Title
Spacetime and Matter in Three-Dimensional Higher Spin Gravity

Author
Perlmutter, Eric

Publication Date
2012

University of California
Los Angeles

Spacetime and Matter in Three-Dimensional Higher Spin Gravity

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Physics

by

Eric Justin Perlmutter

2012

Abstract of the Dissertation

Spacetime and Matter in Three-Dimensional Higher Spin Gravity

by

Eric Justin Perlmutter

Doctor of Philosophy in Physics

University of California, Los Angeles, 2012

Professor Per Kraus, Chair

We investigate the physical content of three-dimensional higher spin gravity, a sophisticated toy model of high energy string theory. Starting from spin-3 gravity and working our way toward the theory of an infinite tower of higher spins coupled to matter, we show how to use higher spin gauge invariance to consistently generalize ordinary conceptions of black holes. We also consider the dynamics of scalar fields in nontrivial higher spin backgrounds. In view of the AdS/CFT correspondence as applied to certain vector-like conformal field theories with extended conformal symmetry, we make several successful comparisons of bulk and boundary computations.

The dissertation of Eric Justin Perlmutter is approved.

Rowan Killip

Michael Gutperle

Eric D'Hoker

Per Kraus, Committee Chair

University of California, Los Angeles

2012

To my infinitely loving and supportive parents.

Contents

Acknowledgments

To begin, I would like to thank my advisor, Per Kraus: for patiently teaching me about physics; for showing me how to do research; and for having faith that I would eventually be worth working with. I would not be in this position without his guidance. I should of course like to thank my other co-authors on the work presented here, Martin Ammon and Michael Gutperle, for their partnership. Thanks to the aforementioned, to Eric D'Hoker and the other members of the UCLA string theory group over the past several years for making UCLA a comfortable place to learn, and for many enjoyable interactions.

Now for some non-physicists. Thank you Gabe, for friendship that continues to define me. Thank you Leslie, for continually opening my eyes to what the world can be. Thank you Dara, for sibling companionship that keeps me from staying inside my own head at all times. And finally, thank you Mom and Dad, for everything that makes my life whole, and for encouraging me to do what I love. I love you all.

Vita

2002	Diploma, Horace Greeley High School, Chappaqua, New York
2006	Sc.B. (Mathematics-Physics), Brown University, Providence, Rhode Island
2008	M.S. (Physics), University of California, Los Angeles, California

Publications

E. Perlmutter, "Hyperscaling violation from supergravity," arXiv:1205.0242 [hep-th].

M. Ammon, P. Kraus and E. Perlmutter, "Scalar fields and three-point functions in D=3 higher spin gravity," arXiv:1111.3926 [hep-th].

P. Kraus and E. Perlmutter, "Partition functions of higher spin black holes and their CFT duals," JHEP **1111**, 061 (2011) [arXiv:1108.2567 [hep-th]].

M. Ammon, M. Gutperle, P. Kraus and E. Perlmutter, "Spacetime Geometry in Higher Spin Gravity," JHEP **1110**, 053 (2011) [arXiv:1106.4788 [hep-th]].

P. Kraus and E. Perlmutter, "Universality and exactness of Schrodinger geometries in string and M-theory," JHEP **1105**, 045 (2011) [arXiv:1102.1727 [hep-th]].

E. Perlmutter,"Domain Wall Holography for Finite Temperature Scaling Solutions," JHEP **1102**, 013 (2011) [arXiv:1006.2124 [hep-th]].

1. Introduction

The primary goal of this dissertation is to elucidate the physical properties of three-dimensional higher spin gravity.

In most contexts in physics, one studies theories of fields with spin two ($s = 2$) or less. Gravitational interactions are mediated by the spin-2 graviton, and one may include other matter fields, for example scalar ($s = 0$) or Yang-Mills ($s = 1$) fields. All theories of tree-level supergravity have fields of spin two or less, in particular those that descend from low-energy limits of string theories; those fields describe the long-range dynamics of the underlying UV complete theory.

There are various no-go theorems stating that subject to certain reasonable assumptions, in flat space, massless fields of $s > 2$ cannot couple consistently to massless fields of $s \leq 2$. More precisely, assuming that the $s < 2$ fields couple in a Lorentz-invariant manner to the graviton with two-derivative couplings, the above statement is true. For recent discussion, see [1].

On the other hand, it has been powerfully demonstrated by Vasiliev and collaborators that if one considers the same problem in curved, maximally symmetric spacetimes, the curvature scale of the spacetime acts as an expansion parameter which one can use to build consistent interactions among fields of all integer and half-integer spins. In particular, we will consider higher spin theories of gravity living in anti-de Sitter space (AdS), for which the full nonlinear equations of motion (but not an action) are known. These highly nonlinear field equations are essentially fixed by the large gauge symmetry of the theory, so-called higher spin gauge symmetry, of which diffeomorphism invariance is but a part. A consistency check on the theory is provided by linearizing around AdS, in which case the spectrum is given by a (generally infinite) tower of free, propagating higher spin fields understood by Frönsdal years before the full nonlinear theory was formulated. For a sampling of some of these foundational works, see [2,3,4,5,6,7]; a nice review of Vasiliev theory is given in [8].

One issue on which we will focus is the fact that the role of spacetime in higher spin theories is rather confusing, relative to traditional gravitational intuition. To begin with, the metric transforms under an arbitrary higher spin gauge transformation, which implies that ordinary notions of geometry become gauge-dependent and hence physically opaque if not altogether meaningless. On the level of the field equations themselves, there is a sense in which the spacetime features of a given solution are not fundamental, emerging only as a sort of gauge choice. This notion has long been around in conceptions of quantum gravity. What *is* fundamental is the internal symmetry, which is realized by a set of noncommutative variables. Thus, we are dealing with a theory which has deep connections between spacetime nonlocality and twistor space noncommutativity, two notions which have appeared widely in string/M/gauge theory contexts over the years (e.g. [9,10]).

We will have much more to say about these features of Vasiliev gravity in what follows. In particular, we choose to focus on the three-dimensional (3d) case [11], which is more amenable to computation, and on non-supersymmetric theories which possess fields of integer spin only. Mainly, this is because a 3d theory of gravity with massless fields of $s \geq 2$ has no local propagating degrees of freedom. This is equivalent to saying that the Weyl tensor vanishes in 3d. As we will soon review, this is intimately connected to the fact that we can reformulate the theory as a direct sum of Chern-Simons gauge theories [12,13]. A further perk is that, unlike in dimensions greater than three, one can truncate to include only a finite set of fields up to some spin $s \leq N$ for some integer N; this provides a stepping stone to studying the theory with an infinite tower of nonlinearly coupled higher spin fields, and we will take advantage of it. Finally, the role of matter fields is closer to our regular intuition, in that they can be coupled in "on top" of the higher spin fields and are not required for the consistency of a given background, again in contrast to higher dimensions. A review of the field content and equations of the 3d theory is given in Appendix II.

Aside from our inherent interest in this extension of ordinary gravity, we are motivated by the fact that Vasiliev gravity has certain features in common with string theory at high energies. It is nonlocal with an enhanced gauge symmetry, perhaps shows signs of black hole de-singularization (for a stringy example of a classical singularity being smoothed out by stringy effects, see [14]), and possesses at least part of the correct spectrum of fields. It has been argued that the tensionless limit of string theory – a high-energy limit in which we take $\alpha' \to \infty$ – should be described by some theory of massless higher spins, in which case we can view the string theory for finite α' as a broken or Higgs-ed phase of this higher spin theory. The string spectrum contains massive higher spin modes which become massless in this limit. While the derivation of such a theory from string theory does not yet exist, it should be an uncontroversial statement that Vasiliev gravity may be a sophisticated toy model of such a theory. Given the fact that the Vasiliev theory's equations are nearly fixed by gauge symmetry, perhaps the connection is even deeper [15]; there are some recent signs of this [16], and some intriguing speculations along these lines are given in [17].

In a modern context, the existence of these theories in AdS obviously begs for application of and to the AdS/CFT correspondence [18,19,20], which conjectures a duality between gravitational theories living in asymptotically AdS spacetimes and quantum field theories living in one fewer spacetime dimension. We cannot do justFice to the power of holography here: its wide impact on theoretical high energy physics, both gravitational and field-theoretic, can hardly be overstated. The fundamental ingredient in the correspondence is the map between bulk and boundary symmetries, likewise between bulk fields and gauge-invariant boundary operators each living in representations thereof. From this, we can deduce that any higher spin example of AdS/CFT must involve a CFT with a tower of conserved higher spin currents, sourced in the bulk by the asymptotic boundary values of massless higher spin fields [21,22,23,15,24].

The first concrete example of higher spin holography was given by Klebanov and Polyakov [25], who conjectured a duality between Vasiliev gravity on AdS_4 and the O(N) vector model. This model has free and critical fixed points, connected by an RG flow triggered by the addition of a relevant double-trace operator to the free O(N) Lagrangian; by the standard story of multi-trace operators in AdS/CFT [26], this is realized in the bulk by the choice of quantization for a scalar field whose mass is (in accord with our previous remarks) fixed by higher spin symmetry to be $(mL)^2 = -2$. Famously, the O(N) vector model has of order N degrees of freedom, hence the name "vector" model, in contrast with the paradigmatic supergravity examples dual to adjoint matter theories with order N^2 degrees of freedom. Further contrast is provided by the fact that the (UV) boundary theory is free and hence trivially soluble, rather than strongly ('t Hooft) coupled.

This conjecture was studied early on by [27,28,29] and all but confirmed by the recent impressive work of [30,31,32] which calculated boundary three-point correlators from the bulk. It was recently proven in [33] if one only assumes the basic tenets of AdS/CFT.

In something of an inverse of the normal use of AdS/CFT, the potential of higher spin holography perhaps lies not in learning about the boundary CFTs – which may be soluble in themselves, at least in principle – but in understanding on a deep level why and how AdS/CFT works at all. Additionally, recent work has unearthed examples of higher spin dualities which are not trivial: the boundary theories are not free, nor are they connected to free theories by RG flows [34,16,35,17].

One of these recent examples involves higher spin gravity in AdS_3. This AdS_3 analog of the Klebanov-Polyakov conjecture is the proposed duality of Gaberdiel and Gopakumar [34] between Vasiliev gravity in 3d coupled to a pair of complex scalar fields and a 't Hooft limit of the W_N minimal model CFT, which has a coset representation

$$\frac{SU(N)_k \oplus SU(N)_1}{SU(N)_{k+1}} \tag{1.1}$$

The 't Hooft limit is defined as

$$N, k \to \infty \ , \quad \lambda \equiv \frac{N}{k+N} \quad \text{fixed} \tag{1.2}$$

and in the limit the central charge behaves as

$$c \approx N(1-\lambda^2) \tag{1.3}$$

The theory has so-called W_N symmetry, a member of a large family of W-algebras which encode extended conformal symmetry. One can think of these as generalizations of Virasoro symmetry, which is actually just the W_2 algebra, and we will discuss this in more detail in section 3.

Notice that $c \sim N$, signaling the vector-like nature of the CFT. This time, however, there is a free parameter λ which takes values $0 \leq \lambda < 1$. Theories with different values of λ are not equivalent, nor are the bulk theories which also carry a λ-dependence in their symmetry algebra and in the scalar mass. Compared to the Klebanov-Polyakov case, there is richer structure to be studied here, and yet the CFT as a coset model retains the virtue of solubility: the theory is not free, yet preserves exact higher spin symmetry.

In the work ahead we will come to bear on this duality and find strong support for it, at zero and finite temperature. There is quite a bit of evidence for this conjecture beyond the present work: this includes matches between bulk and boundary global symmetries [36,37]; the bulk 1-loop determinant and the large N CFT partition function [38]; and certain three-point correlators computed from bulk and boundary [39].

AdS/CFT says that black holes in the bulk map to finite temperature states on the boundary, with the Hawking temperature equal to the temperature of the dual CFT. We have already stressed that ordinary notions of Riemannian geometry are not upheld in higher spin gravity. Is it possible to speak meaningfully about black holes, and to write them down as solutions to the field equations? We will answer this question in the affirmative in what follows, although it will again be crucial that we work in 3d, which is undoubtedly connected to the lack of local degrees of freedom in the gauge sector (i.e. without matter fields turned on). The issue of black holes in 4d is notoriously difficult to study; indeed, there is not yet a satisfactory identification of even a single solution which should rightfully be called a black hole, although there are various proposals for objects which have certain commonalities with black holes, at least in their asymptotic structure [40,41,42]. These works are rather difficult to understand in physical terms in any case. We believe it is fair to say that in 4d, the implications of higher spin gauge invariance for black holes has not yet been understood. The recent work [43] shed some light on these issues in the context of black hole phase transitions in vector-like holography, which introduce subtlety given the $O(N)$ degrees of freedom rather than $O(N^2)$.

1.1. **Summary of chapters**

Chapter 2 will give a brief overview of ordinary 3d gravity. We will discuss concepts of asymptotic symmetry, introduce the AdS solution and the BTZ black hole [44], and its formulation as two copies of SL(2,R) Chern-Simons gauge theory. We will show that in AdS_3, the asymptotic symmetry algebra consists of left- and right-moving Virasoro algebras [45], and the general rotating BTZ black hole carries independent left and right moving Virasoro zero mode charges. The BTZ entropy takes the form of Cardy's formula, and so matches elegantly with the entropy of any CFT dual with the same central charge [46]. The extension of these ideas to the higher spin realm is conceptually pleasing and straightforward [47,48], and we will set the stage for our investigations in subsequent chapters.

Chapter 3 will tackle the simplest case of higher spin gravity, namely Chern-Simons theory with an SL(3,R) Lie algebra[1]. This theory describes the interactions of a graviton with a massless spin-3 field, and has an AdS_3 vacuum with two copies of the W_3 algebra as its asymptotic symmetry group. As ordinary AdS_3 gravity contains BTZ black hole solutions with charge under left- and right-moving Virasoro zero modes it is natural to ask whether generalized SL(3,R) black hole solutions exist that carry charges under the W_3 zero modes too. We will see that they do, and must confront the fact that certain spacetime characteristics that we usually think of as being invariantly defined — such as the event horizon or curvature singularity of a black hole — become gauge dependent.

A key insight in this respect was given in [50], where it was proposed that the holonomies of the Chern-Simons gauge fields around the Euclidean time circle should take the same values as for the BTZ black holes: this defines a gauge-invariant horizon. It was shown that this criterion leads to a sensible thermodynamics obeying a first law: the charges obey an integrability condition allowing one to integrate to find the thermodynamic partition function. The entropy can then be calculated from the first law. We will explicitly show that there exists a gauge in which the solution obeying this constraint is indeed manifestly a black hole. We thus have a very concrete example of how two spacetime metrics with different causal structures can be gauge equivalent in higher spin gravity.

[1] Based on work in [49].

Chapter 4 gives a much more powerful demonstration of the veracity of this method, by extending this logic to the challenging arena of Vasiliev's theory of an infinite tower of higher spins[2]. This theory is based on a one-parameter family of infinite dimensional gauge algebras, denoted hs[λ] [52,53,54,55,56,57]. To access the higher spin sector we turn on a nonzero spin-3 chemical potential, α. Due to the nonlinear structure of the theory, this triggers nonzero values for the entire infinite tower of higher spin charges. The values of these charges, and the full smooth solution, can be determined systematically using perturbation theory in α. Given that this is the theory which might conceivably appear as some subsector of a low-energy limit of string theory, and at the very least is the subject of the intriguing minimal model duality, we find this chapter to be particularly interesting. We will also compare to CFT computations at special values of λ and find agreement; such agreement has been found for all values of λ recently [58].

Chapter 5 changes directions somewhat, leaving black holes behind for computations of scalar wave equations and vacuum three-point functions in the same 3d Vasiliev theory[3]. The former is a crucial ingredient in understanding how consistently coupled matter interacts with higher spin fields, perhaps as a first step toward understanding black hole formation. The latter is a fundamental test of AdS/CFT and will help us come closer to a perturbative proof of the minimal model duality conjecture. Such three-point functions were calculated in the 4d case with much more computational complexity in [30,31,32]; and certain of these correlators were computed in 3d before this work was done. There are two immediate ways the contents of Chapter 5 generalize calculations of three-point correlators that came before it. The bulk computation of [39] was in the "undeformed" Vasiliev theory, that is, at $\lambda = 1/2$ (or $\nu = 0$

[2] Based on work in [51].

[3] Based on work in [59].

in the language of [11]), and we will successfully extend this to arbitrary λ. The minimal model computations of [39,60] only treat the low spins, and we will successfully extend this to arbitrary spin. Our key physical insight is that gauge transformations of AdS can generate nontrivial higher spin backgrounds, and we can extract the three-point correlators by asking how the scalar field in AdS transforms under such an operation.

Chapter 6 closes with some remarks and open questions for the field. We have also included two appendices: Appendix I contains details of the higher spin Lie algebra hs[λ], and Appendix II gives a review of the basic ingredients in 3d Vasiliev gravity.

2. Pure 3d gravity and the generalization to higher spins

We begin with some basic facts about 3d gravity in the absence of higher spin fields. For further details and development beyond the scope of this work, see [61].

The action for this theory with a negative cosmological constant, ignoring boundary terms, is simply

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R + \frac{2}{\ell^2}\right) \tag{2.1}$$

Henceforth we set $\ell = 1$. This theory admits few solutions, though they include black holes. One of them is pure AdS_3 in Poincaré coordinates:

$$ds^2 = -r^2 dt^2 + \frac{dr^2}{r^2} + r^2 d\phi^2 \tag{2.2}$$

where $r \in [0, \infty)$ and we take $\phi \sim \phi + 2\pi$ to be a periodic boundary coordinate. Continuing to Euclidean signature, the conformal boundary at $r \to \infty$ has the topology of a torus. All solutions of this theory are locally AdS_3. We will find it useful to define lightcone coordinates $x^\pm = t \pm \phi$, in which case this metric becomes

$$ds^2 = -r^2 dx^+ dx^- + \frac{dr^2}{r^2} \tag{2.3}$$

We will mostly work in Fefferman-Graham coordinates $\rho = \log r$, in which the metric is written

$$ds^2 = d\rho^2 - e^{2\rho} dx^+ dx^- \tag{2.4}$$

and $\rho \in (-\infty, \infty)$.

The theory also admits an AdS solution in global coordinates,

$$ds^2 = d\rho^2 - (e^\rho + \frac{1}{4}e^{-\rho})^2 dt^2 + (e^\rho - \frac{1}{4}e^{-\rho})^2 d\phi^2 \tag{2.5}$$

which, as we will soon see, is the solution of the theory with the lowest mass.

One might casually remark that 3d gravity is all about boundary conditions. More rigorously, we mean that the specification of boundary conditions is crucial in understanding how a theory that (locally) admits only a single solution can be interesting. One can define a nontrivial notion of asymptotic AdS symmetry as follows. We can consider the set of infinitesimal symmetry transformations – diffeomorphisms – which leaves the leading behavior of the metric (2.5) unaffected, but which generate subleading falloff slow enough to generate a set of nonzero charges at infinity. This set of transformations generates the asymptotic symmetry group.

In the case of ordinary AdS gravity, this analysis was first performed by Brown and Henneaux [45]. The asymptotic symmetry group is two copies of the centrally extended Virasoro algebra: with symmetry generators L_n, the (right-moving) algebra is

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0} \tag{2.6}$$

with central charge c given by

$$c = \frac{3}{2G} \tag{2.7}$$

and similarly for the right-movers $\overline{L}_n$ with central charge $\bar{c}$. This is just the algebra of 2d local conformal transformations. The algebra is equivalently encoded in the OPE (now in Euclidean CFT) between two holomorphic stress tensors $T(z)$:

$$T(z)T(0) \sim \frac{c/2}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z} + (\text{nonsingular}) \tag{2.8}$$

where $(z, \bar{z})$ are complex coordinates on the Euclidean plane, and similarly for $\overline{T}(\bar{z})$. We move back and forth via the mode expansions

$$T(z) = \sum_{n\in Z} \frac{L_n}{z^{n+2}}\ , \quad \overline{T}(\bar{z}) = \sum_{n\in Z} \frac{\overline{L}_n}{\bar{z}^{n+2}} \tag{2.9}$$

(Keep in mind that the anomalous transformation of the stress tensor as we move from the plane to the cylinder induces a shift in the zero mode L_0 by $-c/24$; in the above the modes are defined on the plane by (2.9).)

Thus, from this point of view one can see, independently of AdS/CFT, an equivalence between gravity on an asymptotically AdS_3 background and a 2d conformal field theory with central charge c, where c is given in terms of gravitational quantities in (2.7). Every 2d CFT realizes two copies of the Virasoro symmetry for some central charge $(c, \overline{c})$, which count degrees of freedom[4]; appear as the normalization of the stress tensor two-point functions; parameterize the trace anomaly; and give the coefficient of the thermal free energy. There is one Virasoro copy each for the left-moving (holomorphic) and right-moving (anti-holomorphic) sectors.

Later, we will see how this conformal symmetry becomes extended upon adding higher spins to the bulk (although c will continue to take the Brown-Henneaux value, given that it is defined by the stress tensor transformation law, i.e. the spin-2 gravity sector of the theory). In this context, an "extension" of the conformal symmetry means that there are fields besides the stress tensor which have nontrivial OPEs with the stress tensor and among each other.

2.1. **The BTZ black hole**

Despite containing no local propagating degrees of freedom, the theory does admit rotating black hole solutions, in particular the BTZ black hole [44]. There are two global charges, namely the mass M and angular momentum J, and we can write the metric in their terms as

$$ds^2 = -N(r)^2 dt^2 + \frac{dr^2}{N(r)^2} + r^2(d\phi + N^\phi(r)dt)^2 \tag{2.10}$$

[4] For a diff-invariant theory, $c = \overline{c}$. Relaxing this condition, we encounter a theory with gravitational anomalies, which can be realized holographically by adding bulk gravitational Chern-Simons terms [62] to (2.1). We will often focus solely on the holomorphic sector henceforth, with the anti-holomorphic sector implied.

where

$$N(r) = \sqrt{r^2 - 8GM + \frac{16G^2J^2}{r^2}}$$
$$N^\phi(r) = -\frac{4GJ}{r^2} \ , \quad |J| \leq M \tag{2.11}$$

The inner and outer horizons lie at

$$r_\pm^2 = 4GM\left[1 \pm \sqrt{1 - \left(\frac{J}{M}\right)^2}\right] \tag{2.12}$$

The static case is $J = 0$ – in which case $r_- = 0$ – and the extremal case is $J = M$ – in which case $r_- = r_+$.

Using the Bekenstein-Hawking entropy formula, we compute the entropy as

$$S = \frac{A_H}{4G} = 2\pi\Big(\sqrt{\frac{M+J}{8G}} + \sqrt{\frac{M-J}{8G}}\Big) \tag{2.13}$$

Let us write this metric in two more ways. Having defined the inner and outer horizons, we can write the metric as

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}dt^2 + \frac{r^2}{(r^2 - r_+^2)(r^2 - r_-^2)}dr^2 + r^2(d\phi - \frac{r_+r_-}{lr^2}dt)^2 \tag{2.14}$$

We can also pass to Fefferman-Graham coordinates[5] with lightcone boundary coordinates $x^\pm = t \pm \phi$ once more in which case the metric is

$$ds^2 = d\rho^2 + 8\pi G\Big(\mathcal{L}(dx^+)^2 + \overline{\mathcal{L}}(dx^-)^2\Big) - \Big(e^{2\rho} + (8\pi G)^2\,\mathcal{L}\overline{\mathcal{L}}e^{-2\rho}\Big)dx^+dx^- \tag{2.15}$$

where we have defined linear combinations

$$\mathcal{L} = \frac{M-J}{4\pi} \ , \quad \overline{\mathcal{L}} = \frac{M+J}{4\pi} \tag{2.16}$$

In these coordinates the horizon is at $e^{2\rho_h} = 8\pi G\sqrt{\mathcal{L}\overline{\mathcal{L}}}$.

[5] The explicit coordinate transformation is given in [61].

Thinking of this as a black hole, we pass to Euclidean coordinates and demand the absence of a conical singularity at $\rho = \rho_h$. The Euclidean BTZ black hole is obtained by taking $dx^+ = dz$ and $dx^- = -d\bar{z}$. To avoid a conical singularity at the horizon we need to make the identification $(z, \overline{z}) \cong (z + 2\pi\tau, \overline{z} + 2\pi\overline{\tau})$, with

$$\mathcal{L} = -\frac{k}{8\pi\tau^2} \ , \quad \overline{\mathcal{L}} = -\frac{k}{8\pi\bar{\tau}^2} \tag{2.17}$$

The inverse temperature of this solution, β, and angular velocity of the horizon, Ω, are then given by

$$\tau = \frac{i\beta + i\beta\Omega}{2\pi} \ , \quad \bar{\tau} = \frac{-i\beta + i\beta\Omega}{2\pi} \tag{2.18}$$

Note that Ω should be continued to pure imaginary values in order to obtain a real Euclidean section. The modular parameter τ is identified with that of the boundary torus, and appears in the path integral as a potential:

$$Z(\tau; \bar{\tau}) = \mathrm{Tr}[e^{4\pi^2 i(\tau\mathcal{L} - \bar{\tau}\overline{\mathcal{L}})}] \tag{2.19}$$

The topology of this spacetime is a solid torus, where the radial direction goes "into" the body of the torus. The noncontractible cycle corresponds to the angular direction. We can think of obtaining the BTZ solution as a modular transformation of the thermal AdS spacetime, which is just the Euclidean continuation of the global AdS metric (2.5): for thermal AdS, which is also topologically a solid torus, it is clear upon inspection that the noncontractible cycle is the time direction. So all we have to do to pass from thermal AdS to BTZ is to interchange the time and space directions on the boundary torus: this corresponds to performing a modular S transformation on the torus, which is just an SL(2,Z) symmetry transformation of the torus. A family of SL(2,Z)-inequivalent black holes can be obtained by performing general SL(2,Z) transformations on thermal AdS, which label the two cycles in all ways consistent with SL(2,Z) symmetry.

These definitions allow us to make direct contact with a CFT interpretation. We want to think of the Lorentzian CFT as living on a cylinder. In the language of the Virasoro algebra, $(\mathcal{L}, \overline{\mathcal{L}})$ are left- and right-moving zero mode charges, respectively. The precise relation to the Virasoro zero modes is

$$2\pi\mathcal{L} = L_0 - \frac{c}{24}\ , \quad 2\pi\overline{\mathcal{L}} = \overline{L}_0 - \frac{\overline{c}}{24} \tag{2.20}$$

where $(L_0, \overline{L}_0)$ are defined on the plane. Notice that in terms of $(\mathcal{L}, \overline{\mathcal{L}})$ and $(c, \overline{c})$ as defined by (2.20) and (2.7), the entropy can be written as

$$S = 2\pi\left(\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)} + \sqrt{\frac{\overline{c}}{6}\left(\overline{L}_0 - \frac{\overline{c}}{24}\right)}\right) \tag{2.21}$$

This is precisely the Cardy formula: that is, the high temperature number of states for any[6] 2d CFT with central charges $(c, \overline{c})$ and zero mode charges $(L_0, \overline{L}_0)$. Therefore, we have shown that the entropy of the BTZ black hole with global quantum numbers (M, J) in a theory with Newton's gravitational constant G can be accounted for microscopically by the degrees of freedom of any 2d CFT, subject to definitions (2.7), (2.16) and (2.20) [46].

The value $L_0 = \frac{c}{24}$ defines a black hole threshold; at threshold, we have the extremal BTZ solution $J = M$ (cf. (2.16)), and there is a continuous tower of non-extremal black hole states for all $L_0 > \frac{c}{24}$ with $J < M$. The vacuum value $L_0 = 0$ corresponds to global AdS$_3$; from (2.7), (2.16) and (2.20) we see that we should associate a negative mass $M = -\frac{1}{8G}$ to global AdS$_3$.

2.2. The Chern-Simons formulation

It was discovered over two decades ago that Einstein gravity with a negative cosmological constant can be re-written as a SL(2,R) × SL(2,R) Chern-Simons theory [12,13] . With 1-forms $(A, \bar{A})$ taking values in the Lie algebra of SL(2,R), the action is

$$S = S_{CS}[A] - S_{CS}[\overline{A}] \tag{2.22}$$

[6] This applies upon assuming a gap in the $\mathcal{L}$ spectrum, and unitarity of the CFT.

where

$$S_{CS}[A] = \frac{k}{4\pi} \int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{2.23}$$

The Chern-Simons level k is related to the Newton constant G as

$$k = \frac{1}{4G} \tag{2.24}$$

which is equivalent to

$$c = 6k \tag{2.25}$$

The combination $8\pi G$ that frequently appeared in BTZ calculations becomes $\frac{2\pi}{k}$.

The SL(2,R) commutation relations are

$$[L_m, L_n] = (m - n) L_{m+n} \tag{2.26}$$

We recognize this as the Virasoro algebra restricted to $m = \pm 1, 0$, minus the central term. SL(2,R) is thus the "wedge subalgebra" of the Virasoro algebra; we will see a generalization of this notion in the higher spin context as well.

The Chern-Simons equations of motion correspond to vanishing field strengths,

$$F = dA + A \wedge A = 0 \ , \quad \overline{F} = d\overline{A} + \overline{A} \wedge \overline{A} = 0 \tag{2.27}$$

To relate these to the Einstein equations, we introduce a vielbein e and spin connection ω as

$$A = \omega + e \ , \quad \overline{A} = \omega - e \tag{2.28}$$

and plug into the flatness conditions: the resulting equations are just the Einstein equations with negative cosmological constant written in the frame formalism. Expanding e and ω in a basis of 1-forms dx^μ, the familiar spacetime metric $g_{\mu\nu}$ is defined as

$$g_{\mu\nu} = \frac{1}{2} \mathrm{Tr}(e_\mu e_\nu) \tag{2.29}$$

Let us now write the BTZ solution in this language. One can show that the following flat connections give the metric (2.15) (given some trace normalization for SL(2,R) generators):

$$\begin{aligned} A &= (e^{\rho}L_1 - \frac{2\pi}{k}e^{-\rho}\mathcal{L}L_{-1})dx^+ + L_0 d\rho \\ \overline{A} &= -(e^{\rho}L_{-1} - \frac{2\pi}{k}\overline{\mathcal{L}}e^{-\rho}L_1)dx^- - L_0 d\rho \end{aligned} \tag{2.30}$$

The ρ-dependence can be gauge transformed away by

$$A = b^{-1}(a+d)b\ ,\quad \overline{A} = \overline{b}^{-1}(\overline{A}+d)\overline{b} \tag{2.31}$$

for

$$b = e^{\rho L_0}\ ,\quad \overline{b} = b^{-1} \tag{2.32}$$

in which case flatness is trivial to demonstrate. This deserves comparison to the metric formulation, which requires slightly more work to show that BTZ is a solution. This is especially true for the solution in which $\mathcal{L}(x^+), \overline{\mathcal{L}}(x^-)$ rather that being constants.

We note that as a technical matter, it is easier to find the Virasoro algebra by using a Chern-Simons connection than it is in the metric formulation, where the specification of asymptotic boundary conditions on metric components is somewhat ad hoc. By contrast, one easily recovers the Virasoro algebra by demanding that infinitesimal gauge transformations

$$\delta A = d\lambda + [A, \lambda] \tag{2.33}$$

leave intact the connection (2.30). Assuming that the leading term is to be unmodified – which is equivalent to saying that the leading asymptotic behavior of the metric must not ruin the AdS form – one can always use gauge transformations to put the connection into this form. The set of nontrivial gauge transformations are those that modify $\mathcal{L}$: that is, certain gauge transformations are not trivial at the boundary, but instead moves one around the space of physically distinct solutions [63]. (These are sometimes clumsily called "would-be" gauge transformations.) It is thus a simple matter to recover the Virasoro algebra.

This technique easily generalizes to the higher spin arena [47,48], where a metric-like technique is not available. We will explicitly perform this procedure in the next chapter.

2.3. Higher spin gravity from Chern-Simons theory

The key insight with respect to higher spin gravity, first developed by [48], is that we can consider taking $(A, \overline{A})$ to lie in some other, larger, Lie algebra besides SL(2,R), which we will denote $\mathcal{G}$. Doing so is equivalent to coupling some set of higher spin fields to Einstein gravity, where the rank of $\mathcal{G}$ determines the number of higher spin fields. Taking $\mathcal{G}$ = SL(N,R), for example, one has a theory of Einstein gravity coupled to a tower of symmetric tensor fields of spins $s = 3, 4, \ldots, N$. Taking $\mathcal{G}$ to be an algebra of infinite rank introduces an infinite tower of such spins; the details of the theory depend on which algebra one chooses. In all of these cases, one recovers Einstein gravity upon restriction to a SL(2,R) $\times$ SL(2,R) subalgebra of $\mathcal{G} \times \mathcal{G}$. Note that in general a given $\mathcal{G}$ admits many inequivalent embeddings of SL(2,R); this leads to the appearance of multiple AdS_3 vacua in the theory [49]. Ultimately – with respect to possible relationships to string theory and AdS/CFT duality – we are interested in the theory with algebra $\mathcal{G} = \text{hs}[\lambda]$, which admits a single SL(2,R) subalgebra.

It is worth emphasizing that, on a technical level, to pass to a metric-like formulation is generally difficult. On a basic level, the action of a consistently coupled, infinite tower of higher spin fields – even that of a single spin-3 field coupled to gravity – is not known in the metric like formulation. And while we can define a generalized vielbein which has an expansion in $\mathcal{G}$ generators, it is not even well-understood whether definitions of the metric-like fields – as traces over symmetric products of vielbeins – can be unambiguously fixed (up to field redefinitions) [64]. So our only option at present is to use the Chern-Simons formulation, which fortunately lends itself rather nicely to computation in higher spin gravity.

Furthermore, one of the lessons from recent work on higher spin gravity is that certain quantities are better understood in the Chern-Simons picture, and others in the metric-like picture. To gain any geometric understanding of a given solution – for instance, to probe its causal structure – the Chern-Simons formulation is not helpful. This is true even in the simplest SL(2,R) case, where the BTZ horizon is essentially invisible in the connection itself. On the other hand, we will see that the full gauge invariance of a given higher spin theory is most easily understood in terms of the connections, as is the identification of terms in the connection with CFT currents.

A particularly clean arena in which all of the aforementioned issues – besides the difficulty of passage to the metric-like formulation – are clearly present is the case of spin-3 gravity, which we take to be Chern-Simons theory with an SL(3,R) gauge algebra. We now construct black holes of this theory with spin-3 charge, generalizing BTZ, and tackle the obstacles presented by the enlarged higher spin symmetry to ensuring a consistent thermodynamics.

3. Spin-3 black holes

In preparation for the construction of the spin-3 black hole, we focus on the simplest extension of pure SL(2,R) gravity: namely, SL(3,R) gravity.

In section 3.1 we apply the Chern-Simons formulation to the spin-3 theory and construct the AdS_3 vacuum which has W_3 symmetry following [48]. We also construct a second AdS_3 vacuum based on the non-principal embedding of SL(2,R) in SL(3,R) and show that the asymptotic symmetry algebra is $W_3^{(2)}$. Finally, we construct solutions which can be interpreted as renormalization group flows between the two vacua. In section 3.2, we heat this system up: in particular, we review the black hole solutions in "wormhole" gauge found in [50], and explain the BTZ holonomy prescription for characterizing a gauge-invariant horizon. In section 3.3 we construct the explicit gauge transformation taking the solution from "wormhole" gauge to "black hole" gauge, and establish that the metric and spin-3 field are smooth across the horizon precisely when the holonomy conditions are obeyed. We close with some discussion. Some calculational details, as well as generalizations of some results to the case of general N, are relegated to appendices of this chapter.

3.1. SL(3,R) Chern-Simons theory, spin-3 gravity, and its AdS vacua

In this subsection, we set the stage for the black hole by presenting the two distinct AdS_3 vacua of this theory and their asymptotic symmetry algebras. Finally, we exhibit an RG flow solution interpolating between these vacua, and study some of its properties.

We must preface this discussion with a warning that the asymptotic symmetry of one of these vacua – namely, the $W_3^{(2)}$ symmetry – has recently been shown not to admit any unitary representations [65]. This is a general feature of non-principally embedded AdS vacua for any $\mathcal{G} \supset$ SL(2,R). So while this vacuum does not have desirable physical properties, we discuss it as an entry point to the peculiarities of higher spin gravity; in any case, this feature does not generalize to the hs[λ] case in which we are ultimately most interested.

3.1.1. Connections and metric-like fields

As discussed in the previous section, we consider SL(3,R)$\times$ SL(3,R) Chern-Simons theory, which corresponds to spin-3 gravity in three dimensions with a negative cosmological constant. Our conventions follow those in [50].

The action is

$$S = S_{CS}[A] - S_{CS}[\overline{A}] \tag{3.1}$$

with opposite CS levels $(k, -k)$. The 1-forms A and $\overline{A}$ take values in the Lie algebra of SL(3,R). An explicit representation of the eight generators $L_i,\ i = -1, 0, +1$ and $W_j,\ j = -2, -1, \cdots, +2$, as well as our conventions, is given in appendix 3A. Recall that we set

$$k = \frac{1}{4G} \tag{3.2}$$

With the definition (2.28), the spacetime metric $g_{\mu\nu}$ and spin-3 field $\varphi_{\mu\nu\gamma}$ are identified as

$$g_{\mu\nu} = \frac{1}{2}\mathrm{Tr}(e_\mu e_\nu)\ , \quad \varphi_{\mu\nu\gamma} = \frac{1}{3!}\mathrm{Tr}(e_{(\mu} e_\nu e_{\gamma)}) \tag{3.3}$$

where $\varphi_{\mu\nu\gamma}$ is totally symmetric as indicated. Restricting to the SL(2,R) subalgebra generated by L_i, the flatness conditions (2.27) can be seen to be equivalent to Einstein's equations for the metric $g_{\mu\nu}$ with a torsion free spin-connection. More generally, we find equations describing a consistent coupling of the metric to the spin-3 field.

Acting on the metric and spin-3 field, the SL(3,R) $\times$ SL(3,R) gauge symmetries of the Chern-Simons theory turn into diffeomorphisms along with spin-3 gauge transformations (the Chern-Simons gauge transformation also include frame rotations, which leave the metric and spin-3 field invariant). Under diffeomorphisms, the metric and spin-3 field transform according to the usual tensor transformation rules. The spin-3 gauge transformations are less familiar, as they in general act nontrivially on both the metric and spin-3 field. It is worth noting, though, that if we ignore the spin-3 gauge invariance, then we can view the theory as a particular diffeomorphism invariant theory of a metric and a rank-3 symmetric tensor field.

3.1.2. The W_3 AdS_3 vacuum

We consider the following flat connections

$$\begin{aligned} A_{AdS} &= e^{\rho} L_1 dx^+ + L_0 d\rho \\ \overline{A}_{AdS} &= -e^{\rho} L_{-1} dx^- - L_0 d\rho \end{aligned} \tag{3.4}$$

where $x^{\pm} = t \pm \phi$. Using (3.3), the corresponding metric and spin-3 field are

$$\begin{aligned} ds^2 &= d\rho^2 - e^{2\rho} dx^+ dx^- \\ \varphi_{\alpha\beta\gamma} &= 0 \end{aligned} \tag{3.5}$$

The metric is that of an AdS_3 of unit radius.

More generally, we can consider solutions that approach this vacuum asymptotically, and work out the resulting asymptotic symmetry algebra. The analysis proceeds in a parallel fashion for the barred and unbarred connections, and so we'll just focus on the latter. Also, since we will shortly present a more detailed computation for the second of our AdS_3 vacua, here we just sketch the steps. Following [48], we consider the following form for the connection

$$A = \left(e^{\rho} L_1 - \frac{2\pi}{k} \mathcal{L}(x^+) e^{-\rho} L_{-1} - \frac{\pi}{2k} \mathcal{W}(x^+) e^{-2\rho} W_{-2} \right) dx^+ + L_0 d\rho \tag{3.6}$$

In [48] there is a parameter σ accompanying the W-generators that we are here setting to $\sigma = -1$. Under gauge transformation,

$$A \to A + d\lambda + [A, \lambda] \tag{3.7}$$

One then works out the most general λ that preserves the form (3.6). Under these allowed gauge transformations the functions $\mathcal{L}(x^+)$ and $\mathcal{W}(x^+)$ transform. The asymptotic symmetry algebra is obtained from the Poisson brackets of the charges that generate these transformations; see [48] for details. Alternatively (see section 4 of [50] for details), one can translate these variations into an operator product expansion for the symmetry currents, and the result is the W_3 algebra:

$$\begin{aligned}
T(z)T(0) &\sim \frac{3k}{z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) \\
T(z)\mathcal{W}(0) &\sim \frac{3}{z^2}\mathcal{W}(0) + \frac{1}{z}\partial\mathcal{W}(0) \\
\mathcal{W}(z)\mathcal{W}(0) &\sim -\frac{5k}{\pi^2}\frac{1}{z^6} + \frac{10}{\pi}\frac{1}{z^4}\mathcal{L}(0) + \frac{5}{\pi}\frac{1}{z^3}\partial\mathcal{L}(0) + \frac{3}{2\pi}\frac{1}{z^2}\partial^2\mathcal{L}(0) \\
&\quad - \frac{1}{3\pi}\frac{1}{z}\partial^3\mathcal{L}(0) - \frac{32}{3k}\frac{1}{z}\mathcal{L}(0)\partial\mathcal{L}(0) - \frac{32}{3k}\frac{1}{z^2}\mathcal{L}(0)^2
\end{aligned} \tag{3.8}$$

Here we are using complex coordinates: $z = x^+$. The $T\mathcal{W}$ OPE identifies $\mathcal{W}$ as a spin-3 current, i.e., as a dimension $(3,0)$ primary operator. The same analysis for the barred connection gives rise to anti-holomorphic W_3 algebra with a dimension $(0,3)$ current.

From the TT OPE the central charge is found to be

$$c = 6k \tag{3.9}$$

which, using (2.24), agrees with the usual Brown-Henneaux formula $c = 3l/(2G)$.

3.1.3. The $W_3^{(2)}$ AdS_3 vacuum

The procedure reviewed above for extracting the asymptotic symmetry algebra is an example of the classical Drinfeld-Sokolov procedure, with which one can construct a W-algebra from an embedding of the SL(2,R) algebra inside a Lie algebra. In the example just given, the SL(2,R) algebra was generated by (L_1, L_0, L_{-1}). This is the "principal embedding" in SL(3,R). It is characterized by the fact that the five W generators transform as a spin 2 multiplet with respect to the SL(2,R) algebra. As discussed, this embedding gives rise to the W_3 algebra.

In the case of SL(3,R) there is exactly one additional inequivalent embedding of SL(2,R), up to conjugation[7]. We define rescaled versions of (W_2, L_0, W_{-2}) so that they obey the same commutation relations as (L_1, L_0, L_{-1}),

$$\hat{W}_2 = \frac{1}{4}W_2, \quad \hat{L}_0 = \frac{1}{2}L_0 \ , \quad \hat{W}_{-2} = -\frac{1}{4}W_{-2} \tag{3.10}$$

These have traces

$$\begin{aligned} \mathrm{Tr}(\hat{L}_0\hat{L}_0) &= \frac{1}{2} = \frac{1}{4}\mathrm{Tr}(L_0L_0) \ , \\ \mathrm{Tr}(\hat{W}_2\hat{W}_{-2}) &= -1 = \frac{1}{4}\mathrm{Tr}(L_1L_{-1}) \end{aligned} \tag{3.11}$$

Note that the branching of the adjoint representation of SL(3,R) into SL(2,R) representations is different in this case. There is one spin-0 multiplet, given by W_0, and two spin-1/2 multiplets, given by (W_1, L_{-1}) and (L_1, W_{-1}). (This is in addition to a spin-1 multiplet given by the SL(2,R) generators themselves).

This embedding gives rise to another W-algebra known as $W_3^{(2)}$, sometimes referred to as the Polyakov-Bershadsky algebra [66,67]. Next we work out the corresponding AdS_3 vacuum and its asymptotic $W_3^{(2)}$ symmetry algebra. Its central charge, which we denote by $\hat{c}$, will soon be related to the central charge c of the W_3 vacuum.

The unbarred gauge field includes fields for the highest weight generator in each multiplet

$$A = e^{-\rho\hat{L}_0}\Big(\hat{W}_2 - \mathcal{T}\hat{W}_{-2} + jW_0 + g_1L_{-1} + g_2W_{-1}\Big)e^{\rho\hat{L}_0}dx^+ + \hat{L}_0 d\rho \tag{3.12}$$

We parameterize a general gauge transformation as follows

$$\begin{aligned} \lambda =& e^{-\rho\hat{L}_0}\Big(\epsilon_1\hat{W}_2 + \epsilon_0\hat{L}_0 + \epsilon_{-1}\hat{W}_{-2} \\ &+ \gamma W_0 + \delta_1 L_1 + \delta_{-1}W_{-1} + \rho_1 W_1 + \rho_{-1}L_{-1}\Big)e^{\rho\hat{L}_0} \end{aligned} \tag{3.13}$$

[7] Some relevant facts about SL(2,R) embeddings in SL(N,R) are reviewed in appendix 3E.

Here the x^+ dependence of the fields and the transformation parameters is suppressed. We demand that the gauge transformation

$$\delta A = d\lambda + [A, \lambda] \tag{3.14}$$

respect the form of (3.12). The conditions determine the transformations of the fields (T, j, g_1, g_2) dependent on the parameters $(\epsilon_1, \gamma, \delta_1, \rho_1)$. Defining new fields

$$\begin{aligned} j &= \frac{9}{2\hat{c}} U \\ \mathcal{T} &= -\frac{6}{\hat{c}} T - \frac{27}{\hat{c}^2} U^2 \\ g_1 &= \frac{3}{\sqrt{2}\hat{c}} (G_+ + G_-) \\ g_2 &= \frac{3}{\sqrt{2}\hat{c}} (G_+ - G_-) \end{aligned} \tag{3.15}$$

as well as transformation parameters

$$\begin{aligned} \epsilon_1 &= \epsilon \\ \gamma &= -\frac{1}{2}\eta + \frac{9}{2\hat{c}} U\epsilon \\ \delta_1 &= \frac{1}{2\sqrt{2}} (\alpha_+ + \alpha_-) \\ \rho_1 &= \frac{1}{2\sqrt{2}} (\alpha_+ - \alpha_-) \end{aligned} \tag{3.16}$$

the transformation rules become

$$\begin{aligned} \delta U &= \epsilon' U + \epsilon U' - \alpha_+ G_+ + \alpha_- G_- - \frac{\hat{c}}{9}\eta' \\ \delta T &= \frac{\hat{c}}{12}\epsilon''' + 2\epsilon' T + \epsilon T' + \frac{3}{2}\alpha'_+ G_+ + \frac{1}{2}\alpha_+ G'_+ + \frac{3}{2}\alpha'_- G_- + \frac{1}{2}\alpha_- G'_- + \eta' U \\ \delta G_+ &= \frac{\hat{c}}{6}\alpha''_- + \frac{3}{2}\epsilon' G_+ + \epsilon G'_+ + \alpha_-(T + \frac{18}{\hat{c}} U^2 + \frac{3}{2} U') + 3\alpha'_- U + \eta G_+ \\ \delta G_- &= -\frac{\hat{c}}{6}\alpha''_+ + \frac{3}{2}\epsilon' G_- + \epsilon G'_- - \alpha_+(T + \frac{18}{\hat{c}} U^2 - \frac{3}{2} U') + 3\alpha'_+ U - \eta G_- \end{aligned} \tag{3.17}$$

where the prime denotes a derivative with respect to x^+. These are identical to those generated according to

$$\delta\mathcal{O} = 2\pi \mathrm{Res}\Big(J(z) O(0)\Big) \tag{3.18}$$

with the current (denoting the operators by the same symbols in an abuse of notation)

$$J = \frac{1}{2\pi}\Big(\epsilon T + \eta U + \alpha_+ G_+ + \alpha_- G_-\Big) \tag{3.19}$$

and the operator product expansion

$$\begin{aligned}
U(z)U(0) &\sim -\frac{\hat{c}}{9z^2} \\
U(z)G_\pm(0) &\sim \pm\frac{1}{z}G_\pm(0) \\
T(z)U(0) &\sim \frac{1}{z^2}U(0) + \frac{1}{z}\partial U(0) \\
T(z)G_\pm(0) &\sim \frac{3}{2z^2}G_\pm(0) + \frac{1}{z}\partial G_\pm(0) \\
T(z)T(0) &\sim \frac{\hat{c}}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) \\
G_+(z)G_-(0) &\sim -\frac{\hat{c}}{3z^3} + \frac{3}{z^2}U(0) - \frac{1}{z}(T(0) + \frac{18}{\hat{c}}U(0)^2 - \frac{3}{2}\partial U(0))
\end{aligned} \tag{3.20}$$

This is the classical $W_3^{(2)}$ algebra (see e.g. [68]). Apart from the stress energy tensor T, we have weight 3/2 primaries, $G_\pm$, as well as a weight one current, U. Note that the conformal weight of each field is obtained from the SL(2,R) spin by adding one. This is a general feature as discussed in appendix 3E. Due to the presence of the two spin-3/2 bosonic currents, the $W_3^{(2)}$ algebra can be thought of as a sort of bosonic analog of the $\mathcal{N} = 2$ superconformal algebra. Unlike the latter, the $W_3^{(2)}$ is nonlinear, as seen by the appearance of U^2 in the G_+G_- OPE.

The AdS vacuum for the SL(2,R) embedding (3.10) is given by gauge connections

$$\begin{aligned}
A_{AdS} &= e^\rho \hat{W}_2 dx^+ + \hat{L}_0 d\rho \\
\overline{A}_{AdS} &= -e^\rho \hat{W}_{-2} dx^- - \hat{L}_0 d\rho
\end{aligned} \tag{3.21}$$

which yields

$$\begin{aligned}
ds^2 &= \frac{1}{4}\left(d\rho^2 - e^{2\rho}dx^+dx^-\right) \\
\varphi_{\alpha\beta\gamma} &= 0
\end{aligned} \tag{3.22}$$

The metric now describes an AdS_3 of radius $1/2$.

Note that the central charge $\hat{c}$ was not determined by the procedure leading to (3.20). This is a consequence of the fact that the Chern Simons level k does not enter in (3.14). The central charge can, however, be determined by the following argument.

Comparing the two SL(2,R) embeddings, the generators that appear obey the same SL(2,R) commutation relations. The only difference arises from the rescaled trace relations (3.11). This has the effect of reducing the overall normalization of the Chern-Simons action restricted to the SL(2,R) subalgebra by a factor $1/4$ compared to before, which is to say that k is effectively replaced by $k/4$. The central charge of the $W_3^{(2)}$ vacuum is therefore modified:

$$\hat{c} = \frac{1}{4} c = \frac{3k}{2} \tag{3.23}$$

This result can also be established from the metric point of view. If instead of using (3.3) to define the metric we use $\hat{g}_{\mu\nu} = 2\mathrm{Tr}(e_\mu e_\nu)$, then the $W_3^{(2)}$ AdS_3 vacuum will again have unit radius. The action expressed in terms of the hatted metric will take the same form as in the unhatted case, except for an overall factor of $1/4$ from the rescaled trace. Applying the Brown-Henneaux formula then again yields (3.23).

Yet another way to arrive at this result is to compute the Poisson brackets of the charges generating the asymptotic symmetry transformations. The rescaled trace relations just lead to a factor in front of the action that leads to rescaled versions of the canonical momenta. Once again, the effect is just to replace k by $k/4$.

Given the metric (3.22) with radius $1/2$, one might be tempted to conclude that $\hat{c} = \frac{1}{2} c$ by applying the Brown-Henneaux formula directly. However, a proper analysis has to take into account both the scale size of the AdS_3 and the effective Newton constant, which this argument does not do.

3.1.4. flow between $W_3^{(2)}$ and W_3 vacua

It is very easy to write down a solution that interpolates between the $W_3^{(2)}$ vacuum in the UV and the W_3 vacuum in the IR. We take

$$\begin{aligned} A &= \lambda e^{\rho} L_1 dx^+ + e^{2\rho} \hat{W}_2 dx^- + L_0 d\rho \\ \overline{A} &= -\lambda e^{\rho} L_{-1} dx^- - e^{2\rho} \hat{W}_{-2} dx^+ - L_0 d\rho \end{aligned} \tag{3.24}$$

Note that we have chosen to accompany $d\rho$ by L_0 rather than $\hat{L}_0$, which accounts for the $e^{2\rho}$ factors multiplying the W-generators. Also, for reasons to be explained momentarily, compared to (3.21) we take the $\hat{W}_2$ term in A to be associated with dx^- rather than dx^+. The parameter λ is arbitrary, and can be set to any desired value by shifting ρ and scaling $x^{\pm}$. At large ρ the connections approach (3.21) (after rescaling ρ and exchanging $x^+ \leftrightarrow x^-$), while at small ρ they approach (3.4).

The corresponding metric and spin-3 fields are

$$\begin{aligned} ds^2 &= d\rho^2 - \left(\frac{1}{4} e^{4\rho} + \lambda^2 e^{2\rho}\right) dx^+ dx^- \\ \varphi_{\alpha\beta\gamma} dx^\alpha dx^\beta dx^\gamma &= \frac{1}{3!} \mathrm{Tr}(eee) = -\frac{1}{8}\lambda^2 e^{4\rho} (dx^+)^3 + \frac{1}{8}\lambda^2 e^{4\rho} (dx^-)^3 \end{aligned} \tag{3.25}$$

The metric interpolates between the two AdS_3 vacua at large and small ρ. In an orthonormal frame, the spin-3 field vanishes asymptotically at large and small ρ, but is nonzero in between. Additionally, while the metric preserves Lorentz invariance in $x^{\pm}$, the spin-3 field does not.

We now discuss the CFT interpretation of the RG flow from the standpoints of the UV and IR CFTs. From the standpoint of the UV CFT with $W_3^{(2)}$ symmetry, the RG flow is triggered by the λ-terms in (3.24). In the UV CFT λ is a source for the spin-3/2 operators.[8] This can be seen from the field equations, or by noting that under the UV scale invariance λ has dimension

[8] It was in order to have this interpretation that we associated the L_1 and $\hat{W}_2$ generators in (3.24) with opposite chiralities.

$1/2 = 2 - 3/2$, which is correct for a source conjugate to a dimension $3/2$ operator. Thus, the RG flow is initiated by adding to the UV CFT Lagrangian a relevant operator of dimension $3/2$.

Thinking in terms of the IR CFT with W_3 symmetry, the flow is initiated by adding the $\hat{W}_{\pm 2}$ terms, which correspond to adding to the Lagrangian the spin-3 currents. These are irrelevant dimension 3 operators, and so deform the theory in the UV. They drive the theory to the new UV fixed point with $W_3^{(2)}$ symmetry.

One surprising feature concerns the central charges. We have $c_{UV} = \frac{3k}{2}$ and $c_{IR} = 6k$, and so $c_{IR} > c_{UV}$, which seems at first to be in conflict with the c-theorem. However, there is no actual contradiction since the proof of the c-theorem applies to Lorentz invariant RG flows (rotationally invariant in Euclidean signature), whereas here we are adding non-Lorentz invariant operators. The fixed points of the RG flow are Lorentz invariant field theories, but the full flow is not. While there is thus no immediate conflict with the c-theorem, this result is still somewhat puzzling as it seems at odds with the usual intuition regarding the decrease of degrees of freedom under RG flow. The resolution of this puzzle may involve the fact that the UV CFT really has a family of stress tensors due to the presence of a U(1) current algebra.

While we leave a complete analysis of this and related RG flows to future work, it is useful to present a few results to aid in interpretation. First, we consider linearized perturbation theory around the RG flow background. This will allows us to determine the relation between UV and IR operators. Focussing just on the A-connection, we turn on terms corresponding to nonzero currents in the UV and IR. We thus add to (3.24) the terms

$$\begin{aligned} \delta A = &\left(T_{IR} e^{-\rho} L_{-1} + \mathcal{W}_{IR} e^{-2\rho} W_{-2}\right) dx^+ \\ &+ \left(J_{UV} W_0 + G^{(1)}_{UV} e^{-\rho} L_{-1} + G^{(2)}_{UV} e^{-\rho} W_{-1} + T_{UV} e^{-2\rho} W_{-2}\right) dx^- \end{aligned} \tag{3.26}$$

where all coefficients, T_{IR} etc, are arbitrary functions of (x^+, x^-). Up to normalization, these coefficient functions are the respective currents in the UV/IR.

Solving the linearized field equations, we find

$$\begin{aligned} J_{UV} &= \frac{1}{2\lambda} T_{IR} \\ G^{(1)}_{UV} &= -\frac{2}{\lambda} \mathcal{W}_{IR} \\ G^{(2)}_{UV} &= -\frac{1}{6\lambda^2} \partial_+ T_{IR} \\ T_{UV} &= \frac{1}{24\lambda^3} \partial_+^2 T_{IR} \end{aligned} \tag{3.27}$$

along with

$$\begin{aligned} \partial_- T_{IR} &= -\frac{2}{\lambda} \partial_+ \mathcal{W}_{IR} \\ \partial_- \mathcal{W}_{IR} &= \frac{1}{24\lambda^3} \partial_+^3 T_{IR} \end{aligned} \tag{3.28}$$

From the first two relations in (3.27) we see that the IR currents $(T_{IR}, \mathcal{W}_{IR})$ are locally related to the UV currents $(J_{UV}, G^{(1)}_{UV})$. In particular, the IR stress tensor is locally related to the UV spin-1 current, *not* to the UV stress tensor. The equations in (3.28) show that deep in the IR, which corresponds to large λ, the currents $(T_{IR}, \mathcal{W}_{IR})$ obey chiral conservation equations as expected.

The relation between T_{UV} and T_{IR} indicates that T_{UV}, which is of course a dimension 2 operator in the UV, acquires dimension 4 in the IR. This can be shown in more detail by computing the AdS/CFT two-point correlator $\langle T_{UV} T_{UV} \rangle$. This computation is carried out in appendix 3B, and the result is, in momentum space and up to overall normalization:

$$\langle T_{UV}(p) T_{UV}(-p) \rangle = \frac{p_+^3 p_-}{\lambda^4 - \frac{4}{3}\frac{p_+^4}{p_-^2}} \tag{3.29}$$

At short distances we have $\frac{p_+^4}{p_-^2} \gg \lambda^4$ and so the UV result is

$$\text{UV}: \quad \langle T_{UV}(x) T_{UV}(0) \rangle \sim \int \frac{d^2p}{(2\pi)^2} \frac{p_-^3}{p_+} e^{ip\cdot x} \sim \frac{1}{(x^-)^4} \tag{3.30}$$

At long distances we take $\frac{p_+^4}{p_-^2} \ll \lambda^4$, and expand (3.29) to first subleading order, since the leading order term is polynomial in momentum space and hence a contact term in position space. The leading behavior is thus

$$\mathrm{IR}: \quad \langle T_{UV}(x) T_{UV}(0) \rangle \sim \int \frac{d^2 p}{(2\pi)^2} \frac{p_+^7}{p_-} e^{ip\cdot x} \sim \frac{1}{(x^+)^8} \tag{3.31}$$

This result shows that T_{UV} goes from being a dimension $(2,0)$ operator in the UV, to being a dimension $(0,4)$ operator in the IR. The fact that the operator goes from being rightmoving to leftmoving is of course a consequence of the non-Lorentz invariant character of the RG flow.

3.2. Review of spin-3 black hole solutions in wormhole gauge

In [50] the following solution was proposed to represent black holes carrying spin-3 charge:

$$\begin{aligned}
\mathcal{A} &= \Big(e^{\rho} L_1 - \frac{2\pi}{k}\mathcal{L} e^{-\rho} L_{-1} - \frac{\pi}{2k}\mathcal{W} e^{-2\rho} W_{-2}\Big) dx^+ \\
&\quad + \mu\Big(e^{2\rho} W_2 - \frac{4\pi\mathcal{L}}{k} W_0 + \frac{4\pi^2\mathcal{L}^2}{k^2} e^{-2\rho} W_{-2} + \frac{4\pi\mathcal{W}}{k} e^{-\rho} L_{-1}\Big) dx^- + L_0 d\rho \\
\overline{\mathcal{A}} &= -\Big(e^{\rho} L_{-1} - \frac{2\pi}{k}\overline{\mathcal{L}} e^{-\rho} L_1 - \frac{\pi}{2k}\overline{\mathcal{W}} e^{-2\rho} W_2\Big) dx^- \\
&\quad - \overline{\mu}\Big(e^{2\rho} W_{-2} - \frac{4\pi\overline{\mathcal{L}}}{k} W_0 + \frac{4\pi^2\overline{\mathcal{L}}^2}{k^2} e^{-2\rho} W_2 + \frac{4\pi\overline{\mathcal{W}}}{k} e^{-\rho} L_1\Big) dx^+ - L_0 d\rho
\end{aligned} \tag{3.32}$$

The structure of this solution is easy to understand. Focus on the $\mathcal{A}$-connection. As in (3.6), to add energy and charge density to the W_3 vacuum we should add to $\mathcal{A}_+$ terms involving L_{-1} and W_{-2}, as seen in the top line of (3.32). For black holes, which represent states of thermodynamic equilibrium, the energy and charge should be accompanied by their conjugate thermodynamic potentials, which are temperature and spin-3 chemical potential. We incorporate the former via the periodicity of imaginary time, while the latter was shown in [50] by a Ward identity analysis to correspond to a μW_2 term in $\mathcal{A}_-$. The remaining $\mathcal{A}_-$ terms appearing in (3.32) are then fixed by the equations of motion.

Let us quickly elaborate on the Ward identity analysis of [50]. The conformal dimension of the holomorphic spin-3 current $\mathcal{W}$ is $(3,0)$, which means that the addition of a term

$$\delta S_{CFT} = -\int d^2 z \mu(z,\bar{z}) \mathcal{W}(z,\bar{z}) \tag{3.33}$$

to the CFT action perturbs the CFT by an irrelevant operator, and μ is the chemical potential for spin-3 charge $\mathcal{W}$. (Identical statements are implied about the holomorphic partners $\bar{\mu}, \overline{W}$.) The CFT in question has classical W_3 symmetry, which is fully encoded in the OPEs among the stress tensor T and spin-3 field $\mathcal{W}$:

$$\begin{aligned}
T(z)T(0) &\sim \frac{c/2}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z} + \ldots \\
T(z)\mathcal{W}(0) &\sim \frac{3\mathcal{W}(0)}{z^2} + \frac{\partial \mathcal{W}(0)}{z} + \ldots \\
\mathcal{W}(z)\mathcal{W}(0) &\sim \frac{c/3}{z^6} + \frac{2\,T(0)}{z^4} + \frac{\partial T(0)}{z^3} + \frac{1}{z^2}\left(2\beta\Lambda(0) + \frac{3}{10}\,\partial^2 T(0)\right) \\
&\quad + \frac{1}{z}\left(\beta\,\partial\Lambda(0) + \frac{1}{15}\,\partial^2 T(0)\right)
\end{aligned} \tag{3.34}$$

where the parameter β is given by

$$\beta = \frac{16}{5c} \tag{3.35}$$

and the operator Λ is defined as follows

$$\Lambda(w) =: T(w)T(w) : -\frac{3}{10}\,\partial_w^2 T(w) \tag{3.36}$$

Note that we do not claim the bulk theory and W_3 CFT to be dual: this is merely a statement about asymptotic symmetry in 3d classical gravity. We have not specified a particular CFT with some central charge, etc, and in fact there is no known W_3-symmetric CFT with large central charge, a prerequisite to mapping classical gravity calculations to those of CFT.

For us to consider μ as a source, we must show that certain Ward identities among the potentials and charges are obeyed. On the CFT side, these can be derived by computing expectation values of $\partial_{\overline{z}}\mathcal{L}$ and $\partial_{\overline{z}}\mathcal{W}$ in the presence of the operator insertion (3.33), and using the OPEs. We can map these Ward identities to bulk equations of motion evaluated on *some* connection, as was shown in [69] for the case of ordinary gravity with a spin-2 source. The question is, "what connection?" Following [69], the answer is to write the most general possible connection in so-called "highest weight gauge", with each component an arbitrary function of $(z, \overline{z})$, and solve the field equations. This connection contains a piece $\mathcal{A}_- \supset \mu(z, \overline{z})e^{2\rho}W_2$ – this is the source term. Upon making a precise identification between the bulk charges $(\mathcal{L}, \mathcal{W})$ and the boundary charges $(T, \mathcal{W})$, one can show that the Ward identities are satisfied. See [50] for further details.

The upshot is that the connection (3.32) is the most general highest-weight gauge connection one can write with a constant source μ, where μ appears as a leading term in $\mathcal{A}_-$. The fact that μ multiples a W_2 generator means that this connection violates the W_3 AdS boundary conditions, since that piece grows faster than e^{ρ}: this comports with the fact that it is the bulk realization of an irrelevant operator deformation of the CFT.

We have now established the baseline argument for why the connection (3.32) may represent a bulk state of thermodynamic equilibrium with nonzero spin-3 charge. Let us now try to understand the geometric implications.

3.2.1. Geometry, thermodynamics and holonomy

The metric and spin-3 field are extracted using (3.3); the metric, for example, looks like

$$
\begin{aligned}
ds^2 = d\rho^2 + 4\Big(&\mu e^{2\rho}dx^- - \frac{\pi}{2k}\overline{\mathcal{W}}e^{-2\rho}dx^- + \frac{4\pi^2}{k^2}\overline{\mu}\overline{\mathcal{L}}^2e^{-2\rho}dx^+\Big) \\
&\times\left(\overline{\mu}e^{2\rho}dx^+ - \frac{\pi}{2k}\mathcal{W}e^{-2\rho}dx^+ + \frac{4\pi^2}{k^2}\mu\mathcal{L}^2e^{-2\rho}dx^-\right) \\
-&\left(e^\rho dx^+ - \frac{2\pi}{k}\overline{\mathcal{L}}e^{-\rho}dx^- + \frac{4\pi}{k}\overline{\mu}\overline{\mathcal{W}}e^{-\rho}dx^+\right) \\
&\times\left(e^\rho dx^- - \frac{2\pi}{k}\mathcal{L}e^{-\rho}dx^+ + \frac{4\pi}{k}\mu\mathcal{W}e^{-\rho}dx^-\right) \\
+&\frac{1}{3}\left(\frac{4\pi}{k}\right)^2(\mu\mathcal{L}dx^- + \overline{\mu}\overline{\mathcal{L}}dx^+)^2
\end{aligned}
\tag{3.37}
$$

It is already clear that the solution is cleanest in the Chern-Simons picture, though we still must pass to the metric-like formulation to understand various physical features of the solution. For instance, the spin-2 charges $(\mathcal{L}, \overline{L})$, equivalently the mass and angular momentum, correspond to simple coefficients in the connection, whereas in the metric they appear in more convoluted fashion. (This is true of the BTZ solution as well.)

We will henceforth restrict attention to the nonrotating case:

$$
\overline{\mathcal{L}} = \mathcal{L}\ , \quad \overline{\mathcal{W}} = -\mathcal{W}\ , \quad \overline{\mu} = -\mu \tag{3.38}
$$

for which the metric is, in terms of (ϕ, t) coordinates,

$$
\begin{aligned}
ds^2 = d\rho^2 - \Big\{&(2\mu e^{2\rho} + \frac{\pi}{k}\mathcal{W}e^{-2\rho} - \frac{8\pi^2}{k^2}\mu\mathcal{L}^2e^{-2\rho})^2 \\
&+ (e^\rho - \frac{2\pi}{k}\mathcal{L}e^{-\rho} + \frac{4\pi}{k}\mu\mathcal{W}e^{-\rho})^2\Big\}dt^2 \\
&+ \Big\{(e^\rho + \frac{2\pi}{k}\mathcal{L}e^{-\rho} + \frac{4\pi}{k}\mu\mathcal{W}e^{-\rho})^2 + 4(\mu e^{2\rho} + \frac{\pi}{2k}\mathcal{W}e^{-2\rho} + \frac{4\pi^2}{k^2}\mu\mathcal{L}^2e^{-2\rho})^2 \\
&\quad + \frac{4}{3}\left(\frac{4\pi}{k}\right)^2\mu^2\mathcal{L}^2\Big\}d\phi^2
\end{aligned}
\tag{3.39}
$$

If we set $\mu = \mathcal{W} = 0$, then the connections become those corresponding to a BTZ black hole asymptotic to the W_3 vacuum. From the standpoint of this

CFT, the $\mu e^{2\rho} W_2 dx^-$ term represents a chemical potential for spin-3 charge, and the $\mathcal{W} e^{-2\rho} W_{-2} dx^+$ term gives the value of the spin-3 charge. This solution is therefore interpreted as a generalization of the BTZ black hole to include nonzero spin-3 charge and chemical potential.

Another useful special case to consider is $\mathcal{L} = \mathcal{W} = 0$ with $\mu \neq 0$. After shifting ρ, the resulting connection is identified with the RG flow solution (3.24) with $\lambda = \frac{1}{2\sqrt{\mu}}$. So for small $(\mathcal{L}, \mathcal{W})$ and finite μ, we can think of this solution as representing an excited version of the RG flow.

Suppose one wants to interpret the connections (3.32) as describing a smooth black hole. We should first demand that as $\mu \to 0$, the metric turn over to that of the BTZ black hole. Secondly, the Euclidean geometry should be smooth at the horizon, and the accompanying spin-3 field non-singular. In ordinary gravity, we enforce this by demanding the absence of a conical singularity at the Euclidean black hole horizon; *a priori*, this should not be expected to work in this spin-3 setting where that notion is not gauge-invariant. Finally, the black hole must define a consistent thermodynamics. In ordinary gravity, this is guaranteed by enforcing smoothness, but this is not so here.

To elaborate, the general non-rotating solution (3.32)-(3.39), as written, can be thought of as depending on four free parameters: three of these are $(\mathcal{L}, \mathcal{W}, \mu)$, and the fourth is the inverse temperature β, corresponding to the periodicity of imaginary time, $t \cong t + i\beta$. However, we expect that there should only be a two-parameter family of physically admissible solutions: once one has specified the temperature and chemical potential the values of the energy and charge should be determined thermodynamically. For the uncharged BTZ solution the relation between the energy and the temperature is obtained by demanding the absence of a conical singularity at the horizon in Euclidean signature. The analogous procedure in the presence of spin-3 charge is more subtle, as was discussed in detail in [50].

We want to think of the black hole as a saddle point contribution to a partition function of the form

$$Z(\tau,\alpha) = \mathrm{Tr}[e^{8\pi^2 i(\tau\mathcal{L}+\alpha\mathcal{W})}] \tag{3.40}$$

But since this implies $\mathcal{L} \sim \partial Z/\partial\tau$ and $\mathcal{W} \sim \partial Z/\partial\alpha$, Z will exist only if the charge assignments obey the integrability condition

$$\frac{\partial\mathcal{L}}{\partial\alpha} = \frac{\partial\mathcal{W}}{\partial\tau} \tag{3.41}$$

Therefore, our task is to come up with a prescription for defining the charges, consistent with a definition of smoothness, that leads to (3.41).

If we naively try to interpret the metric (3.39) directly as a black hole, the zeroth order thing to check is whether there is a horizon. Enforcing $g_{tt} = 0$ in (3.39) implies that a sum of squares must vanish; in addition, these Fefferman-Graham coordinates require that $g_{tt}(\rho_h) = \partial_\rho g_{tt}(\rho_h) = 0$. The only solutions to the resulting equations do not define a consistent thermodynamics, nor do they have a smooth $\mu \to 0$ limit in which we should hope to recover the BTZ solution. Thus, (3.39) looks like a traversable wormhole, interpolating between two $W_3^{(2)}$ AdS vacua. Is there a black hole hiding somewhere?

The answer, as we will show explicitly, is yes. The key point is that we should not have expected our naive treatment to work: such a procedure is not gauge-invariant! This theory includes spin-3 gauge transformations which modify the metric, and so any definition of a black hole which relies on computations of actual metric components in a fixed gauge is bound to fail. Instead, a gauge-invariant proposal was put forth in [50]. It was proposed there that one should compute the holonomy of the Chern-Simons connection around the Euclidean time circle, and demand that its eigenvalues take the fixed values $(0, 2\pi i, -2\pi i)$. This was proposed as the gauge invariant characterization of the condition for the Euclidean time circle to smoothly close off at the horizon. This condition has the virtue of being gauge invariant, reproducing known

BTZ results in the uncharged limit, and, as shown in [50], being compatible with the first law of thermodynamics. In the next section we will find strong independent evidence for the validity of this proposal: in particular, we will see that *this metric can be gauge-transformed to look like a black hole, whose smoothness condition as determined by the usual Euclidean prescription gives the same charge assignments as the holonomy condition.*

Writing

$$\tau = \frac{i\beta}{2\pi} \tag{3.42}$$

corresponding to the modular parameter of the torus, the holonomy around the time circle is[9]

$$\omega = 2\pi(\tau\mathcal{A}_+ - \overline{\tau}\mathcal{A}_-) \tag{3.43}$$

For the BTZ black hole, the conditions on its eigenvalues can be recast as

$$\det(\omega) = 0\ , \quad \mathrm{Tr}(\omega^2) + 8\pi^2 = 0 \tag{3.44}$$

which, enforced for the connection (3.32), become explicitly

$$\begin{aligned} 0 &= -2048\pi^2\mu^3\mathcal{L}^3 + 576\pi k\mu\mathcal{L}^2 - 864\pi k\mu^2\mathcal{W}\mathcal{L} + 864\pi k\mu^3\mathcal{W}^2 - 27k^2\mathcal{W} \\ 0 &= 256\pi^2\mu^2\mathcal{L}^2 + 24\pi k\mathcal{L} - 72\pi k\mu\mathcal{W} + \frac{3k^2}{\tau^2} \end{aligned} \tag{3.45}$$

It was shown in [50] that, upon taking $\alpha = \beta\mu$ (cf. (3.40)), one indeed finds that the conditions (3.45) imply integrability. This is equivalent to saying that we can define an entropy that is consistent with the first law of thermodynamics. In contrast to ordinary gravity where we can use either the area law or the value of the Euclidean action to compute the entropy, neither of these approaches are immediately applicable in the higher spin case, and so we need to proceed as described here. Note also that the relation $\alpha = \beta\mu$ should be thought of as being determined self-consistently together with the charge assignments.

[9] For completeness, in appendix 3C we provide the holonomy along the angular direction.

In the next section we find the explicit gauge transformation needed to take the wormhole into a black hole, along with the explicit black hole metric. This metric is completely smooth, as is the accompanying spin-3 field, and has AdS_3 asymptotics. Furthermore, at least within our ansatz, there is a unique such black hole solution, and its smoothness requires that the holonomy conditions (3.45) be obeyed.

3.3. Gauge transforming the wormhole into a black hole

In this section we describe the solution to the following problem. We start from the static wormhole solution reviewed in the last section:

$$\begin{aligned}
\mathcal{A} &= \Big(e^{\rho}L_1 - \frac{2\pi}{k}\mathcal{L}e^{-\rho}L_{-1} - \frac{\pi}{2k}\mathcal{W}e^{-2\rho}W_{-2}\Big)dx^+ \\
&\quad + \mu\Big(e^{2\rho}W_2 - \frac{4\pi\mathcal{L}}{k}W_0 + \frac{4\pi^2\mathcal{L}^2}{k^2}e^{-2\rho}W_{-2} + \frac{4\pi\mathcal{W}}{k}e^{-\rho}L_{-1}\Big)dx^- + L_0 d\rho \\
\overline{\mathcal{A}} &= -\Big(e^{\rho}L_{-1} - \frac{2\pi}{k}\mathcal{L}e^{-\rho}L_1 + \frac{\pi}{2k}\mathcal{W}e^{-2\rho}W_2\Big)dx^- \\
&\quad + \mu\Big(e^{2\rho}W_{-2} - \frac{4\pi\mathcal{L}}{k}W_0 + \frac{4\pi^2\mathcal{L}^2}{k^2}e^{-2\rho}W_2 - \frac{4\pi\mathcal{W}}{k}e^{-\rho}L_1\Big)dx^+ - L_0 d\rho
\end{aligned} \tag{3.46}$$

We then consider new connections related to these by SL(3,R) gauge transformations:

$$\begin{aligned}
A &= g^{-1}(\rho)\mathcal{A}(\rho)g(\rho) + g^{-1}(\rho)dg(\rho) \\
\overline{A} &= g(\rho)\overline{\mathcal{A}}(\rho)g^{-1}(\rho) - dg(\rho)g^{-1}(\rho)
\end{aligned} \tag{3.47}$$

with $g(\rho) \in$ SL(3,R). The relative gauge transformation for $\overline{A}$ versus A is taken to maintain a non-rotating ansatz. The metric and spin-3 field corresponding to $(A, \overline{A})$ will take the form

$$\begin{aligned}
ds^2 &= g_{\rho\rho}(\rho)d\rho^2 + g_{tt}(\rho)dt^2 + g_{\phi\phi}(\rho)d\phi^2 \\
\varphi_{\alpha\beta\gamma}dx^{\alpha}dx^{\beta}dx^{\gamma} &= \varphi_{\phi\rho\rho}(\rho)d\phi d\rho^2 + \varphi_{\phi tt}(\rho)d\phi dt^2 + \varphi_{\phi\phi\phi}(\rho)d\phi^3
\end{aligned} \tag{3.48}$$

We demand that this solution describe a smooth black hole with event horizon at $\rho = \rho_+$, or at $r = 0$ with

$$r = \rho - \rho_+ \tag{3.49}$$

Assuming that $g_{rr}(0) > 0$, as will be the case, this first of all requires $g_{tt}(0) = g'_{tt}(0) = 0$ and $g_{\phi\phi}(0) > 0$, so that after rotating to imaginary time the metric expanded around $r = 0$ will look locally like $R^2 \times S^1$:

$$ds^2 \approx g_{rr}(0)dr^2 - \frac{1}{2}g''_{tt}(0)r^2dt_E^2 + g_{\phi\phi}(0)d\phi^2 \tag{3.50}$$

In order for the metric to avoid a conical singularity at $r = 0$ we need to identify $t_E \cong t_E + \beta$ with

$$\beta = 2\pi\sqrt{\frac{2g_{rr}(0)}{-g''_{tt}(0)}} \tag{3.51}$$

Having done so, we can switch to Cartesian coordinates near $r = 0$ and the metric will be smooth.

The same smoothness considerations apply to the spin-3 field. Noticing the parallel structure, we see that we should demand $\varphi_{\phi tt}(0) = \varphi'_{\phi tt}(0) = 0$, and

$$\beta = 2\pi\sqrt{\frac{2\varphi_{\phi rr}(0)}{-\varphi''_{\phi tt}(0)}} \tag{3.52}$$

with the same β as in (3.51).

There is still one more condition to impose to ensure that the solution is completely smooth at the horizon. If we work in Cartesian coordinates (x, y) around $r = 0$, we should demand that all functions are infinitely differentiable with respect to both x and y. If this is not the case then some curvature invariant (or spin-3 quantity) involving covariant derivatives will diverge. Given the rotational symmetry, this condition implies that the series expansion of all functions should only involve non-negative even powers of r. We impose this by demanding that all functions be smooth at the horizon, and even under reflection about the horizon:

$$\begin{gathered} g_{rr}(-r) = g_{rr}(r)\ , \quad g_{tt}(-r) = g_{tt}(r)\ , \quad g_{\phi\phi}(-r) = g_{\phi\phi}(r) \\ \varphi_{\phi rr}(-r) = \varphi_{\phi rr}(r)\ , \quad \varphi_{\phi tt}(-r) = \varphi_{\phi tt}(r)\ , \quad \varphi_{\phi\phi\phi}(-r) = \varphi_{\phi\phi\phi}(r) \end{gathered} \tag{3.53}$$

We now summarize the solution to this problem. More details are provided in appendix 3D. The symmetry conditions (3.53) can be enforced by demanding

$$\begin{aligned} e_t(-r) &= -h(r)^{-1} e_t(r) h(r) \\ e_\phi(-r) &= h(r)^{-1} e_\phi(r) h(r) \\ e_r(-r) &= h(r)^{-1} e_r(r) h(r) \end{aligned} \tag{3.54}$$

with $h(r) \in$ SL(3,R), and similar conditions on the spin-connection. The BTZ solution has $h(r) = \mathbf{1}$, so we can think of these conditions as a "twist" of the BTZ vielbein reflection symmetries. In addition, $h(0) = \mathbf{1}$, implying that $e_t(0) = 0$, a feature of the BTZ solution that persists in the spin-3 case.[10]

To gain some insight into the form of $g(r)$ and $h(r)$ we can start with the BTZ solution and then carry out the gauge transformation perturbatively in the charge. These considerations lead us to the ansatz

$$\begin{aligned} g(r) &= e^{F(r)(W_1 - W_{-1}) + G(r) L_0} \\ h(r) &= e^{H(r)(W_1 + W_{-1})} \end{aligned} \tag{3.55}$$

for some functions F, G and H. Perturbation theory suggests that this ansatz gives the unique solution to our problem, although we have not proven this. On the other hand, having assumed the ansatz (3.55) the remaining analysis definitely has a unique solution.

Even after assuming this ansatz, finding a solution that satisfies all the smoothness conditions involves a surprisingly large amount of complicated algebra requiring extensive use of Mathematica and Maple. Some details are provided in the appendix. As we have already mentioned, a crucial point is that the solution to our problem requires that the holonomy conditions (3.45) are obeyed; equivalently, we can derive the holonomy conditions by requiring the existence of a smooth black hole solution.

[10] Moreover, it is both surprising and convenient that the location of the horizon $r = 0$ turns out to be at $\rho = \rho_+$, with ρ_+ given by the *same* expression as for BTZ: $e^{2\rho_+} = \frac{2\pi\mathcal{L}}{k}$.

Here we just present the final form for the transformed metric. It will be convenient to define dimensionless versions of the charge and chemical potential:

$$\zeta = \sqrt{\frac{k}{32\pi\mathcal{L}^3}}\mathcal{W} \ , \quad \gamma = \sqrt{\frac{2\pi\mathcal{L}}{k}}\mu \tag{3.56}$$

as well as a parameter C defined as

$$\zeta = \frac{C-1}{C^{3/2}} \tag{3.57}$$

The metric takes the form (3.48) with

$$\begin{aligned} g_{rr} &= \frac{(C-2)(C-3)}{\left(C-2-\cosh^2(r)\right)^2} \\ g_{tt} &= -\left(\frac{8\pi\mathcal{L}}{k}\right)\left(\frac{C-3}{C^2}\right)\frac{(a_t+b_t\cosh^2(r))\sinh^2(r)}{\left(C-2-\cosh^2(r)\right)^2} \\ g_{\phi\phi} &= \left(\frac{8\pi\mathcal{L}}{k}\right)\left(\frac{C-3}{C^2}\right)\frac{(a_\phi+b_\phi\cosh^2(r))\sinh^2(r)}{\left(C-2-\cosh^2(r)\right)^2} \\ &\quad + \left(\frac{8\pi\mathcal{L}}{k}\right)(1+\frac{16}{3}\gamma^2+12\gamma\zeta) \end{aligned} \tag{3.58}$$

The coefficients $a_{t,\phi}$ and $b_{t,\phi}$ are functions of γ and C, and are displayed in appendix 3C along with the spin-3 field.

With these results in hand, we demand a smooth horizon via (3.51) and (3.52). Using the definition (3.57), the resulting equations can be written as

$$\begin{aligned} 1728\gamma^3\zeta^2-(432\gamma^2+27)\zeta-128\gamma^3+72\gamma &= 0 \\ \left(1+\frac{16}{3}\gamma^2-12\gamma\zeta\right)\mathcal{L}-\frac{\pi k}{2\beta^2} &= 0 \end{aligned} \tag{3.59}$$

These are precisely the holonomy conditions (3.45), merely rewritten in the (ζ,γ) variables!

3.3.1. Limits and asymptotics

Solution of equations (3.59) for the charge ζ and inverse temperature β yields

$$\begin{aligned}\zeta &= \frac{1+16\gamma^2-\left(1-\frac{16}{3}\gamma^2\right)\sqrt{1+\frac{128}{3}\gamma^2}}{128\gamma^3} \\ \beta &= \frac{\sqrt{\frac{\pi k}{2\mathcal{L}}}}{\sqrt{1+\frac{16}{3}\gamma^2-12\gamma\zeta}}\end{aligned} \tag{3.60}$$

We have chosen the branch of ζ consistent with recovery of the BTZ solution in the $\gamma \to 0$ limit, namely the one with a power series expansion in positive (odd) powers of γ.

The uncharged BTZ limit corresponds to taking $\zeta, \gamma \to 0$, and $C \to \infty$. On the other hand, the maximal value of (ζ, γ) obtained from (3.60) is

$$\zeta_{max} = \sqrt{\frac{4}{27}}\,, \quad \gamma_{max} = \sqrt{\frac{3}{16}} \tag{3.61}$$

and this corresponds to $C = 3$. Therefore, we can take C to lie in the range $3 \le C < \infty$.

The extremal lower bound can also be seen directly on the level of the metric (3.58), which degenerates at $C = 3$. This is a virtue of the manifestly smooth horizon of the black hole, in contrast to the wormhole geometry which has no limiting form at this value of the charge, and hence one must resort to the holonomy to find it instead.

The metric coefficients diverge at $r = r_\star$, where

$$\cosh^2(r_\star) = C - 2 \tag{3.62}$$

The leading behavior of the metric near $r_\star$ is

$$\begin{aligned}ds^2 \approx\ & \frac{1}{4}\frac{dr^2}{(r_\star - r)^2} \\ & + \left(\frac{2\pi\mathcal{L}}{k}\right)\left(\frac{C-3}{C^2(C-2)}\right)\frac{-[a_t + b_t(C-2)]dt^2 + [a_\phi + b_\phi(C-2)]d\phi^2}{(r_\star - r)^2}\end{aligned} \tag{3.63}$$

This gives AdS_3 with radius 1/2, and thus our transformed black hole solutions are asymptotically AdS_3. In addition, the spin-3 field expressed in an orthonormal basis goes to zero at $r_\star$.

As C approaches 3 from above we see from (3.62) that $r_\star \to 0$. Nonetheless we can extract the extremal asymptotics by scaling the coordinates as we take $C \to 3$. This has the effect of separating $r_\star$ from the horizon and stretching the region in between.

Expanding around extremality by defining

$$\gamma = \gamma_{ext} - \delta \tag{3.64}$$

we define asymptotic coordinates

$$r = \sqrt{\delta}\tilde{r} \ , \quad t = \frac{\tilde{t}}{\delta} \tag{3.65}$$

Expanding all quantities and taking the $\delta \to 0$ limit, one finds a metric

$$\begin{aligned} ds^2 \approx \frac{\tilde{r}_\star^2}{(\tilde{r}_\star^2 - \tilde{r}^2)^2} d\tilde{r}^2 - \frac{512\pi\mathcal{L}}{9k} \frac{\tilde{r}_\star^2}{(\tilde{r}_\star^2 - \tilde{r}^2)^2} \tilde{r}^2 d\tilde{t}^2 \\ + \left(\frac{32\pi\mathcal{L}}{k} + \frac{96\pi\mathcal{L}}{k} \frac{\tilde{r}_\star^2}{(\tilde{r}_\star^2 - \tilde{r}^2)^2} \tilde{r}^2 \right) d\phi^2 \end{aligned} \tag{3.66}$$

where $\tilde{r}_\star = \sqrt{\frac{8\sqrt{3}}{3}}$. This metric has the same AdS_3 asymptotics as the nonextremal metric (3.63), which becomes evident upon defining a Fefferman-Graham coordinate $\tilde{r} = \tilde{r}_\star \tanh(\eta)$.

The coordinate r appearing in (3.58) only covers the region outside the event horizon; it is analogous to $\sqrt{r - 2M}$ for the Schwarzschild solution. The region inside the horizon is obtained by continuing r to pure imaginary values.

3.3.2. Black hole entropy

In [50], the entropy of the black hole was found to be

$$S = 4\pi\sqrt{2\pi k\mathcal{L}} f(y) \tag{3.67}$$

where

$$f(y) = \cos\theta\,, \quad \theta = \frac{1}{6}\arctan\left(\frac{\sqrt{y(2-y)}}{1-y}\right)\,, \quad 0 \le \theta \le \frac{\pi}{6} \tag{3.68}$$

and $y = \frac{27}{2}\zeta^2$. (We have specialized to the static case.) This was originally found by solving a first-order nonlinear ODE, derived by combining the definition of the partition function with the holonomy conditions.

The function $f(y)$ takes a pleasantly simple form upon plugging in for ζ as a function of C, using (3.57). This step yields

$$\theta = \frac{1}{6}\arctan\left(\Lambda(C)\sqrt{1-\frac{3}{4C}}\right) \tag{3.69}$$

with

$$\Lambda(C) \equiv \frac{6\sqrt{3C}(C-1)(C-3)}{(2C-3)(C^2-12C+9)} \tag{3.70}$$

Surprisingly, taking the cosine of this angle yields the simple expression

$$f(y) = \sqrt{1-\frac{3}{4C}} \tag{3.71}$$

As with other quantities in our analysis, we see that the entropy is most simply expressed in terms of C. The extremal and zero charge limits are recovered upon inspection.

It is of course natural to wonder if the black hole entropy can be expressed in terms of a geometrical property of the horizon. There is of course no reason to expect that the Bekenstein-Hawking area law holds, since the spin-3 field is nonzero at the horizon, and indeed one easily checks that $S \neq A/4G$. This is related to one of the primary challenges inherent in higher spin gravity: the enlarged gauge invariance renders familiar geometric quantities, such as the horizon area, non-invariant under higher spin gauge transformations. Perhaps there exists a higher spin version of the Wald entropy formula [70] that is fully gauge invariant.

3.4. Discussion

Let us close with some open issues. We address the black hole, the RG flow, and generalization to other higher spin theories in turn.

Our main result was showing explicitly how to gauge transform between the wormhole and the black hole. The overall logical structure is very tight: the existence of the gauge transformation is contingent upon the holonomy conditions being satisfied, and there then exists a unique gauge transformation and smooth black hole metric.[11] The wormhole and black hole have dramatically different causal structures, and so it is of course conceptually interesting that they can be related by a higher spin gauge transformation.

This situation has no analog in ordinary gravity. By adding in matter to probe the solution, one can map out the light cones and determine the causal structure in a unique fashion. But in our spin-3 theory it is not possible to simply throw in a minimally coupled scalar field to probe the solution, as there is no obvious way to do so that is compatible with the full higher spin gauge invariance. In this respect, the situation is analogous to string theory, where it is also not possible to add in matter arbitrarily. However, it is known how to couple in propagating scalar fields to the large N limit of these higher spin theories [11,71][12], and such scalars play an important role in the conjecture of [34]. In Chapter 5, we will consider this problem. In that context, it therefore seems possible to compute AdS/CFT correlators between the two asymptotic boundaries of the wormhole/blackhole. One could then determine whether or not these two boundaries are causally connected, by seeing whether such correlators exhibit lightcone singularities.

We have already noted that it would be very useful to have a higher spin

---

[11] Strictly speaking, these statements are subject to the qualifications noted below (3.55).

[12] Using this formalism massless fields in the background of the BTZ black hole were studied in [72].

version of Wald's entropy formula at our disposal to gain a better geometric understanding of the theory, to the extent that this is possible. Our entropy formula (3.71) takes a surprisingly simple form (as compared to the algebra leading to it), and is perhaps suggestive of a geometrical interpretation. Also, the twisted vielbein reflection conditions (3.54) deserve to be better understood.

While the existence of the black hole gauge is extremely useful for conceptual and interpretational purposes, it is likely to be the case that the wormhole gauge is more convenient for doing computations. In the wormhole gauge we know how to read off the charges and symmetries from the asymptotic form of the connection. In fact, this can be done in either of two ways: by viewing the solution in terms of the W_3 CFT deformed by an irrelevant spin-3 operator, or in terms of the $W_3^{(2)}$ CFT deformed by a relevant spin-3/2 operator. On the other hand, in the black hole gauge, the gauge field near $r = r_\star$ does not take a form in which we know how to identify the CFT data, cf. (3.12). It would be convenient and perhaps enlightening if we could understand the black hole asymptotics better. Such an analysis would likely need to involve the subleading terms near the AdS_3 boundary at $r = r_\star$.

The existence of multiple AdS_3 vacua in this theory, both of which have vanishing spin-3 field, is another intriguing and novel feature due to the inclusion of higher spin. We were led to a simple RG flow solution between these vacua triggered by a spin-3/2, Lorentz symmetry-breaking CFT operator. We would like to improve our understanding of the behavior of the central charge under the flow, and in particular why it increases towards the IR. As noted earlier, the Lorentz symmetry-breaking nature of this RG flow places it outside the assumptions of the c-theorem; maybe another quantity can be constructed which monotonically decreases along RG flows from UV to IR.

Perhaps most conceptually interesting is the extension of these ideas to bulk theories with larger gauge groups. Let us consider SL(N,R) $\times$ SL(N,R) for now. When $N > 3$, there are then multiple higher spin charges and potentials to play with, and there could be interesting interplay among them. The ideas of this chapter were directly generalized to the spin-4 case in [73], and a similarly consistent thermodynamic story was found. Also, to each embedding of SL(2,R) in SL(N,R) is associated an AdS_3 vacuum with asymptotic symmetry given by the W-algebra obtained by classical Drinfeld-Sokolov reduction. Modulo issues of interpretation described above, the central charge of the Virasoro algebra coming from the principal embedding will always be the largest. That of the other vacua will have an inverse relation to the index of the SL(2,R) embedding used to construct the vacuum. Some relevant group theoretic details are presented in appendix 3E. It is easy to write these solutions down in the Chern-Simons language once one has the explicit SL(2,R) embedding: simply take ansatz (3.21) and replace $\hat{W}_2$ and $\hat{L}_0$ by the generators of the new SL(2,R) embedding. It is equally straightforward to construct RG flows between these vacua, and hence between CFTs with different symmetries, by altering (3.24) in the same manner. The set of allowed RG flows depends on the details of the Lie algebra, but one can see that in general the vacuum with largest (smallest) AdS radius has no relevant (irrelevant) operators, and so is IR (UV) stable.

However, as we mentioned earlier, all AdS vacua with connections valued in non-principally embedded SL(2,R) subalgebras have asymptotic symmetries which do not admit unitary representations. Furthermore, when we soon generalize to the hs[λ] theory which has an infinite tower of higher spins, there is only a single SL(2,R) subalgebra. We prefer to view the existence of multiple AdS "vacua" in the SL(N,R) theories as a peculiarity that shouldn't be taken too seriously. In combination with the fact that we do not know whether one can consistently couple matter to these theories, it seems that the SL(N,R) theories lack certain features which we wish to see in an interesting theory of higher spin gravity. On the other hand, the SL(N,R) theories appear to have some features which lend themselves to interpretation of the puzzle of the preponderance of "light states" in the W_N minimal model CFT at finite N [74]; so perhaps we should not cast them off as physically irrelevant.

Appendix 3A. SL(3,R) generators

As in [48] with $\sigma = -1$, we use the following basis of SL$(3, R)$ generators

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad L_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$W_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad W_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad W_0 = \frac{2}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$W_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad W_{-2} = \begin{pmatrix} 0 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3A.1}$$

The generators obey the following commutation relations

$$\begin{aligned} [L_i, L_j] &= (i-j)L_{i+j} \\ [L_i, W_m] &= (2i-m)W_{i+m} \\ [W_m, W_n] &= -\frac{1}{3}(m-n)(2m^2+2n^2-mn-8)L_{m+n} \end{aligned} \tag{3A.2}$$

and trace relations

$$\begin{aligned} &\mathrm{Tr}(L_0L_0) = 2\ , \quad \mathrm{Tr}(L_1L_{-1}) = -4 \\ &\mathrm{Tr}(W_0W_0) = \frac{8}{3}\ , \quad \mathrm{Tr}(W_1W_{-1}) = -4\ , \quad \mathrm{Tr}(W_2W_{-2}) = 16 \end{aligned} \tag{3A.3}$$

All other traces involving a product two generators vanish.

Appendix 3B. Stress tensor correlator in RG flow solution

In this appendix we compute the AdS/CFT two-point correlator of the UV stress tensor T_{UV} in the background of the RG flow solution (3.24). To compute the correlator we need to turn on the source conjugate to T_{UV} and then compute the linearized response. The coefficient relating T_{UV} to the source is the two-point function. In this computation we will not pay attention to overall normalization factors.

In studying linearized fluctuations around the RG solution, the following ansatz turns out to be appropriate

$$\begin{aligned}
A &= \left(\mu e^{2\rho}W_2 + \lambda e^{\rho}L_1 + l_0 L_0 + h_1 e^{-\rho}L_{-1} + h_2 e^{-2\rho}W_{-2}\right)dx^+ \\
&\quad + \left(W_2 e^{2\rho} + J_{UV}W_0 + G^{(1)}_{UV}e^{-\rho}L_{-1} + G^{(2)}_{UV}e^{-\rho}W_{-1} + T_{UV}e^{-2\rho}W_{-2}\right)dx^- \\
&\quad + L_0 d\rho \\
\overline{A} &= e^{2\rho}W_{-2}dx^+ - \lambda e^{\rho}L_{-1}dx^- - L_0 d\rho
\end{aligned} \tag{3B.1}$$

Here λ is constant, while all other coefficients are arbitrary functions of $x^\pm$. The leading large ρ behavior of the metric derived from this connection is

$$ds^2 \approx d\rho^2 - 4e^{4\rho}dx^+dx^- - 4e^{4\rho}\mu(dx^+)^2 \tag{3B.2}$$

From this we see that μ acts as a source for $T_{++} = T_{UV}$, and so we need to work out the relation between T_{UV} and μ. Solving the linearized flatness conditions, we find

$$\begin{aligned}
G^{(2)}_{UV} &= -\frac{1}{3\lambda}\partial_+ J_{UV} \\
l_0 &= -\frac{1}{2}\partial_-\mu \\
h_1 &= \frac{\lambda}{2}J_{UV} \\
h_2 &= -\frac{1}{32}\partial_-^2\mu - \frac{\lambda}{8}G^{(1)}_{UV}
\end{aligned} \tag{3B.3}$$

along with

$$
\begin{aligned}
&\partial_+ G^{(1)}_{UV} = \frac{\lambda}{2}\partial_- J_{UV} \\
&8\partial_+^3 J_{UV} + 12\lambda^3\partial_- G^{(1)}_{UV} = -3\lambda^2\partial_-^3\mu \\
&T_{UV} = \frac{1}{12\lambda^2}\partial_+^2 J_{UV}
\end{aligned} \tag{3B.4}
$$

We solve (3B.4) in momentum space, assuming dependence $e^{ip_+x^+ + ip_-x^-}$, which gives

$$
T_{UV} = -\frac{1}{24}\frac{p_+^3 p_-}{\lambda^4 - \frac{4}{3}\frac{p_+^4}{p_-^2}}\mu \tag{3B.5}
$$

which implies the result quoted in (3.29).

Appendix 3C. Black hole and spin-3 field parameters

3C.1. Metric and spin-3 field in black hole gauge

The coefficients $a_{t,\phi}$ and $b_{t,\phi}$ in the black hole metric (3.58) are as follows:

$$
\begin{aligned}
a_t &= (C-1)^2\left(4\gamma - \sqrt{C}\right)^2 \\
a_\phi &= (C-1)^2\left(4\gamma + \sqrt{C}\right)^2 \\
b_t &= 16\gamma^2(C-2)(C^2-2C+2) - 8\gamma\sqrt{C}(2C^2-6C+5) + C(3C-4) \\
b_\phi &= 16\gamma^2(C-2)(C^2-2C+2) + 8\gamma\sqrt{C}(2C^2-6C+5) + C(3C-4)
\end{aligned} \tag{3C.1}
$$

The t and ϕ coefficients are related by flipping the sign of γ, though this is not a bonafide sign flip of the charge, under which C would also transform.

The spin-3 field has components

$$
\begin{aligned}
\varphi_{\phi rr} &= \frac{2}{3}\sqrt{\frac{2\pi\mathcal{L}}{k}}\frac{(C-3)(4\gamma(C^2-5C+3)-3\sqrt{C})}{C(C-2-\cosh^2(r))^2} \\
\varphi_{\phi tt} &= -\frac{16\sqrt{2}}{3}\left(\frac{\pi\mathcal{L}}{k}\right)^{3/2}\left(\frac{C-3}{C^3}\right)\frac{(a_{t,3}+b_{t,3}\cosh^2(r))\sinh^2(r)}{(C-2-\cosh^2(r))^2} \\
\varphi_{\phi\phi\phi} &= 16\sqrt{2}\left(\frac{\pi\mathcal{L}}{k}\right)^{3/2}\left(\frac{C-3}{C^3}\right)\frac{(a_{\phi,3}+b_{\phi,3}\cosh^2(r))\sinh^2(r)}{(C-2-\cosh^2(r))^2} + \varphi_{\phi\phi\phi}(0)
\end{aligned} \tag{3C.2}
$$

where

$$
\begin{aligned}
a_{t,3} &= a_t \cdot \Big(4\gamma(3-2C) - 3\sqrt{C}\Big) \\
a_{\phi,3} &= a_\phi \cdot \Big(4\gamma(3-2C) - 3\sqrt{C}\Big) \\
b_{t,3} &= 64\Big(C(C(C^2-10C+30)-37)+15\Big)\gamma^3 \\
&- 16\sqrt{C}(C(10C^2-57C+88)-36)\gamma^2 \\
&+ 4C(C(9C^2-42C+62)-36)\gamma \\
&- 3C^{3/2}(2C^2-6C+5) \\
b_{\phi,3} &= 64\Big(C(C(C^2-10C+30)-37)+15\Big)\gamma^3 \\
&- 16\sqrt{C}(C(2C^2+3C-16)+12)\gamma^2 \\
&+ 4C(C(-3C^2+6C-10)+12)\gamma \\
&- 3C^{3/2}(2C^2-6C+5)
\end{aligned}
\tag{3C.3}
$$

and

$$
\begin{aligned}
\varphi_{\phi\phi\phi}(0) = \frac{16\sqrt{2}}{9C^3}\left(\frac{\pi\mathcal{L}}{k}\right)^{3/2}\Big(4\gamma(2C-3)+3\sqrt{C}\Big)\cdot \\
\Big(16\gamma^2(C^2-12C+9)+12\gamma\sqrt{C}(3-5C)-9C(C-1)\Big)
\end{aligned}
\tag{3C.4}
$$

3C.2. Horizon holonomy

For completeness, we provide the holonomy around the ϕ circle, which can be viewed as another piece of gauge-invariant information characterizing the effect of spin-3 charge. The holonomy matrix is just A_ϕ itself. We again work with the trace squared and determinant to obtain

$$
\begin{aligned}
\det(A_\phi) &= -16\left(\frac{2\pi\mathcal{L}}{k}\right)^{3/2}(1+16\gamma^2)\zeta \\
\mathrm{Tr}(A_\phi^2) &= \left(\frac{16\pi\mathcal{L}}{k}\right)(1+\frac{16}{3}\gamma^2+12\gamma\zeta)
\end{aligned}
\tag{3C.5}
$$

The holonomy for the barred gauge field is

$$
\det(\bar{A}_\phi) = -\det(A_\phi)\ , \quad \mathrm{Tr}(\bar{A}_\phi^2) = \mathrm{Tr}(A_\phi^2) \tag{3C.6}
$$

Note from (3.58) that $\mathrm{Tr}(A_\phi^2)$ is directly related to the area of the event horizon, since $g_{\phi\phi}(0) = \frac{1}{2}\mathrm{Tr}(A_\phi^2)$.

Appendix 3D. From wormhole to black hole

We begin by recalling the vielbein reflection equations (3.54):

$$\begin{aligned} e_t(-r) &= -h(r)^{-1} e_t(r) h(r) \\ e_\phi(-r) &= h(r)^{-1} e_\phi(r) h(r) \\ e_r(-r) &= h(r)^{-1} e_r(r) h(r) \end{aligned} \tag{3D.1}$$

where $h(r) \in SL(3,R)$. We obtained these by solving for the horizon geometry perturbatively in the charge, noticing that $e_t(0) = 0$, and demanding reflection symmetry of the metric and spin-3 field as one moves away from the horizon. Consideration of the spin-connection, for which only $\omega_\phi(0) = 0$, allows us to convert these to statements about the gauge fields which are simpler to work with:

$$\begin{aligned} A_+(-r) &= h^{-1}(r) \overline{A}_-(r) h(r) \\ \overline{A}_+(-r) &= h^{-1}(r) A_-(r) h(r) \\ A_r(-r) - \overline{A}_r(-r) &= h^{-1}(r)[A_r(r) - \overline{A}_r(r)] h(r) \\ A_r(-r) + \overline{A}_r(-r) &= \alpha(r) h^{-1}(r)[A_r(r) + \overline{A}_r(r)] h(r) \end{aligned} \tag{3D.2}$$

$\alpha(r)$ is some function of r and the charge which will not be needed.

Recalling that the gauge field A is related to the wormhole gauge field $\mathcal{A}$ by (3.47), our goal is to solve equations (3D.2) for $g(r)$ while solving for $h(r)$ along the way. We will solve the first of these equations, after which we find that the remaining three are satisfied automatically.

As stated in the text, perturbation theory indicates that $g(r)$ and $h(r)$ take the following simple forms:

$$\begin{aligned} g(r) &= e^{F(r)(W_1 - W_{-1}) + G(r) L_0} \\ h(r) &= e^{H(r)(W_1 + W_{-1})} \end{aligned} \tag{3D.3}$$

This ansatz gives a metric and spin-3 field which respect the symmetries of the static BTZ solution around which we perturb. F and H are odd in γ, and G is even. In addition, H is odd under reflection through the horizon, consistent with

$$h^{-1}(-r) = h(r) \equiv h \tag{3D.4}$$

which is implied by (3D.1). Notice that the problem is now highly overconstrained: we are solving for three functions (F, G, H), but we have four 3×3 matrix equations to solve.

Let us rewrite the first of equations (3D.2) in terms of $\mathcal{A}$:

$$\mathcal{A}_+(-r) = M^{-1}\overline{\mathcal{A}}_-(r)M \ , \quad M = g^{-1}(r)h(r)g^{-1}(r) \tag{3D.5}$$

There are five independent components of this matrix equation, which we solve directly. Expanded in generators, M includes pieces proportional to each element of SL(3,R) and the identity, making a mess of algebra. One is aided by defining redundant variables, solving for them, and reinserting these definitions to solve for (F, G, H). To that end, we define the variables (X, Y) as

$$Y = -\frac{\sqrt{4F^2 + G^2}}{G} \ , \quad X = e^{-GY} \tag{3D.6}$$

These combinations are ubiquitous in the explicit form of these equations.

After a display of brute force, one can reduce these five equations to the following three:

$$\begin{aligned}
\zeta &= \frac{Y^2 - (1 + \cosh^2(r))}{(Y^2 - 1)^{3/2}} \cosh(r) \\
X &= \sqrt{\frac{Y - 1}{Y + 1}} \sqrt{\frac{Y + 1 + \cosh^2(r)}{Y - (1 + \cosh^2(r))}} \\
\tan H &= -\frac{\sinh(r) \cosh(r)}{\sqrt{Y^2 - (1 + \cosh^2(r))^2}}
\end{aligned} \tag{3D.7}$$

The sign of H correlates with the convention $\mu > 0$.

One can solve the first equation by taking

$$Y^2 = 1 + C\cosh^2(r) \tag{3D.8}$$

where C is defined by

$$\zeta = \frac{C-1}{C^{3/2}} \tag{3D.9}$$

as in (3.57). The final expression for X, and then for (F, G, H), can be written most compactly as

$$\begin{aligned} X &= \sqrt{\frac{C+Y-1}{C-Y-1}} \\ G &= -\frac{1}{Y}\log(X) \\ \frac{F}{G} &= \frac{\sqrt{C}}{2}\cosh(r) \\ \tan H &= -\frac{\sinh(r)}{\sqrt{C-2-\cosh^2(r)}} \end{aligned} \tag{3D.10}$$

Remarkably, this solves all of the reflection symmetry equations (3D.2).

Let us state some of the parameter ranges. Recalling the holonomy conditions, for example, we know that $3 \leq C < \infty$. This implies $|Y| \geq 2$, so choosing the branch $Y > 0$, we see that $X \geq 1$. This in turn implies that $(F, G) < 0$.

We can clearly see the divergence that feeds down to the metric and spin-3 field. Using the definition of Y, we note that

$$\cosh^2(r_\star) = C - 2 \quad \Leftrightarrow \quad Y(r_\star) = C - 1 \tag{3D.11}$$

In the zero charge limit, $C \to \infty$, $X \to 1$, and $(F, G, H) \to 0$ as we recover the BTZ black hole. In the extremal limit, $C \to 3$, $X \to \infty$, and $(F, G) \to -\infty$. This explains the divergence of the metric and spin-3 field in the extremal limit: the gauge parameters are breaking down. For convenience, we present the first few terms in a perturbative expansion of F and G. With

$$\begin{aligned} F &= f_1\gamma + f_3\gamma^3 + f_5\gamma^5 + \ldots \\ G &= g_2\gamma^2 + g_4\gamma^4 + g_6\gamma^6 + \ldots \end{aligned} \tag{3D.12}$$

we have

$$
\begin{aligned}
f_1 &= -\frac{4}{3}\cosh(r) \\
f_3 &= -\frac{128}{81}\cosh(r)\left(2\cosh^2(r)-3\right) \\
f_5 &= -\frac{8192}{3645}\cosh(r)\left(6\cosh^4(r)-15\cosh^2(r)+40\right)
\end{aligned}
\tag{3D.13}
$$

and

$$
\begin{aligned}
g_2 &= -\frac{64}{9} \\
g_4 &= -\frac{4096}{243}\left(\cosh^2(r)-6\right) \\
g_6 &= -\frac{65536}{10935}\left(12\cosh^4(r)-60\cosh^2(r)+395\right)
\end{aligned}
\tag{3D.14}
$$

Appendix 3E. Generalization to SL(N,R)

In this appendix we will briefly discuss (mainly following [75,76]) how one determines the asymptotic symmetry algebras for the SL(N,R) vacua. As shown in [78] the inequivalent SL(2,R) embeddings in SL(N,R) are uniquely determined by the branching of the fundamental representation of SL(N,R) into n_i dimensional representations of SL(2,R). The branchings are given by the partitions $\{n_1, n_2, \cdots, n_l\}$ of N. From this one can determine how the adjoint, i.e. the algebra itself, decomposes into representations of the SL(2,R) algebra. In general there are representations of spin $s = 0$ up to spin $s = N-1$. We denote the number of spin s representations by m_s (of which some can be zero). For example the principal embedding is given by the partition $\{N\}$, hence the adjoint representation decomposes as

$$
m_1 = 1, m_2 = 1, \cdots, m_{n-1} = 1 \tag{3E.1}
$$

We denote the SL(2,R) generators as $(\hat{L}_+, \hat{L}_-, \hat{L}_0)$ (corresponding to a spin s=1 multiplet) and the generators of spin s_i as $(W^{(i)}_{-s_i}, W^{(i)}_{-s_i+1}, \cdots, W^{(i)}_{s_i})$. The ansatz for the SL(N,R) connection is a "highest weight" gauge, where we associate a field with the $W^{(i)}_{-s_i}$ for each spin i.

$$A = e^{-\rho \hat{L}_0}\Big(\hat{L}_1 + \mathcal{T}(x^+)\hat{L}_{-1} + \sum_i J^{(i)}(x^+)W^{(i)}_{-s_i}\Big)e^{\rho\hat{L}_0}dx^+ + \hat{L}_0 d\rho \qquad (3E.2)$$

Here $\mathcal{T}$ is related to the stress energy tensor of the conformal algebra. A general gauge transformation is given by

$$\lambda = e^{-\rho \hat{L}_0}\Big(\epsilon_1\hat{L}_1 + \epsilon_0\hat{L}_0 + \epsilon_{-1}\hat{L}_{-1} + \sum_i \sum_{j=0}^{2s_i} \alpha^{(i)}_j W^{(i)}_{-s_i+j}\Big)e^{\rho\hat{L}_0} \qquad (3E.3)$$

By using the fact that the $W^{(i)}_s$ transform in spin s_i representations of the SL(2,R) and following the general strategy of considering gauge transformations which preserve the gauge choice (3E.3), one can establish the following facts:

First, for each field $J^{(i)}$ with spin zero (i.e. $s_i = 0$) to transform like a conformal primary, the relation of $\mathcal{T}$ to the stress tensor and the transformation parameter have to be modified. Temporarily referring only to the $\{J^{(i)}, \alpha^{(i)}_0\}$ of the spin-0 representations, we must make the redefinitions

$$T = \mathcal{T} + \sum_i \frac{1}{2}(J^{(i)})^2, \qquad \alpha^{(i)}_0 = \tilde{\alpha}^{(i)}_0 + \epsilon_1 J^{(i)} \qquad (3E.4)$$

With this improvement $J^{(i)}$ is associated with a weight one primary, and T is the stress tensor up to constant rescaling. Second, the fields $J^{(k)}$ with spin $s_k > 0$ then all transform like conformal primaries of weight $s_k + 1$.

For the principal embedding, (3E.1) implies that one has a (quasi) primary of weight 2, i.e. the stress tensor, and conformal primaries of weight $3, 4, \cdots, n$. This is indeed the field content of the W_n algebra and the classical W_n algebra can be obtained this way.

Other SL(2,R) embeddings will lead to different W algebras. For example, for SL(3,R) the only other partition is given by $\{2, 1\}$, and (3E.1) becomes

$$m_0 = 1,\ m_{1/2} = 2,\ m_1 = 1 \qquad (3E.5)$$

This gives one weight 1, two weight 3/2 and one weight 2 primary of the $W_3^{(2)}$ algebra, as was established in more detail in the main part of the paper.

The number of embeddings quickly grows with larger N and we will not discuss these cases here.

4. hs[λ] black holes in 3d Vasiliev gravity

Having constructed spin-3 black holes in the relatively simple SL(3,R) theory of higher spin gravity, we now apply our previous logic to the Vasiliev theories containing an infinite tower of higher spin fields [11]. These theories are based on a one-parameter family of infinite dimensional gauge algebras, denoted hs[λ] [57]. The BTZ black hole is still a solution of this theory, but rather too simple as it carries vanishing values for all higher spin charges. To access the higher spin sector we turn on a nonzero spin-3 chemical potential, α. Due to the nonlinear structure of the theory, this triggers nonzero values for the entire infinite tower of higher spin charges. The values of these charges, and the full smooth solution, can be determined systematically using perturbation theory in α.

As in the SL(3,R) case studied in [79,49], crucial input is provided by demanding a gauge invariant smooth horizon, as expressed in terms of the holonomies. The main output of this procedure is a result for the black hole partition function, $Z(\tau, \alpha; \overline{\tau}, \overline{\alpha})$. Here τ is the modular parameter of the torus that describes the boundary of the Euclidean black hole geometry, and α is the leftmoving spin-3 chemical potential, as noted above. Similarly, $\overline{\alpha}$ is the rightmoving analog of α. As usual, given the partition function, other thermodynamical quantities such as the energy and entropy can be obtained by suitable differentiation. Our result for the black hole partition function, up to order α^8, is

$$\begin{aligned}\ln Z = \frac{i\pi k}{2\tau}\Bigg[1 - \frac{4}{3}\,\frac{\alpha^2}{\tau^4} + \frac{400}{27}\,\frac{\lambda^2-7}{\lambda^2-4}\,\frac{\alpha^4}{\tau^8} - \frac{1600}{27}\,\frac{5\lambda^4-85\lambda^2+377}{(\lambda^2-4)^2}\,\frac{\alpha^6}{\tau^{12}} \\ + \frac{32000}{81}\,\frac{20\lambda^6-600\lambda^4+6387\lambda^2-23357}{(\lambda^2-4)^3}\,\frac{\alpha^8}{\tau^{16}}\Bigg] + \cdots \\ + \text{rightmoving}\end{aligned} \tag{4.1}$$

where the rightmoving part is obtained by replacing τ and α by $\overline{\tau}$ and $\overline{\alpha}$ in the obvious way. The leading term in (4.1) is the usual BTZ result. We note that the factor of $\lambda^2 - 4$ appearing in the denominators is just due to our normalization convention for the spin-3 charge, and has no special significance. The entropy formula obtained from this partition function can be thought of as a generalized version of Cardy's formula to include higher spin charge.

As we now discuss, this result can be used to test the AdS/CFT duality conjectured recently by Gaberdiel and Gopakumar [34], which we discussed in the introduction. Recall that they propose to consider the $\mathcal{W}_N$ minimal model coset CFT

$$\frac{SU(N)_k \oplus SU(N)_1}{SU(N)_{k+1}} \tag{4.2}$$

The 't Hooft limit is defined as

$$N, k \to \infty \ , \quad \lambda \equiv \frac{N}{k+N} \quad \text{fixed} \tag{4.3}$$

Gaberdiel and Gopakumar conjecture that this theory in the 't Hooft limit is dual to the bulk higher spin theory based on the algebra hs[λ], along with some additional scalar fields that will play no role in the present discussion.

To make contact with our black hole result, we should consider the partition function of this theory with the insertion of a spin-3 chemical potential,[13]

$$Z_{CFT}(\tau, \alpha; \overline{\tau}, \overline{\alpha}) = \mathrm{Tr}\Big[e^{4\pi^2 i(\tau\hat{\mathcal{L}} + \alpha\hat{\mathcal{W}} - \overline{\tau}\hat{\overline{\mathcal{L}}} - \overline{\alpha}\hat{\overline{\mathcal{W}}})}\Big] \tag{4.4}$$

where the operators denote the suitably normalized Virasoro and spin-3 zero modes. As is standard, to compare with the black hole side we should consider the leading high temperature asymptotics, defined here by taking $\tau, \alpha \to 0$ with α/τ^2 fixed. This is a version of the Cardy limit, generalized to include the higher spin chemical potential. In this limit, the duality conjecture asserts that (4.1) and (4.4) should agree.

[13] In general, one might wonder whether such traces are convergent. In the following we will be considering perturbation theory in α and $\overline{\alpha}$, and such issues will not arise.

While it should be possible to test this prediction for general λ, in this paper we will only carry out the CFT computation for the special values $\lambda = 0, 1$. The reason why these values are more tractable is as follows. In general, the symmetry algebra controlling the coset theory in the 't Hooft limit is believed to be the infinite dimensional algebra $\mathcal{W}_\infty[\lambda]$. This needs to be so in order for the duality conjecture to be true — for instance, the $\mathcal{W}_\infty[\lambda]$ algebra is the asymptotic symmetry algebra of hs$[\lambda]$ gravity on AdS_3 [36,47,48,64] — but independent evidence is also available [36,38]. At $\lambda = 0, 1$ these algebras simplify. At $\lambda = 1$, after a change of basis, the algebra turns into the *linear* algebra $\mathcal{W}_\infty^{\mathrm{PRS}}$ [80]. Importantly for us, this algebra can be represented in terms of a collection of free bosons, with the higher spin currents being quadratic in the bosons [81,82,83]. Since the bosons are free, we can of course compute (4.4) exactly for this theory. If we make the plausible assumption (justified in more detail in the text) that this free boson theory should share the same high temperature partition function as the coset theory at $\lambda = 1$, then we arrive at the striking prediction that our black hole result should match a certain free boson partition function. Up to the order that we have checked, this turns out to be correct: we find precise agreement with (4.1) at $\lambda = 1$!

An analogous story holds at $\lambda = 0$, but now in terms of free fermions. At $\lambda = 0$ the $\mathcal{W}_\infty[\lambda]$ algebra is related to the algebra $\mathcal{W}_{1+\infty}$ [84] by a constraint that removes the spin-1 current. Since the $\mathcal{W}_{1+\infty}$ algebra can be represented by free fermions [85], we can compute its partition function with the spin-1 constraint imposed. We then find precise agreement with (4.1) at $\lambda = 0$.

We view these results as providing strong evidence for the validity of our rules for treating black holes in higher spin gravity, and for applying them to the conjecture of Gaberdiel and Gopakumar. Further tests along these lines are clearly possible, perhaps by pushing the comparison to higher all orders in α. Another useful generalization would be to turn on additional chemical potentials. More ambitiously, it seems reasonable to hope that these comparisons will lead to a deeper understanding of how the duality is working at a fundamental level.

The remainder of this chapter is organized as follows. In section 4.1, we generalize the results of the previous chapter to a set of rules for constructing higher spin black hole solutions. These are applied to the hs[λ] theories in section 4.2, and the black hole partition function is computed. In section 4.3 we compute the partition functions for free bosons and fermions, and demonstrate agreement with the bulk result for $\lambda = 0, 1$. Section 4.4 contains a discussion of the implications of our results for the AdS/CFT correspondence. In appendix 4A we display certain holonomy equations in detail.

4.1. **How to make higher spin black holes**

We would like to consider the general problem of constructing consistent black holes in a higher spin theory built upon an arbitrary Lie algebra $\mathcal{G}$ which contains a SL(2,R) subalgebra. Based on the lessons of the previous chapter's spin-3 toy model (and the success of the same construction at spin-4 [73]), a reasonable prescription for building smooth black holes with higher spin charge is as follows:

1. Write down a BTZ solution.
2. Compute the BTZ time circle holonomy eigenvalues.
3. Write down a flat connection that includes nonzero chemical potentials for some chosen set of higher spin charges.
4. Fix the charges in the solution by demanding that the holonomy of the solution around the time circle agrees with that of BTZ.

The resulting solution will represent a black hole in the sense described above. Note that if $\mathcal{G}$ is of infinite rank, there will be an infinite number of holonomy constraints. Note also that the these solutions are not in general gauge equivalent to BTZ, since the holonomies around the angular circle will differ.

In the above algorithm, step 3 is stated the least explicitly. Fortunately, our spin-3 example suggests a straightforward way to find the relevant connections. To explain this, consider the explicit spin-3 connection used in [79,49][14]

$$A = \Big(e^\rho L_1 - \frac{2\pi}{k}\mathcal{L}e^{-\rho}L_{-1} - \frac{\pi}{2k}\mathcal{W}e^{-2\rho}W_{-2}\Big)dx^+ \\ + \mu\Big(e^{2\rho}W_2 - \frac{4\pi\mathcal{L}}{k}W_0 + \frac{4\pi^2\mathcal{L}^2}{k^2}e^{-2\rho}W_{-2} + \frac{4\pi\mathcal{W}}{k}e^{-\rho}L_{-1}\Big)dx^- + L_0 d\rho \tag{4.5}$$

with an analogous formula for $\overline{A}$. The corresponding metric has no event horizon, but when the holonomy conditions are obeyed it is gauge-equivalent to one that does [49].

Written in the form (4.5) the solution appears rather complicated, but in fact the structure is quite simple. First, following [48] we note that we can write

$$A = b^{-1}ab + b^{-1}db \tag{4.6}$$

where $b = e^{\rho L_0}$, and a is obtained from A by setting $\rho = d\rho = 0$. In terms of a, the flatness equations are simply $[a_+, a_-] = 0$. To exhibit flatness we need only observe that

$$a_- = 2\mu\left[(a_+)^2 - \frac{1}{3}\mathrm{Tr}(a_+)^2\right] \tag{4.7}$$

The form of A_+ corresponds to choosing the “highest weight gauge”. Namely, if we assume that A_+ grows as e^ρ, then by a gauge transformation it can always be put into the form in (4.5) [48]. Finally, as shown in [79] by a Ward identity analysis, the $\mu e^{2\rho}W_2$ term in A_- gives rise to a chemical potential μ conjugate to spin-3 charge.

[14] The generators $\{W_{\pm 2}, W_{\pm 1}, W_0\}$ transform in the 5-dimensional representation under the adjoint action of $\{L_{\pm 1}, L_0\}$. Also, as in [79,49] we are here using a representation of the SL(3,R) generators in terms of 3×3 traceless matrices.

This discussion suggests a simple way to write down solutions that incorporate chemical potentials for higher-spin charges for any $\mathcal{G}$: to turn on potentials μ_s for fields of spin s, simply take

$$
\begin{aligned}
A_+ &= A_+^{BTZ} + (\text{higher spin charges}) \\
A_- &\sim \sum_s \mu_s \Big[(A_+)^{s-1} - \text{trace}\Big] \\
A_\rho &= L_0
\end{aligned}
\tag{4.8}
$$

where multiplication is defined by the chosen matrix representation of the Lie algebra $\mathcal{G}$. L_0 is the diagonal element of the SL(2,R) embedding into $\mathcal{G}$ used to construct the BTZ solution. As usual, the terms in A_- incorporate the sources, and those in A_+ encode the charges. Exactly which charges one must turn on in order to have a consistent solution depends on the theory in question, and is determined by solution of the holonomy equations.

As already emphasized, the metric derived from (4.8) may not possess a horizon, but based on our study of the SL(3,R) theory we expect that there exists another connection lying on the same gauge orbit that does yield a black hole metric. Finding the explicit gauge transformation will typically be quite involved and $\mathcal{G}$-dependent, but for purposes of interpretation, we merely require its existence.

4.2. The $\mathcal{W}_\infty[\lambda]$ black hole

As an intermediate step before writing down the black hole solutions, we review the features of hs[λ] that we will need. In particular, we introduce an associative multiplication known as the “lone-star product” [57], the antisymmetric part of which yields the hs[λ] Lie algebra.

4.2.1 hs[λ] from an associative multiplication

The hs[λ] Lie algebra is spanned by generators labeled by a spin and a mode index. We use the notation of [36], in which a generator is represented as

$$V_m^s \ , \quad s \geq 2 \ , \quad |m| < s \tag{4.9}$$

The commutation relations are

$$[V_m^s, V_n^t] = \sum_{u=2,4,6,\ldots}^{s+t-|s-t|-1} g_u^{st}(m,n;\lambda) V_{m+n}^{s+t-u} \tag{4.10}$$

with structure constants defined in appendix I.

The generators with $s = 2$ form an SL(2,R) subalgebra, and the remaining generators transform simply under the adjoint SL(2,R) action as

$$[V_m^2, V_n^t] = (m(t-1) - n) V_{m+n}^t \tag{4.11}$$

These SL(2,R) generators will be relevant in construction of the BTZ solution.

When $\lambda = 1/2$, this algebra is isomorphic to hs(1,1), the commutator of which can be written as the antisymmetric part of the Moyal product. Similarly, the general λ commutation relations (4.10) can be realized as

$$[V_m^s, V_n^t] = V_m^s \star V_n^t - V_n^t \star V_m^s \tag{4.12}$$

if we define the associative product

$$V_m^s \star V_n^t \equiv \frac{1}{2} \sum_{u=1,2,3,\ldots}^{s+t-|s-t|-1} g_u^{st}(m,n;\lambda) V_{m+n}^{s+t-u} \tag{4.13}$$

This is known as the "lone star product" [57], and (4.12) follows upon using the fact that

$$g_u^{st}(m,n;\lambda) = (-1)^{u+1} g_u^{ts}(n,m;\lambda) \tag{4.14}$$

The odd values of u drop out of the commutator, leaving (4.10). In the remainder of the paper we may resort to the shorthand

$$\underbrace{\Gamma \star \Gamma \star \ldots \star \Gamma}_{N} \equiv (\Gamma)^N \tag{4.15}$$

for some hs[λ]-valued element Γ.

Formally, V_0^1 is the identity element. Thus, to extract the trace from a product of generators, one picks out the $u = s + t - 1$ part of (4.13), up to some normalization:

$$\mathrm{Tr}(V_m^s V_n^t) \propto g_{s+t-1}^{st}(m, n; \lambda)\delta^{st}\delta_{m,-n} \tag{4.16}$$

In order to facilitate easy comparison to the SL(3,R) conventions of [79,49], we choose to define

$$\mathrm{Tr}(V_m^s V_{-m}^s) = \frac{12}{(\lambda^2 - 1)} g_{2s-1}^{ss}(m, -m; \lambda) \tag{4.17}$$

which implies the SL(2,R) traces

$$\mathrm{Tr}(V_1^2 V_{-1}^2) = -4\ , \quad \mathrm{Tr}(V_0^2 V_0^2) = 2 \tag{4.18}$$

in agreement with the basis used in [79,49].

A convenient property of the hs[λ] Lie algebra is that when $\lambda = N$ for integer $N \geq 2$, one can consistently set all generators with $s > N$ to zero (i.e. factor out the ideal of the Lie algebra), and the algebra reduces to SL(N,R). This implies a similar truncation of the boundary symmetry: that is, $\mathcal{W}_\infty[N] = \mathcal{W}_N$ upon constraining all fields of spin $s > N$ to vanish. Factoring out the ideal is automatic on the level of the trace:

$$\mathrm{Tr}(V_m^s V_{-m}^s) \propto \prod_{\sigma=2}^{s-1} (\lambda^2 - \sigma^2) \tag{4.19}$$

Therefore, as regards the construction of black holes, the holonomy conditions reduce to those of SL(N,R) when $\lambda = N$; this will be a useful check for us.

Another aspect of the lone star product that we wish to highlight is the following simple result for products of the highest weight SL(2,R) generator:

$$(V_1^2)^{s-1} = V_{s-1}^s \tag{4.20}$$

A look back at (4.8) shows that this relation makes it easy to read off the leading behavior of A_- from that of A_+.

In what follows, we work with a flat connection a (and, implicitly, $\bar{a}$) that has no ρ-dependence nor ρ component, as in (4.7), by writing

$$\begin{aligned} A &= b^{-1}ab + b^{-1}db \\ \overline{A} &= b\bar{a}b^{-1} + bdb^{-1} \end{aligned} \tag{4.21}$$

with

$$b = e^{\rho V_0^2} \tag{4.22}$$

Conjugation by b of a generator with mode index m produces a factor $e^{m\rho}$.

4.2.2. The BTZ black hole

We now follow the prescription described in section 2.3 for constructing the higher spin black hole. In the hs[λ] theory, the BTZ black hole has the connection

$$\begin{aligned} a_+ &= V_1^2 + \frac{1}{4\tau^2}V_{-1}^2 \\ a_- &= 0 \end{aligned} \tag{4.23}$$

This is straightforward, as the $V_{\pm 1}^2$ are SL(2,R) elements. The BTZ holonomy can be encoded in the infinite set of traces

$$\mathrm{Tr}(\omega_{BTZ}^n)\ , \quad n = 2, 3, \ldots \tag{4.24}$$

where

$$\omega_{BTZ} = 2\pi\tau\left(V_1^2 + \frac{1}{4\tau^2}V_{-1}^2\right) \tag{4.25}$$

All odd-n traces vanish. The lowest even-n traces are

$$\begin{aligned} \mathrm{Tr}(\omega_{BTZ}^2) &= -8\pi^2 \\ \mathrm{Tr}(\omega_{BTZ}^4) &= \frac{8\pi^4}{5}(3\lambda^2 - 7) \\ \mathrm{Tr}(\omega_{BTZ}^6) &= -\frac{8\pi^6}{7}(3\lambda^4 - 18\lambda^2 + 31) \end{aligned} \tag{4.26}$$

4.2.3. The $\mathcal{W}_\infty[\lambda]$ black hole

Our ansatz for a black hole with spin-3 chemical potential is

$$\begin{aligned} a_+ &= V^2_1 - \frac{2\pi\mathcal{L}}{k}V^2_{-1} - N(\lambda)\frac{\pi\mathcal{W}}{2k}V^3_{-2} + J \\ a_- &= \mu N(\lambda)\left(a_+ \star a_+ - \frac{2\pi\mathcal{L}}{3k}(\lambda^2-1)\right) \end{aligned} \tag{4.27}$$

where

$$J = J_4 V^4_{-3} + J_5 V^5_{-4} + \ldots \tag{4.28}$$

allows for an infinite series of higher-spin charges. The solution is accompanied by the analogous barred connection. $N(\lambda)$ is a normalization factor,

$$N(\lambda) = \sqrt{\frac{20}{(\lambda^2-4)}} \tag{4.29}$$

chosen to simplify comparison to the SL(3,R) results of [79,49]. In particular, truncating all spins $s > 3$ gives a solution with the same generator normalizations and bilinear traces as the spin-3 black hole (4.5) of the SL(3,R) theory.[15]

Suppressing the dependence on barred quantities, we think of this black hole as a saddle point contribution to the partition function

$$Z(\tau,\alpha) = \mathrm{Tr}\left[e^{4\pi^2 i(\tau\mathcal{L}+\alpha\mathcal{W})}\right] \tag{4.30}$$

where we continue to define the potential as

$$\alpha = \overline{\tau}\mu \tag{4.31}$$

where τ is the modular parameter of the boundary torus, defined via the identification $(z,\overline{z}) \cong (z+2\pi\tau, \overline{z}+2\pi\overline{\tau})$, with $x^+ = z,\ x^- = -\overline{z}$. This will once again be justified upon solving the holonomy equations, as the charges will satisfy the integrability condition

$$\frac{\partial\mathcal{L}}{\partial\alpha} = \frac{\partial\mathcal{W}}{\partial\tau} \tag{4.32}$$

[15] To be clear, there is no pathology for $\lambda \leq 2$: we could easily rescale generators to eliminate any troublesome factors of $\lambda^2 - 4$.

The structure of this ansatz is understood as follows: a_+ is the asymptotically AdS$_3$ connection written in the "highest weight gauge" that was used to reveal the asymptotic $\mathcal{W}_\infty[\lambda]$ symmetry in [47,48]. The component a_- is a traceless source term that deforms the UV asymptotics: by (4.20),

$$a_- = \mu N(\lambda) V_2^3 + \text{(subleading)} \tag{4.33}$$

Though similar in some ways, this hs[λ] black hole has some properties that are quite different from its SL(3,R) counterpart. First, there is an infinite set of holonomy constraints to solve, corresponding to enforcing smoothness across the horizon of the metric and higher spin fields. Furthermore, solution of these constraints demands that *all* higher-spin charges are turned on. This is due to the structure of the $\mathcal{W}_\infty[\lambda]$ algebra. For instance, the $\mathcal{W}\mathcal{W}$ OPE has a term

$$\mathcal{W}(z)\mathcal{W}(0) \sim \ldots + \frac{\mu J_4(0)}{z^2} + \ldots \tag{4.34}$$

and likewise for OPEs of higher spin currents.

This sourcing of ever-higher spins is nicely on display in the bulk: we will find that without all spins turned on, (4.32) is not satisfied by the solution of the holonomy equations.

4.2.4. Holonomy

For the $\mathcal{W}_\infty[\lambda]$ black hole ansatz (4.27), the holonomy matrix is

$$\omega = 2\pi \left[\tau a_+ - \alpha N(\lambda) \left(a_+ \star a_+ - \frac{2\pi\mathcal{L}}{3k}(\lambda^2 - 1) \right) \right] \tag{4.35}$$

with a_+ and $N(\lambda)$ as in (4.27) and (4.29), respectively. The holonomy constraints are

$$\text{Tr}(\omega^n) = \text{Tr}(\omega^n_{BTZ}) \ , \quad n = 2, 3, \ldots \tag{4.36}$$

We proceed to solve (4.36) perturbatively in α. We assume a perturbative expansion of the form

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \alpha^2 \mathcal{L}_2 + \dots \\
\mathcal{W} &= \alpha \mathcal{W}_1 + \alpha^3 \mathcal{W}_3 + \dots \\
J_4 &= \alpha^2 J_4^{(2)} + \alpha^4 J_4^{(4)} + \dots \\
J_5 &= \alpha^3 J_5^{(3)} + \alpha^5 J_5^{(5)} + \dots
\end{aligned}
\tag{4.37}
$$

and so on for higher spins. The OPEs tell us that the spin-3 chemical potential μ sources the spin-4 current at $O(\mu^2)$, the spin-5 current at $O(\mu^3)$, and onwards, which this ansatz incorporates. Note the parity under sign flip of α.

To exhibit the structure of the holonomy equations, and to set up our perturbative solution, we write the terms that contribute at lowest perturbative order for each charge that appears in a given equation, ignoring all of the coefficients and displaying just the first four equations:

$$
\begin{aligned}
n=2: \quad & C^{(2)}_{BTZ} = \mathcal{L} + \alpha \mathcal{W} + \alpha^2 J_4 + \dots \\
n=3: \quad & C^{(3)}_{BTZ} = \alpha \mathcal{L}^2 + \mathcal{W} + \alpha J_4 + \alpha^2 J_5 + \alpha^3 J_6 + \dots \\
n=4: \quad & C^{(4)}_{BTZ} = \mathcal{L}^2 + \alpha \mathcal{W}\mathcal{L} + J_4 + \alpha J_5 + \alpha^2 J_6 + \alpha^3 J_7 + \alpha^4 J_8 + \dots \\
n=5: \quad & C^{(5)}_{BTZ} = \alpha \mathcal{L}^3 + \mathcal{W}\mathcal{L} + \alpha J_4 \mathcal{L} + J_5 + \alpha J_6 + \alpha^2 J_7 + \alpha^3 J_8 + \dots
\end{aligned}
\tag{4.38}
$$

$C^{(n)}_{BTZ}$ stand for the BTZ holonomies (4.26) that are of course of order α^0. The "…" denote terms that contribute at higher perturbative order (for instance, an $\alpha^2\mathcal{L}^2$ term at $n=2$). At each value of n, two more charges enter at ever-higher orders in α. In appendix 4A we write out the all-order holonomy equations up to $n=4$, with spins J_5 and higher set to zero for simplicity.

In the case that the gauge algebra is SL(N,R), as obtained by setting $\lambda = N$, the system of equations terminates at $n = N$. For the SL(3,R) theory studied in [79,49] this allowed the holonomy equations to be solved exactly as the solution of a cubic equation. For general λ we instead must proceed perturbatively. In examining the structure of the equations (4.38) one might be concerned by the fact that at even(odd) orders in α, *all* even(odd) n equations contribute. This implies that the system is highly overconstrained, infinitely so, in fact; nevertheless it turns out that there is a consistent solution that satisfies the integrability condition, at least as far as we have checked.

We solve through $O(\alpha^8)$. Combining (4.37) and (4.38), we see that we only need work up to $n = 6$. The solution is

$$\begin{aligned}
\mathcal{L} &= -\frac{k}{8\pi\tau^2} + \frac{5k}{6\pi\tau^6}\alpha^2 - \frac{50k}{3\pi\tau^{10}}\,\frac{\lambda^2-7}{\lambda^2-4}\alpha^4 + \frac{2600k}{27\pi\tau^{14}}\,\frac{5\lambda^4-85\lambda^2+377}{(\lambda^2-4)^2}\alpha^6 \\
&\quad - \frac{68000k}{81\pi\tau^{18}}\,\frac{20\lambda^6-600\lambda^4+6387\lambda^2-23357}{(\lambda^2-4)^3}\alpha^8 + \ldots \\
\mathcal{W} &= -\frac{k}{3\pi\tau^5}\alpha + \frac{200k}{27\pi\tau^9}\,\frac{\lambda^2-7}{\lambda^2-4}\alpha^3 - \frac{400k}{9\pi\tau^{13}}\,\frac{5\lambda^4-85\lambda^2+377}{(\lambda^2-4)^2}\alpha^5 \\
&\quad + \frac{32000k}{81\pi\tau^{17}}\,\frac{20\lambda^6-600\lambda^4+6387\lambda^2-23357}{(\lambda^2-4)^3}\alpha^7 + \ldots \\
J_4 &= \frac{35}{9\tau^8}\,\frac{1}{\lambda^2-4}\alpha^2 - \frac{700}{9\tau^{12}}\,\frac{2\lambda^2-21}{(\lambda^2-4)^2}\alpha^4 + \frac{2800}{9\tau^{16}}\,\frac{20\lambda^4-480\lambda^2+3189}{(\lambda^2-4)^3}\alpha^6 + \ldots \\
J_5 &= \frac{100\sqrt{5}}{9\tau^{11}}\,\frac{1}{(\lambda^2-4)^{3/2}}\alpha^3 - \frac{400\sqrt{5}}{27\tau^{15}}\,\frac{44\lambda^2-635}{(\lambda^2-4)^{5/2}}\alpha^5 + \ldots \\
J_6 &= \frac{14300}{81\tau^{14}}\,\frac{1}{(\lambda^2-4)^2}\alpha^4 + \ldots
\end{aligned} \tag{4.39}$$

When solving these equations the coefficients are obtained in a zigzag pattern: first solve for the leading term in $\mathcal{L}$, then that of $\mathcal{W}$, then the subleading term in $\mathcal{L}$, then the leading term in J_4, and so on. From the solutions of $\mathcal{L}$ and $\mathcal{W}$ we readily confirm that the integrability equation (4.32) is obeyed, although it has to be said that at our current level of understanding this appears as a minor miracle. We take this to be powerful evidence that the holonomy prescription is the correct one for defining higher spin black holes with consistent thermodynamics.

To obtain the partition function we can integrate either one of the equations

$$\frac{\partial \ln Z(\tau,\alpha)}{\partial \tau} = 4\pi^2 i\mathcal{L} \ , \qquad \frac{\partial \ln Z(\tau,\alpha)}{\partial \alpha} = 4\pi^2 i\mathcal{W} \tag{4.40}$$

and thereby arrive at the result quoted in the introduction:

$$\begin{aligned}\ln Z(\tau,\alpha) = \frac{i\pi k}{2\tau}\Bigg[1 - \frac{4}{3}\frac{\alpha^2}{\tau^4} + \frac{400}{27}\frac{\lambda^2-7}{\lambda^2-4}\frac{\alpha^4}{\tau^8} - \frac{1600}{27}\frac{5\lambda^4-85\lambda^2+377}{(\lambda^2-4)^2}\frac{\alpha^6}{\tau^{12}} \\ + \frac{32000}{81}\frac{20\lambda^6-600\lambda^4+6387\lambda^2-23357}{(\lambda^2-4)^3}\frac{\alpha^8}{\tau^{16}}\Bigg] + \dots\end{aligned} \tag{4.41}$$

This partition function is the main result of our bulk analysis. The black hole entropy S can be obtained by applying standard thermodynamics:

$$S = \ln Z(\tau,\alpha) - 4\pi^2 i(\tau\mathcal{L} + \alpha\mathcal{W} - \overline{\tau}\overline{\mathcal{L}} - \overline{\alpha}\overline{\mathcal{W}}) \tag{4.42}$$

Since the formula for the entropy does not appear particularly illuminating we refrain from writing it. Suffice it to say that at order α^0 the the entropy is $A/4G$, where A is the area of the BTZ horizon, but at higher orders in α no geometric interpretation is evident.

4.2.5. Comments and two checks

Ignoring the factors of λ^2-4 for the moment — which, we recall, can be normalized away — the charges all take the form

$$J_s = \tau^{-s}\sum_{n=0}^{\infty}\left(\frac{\alpha}{\tau^2}\right)^{2n+s-2} P_n^{(s)}(\lambda^2) \ , \quad s \geq 3 \tag{4.43}$$

where we include $\mathcal{W} \equiv J_3$. The degree n polynomials $P_n^{(s)}(\lambda^2)$ have zeroes that do not coincide with those of other values of n or s, and so are unlikely to carry any significance.

A first check on our result (4.39) is the λ-independence of the first correction to $\mathcal{W}$ and, by integrability, to $\mathcal{L}$. This can be understood as following from the universal leading coefficient of the $\mathcal{W}\mathcal{W}$ OPE, which in our normalization is[16]

$$\mathcal{W}(z)\mathcal{W}(0) \sim -\frac{5k}{\pi^2}\frac{1}{z^6} + \dots \tag{4.44}$$

In addition, we recall that the hs[λ] trace automatically mods out the effects of higher spin generators upon taking $\lambda = N$ for integer $N \geq 3$. So computing the holonomy in hs[λ] and then evaluating at $\lambda = N$ is identical to computing the holonomy in the SL(N,R) theory from the start.

To verify this, we have embedded the black hole with spin-3 chemical potential in the theories with Lie algebras $\mathcal{G}$ =SL(3,R), SL(4,R), SL(5,R). The results of these investigations match (4.39) exactly (modulo the non-existence of some of the J charges in these cases). Furthermore, we have confirmed that the holonomy equations themselves reduce to those of SL(N,R) non-perturbatively (see appendix 4A for an example).

Another useful check is to consider hs[$\frac{1}{2}$]. In this case, the gauge algebra can be represented in terms of a Moyal product, with generators being even degree polynomials in two spinor variables [11]. All computations can then be carried out in this framework, and the results precisely agree with (4.39) at $\lambda = 1/2$.

4.3. $\lambda = 0, 1$: comparison to CFT

Recall that $\mathcal{W}_\infty[\lambda]$ is the asymptotic symmetry of the AdS$_3$ vacuum of the hs[λ] theory, and possesses hs[λ] as its wedge subalgebra in the $c \to \infty$ limit, as discussed in [36]. In anticipation of an application of our results to the holographic realm, we switch gears and study two CFT realizations of $\mathcal{W}_\infty[\lambda]$ symmetry. One is a theory of free bosons at $\lambda = 1$, and the other, a theory of free fermions at $\lambda = 0$. We compute their exact partition functions

16 The sign convention adopted here differs from that in most CFT references, but turns out to be more convenient.

in the presence of a spin-3 chemical potential; the perturbative expansions match (4.41). We defer a discussion of why this should be, and its intriguing implications, to the next section.

For easy reference, the gravity results for the leftmoving part of the partition function through $O(\alpha^8)$ are

$$\begin{aligned}
\lambda = 1 : \ln Z(\tau,\alpha) &= \frac{i\pi k}{\tau}\left[\frac{1}{2} - \frac{2}{3}\frac{\alpha^2}{\tau^4} + \frac{400}{27}\frac{\alpha^4}{\tau^4} - \frac{8800}{9}\frac{\alpha^6}{\tau^{12}} + \frac{10400000}{81}\frac{\alpha^8}{\tau^{16}} + \dots\right] \\
\lambda = 0 : \ln Z(\tau,\alpha) &= \frac{i\pi k}{\tau}\left[\frac{1}{2} - \frac{2}{3}\frac{\alpha^2}{\tau^4} + \frac{350}{27}\frac{\alpha^4}{\tau^8} - \frac{18850}{27}\frac{\alpha^6}{\tau^{12}} + \frac{5839250}{81}\frac{\alpha^8}{\tau^{16}} + \dots\right]
\end{aligned} \tag{4.45}$$

We ignore the rightmoving part henceforth and focus on a single chiral sector.

4.3.1. $\lambda = 1$: Free bosons

A theory of D free complex bosons has global $\mathcal{W}_\infty[1]$ symmetry with central charge $c = 2D$ [81,82,83]. The symmetry algebra is also known [57] as $\mathcal{W}_\infty^{\mathrm{PRS}}$, and can be written in a linear basis, unlike the $\mathcal{W}_\infty[\lambda]$ algebra for other values of λ.

The complex bosons have OPE

$$\partial\overline{\phi}^i(z_1)\partial\phi_j(z_2) \sim -\frac{\delta^i_j}{(z_1-z_2)^2}\ , \quad i,j = 1,2,\dots D \tag{4.46}$$

The stress tensor and spin-3 current are (summations implied)

$$\begin{aligned}
T &= -\partial\overline{\phi}^i\partial\phi_i \\
\mathcal{W} &= ia\left(\partial^2\overline{\phi}^i\partial\phi_i - \partial\overline{\phi}^i\partial^2\phi_i\right)
\end{aligned} \tag{4.47}$$

where a is some normalization constant. $\mathcal{W}$ is Virasoro primary, as it should be to match with the spin-3 current of our bulk theory. The other higher spin currents are also quadratic in the scalars but include more derivatives; the linearity of the symmetry algebra follows from the quadratic nature of these currents.

To fix a, we compute the leading part of the $\mathcal{W}\mathcal{W}$ OPE:

$$\mathcal{W}(z)\mathcal{W}(0) \sim -\frac{4a^2 D}{z^6} + \cdots \tag{4.48}$$

Since our standard normalization is

$$\mathcal{W}(z)\mathcal{W}(0) \sim -\frac{5k}{\pi^2 z^6} \tag{4.49}$$

to match up we take

$$a = \sqrt{\frac{5k}{4\pi^2 D}} \tag{4.50}$$

Trading D for k according to $c = 6k = 2D$, we have

$$a = \sqrt{\frac{5}{12\pi^2}} \tag{4.51}$$

Now we write down mode expansions. We work on the cylinder with coordinate $w = \sigma_1 + i\sigma_2$, related to the plane coordinate z as $z = e^{-iw}$. We suppress the i indices for the time being. We write

$$\partial\phi(w) = -\sum_m \beta_m e^{imw} \;, \quad \partial\overline{\phi}(w) = -\sum_m \overline{\beta}_m e^{imw} \tag{4.52}$$

where the modes obey

$$[\overline{\beta}_m, \beta_n] = m\delta_{m,-n} = [\beta_m, \overline{\beta}_n] \tag{4.53}$$

The normal ordered stress tensor is

$$T = -\sum_{m=1}^{\infty} \left(\overline{\beta}_{-m}\beta_m + \beta_{-m}\overline{\beta}_m\right) + \frac{k}{4} + \text{nonconstant} \tag{4.54}$$

Since what appears in the partition function is the zero mode of the stress tensor, only the constant terms are relevant here. In the remainder of this section we drop the ground state $k/4$ term, as it plays no role in the high temperature expansion that we are interested in.

As in [79], the quantity $\mathcal{L}$ is related to the constant part of the stress tensor as

$$\mathcal{L} = -\frac{1}{2\pi}T \tag{4.55}$$

and so we have

$$\mathcal{L} = \frac{1}{2\pi}\sum_{m=1}^{\infty}\left(\overline{\beta}_{-m}\beta_m + \beta_{-m}\overline{\beta}_m\right) \tag{4.56}$$

Now consider the mode expansion of the spin-3 charge, which is the constant part of $\mathcal{W}$. We get, after normal ordering,

$$\mathcal{W} = 2a\sum_{m=1}^{\infty} m\left(\overline{\beta}_{-m}\beta_m - \beta_{-m}\overline{\beta}_m\right) \tag{4.57}$$

We can think of the states as being described by arbitrary numbers of positively and negatively charged particles. For example, a state of the form

$$|n_m, \overline{n}_m\rangle = (\beta_{-m})^{n_m}(\overline{\beta}_{-m})^{\overline{n}_m}|0\rangle \tag{4.58}$$

obeys

$$\begin{aligned}\mathcal{L}|n_m, \overline{n}_m\rangle &= \frac{1}{2\pi}m(\overline{n}_m + n_m)|n_m, \overline{n}_m\rangle \\ \mathcal{W}|n_m, \overline{n}_m\rangle &= 2am^2(\overline{n}_m - n_m)|n_m, \overline{n}_m\rangle\end{aligned} \tag{4.59}$$

It is now elementary to compute the partition function

$$Z(\tau, \alpha) = \mathrm{Tr}\left[e^{4\pi^2 i(\tau\mathcal{L} + \alpha\mathcal{W})}\right] \tag{4.60}$$

and we obtain

$$\ln Z(\tau, \alpha) = -D\sum_{m=1}^{\infty}\left[\ln\left(1 - e^{2\pi i\tau m - 8\pi^2 ia\alpha m^2}\right) + \ln\left(1 - e^{2\pi i\tau m + 8\pi^2 ia\alpha m^2}\right)\right] \tag{4.61}$$

In the high temperature regime, $\tau_2 \to 0$, we can convert the sum to an integral. This gives

$$\ln Z(\tau, \alpha) = -\frac{3ik}{2\pi\tau}\int_0^{\infty}dx\left[\ln\left(1 - e^{-x + \frac{2ia\alpha}{\tau^2}x^2}\right) + \ln\left(1 - e^{-x - \frac{2ia\alpha}{\tau^2}x^2}\right)\right] \tag{4.62}$$

where we also used $D = 3k$.

It is straightforward to expand in powers of α and do the integrals:

$$\ln Z(\tau, \alpha) = \frac{i\pi k}{2\tau} - \frac{2i\pi k}{3}\frac{\alpha^2}{\tau^5} + \frac{400i\pi k}{27}\frac{\alpha^4}{\tau^9} - \frac{8800i\pi k}{9}\frac{\alpha^6}{\tau^{13}} + \frac{10400000i\pi k}{81}\frac{\alpha^8}{\tau^{17}} + \cdots \tag{4.63}$$

This agrees precisely with the gravity result (4.45).

4.3.2. $\lambda = 0$: Free fermions

A theory of D free complex fermions furnishes $\mathcal{W}_{1+\infty}$ symmetry with central charge $c = D$ [85]. The $\mathcal{W}_{1+\infty}$ algebra has spins $s = 1, 2, 3, \ldots$, and is related to $\mathcal{W}_\infty[0]$ by a constraint that eliminates the spin-1 current. In the following we will proceed by using a chemical potential to demand that the spin-1 charge is set to zero in the partition function.

The fermions have OPE

$$\overline{\psi}^i(z_1)\psi_j(z_2) \sim \frac{\delta^i_j}{z_1 - z_2} \tag{4.64}$$

The stress tensor is

$$T = -\frac{1}{2}\overline{\psi}^i\partial\psi_i - \frac{1}{2}\psi_i\partial\overline{\psi}^i \tag{4.65}$$

According to [85] the relevant spin-3 current is, up to a normalization constant b that we will fix in a moment,

$$\mathcal{W} = ib(\partial^2\overline{\psi}^i\psi_i - 4\partial\overline{\psi}^i\partial\psi_i + \overline{\psi}^i\partial^2\psi_i) \tag{4.66}$$

As noted in [84], this current is not primary, as the $T\mathcal{W}$ OPE contains an extra term of the form J/z^4 where J is the spin-1 current. From this point of view it is clear that to compare with the bulk we need to set $J = 0$ so that $\mathcal{W}$ will appear as effectively primary.

Now we consider the leading part of the $\mathcal{W}\mathcal{W}$ OPE, which is

$$\mathcal{W}(z)\mathcal{W}(0) \sim -\frac{24Db^2}{z^6} + \cdots \tag{4.67}$$

where we've now taken D complex fermions. To match to our usual normalization we should take

$$b = \sqrt{\frac{5}{144\pi^2}} \tag{4.68}$$

where we've used $D = c = 6k$.

The mode expansion on the cylinder is[17]

$$\overline{\psi}(w) = e^{-\frac{i\pi}{4}} \sum_m \overline{b}_m e^{imw} \ , \quad \psi(w) = e^{-\frac{i\pi}{4}} \sum_m b_m e^{imw} \tag{4.69}$$

with

$$\{\overline{b}_m, b_n\} = \delta_{m,-n} \tag{4.70}$$

Ignoring the zero point energy and nonconstant terms, the normal ordered stress tensor is

$$T(w) = -\sum_{m=1}^{\infty} m(\overline{b}_{-m} b_m + b_{-m}\overline{b}_m) \tag{4.71}$$

so

$$\mathcal{L} = \frac{1}{2\pi} \sum_{m=1}^{\infty} m(\overline{b}_{-m} b_m + b_{-m}\overline{b}_m) \tag{4.72}$$

Similarly, the spin-3 current is

$$\mathcal{W} = -6b \sum_{m=1}^{\infty} m^2(\overline{b}_{-m} b_m - b_{-m}\overline{b}_m) \tag{4.73}$$

A state of the form

$$|n_m, \overline{n}_m\rangle = (b_{-m})^{n_m} (\overline{b}_{-m})^{\overline{n}_m} |0\rangle \tag{4.74}$$

obeys

$$\begin{aligned} \mathcal{L}|n_m, \overline{n}_m\rangle &= \frac{1}{2\pi} m(\overline{n}_m + n_m)|n_m, \overline{n}_m\rangle \\ \mathcal{W}|n_m, \overline{n}_m\rangle &= -6bm^2(\overline{n}_m - n_m)|n_m, \overline{n}_m\rangle \end{aligned} \tag{4.75}$$

[17] We suppress the i indices.

Finally, we need to consider the spin-1 charge operator, which is $Q \sim \int \overline{\psi}\psi$. Our precise definition of the charge operator is

$$Q|n_m, \overline{n}_m\rangle = (\overline{n}_m - n_m)|n_m, \overline{n}_m\rangle \tag{4.76}$$

In the thermodynamic (high temperature) limit it won't matter whether we impose $Q = 0$ as an exact condition on states or as an expectation value; the latter is more convenient since it can be imposed by including a chemical potential for Q and tuning it appropriately. The partition function including a chemical potential for Q is

$$Z(\tau, \alpha, \gamma) = \mathrm{Tr}\left[e^{4\pi^2 i[\tau\mathcal{L}+\alpha\mathcal{W}]+i\gamma Q}\right] \tag{4.77}$$

We calculate this to be

$$\begin{aligned}\ln Z(\tau, \alpha, \gamma) = D\sum_{m=1}^{\infty}\Big[&\ln\left(1 + e^{(2\pi i\tau m - 24\pi^2 ib\alpha m^2 + i\gamma)}\right)\\ &+ \ln\left(1 + e^{(2\pi i\tau m + 24\pi^2 ib\alpha m^2 - i\gamma)}\right)\Big]\end{aligned} \tag{4.78}$$

Converting the sum to an integral we have

$$\ln Z(\tau, \alpha, \gamma) = \frac{3ik}{\pi\tau}\int_0^\infty dx\left[\ln\left(1 + e^{-x+\frac{6ib\alpha}{\tau^2}x^2+i\gamma}\right) + \ln\left(1 + e^{-x-\frac{6ib\alpha}{\tau^2}x^2-i\gamma}\right)\right] \tag{4.79}$$

We now fix γ by demanding charge neutrality. The charge is obtained by differentiating with respect to γ and so we need

$$0 = \int_0^\infty dx\left[\frac{1}{e^{-x-i\epsilon x^2-i\gamma}+1} - \frac{1}{e^{-x+i\epsilon x^2+i\gamma}+1}\right] \tag{4.80}$$

where we defined

$$\epsilon = \frac{6b\alpha}{\tau^2} \tag{4.81}$$

Solving perturbatively gives

$$\gamma = -\frac{\pi^2}{3}\epsilon + \frac{16\pi^4}{9}\epsilon^3 - \frac{448\pi^6}{9}\epsilon^5 + \frac{1254656\pi^8}{405}\epsilon^7 + \cdots \tag{4.82}$$

We now plug this into (4.79), expand in ϵ, and compute the integrals. After inserting the value of b given above, we find

$$\ln Z(\tau,\alpha) = \frac{i\pi k}{2\tau} - \frac{2i\pi k}{3}\frac{\alpha^2}{\tau^5} + \frac{350i\pi k}{27}\frac{\alpha^4}{\tau^9} - \frac{18850i\pi k}{27}\frac{\alpha^6}{\tau^{13}} + \frac{5839250i\pi k}{81}\frac{\alpha^8}{\tau^{17}} + \cdots \tag{4.83}$$

This agrees precisely with the gravity result (4.45).

4.4. Implications for higher spin $\mathbf{AdS_3/CFT_2}$ duality

We now consider what lessons can be drawn from the agreement between our bulk gravity computations and those for free bosons and fermions. For this discussion, let us make the assumption that the agreement will persist to all order in α.

Symmetry obviously plays a powerful role in determining these partition functions. The most likely explanation for why we see agreement is that the answer is fixed by symmetry. On the bulk side, our black hole solutions just involve the non-propagating bulk fields described by the Chern-Simons action, and not the additional scalar fields that arise in the context of the conjecture [34]. Since the topological sector is what gives rise to the asymptotic symmetry algebra, it seems plausible that the physical properties of solutions that lie in this sector are fixed by symmetry.

On the CFT side, a symmetry argument proceeds along the following lines.[18] The partition functions we compute are determined, at order α^n, by the n-point correlation functions of spin-3 currents on the torus. We are interested in the high temperature behavior of these correlation functions. Performing modular transformations term by term, the leading high temperature behavior will be related to correlation functions at low temperature, which are evaluated on the infinite cylinder, or equivalently the plane. Finally, the correlation functions of spin-3 currents on the plane can be computed from the

[18] We thank Matthias Gaberdiel for discussions of these matters.

OPEs. Thus, given the OPEs, we expect that we should be able to compute the partition function in the high temperature limit, and it should agree with our gravity result. Indeed, this computation has now been done on the CFT side [58] to $O(\alpha^6)$, to agreement with the results herein. This is a powerful check, as the mechanics of the CFT computation look nothing like those on the gravity side; it is an outstanding issue to understand in a deeper sense *why* these two diverse approaches match.

As was already mentioned in the introduction, our final result should be thought of as a Cardy formula for CFTs with $\mathcal{W}_\infty[\lambda]$ symmetry and with large central charge.

Although we have argued that our successful matching of partition functions in terms of free fermions and bosons is a consequence of symmetry, it is interesting to note that for $\lambda = 0$ it is believed that the theory (4.2) is in fact fully equivalent to free fermions with a singlet constraint. It would therefore be very interesting to carry out further tests of the AdS/CFT duality [34] at $\lambda = 0$. In fact, a recent construction in [86] has cast the theory at $\lambda = 0$ as the untwisted sector of a theory of free *bosons* subject to a continuous orbifold. The fact that bosons, not fermions, were utilized in that construction is not fixed, however.[18]

We conclude with one final simple observation. Both for $\lambda = 0$ and $\lambda = 1$, in the free fermion/boson theories, there are natural candidates for operators dual to the scalar fields that appear on the bulk side of the duality [34]. These bulk scalars are dual to spinless operators in the CFT of dimension $\Delta = 1 \pm \lambda$. At $\lambda = 0$ we have the free fermion operator $\psi(z)\tilde{\psi}(\overline{z})$, and at $\lambda = 1$ we have the free boson operator $\partial\phi(z)\overline{\partial}\phi(\overline{z})$, both of which have the appropriate dimension.

Appendix 4A. Holonomy equations with $J_4 \neq 0$

Here we present the holonomy equations, $\mathrm{Tr}(\omega^n) = \mathrm{Tr}(\omega^n_{BTZ})$, for the black hole connection (4.27), up to $n = 4$. In the interest of clarity and space, we only include charges up to J_4. We find

$$\begin{aligned}
n = 2:\ \ 0 &= \alpha^2 J_4(144k^2(\lambda^2 - 9)) - 1792\pi^2\alpha^2\mathcal{L}^2 - 504\pi\alpha\tau k\mathcal{W} \\
&\quad - 168\pi\tau^2 k\mathcal{L} - 21k^2 \\
n = 3:\ \ 0 &= \alpha J_4\Big(36k^2(\lambda^2 - 4)(\lambda^2 - 9)[80\pi\alpha^2\mathcal{L}(\lambda^2 - 16) + 9k\tau^2(\lambda^2 - 4)]\Big) \\
&\quad - 40\alpha^3\pi^2\Big[45\mathcal{W}^2 k(5\lambda^4 - 65\lambda^2 + 264) + 256\mathcal{L}^3\pi(\lambda^2 - 4)(\lambda^2 - 16)\Big] \\
&\quad - 4320\alpha^2\mathcal{W}\mathcal{L}\pi^2\tau k(\lambda^2 - 4)(4\lambda^2 - 29) \\
&\quad - 4032\alpha\mathcal{L}^2\pi^2\tau^2 k(\lambda^2 - 4)^2 \\
&\quad - 189\mathcal{W}\pi\tau^3 k^2(\lambda^2 - 4)^2 \\
n = 4:\ \ 0 &= \alpha^4 J_4^2\Big(57600k^4(\lambda^2 - 4)(\lambda^2 - 9)[35\lambda^6 - 1330\lambda^4 + 21707\lambda^2 \\
&\quad - 134748]\Big) - J_4\Big(624k^2(\lambda^2 - 4)(\lambda^2 - 9)\Big[12800\alpha^4\mathcal{L}^2\pi^2(7\lambda^4 - 199\lambda^2 \\
&\quad + 1788) + 8400\alpha^3\mathcal{W}\pi\tau k(5\lambda^4 - 95\lambda^2 + 636) \\
&\quad + 23760\alpha^2\mathcal{L}\pi\tau^2 k(\lambda^2 - 4)(\lambda^2 - 11) + 99\tau^4 k^2(\lambda^2 - 4)^2\Big]\Big) \\
&\quad + 665600\alpha^4\mathcal{L}\pi^3\Big[75\mathcal{W}^2 k(\lambda^2 - 9)(5\lambda^4 - 95\lambda^2 + 636) \\
&\quad + 352\mathcal{L}^3\pi(\lambda^2 - 4)(\lambda^4 - 17\lambda^2 + 100)\Big] \\
&\quad + 131788800\alpha^3\mathcal{W}\mathcal{L}^2\pi^3\tau k(\lambda^2 - 4)\Big[3\lambda^4 - 51\lambda^2 + 244\Big] \\
&\quad + 137280\alpha^2\pi^2\tau^2 k(\lambda^2 - 4)\Big[64\mathcal{L}^3\pi(\lambda^2 - 4)(11\lambda^2 - 71) \\
&\quad + 45\mathcal{W}^2 k(5\lambda^4 - 65\lambda^2 + 264)\Big] \\
&\quad + 823680\alpha\mathcal{W}\mathcal{L}\pi^2\tau^3 k^2(\lambda^2 - 4)^2\Big[23\lambda^2 - 123\Big] \\
&\quad - 9009k^2(3\lambda^2 - 7)(\lambda^2 - 4)^2(k + 8\pi\mathcal{L}\tau^2)(k - 8\pi\mathcal{L}\tau^2)
\end{aligned} \tag{4A.1}$$

We have organized these equations so as to reveal the J_4 dependence. Comparison to (4.38) reveals the underlying structure discussed in the main text.

It is instructive to ask what happens when we take $\lambda = 3$. Since this reduces to the SL(3,R) case, in which there are only two independent holonomy equations, one requires that the $n = 4$ equation should vanish on account of the other two, and that any J_4 dependence should drop out of these equations.

The latter is evident upon inspection. And indeed, taking $\lambda = 3$ reduces the mess of $n = 4$ to be proportional to the $n = 2$ equation by a finite factor. Both the $n = 2$ and $n = 3$ equations reduce to those in [79] (see e.g. equation (5.14) there).

5. Scalar fields and three-point functions in 3d Vasiliev gravity

We now switch gears from finite to zero temperature physics. One of the motivations for constructing the hs[λ] black holes was to construct the (conjectured) bulk dual to finite temperature states in the minimal model CFT. There are many other checks of this conjecture that can be done, some of them prior, in a sense, to the construction of black holes. For instance, a more basic quantity available for comparison is the set of boundary correlation functions obtained from the Vasiliev theory. The simplest correlators are three-point functions, and in what follows we calculate these correlators from the bulk and compare to CFT: we will find perfect agreement between bulk and boundary.

Along the way, we present techniques for writing generalized bulk scalar wave equations in arbitrary on-shell higher spin backgrounds, which have interesting applications beyond the present context which we shall touch upon in the next chapter.

Let us be more specific. On the CFT side we proceed under the assumption that the theory has $\mathcal{W}_\infty[\lambda]$ symmetry. A separate question, not addressed here, is whether this is indeed true of the minimal model CFTs in the 't Hooft limit; evidence in favor appears in [36,38,60].

We now state our result for the three-point correlators. First, in the deformed bulk theory, we calculate the three-point function of two scalar fields and a higher spin field of arbitrary spin s. Recall that there are two complex scalar fields in the bulk, each with $m^2 = \lambda^2 - 1$. Taking one of them to be dual to an operator $\mathcal{O}$ and its complex conjugate $\overline{\mathcal{O}}$, our result for the three-point function in terms of the scalar-scalar two-point function is

$$\frac{\langle \mathcal{O}_\pm(z_1)\overline{\mathcal{O}}_\pm(z_2)J^{(s)}(z_3)\rangle}{\langle \mathcal{O}_\pm(z_1)\overline{\mathcal{O}}_\pm(z_2)\rangle} = \frac{(-1)^{s-1}}{2\pi}\frac{\Gamma(s)^2}{\Gamma(2s-1)}\frac{\Gamma(s\pm\lambda)}{\Gamma(1\pm\lambda)}\left(\frac{z_{12}}{z_{13}z_{23}}\right)^s \tag{5.1}$$

where the subscript denotes standard (+) or alternative (−) quantization of the scalar. The higher spin curent normalization is specified in what follows.

The same correlator for the other scalar field, dual to an operator $\widetilde{\mathcal{O}}$ and its complex conjugate $\overline{\widetilde{\mathcal{O}}}$, is identical to (5.1) but absent the $(-1)^s$ prefactor. In the context of the duality [34], these operators should be assigned to take opposite quantization to one another.

Let us now mention a few of the main insights that allowed us to compute these correlators relatively easily. First, it is well known that if the scalar fields are set to zero the Vasiliev theory is equivalent to a Chern-Simons theory with gauge algebra hs[λ]$\oplus$hs[λ]. To compute correlators of the type (5.1) we need to couple a free scalar field to this theory. The general rules for incorporating scalar fields into the higher spin theory are complicated but known (see [11] and appendix II). However, free scalar field equations can be derived from the elegant equation[19]

$$dC + A \star C - C \star \overline{A} = 0 \tag{5.2}$$

where $(A, \overline{A})$ denote the hs[λ]$\oplus$hs[λ] gauge fields. As we will explain, C is a "master field" that takes values in the Lie algebra hs[λ] supplemented with an identity element, and the scalar is the part of C proportional to the identity. This equation reduces to the Klein-Gordon equation for a scalar of mass $m^2 = \lambda^2 - 1$ when evaluated in AdS, and in general gives the coupling of higher spin gauge fields to the linearized scalar. An important role is played by the star-product appearing in (5.2), which is an associative multiplication known as the "lone-star product" [57]. In the original paper [11] this product is realized in terms of the Moyal product applied to symmetric polynomials of "deformed oscillators." However, the deformed oscillator approach is rather inconvenient for present purposes, as one has to deal with the tedious procedure of resymmetrizing strings of oscillators. By contrast, the lone-star product gives us a closed form expression for the multiplication rules, and turns out to

[19] A second type of scalar, dual to the $\widetilde{\mathcal{O}}$ operators noted above, obeys the same equation but with A and $\overline{A}$ interchanged.

be much simpler to work with.

The second key insight is to put the higher spin gauge invariance at center stage. To compute three-point correlators of the type (5.1) we need to solve (5.2) in the presence of flat connections A and $\overline{A}$ representing higher spin gauge fields with prescribed asymptotics. Such flat connections can be generated by gauge transformations. Therefore, if we start from a solution for the free scalar in AdS$_3$ and then act with the gauge transformation, we generate a new scalar solution in the presence of the higher spin gauge fields. In this way, rather than having to first work out the perturbed scalar equation and then solve it, we can generate the solution in one step, which is a huge simplification.

On the CFT side, our starting point is the assumption that in the 't Hooft limit the W_N coset CFT has $\mathcal{W}_\infty[\lambda]$ global symmetry. While this is unproven, it is a prerequisite for the duality to hold in the pure gauge sector. Previous calculations [39,60] took the tack of computing the $s = 2, 3, 4$ correlators (5.1) at finite N in the CFT, and taking the 't Hooft limit afterwards; this serves as good evidence that $\mathcal{W}_\infty[\lambda]$ really does emerge in the 't Hooft limit, but the complications of finite N are not required if one wants to ask questions about the scalar sector. Thus, for the purposes of calculating the correlator (5.1), we believe that the most direct strategy begins with the assumed symmetry of the 't Hooft limit, and asks to what extent the representation theory of $\mathcal{W}_\infty[\lambda]$ — in particular, of its large k wedge subalgebra hs$[\lambda]$ — fixes the correlator. Generalizing results in [36], we show that, in fact, it is enough to reproduce the result (5.1) and the accompanying result for the second scalar, providing perfect agreement with the bulk.

Our results for bulk and boundary correlators reduce to previous computations [39,60] in the appropriate limits. In all, we find this to be a significant step toward verifying the duality proposal [34].

Our techniques will also have application to computing other correlation functions in these theories. For instance, there is by now a good understanding of higher spin black holes in D=3 [79,49,51,87,73]; in particular their entropy is known to match that of the dual CFT in the high temperature limit [51]. Using the results of this paper, it is now quite feasible to compute scalar correlators in the background of a higher spin black hole and compare with CFT.

The remainder of this chapter is organized as follows. In section 5.1 we go through the main steps involved in deriving the equation (5.2) from the general formulation of the Vasiliev theory. Further details are provided in appendix II of the manuscript. In section 5.2 we show how to work out the explicit form of the scalar wave equation in the presence of higher spin gauge fields. In section 5.3, which is the core of the chapter, we show how to use gauge invariance to generate solutions of the scalar wave equations, and then read off the desired correlation functions. In section 5.4 we compute these correlators on the CFT side under the assumption of $\mathcal{W}_\infty[\lambda]$ symmetry, and demonstrate perfect agreement with the bulk. We conclude with some comments in section 5.5. In Appendix 5A we derive a result needed in the text, and Appendix 5B provides some useful explicit expressions for comparison with a formula derived in section 4. Finally, Appendix I of the manuscript presents some evidence for the isomorphism between the lone-star product and the Moyal product acting on deformed oscillators.

5.1. Matter fields in Vasiliev gravity

We begin with a review of the formulation of 3d higher spin gravity due to Vasiliev and collaborators, as presented in [11]. We first recall how to write the gauge sector of this theory as a hs[λ]$\oplus$hs[λ] Chern-Simons theory, and then show how to introduce linearized scalar fields in the Chern-Simons language. Seeing as we will not need all of the details of the theory's construction, we present an abridged discussion; the reader who would prefer not to take anything on faith is referred to appendix II.

Vasiliev gravity contains one higher spin gauge field for each integer spin $s \geq 2$, coupled to some number of matter multiplets. There are various ingredients, foremost among them a set of "master fields:" a spacetime 1-form $W = W_\nu dx^\nu$ as well as spacetime 0-forms B and S_α. Besides the spacetime coordinates x, the generating functions W, B and S_α also depend on auxiliary bosonic twistor variables[20] z_α, y_α where $\alpha = 1, 2$, as well as on two pairs of Clifford elements: $\psi_{1,2}$, and k, ρ. That is,

$$\{\psi_i, \psi_j\} = 2\delta_{ij} \ , \quad k\rho = -\rho k \ , \quad k^2 = \rho^2 = 1 \tag{5.3}$$

Whereas $\psi_{1,2}$ commute with all other auxiliary variables, k and ρ have the properties

$$\begin{aligned} k y_\alpha &= -y_\alpha k, \quad & k z_\alpha &= -z_\alpha k \\ \rho y_\alpha &= y_\alpha \rho, \quad & \rho z_\alpha &= z_\alpha \rho \end{aligned} \tag{5.4}$$

Roughly speaking, W encodes the gauge sector, B parameterizes the AdS vacua of the theory and is used to introduce propagating matter fields, and S_α ensures that the theory has the correct internal symmetries. The elements $\{z_\alpha, y_\alpha, k, \rho\}$ are ingredients in the realization of these symmetries, and the ψ_i are required only when writing down solutions.

Twistor indices are raised and lowered by the rank two antisymmetric tensor $\epsilon_{\alpha\beta}$:

$$z^\alpha = \epsilon^{\alpha\beta} z_\beta \,, \qquad z_\alpha = z^\beta \epsilon_{\beta\alpha} \tag{5.5}$$

where we use the convention, following [11], $\epsilon^{12} = \epsilon_{12} = 1$. Functions of the twistors z_α, y_α are multiplied by the Moyal product:

$$f(z,y) \star g(z,y) = \frac{1}{(2\pi)^2} \int d^2u \int d^2v \; e^{i u_\alpha v^\alpha} f(z+u, y+u) g(z-v, y+v) \tag{5.6}$$

We can combine the oscillators and various other ingredients to construct two sets of so-called "deformed" oscillators, denoted $\tilde{z}_\alpha, \tilde{y}_\alpha$. The $\tilde{y}_\alpha$, which

20 The z_α, y_α are sometimes referred to as "oscillators."

will be more important for our work here, obey deformed oscillator star-commutation relations

$$[\tilde{y}_\alpha, \tilde{y}_\beta]_\star = 2i\epsilon_{\alpha\beta}(1 + \nu k) \tag{5.7}$$

These deformed oscillators give rise to the higher spin algebra hs$[\lambda]$ as follows. Define elements of the algebra to consist of symmetrized, positive even-degree polynomials in $\tilde{y}_\alpha$. Multiplying these elements using (5.7), and projecting onto $k = \pm 1$, the commutation relations are those of hs$[\lambda]$, with $\lambda = \frac{1}{2}(1 \mp \nu)$.[21]

This deformed oscillator algebra plays a central role in the study of AdS vacua in Vasiliev theory. It emerges dynamically from the field equations of the full nonlinear system of higher spins. The parameter ν encodes the deformation, and in the event that $\nu = 0$, the theory is said to be undeformed. ν also plays two other important roles: it parameterizes a family of inequivalent AdS vacua, and sets the mass of any scalar fields that we introduce to the theory. These connections stem from the structure of the field equations, themselves tightly constrained by higher spin gauge invariance.

To see how ν appears in connection with the AdS vacua, we examine the field equations. There are five equations in terms of the master fields W, B and S_α, and we present two of them here (the others are written in the appendix):

$$\begin{aligned} dW &= W \wedge \star W \\ dB &= W \star B - B \star W \end{aligned} \tag{5.8}$$

The equations that we have omitted all depend on S_α. At the order to which we will be working in this paper — namely, linear in the scalar fields — the entire effect of S_α is that it forces W and B to be independent of k and $\tilde{z}_\alpha$. With this in hand, we can proceed by focussing on (5.8).

[21] More precisely, we consider polynomials of degree two and higher, since the constant term star-commutes with everything. We also note that one can enlarge hs$[\lambda]$ by including odd powers of the $\tilde{y}_\alpha$ in the polynomials, though our interests here are in purely bosonic Vasiliev theory.

We first consider solutions with vanishing scalar field. This corresponds to taking a constant background value for B,

$$B = \nu \tag{5.9}$$

where the constant ν is fixed by the omitted S_α equations to be the same parameter as appears in the deformed oscillator expressions.

Now, the first equation in (5.8) can be written as a flatness condition for two Chern-Simons gauge fields, each taking values in the Lie algebra hs[λ]. In order to see this we introduce gauge fields A and $\overline{A}$ by

$$W = -\mathcal{P}_+ A - \mathcal{P}_- \overline{A} \tag{5.10}$$

where A and $\overline{A}$ are functions of $\tilde{y}_\alpha$ and the spacetime coordinates x^μ. Here we have introduced the projection operators

$$\mathcal{P}_\pm = \frac{1 \pm \psi_1}{2} \tag{5.11}$$

obeying

$$\mathcal{P}_\pm \psi_1 = \psi_1 \mathcal{P}_\pm = \pm \mathcal{P}_\pm, \qquad \mathcal{P}_\pm \psi_2 = \psi_2 \mathcal{P}_\mp \tag{5.12}$$

Plugging (5.10) into (5.8) yields

$$\begin{aligned} dA + A \wedge \star A &= 0 \\ d\overline{A} + \overline{A} \wedge \star \overline{A} &= 0 \end{aligned} \tag{5.13}$$

From our earlier remarks, we see that if A and $\overline{A}$ are taken to be polynomials of positive even degree in symmetrized products of $\tilde{y}_\alpha$, then (5.13) are equivalent to the field equations of hs[λ]$\oplus$hs[λ] Chern-Simons theory. Since SL(2) is a subalgebra of hs[λ], this theory includes ordinary Einstein gravity with a negative comological constant as a consistent truncation.

Before introducing the scalar fields, let us say a bit more about hs[λ]. The hs[λ] Lie algebra is spanned by generators labeled by a spin index s and a mode index m. We use the notation of [36], in which a generator is represented as

$$V^s_m \ , \quad s \geq 2 \ , \quad |m| < s \tag{5.14}$$

The commutation relations are

$$[V^s_m, V^t_n] = \sum_{u=2,4,6,\ldots}^{s+t-|s-t|-1} g^{st}_u(m,n;\lambda) V^{s+t-u}_{m+n} \tag{5.15}$$

with structure constants defined in appendix I. The generators with $s = 2$ form an SL(2,R) subalgebra, and the remaining generators transform simply under the adjoint SL(2,R) action as

$$[V^2_m, V^t_n] = (m(t-1) - n)V^t_{m+n} \tag{5.16}$$

The parameter λ can be mapped to the parameters of the oscillator formulation as

$$\lambda = \frac{1 - \nu k}{2} \tag{5.17}$$

where $k = \pm 1$. When $\lambda = 1/2$, the theory is undeformed and this algebra is isomorphic to hs(1,1) [88].

To summarize what we have found so far, the gauge sector of the Vasiliev theory boils down to hs[λ]$\oplus$hs[λ] Chern-Simons theory. This sector of the theory has no propagating degrees of freedom.

To introduce propagating scalar fields we study a linearized fluctuation of B around its vacuum value (5.9),

$$B = \nu + \mathcal{C}(x, \psi_i, \tilde{y}_\alpha) \tag{5.18}$$

The bosonic field $\mathcal{C}$ is taken to have an expansion in even-degree symmetrized products of the deformed oscillators $\tilde{y}_\alpha$. The lowest term in the expansion, with no deformed oscillators, will be identified with the physical scalar field. $\mathcal{C}$ obeys

$$d\mathcal{C} - W \star \mathcal{C} + \mathcal{C} \star W = 0 \tag{5.19}$$

We decompose $\mathcal{C}$ as

$$\mathcal{C} = \mathcal{P}_+ \psi_2 C(x, \tilde{y}_\alpha) + \mathcal{P}_- \psi_2 \tilde{C}(x, \tilde{y}_\alpha) \tag{5.20}$$

Plugging into (5.19) we find

$$\begin{aligned} dC + A \star C - C \star \overline{A} &= 0 \\ d\tilde{C} + \overline{A} \star \tilde{C} - \tilde{C} \star A &= 0 \end{aligned} \tag{5.21}$$

As shown in the next section, expanded around AdS each of these equations reduces to the Klein-Gordon equation for a scalar field of mass $m^2 = \lambda^2 - 1$. More generally, these equations capture the interaction of the linearized scalars with an arbitrary higher spin background. For example, they can be used to study the propagation of a scalar field in the higher spin black hole of [51].

We note that the two equations (5.21) are related by $A \leftrightarrow \overline{A}$, $C \leftrightarrow \tilde{C}$. This is interpreted as a "charge conjugation" operation that flips the sign of all odd spin tensor gauge fields. Another notable feature is that the equations (5.21) are only sensible for A and $\overline{A}$ on-shell, i.e. satisfying equation (5.13). This can be seen by taking d of these equations; if the connections are not flat this leads to extra constraints on C and $\tilde{C}$ with no interpretation in terms of propagating scalar fields.

Recapping, we have now reduced the system of equations down to (5.13) and (5.21). These equations describe the propagation of linearized scalar fields in an arbitrary on-shell higher spin background. They do not capture back-reaction of the scalar on the higher spin fields, or self-interactions among the scalars. Neither of these effects is needed for the computation of the three-point correlators herein.

We now turn to solving these equations, and introducing efficient tools for this purpose. We focus on the C equation; results for $\tilde{C}$ then follow by charge conjugation.

5.2. Generalized Klein-Gordon equations in higher spin backgrounds

Starting from

$$dC + A \star C - C \star \overline{A} = 0 \tag{5.22}$$

for some higher spin background determined by hs[λ]-valued connections $(A, \overline{A})$, we want to extract the generalized Klein-Gordon equation hiding within.

As we said, in the traditional formalism of bosonic Vasiliev theory, the master field C is expanded in deformed oscillators $\tilde{y}_\alpha$ as

$$C = C_0^1 + C^{\alpha\beta}\tilde{y}_\alpha\tilde{y}_\beta + C^{\alpha\beta\sigma\lambda}\tilde{y}_\alpha\tilde{y}_\beta\tilde{y}_\sigma\tilde{y}_\lambda + \dots \tag{5.23}$$

where the star product is implied and all components of C are symmetric in twistor indices. This separates the components of the master field into the physical scalar field, which is the lowest component C_0^1, and the remaining components related on-shell to C_0^1 by derivatives. Plugging (5.23) into (5.22) leads, after much work, to the scalar equations.

The most tedious part of this computation is multiplying the deformed oscillators. We need to take a pair of symmetrized combinations of oscillators, multiply them, and then resymmetrize using (5.7). Rather than carrying out this procedure each time, it would be much more convenient if we had a closed-form expression for the multiplication rules. Recall that the Lie algebra obtained via star-commutation of these elements is hs[λ]. Now, underlying hs[λ] is an associative product, under which the hs[λ] Lie bracket becomes the commutator (5.15). This "lone-star product" is defined as

$$V^s_m \star V^t_n \equiv \frac{1}{2}\sum_{u=1,2,3,\dots}^{s+t-|s-t|-1} g^{st}_u(m,n;\lambda)V^{s+t-u}_{m+n} \tag{5.24}$$

One recovers (5.15) upon using the fact that

$$g^{st}_u(m,n;\lambda) = (-1)^{u+1}g^{ts}_u(n,m;\lambda) \tag{5.25}$$

This was originally presented in an early paper on W_∞ algebras [57], and used more recently by two of the present authors [51] to compute black hole partition functions in hs[λ] gravity.

It is then very natural to suspect an isomorphism between the product rules for symmetrized oscillator combinations and those of the lone-star product. In appendix I we present strong evidence at low spins that, indeed, the lone-star product acting on hs[λ] generators is isomorphic to the product involving the deformed oscillators, with a specific identification between generators and oscillator polynomials.

Proceeding under this assumption, which will be well justified by the consistency of all results, provides a major technical simplification. One trades the tedious symmetrization procedure for the known and easily manipulated hs[λ] structure constants.

In this language, we expand the master field C as follows:

$$C = \sum_{s=1}^{\infty} \sum_{|m|<s} C^s_m V^s_m \tag{5.26}$$

This maps to (5.23) under the identifications

$$C^s_m \sim C^{\alpha_1 \alpha_2 \cdots \alpha_{2s-2}} \tag{5.27}$$

with the index m related to the number of oscillators $\tilde{y}_1$ versus $\tilde{y}_2$ as $2m = N_1 - N_2$. (For more details see, appendix I.) The functions C^s_m are functions of spacetime coordinates x, and the auxiliary tensor structure has been absorbed into (s, m). The gauge fields are expanded similarly,

$$A = \sum_{s=2}^{\infty} \sum_{|m|<s} A^s_m V^s_m \ , \quad \overline{A} = \sum_{s=2}^{\infty} \sum_{|m|<s} \overline{A}^s_m V^s_m \tag{5.28}$$

The lowest scalar component, C^1_0, will be our physical scalar field, and the remaining C^s_m will be related to derivatives of C^1_0. Upon plugging into (5.22) we can obtain the equation of motion for C^1_0.

Let us begin by solving the equation (5.22) in AdS; aside from being an instructive exercise, this will lay the foundation for what follows. The same computation has been performed in various places (e.g. [11,39]) in the oscillator language. Because there are no higher spin fields turned on in the vacuum, C_0^1 should obey the ordinary Klein-Gordon equation and the remaining components should be fixed in terms of C_0^1. In addition, previous literature on the subject tells us that the scalar mass squared is $m^2 = \lambda^2 - 1$. An attractive feature of the theory is that the value of the mass is fixed by the gauge algebra.

5.2.1. AdS: Recovering the Klein-Gordon equation

The AdS connection is constructed out of the spin-2 generators alone, namely those forming an SL(2,R) subalgebra of hs[λ]. We work in Euclidean signature in Fefferman-Graham gauge, with radial coordinate ρ and boundary coordinates $(z, \bar{z})$. The connection is

$$\begin{aligned} A &= e^{\rho} V_1^2 dz + V_0^2 d\rho \\ \overline{A} &= e^{\rho} V_{-1}^2 d\bar{z} - V_0^2 d\rho \end{aligned} \tag{5.29}$$

giving rise to a metric[22]

$$ds^2 = d\rho^2 + e^{2\rho} dz d\bar{z} \tag{5.30}$$

and vanishing higher spin fields.

Due to the simplicity of this connection — the small number of generators, their low spin and the symmetric appearance of A and $\overline{A}$ — the C equations (5.22) are simple to write. Decomposing along both spacetime and internal hs[λ] space, and using the lone-star product, one finds the following linear

[22] In this work, we will not need the prescription to pass from Chern-Simons to metric language; suffice it to say that in writing this metric, we have chosen a particular normalization of the hs[λ] trace. See [51] for conventions used here.

combinations to equal zero [23]

$$
\begin{aligned}
&V^s_{m,\rho} : \partial_\rho C^s_m + 2C^{s-1}_m + C^{s+1}_m g_3^{(s+1)2}(m,0) \\
&V^s_{m,z} : \partial C^s_m + e^\rho \left[C^{s-1}_{m-1} + \frac{1}{2} g_2^{2s}(1,m-1) C^s_{m-1} + \frac{1}{2} g_3^{2(s+1)}(1,m-1) C^{s+1}_{m-1} \right] \\
&V^s_{m,\bar{z}} : \overline{\partial} C^s_m \\
&\qquad - e^\rho \left[C^{s-1}_{m+1} - \frac{1}{2} g_2^{2s}(-1,m+1) C^s_{m+1} + \frac{1}{2} g_3^{2(s+1)}(-1,m+1) C^{s+1}_{m+1} \right]
\end{aligned}
\tag{5.31}
$$

where $|m| < s$ and $\partial = \partial_z, \overline{\partial} = \partial_{\bar{z}}$. (Here and henceforth, we suppress the λ-dependence of the structure constants $g^{st}_u(m,n)$.)

It is now easy to obtain the Klein-Gordon equation. Writing out a handful of equations at $s = 1, 2$, one finds four that form a closed set for components $\{C^1_0, C^2_0, C^3_0, C^2_1\}$:

$$
\begin{aligned}
V^1_{0,\rho} :&\quad \partial_\rho C^1_0 + C^2_0 \cdot \frac{\lambda^2 - 1}{6} = 0 \\
V^1_{0,\bar{z}} :&\quad \overline{\partial} C^1_0 + e^\rho C^2_1 \cdot \frac{\lambda^2 - 1}{6} = 0 \\
V^2_{1,z} :&\quad \partial C^2_1 + e^\rho C^1_0 + \frac{e^\rho}{2} C^2_0 - e^\rho C^3_0 \cdot \frac{\lambda^2 - 4}{30} = 0 \\
V^2_{0,\rho} :&\quad \partial_\rho C^2_0 + 2C^1_0 + C^3_0 \cdot \frac{2(\lambda^2 - 4)}{15} = 0
\end{aligned}
\tag{5.32}
$$

Solving for the higher components and plugging back in yields

$$
\left[\partial^2_\rho + 2\partial_\rho + 4e^{-2\rho} \partial \overline{\partial} - (\lambda^2 - 1) \right] C^1_0 = 0 \tag{5.33}
$$

This is the Klein-Gordon equation in the background (5.30) with the correct scalar mass.

In order for the entire set of unfolded equations to be consistent, all components of C must have a smooth solution in terms of C^1_0.[24] We delay presentation of the solution for the full master field C in AdS until section 4, where

[23] V^s_{m,x^μ} is a short-hand notation for the component along $V^s_m dx^\mu$.

[24] Upon solving for the C^s_m in terms of C^1_0, one observes the following pole structure in the λ

we will need it to compute the three-point functions. There, we will also show that for any connection related to AdS by a non-singular gauge transformation, the linearized matter equations (5.22) admit a consistent solution for the full master field C.

Before moving on to higher spin deformations of AdS, let us elucidate the structure of (5.31) and present a systematic strategy for isolating the minimal set of equations needed to solve for C^1_0. We wish to highlight a special type of component of C, namely those which are of the form $C^{m+1}_{\pm m}$ and hence have the smallest possible spin for fixed mode m: $C^1_0, C^2_{\pm 1}$, etc. We call these "minimal" components.

Starting with the $V^s_{m,\rho}$ equations, it is clear that for fixed mode m, one can solve these recursively for all non-minimal components in terms of $C^{m+1}_{\pm m}$ and ρ derivatives thereof. This is a consequence of being in Fefferman-Graham gauge, whereby $A_\rho = -\overline{A}_\rho = V^2_0$, and we will remain in this gauge throughout this paper. Having solved for the minimal components $C^{m+1}_{\pm m}$, one should view the $V^s_{m,z}$ and $V^s_{m,\bar{z}}$ equations as determining these in terms of C^1_0 and $(z,\bar{z})$ derivatives thereof.

This reveals a useful strategy for extracting the smallest possible set of equations to obtain the scalar equation in any background. We think of the ρ equations as implicitly solved. Then, along $(z,\bar{z})$, one need only keep track of the mode indices appearing in any given equation, and one need only look at equations along minimal directions.

---

plane:

$$C^s_m \sim (\cdots)\prod_{p=1}^{s-1}(\lambda^2 - p^2)^{-1} \tag{5.34}$$

This is evident in (5.32), for example. These are not problematic for $p \neq 1$, as hs[λ] degenerates to SL(N,R) at integer values of $\lambda \geq 2$, and the spin $s > \lambda$ fields do not exist. The singularity at $\lambda = 1$ has a different role, but is a reflection of the fact that hs[1], in the absence of a rescaling of generators, becomes similarly degenerate as many structure constants vanish.

Let us demonstrate with the AdS connection (5.29). We use the following heuristic for which components appear in which equations (structure constants implied): for modes $m = 0, 1, 2,$

$$\begin{aligned}
&\vdots && \vdots \\
V_{0,\bar{z}} &\sim \overline{\partial} C_0 + C_1 \ , \quad & V_{0,z} &\sim \ \partial C_0 + C_{-1} \\
V_{1,\bar{z}} &\sim \overline{\partial} C_1 + C_2 \ , \quad & V_{1,z} &\sim \ \partial C_1 + C_0 \\
V_{2,\bar{z}} &\sim \overline{\partial} C_2 + C_3 \ , \quad & V_{2,z} &\sim \ \partial C_2 + C_1 \\
&\vdots && \vdots
\end{aligned} \tag{5.35}$$

The equations $V^1_{0,\bar{z}}, V^2_{1,z}$ form a closed set among components with $m = 0, 1$, and so will be enough, along with whatever ρ equations we need, to find the Klein-Gordon equation. This is exactly what we presented in (5.32).

5.2.2. Chiral higher spin deformations of AdS

To warm up to the higher spin connections we will ultimately consider, we present the simplest possible higher spin deformation of AdS: a constant, chiral spin-3 deformation,

$$\begin{aligned}
A &= e^{\rho} V^2_1 dz - \eta e^{2\rho} V^3_2 d\bar{z} + V^2_0 d\rho \\
\overline{A} &= e^{\rho} V^2_{-1} d\bar{z} - V^2_0 d\rho
\end{aligned} \tag{5.36}$$

From the point of view of the boundary CFT, this corresponds to adding a dimension-3 operator to the CFT, with constant coupling η.

Using our mnemonic of the previous subsection, we make quick work of this connection. Again writing the C master field equation in spacetime and gauge components, we show some of the $\bar{z}$ equations:

$$\begin{aligned}
V_{-1,\bar{z}} &\sim \overline{\partial} C_{-1} + C_0 + \eta C_{-3} \\
V_{0,\bar{z}} &\sim \overline{\partial} C_0 + C_1 + \eta C_{-2} \\
V_{1,\bar{z}} &\sim \overline{\partial} C_1 + C_2 + \eta C_{-1} \\
V_{2,\bar{z}} &\sim \overline{\partial} C_2 + C_3 + \eta C_0
\end{aligned} \tag{5.37}$$

The components along z are unchanged from the AdS case. Reinserting the spin indices, the minimal closed set of equations is $\{V^1_{0,z}, V^2_{1,z}, V^3_{2,z}, V^2_{1,\bar{z}}\}$, along with any ρ equations necessary. Solving this system gives the following equation for C^1_0:

$$\left[\partial_\rho^2 + 2\partial_\rho + 4e^{-2\rho}(\partial\overline{\partial} - \eta\partial^3) - (\lambda^2 - 1)\right] C^1_0 = 0 \tag{5.38}$$

There is now a three-derivative term, as one would expect from dimensional analysis.

This can be extended to a chiral spin-s deformation: for a connection

$$\begin{aligned} A &= e^\rho V^2_1 dz - \eta e^{(s-1)\rho} V^s_{s-1} d\bar{z} + V^2_0 d\rho \\ \overline{A} &= e^\rho V^2_{-1} d\bar{z} - V^2_0 d\rho \end{aligned} \tag{5.39}$$

the generalized Klein-Gordon equation is

$$\left[\partial_\rho^2 + 2\partial_\rho + 4e^{-2\rho}(\partial\overline{\partial} + \eta(-\partial)^s) - (\lambda^2 - 1)\right] C^1_0 = 0 \tag{5.40}$$

This example nicely captures the primary general feature of higher spin deformations: higher derivative terms enter the generalized wave equation.

One can extend these methods to any connection — black holes or RG flows, for instance — although the difficulty in solving the resulting set of equations increases rather quickly with the number of generators. We now study a slightly more complicated connection, relevant for computation of correlation functions in section 4.

5.2.3. Higher spin currents in AdS

Starting from (5.36), we wish to allow η to have arbitrary dependence on $(z, \bar{z})$.[25] This will act as a source for spin-3 charge $J^{(3)}$, enabling us to compute the correlator $\langle \mathcal{O}(z_1)\overline{\mathcal{O}}(z_2)J^{(3)}(z_3)\rangle$, where $\mathcal{O}$ is a scalar primary dual to C^1_0.

[25] When writing the functional dependence of fields and operators on $(z, \bar{z})$, we temporarily use the notation $z \equiv (z, \bar{z})$.

To do so, we will need to obtain the scalar equation to linear order in the spin-3 source (which we now label $\mu^{(3)}$).

Previous work on higher spin gravity [48,79], following earlier work in pure gravity [69], laid out a dictionary for relating sources and charges to components of the Chern-Simons connection, and we will apply and recapitulate those techniques here.

The following is a flat connection, to linear order in the source $\mu^{(3)}(z)$:

$$
\begin{aligned}
A_z &= e^{\rho} V_1^2 + \frac{1}{B^{(3)}} J^{(3)}(z) e^{-2\rho} V_{-2}^3 \\
A_{\bar{z}} &= -\sum_{n=0}^{4} \frac{1}{n!}((-\partial)^n \mu^{(3)}(z)) e^{(2-n)\rho} V_{2-n}^3 \\
\overline{A}_{\bar{z}} &= e^{\rho} V_{-1}^2
\end{aligned}
\tag{5.41}
$$

along with the usual $A_\rho = -\overline{A}_\rho = V_0^2$, subject to

$$
\bar{\partial} J^{(3)}(z) = -\frac{B^{(3)}}{4!} \partial^5 \mu^{(3)}(z) \tag{5.42}
$$

This is the same connection as first presented in [79], now embedded in hs[λ] instead of SL(3,R) and with vanishing stress tensor. The leading term in $A_{\bar{z}}$, namely $-\mu^{(3)} e^{2\rho} V_2^3$, is the source, dual to the charge term $\frac{1}{B^{(3)}} J^{(3)} e^{-2\rho} V_{-2}^3$ in A_z. The remaining terms in $A_{\bar{z}}$ are required for flatness. We have included a normalization constant $B^{(3)}$ in the definition of the current. While its actual value is unimportant for the calculations in this paper[26], we include it to stress that we are inserting the factor

$$
e^{\int d^2 z\ \mu^{(3)} J^{(3)}} \tag{5.43}
$$

in the CFT path integral. Writing the connection with unit coefficient (up to a sign) for the chemical potential $\mu^{(3)}$ then fixes the other free coefficient, and

[26] For an explicit formula for this coefficient for any spin, see [36], equation (A.4), where $B^{(s)} = -\frac{k}{2\pi} N_s$ in their notation.

$B^{(3)}$ is fixed by the Ward identity (5.42), equivalently by the OPE

$$J^{(3)}(z)J^{(3)}(0) \sim \frac{5B^{(3)}}{2\pi}\frac{1}{z^6} \tag{5.44}$$

Passing to the metric-like formulation of the spin-3 field using $\varphi_{\mu\nu\sigma} \sim \mathrm{Tr}(e_{(\mu}e_\nu e_{\gamma)})$, one can explicitly check that this connection turns on various components of $\varphi_{\mu\nu\sigma}$: for instance, the component

$$\varphi_{\bar{z}\bar{z}\bar{z}} \sim \mu^{(3)} e^{4\rho}\mathrm{Tr}(V^3_2 V^2_{-1} V^2_{-1}) \tag{5.45}$$

makes it clear that we have turned on a spin-3 source that grows toward the boundary.

Following our prior method, one finds that the following set of equations forms the minimal closed set needed to determine C^1_0:

$$V^2_{1,z}\ ,\quad V^1_{0,z}\ ,\quad V^2_{-1,z}\ ,\quad V^3_{-2,z}\ ,\quad V^2_{1,\bar{z}}\ ,\quad V^1_{0,\bar{z}}\ ,\quad V^2_{-1,\bar{z}} \tag{5.46}$$

As always, these should be accompanied by some number of $V^s_{m,\rho}$ equations required to eliminate non-minimal components of C. Solving these perturbatively, one finds the following scalar equation to linear order in $\mu^{(3)} \equiv \mu$:

$$\left(\Box_{KG} + \Box_\mu\right) C^1_0 = 0 \tag{5.47}$$

where

$$\begin{aligned}
\Box_{KG} &= \partial_\rho^2 + 2\partial_\rho + 4e^{-2\rho}\partial\bar{\partial} - (\lambda^2-1)\\
\Box_\mu &= -\frac{1}{6}e^{-6\rho}\left(\partial^5\mu\bar{\partial}^2 + \partial^4\mu\partial\bar{\partial}^2\right) + \frac{4}{B^{(3)}}J^{(3)}e^{-4\rho}\bar{\partial}^3\\
&\quad + \frac{1}{3}e^{-4\rho}\left(\partial^4\mu\bar{\partial} - \partial^4\mu\bar{\partial}\partial_\rho + \partial^3\mu\partial\bar{\partial} - \partial^3\mu\partial\bar{\partial}\partial_\rho\right)\\
&\quad - \frac{1}{6}e^{-2\rho}\Big(3\partial^3\mu\partial_\rho^2 - (\lambda^2-1)\partial^3\mu + 3\partial^2\mu\partial\partial_\rho^2 - 12\partial^2\mu\partial\partial_\rho\\
&\quad - (\lambda^2-13)\partial^2\mu\partial + 36\partial\mu\partial^2 - 12\partial\mu\partial^2\partial_\rho + 24\mu\partial^3\Big)
\end{aligned} \tag{5.48}$$

This equation may be used to infer the cubic vertex between two scalars and the metric-like spin-3 field. On the other hand, it does not clearly suggest what the analogous equation would look like for higher spin sources. What is clear is that the number of terms will grow with the spin.

To solve this using standard AdS/CFT methodology, we write

$$C_0^1(\rho, z; z_1) = G_{b\partial}(\rho, z; z_1) + \phi_\mu(\rho, z; z_1) \tag{5.49}$$

where the subscript denotes the order in μ. The first term is the bulk-to-boundary propagator of a scalar in AdS with $m^2 = \lambda^2 - 1$, and with standard $(+)$ or alternative $(-)$ quantization:

$$G_{b\partial}(\rho, z; z_1) = \pm\frac{\lambda}{\pi}\left(\frac{e^{-\rho}}{e^{-2\rho} + |z - z_1|^2}\right)^{1\pm\lambda} \tag{5.50}$$

The solution to (5.47) to linear order in μ is

$$\phi_\mu(\rho, z; z_1) = -\int d^2z' d\rho' e^{2\rho'} G_{bb}(\rho, z; \rho', z')\Box'_\mu G_{b\partial}(\rho', z'; z_1) \tag{5.51}$$

where $G_{bb}(\rho, z; \rho', z')$ is the bulk-to-bulk propagator obeying

$$\Box_{KG} G_{bb}(\rho, z; \rho', z') = e^{-2\rho}\delta(\rho - \rho')\delta^{(2)}(z - z') \tag{5.52}$$

By judicious use of integration by parts, one can reduce the integral to a boundary term and read off the three-point function. This was the strategy employed in [39]. Instead, we will step back and discuss a simpler method that makes full use of higher spin gauge invariance from the outset. With this approach we bypass the need to first find the modified scalar equation and then solve it, and instead contruct the needed solution directly.

5.3. Three-point correlators from the bulk

We now turn to the main focus of this paper: the efficient computation of three-point correlation functions involving two scalar operators and one higher spin current. Our basic observation is that starting from the solution for a free scalar field in AdS we can generate a new solution by performing a higher spin gauge transformation. This essentially reduces the whole problem to determining how the scalar field transforms under the gauge transformation.

5.3.1. Spin-1 example

To illustrate our general approach in the simplest context, in this section we compute the three-point function of two scalar operators and a spin-1 current. Rather than working in the Vasiliev theory, here we take the bulk action to be a complex scalar field of mass $m^2 = \lambda^2 - 1$ coupled to a U(1) Chern-Simons gauge field,

$$S = \frac{k}{4\pi} \int A \wedge dA + \frac{1}{2} \int d^3x \sqrt{g} \Big(|D^\mu \phi|^2 + (\lambda^2 - 1)|\phi|^2 \Big) \tag{5.53}$$

with $D_\mu = \partial_\mu + A_\mu$. To compute the correlator $\langle \mathcal{O}(z_1) \overline{\mathcal{O}}(z_2) J^{(1)}(z_3) \rangle$ we proceed as follows. We insert delta function sources at z_2 and z_3 by imposing the following asymptotic behavior on the scalar and gauge field[27]

$$\widehat{\phi}(\rho, z) \sim \mu_\phi \delta^{(2)}(z - z_2) e^{-(1-\lambda)\rho} \ , \quad \widehat{A}_{\overline{z}}(\rho, z) \sim \mu_A \delta^{(2)}(z - z_3) \ , \quad \rho \to \infty \tag{5.54}$$

This form for the gauge field corresponds to a source for the boundary current, as explained in [89,90,61]. We then need to find the order $\mu_\phi \mu_A$ contribution to the vev $\mathcal{O}(z_1)$, which also can be read off from the scalar field asymptotics,

$$\widehat{\phi}(\rho, z) \sim \frac{\mathcal{O}(z)}{B_\phi} e^{-(1+\lambda)\phi} \ , \quad \rho \to \infty, \quad z \neq z_{2,3} \tag{5.55}$$

We will keep the constant B_ϕ unspecified, though we note that a consistent holographic dictionary fixes $B_\phi = 2\lambda$ (for $\lambda \neq 0$) [91]. The three-point function is then given by

$$\mathcal{O}(z_1) = \mu_\phi \mu_A \langle \mathcal{O}(z_1) \overline{\mathcal{O}}(z_z) J^{(1)}(z_3) \rangle + \cdots \tag{5.56}$$

[27] The reason for the hats will be clear momentarily. Also, note here that we are using standard quantization for the scalar.

where $\cdots$ denote terms of other order in $\mu_{\phi,A}$.

Since the gauge field has no propagating degrees of freedom we can generate the required solution by a gauge transformation. In particular, we start by solving for the scalar field with $A = 0$, using the bulk-boundary propagator

$$\phi(\rho, z) = \int d^2 z' G_{b\partial}(\rho, z; z')\phi_-(z') \tag{5.57}$$

This solution corresponds to an arbitrary source $\phi_-(z)$. To generate the desired gauge field solution we apply a gauge transformation

$$A_\mu = \partial_\mu \Lambda \ , \quad \Lambda(z) = \frac{\mu_A}{2\pi} \frac{1}{z - z_3} \tag{5.58}$$

where we use the formula $\partial_{\overline{z}} \left(\frac{1}{z}\right) = 2\pi\delta^{(2)}(z)$. The gauge transformation acts on the scalar field as

$$\begin{aligned} \phi(\rho, z) \to \widehat{\phi}(\rho, z) &= (1 - \Lambda(z))\phi(\rho, z) \\ &= (1 - \Lambda(z)) \int d^2 z' G_{b\partial}(\rho, z; z_2)\phi_-(z') \end{aligned} \tag{5.59}$$

The leading asymptotic behavior of the transformed scalar solution is

$$\widehat{\phi}(\rho, z) \sim (1 - \Lambda(z))\phi_-(z)e^{-(1-\lambda)\rho} \ , \quad \rho \to \infty \tag{5.60}$$

from which we read off the relation between the original and transformed sources

$$\widehat{\phi}_-(z) = (1 - \Lambda(z))\phi_-(z) \tag{5.61}$$

Inverting to first order in Λ, taking $\widehat{\phi}_-(z) = \mu_\phi \delta^{(2)}(z - z_2)$ gives us an expression for the original source

$$\phi_-(z) = \mu_\phi(1 + \Lambda(z))\delta^{(2)}(z - z_2) \tag{5.62}$$

We now insert this result in (5.59) and compute the asymptotic behavior for $z \neq z_{2,3}$. Retaining just the term of order $\mu_\phi \mu_A$, we find

$$\widehat{\phi}(\rho, z) \sim \frac{\lambda}{\pi}\mu_\phi \left(\frac{\Lambda(z_2)}{|z - z_2|^{2(1+\lambda)}} - \frac{\Lambda(z)}{|z - z_2|^{2(1+\lambda)}} \right) e^{-(1+\lambda)\rho} \ , \quad \rho \to \infty \tag{5.63}$$

from which we read off

$$\mathcal{O}(z_1) = \frac{\lambda\mu_\phi B_\phi}{\pi}\left(\frac{\Lambda(z_2)-\Lambda(z_1)}{|z_{12}|^{2(1+\lambda)}}\right) \tag{5.64}$$

Inserting the formula for Λ given in (5.58) and using (5.56) we arrive at

$$\begin{aligned}\langle\mathcal{O}(z_1)\overline{\mathcal{O}}(z_2)J^{(1)}(z_3)\rangle &= \frac{\lambda B_\phi}{2\pi^2}\left(\frac{z_{12}}{z_{13}z_{23}}\right)\frac{1}{|z_{12}|^{2(1+\lambda)}} \\ &= \frac{1}{2\pi}\left(\frac{z_{12}}{z_{13}z_{23}}\right)\langle\mathcal{O}(z_1)\overline{\mathcal{O}}(z_2)\rangle\end{aligned} \tag{5.65}$$

This simple example illustrates the power of our approach. Starting from the solution for a free scalar in AdS$_3$, as given by the bulk to boundary propagator, all we need to do is to perform a gauge transformation to generate a solution with the required asymptotic behavior of the gauge field. From this solution we read off the scalar vev, and thence the three-point function.

5.3.2. General spin correlators

We now establish an algorithm for computing $\langle\mathcal{O}_\pm(z_1)\overline{\mathcal{O}}_\pm(z_2)J^{(s)}(z_3)\rangle$ for arbitrary s, where we take $\mathcal{O}_\pm$ and its complex conjugate to be dual to the complex scalar field C^1_0 in either standard $(+)$ or alternate $(-)$ quantization. We will work in the standard quantization throughout and include the alternate quantization at the end by taking $\lambda \to -\lambda$. In addition, the case of the second complex bulk scalar $\widetilde{C}^1_0$, dual to an operator $\widetilde{\mathcal{O}}_\pm$ and its complex conjugate, will be read off afterwards.

Our starting point is the equation (5.22) which we reproduce here:

$$dC + A\star C - C\star\overline{A} = 0 \tag{5.66}$$

This equation is invariant under the hs[λ]$\oplus$hs[λ] gauge invariance

$$\begin{aligned}&A \to A + d\Lambda + [A,\Lambda]_\star \\ &\overline{A} \to \overline{A} + d\overline{\Lambda} + [\overline{A},\overline{\Lambda}]_\star \\ &C \to C + C\star\overline{\Lambda} - \Lambda\star C\end{aligned} \tag{5.67}$$

Starting from AdS, we introduce a spin-s source in the unbarred sector by performing a chiral gauge transformation with parameter

$$\Lambda(\rho, z) = \sum_{n=1}^{2s-1} \frac{1}{(n-1)!}(-\partial)^{n-1}\Lambda^{(s)}(z)e^{(s-n)\rho}V^s_{s-n} \tag{5.68}$$

This generates the desired source term in $A_{\bar{z}}$,

$$\delta A_{\bar{z}} = \partial_{\bar{z}}\Lambda^{(s)}e^{(s-1)\rho}V^s_{s-1} + \cdots \tag{5.69}$$

and a conjugate current in A_z,

$$\delta A_z = \frac{1}{(2s-2)!}\partial^{2s-1}\Lambda^{(s)}e^{-(s-1)\rho}V^s_{-(s-1)} \tag{5.70}$$

Isolating the lowest component of the transformed C field using (5.67), which we denote with a hat, gives

$$\widehat{C}^1_0 = C^1_0 + (\delta C)^1_0 = C^1_0 - (\Lambda \star C)^1_0 \tag{5.71}$$

Using (5.68) and (5.24) we can compute $(\delta C)^1_0$:

$$(\delta C)^1_0 = -\sum_{n=1}^{2s-1} \frac{1}{(n-1)!}(-\partial)^{n-1}\Lambda^{(s)} \cdot \frac{1}{2}g^{ss}_{2s-1}(s-n, n-s)C^s_{-(s-n)}e^{(s-n)\rho} \tag{5.72}$$

This depends on an arbitrary component $C^s_{-(s-n)}$ of the master field C in AdS. As discussed previously, these are all fixed on-shell in terms of C^1_0 and its derivatives. Once we write $(\delta C)^1_0$ in terms of the AdS scalar C^1_0 to obtain an expression analogous to (5.59), the remaining work follows the spin-1 example of subsection 4.1.

To organize the calculation, we first derive a general formula for the correlator, delaying presentation of the explicit formulae for $C^s_{-(s-n)}$ to the next subsection. We can rewrite (5.72) as

$$(\delta C)^1_0 = D^{(s)}C^1_0 \tag{5.73}$$

for some s-dependent differential operator $D^{(s)}$ which contains derivatives $(\partial, \partial_\rho)$. Substitution for the $C^s_{-(s-n)}$ will reveal[28] that in the sum (5.72), only the terms for which $n \leq s$ will be needed for our computation: these have no ρ-dependence, while the $n > s$ terms decay at the AdS boundary. This will imply that $D^{(s)}$ is of order $s-1$ in derivatives ∂, so we can decompose $D^{(s)}$ as

$$D^{(s)} = \sum_{n=1}^{s} f^{s,n}(\lambda, \partial_\rho)\partial^{n-1}\Lambda^{(s)}\partial^{s-n} \tag{5.74}$$

All of the nontrivial information about the higher spin deformation is hidden in the functions $f^{s,n}(\lambda, \partial_\rho)$.

We now switch to the notation

$$C^1_0 \equiv \phi \tag{5.75}$$

The leading asymptotic behavior of the transformed scalar is

$$\widehat{\phi}(\rho, z) \sim (1 + D^{(s)})e^{-(1-\lambda)\rho}\phi_-(z) \ , \quad \rho \to \infty \tag{5.76}$$

To move the $D^{(s)}$ through the ρ-dependent prefactor, we define

$$D^{(s)}_\pm \equiv D^{(s)}(\partial_\rho \to -(1 \pm \lambda)) \tag{5.77}$$

and likewise for $f^{s,n}_\pm(\lambda)$. Then setting the transformed source $\widehat{\phi}_-(z) = \mu_\phi \delta^{(2)}(z - z_2)$ and inverting to linear order,

$$\phi_-(z) = \mu_\phi(1 - D^{(s)}_-)\delta^{(2)}(z - z_2) \tag{5.78}$$

The asymptotic behavior of the transformed scalar for $z \neq z_2$ now reads, omitting the leading part that is local in the higher spin source,

$$\widehat{\phi}_+(z) = \frac{\lambda\mu_\phi}{\pi}\int d^2z'(1+D^{(s)}_+(z))\frac{(1 - D^{(s)}_-(z'))\delta^{(2)}(z' - z_2)}{|z - z'|^{2(1+\lambda)}} \ , \quad \rho \to \infty \tag{5.79}$$

[28] See equations 4.40, 4.42.

It is now important to know what coordinates derivatives are acting on, so we re-institute the subscript on derivatives: $\partial \to \partial_z$. Isolating the piece of order $\mu_\phi D^{(s)}$ and placing our scalar operator at the boundary point $z = z_1$, we have

$$\widehat{\phi}_+(z_1) = \frac{\lambda \mu_\phi}{\pi} \left[D_+^{(s)}(z_1) \cdot \frac{1}{|z_{12}|^{2(1+\lambda)}} - \int d^2z' \frac{D_-^{(s)}(z')\delta^{(2)}(z'-z_2)}{|z_1 - z'|^{2(1+\lambda)}} \right] \tag{5.80}$$

We want to integrate the second piece by parts, which was the point of the definition (5.74). Writing the integral as

$$\int d^2z' \frac{D_-(z')\delta^{(2)}(z'-z_2)}{|z_1-z'|^{2(1+\lambda)}} = \sum_{n=1}^{s} f_-^{s,n}(\lambda) \int d^2z' \frac{\partial_{z'}^{n-1} a \partial_{z'}^{s-n} \delta^{(2)}(z'-z_2)}{|z_1-z'|^{2(1+\lambda)}} \tag{5.81}$$

integrating by parts and ignoring boundary terms, the n'th term is

$$\begin{aligned} &(-1)^{s-n} f_-^{s,n}(\lambda) \partial_{z_2}^{s-n} \left(\frac{\partial_{z_2}^{n-1} \Lambda^{(s)}}{|z_{12}|^{2(1+\lambda)}} \right) \\ =&(-1)^{s-n} f_-^{s,n}(\lambda) \sum_{j=0}^{s-n} \binom{s-n}{j} [\partial_{z_2}^{n-1+j} \Lambda^{(s)}] \partial_{z_2}^{s-n-j} \frac{1}{|z_{12}|^{2(1+\lambda)}} \end{aligned} \tag{5.82}$$

Making use of the identity,

$$\begin{aligned} \partial_{z'}^n \frac{1}{|z_1-z'|^{2(1+\lambda)}} &= (-1)^n \partial_{z_1}^n \frac{1}{|z_1-z'|^{2(1+\lambda)}} \\ &= \frac{\Gamma(\lambda+n+1)}{\Gamma(\lambda+1)} \frac{1}{(z_1-z')^n} \frac{1}{|z_1-z'|^{2(1+\lambda)}} \end{aligned} \tag{5.83}$$

one can reduce (5.80) to the following expression in terms of the $f_\pm^{s,n}(\lambda)$ and the transformation parameter $\Lambda^{(s)}$:

$$\begin{aligned} \widehat{\phi}_+(z_1) = \frac{\lambda \mu_\phi}{\pi |z_{12}|^{2(1+\lambda)}} \Bigg[\sum_{n=1}^{s} \left(\frac{-1}{z_{12}} \right)^{s-n} \Bigg\{ & f_+^{s,n}(\lambda) \frac{\Gamma(\lambda+s-n+1)}{\Gamma(\lambda+1)} \partial_{z_1}^{n-1} \Lambda^{(s)} \\ - f_-^{s,n}(\lambda) \sum_{j=0}^{s-n} \binom{s-n}{j} & \frac{\Gamma(\lambda+s-n-j+1)}{\Gamma(\lambda+1)} (\partial_{z_2}^{n-1+j} \Lambda^{(s)}) z_{12}^j \Bigg\} \Bigg] \end{aligned} \tag{5.84}$$

(5.84) is the general expression for the scalar vev in the presence of a higher spin perturbation, to first order in each of the scalar and higher spin sources. For a higher spin delta function source at z_3, we take (cf. (5.69))

$$\Lambda^{(s)} = \frac{1}{2\pi} \frac{1}{z - z_3} \tag{5.85}$$

and (5.84) becomes

$$\begin{aligned}\widehat{\phi}_+(z_1) = \frac{(-1)^{s-1}\lambda\mu_\phi}{2\pi^2|z_{12}|^{2(1+\lambda)}} \sum_{n=1}^{s} \frac{1}{z_{12}^{s-n}} \Bigg\{ & f_+^{s,n}(\lambda) \frac{\Gamma(\lambda+s-n+1)}{\Gamma(\lambda+1)} (n-1)! \frac{1}{z_{13}^n} \\ & - f_-^{s,n}(\lambda) \frac{1}{z_{23}^n} \sum_{j=0}^{s-n} (-1)^j \binom{s-n}{j} \frac{\Gamma(\lambda+s-n-j+1)}{\Gamma(\lambda+1)} (n-1+j)! \left(\frac{z_{12}}{z_{23}}\right)^j \Bigg\}\end{aligned} \tag{5.86}$$

Therefore, reading off the three-point function $\langle \mathcal{O}_+(z_1)\overline{\mathcal{O}}_+(z_2)J^{(s)}(z_3)\rangle$ boils down to knowing the functions $f_\pm^{s,n}(\lambda)$ encoding the change in the scalar under gauge transformation.

Before turning to the problem of determining these functions $f_\pm^{s,n}(\lambda)$, we note that a conformally symmetric result has the property

$$\langle \mathcal{O}(z_1)\overline{\mathcal{O}}(z_2)J^{(s)}(z_3)\rangle = (-1)^s \langle \mathcal{O}(z_2)\overline{\mathcal{O}}(z_1)J^{(s)}(z_3)\rangle \tag{5.87}$$

This property is not manifest in our formula (5.86), but it implies that, resorting to the notation of (5.80), the correct solution is given by

$$\widehat{\phi}_+(z_1) = \frac{\lambda\mu_\phi}{\pi} \left[D_+^{(s)}(z_1) + (-1)^s D_+^{(s)}(z_2) \right] \cdot \frac{1}{|z_{12}|^{2(1+\lambda)}} \tag{5.88}$$

In terms of the $f_\pm^{s,n}(\lambda)$ this looks like

$$\widehat{\phi}_+(z_1) = \frac{(-1)^{s-1}\lambda\mu_\phi}{2\pi^2|z_{12}|^{2(1+\lambda)}} \sum_{n=1}^{s} \frac{f_+^{s,n}(\lambda)}{z_{12}^{s-n}} \frac{\Gamma(\lambda+s-n+1)}{\Gamma(\lambda+1)} (n-1)! \left(\frac{1}{z_{13}^n} + \frac{(-1)^n}{z_{23}^n} \right) \tag{5.89}$$

where the extra $(-1)^n$ comes from sign changes under derivatives acting on $\frac{1}{|z_{12}|}$ (cf. (5.83)). One can show, by using (5.86) and (5.89) and equating terms with the same number of powers of z_{12}, that this in turn implies the following unobvious relations:

$$f_{+}^{s,j}(\lambda) = -\sum_{n=1}^{s}(-1)^{n} f_{-}^{s,n}(\lambda)\binom{s-n}{j-n} \tag{5.90}$$

for some fixed j.

Upon solving for the $f_{\pm}^{s,n}(\lambda)$ generated by the gauge transformation (5.68), we will show that (5.90) is indeed satisfied.

5.3.3. The AdS master field C

To write the $f_{\pm}^{s,n}(\lambda)$ as defined by (5.72), (5.73) and (5.74), we need a formula for all components of the master field C in AdS, written in terms of $C_0^1 \equiv \phi$. This amounts to solving (5.31). We first solve for the minimal components $C_{\pm m}^{m+1}$ in terms of ϕ, then for the non-minimal components $C_{\pm m}^{s\neq m+1}$ in terms of the $C_{\pm m}^{m+1}$, and finally we put the two together.

Minimal components $C_{\pm m}^{m+1}$:

Taking $m \to -m$ and $s = m+1$ in the second equation in (5.31) gives

$$\partial_z C_{-m}^{m+1} + \frac{e^{\rho}}{2} g_3^{2(m+2)}(1,-m-1) C_{-m-1}^{m+2} = 0 \tag{5.91}$$

where we recall that for some component C_n^s one needs $|n| \leq s-1$. Solving recursively yields the following expression:

$$C_{-m}^{m+1} = \left(\prod_{p=2}^{m+1} g_3^{2p}(1,1-p)\right)^{-1} (-2e^{-\rho}\partial_z)^m \phi \tag{5.92}$$

A similar analysis along $\bar{z}$ yields

$$C_{m}^{m+1} = \left(\prod_{p=2}^{m+1} g_3^{2p}(-1,p-1)\right)^{-1} (2e^{-\rho}\partial_{\bar{z}})^m \phi \tag{5.93}$$

where we have used the fact that $g_3^{st}(m,n) = g_3^{ts}(n,m)$.

Non-minimal components $C^{s\neq m+1}_{\pm m}$:

From the ρ equations in (5.31), one can see that, say, the component C^2_0 will have the same structure, when written in terms of ϕ, as C^3_1 in terms of C^2_1, and so on. In general, components with fixed $s-|m|$ when expressed as a function of their respective minimal components will have the same form in terms of structure constants.

The solution is

$$\begin{aligned} C^s_{\pm m} &= (-1)^{s-1-m}\left(\prod_{p=2+m}^{s} g_3^{p2}(m,0)\right)^{-1} \\ &\times\left[\sum_{\alpha=0}^{\lfloor\frac{s-1-m}{2}\rfloor} A_\alpha(s,m)\partial_\rho^{s-2\alpha-m-1}\right] C^{m+1}_{\pm m} \end{aligned} \tag{5.94}$$

The $A_\alpha(s,m)$ are defined as

$$A_\alpha(s,m) = (-2)^\alpha \sum_{i_1\ldots i_\alpha}\prod_{k=1}^{\alpha} g_3^{i_k 2}(m,0) \tag{5.95}$$

with indices subject to

$$\begin{aligned} &2k+m\leq i_k\leq 2k+s-1-2\alpha \\ &i_k\geq i_{k-1}+2\ , \quad \forall\ k\geq 2 \end{aligned} \tag{5.96}$$

In appendix 5B, we present expressions for several of the C^s_0 obtained from solving (5.31) directly to facilitate easy comparison with (5.94).

Via (5.92), (5.93) and (5.94), one has all components of C in terms of the fundamental scalar field, ϕ. These formulae also justify our previous statements about subleading terms, and about $D^{(s)}$ being order $s-1$ in derivatives ∂.

With these results in hand, we combine (5.92) and (5.94) at $m=s-n$ to write $C^s_{-(s-n)}$ in terms of ϕ, and plug into (5.72). The result can be written

$$\begin{aligned} (\delta C)^1_0 &= \sum_{n=1}^{s} f^{s,n}_\pm(\lambda)\partial_z^{n-1}\Lambda^{(s)}\partial_z^{s-n}\phi\ +(\text{subleading}) \\ &= \sum_{n=1}^{s}(-1)^s\frac{2^{s-n-1}}{(n-1)!}\mathcal{F}_\pm(s,s-n;\lambda)\partial_z^{n-1}\Lambda^{(s)}\partial_z^{s-n}\phi \end{aligned} \tag{5.97}$$

where

$$\mathcal{F}_{\pm}(s,s-n;\lambda) = g^{ss}_{2s-1}(s-n,n-s) \\ \times \left(\prod_{p=2+s-n}^{s} g^{p2}_{3}(s-n,0)\right)^{-1} \left(\prod_{p=2}^{s-n+1} g^{p2}_{3}(1-p,1)\right)^{-1} \\ \times \left[\sum_{\alpha=0}^{\lfloor\frac{n-1}{2}\rfloor} A_{\alpha}(s,s-n)(\pm\lambda+s-n+1)^{n-1-2\alpha}\right] \tag{5.98}$$

with $A_{\alpha}(s,s-n)$ defined as in (5.95) and (5.96). We have dropped the sub-leading terms in the second line of (5.97). To obtain this result we have used the replacement $\partial_{\rho} \to -(1\pm\lambda)$ and the identity

$$\sum_{\chi=0}^{n-1-2\alpha} \binom{n-1-2\alpha}{\chi} (s-n)^{n-1-2\alpha-\chi}(1\pm\lambda)^{\chi} = (\pm\lambda+s-n+1)^{n-1-2\alpha} \tag{5.99}$$

From (5.97), we easily read off the reduced expression for the functions $f^{s,n}_{\pm}(\lambda)$:

$$f^{s,n}_{\pm}(\lambda) = (-1)^{s}\frac{2^{s-n-1}}{(n-1)!}\mathcal{F}_{\pm}(s,s-n;\lambda) \tag{5.100}$$

We can now plug this into our general formula (5.86) and read off the correlator. If we are to obtain a conformally invariant result, it must also satisfy the conformal identity (5.90), in which case formula (5.89) is equally valid.

Of course, (5.100) is a rather complicated function when expressed in terms of structure constants, and combined with (5.89) it is not at all clear that we will arrive at our desired result. Accordingly we expect dramatic simplification of the $f^{s,n}_{\pm}(\lambda)$ when the structure constants are written out explicitly.

5.3.4. Final result

We set out to compute the $f^{s,n}_{\pm}(\lambda)$ for low values of n, using the formulae (5.98), (5.100). The results up to $s=8$ are:

$$
\begin{aligned}
f_{\pm}^{s,1}(\lambda) &= (-1)^s \\
f_{\pm}^{s,2}(\lambda) &= \frac{(-1)^s}{2}\frac{\Gamma(s\pm\lambda)}{\Gamma(s-1\pm\lambda)} \\
f_{\pm}^{s,3}(\lambda) &= \frac{(-1)^s}{4}\frac{s-2}{2s-3}\frac{\Gamma(s\pm\lambda)}{\Gamma(s-2\pm\lambda)} \\
f_{\pm}^{s,4}(\lambda) &= \frac{(-1)^s}{24}\frac{s-3}{2s-3}\frac{\Gamma(s\pm\lambda)}{\Gamma(s-3\pm\lambda)} \\
f_{\pm}^{s,5}(\lambda) &= \frac{(-1)^s}{96}\frac{(s-3)(s-4)}{(2s-3)(2s-5)}\frac{\Gamma(s\pm\lambda)}{\Gamma(s-4\pm\lambda)} \\
f_{\pm}^{s,6}(\lambda) &= \frac{(-1)^s}{960}\frac{(s-4)(s-5)}{(2s-3)(2s-5)}\frac{\Gamma(s\pm\lambda)}{\Gamma(s-5\pm\lambda)} \\
f_{\pm}^{s,7}(\lambda) &= \frac{(-1)^s}{5760}\frac{(s-4)(s-5)(s-6)}{(2s-3)(2s-5)(2s-7)}\frac{\Gamma(s\pm\lambda)}{\Gamma(s-6\pm\lambda)} \\
f_{\pm}^{s,8}(\lambda) &= \frac{(-1)^s}{80640}\frac{(s-5)(s-6)(s-7)}{(2s-3)(2s-5)(2s-7)}\frac{\Gamma(s\pm\lambda)}{\Gamma(s-7\pm\lambda)}
\end{aligned}
\tag{5.101}
$$

Indeed, these are compact expressions. By induction, we obtain a tidy formula for these functions:

$$
f_{\pm}^{s,n}(\lambda) = (-1)^s\frac{\Gamma(s\pm\lambda)}{\Gamma(s-n+1\pm\lambda)}\frac{1}{2^{\lfloor\frac{n}{2}\rfloor}(n-1)!}\prod_{j=1}^{\lfloor\frac{n-1}{2}\rfloor}\frac{s+j-n}{2s-2j-1} \tag{5.102}
$$

Encouragingly, one can check spin-by-spin that this expression indeed satisfies the conformal identity (5.90). Plugging this into (5.89) and including the alternate quantization $\lambda\to-\lambda$ then yields the final answer for the correlator:

$$
\langle\mathcal{O}_{\pm}(z_1)\overline{\mathcal{O}}_{\pm}(z_2)J^{(s)}(z_3)\rangle = \frac{(-1)^{s-1}B_\phi}{2\pi^2|z_{12}|^{2(1\pm\lambda)}}\frac{\Gamma(s)^2}{\Gamma(2s-1)}\frac{\Gamma(s\pm\lambda)}{\Gamma(\pm\lambda)}\left(\frac{z_{12}}{z_{13}z_{23}}\right)^s \tag{5.103}
$$

That is,

$$
\frac{\langle\mathcal{O}_{\pm}(z_1)\overline{\mathcal{O}}_{\pm}(z_2)J^{(s)}(z_3)\rangle}{\langle\mathcal{O}_{\pm}(z_1)\overline{\mathcal{O}}_{\pm}(z_2)\rangle} = \frac{(-1)^{s-1}}{2\pi}\frac{\Gamma(s)^2}{\Gamma(2s-1)}\frac{\Gamma(s\pm\lambda)}{\Gamma(1\pm\lambda)}\left(\frac{z_{12}}{z_{13}z_{23}}\right)^s \tag{5.104}
$$

This is manifestly conformally invariant, and the overall coefficient is the main result.

While we lack a rigorous proof of (5.102) and (5.103), we have confirmed (5.102) up to $n = 13$; assuming (5.102), it is simple to confirm (5.103) at any desired spin.

Recalling the discussion at the end of section 2, there exists a second projection of the master field C that gives rise to the equation

$$d\widetilde{C} + \overline{A} \star \widetilde{C} - \widetilde{C} \star A = 0 \tag{5.105}$$

The lowest component of $\widetilde{C}$ is also a complex scalar field, dual to a scalar CFT operator $\widetilde{\mathcal{O}}$ and its complex conjugate. This equation is simply related to (5.66) by the exchange $A \leftrightarrow \overline{A}$, which flips the sign of all odd spin tensors. Therefore, the result for the tilded correlator is simply our result (5.103) with a $(-1)^s$ removed:

$$\frac{\langle \widetilde{\mathcal{O}}_{\pm}(z_1)\overline{\widetilde{\mathcal{O}}}_{\pm}(z_2)J^{(s)}(z_3)\rangle}{\langle \widetilde{\mathcal{O}}_{\pm}(z_1)\overline{\widetilde{\mathcal{O}}}_{\pm}(z_2)\rangle} = -\frac{1}{2\pi}\frac{\Gamma(s)^2}{\Gamma(2s-1)}\frac{\Gamma(s\pm\lambda)}{\Gamma(1\pm\lambda)}\left(\frac{z_{12}}{z_{13}z_{23}}\right)^s \tag{5.106}$$

In the context of the W_N coset CFT duality conjecture, one scalar is in standard quantization and the other in alternate quantization. The operation of flipping the sign of odd spin fields corresponds to a charge conjugation operation in the CFT.

These results match and extend the bulk calculations of [39] which were restricted to $\lambda = 1/2$. To further compare to the CFT, we now compute the same correlators using CFT considerations, again for all λ and s, and find perfect agreement with the bulk.

5.4. Three-point correlators from CFT

We now shift focus and consider the constraints on three-point functions due to the existence of a higher-spin current algebra. Our considerations will be entirely based on symmetry, and in particular on the existence of $\mathcal{W}_\infty[\lambda]$

current algebra. If these currents are not present in the CFT, then even before considering scalar operators there will be a mismatch between bulk and boundary correlators involving only currents. So we will assume the existence of this symmetry algebra, and then see what constraints this imposes on the scalar-scalar-current three-point functions. Our computations in this section are along the same lines as in [36], but generalized to arbitrary s.

Suppose we have a spin-s current $J^{(s)}(z)$, and scalar primary operator $\mathcal{O}(z,\overline{z})$,[29] whose OPE has the following leading singularity,

$$J^{(s)}(z)\mathcal{O}(0) \sim \frac{A(s)}{z^s}\mathcal{O}(0) + \cdots \tag{5.107}$$

The three-point function is then related to the scalar two-point function as

$$\langle \mathcal{O}(z_1)\overline{\mathcal{O}}(z_2)J^{(s)}(z_3)\rangle = A(s)\left(\frac{z_{12}}{z_{13}z_{23}}\right)^s \langle \mathcal{O}(z_1)\overline{\mathcal{O}}(z_2)\rangle \tag{5.108}$$

Writing the mode expansion

$$J^{(s)}(z) = -\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\frac{J_m^{(s)}}{z^{m+s}} \tag{5.109}$$

the zero modes act on the primary state $|\mathcal{O}\rangle$ as

$$J_0^{(s)}|\mathcal{O}\rangle = -2\pi A(s)|\mathcal{O}\rangle \tag{5.110}$$

In the case of interest, the currents $J^{(s)}(z)$, $s = 2, 3, \ldots$, obey the $\mathcal{W}_\infty[\lambda]$ current algebra. The wedge modes are defined to be those that annihilate the vacuum state, and are given by

$$V_m^s = J_m^{(s)}\ , \quad |m| < s \tag{5.111}$$

In general, these wedge modes do not yield a closed subalgebra of $\mathcal{W}_\infty[\lambda]$, as their commutators yield modes outside the wedge; in particular, this is

[29] We use the shorthand $\mathcal{O}(z,\overline{z}) = \mathcal{O}(z)$ in what follows.

due to the nonlinearities present in $\mathcal{W}_\infty[\lambda]$. However, the nonlinear terms are suppressed at large central charge, and so in the limit $c \to \infty$ the wedge modes do define a subalgebra of $\mathcal{W}_\infty[\lambda]$ – the wedge subalgebra. This subalgebra is hs$[\lambda]$. To exploit this simplification, for the remainder of this section we will assume that we are working in the limit of large central charge. See [36] for further discussion.

Furthermore, these considerations fix the relative normalization of the bulk and boundary currents. In particular, we have defined our conventions such that the currents (5.109) are equal to the currents derived from the bulk.

Acting on $|\mathcal{O}\rangle$ with the wedge modes, we obtain a representation of hs$[\lambda]$, and we can therefore use the representation theory of hs$[\lambda]$ to determine the zero mode eigenvalues appearing in (5.110), and thence the three point function (5.108). The Virasoro zero mode eigenvalue is fixed by the scaling dimension of $\mathcal{O}$,

$$V_0^2|\mathcal{O}_\pm\rangle = \frac{1}{2}(1 \pm \lambda)|\mathcal{O}_\pm\rangle \tag{5.112}$$

where we've now introduced the pair of scalar operators $\mathcal{O}_\pm$. We need to compute the remaining eigenvalues.

To proceed, it is useful to build up the hs$[\lambda]$ generators in terms of SL(2) generators. We write $V_1^2 = L_1, V_0^2 = L_0, V_{-1}^2 = L_{-1}$, which obey the SL(2) algebra

$$[L_1, L_{-1}] = 2L_0 \ , \quad [L_0, L_1] = -L_1 \ , \quad [L_0, L_{-1}] = L_{-1} \tag{5.113}$$

We then construct the V_m^s generators as[30]

$$V_m^s = (-1)^{m-1}\frac{(m+s-1)!}{(2s-2)!}\underbrace{[L_{-1}, \ldots [L_{-1}, [L_{-1},}_{s-1-m} L_1^{s-1}]]] \tag{5.114}$$

[30] As will be discussed at the end of this section, we obtain a second inequivalent representation by appending a factor of $(-1)^s$ to the generators.

where in addition we mod out by the ideal obtained by fixing the SL(2) quadratic Casimir as

$$C_2 = L_0^2 - \frac{1}{2}(L_1 L_{-1} + L_{-1} L_1) = \frac{1}{4}(\lambda^2 - 1) \tag{5.115}$$

We can use this to work out the zero mode eigenvalues, as illustrated by the first nontrivial case at $s = 3$, where V_0^3 acts on $|\mathcal{O}_\pm\rangle$:

$$V_0^3 = -\frac{1}{12}[L_{-1}, [L_{-1}, L_1^2]] = \left(\frac{1}{3}C_2 - L_0^2\right) = -\frac{1}{6}(\lambda \pm 2)(\lambda \pm 1) \tag{5.116}$$

In the last step we used (5.112) and (5.114). Working out further examples by brute force quickly gets tedious, so we adopt a more indirect approach.

First, it's easy to see that V_0^s will be a polynomial in λ of degree $s-1$, as illustrated in (5.116). This follows, since the terms in (5.114) obtained after working out all the commutators will each have $s-1$ generators, and using either (5.115) or $L_0 = \frac{1}{2}(1 \pm \lambda)$ will convert a generator into at most one power of λ.

Next consider taking $\lambda = N$, a positive integer. In this case, after factoring out an ideal, hs[λ] becomes SL(N), which we can represent in terms of $N \times N$ matrices. In this representation the generators V_m^s with $s > N$ all vanish identically when they are constructed using (5.114), and in particular this holds for the zero modes V_0^s. We also note that the eigenvalues of $L_0 = V_0^2$ in the $N \times N$ matrix representation are: $-\frac{N-1}{2}, -\frac{N-1}{2} + 1, \ldots, \frac{N-1}{2}$. Note that the smallest eigenvalue coincides with $\frac{1}{2}(1-\lambda)$, i.e. with the eigenvalue of L_0 acting on $|\mathcal{O}_-\rangle$. Together, these facts imply that $V_0^s|\mathcal{O}_-\rangle = 0$ for $\lambda = 1, 2, \ldots, s-1$. Combining this with the statement in the previous paragraph, we fix the λ dependence of the zero-mode eigenvalues to be

$$\begin{aligned} V_0^s|\mathcal{O}_-\rangle &= N(s)[\lambda - (s-1)] \cdots [\lambda - 2][\lambda - 1]|\mathcal{O}_-\rangle \\ &= N(s)(-1)^{s-1}\frac{\Gamma(s-\lambda)}{\Gamma(1-\lambda)}|\mathcal{O}_-\rangle \end{aligned} \tag{5.117}$$

for some prefactor $N(s)$. To obtain the eigenvalues for $|\mathcal{O}_+\rangle$ we simply flip the sign of λ, and we arrive at

$$V_0^s|\mathcal{O}_\pm\rangle = N(s)(-1)^{s-1}\frac{\Gamma(s\pm\lambda)}{\Gamma(1\pm\lambda)}|\mathcal{O}_\pm\rangle \tag{5.118}$$

To fix $N(s)$ we can pick some convenient value of λ. If we take $\lambda = 1/2$ then we can represent SL(2) as

$$L_0 = -\frac{1}{4}(x\partial_x + \partial_x x)\ , \quad L_{-1} = \frac{1}{2}\partial_x^2\ , \quad L_1 = \frac{1}{2}x^2 \tag{5.119}$$

We can then construct hs$[\frac{1}{2}]$ using (5.114).

Now, at $\lambda = 1/2$ we have $L_0|\mathcal{O}_+\rangle = \frac{3}{4}|\mathcal{O}_+\rangle$, which in the representation (5.119) is achieved by taking L_0 to act on the function x^{-2}. As shown in appendix 5A the eigenvalues of the other zero mode generators are then computed to be

$$V_0^s x^{-2} = \frac{(-1)^s[(s-1)!]^2(2s-1)!!}{2^{s-1}(2s-2)!}x^{-2} \tag{5.120}$$

Comparing this with the $\mathcal{O}_+$ version of (5.118) at $\lambda = 1/2$ yields

$$N(s) = -\frac{\Gamma(s)^2}{\Gamma(2s-1)} \tag{5.121}$$

which gives us our final result for the zero mode eigenvalues

$$V_0^s|\mathcal{O}_\pm\rangle = (-1)^s\frac{\Gamma(s)^2}{\Gamma(2s-1)}\frac{\Gamma(s\pm\lambda)}{\Gamma(1\pm\lambda)}|\mathcal{O}_\pm\rangle \tag{5.122}$$

Using (5.107) and (5.110), we have therefore fixed the three-point function to be

$$\frac{\langle\mathcal{O}_\pm(z_1)\overline{\mathcal{O}}_\pm(z_2)J^{(s)}(z_3)\rangle}{\langle\mathcal{O}_\pm(z_1)\overline{\mathcal{O}}_\pm(z_2)\rangle} = \frac{(-1)^{s-1}}{2\pi}\frac{\Gamma(s)^2}{\Gamma(2s-1)}\frac{\Gamma(s\pm\lambda)}{\Gamma(1\pm\lambda)}\left(\frac{z_{12}}{z_{13}z_{23}}\right)^s \tag{5.123}$$

This agrees perfectly with the correlator obtained from the bulk, (5.103).

Now let us come back to the footnote accompanying (5.114). hs$[\lambda]$ admits the automorphism $V_m^s \to (-1)^s V_m^s$, which can be thought of as charge conjugation. If we had instead used the charge conjugate generators in (5.114) then a factor of $(-1)^s$ would have propagated through to the final result (5.123). Equivalently, starting from $\mathcal{O}_\pm$ we can consider operators $\widetilde{\mathcal{O}}_\pm$ that transform as their charge conjugates. The three-point function of such operators is therefore

$$\frac{\langle \widetilde{\mathcal{O}}_\pm(z_1)\overline{\widetilde{\mathcal{O}}}_\pm(z_2)J^{(s)}(z_3)\rangle}{\langle \widetilde{\mathcal{O}}_\pm(z_1)\overline{\widetilde{\mathcal{O}}}_\pm(z_2)\rangle} = -\frac{1}{2\pi}\frac{\Gamma(s)^2}{\Gamma(2s-1)}\frac{\Gamma(s\pm\lambda)}{\Gamma(1\pm\lambda)}\left(\frac{z_{12}}{z_{13}z_{23}}\right)^s \tag{5.124}$$

This agrees with the bulk result (5.106).

5.5. Discussion

5.5.1. Comparison with previous results

The results (5.123)-(5.124) derived from CFT considerations agree perfectly with the correlators derived from the higher spin theory in the bulk, namely (5.103) and (5.106). Before discussing the implications of this agreement, let us compare to previous work. To this end, we note that in our normalization the current-current two-point function is

$$\langle J^{(s)}(z)J^{(s)}(0)\rangle = \frac{3k}{2^{2s-1}\pi^{5/2}}\frac{\sin(\pi\lambda)}{\lambda(1-\lambda^2)}\frac{\Gamma(s)\Gamma(s-\lambda)\Gamma(s+\lambda)}{\Gamma(s-\frac{1}{2})}\frac{1}{z^{2s}} \tag{5.125}$$

The derivation of this result proceeds in the same fashion as led to (5.44). This normalization is the "natural" one, since the wedge modes defined in (5.109) then obey the hs[λ] algebra with standard normalization.

In [39] three-point correlators were computed from the bulk for arbitrary s and $\lambda = 1/2$; and in the 't Hooft limit of the W_N minimal model for $s = 3$ and arbitrary λ. Their currents are normalized to $\langle J^{(s)}(z)J^{(s)}(0)\rangle = z^{-2s}$. Under the identification $\mathcal{O}_+^{\text{here}} = \overline{\mathcal{O}}_+^{\text{CY}}$ and $\widetilde{\mathcal{O}}_-^{\text{here}} = \overline{\mathcal{O}}_-^{\text{CY}}$, and taking into account the different normalizations for the currents, we verify that our results reduce to those of [39] for the special values of s and λ.

In [60] the three-point function for $s = 4$ was computed in the 't Hooft limit of the W_N minimal model. The normalization of the current was not specified, but a normalization independent ratio was obtained (see equation 3.35 therein). The corresponding ratio obtained from our result is

$$\frac{\langle\mathcal{O}_+(z_1)\overline{\mathcal{O}}_+(z_2)J^{(4)}(z_3)\rangle}{\langle\widetilde{\mathcal{O}}_-(z_1)\overline{\widetilde{\mathcal{O}}}_-(z_2)J^{(4)}(z_3)\rangle}=\frac{(1+\lambda)(2+\lambda)(3+\lambda)}{(1-\lambda)(2-\lambda)(3-\lambda)} \tag{5.126}$$

which agrees with [60].

5.5.2. Comments

We now discuss the implications of our agreement between bulk and boundary. On the CFT side, what went into the computation was the assumption that the CFT has a symmetry algebra containing hs$[\lambda]$, along with scalar operators of the correct dimension; everything else followed from hs$[\lambda]$ representation theory. So any CFT with these properties will have three-point functions that match those of the bulk. One way that hs$[\lambda]$ symmetry can emerge is if the CFT has $\mathcal{W}_\infty[\lambda]$ symmetry, and the central charge is taken to infinity. Then hs$[\lambda]$ is identified with the wedge subalgebra of $\mathcal{W}_\infty[\lambda]$.

Now consider the case of the $\mathcal{W}_N$ minimal models proposed by Gaberdiel and Gopakumar as CFT duals of the bulk higher spin theory. As we have stressed, even before considering the scalars, it is necessary that in the 't Hooft limit the CFT acquire $\mathcal{W}_\infty[\lambda]$ symmetry if it is to have a chance of matching with the bulk. The bulk theory has such a symmetry, and this fixes the form of all correlation functions on the plane involving just currents. These will not match with the CFT unless the latter also has $\mathcal{W}_\infty[\lambda]$ symmetry.

Assuming that the CFT does indeed exhibit hs$[\lambda]$ symmetry, let's consider the scalars. The results of this paper establish that if the CFT has scalar operators of the correct dimension, $\Delta = 1 \pm \lambda$, then the scalar-scalar-current three-point functions on the plane will match between the bulk and boundary.

As an illustration of these comments, we now establish that a theory of free bosons has correlators that match those of the corresponding bulk theory.

Here by "correlators" we mean those discussed above: namely pure current correlators, and scalar-scalar-current correlators, all evaluated on the plane. Consider the following currents

$$J^{(s+2)} = -\frac{1}{2\pi}\frac{2^{-s-1}(s+2)!}{(2s+1)!!}\sum_{k=0}^{s}(-1)^k\frac{1}{s+1}\begin{pmatrix} s+1 \\ k \end{pmatrix}\begin{pmatrix} s+1 \\ k+1 \end{pmatrix}\partial^{s-k+1}\overline{\phi}\partial^{k+1}\phi \tag{5.127}$$

where ϕ is a complex free boson. These currents yield the linear algebra $\mathcal{W}_\infty^{\rm PRS}$ [57] at $c=2$ [81,85]; the generalization to higher c is obtained by introducing additional copies of the free boson. After a nonlinear redefinition of the currents, $\mathcal{W}_\infty^{\rm PRS}$ becomes equivalent to $\mathcal{W}_\infty[1]$ [92,36]. For $\lambda=1$ the bulk scalar is dual to a CFT operator of dimension 2, and this is $\mathcal{O}=\partial\phi\overline{\partial\phi}$. The results of this paper show that the scalar-scalar-current three-point functions of this free boson theory will match those of the higher spin theory in the bulk at $\lambda=1$. For instance, it is simple to check this explicitly for the case of the spin-3 current.

A related situation occurs with complex free fermions at $\lambda=0$. The following currents [85]

$$J^{(s+2)} = \frac{1}{2\pi}\frac{2^{-s-1}(s+1)!}{(2s+1)!!}\sum_{k=0}^{s+1}(-1)^k\begin{pmatrix} s+1 \\ k \end{pmatrix}^2\partial^{s-k+1}\overline{\psi}\partial^k\psi \tag{5.128}$$

realize the algebra $\mathcal{W}_{1+\infty}$ [84] at $c=1$. Although $\mathcal{W}_{1+\infty}$ is not equivalent to $\mathcal{W}_\infty[0]$ due to the presence of the spin-1 current in $\mathcal{W}_{1+\infty}$, the wedge subalgebra of $\mathcal{W}_{1+\infty}$ yields hs[0]; the spin-1 zero mode just yields an operator that commutes with all the other wedge modes. The scalar operator in this theory is $\mathcal{O}=\tilde{\psi}\psi$. As we have discussed, this is enough structure to guarantee that the scalar-scalar-current correlators (for spin greater than 1) will match those of the bulk theory at $\lambda=0$, and verifying this is straightforward.

Note that we are not making any claims here about a full duality between these free boson/fermion theories and their bulk counterparts. Indeed, without further ingredients it seems clear that the theories cannot be equivalent: if we simply add N copies of the free fields the CFT will have a $U(N)$ symmetry along with various nonsinglet operators, none of which appear to be present in the bulk, at least classically.

Appendix 5A. Derivation of (5.120)

Given

$$L_0 = -\frac{1}{4}(x\partial_x + \partial_x x) \ , \quad L_{-1} = \frac{1}{2}\partial_x^2 \ , \quad L_1 = \frac{1}{2}x^2 \tag{5A.1}$$

and

$$V_m^s = (-1)^{m-1}\frac{(m+s-1)!}{(2s-2)!}\underbrace{[L_{-1},\dots[L_{-1},[L_{-1},}_{s-1-m}L_1^{s-1}]]] \tag{5A.2}$$

we need to show

$$V_0^s x^{-2} = \frac{(-1)^s[(s-1)!]^2(2s-1)!!}{2^{s-1}(2s-2)!}x^{-2} \tag{5A.3}$$

We start from

$$e^{tL_{-1}}f(x)e^{-tL_{-1}} = f(x+t\partial_x) \tag{5A.4}$$

which follows by thinking of L_{-1} as the Hamiltonian for a free particle. In particular, this implies

$$e^{tL_{-1}}(L_1)^{s-1}e^{-tL_{-1}} = \frac{1}{2^{s-1}}(x+t\partial_x)^{2s-2} \tag{5A.5}$$

Expanding the left hand side in powers of t, the desired term yielding V_0^s is the t^{s-1} term. This term preserves the power of x, and so we can write

$$V_0^s x^{-2} = -\left[\frac{[(s-1)!]^2}{2^{s-1}(2s-2)!}(x+\partial_x)^{2s-2}x^{-2}\right]\Bigg|_{x^{-2}\ \text{term}} \tag{5A.6}$$

To extract the x^{-2} term on the right hand side we write a contour integral and integrate by parts,

$$(x+\partial_x)^{2s-2}x^{-2}\Big|_{\text{coeff of } x^{-2}} = \frac{1}{2\pi i}\oint dzz(z+\partial)^{2s-2}\frac{1}{z^2} = \frac{1}{2\pi i}\oint dz\frac{1}{z^2}(z-\partial)^{2s-2}z \tag{5A.7}$$

Now write $(z-\partial)^{2s-2}$ as the t^{2s-2} term in $e^{t(z-\partial)} = e^{-t^2/2}e^{tz}e^{-t\partial}$ and perform the integral. This yields

$$(x+\partial_x)^{2s-2}x^{-2}\Big|_{\text{coeff. of } x^{-2}} = (-1)^{s-1}(2s-1)!! \tag{5A.8}$$

Plugging this result into (5A.6) yields our desired formula (5A.3).

Appendix 5B. Low spin results: C_0^s in AdS

We present the explicit formulae for the components C_0^s of the master field in AdS, through $s = 8$, obtained by recursive solution of the $V_{0,\rho}^s$ equations in AdS:

$$\partial_\rho C_0^s + 2C_0^{s-1} + C_0^{s+1}g_3^{(s+1)2}(0,0) = 0 \tag{5B.1}$$

These can be compared to the spin s formula (5.94). We use the temporary notation

$$g_3^{s2}(0,0) \equiv g^s \tag{5B.2}$$

and the following expressions act on C_0^1:

$$
\begin{aligned}
C_0^2 &= -(g^2)^{-1}\partial_\rho \\
C_0^3 &= (g^2g^3)^{-1}\cdot(\partial_\rho^2 - 2g^2) \\
C_0^4 &= (g^2g^3g^4)^{-1}\cdot\Big(-\partial_\rho^3 + 2(g^2+g^3)\partial_\rho\Big) \\
C_0^5 &= (g^2g^3g^4g^5)^{-1}\cdot\Big(\partial_\rho^4 - 2(g^2+g^3+g^4)\partial_\rho^2 + 4g^2g^4\Big) \\
C_0^6 &= (g^2g^3g^4g^5g^6)^{-1}\cdot\Big(-\partial_\rho^5 + 2(g^2+g^3+g^4+g^5)\partial_\rho^3 - 4(g^5(g^3+g^2) \\
&\quad + g^4g^2)\partial_\rho\Big) \\
C_0^7 &= (g^2g^3g^4g^5g^6g^7)^{-1}\cdot\Big(\partial_\rho^6 - 2(g^2+g^3+g^4+g^5+g^6)\partial_\rho^4 \\
&\quad + 4(g^6(g^4+g^3+g^2) + g^5(g^3+g^2) + g^4g^2)\partial_\rho^2 - 8g^2g^4g^6\Big) \\
C_0^8 &= (g^2g^3g^4g^5g^6g^7g^8)^{-1}\cdot\Big(-\partial_\rho^7 + 2(g^2+g^3+g^4+g^5+g^6+g^7)\partial_\rho^5 \\
&\quad - 4(g^7(g^5+g^4+g^3+g^2) + g^6(g^4+g^3+g^2) + g^5(g^3+g^2) + g^4g^2)\partial_\rho^3 \\
&\quad + 8(g^7(g^5(g^3+g^2) + g^4g^2) + g^6g^4g^2)\Big)\partial_\rho
\end{aligned}
\tag{5B.3}
$$

6. Conclusion and future directions

A major theme of this work is the importance of gauge invariance. To be sure, this is an old theme in theoretical physics, but in the higher spin context there remains much to learn about its constraints on the physics. We have shown that in the 3d case where the theory can be cast essentially as a pure gauge theory coupled to matter, a clever use of gauge invariance alone can determine much of the theory's physical content.

We can use this gauge invariance to study the behavior of a scalar field in the hs[λ] black hole background. This work is underway, and should tell us something about the causal structure of this spacetime, i.e. whether it is "more like" a black hole or a wormhole. We should also note, on the subject of 3d black holes, that there is a supersymmetric version of Vasiliev theory which has recently been studied in a holographic context [93,94,95,96]. The gauge sector is described not by hs[λ] but by a super-version known as shs[λ] [97]. Perhaps this theory contains interesting black hole solutions above and beyond those of the bosonic Vasiliev theory and supersymmetric ordinary gravity.

On the subject of correlation functions and the deep lessons of higher spin AdS/CFT duality, in the 4d three-point function calculations of Giombi and Yin, especially in [31], gauge invariance plays a similar role as in our Chapter 5. They utilize what is called a "$W = 0$ gauge" to efficiently compute the correlators of three fields of arbitrary spin. The $W = 0$ gauge master field equations are much simpler to solve – all spacetime coordinate dependence has been gauged away, leaving only certain functions of the internal spinor variables – and upon a suitable gauge transformation back to the "physical gauge," one can essentially read off correlators. The key is to find the correct ansatz for the nonzero master fields (B, S_α) in the $W = 0$ gauge.

[31] only computed O(N) vector model correlators this way, but we know that the 4d Vasiliev theory contains a self-interaction ambiguity of the scalar

master field which can, for generic choices of parameters, break parity [29]. It is natural to suggest that these theories are dual to parity-breaking CFTs. There is limited evidence for this idea in the case of a dual Chern-Simons-fermion CFT [16], but also evidence against this idea based on unpublished bulk computations of correlators by the same authors as [16]. We expect that some variant of our approach in 3d, whereby we generated the three-point correlators by gauge transformation, may work in the 4d case; this is motivated partly by considerations of [17] who showed that even in these parity-breaking cases, correlators are fully fixed by the “almost conserved” higher spin symmetry.

In any case, we now sense a greater set of possibilities for higher spin duality in 4d, in part motivated by [17]. This may also imply certain equivalences among field theories, e.g. the critical O(N) model of bosons and the $\lambda \to 1$ limit of the Chern-Simons-fermion theory.

The issue of black holes in 4d Vasiliev theory is also an important outstanding issue, as is the question of how to introduce a Higgs mechanism into Vasiliev theory to explicitly study the $1/N$ higher spin symmetry breaking. Any connection to string theory will require understanding of both of these issues. For instance, extrapolating the $\mathcal{N} = 4$ SYM AdS/CFT duality to the $\lambda \to 0$ limit will require an understanding of how the higher spin fields become massless in this limit. Perhaps it will be more profitable to focus on 3d rather than 4d Vasiliev theory in an effort to find higher spins from string theory.

Appendix I. hs[λ], and Moyal vs. lone-star products

This appendix is meant as a self-contained review of salient features of the hs[λ] Lie algebra, its associative "lone star product" multiplication, and their relation to the deformed oscillator algebra used by Vasiliev.

The hs[λ] commutation relations are

$$[V_m^s, V_n^t] = \sum_{u=2,4,6,...}^{s+t-|s-t|-1} g_u^{st}(m,n;\lambda) V_{m+n}^{s+t-u} \tag{I.1}$$

with structure constants are

$$g_u^{st}(m,n;\lambda) = \frac{q^{u-2}}{2(u-1)!} \phi_u^{st}(\lambda) N_u^{st}(m,n) \tag{I.2}$$

where

$$\begin{aligned} N_u^{st}(m,n) &= \sum_{k=0}^{u-1} (-1)^k \begin{pmatrix} u-1 \\ k \end{pmatrix} \\ &\quad \times [s-1+m]_{u-1-k}[s-1-m]_k[t-1+n]_k[t-1-n]_{u-1-k} \\ \phi_u^{st}(\lambda) &= {}_4F_3 \left[\begin{matrix} \frac{1}{2}+\lambda \ , & \frac{1}{2}-\lambda \ , & \frac{2-u}{2} \ , & \frac{1-u}{2} \\ \frac{3}{2}-s \ , & \frac{3}{2}-t \ , & \frac{1}{2}+s+t-u & \end{matrix} \middle| 1 \right] \end{aligned} \tag{I.3}$$

We make use of the descending Pochhammer symbol,

$$[a]_n = a(a-1)...(a-n+1) \tag{I.4}$$

q is a normalization constant that can be scaled away by taking $V_m^s \to q^{s-2} V_m^s$. As in much of the existing literature, we choose to set $q = 1/4$.

We note a handful of useful properties of the structure constants:

$$\begin{aligned} &\phi_u^{st}\left(\frac{1}{2}\right) = \phi_2^{st}(\lambda) = 1 \\ &N_u^{st}(m,n) = (-1)^{u+1} N_u^{ts}(n,m) \\ &N_u^{st}(0,0) = 0 \ , \quad u \text{ even} \\ &N_u^{st}(n,-n) = N_u^{ts}(n,-n) \end{aligned} \tag{I.5}$$

The first three of these imply the isomorphism hs$[\frac{1}{2}] \cong$ hs(1,1); that the lone star product can be used to define the hs$[\lambda]$ Lie algebra; and that all zero modes commute.

The generators with $s = 2$ form an SL(2,R) subalgebra, and the remaining generators transform simply under the adjoint SL(2,R) action as

$$[V^2_m, V^t_n] = (m(t-1) - n)V^t_{m+n} \tag{I.6}$$

These SL(2,R) generators are those used in construction of the AdS and BTZ solutions.

When $\lambda = 1/2$, this algebra is isomorphic to hs(1,1), the commutator of which can be written as the antisymmetric part of the Moyal product. Similarly, the general λ commutation relations (5.15) can be realized as

$$[V^s_m, V^t_n] = V^s_m \star V^t_n - V^t_n \star V^s_m \tag{I.7}$$

if we define the associative product

$$V^s_m \star V^t_n \equiv \frac{1}{2} \sum_{u=1,2,3,\ldots}^{s+t-1} g^{st}_u(m,n;\lambda) V^{s+t-u}_{m+n} \tag{I.8}$$

This is known as the "lone star product" [57], and (I.7) follows upon using the fact that

$$g^{st}_u(m,n;\lambda) = (-1)^{u+1} g^{ts}_u(n,m;\lambda) \tag{I.9}$$

The odd values of u drop out of the commutator, leaving (5.15). In parts of the preceding, we have resorted to the shorthand

$$\underbrace{\Gamma \star \Gamma \star \ldots \star \Gamma}_{N} \equiv (\Gamma)^N \tag{I.10}$$

for some hs$[\lambda]$-valued element Γ.

Formally, V^1_0 is the identity element. Thus, to extract the trace from a product of generators, one picks out the $u = s + t - 1$ part of (I.8), up to some normalization:

$$\begin{aligned} \mathrm{Tr}(V^s_m V^t_n) &\propto g^{st}_{s+t-1}(m,n;\lambda)\delta^{st}\delta_{m,-n} \\ &\propto g^{ss}_{2s-1}(m,-m;\lambda) \end{aligned} \tag{I.11}$$

I.1. **Relation between hs[λ] and deformed oscillators**

Recall the deformed oscillator commutation relations:

$$[\tilde{y}_\alpha, \tilde{y}_\beta]_\star = 2i\epsilon_{\alpha\beta}(1+\nu k) \tag{I.12}$$

which in our conventions $\epsilon_{12} = \epsilon^{12} = 1$ is

$$[\tilde{y}_1, \tilde{y}_2]_\star = 2i(1+\nu k) \tag{I.13}$$

The beauty of the deformed oscillators is that under the action of the Moyal product[31], their star commutator gives the oscillator algebra (I.12).

To compute the Moyal product of two symmetric, even-degree oscillator polynomials, one uses (I.12) to symmetrize the product. To compute the lone-star product of two hs[λ] generators, one plugs into formula (5.24). The purpose of this section is to provide evidence that these two multiplications are isomorphic upon identifying the map between the generator and polynomial bases, and using the relation

$$\lambda = \frac{1-\nu k}{2} \tag{I.14}$$

where $k^2 = 1$ is the Clifford element defined in section 2 and in appendix A. To our knowledge, this has not been proven in the literature.

Let us begin with the SL(2,R) subalgebra spanned by symmetric polynomials

$$S_{\alpha\beta} = \tilde{y}_{(\alpha}\tilde{y}_{\beta)} \tag{I.15}$$

These obey commutation relations

$$\begin{aligned}
[S_{11}, S_{22}] &= 8iS_{12} \\
[S_{11}, S_{12}] &= 4iS_{11} \\
[S_{12}, S_{22}] &= 4iS_{22}
\end{aligned} \tag{I.16}$$

[31] In what follows, every product of $\tilde{y}$ is implicitly a Moyal product.

Comparing with the SL(2,R) subalgebra of hs[λ] canonically normalized as

$$\begin{aligned}
[V^2_1, V^2_{-1}] &= 2V^2_0 \\
[V^2_1, V^2_0] &= V^2_1 \\
[V^2_0, V^2_{-1}] &= V^2_{-1}
\end{aligned} \tag{I.17}$$

these are equivalent under the assignment

$$V^2_1 = \left(\frac{-i}{4}\right) S_{11} \ , \quad V^2_0 = \left(\frac{-i}{4}\right) S_{12} \ , \quad V^2_{-1} = \left(\frac{-i}{4}\right) S_{22} \tag{I.18}$$

Having fixed (I.18) we can compare, on the one hand, the Moyal product of two $S_{\alpha\beta}$, and on the other, the lone-star product between two of the V^2_m, and so on for higher spins. The work comes in symmetrizing the oscillator products, through tedious but straightforward application of (I.12). We present results through spin-4:

$$\begin{aligned}
\tilde{y}_1\tilde{y}_2 &= S_{12} + i(\nu k + 1) \\
\tilde{y}_1\tilde{y}_1\tilde{y}_1\tilde{y}_2 &= S_{1112} + i(\nu k + 3)S_{11} \\
\tilde{y}_1\tilde{y}_1\tilde{y}_2\tilde{y}_2 &= S_{1122} + 4iS_{12} + \frac{2}{3}(\nu k + 1)(\nu k - 3) \\
\tilde{y}_1\tilde{y}_1\tilde{y}_1\tilde{y}_1\tilde{y}_1\tilde{y}_2 &= S_{111112} + i(\nu k + 5)S_{1111} \\
\tilde{y}_1\tilde{y}_1\tilde{y}_1\tilde{y}_1\tilde{y}_2\tilde{y}_2 &= S_{111122} + 8iS_{1112} + \frac{4}{5}(\nu k + 3)(\nu k - 5)S_{11} \\
\tilde{y}_1\tilde{y}_1\tilde{y}_1\tilde{y}_2\tilde{y}_2\tilde{y}_2 &= S_{111222} + i(\nu k + 9)S_{1122} + \frac{2}{5}(\nu k + 3)(\nu k - 15)S_{12} \\
&\quad + \frac{2i}{3}(\nu k + 3)(\nu k + 1)(\nu k - 3)
\end{aligned} \tag{I.19}$$

All remaining products can be found by commutation or taking the adjoint, $\tilde{y}_1 \leftrightarrow \tilde{y}_2, i \to -i$.

Using these, one can show by explicit computation that, at least through spin-4, any Moyal product of oscillator polynomials maps to the lone-star product of hs[λ] generators upon making the identification

$$V^s_m = \left(\frac{-i}{4}\right)^{s-1} S^s_m \tag{I.20}$$

where S^s_m is the symmetrized product of $2s-2$ oscillators, with m defined as

$$2m = N_1 - N_2 \tag{I.21}$$

The prefactor depends on the hs[λ] normalization factor $q = 1/4$ which we have been using.

As a first check, (I.18) along with the fact that

$$(V^2_{\pm 1})^{s-1} = V^s_{\pm(s-1)} \tag{I.22}$$

implies that this is trivially true for elements with $m = \pm(s-1)$.

Let us demonstrate this equivalence for a few examples. Using (I.19) one obtains the Moyal products of spin-2 polynomials:

$$\begin{aligned}
S_{11} \star S_{11} &= S_{1111} \\
S_{22} \star S_{22} &= S_{2222} \\
S_{12} \star S_{11} &= S_{1112} - 2iS_{11} \\
S_{12} \star S_{22} &= S_{1222} + 2iS_{22} \\
S_{11} \star S_{22} &= S_{1122} + 4iS_{12} + \frac{2}{3}(\nu k + 1)(\nu k - 3) \\
S_{12} \star S_{12} &= S_{1122} - \frac{1}{3}(\nu k + 1)(\nu k - 3)
\end{aligned} \tag{I.23}$$

We now compare this to the lone-star products of spin-2 generators:

$$\begin{aligned}
V^2_1 \star V^2_1 &= V^3_2 \\
V^2_{-1} \star V^2_{-1} &= V^3_{-2} \\
V^2_0 \star V^2_1 &= V^3_1 - \frac{1}{2}V^2_1 \\
V^2_0 \star V^2_{-1} &= V^3_{-1} + \frac{1}{2}V^2_{-1} \\
V^2_1 \star V^2_{-1} &= V^3_0 + V^2_0 - \left(\frac{\lambda^2 - 1}{6}\right) \\
V^2_0 \star V^2_0 &= V^3_0 + \frac{\lambda^2 - 1}{12}
\end{aligned} \tag{I.24}$$

These are isomorphic under (I.14) and (I.20).

A less trivial example is the product

$$\begin{aligned} S_{1112} \star S_{22} &= \frac{1}{4}(\tilde{y}_1\tilde{y}_1\tilde{y}_1\tilde{y}_2 + \tilde{y}_1\tilde{y}_1\tilde{y}_2\tilde{y}_1 + \tilde{y}_1\tilde{y}_2\tilde{y}_1\tilde{y}_1 + \tilde{y}_2\tilde{y}_1\tilde{y}_1\tilde{y}_1)\tilde{y}_2\tilde{y}_2 \\ &= S_{111222} + 6iS_{1112} + \frac{2}{5}(\nu k + 3)(\nu k - 5)S_{12} \end{aligned} \tag{I.25}$$

Compare to the lone-star product

$$V_1^3 \star V_{-1}^2 = V_0^4 + \frac{3}{2}V_0^3 - \left(\frac{\lambda^2 - 4}{10}\right) V_0^2 \tag{I.26}$$

These are isomorphic under (I.14) and (I.20).

We conjecture that the isomorphism is valid for all spins under the identification (I.20).

Appendix II. Lightning review of 3d higher spin gravity coupled to scalars

In this appendix we present all details necessary for the bulk theory's construction, following [11]. In particular we derive the equations (5.13) and (5.21).

According to [11], the full non-linear system of equations governing the interaction of matter with higher spin gauge fields is formulated in terms of the following generating functions: a spacetime 1-form $W = W_\nu dx^\nu$ as well as spacetime 0-forms B and S_α. The generating functions W, B and S_α depend on spacetime coordinates x, on auxiliary bosonic twistor variables z_α and y_α ($\alpha = 1, 2$) as well as on two pairs of Clifford elements, $\psi_{1,2}$ and k, ρ :

$$\{\psi_i, \psi_j\} = 2\delta_{ij} \ , \quad k\rho = -\rho k \ , \quad k^2 = \rho^2 = 1 \tag{II.1}$$

Moreover, $\psi_{1,2}$ commute with all other auxiliary variables, and k, ρ obey

$$\begin{aligned} ky_\alpha &= -y_\alpha k, \quad kz_\alpha = -z_\alpha k \\ \rho y_\alpha &= y_\alpha \rho, \qquad \rho z_\alpha = z_\alpha \rho \end{aligned} \tag{II.2}$$

Indices on z_α and y_α are raised by $\epsilon^{\alpha\beta}$ and lowered by the rank two antisymmetric tensor $\epsilon_{\beta\alpha}$,

$$z^\alpha = \epsilon^{\alpha\beta} z_\beta \,, \qquad\qquad z_\alpha = z^\beta \epsilon_{\beta\alpha} \tag{II.3}$$

with $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = -\delta^\alpha_\gamma$. We follow the convention $\epsilon^{12} = \epsilon_{12} = 1$.

Using these properties of the auxiliary variabls the basic fields W_ν, B and S_α can be expanded in the form

$$\begin{aligned} A(z, y; \psi_{1,2}, k, \rho|x) = \sum_{b,c,d,e=0}^{1} \sum_{m,n=0}^{\infty} \frac{1}{m!n!} A_{bcde}^{\alpha_1\ldots\alpha_m\beta_1\ldots\beta_n}(x)\; k^b \rho^c \psi_1^d \psi_2^e \\ \times\;\; z_{\alpha_1}\ldots z_{\alpha_m} y_{\beta_1}\ldots y_{\beta_n} \end{aligned} \tag{II.4}$$

where A is either W_ν, B or S_α. The expression $A_{bcde}^{\alpha_1\ldots\alpha_m\beta_1\ldots\beta_n}(x)$ in equation (II.4) is an ordinary spacetime function. Note that $A_{bcde}^{\alpha_1\ldots\alpha_m\beta_1\ldots\beta_n}(x)$ can be choosen to be symmetric in the indices $(\alpha_1\ldots\alpha_m)$ and in the indices $(\beta_1\ldots\beta_n)$.

In order to formulate the equations of motion we use the Moyal $\star$-product which acts on the twistors y and z in the following way

$$f(z, y) \star g(z, y) = \frac{1}{(2\pi)^2} \int d^2u \int d^2v \; e^{i(uv)} f(z + u, y + u) g(z - v, y + v) \tag{II.5}$$

where uv is a short-hand notation, $uv = u_\alpha v^\alpha$. We can verify the following commutation relations:

$$[y_\alpha, y_\beta]_\star = -[z_\alpha, z_\beta]_\star = 2i\epsilon_{\alpha\beta}, \qquad [y_\alpha, z_\beta]_\star = 0 \tag{II.6}$$

where $[a, b]_\star \equiv a \star b - b \star a$ is the commutator with respect to the $\star$-product.

In terms of the generating functions $W = W_\nu dx^\nu, B, S_\alpha$ we are now ready to write down the full non-linear equations of motion [11]:

$$\begin{aligned} dW &= W \wedge \star W \\ dB &= W \star B - B \star W \\ dS_\alpha &= W \star S_\alpha - S_\alpha \star W \\ S_\alpha \star S^\alpha &= -2i(1 + B \star K) \\ S_\alpha \star B &= B \star S_\alpha \end{aligned} \tag{II.7}$$

Here, K – the so-called Kleinian – is given by

$$K = ke^{i(zy)} \tag{II.8}$$

These equations (II.7) are invariant under the infinitesimal higher spin gauge transformation

$$\begin{aligned} \delta W &= d\epsilon + \epsilon \star W - W \star \epsilon \\ \delta B &= \epsilon \star B - B \star \epsilon \\ \delta S_\alpha &= \epsilon \star S_\alpha - S_\alpha \star \epsilon \end{aligned} \tag{II.9}$$

where ϵ is the infinitesimal gauge parameter which does not depend on ρ, i.e.

$$\epsilon = \epsilon(z, y; \psi_{1,2}, k|x) \tag{II.10}$$

We will see that W is the generating function for higher spin gauge fields whereas B is the generating function for the matter fields. S_α will describe auxiliary degrees of freedom.

Since the equations of the motion (II.7) possess the symmetry $\rho \to -\rho$ and $S_\alpha \to -S_\alpha$ we can truncate the system to the so-called "reduced" system, in which W_ν and B are independent of ρ, while S_α is linear in ρ. In this paper we consider the reduced system.

II.1. Vacuum solutions

Here we consider vacuum solutions of the equations of motion (II.7). The fields B, W and S_α of the vacuum solution are denoted by $B^{(0)}, W^{(0)}$ and $S_\alpha^{(0)}$, respectively. In particular we take $B^{(0)}$ to be constant, i.e.

$$B^{(0)} \equiv \nu \tag{II.11}$$

Plugging this ansatz into (II.7) we obtain the following three equations

$$\begin{aligned} dW^{(0)} &= W^{(0)} \star \wedge W^{(0)} \\ dS_\alpha^{(0)} &= W^{(0)} \star S_\alpha^{(0)} - S_\alpha^{(0)} \star W^{(0)} \\ S_\alpha^{(0)} S^{(0)\,\alpha} &= -2i(1 + \nu K) \end{aligned} \tag{II.12}$$

Note that the other two equations of (II.7) are automatically satisfied by the ansatz (II.11).

First, let us discuss the third equation of (II.12). We already mentioned that S_α and therefore also $S_\alpha^{(0)}$ is linear in ρ. For the case $\nu = 0$ we can choose $S_\alpha^{(0)} = \rho z_\alpha$, cf. (II.6). For general ν, $S_\alpha^{(0)}$ can be given by

$$S_\alpha^{(0)} = \rho \tilde{z}_\alpha \tag{II.13}$$

where we have introduced new auxiliary twistor variables $\tilde{z}_\alpha$ and $\tilde{y}_\alpha$ which are also known as "deformed oscillators,"

$$\begin{aligned} \tilde{z}_\alpha &= z_\alpha + \nu\, w_\alpha k \\ \tilde{y}_\alpha &= y_\alpha + \nu\, w_\alpha \star K \\ w_\alpha &= (z_\alpha + y_\alpha) \int_0^1 dt\, t e^{it(zy)} \end{aligned} \tag{II.14}$$

The deformed oscillators $\tilde{y}_\alpha$ and $\tilde{z}_\alpha$ satisfy the commutation relations

$$\begin{aligned} [\tilde{y}_\alpha, \tilde{y}_\beta]_\star &= 2i\epsilon_{\alpha\beta}(1 + \nu k) \\ [\rho\tilde{z}_\alpha, \rho\tilde{z}_\beta]_\star &= -2i\epsilon_{\alpha\beta}(1 + \nu K) \\ [\rho\tilde{z}_\alpha, \tilde{y}_\beta]_\star &= 0 \end{aligned} \tag{II.15}$$

and therefore it is straightforward to verify that $S_\alpha^{(0)}$, given by equation (II.13), indeed satisfies the third equation of (II.12).

As $dS_\alpha^{(0)} = 0$, the second equation of (II.12) implies that $W^{(0)}$ should commute with $S_\alpha^{(0)}$, which according to the third line of (II.15) can be achieved by taking $W^{(0)}$ to be independent of k and $\tilde{z}_\alpha$. Therefore $W^{(0)}$ is only a function of $x, \psi_{1,2}$ and $\tilde{y}$ and can be expanded as

$$W^{(0)}(\tilde{y}; \psi_{1,2}|x) = \sum_{d,e=0}^{1} \sum_{n=0}^{\infty} \frac{1}{n!} W_{00de}^{\beta_1 \ldots \beta_n}(x) \psi_1^d \psi_2^e \; \tilde{y}_{\beta_1} \star \ldots \star \tilde{y}_{\beta_n} \tag{II.16}$$

It turns out that we only have to consider symmetric products in $\tilde{y}_{\beta_1} \star \ldots \star \tilde{y}_{\beta_n}$. Moreover, here we will consider only products with an even number of $\tilde{y}$. Under the star product the auxiliary variables $\tilde{y}_\alpha$ generate the three-dimensional higher spin algebra hs[λ]. To be more precise, a symmetric product $\tilde{y}_{\beta_1}\star\ldots\star\tilde{y}_{\beta_{2n}}$ corresponds to a generator of hs[λ] with spin $n+1$. In particular the generators $T_{\alpha\beta}$ of the SL(2) subalgebra are given by

$$T_{\alpha\beta} = -\frac{i}{4}\{\tilde{y}_\alpha, \tilde{y}_\beta\}_\star \tag{II.17}$$

Multiplying symmetrized even-degree polynomials in $\tilde{y}$, using the commutation relations as given in the first line of equation (II.15) and finally projecting on $k = \mp 1$, the commutation relation are those of hs[λ] with $\lambda = \frac{1}{2}(1 \pm \nu)$. More details, including the hs[λ] the structure constants, can be found in appendix B.

Finally, let us consider the last equation of motion which we have to solve:

$$dW^{(0)} = W^{(0)} \star \wedge W^{(0)}. \tag{II.18}$$

This equation can be written as a flatness condition of a Chern-Simons theory with gauge group hs[λ]$\oplus$hs[λ] . In order to see this we introduce hs[λ]-valued gauge fields A and $\overline{A}$ by

$$W^{(0)} = -\mathcal{P}_+ A - \mathcal{P}_- \overline{A} \tag{II.19}$$

where A and $\overline{A}$ are functions of x and $\tilde{y}$. We have introduced projection operators

$$\mathcal{P}_\pm = \frac{1 \pm \psi_1}{2} \tag{II.20}$$

obeying

$$\mathcal{P}_\pm \psi_1 = \psi_1 \mathcal{P}_\pm = \pm \mathcal{P}_\pm, \qquad \mathcal{P}_\pm \psi_2 = \psi_2 \mathcal{P}_\mp \tag{II.21}$$

Using (II.19) we can rewrite (II.18) in the following form

$$dA + A \wedge \star A = 0\,, \qquad d\overline{A} + \overline{A} \wedge \star \overline{A} = 0 \tag{II.22}$$

which is precisely equation (5.13). Therefore the equations of motion for $W^{(0)}$ are equivalent to flatness conditions of gauge fields A and $\overline{A}$ defined by equation (II.19).

II.2. Matter equations

Let us now linearize the equations of motion (II.7) around the vacuum solution constructed in the last section. In particular, in this paper we are interested in fluctuations of the field B around the constant background $B^{(0)} = \nu$. The fluctuations of B are denoted by $\mathcal{C}$, i.e.

$$B = \nu + \mathcal{C} \tag{II.23}$$

For W and S_α we do not consider any fluctuations. Substituting this ansatz (II.23) into the equations of motion (II.7) and using the equations (II.12), we obtain two non-trivial equations for $\mathcal{C}$

$$\begin{aligned} d\mathcal{C} - W^{(0)} \star \mathcal{C} + \mathcal{C} \star W^{(0)} &= 0 \\ \left[S_\alpha^{(0)}, \mathcal{C} \right]_\star &= 0 \end{aligned} \tag{II.24}$$

Since $S_\alpha^{(0)}$ is given by equation (II.13) we can satisfy the second line of equation (II.24) by demanding that $\mathcal{C}$ does not depend on $\tilde{z}_\alpha$ nor on k. Therefore $\mathcal{C}$ is only a function of $\tilde{y}_\alpha, \psi_{1,2}$ and x. Typically, $\mathcal{C}$ is decomposed as

$$\mathcal{C}(\tilde{y}; \psi_{1,2}|x) = \mathcal{C}^{aux}(\tilde{y}; \psi_1|x) + \mathcal{C}^{dyn}(\tilde{y}; \psi_1|x)\psi_2 \tag{II.25}$$

It turns out [11] that $\mathcal{C}^{aux}$ gives rise to an auxiliary set of fields that can be set to zero consistently. By abuse of notation, we will use $\mathcal{C}$ instead of $\mathcal{C}^{dyn}$ to simplify the notation. Moreover we will decompose $\mathcal{C}$ under the projection operators $\mathcal{P}_\pm$ as given in equation (II.20)

$$\mathcal{C} = C(\tilde{y}|x)\, \psi_2 + \tilde{C}(\tilde{y}|x)\, \psi_2 \tag{II.26}$$

where $C = \mathcal{P}_+ \mathcal{C}$ and $\tilde{C} = \mathcal{P}_- \mathcal{C}$. If we also express $W^{(0)}$ in terms of gauge fields, eq. (II.19), we can rewrite the second line of (II.24) in the form

$$\begin{aligned} dC + A \star C - C \star \overline{A} &= 0 \\ d\tilde{C} + \overline{A} \star \tilde{C} - \tilde{C} \star A &= 0 \end{aligned} \tag{II.27}$$

These are the equations of linearized matter interacting with an arbitrary higher spin background.

Comparing the first and second line of equation (II.27) we see that the equations of motion for C and $\tilde{C}$ are related to each other by exchanging A and $\overline{A}$. Note that A and $\overline{A}$ can be written in terms of the generalized vielbein e and generalized spin connection ω,

$$A = \omega + e \qquad \overline{A} = \omega - e \tag{II.28}$$

Under the exchange of A and $\overline{A}$ the generalized vielbein e is odd. Therefore the sign of the generalized vielbein and hence the sign of any metric-like tensor field of odd spin is flipped under this operation.

References

[1] X. Bekaert, N. Boulanger and P. Sundell, "How higher-spin gravity surpasses the spin two barrier: no-go theorems versus yes-go examples," [arXiv:1007.0435 [hep-th]].

[2] C. Fronsdal, "Massless Fields With Integer Spin," Phys. Rev. D **18**, 3624 (1978).

[3] E. S. Fradkin and M. A. Vasiliev, "On the Gravitational Interaction of Massless Higher Spin Fields," Phys. Lett. B **189**, 89 (1987).

[4] E. S. Fradkin and M. A. Vasiliev, "Cubic Interaction in Extended Theories of Massless Higher Spin Fields," Nucl. Phys. B **291**, 141 (1987).

[5] M. A. Vasiliev, "Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions," Phys. Lett. B **243**, 378 (1990).

[6] M. A. Vasiliev, "Closed equations for interacting gauge fields of all spins," JETP Lett. **51**, 503 (1990) [Pisma Zh. Eksp. Teor. Fiz. **51**, 446 (1990)].

[7] M. A. Vasiliev, "More On Equations Of Motion For Interacting Massless Fields Of All Spins In (3+1)-Dimensions," Phys. Lett. B **285**, 225 (1992).

[8] M. A. Vasiliev, "Higher spin gauge theories: Star product and AdS space," arXiv:hep-th/9910096.

[9] A. Connes, M. R. Douglas, A. S. Schwarz, "Noncommutative geometry and matrix theory: Compactification on tori," JHEP **9802**, 003 (1998). [hep-th/9711162].

[10] N. Seiberg, E. Witten, "String theory and noncommutative geometry," JHEP **9909**, 032 (1999). [hep-th/9908142].

[11] S. F. Prokushkin and M. A. Vasiliev, "Higher spin gauge interactions for massive matter fields in 3-D AdS space-time," Nucl. Phys. B **545**, 385 (1999) [arXiv:hep-th/9806236].

[12] A. Achucarro and P. K. Townsend, "A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories," Phys. Lett. B **180**, 89 (1986).

[13] E. Witten, "(2+1)-Dimensional Gravity as an Exactly Soluble System," Nucl. Phys. B **311**, 46 (1988).

[14] A. Dabholkar, R. Kallosh and A. Maloney, "A Stringy cloak for a classical singularity," JHEP **0412**, 059 (2004). [hep-th/0410076].

[15] B. Sundborg, "Stringy gravity, interacting tensionless strings and massless higher spins," Nucl. Phys. Proc. Suppl. **102**, 113 (2001). [hep-th/0103247].

[16] S. Giombi, S. Minwalla, S. Prakash, S. P. Trivedi, S. R. Wadia and X. Yin, "Chern-Simons Theory with Vector Fermion Matter," [arXiv:1110.4386 [hep-th]].

[17] J. Maldacena and A. Zhiboedov, "Constraining conformal field theories with a slightly broken higher spin symmetry," [arXiv:1204.3882 [hep-th]].

[18] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[19] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[20] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].

[21] P. Haggi-Mani and B. Sundborg, "Free large N supersymmetric Yang-Mills theory as a string theory," JHEP **0004**, 031 (2000) [arXiv:hep-th/0002189].

[22] S. E. Konstein, M. A. Vasiliev and V. N. Zaikin, "Conformal higher spin currents in any dimension and AdS/CFT correspondence," JHEP **0012**, 018 (2000) [arXiv:hep-th/0010239].

[23] E. Sezgin and P. Sundell, "Massless higher spins and holography," Nucl. Phys. B **644**, 303 (2002) [Erratum-ibid. B **660**, 403 (2003)] [arXiv:hep-th/0205131].

[24] A. Mikhailov, "Notes on higher spin symmetries," arXiv:hep-th/0201019.

[25] I. R. Klebanov and A. M. Polyakov, "AdS dual of the critical O(N) vector model," Phys. Lett. B **550**, 213 (2002) [arXiv:hep-th/0210114].

[26] E. Witten, "Multitrace operators, boundary conditions, and AdS / CFT correspondence," [hep-th/0112258].

[27] A. C. Petkou, "Evaluating the AdS dual of the critical O(N) vector model," JHEP **0303**, 049 (2003). [hep-th/0302063].

[28] R. G. Leigh, A. C. Petkou, "Holography of the N=1 higher spin theory on AdS(4)," JHEP **0306**, 011 (2003). [hep-th/0304217].

[29] E. Sezgin, P. Sundell, "Holography in 4D (super) higher spin theories and a test via cubic scalar couplings," JHEP **0507**, 044 (2005). [hep-th/0305040].

[30] S. Giombi and X. Yin, "Higher Spin Gauge Theory and Holography: The Three-Point Functions," JHEP **1009**, 115 (2010) [arXiv:0912.3462 [hep-th]].

[31] S. Giombi and X. Yin, "Higher Spins in AdS and Twistorial Holography,"arXiv:1004.3736 [hep-th].

[32] S. Giombi, X. Yin, "On Higher Spin Gauge Theory and the Critical O(N) Model," [arXiv:1105.4011 [hep-th]].

[33] J. Maldacena and A. Zhiboedov, "Constraining Conformal Field Theories with A Higher Spin Symmetry," [arXiv:1112.1016 [hep-th]].

[34] M. R. Gaberdiel and R. Gopakumar, "An AdS_3 Dual for Minimal Model CFTs," arXiv:1011.2986 [hep-th].

[35] O. Aharony, G. Gur-Ari and R. Yacoby, "d=3 Bosonic Vector Models Coupled to Chern-Simons Gauge Theories," JHEP **1203**, 037 (2012). [arXiv:1110.4382 [hep-th]].

[36] M. R. Gaberdiel and T. Hartman, "Symmetries of Holographic Minimal Models," arXiv:1101.2910 [hep-th].

[37] M. R. Gaberdiel and R. Gopakumar, "Triality in Minimal Model Holography," [arXiv:1205.2472 [hep-th]].

[38] M. R. Gaberdiel, R. Gopakumar, T. Hartman and S. Raju, "Partition Functions of Holographic Minimal Models," arXiv:1106.1897 [hep-th].

[39] C. M. Chang and X. Yin, "Higher Spin Gravity with Matter in AdS_3 and Its CFT Dual," arXiv:1106.2580 [hep-th].

[40] V. E. Didenko and M. A. Vasiliev, "Static BPS black hole in 4d higher-spin gauge theory," Phys. Lett. B **682**, 305 (2009) [arXiv:0906.3898 []].

[41] A. Castro, A. Lepage-Jutier and A. Maloney, "Higher Spin Theories in AdS_3 and a Gravitational Exclusion Principle," JHEP **1101**, 142 (2011) [arXiv:1012.0598 [hep-th]].

[42] C. Iazeolla, P. Sundell, “Families of exact solutions to Vasiliev’s 4D equations with spherical, cylindrical and biaxial symmetry,” [arXiv:1107.1217 [hep-th]].

[43] S. H. Shenker and X. Yin, “Vector Models in the Singlet Sector at Finite Temperature,” [arXiv:1109.3519 [hep-th]].

[44] M. Banados, C. Teitelboim and J. Zanelli, “The Black hole in three-dimensional space-time,” Phys. Rev. Lett. **69**, 1849 (1992) [arXiv:hep-th/9204099].

[45] J. D. Brown and M. Henneaux, “Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity,” Commun. Math. Phys. **104**, 207 (1986).

[46] A. Strominger, “Black hole entropy from near horizon microstates,” JHEP **9802**, 009 (1998) [arXiv:hep-th/9712251].

[47] M. Henneaux and S. J. Rey, “Nonlinear W(infinity) Algebra as Asymptotic Symmetry of Three-Dimensional Higher Spin Anti-de Sitter Gravity,” JHEP **1012**, 007 (2010) [arXiv:1008.4579 [hep-th]].

[48] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, “Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields,” JHEP **1011**, 007 (2010) [arXiv:1008.4744 [hep-th]].

[49] M. Ammon, M. Gutperle, P. Kraus and E. Perlmutter, “Spacetime Geometry in Higher Spin Gravity,” arXiv:1106.4788 [hep-th].

[50] M. Gutperle and P. Kraus, “Higher Spin Black Holes,” JHEP **1105**, 022 (2011) [arXiv:1103.4304 [hep-th]].

[51] P. Kraus, E. Perlmutter, “Partition functions of higher spin black holes and their CFT duals,” [arXiv:1108.2567 [hep-th]].

[52] E. Bergshoeff, M. P. Blencowe and K. S. Stelle, “Area Preserving Diffeomorphisms And Higher Spin Algebra,” Commun. Math. Phys. **128**, 213 (1990).

[53] M. Bordemann, J. Hoppe and P. Schaller, “Infinite dimensional matrix algebras,” Phys. Lett. B **232**, 199 (1989).

[54] M. A. Vasiliev, “Quantization on sphere and high spin superalgebras,” JETP Lett. **50**, 374 (1989) [Pisma Zh. Eksp. Teor. Fiz. **50**, 344 (1989)].

[55] M. A. Vasiliev, "Higher Spin Algebras and Quantization on the Sphere and Hyperboloid," Int. J. Mod. Phys. A **6**, 1115 (1991).

[56] E. S. Fradkin and V. Y. Linetsky, "Supersymmetric Racah basis, family of infinite dimensional superalgebras, SU(infinity + 1—infinity) and related 2-D models," Mod. Phys. Lett. A **6**, 617 (1991).

[57] C. N. Pope, L. J. Romans and X. Shen, "W(infinity) AND THE RACAH-WIGNER ALGEBRA," Nucl. Phys. B **339**, 191 (1990).

[58] M. R. Gaberdiel, T. Hartman and K. Jin, "Higher Spin Black Holes from CFT," JHEP **1204**, 103 (2012). [arXiv:1203.0015 [hep-th]].

[59] M. Ammon, P. Kraus and E. Perlmutter, "Scalar fields and three-point functions in D=3 higher spin gravity," [arXiv:1111.3926 [hep-th]].

[60] C. Ahn, "The Coset Spin-4 Casimir Operator and Its Three-Point Functions with Scalars," [arXiv:1111.0091 [hep-th]].

[61] P. Kraus, "Lectures on black holes and the AdS(3)/CFT(2) correspondence," Lect. Notes Phys. **755**, 193 (2008) [arXiv:hep-th/0609074].

[62] P. Kraus and F. Larsen, "Holographic gravitational anomalies," JHEP **0601**, 022 (2006). [hep-th/0508218].

[63] M. Banados, "Three-dimensional quantum geometry and black holes," arXiv: hep-th/9901148.

[64] A. Campoleoni, S. Fredenhagen and S. Pfenninger, "Asymptotic W- symmetries in three-dimensional higher-spin gauge theories," arXiiv:1107.0290 [hep-th].

[65] A. Castro, E. Hijano and A. Lepage-Jutier, "Unitarity Bounds in AdS_3 Higher Spin Gravity," [arXiv:1202.4467 [hep-th]].

[66] M. Bershadsky, "Conformal field theories via Hamiltonian reduction," Commun. Math. Phys. **139**, 71 (1991).

[67] A. M. Polyakov, "Gauge Transformations and Diffeomorphisms," Int. J. Mod. Phys. A **5**, 833 (1990).

[68] A. Bilal, "W algebras from Chern-Simons theory," Phys. Lett. B **267**, 487 (1991).

[69] M. Banados and R. Caro, "Holographic Ward identities: Examples from 2+1 gravity," JHEP **0412**, 036 (2004) [arXiv:hep-th/0411060].

[70] R. M. Wald, "Black hole entropy is the Noether charge," Phys. Rev. D **48**, 3427 (1993) [arXiv:gr-qc/9307038].

[71] S. Prokushkin and M. A. Vasiliev, "3-d higher spin gauge theories with matter," arXiv:hep-th/9812242.

[72] V. E. Didenko, A. S. Matveev and M. A. Vasiliev, "BTZ Black Hole as Solution of 3-D Higher Spin Gauge Theory," Theor. Math. Phys. **153**, 1487 (2007) [Teor. Mat. Fiz. **153**, 158 (2007)] [arXiv:hep-th/0612161].

[73] H. S. Tan, "Aspects of Three-dimensional Spin-4 Gravity," arXiv:1111.2834 [hep-th].

[74] A. Castro, R. Gopakumar, M. Gutperle and J. Raeymaekers, "Conical Defects in Higher Spin Theories," JHEP **1202**, 096 (2012). [arXiv:1111.3381 [hep-th]].

[75] F. A. Bais, T. Tjin and P. van Driel, "Covariantly coupled chiral algebras," Nucl. Phys. B **357**, 632 (1991).

[76] P. Forgacs, A. Wipf, J. Balog, L. Feher and L. O'Raifeartaigh, "Liouville and Toda Theories as Conformally Reduced WZNW Theories," Phys. Lett. B **227**, 214 (1989).

[77] J. de Boer and T. Tjin, "The Relation between quantum W algebras and Lie algebras," Commun. Math. Phys. **160**, 317 (1994) [arXiv:hep-th/9302006].

[78] E. B. Dynkin, "Semisimple subalgebras of semisimple Lie algebras," Trans. Am. Math. Soc. **6**, 111 (1957).

[79] M. Gutperle and P. Kraus, "Higher Spin Black Holes," JHEP **1105**, 022 (2011) [arXiv:1103.4304 [hep-th]].

[80] C. N. Pope, L. J. Romans and X. Shen, "The Complete Structure of W(Infinity)," Phys. Lett. B **236**, 173 (1990).

[81] I. Bakas and E. Kiritsis, "BOSONIC REALIZATION OF A UNIVERSAL W ALGEBRA AND Z(infinity) PARAFERMIONS," Nucl. Phys. B **343**, 185 (1990) [Erratum-ibid. B **350**, 512 (1991)].

[82] C. M. Hull, "W gravity anomalies 1: Induced quantum W gravity," Nucl. Phys. B **367**, 731 (1991).

[83] C. M. Hull, “Lectures on W gravity, W geometry and W strings,” arXiv:hep-th/9302110.

[84] C. N. Pope, L. J. Romans and X. Shen, “A NEW HIGHER SPIN ALGEBRA AND THE LONE STAR PRODUCT,” Phys. Lett. B **242**, 401 (1990).

[85] E. Bergshoeff, C. N. Pope, L. J. Romans, E. Sezgin and X. Shen, “THE SUPER W(infinity) ALGEBRA,” Phys. Lett. B **245**, 447 (1990).

[86] M. R. Gaberdiel and P. Suchanek, “Limits of Minimal Models and Continuous Orbifolds,” JHEP **1203**, 104 (2012). [arXiv:1112.1708 [hep-th]].

[87] A. Castro, E. Hijano, A. Lepage-Jutier and A. Maloney, “Black Holes and Singularity Resolution in Higher Spin Gravity,” arXiv:1110.4117 [hep-th].

[88] M. P. Blencowe, “A Consistent Interacting Massless Higher Spin Field Theory In D = (2+1),” Class. Quant. Grav. **6**, 443 (1989).

[89] J. Hansen, P. Kraus, “Generating charge from diffeomorphisms,” JHEP **0612**, 009 (2006). [hep-th/0606230].

[90] P. Kraus and F. Larsen, “Partition functions and elliptic genera from supergravity,” JHEP **0701**, 002 (2007) [arXiv:hep-th/0607138].

[91] D. Z. Freedman, S. D. Mathur, A. Matusis, L. Rastelli, “Correlation functions in the CFT(d) / AdS(d+1) correspondence,” Nucl. Phys. **B546**, 96-118 (1999). [hep-th/9804058].

[92] J. M. Figueroa-O’Farrill, J. Mas, E. Ramos, “A One parameter family of Hamiltonian structures for the KP hierarchy and a continuous deformation of the nonlinear W(KP) algebra,” Commun. Math. Phys. **158**, 17-44 (1993). [hep-th/9207092].

[93] T. Creutzig, Y. Hikida and P. B. Ronne, “Higher spin AdS$_3$ supergravity and its dual CFT,” JHEP **1202**, 109 (2012). [arXiv:1111.2139 [hep-th]].

[94] C. Candu and M. R. Gaberdiel, “Supersymmetric holography on AdS_3.”

[95] M. Henneaux, G. Lucena Gomez, J. Park and S. -J. Rey, “Super-W(infinity) Asymptotic Symmetry of Higher-Spin AdS_3 Supergravity,” [arXiv:1203.5152 [hep-th]].

[96] K. Hanaki and C. Peng, “Symmetries of Holographic Super-Minimal Models,” [arXiv:1203.5768 [hep-th]].

[97] E. Bergshoeff, B. de Wit and M. A. Vasiliev, "The Structure of the superW(infinity) (lambda) algebra," Nucl. Phys. B **366**, 315 (1991).