|_Y$ is torsion. We define the normalized relative Bauer-Furuta invariant

$$\underline{\mathrm{BF}}^A(X,\hat{\mathfrak{s}},\hat{o},[\vec{\eta}];S^1) \in \mathrm{mor}_{\mathfrak{S}}(\Sigma^{-I_+(X)}\tilde{T}(X,\hat{\mathfrak{s}},\hat{o};S^1),\underline{\mathrm{SWF}}^A(Y,\mathfrak{s};S^1))$$

as the morphism represented by the desuspension $(\psi_m^{n_m,+},0,n(Y,\mathfrak{s},A_0,g))$.

When $\hat{\mathfrak{s}}$ is a spin structure, the desuspension $\Sigma^{-\bar{V}^0_{\lambda_n}}\Sigma^{-U_n}((\mathrm{id}\wedge l^+_{n,m,\epsilon})\circ\nu(n,\epsilon))$ can also be treated as an element of $\mathrm{mor}_{\mathfrak{C}_{Pin(2)}}(\Sigma^{-I_+(X)}T(X,\hat{\mathfrak{s}},A_0,g,\hat{o};Pin(2)),I_m^{n,+}(Pin(2)))$. We denote it by $\psi^{n,+}_{Pin(2),m}$

Definition 7.4.7. When $\hat{\mathfrak{s}}$ is spin, we define the normalized Pin(2)-equivariant relative Bauer-Furuta invariant

$$\underline{\mathrm{BF}}^A(X,\hat{\mathfrak{s}},\hat{o},[\vec{\eta}];Pin(2)) \in \mathrm{mor}_{\mathfrak{S}_{Pin(2)}}(\Sigma^{-I_+(X)}\tilde{T}(X,\hat{\mathfrak{s}},\hat{o};Pin(2)),\underline{\mathrm{SWF}}^A(Y,\mathfrak{s};Pin(2)))$$

as the morphism represented by the desuspension $(\psi^{n,+}_{Pin(2),m},0,\frac{n(Y,\mathfrak{s},A_0,g)}{2})$ for any $m \geq m_0$.

7.4.3 Type-R relative Bauer-Furuta invariant

Now we turn to the repeller case. Let $\tilde{R}$ be chosen as last subsection. For $m \in \mathbb{N}$, we let $R(J_m^-)$ be the constant provided by Proposition 7.3.9 with $S = J_m^-$. For any $n, m \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{>0}$, we consider the following two compact subsets of $J_m^{n,-}$

$$K_1^-(n,m,\epsilon) := J_m^{n,-} \cap p^{\mu_n}_{-\infty} \circ \tilde{r}(\widetilde{SW_n}^{-1}(B(U_n,\epsilon)) \cap B(W_n,R(J_m^-))),$$

$$K_2^-(n,m,\epsilon) := (K_1^-(n,m,\epsilon)\cap\partial J_m^{n,-})\cup(J_m^{n,-}\cap p_{-\infty}^{\mu_n}\circ\tilde{r}(\widetilde{SW_n}^{-1}(B(U_n,\epsilon))\cap S(W_n,R(J_m^-)))).$$

We define the map

$$\begin{aligned}\upsilon^-(n,m,\epsilon): B(W_n,R(J_m^{n,-}))/S(W_n,R(J_m^{n,-}))\\ \to (B(U_n,\epsilon)/S(U_n,\epsilon))\wedge(K_1^-(n,m,\epsilon)/K_2^-(n,m,\epsilon))\end{aligned}$$

by the formula

$$\upsilon^-(n,m,\epsilon)(x) := \begin{cases} (\widetilde{SW_n}(x), p_{-\infty}^{\mu_n}\circ\tilde{r}(x)) & \text{if } \|\widetilde{SW_n}(x)\|_{L^2_{k-1/2}}\le\epsilon,\ p_{-\infty}^{\mu_n}\circ\tilde{r}(x)\in J_m^{n,-} \\ \text{basepoint} & \text{otherwise}\end{cases}.$$

One can check that $\upsilon^-(n,m,\epsilon)$ is always well-defined and continuous.

Lemma 7.4.8. *For any $m\in\mathbb{N}$, there exists $N_m\in\mathbb{N}$ and $T_m,\epsilon_m\in\mathbb{R}^{>0}$ such that for any $n\ge N_m$ and $\epsilon\le\epsilon_m$, we have*

(1) $J_m^{n,-}$ is an T_m-tame isolating neighborhood under the flow φ_m^n;

(2) $(K_1^-(n,m,\epsilon),K_2^-(n,m,\epsilon))$ is a T_m-tame pre-index pair in $J_m^{n,-}$.

Proof. The proof is identical with Lemma 7.4.3 except that we use Lemma 7.3.9 instead of Lemma 7.3.8. □

For each $m\in\mathbb{N}$ and $n\ge N_m$, we choose $\epsilon<\epsilon_m$ and let

$$l^-_{n,m,\epsilon}: K_1^-(n,m,\epsilon)/K_2^-(n,m,\epsilon)\to I_{S^1}(\varphi_m^n,\operatorname{inv}(J_m^{n,-}))$$

be the corresponding canonical map. Then as in the attractor case, we consider the following morphism in $\mathfrak{C}$

$$\begin{aligned}\psi_m^{n,-} := \Sigma^{-V^0_{\lambda_n}}\Sigma^{-U_n}((\mathrm{id}\wedge l^-_{n,m,\epsilon})\circ\upsilon^-(n,m,\epsilon)):\\ \Sigma^{-(I_+(X)\oplus\operatorname{coker}(H^1(X;\mathbb{R})\to H^1(Y;\mathbb{R})))}T(X,\hat{\mathfrak{s}},A_0,g,\hat{o};S^1)\to I_m^{n,-}.\end{aligned}$$

Here we made use of the isomorphism

$$\begin{aligned}\Sigma^{-V^0_{\lambda_n}}\Sigma^{-U_n}(B(W_n,R(J_m^-))/S(W_n,R(J_m^-)))\\ \cong\Sigma^{-(I_+(X)\oplus\operatorname{coker}(H^1(X;\mathbb{R})\to H^1(Y;\mathbb{R})))}T(X,\hat{\mathfrak{s}},A_0,g,\hat{o};S^1)\end{aligned}$$

provided by Lemma 7.4.2.

Lemma 7.4.9. *For any $m \in \mathbb{N}$ and any n large enough relative to m, we have*

$$\tilde{i}_m^{n,-} \circ \psi_{m+1}^{n,-} = \psi_m^{n,-}, \tag{7.19}$$

where $\tilde{i}_m^{n,-} = \Sigma^{-V_{\lambda_n}^0} i_{m+1}^{n,-} \in \mathrm{mor}_{\mathfrak{C}}(I_{m+1}^{n,-}, I_m^{n,-})$ is given in Proposition 5.1.8.

Proof. The proof is identical with Lemma 7.4.4 except that we use Proposition 3.2.19 instead of Proposition 3.2.18. □

Lemma 7.4.10. *For any $m \in \mathbb{N}$ and any n large enough relative to m, we have*

$$\tilde{\rho}_m^{n,-} \circ \psi_m^{n,-} = \psi_m^{n+1,-}. \tag{7.20}$$

where $\tilde{\rho}_m^{n,-}$ is the isomorphism given in Proposition 5.1.8.

Proof. The proof is identical with Lemma 7.4.5. □

Recall that the type-R spectrum invariant $\underline{\mathrm{swf}}^R(Y, \mathfrak{s}, A_0, g)$ is defined as the projective system

$$I_1^{n_1,-} \xleftarrow{\eta_1^-} I_2^{n_2,-} \xleftarrow{\eta_2^-} I_3^{n_3,-} \xleftarrow{\eta_3^-} \cdot$$

with the connecting isomorphism $\eta_m^- = (\tilde{\rho}_m^{n_{m+1}-1,-} \circ \cdots \circ \tilde{\rho}_m^{n_m,-})^{-1} \circ \tilde{i}_m^{n_{m+1},-}$. Suppose we choose n_m large enough relative to m such that (7.19) and (7.20) holds for all $m \in \mathbb{N}$. Then we have

$$\eta_m^- \circ \psi_{m+1}^{n_{m+1},-} = \psi_m^{n_m,-}.$$

As a result, $\{\psi_m^{n_m,-}\}_{m \in \mathbb{N}}$ gives a well-defined element in

$$\mathrm{mor}_{\mathfrak{S}^*}(\Sigma^{-(I_+(X) \oplus \mathrm{coker}(H^1(X;\mathbb{R}) \to H^1(Y;\mathbb{R})))} T(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}; S^1), \underline{\mathrm{swf}}^R(Y, \mathfrak{s}, A_0, g)).$$

Definition 7.4.11. (1) The S^1-equivariant relative Bauer-Furuta invariant

$$\underline{\mathrm{bf}}^R(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}, [\vec{\eta}]; S^1) \in \mathrm{mor}_{\mathfrak{S}^*}(\Sigma^{-(I_+(X) \oplus \mathrm{coker}(H^1(X;\mathbb{R}) \to H^1(Y;\mathbb{R})))} T(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}; S^1), \underline{\mathrm{swf}}^R(Y, \mathfrak{s}, A_0, g; S^1))$$

is defined as the morphism given by $\{\psi_m^{n_m,-}\}_{m \in \mathbb{N}}$.

(2) When $\mathfrak{s} = \hat{\mathfrak{s}}|_Y$ is torsion. We define the normalized relative Bauer-Furuta invariant

$$\underline{\mathrm{BF}}^R(X, \hat{\mathfrak{s}}, \hat{o}, [\vec{\eta}]; S^1)$$
$$\in \mathrm{mor}_{\mathfrak{S}^*}(\Sigma^{-(I_+(X)\oplus \mathrm{coker}(H^1(X;\mathbb{R})\to H^1(Y;\mathbb{R})))}\tilde{T}(X, \hat{\mathfrak{s}}, \hat{o}; S^1), \underline{\mathrm{SWF}}^R(Y, \mathfrak{s}; S^1))$$

as the morphism given by $\{(\psi_m^{n_m,-}, 0, n(Y, \mathfrak{s}, A_0, g))\}_{m\in\mathbb{N}}$.

In the case that $\hat{\mathfrak{s}}$ is a spin structure, we can define the morphism $\psi_{Pin(2),m}^{n_m,-}$ from the object

$$\Sigma^{-(I_+(X)\oplus \mathrm{coker}(H^1(X;\mathbb{R})\to H^1(Y;\mathbb{R})))}T(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}; Pin(2))$$

to the object $I_m^{n_m,-}(Pin(2))$, in same manner as $\psi_{Pin(2),m}^{n_m,+}$.

Definition 7.4.12. When $\hat{\mathfrak{s}}$ is a spin structure, we define the normalized Pin(2)-equivariant relative Bauer-Furuta invariant

$$\underline{\mathrm{BF}}^R(X, \hat{\mathfrak{s}}, \hat{o}, [\vec{\eta}]; Pin(2))$$
$$\in \mathrm{mor}_{\mathfrak{S}^*_{Pin(2)}}(\Sigma^{-(I_+(X)\oplus \mathrm{coker}(H^1(X;\mathbb{R})\to H^1(Y;\mathbb{R})))}\tilde{T}(X, \hat{\mathfrak{s}}, \hat{o}; Pin(2)), \underline{\mathrm{SWF}}^R(Y, \mathfrak{s}; Pin(2)))$$

as the morphism given by $\{(\psi_{Pin(2),m}^{n_m,-}, 0, \frac{n(Y,\mathfrak{s},A_0,g)}{2})\}_{m\in\mathbb{N}}$.

CHAPTER 8

Further developments

In this chapter, we will discuss some further developments of the theory. Most of the results in this chapter will be stated without proof. We refer our paper [28], [26] and [27] for complete proofs. Although some results can be extended to the general case, for simplicity, **we will assume that all spin^c structures on 3-manifolds are torsion throughout this chapter.**

8.1 Calculations

As mentioned in the introduction, our spectrum invariants are quite difficult to compute. However, by using Mrowka-Ozváth-Yu's explicit description of the Seiberg-Witten moduli space for Seifert manifolds [48] and a refinement of the rescaling technique developed by Khandhawit [29], we are able to give explicit computation of the invariants in torsion cases of some Seifert manifolds. We summarize these examples in this section. See [28] for the derivations.

8.1.1 $S^2 \times S^1$

Since the manifold $S^2 \times S^1$ admits a metric with positive scalar structure. The spectrum invariants for the torsion spin^c structure are just the sphere spectra.

More precisely, let $\mathfrak{s}$ be the unique torsion spin structure on $S^2 \times S^1$. Then we have

$$\underline{\mathrm{SWF}}^A(S^2 \times S^1, \mathfrak{s}; S^1) \cong (S^0, 0, 0),\ \underline{\mathrm{SWF}}^R(S^2 \times S^1, \mathfrak{s}; S^1) \cong (S^0, 0, 0).$$

Note that $\mathfrak{s}$ can be lifted to two spin structures, denoted by $\mathfrak{s}^j$ $(j = 1, 2)$. We have

$$\underline{\mathrm{SWF}}^A(S^2 \times S^1, \mathfrak{s}^j; Pin(2)) \cong (S^0, 0, 0),\ \underline{\mathrm{SWF}}^R(S^2 \times S^1, \mathfrak{s}^j; Pin(2)) \cong (S^0, 0, 0).$$

8.1.2 Large degree circle bundle over surfaces

Let N_d be a complex line bundle of degree $d > 0$ over a surface Σ of genus g and let $Y = S(N_d)$ be the corresponding circle bundle. The torsion spin^c structures on Y can be identified with $\mathbb{Z}/d\mathbb{Z}$ in a canonical way and we denote them by $\mathfrak{s}_0, \mathfrak{s}_1, \cdots, \mathfrak{s}_{d-1}$.

In the case that $0 < g \leq q < d$, we have the following results:

$$\begin{aligned} \underline{\mathrm{SWF}}^A(S(N_d), \mathfrak{s}_{\mathrm{q}}; S^1) &\cong (S^0, 0, c(g,d,q)), \\ \underline{\mathrm{SWF}}^R(S(N_d), \mathfrak{s}_{\mathrm{q}}; S^1) &\cong (S^0, 0, c(g,d,q)), \end{aligned} \tag{8.1}$$

where

$$c(g,d,q) = \frac{d-1}{8} + \frac{(g-1-q)(d+g-1-q)}{2d}.$$

A particularly interesting case is when d is an even number greater than $2g-2$. Setting q to be $\frac{d}{2} + g - 1$, the spin^c structure $\mathfrak{s}_q$ has vanishing c_1 and therefore can be lifted to 2^{2g} spin structures, denoted by $\mathfrak{s}_q^j$ ($j = 1, 2, \cdots, 2^{2g}$). We have

$$\begin{aligned} \underline{\mathrm{SWF}}^A(S(N_d), \mathfrak{s}_{\mathrm{q}}^{\mathrm{j}}; Pin(2)) &\cong (S^0, 0, -\frac{1}{16}), \\ \underline{\mathrm{SWF}}^R(S(N_d), \mathfrak{s}_{\mathrm{q}}^{\mathrm{j}}; Pin(2)) &\cong (S^0, 0, -\frac{1}{16}). \end{aligned} \tag{8.2}$$

8.1.3 Circle bundles over torus

When $g = 1$ and $d \neq 0$, the circle bundle $Y = S(N_d)$ is a nil manifold and the spectrum invariants Y for torsion spin^c structure can be explicitly computed.

We focus on the case $d > 0$ (the $d < 0$ case is similar). The spectrum invariants for $\mathfrak{s}_1, \cdots, \mathfrak{s}_{d-1}$ can be computed by formula (8.1). Hence we are left with $\mathfrak{s}_0$, whose spectrum invariants are actually not suspensions of the sphere spectrum. To state the result, we give the following notation.

Notation. Let A be a topological space and B be a subset of A. We denote by $\bigvee_B^n A$ the topological space obtained from gluing n copies of A together along their subset B.

We have

$$\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}_0; S^1) \cong \left(\mathbb{C}^+ \to \bigvee_{S^0}^2 \mathbb{C}^+ \to \bigvee_{S^0}^3 \mathbb{C}^+ \to \cdots, 0, \frac{d-17}{8} \right),$$

$$\underline{\mathrm{SWF}}^R(Y,\mathfrak{s}_0;S^1) \cong \left((\mathbb{C}^2)^+ \setminus (S^1) \leftarrow (\mathbb{C}^2)^+ \setminus (\coprod^2 S^1) \leftarrow (\mathbb{C}^2)^+ \setminus (\coprod^3 S^1) \leftarrow \cdots, 0, \frac{d-1}{8}\right),$$

where in the second formula, $\coprod^n S^1$ denotes the union of n disjoint orbits in the S^1-space $(\mathbb{C}^2)^+$. The connecting morphisms are given by the obvious inclusions.

The spin^c structure $\mathfrak{s}_0$ can be lifted to four spin structures, denoted by $\mathfrak{s}_0^j$ ($j = 0,1,2,3$). The invariants for $\mathfrak{s}_0^1$, $\mathfrak{s}_0^2$, $\mathfrak{s}_0^3$ are isomorphic so we only consider $\mathfrak{s}_0^1$. We have the following result:

$$\underline{\mathrm{SWF}}^A(Y,\mathfrak{s}_0^1;Pin(2)) \cong \left(\Sigma(\coprod^\infty Pin(2)), 0, \frac{d-17}{16}\right),$$

where $\Sigma(\coprod^\infty Pin(2))) \in \mathrm{ob}(\mathfrak{S}(Pin(2)))$ denotes the direct system

$$\Sigma Pin(2) \to \Sigma(Pin(2) \amalg Pin(2)) \to \Sigma(Pin(2) \amalg Pin(2) \amalg Pin(2)) \to \cdots,$$

with the connecting morphisms given by natural inclusions and $\Sigma *$ denoting the unreduced suspensions. We also have

$$\underline{\mathrm{SWF}}^R(Y,\mathfrak{s}_0^1;Pin(2)) \cong \left(\mathbb{H}^+ \setminus \coprod^\infty Pin(2), 0, \frac{d-1}{16}\right).$$

where $\mathbb{H}^+ \setminus \coprod^\infty Pin(2)$ denotes the inverse system

$$\mathbb{H}^+ \setminus Pin(2) \leftarrow \mathbb{H}^+ \setminus \coprod^2 Pin(2) \leftarrow \mathbb{H}^+ \setminus \coprod^3 Pin(2) \leftarrow \cdots.$$

(As before, $\coprod^n Pin(2)$ denotes the union of n disjoint orbits in the Pin(2)-space $\mathbb{H}^+$.)

As for the spin structure $\mathfrak{s}_0^0$, the spectrum invariants are as follows

$$\underline{\mathrm{SWF}}^A(Y,\mathfrak{s}^0;Pin(2)) \cong \left(\Sigma\left(S(\mathbb{H}) \underset{Pin(2)}{\vee} \bigvee_{Pin(2)}^{\infty} (\tilde{Z}_2 \times S(\mathbb{H}))\right), 0, \frac{d-9}{16}\right),$$

$$\underline{\mathrm{SWF}}^R(Y,\mathfrak{s}^0;Pin(2)) \cong \left((\mathbb{H}^2)^+ \setminus D_\infty, 0, \frac{d+7}{16}\right).$$

Let us explain the notations above: In the first formula, $S(\mathbb{H})$ denotes the unit sphere in $\mathbb{H}$; $\tilde{Z}_2$ denotes the two-point space with nontrivial Pin(2)-action and $S(\mathbb{H}) \underset{Pin(2)}{\vee} (\bigvee_{Pin(2)}^{\infty} (\tilde{Z}_2 \times$

$S(\mathbb{H})))$ denotes the direct system

$$S(\mathbb{H}) \underset{Pin(2)}{\vee} (\tilde{Z}_2 \times S(\mathbb{H})) \to S(\mathbb{H}) \underset{Pin(2)}{\vee} \left(\bigvee_{Pin(2)}^{2} (\tilde{Z}_2 \times S(\mathbb{H})) \right) \to$$
$$S(\mathbb{H}) \underset{Pin(2)}{\vee} \left(\bigvee_{Pin(2)}^{3} (\tilde{Z}_2 \times S(\mathbb{H})) \right) \to \cdots .$$

In the second formula, D_m is the subset of $S(\mathbb{H}^2)$ defined as

$$D_m := \bigcup_{n=0}^{m} \{(z_1 + jz_2, z_3 + jz_4) \in S(\mathbb{H}^2) | z_3 = -\bar{z}_4 = nz_1 \text{ or } z_3 = \bar{z}_4 = n\bar{z}_2\}$$

and $(\mathbb{H}^2)^+ \setminus D_\infty$ denotes the inverse system

$$(\mathbb{H}^2)^+ \setminus D_1 \leftarrow (\mathbb{H}^2)^+ \setminus D_2 \leftarrow (\mathbb{H}^2)^+ \setminus D_3 \leftarrow \cdots .$$

8.1.4 Other nil manifolds

Any manifold Y can be realized as a nonzero-degree S^1-bundle over an orbifold Σ with Euler characteristic $\chi(\Sigma) = 0$. When $b_1(Y) > 0$, the orbifold Σ is either a torus or a Klein bottle. The torus case has been done in last subsection and the Klein bottle case is very similar.

We are left with the case $b_1(Y) = 0$. Since Y is a rational homology sphere, type-A and type-R invariants are both isomorphic the Manolescu's original definition of the spectrum invariant. In this case, we actually have

$$\mathrm{SWF}(Y, \mathfrak{s}; S^1) \cong (S^0, 0, c(Y, \mathfrak{s})) \in \mathrm{ob}\, \mathfrak{C} \tag{8.3}$$

where $c(Y, \mathfrak{s}) \in \mathbb{Q}$ is a constant whose explicit formula can be obtained in the same fashion as the constant $c(g, d, q)$ in formula (8.1), we refer to [28] for a model calculation.

8.1.5 Flat manifolds except T^3

There are five manifolds belonging to this class: four of them are T^2-bundles over S^1 with monodromy automorphism fixing a point and having orders $2, 3, 4, 6$, and the last of them is the Hantzsche-Wendt manifold. By the Weitzenböck formula, for any torsion spin^c structure $\mathfrak{s}$ on Y, the functional CSD has only reducible critical points.

The Hantzsche-Wendt manifold is a rational homology sphere. Therefore, its spectrum invariant is also of the form (8.3).

Now we consider the T^2-bundles over S^1 whose monodromies are automorphisms $\tau\colon T^2 \to T^2$ of order 2 (i.e. the hyperelliptic involution on T^2). The situations for the cases of order 3,4 or 6 are very similar, so we will focus our attention to this case of order 2.

Y has four spin^c structures $\mathfrak{s}_j$ $(j = 0, 1, 2, 3)$. Three of them have simple spectrum invariants:

$$\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}_j; S^1) \cong (S^0, 0, 0),\ \underline{\mathrm{SWF}}^R(Y, \mathfrak{s}_j; S^1) \cong (S^0, 0, 0) \text{ for } j = 1, 2, 3.$$

Similarly, the Pin(2)-spectrum invariants for spin structures lifting $\mathfrak{s}_j$ $(j = 1, 2, 3)$ are all isomorphic to the sphere spectra $(S^0, 0, 0)$.

As for $\mathfrak{s}_0$, the results are given as follows

$$\begin{aligned}
\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}_0; S^1) &\cong \left(\bigvee_{S^0}^{\infty} \mathbb{C}^+, 0, \frac{1}{2} \right), \\
\underline{\mathrm{SWF}}^R(Y, \mathfrak{s}_0; S^1) &\cong \left((\mathbb{C}^2)^+ \setminus (\coprod^{\infty} S^1), 0, \frac{3}{2} \right), \\
\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}_0^0; Pin(2)) &\cong \left(\Sigma \left(S(\mathbb{H}) \underset{Pin(2)}{\vee} \bigvee_{Pin(2)}^{\infty} (\tilde{Z}_2 \times S(\mathbb{H})) \right), 0, \frac{3}{4} \right), \\
\underline{\mathrm{SWF}}^R(Y, \mathfrak{s}_0^0; Pin(2)) &\cong \left((\mathbb{H}^2)^+ \setminus D_\infty, 0, \frac{5}{4} \right), \\
\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}_0^1; Pin(2)) &\cong \left(\Sigma(\coprod^{\infty} Pin(2)), 0, \frac{1}{4} \right), \\
\underline{\mathrm{SWF}}^R(Y, \mathfrak{s}_0^1; Pin(2)) &\cong \left(\mathbb{H}^+ \setminus \coprod^{\infty} Pin(2), 0, \frac{3}{4} \right),
\end{aligned}$$

where $\mathfrak{s}_0^j$ $(j = 0, 1)$ are the two spin structures lifting $\mathfrak{s}_0$.

8.2 The homology cobordism invariants

In this section, we will define all kinds of Frøyshov-type numerical invariants for a general spin^c 3-manifold by applying different generalized cohomology functors to our spectrum invariants. (Recall that we only consider torsion spin^c structures on 3-manifolds in this chapter.) These invariants are all invariant under $\mathbb{Q}$-homology cobordisms. Recall that two spin^c (resp. spin) 3-manifolds $(Y_j, \mathfrak{s}_j)$ are $\mathbb{Q}$-homology cobordism to each other if $(Y_1, \mathfrak{s}_1) \cup (Y_2, \mathfrak{s}_2)$ bounds a spin 4-manifold $(X, \hat{\mathfrak{s}})$ satisfying $H^*(X, Y_j; \mathbb{Q}) = 0$ $(j = 1, 2)$. These homology cobordism invariants give interesting constrains on the intersection forms of 4-manifolds with boundary.

We start with the following theorem describing the behavior of the spectrum invariants under the fixed point functors $\Phi_*^{S^1}$ defined in Section 3.1.2.

Theorem 8.2.1. *(1) For any torsion spin^c structure $\mathfrak{s}$ on a 3-manifold Y, we have natural isomorphisms*

$$\tau_+^{S^1} : (S^0, 0) \xrightarrow{\cong} \Phi_{\mathfrak{S}}^{S^1}(\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}; S^1)) \in \mathrm{ob}(\bar{\mathfrak{S}})$$

and

$$\tau_-^{S^1} : (S^0, 0) \xrightarrow{\cong} \Phi_{\mathfrak{S}^*}^{S^1}(\underline{\mathrm{SWF}}^R(Y, \mathfrak{s}; S^1)) \in \mathrm{ob}(\bar{\mathfrak{S}}^*).$$

(2) For any spin structure $\mathfrak{s}$ on a 3-manifold Y, we have natural isomorphisms

$$\tau_+^{Pin(2)} : (S^0, 0) \xrightarrow{\cong} \Phi_{\mathfrak{S}_{Pin(2)}}^{S^1}(\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}; Pin(2))) \in \mathrm{ob}(\bar{\mathfrak{S}}_{\mathbb{Z}_2})$$

and

$$\tau_-^{Pin(2)} : (S^0, 0) \xrightarrow{\cong} \Phi_{\mathfrak{S}^*_{Pin(2)}}^{S^1}(\underline{\mathrm{SWF}}^R(Y, \mathfrak{s}; Pin(2))) \in \mathrm{ob}(\bar{\mathfrak{S}}^*_{\mathbb{Z}_2}).$$

Now we start defining the numerical invariants in each case separately. The constructions are very similar to each other because they all follow from the same idea: by composing the isomorphisms provided by Theorem 8.2.1 with the inclusion of the S^1-fixed point spectra, we get morphisms from the sphere spectrum into our spectrum invariants. Then we consider the maps on generalized cohomology induced by these morphisms and extract numerical invariants from the images of these maps.

8.2.1 d invariants for torsion spin^c structures

We will define the invariants d^A, d^R by applying the functor of S^1-equivariant cohomology to our spectrum invariants.

Consider the type-A case first. We let $\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}; S^1)$ be represented by the inductive system

$$I_1^+ \xrightarrow{i_1^+} I_2^+ \xrightarrow{i_2^+} I_3^+ \ldots.$$

with $I_m^+ = (A_{m,+}, a_{m,+}, b_{m,+})$ for $m > 0$. The isomorphism $\tau_+^{S^1}$ in Theorem 8.2.1 can be represented by a morphism $\tau_{m,+}^{S^1} \in \mathrm{mor}_{\tilde{\mathfrak{C}}}((S^0, 0), (A_{m,+}^{S^1}, a_{m,+}))$ for all m greater or equal to some constant m_0. These morphisms satisfy the relation

$$i_m^+ \circ \tau_{m,+}^{S^1} = \tau_{m+1,+}^{S^1} \text{ for any } m \geq m_0. \tag{8.4}$$

We consider the stable map $\rho_{m,+}$ given by the composition

$$(\mathbb{R}^{a_{m,+}})^+ \to A_{m,+}^{S^1} \to A_{m,+}$$

where the first arrow is given by the stable map representing $\tau_{m,+}^{S^1}$ and the second arrow is the natural inclusion. For each $m \geq m_0$, we define a number

$$d_{m,+} := \inf\{p \in \mathbb{N} \mid \rho_{m,+}^*(\tilde{H}_{S^1}^p(A_{m,+}; \mathbb{Z}_2)) \neq 0\} - a_{m,+} - 2b_{m,+}$$

where $\tilde{H}_{S^1}^p$ denotes the reduced S^1-equivariant cohomology of degree p. Using the relation (8.4), one can prove that $d_m \leq d_{m+1}$ for any $m \geq m_0$. We define the type-A d-invariant for $(Y, \mathfrak{s})$ as

$$d^A(Y, \mathfrak{s}) := \sup\{d_{m,+} \mid m \geq m_0\} - \frac{b_1(Y)}{2} \in \mathbb{Q} \cup \{+\infty\}.$$

Now we turn to the type-R case. Suppose $\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}; S^1)$ is represented by the projective system

$$I_1^- \xleftarrow{i_1^-} I_2^- \xleftarrow{i_2^-} I_3^- \cdots$$

with $I_m^- = (A_{m,-}, a_{m,-}, b_{m,-})$. The isomorphism $\tau_-^{S^1}$ in Theorem 8.2.1 can be represented by a collection $\{\tau_{m,-}^{S^1}\}_{m \geq 0}$ of morphisms with $\tau_{m,-}^{S^1} \in \mathrm{mor}_{\tilde{\mathfrak{C}}}((S^0, 0), I_m^-)$. Similar to the previous case, we consider stable map $\rho_{m,-}$ given by the composition

$$(\mathbb{R}^{a_{m,-}})^+ \to A_{m,-}^{S^1} \to A_{m,-}$$

and define the number

$$d_{m,-} := \inf\{p \in \mathbb{N} \mid \rho^*_{m,-}(\tilde{H}^p_{S^1}(A_{m,-};\mathbb{Z}_2)) \neq 0\} - a_{m,-} - 2b_{m,-}$$

for each $m \geq 0$. It can be proved that $d_{m+1,-} \leq d_{m,-}$ for any $m > 0$. We define the type-R d-invariant for $(Y, \mathfrak{s})$ as

$$d^R(Y,\mathfrak{s}) := \inf\{d_{m,-} \mid m > 0\} + \frac{b_1(Y)}{2} \in \mathbb{Q} \cup \{-\infty\}.$$

Theorem 8.2.2. *Both the numbers $d^A(Y,\mathfrak{s})$ and $d^R(Y,\mathfrak{s})$ are topological invariants of the pair $(Y,\mathfrak{s})$. Moreover, they satisfy the following properties:*

(i) $d^A(-Y,\mathfrak{s}) = -d^R(Y,\mathfrak{s})$.

(ii) Suppose $(Y_j,\mathfrak{s}_j)$ $(j = 1,2)$ are $\mathbb{Q}$-homology cobordant to each other. Then we have

$$d^A(Y_1,\mathfrak{s}_1) = d^A(Y_2,\mathfrak{s}_2) \text{ and } d^R(Y_1,\mathfrak{s}_1) = d^R(Y_2,\mathfrak{s}_2).$$

(iii) Suppose $(Y,\mathfrak{s})$ bounds a spin^c 4-manifold $(X,\hat{\mathfrak{s}})$ with $b_2^+(X) = 0$. Then we have

$$c_1(\hat{\mathfrak{s}})^2 + b_2^-(X) \leq 4d^A(Y,\mathfrak{s}) + 2b_1(Y).$$

If we further assume that map $H^1(X;\mathbb{Q}) \to H^1(Y;\mathbb{Q})$ is surjective, then we also have

$$c_1(\hat{\mathfrak{s}})^2 + b_2^-(X) \leq 4d^R(Y,\mathfrak{s}) - 2b_1(Y).$$

Remark. From our definition, $d^A(Y,\mathfrak{s})$ (resp. $d^R(Y,\mathfrak{s})$) can be $+\infty$ (resp. $-\infty$). However, we have good reason to believe that this never happens. Actually, it is possible to prove this fact by considering periodic structure of the spectrum invariants under the action of $\mathcal{G}^{h,o}_Y$. Similar remarks applies for other invariants we construct in this section.

Remark. Behrens and Golla [63] defined a correction term $\underline{d}(Y,\mathfrak{s})$ using Heegaard Floer homology with fully twist coefficients. Their invariant corresponds to the invariant $d^A(Y,\mathfrak{s})$ here. In [52], Ozváth and Szabó defined two correction terms $d_{\pm\frac{1}{2}}$ for 3-manifolds with $b_1 = 1$ (and more generally, 3-manifolds with "standard HF^∞"). In principle, $d_{+\frac{1}{2}}$ and $d_{-\frac{1}{2}}$ correspond to the invariants d^R and d^A here. However, since Ozváth and Szabó do not use twisted coefficient in the construction, we do not know the exact relation between these invariants.

8.2.2 α, β, γ invariants for spin structures

Now we turn to the situation that $\mathfrak{s}$ is a spin structure and define the homology cobordism invariants $\alpha^A, \beta^A, \gamma^A, \alpha^R, \beta^R, \gamma^R$. As in [43], they are defined using Pin(2)-equivariant cohomology.

To begin with, we let $\underline{\mathrm{SWF}}^A(Y, \mathfrak{s}; Pin(2))$ be represented by

$$\tilde{I}_1^+ \to \tilde{I}_2^+ \to I_3^+ \to \cdots$$

and $\underline{\mathrm{SWF}}^R(Y, \mathfrak{s}; Pin(2))$ be represented by

$$\tilde{I}_1^- \leftarrow \tilde{I}_2^- \leftarrow I_3^- \leftarrow \cdots$$

with $\tilde{I}_m^{\pm} = (\tilde{A}_{m,\pm}, \tilde{a}_{m,\pm}, \tilde{b}_{m,\pm})$. We can represent isomorphism $\tau_+^{Pin(2)}$ in Theorem 8.2.1 by a morphism $\tau_{m,+}^{Pin(2)}$ for any m greater or equal to a constant m_0, while $\tau_-^{Pin(2)}$ is represented by the whole collection $\{\tau_{m,-}^{Pin(2)}\}_{m>0}$ of morphisms. Here $\tau_{m,\pm}^{Pin(2)}$ belongs to $\mathrm{mor}_{\tilde{\mathfrak{C}}_{\mathbb{Z}_2}}((S^0, 0), (\tilde{A}_{m,\pm}^{S^1}, \tilde{a}_{m,\pm}))$. Now consider the Pin(2)-equivariant stable map $\tilde{\rho}_{m,\pm}$ as the composition

$$(\tilde{\mathbb{R}}^{\tilde{a}_{m,\pm}})^+ \to \tilde{A}_{m,\pm}^{S^1} \to \tilde{A}_{m,\pm}, \tag{8.5}$$

with the first arrow given by the stable map representing $\tau_{m,\pm}^{Pin(2)}$ and the second arrow being the natural inclusion. We define the numbers

$$\alpha_{m,\pm} := \inf\{p \mid p \equiv \tilde{a}_{m,\pm} \pmod 4,\ \tilde{\rho}_{m,\pm}^*(\tilde{H}_{Pin(2)}^p(\tilde{A}_{m,\pm}; \mathbb{Z}_2)) \neq 0\} - \tilde{a}_{m,\pm} - 4\tilde{b}_{m,\pm};$$

$$\beta_{m,\pm} := \inf\{p \mid p \equiv \tilde{a}_{m,\pm} + 1 \pmod 4,\ \tilde{\rho}_{m,\pm}^*(\tilde{H}_{Pin(2)}^p(\tilde{A}_{m,\pm}; \mathbb{Z}_2)) \neq 0\} - \tilde{a}_{m,\pm} - 4\tilde{b}_{m,\pm} - 1;$$

$$\gamma_{m,\pm} := \inf\{p \mid p \equiv \tilde{a}_{m,\pm} + 2 \pmod 4,\ \tilde{\rho}_{m,\pm}^*(\tilde{H}_{Pin(2)}^p(\tilde{A}_{m,\pm}; \mathbb{Z}_2)) \neq 0\} - \tilde{a}_{m,\pm} - 4\tilde{b}_{m,\pm} - 2.$$

(Note that $\alpha_{m,+},\ \beta_{m,+},\ \gamma_{m,+}$ are only defined for $m \geq m_0$.) One can prove the following inequalities:

$$\alpha_{m+1,+} \geq \alpha_{m,+};\ \beta_{m+1,+} \geq \beta_{m,+};\ \beta_{m+1,+} \geq \beta_{m,+} \text{ for any } m \geq m_0$$

$$\alpha_{m+1,-} \leq \alpha_{m,-};\ \beta_{m+1,-} \leq \beta_{m,-};\ \beta_{m+1,-} \leq \beta_{m,-} \text{ for any } m > 0.$$

As before, we define our invariants by passing to the limit as $m \to +\infty$ and doing suitable normalization as follows

$$\begin{aligned}
\alpha^A(Y,\mathfrak{s}) &:= \frac{1}{2}\sup\{\alpha_{m,+} \mid m \geq m_0\} - \frac{b_1(Y)}{4} \in \mathbb{Q}\cup\{+\infty\};\\
\beta^A(Y,\mathfrak{s}) &:= \frac{1}{2}\sup\{\beta_{m,+} \mid m \geq m_0\} - \frac{b_1(Y)}{4} \in \mathbb{Q}\cup\{+\infty\};\\
\gamma^A(Y,\mathfrak{s}) &:= \frac{1}{2}\sup\{\gamma_{m,+} \mid m \geq m_0\} - \frac{b_1(Y)}{4} \in \mathbb{Q}\cup\{+\infty\};\\
\alpha^R(Y,\mathfrak{s}) &:= \frac{1}{2}\inf\{\alpha_{m,-} \mid m > 0\} + \frac{b_1(Y)}{4} \in \mathbb{Q}\cup\{-\infty\};\\
\beta^R(Y,\mathfrak{s}) &:= \frac{1}{2}\inf\{\beta_{m,-} \mid m > 0\} + \frac{b_1(Y)}{4} \in \mathbb{Q}\cup\{-\infty\};\\
\gamma^R(Y,\mathfrak{s}) &:= \frac{1}{2}\inf\{\gamma_{m,-} \mid m > 0\} + \frac{b_1(Y)}{4} \in \mathbb{Q}\cup\{-\infty\}.
\end{aligned}$$

We summarize the important properties of these numbers in the following theorem:

Theorem 8.2.3. *The numbers α^A, β^A, γ^A, α^R, β^R, γ^R are all topological invariants for the pair $(Y,\mathfrak{s})$. Moreover, they satisfies the following properties:*

(i) All these invariants are invariant under $\mathbb{Q}$-homology spin cobordisms;

(ii) Under the change of orientation, these invariants behave as follows

- $-\alpha^A(Y,\mathfrak{s}) = \gamma^R(-Y,\mathfrak{s})$,
- $-\beta^A(Y,\mathfrak{s}) = \beta^R(-Y,\mathfrak{s})$,
- $-\gamma^A(Y,\mathfrak{s}) = \alpha^R(-Y,\mathfrak{s})$;

(iii) Suppose $(Y,\mathfrak{s})$ bounds a spin 4-manifold $(X,\hat{\mathfrak{s}})$ with $b_2^+(X) = 0$. Then we have

$$\frac{b_2^-(X) - 2b_1(Y)}{8} \leq \min(\alpha^A(Y,\mathfrak{s}), \beta^A(Y,\mathfrak{s}), \gamma^A(Y,\mathfrak{s})).$$

If we further assume that the map $H^1(X;\mathbb{Q}) \to H^1(Y;\mathbb{Q})$ induced by the inclusion of boundary is surjective, then we also have

$$\frac{b_2^-(X) + 2b_1(Y)}{8} \leq \min(\alpha^R(Y,\mathfrak{s}), \beta^R(Y,\mathfrak{s}), \gamma^R(Y,\mathfrak{s})).$$

(iv) In the case $b_1(Y) = 0$, we have $\beta^A(Y,\mathfrak{s}) = \beta^R(Y,\mathfrak{s}) = \beta(Y,\mathfrak{s})$, where $\beta(Y,\mathfrak{s})$ is the invariant defined by Manolescu [43]. Similar results hold for the α and γ invariants.

8.2.3 κ invariants for spin structures

Now we define the invariants κ^A and κ^R. They are obtained by applying Pin(2)-equivariant K-theory on the spectrum invariants. As before, let $\tilde{\rho}_{m,\pm} : (\tilde{\mathbb{R}}^{\tilde{a}_{m,\pm}})^+ \to \tilde{A}_{m,\pm}$ be the stable maps given by the composition 8.5.

Recall that the numbers $\tilde{a}_{m,\pm}$ are always even by our definition of the category $\mathfrak{C}_{Pin(2)}$. By the Bott periodicity theorem, there exists an isomorphism

$$\tilde{K}_{Pin(2)}((\tilde{\mathbb{R}}^{\tilde{a}_{m,\pm}})^+) \cong R(Pin(2)),$$

where $R(Pin(2))$ denotes the representation ring of Pin(2). Under this isomorphism, the image of the map

$$\tilde{\rho}^*_{m,\pm} : \tilde{K}_{Pin(2)}(\tilde{A}_{m,\pm}) \to \tilde{K}_{Pin(2)}((\tilde{\mathbb{R}}^{\tilde{a}_{m,\pm}})^+)$$

is an ideal of $R(Pin(2))$. By taking the character of the element $j \in Pin(2)$, we get an surjective ring homomorphism $\tau : R(Pin(2)) \to \mathbb{Z}$. Then $\tau(\operatorname{im} \tilde{\rho}^*_{m,\pm})$ is an ideal of $\mathbb{Z}$.

The following lemma is proved using the fact that the augmentation ideal of $R(Pin(2))$ acts nilpotently on $\tilde{K}^*_{Pin(2)}(\tilde{A}_{m,\pm}/\tilde{A}^{S^1}_{m,\pm})$ (see [41, Fact 2.11]).

Lemma 8.2.4. *For any $m \geq m_0$, the ideal $\tau(\operatorname{im} \tilde{\rho}^*_{m,+})$ is generated by $2^{c_{m,+}}$ for some $c_{m,+} \in \mathbb{N}$. Similarly, for any $m > 0$, the ideal $\tau(\operatorname{im} \tilde{\rho}^*_{m,-})$ is generated by $2^{c_{m,-}}$ for some $c_{m,-} \in \mathbb{N}$.*

We define $\kappa_{m,\pm}$ as $c_{m,\pm} - \tilde{b}_{m,\pm}$, where $\tilde{b}_{m,\pm}$ is the second desuspension index of the object $\tilde{I}^{\pm}_m$. It can be proved that $\kappa_{m+1,+} \geq \kappa_{m,+}$ for any $m \geq m_0$, while $\kappa_{m+1,-} \leq \kappa_{m,-}$ for any $m > 0$. We define our invariants by taking the limit of $\kappa_{m,\pm}$:

$$\kappa^A(Y, \mathfrak{s}) := 2 \sup\{\kappa_{m,+} \mid m \geq m_0\} \in \mathbb{Q} \cup \{+\infty\};$$
$$\kappa^R(Y, \mathfrak{s}) := 2 \inf\{\kappa_{m,-} \mid m > 0\} \in \mathbb{Q} \cup \{-\infty\}.$$

We summarize the important properties of $\kappa^A(Y, \mathfrak{s})$ and $\kappa^R(Y, \mathfrak{s})$ in the following theorem.

Theorem 8.2.5. *The numbers $\kappa^A(Y,\mathfrak{s})$ and $\kappa^R(Y,\mathfrak{s})$ are topological invariants of the pair $(Y,\mathfrak{s})$. They satisfy the following properties:*

(i) Both invariants are invariant under $\mathbb{Q}$-homology spin cobordisms;

(ii) Let $(X,\hat{\mathfrak{s}})$ be a spin 4-manifold bounded by $(Y,\mathfrak{s})$. Then we have the following inequalities

$$-\frac{\sigma(X)}{8} \le b_2^+(X) + \xi(b_2^+(X)) + \kappa^A(Y,\mathfrak{s});$$
$$-\frac{\sigma(X)}{8} \le b_2^+(X) + b + \xi(b_2^+(X)+b) + \kappa^R(Y,\mathfrak{s}).$$

Here $b = \dim_{\mathbb{R}} \operatorname{coker}(H^1(X;\mathbb{R}) \to H^1(Y;\mathbb{R}))$ and the function $\xi : \mathbb{Z}^{\ge 0} \to \mathbb{Z}$ is given by the formula

$$\xi(a) := \begin{cases} 0 & a = 0 \\ -1 & a \text{ odd} \\ -2 & a > 0, \text{ even} \end{cases}.$$

(iii) In the special case that $b_1(Y) = 0$, we have $\kappa^A(Y,\mathfrak{s}) = \kappa^R(Y,\mathfrak{s}) = \kappa(Y,\mathfrak{s})$. Here $\kappa(Y,\mathfrak{s})$ denotes the original κ invariant defined by Manolescu [41].

8.3 The gluing theorem and its corollaries

In this section, we will give the statement of the gluing theorem and discuss its various applications. To keep our discussion comprehensible, we focus on the simplest case of gluing theorem instead of giving its most general form. Also, we only talk about the S^1 version and the argument can be easily adapted to the Pin(2) version when $\mathfrak{s}$ is a spin structure.

We begin with the following theorem on the behavior of the spectrum invariants under orientation reversal.

Theorem 8.3.1. *Let $\mathfrak{s}$ be a torsion spin^c structure on Y. Then the spectrum invariants $\underline{\mathrm{SWF}}^A(Y,\mathfrak{s};S^1)$ and $\underline{\mathrm{SWF}}^R(-Y,\mathfrak{s};S^1)$ are S^1-equivariant Spanier-Whitehead dual to* *each other in the sense of Definition 3.1.5. Similar result holds for Pin(2)-spectrum invariants for spin manifolds.*

Now let $(X, \hat{s})$ be a closed spin^c 4-manifold. We assume that X can be decomposed as $X_1 \cup_Y X_2$ with Y being a *connected* 3-manifold. Denote by $\hat{\mathfrak{s}}_1, \hat{\mathfrak{s}}_2$ and $\mathfrak{s}$ the restriction of $\hat{\mathfrak{s}}$ on X_1, X_2 and Y respectively.

We choose the base point $\hat{o} \in X$ to be on Y. The based path data for X_j $(j = 1, 2)$ is set to be represented by the constant path. With these topological inputs, the (relative) Bauer-Furuta invariants for X_1, X_2 and X are defined. Recall that the Bauer-Furuta invariant for X, denoted by $\mathrm{BF}(X, \hat{\mathfrak{s}}; S^1)$, is an equivariant stable homotopy class of maps defined on the Thom space of a virtual index bundle over the Picard torus $Pic^0(X)$. Therefore, we can restrict $\mathrm{BF}(X, \hat{\mathfrak{s}}; S^1)$ to the subtorus

$$Pic^0(X, Y) := \ker(H^1(X, \mathbb{R}) \to H^1(Y; \mathbb{R}))/\ker(H^1(X, \mathbb{Z}) \to H^1(Y; \mathbb{Z})).$$

We denote this restriction by $\mathrm{BF}(X, \hat{\mathfrak{s}}; S^1)|_{Pic^0(X,Y)}$.

Theorem 8.3.2. *Suppose the following two conditionns are satisfied*

- $\mathfrak{s}$ *is torsion;*
- $im(H^1(X_1; \mathbb{R}) \to H^1(Y; \mathbb{R})) \subset im(H^1(X_2; \mathbb{R}) \to H^1(Y; \mathbb{R}))$.

Then under the natural identification $Pic^0(X, Y) \cong Pic^0(X_1, Y) \times Pic^0(X_2, Y)$ *provided by the Mayer-Vietoris sequence, we have*

$$\mathrm{BF}(X, \hat{\mathfrak{s}}; S^1)|_{Pic^0(X,Y)} = \tilde{\epsilon}(\underline{\mathrm{BF}}^A(X_1, \hat{\mathfrak{s}}_1; S^1), \underline{\mathrm{BF}}^R(X_2, \hat{\mathfrak{s}}_2; S^1)),$$

where $\tilde{\epsilon}(\cdot, \cdot)$ *denotes the smashing operation given in Definition 3.1.6.*

Remark. When $im(H^1(X_1; \mathbb{R}) \to H^1(Y; \mathbb{R})) = 0$, the second condition in Theorem 8.3.2 is automatically satisfied. Moreover, in this case we have $Pic^0(X, Y) = Pic^0(X)$ and the full Bauer-Furuta invariant can be recovered from this Theorem.

Note that in the current situation, the explicit description of the smashing operation $\tilde{\epsilon}$ is given with the help of the manifold Conley index pairs. (Roughly speaking, a manifold Conley index pair is an index pair (N, L) such that N is a manifold with boundary and L is a submanifold (also with boundary) of ∂N. See [41] for the precise definition.) Recall that in the definition of the relative Bauer-Furuta invariants, we choose index pairs containing some specific pre-index pairs. In general, these index pairs can not be assumed to be manifold index pairs. To prove the gluing theorem, we have to use the flow maps to relate our index pairs with the manifold index pairs. This actually corresponds to the "neck-stretching argument" in the proof of gluing theorem for Floer homologies. We mention that a major difficulty in the proof of Theorem 8.3.2 is to control the time of the flow map (which corresponds to the length of the "neck") so that it does not goes to infinity as we do finite dimension approximations. Theorem 3.2.14 plays an important role here. We refer to our paper [27] for a complete proof.

We end this section by a few corollaries of the gluing theorem. The proof of these corollaries are straightforward applications of Theorem 8.3.2, together with the knowledge about the spectrum invariants of simple examples (such as $S^2 \times S^1$).

Corollary 8.3.3. *Let γ be an embedded loop in a closed spin^c 4-manifold $(X, \hat{\mathfrak{s}})$. We assume that γ represent a nonzero class in $H^1(X; \mathbb{R})$. By removing a regular neighborhood $D^3 \times S^1$ of γ and attaching a copy of $D^2 \times S^2$, we obtain a new spin^c 4-manifold $(X', \hat{\mathfrak{s}}')$. Under the canonical embedding $Pic^0(X') \hookrightarrow Pic^0(X)$, we have the following equality*

$$\mathrm{BF}(X, \hat{\mathfrak{s}}; S^1)|_{Pic^0(X')} = \mathrm{BF}(X', \hat{\mathfrak{s}}'; S^1).$$

In particular, this implies that the fiberwise Bauer-Furuta invariant does not change under surgeries along loops.

Corollary 8.3.4. *Suppose $(X, \hat{\mathfrak{s}})$ is a closed spin^c 4-manifold with non-zero Bauer-Furuta invariant. Then $(X, \hat{\mathfrak{s}})$ can not be obtained as the fiber sum of two closed spin^c 4-manifold $(X_j, \hat{\mathfrak{s}}_j)$ $(j = 1, 2)$ along embedded Klein bottles K_j with self-intersection number 0. (Fiber sum here means removing a regular neighborhood of K_j and gluing the resulting manifolds by an orientation reversing diffeomorphism between their boundaries.)*

The following result is a special case of [18, Theorem 1.1] and our gluing theorem provides an alternative proof.

Corollary 8.3.5. *Suppose X is a closed, smooth 4-manifold with non-zero Bauer-Furuta invariant. Then we have the following results:*

- *X can not be obtained as the union of X_1, X_2 along their common boundaries $S^2 \times S^1$ such that the homology class of S^2 is non-zero in $H^2(X;\mathbb{R})$;*
- *X does not contain any smoothly embedded S^2 that has self-intersection number 0 and represents a non-zero class in $H^2(X;\mathbb{R})$.*

8.4 The connected sum formula for Manolescu's spectrum

A natural question in Seiberg-Witten Floer theory is: how does the Seiberg-Witten Floer homology behave under the connected-sum operation. In the contexts of Heegaard Floer homology and monopole Floer homology, a Künneth formula for connected sums is proved respectively by Ozsváth-Szabó [53] and Baldwin-Bloom [6]. A spectral sequence for Pin(2)-equivariant Seiberg-Witten Floer homology of connected sums is proved by Lin [38]. In this section, we sketch the proof of the following connected sum formula for Manolescu's spectrum invariant. See [27] for the detailed proof.

Theorem 8.4.1. *Suppose Y_1, Y_2 are two rational homology 3-spheres. Then we have the following results:*

- *Let $\mathfrak{s}_j$ be a spin^c structure on Y_j. Then we have*
$$\mathrm{SWF}(Y_1\#Y_2, \mathfrak{s}_1\#\mathfrak{s}_2; S^1) \cong \mathrm{SWF}(Y_1, \mathfrak{s}_1; S^1) \wedge \mathrm{SWF}(Y_2, \mathfrak{s}_2; S^1).$$
- *Let $\tilde{\mathfrak{s}}_j$ be a spin structure on Y_j. Then we have*
$$\mathrm{SWF}(Y_1\#Y_2, \tilde{\mathfrak{s}}_1\#\tilde{\mathfrak{s}}_2; Pin(2)) \cong \mathrm{SWF}(Y_1, \tilde{\mathfrak{s}}_1; Pin(2)) \wedge \mathrm{SWF}(Y_2, \tilde{\mathfrak{s}}_2; Pin(2)).$$

We need the following proposition in order to prove Theorem 8.4.1. Although the result of this proposition seems natural, the proof is actually highly nontrivial. Again, we refer to [27] for the proof.

Proposition 8.4.2. *Let $(Y, \mathfrak{s})$ be a spin^c 3-manifold whose components are all rational homology spheres. Then the trivial cobordism $X = [0,1] \times Y$ induces the identity morphism on* $\mathrm{SWF}(Y, \mathfrak{s}; S^1)$*. Similar result holds for the Pin(2)-spectrum invariant.*

Proof of Theorem 8.4.1 (Sketch). We focus on the S^1 case and the Pin(2) case is similar. By adding an 1-handle on the manifold $[0,1] \times (Y_1 \cup Y_2)$, we get a cobordism $(X_1, \hat{\mathfrak{s}}_1)$ from the disjoint union $(Y_1 \cup Y_2, \mathfrak{s}_1 \cup \mathfrak{s}_2)$ to the connected sum $(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2)$. By reversing the orientation, we get a cobordism $(X_2, \hat{\mathfrak{s}}_2)$ from $(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2)$ to $(Y_1 \cup Y_2, \mathfrak{s}_1 \cup \mathfrak{s}_2)$. It is not hard to prove that the spectrum invariant of the disjoint union $(Y_1 \cup Y_2, \mathfrak{s}_1 \cup \mathfrak{s}_2)$ is isomorphic to the smash product $\mathrm{SWF}(Y_1, \mathfrak{s}_1; S^1) \wedge \mathrm{SWF}(Y_2, \mathfrak{s}_2; S^1)$. Therefore, the cobordism $(X_1, \hat{\mathfrak{s}}_1)$ induces a morphism

$$\rho_1 : \mathrm{SWF}(Y_1, \mathfrak{s}_1; S^1) \wedge \mathrm{SWF}(Y_2, \mathfrak{s}_2; S^1) \to \mathrm{SWF}(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2; S^1)$$

and the cobordism $(X_2, \hat{\mathfrak{s}}_2)$ induces the morphism

$$\rho_2 : \mathrm{SWF}(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2; S^1) \to \mathrm{SWF}(Y_1, \mathfrak{s}_1; S^1) \wedge \mathrm{SWF}(Y_2, \mathfrak{s}_2; S^1).$$

We need to show that ρ_1, ρ_2 are inverse to each other, i.e., $\rho_1 \circ \rho_2$ and $\rho_2 \circ \rho_1$ are respective identity morphisms.

Consider $\rho_2 \circ \rho_1$ first. By the Manolescu's original version of gluing theorem for relative Bauer-Furuta invariant (see [41]), the composition $\rho_2 \circ \rho_1$ equals the morphism $\rho_{1,2}$ induced by the 4-manifold

$$X_{1,2} = X_1 \cup_{Y_1 \# Y_2} X_2.$$

Since $X_{1,2}$ can be obtained from taking connected sum of the two components of the trivial cobordism $X'_{1,2} = [0,1] \times (Y_1 \cup Y_2)$, by the gluing theorem again, $\rho_{1,2}$ equals the morphism $\rho'_{1,2}$ induced by $X'_{1,2}$. By Proposition 8.4.2, we see that $\rho'_{1,2}$ equals the identity morphism on $\mathrm{SWF}(Y_1, \mathfrak{s}_1; S^1) \wedge \mathrm{SWF}(Y_2, \mathfrak{s}_2; S^1)$.

Now consider $\rho_2 \circ \rho_1$. Note that because $Y_1 \cup Y_2$ is not connected, we can not apply the Manolescu's gluing theorem to the composed cobordism $X_{2,1} = X_2 \cup_{Y_1 \cup Y_2} X_1$ to show that $\rho_2 \circ \rho_1$ equals the morphism $\rho_{2,1}$ induced by $X_{2,1}$. However, a variation of this theorem can be proved similarly. It asserts that the composition $\rho_2 \circ \rho_1$ equals $\tilde{\rho}_{2,1}$, the restriction of $\rho_{2,1}$ on a single fiber of Ind $\not{D}(X_{2,1})$ (the virtual index bundle over $Pic^0(X_{2,1}) \cong S^1$ of the Dirac operator). Notice that we can obtain the trivial cobordism $X'_{2,1} = [0,1] \times (Y_1 \# Y_2)$ from $X_{2,1}$ by doing a surgery along a loop generating $H_1(X_{2,1}; \mathbb{Z})$. A slight generalization of Corollary 8.3.3 (to the case of 4-manifolds with boundary) can be used to show that $\tilde{\rho}_{2,1}$ equals $\rho'_{2,1}$, the morphism induced by $X'_{2,1}$. By Proposition 8.4.2 again, we see that $\rho'_{2,1}$ equals the identity morphism on $\mathrm{SWF}(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2; S^1)$. This finishes the proof of the theorem. □

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