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EQUIVALENT-POTENTIAL CALCULATION OF πN SCATTERING

Jerome Finkelstein

August 26, 1966

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ABSTRACT

The equivalent-potential method is extended to permit a calculation of $(\pi N)'$ states with $J = \ell + \frac{1}{2}$. With the potential given by nucleon and $N^*(1238)$ exchange, there are no free parameters in the calculation. The N^* is then "predicted" at a mass of 1100 MeV. The nonresonant phase shifts also agree in a general way with the results of phase-shift analysis; in particular, the S_{11} scattering length has the correct sign, while N/D calculations produce the wrong sign. It is argued that this result indicates that the force due to iteration of the potential, included in this method but not in N/D, can be important.

I. INTRODUCTION

An important problem in strong interaction physics has been the calculation of scattering amplitudes from input "forces," which are assumed known. One way of doing this is the N/D approximation. A few years ago Charap and Fubini[1] and Balázs[2] suggested an alternative procedure for these calculations. In a previous paper[3] I reported an application of this procedure to $\pi\pi$ and πK scattering. In this paper I extend the method to apply to the πN amplitude.

At moderate energies the "force," or at least the long-range part of it, may be considered to arise from the exchange of simple systems in the crossed channel. Let us suppose that we have written down a satisfactory representation of the force, and wish to construct the corresponding scattering amplitude. More precisely, suppose that we believe we know the nearby t discontinuity $A_t^B(s,t)$ (where s and t are the squares of the center-of-mass energy and of the momentum transfer, respectively, and all amplitudes have definite exchange parity), and that we will settle for an amplitude $A(s,t)$ which (a) satisfies elastic unitarity, and (b) has its t discontinuity equal to A_t^B, at least for small t. The reason that such an amplitude might be satisfactory is, of course, that the closest singularities are included correctly. We can certainly use the N/D equations to insure requirements (a) and (b). However, since A_t^B does not contain the iterations of the input forces, the amplitude we obtain will not have some of the properties that

it would if the iterations were included: for example, correct threshold behavior will not automatically appear.

As an alternative to the N/D approximation, we could consider, for spinless external particles, the amplitude $f(s,t)$ obtained from a Schrödinger equation with the energy-dependent potential

$$V(r,s) = \frac{-1}{2\pi\mu_R} \int_{t_0}^{\infty} dt\, 2s^{-1/2}\, A_t^B(s,t)\, e^{-r\sqrt{t}}/r \,, \qquad (1)$$

μ_R being the reduced mass. This potential is actually the first term (the long-range part) of an iteratively constructed potential discussed in Refs. 1 and 2 which, when inserted into the Schrödinger equation, would reproduce exactly the scattering amplitude generated by the Mandelstam unitarity iteration.[4] However, the amplitude f already satisfies requirements (a) and (b); moreover, since it comes from a Schrödinger equation, it will, under one stipulation, have correct threshold behavior and the structure in t which we expect from the Mandelstam representation. This stipulation is that $V(r,s)$ be less singular than r^{-2} at the origin, which from (1) means A_t^B goes to zero faster than $t^{-1/2}$ at large t. Whenever the input forces are given as pole terms, this requirement is surely satisfied.

Although solving a Schrödinger equation implies going off the mass shell, it is not necessary in this method to assume anything

about off-shell scattering, since the only input is the on-shell potential A_t^B. Blankenbecler and Sugar[5] have recently proposed a method of making dynamical calculations which shares with this one the feature of including the force due to iterated exchange, but which requires the off-shell potential.

If the external particles do have spin, there may be several coupled amplitudes, in which case Eq. (1) becomes more complicated. Balázs has tried to write a Schrödinger equation for the πN amplitude which is a matrix in spin space.[6] He found that the attractive potential corresponding to nucleon exchange behaved like r^{-3} at the origin, which he proposed to handle with a cutoff. My approach will be different: to define a single, unitary amplitude, which I will call $F(s,t)$, and its nearby t discontinuity F_t^B. I can then use (1) together with the Schrödinger equation to recover F from a knowledge of F_t^B. In Section II below I construct this amplitude, discuss its crossing relations, and discover that my method is applicable to πN states with $J = \ell + \frac{1}{2}$ but not those with $J = \ell - \frac{1}{2}$. In Section III I display the potentials corresponding to N and N^* that I have used, and in Section IV present numerical results and a few conclusions.

II. SPIN-INDEPENDENT AMPLITUDES

The kinematics of πN scattering have been summarized by, for example, Frautschi and Walecka,[7] whose notation I shall use. Consider the partial-wave amplitudes $f_{\ell}^{\pm}(s)$ for orbital angular momentum ℓ and $J = \ell \pm \frac{1}{2}$, which are normalized to $e^{i\delta} \sin \delta/q$ under the assumption of elastic unitarity. Then let $F_{\pm}$ be defined by

$$F_{\pm}(s,t) = \sum (2\ell + 1)\, f_{\ell}^{\pm}(s)\, P_{\ell}\,(1 + t/2q^2) \qquad (2)$$

in the physical region, and by analytic continuation wherever the sum in (2) does not converge. The sum defining F_+ begins at $\ell = 0$, the sum for F_- at $\ell = 1$. Since the $f_{\ell}^{\pm}$ satisfy the unitarity of spinless particles, so will F_+ and F_-. That is, we have the familiar relations

$$\operatorname{Im} F_{\pm}(s,t) = (q/4\pi) \int d\Omega\, F_{\pm}(s,t')\, F_{\pm}^{*}(s,t'') \;; \qquad (3)$$

F_+ and F_- are not coupled by unitarity.

The price we pay for eliminating spin from the unitarity equation is that the crossing relations become complicated. In the Appendix, the following points are established: the exchange of a particle contributes to $F_{\pm}(s,t)$ not only a pole, but also a cut extending from the pole position to $t = +\infty$. Thus $F_{t\pm}^{B}$ corresponding to a single-particle exchange force can be written

$$F_{t\pm}{}^{B}(s,t) = g_{\pm}(s)\ \delta(t - t_p) + h_{\pm}(s)\ \phi_{\pm}(s,t)\ , \qquad (4)$$

where $g_{\pm}$ and $h_{\pm}$ are kinematic factors, t_p is the position of the pole in $F_{\pm}$, and $\phi_{\pm}$ is zero if $t < t_p$. Also, at large t, $|\phi_{+}(s,t)| < \text{const.} \times t^{-3/2}$ and $\phi_{-}(s,t) > \text{const.} > 0$.

From Eq. (1), this means that the potential corresponding to $F_{t-}{}^{B}$ behaves like r^{-3} at the origin. Thus I cannot use a (nonsingular) Schrödinger equation to satisfy requirements (a) and (b) applied to F_{-} (as could also have been seen from the fact that F_{-}' has no s wave). For this reason my method does not enable me to calculate scattering in those states with $J = \ell - \frac{1}{2}$. However, the potential corresponding to $F_{t+}{}^{B}$ behaves like r^{-1} at the origin, and so I am able to produce an amplitude satisfying (a) and (b). In fact, it will turn out that for the range of energies considered, the effect on the amplitude of ϕ_{+} is very small and might as well have been neglected; this is consistent with the hope that for moderate energies we need only consider the long-range parts of forces.

To summarize this section, the plan is as follows: to construct the potential corresponding to particle exchange according to Eqs. (1) and (4) with the + sign, and to numerically solve the resulting Schrödinger equation for the amplitude f. This amplitude will, within the limitations of the approximations implicit in my method, coincide with $F_{+}(s,t)$, whose partial waves are the physical partial waves $F_{\ell+}$.

III. THE POTENTIAL

I consider the forces due to the exchange of the nucleon and the $N^*(1238)$. The couplings of the ρ are somewhat uncertain, and in any case the ρ contribution is expected to be small.[8] Since one of the virtues of the method I use is that there need be no adjustable parameters, I simply neglect the ρ force.

The contributions of N and N^* exchange in the u channel to the invariant amplitudes A and B have been given by Ball and Wong.[9] For the amplitudes of isotopic spin $(\frac{1}{2}, \frac{3}{2})$ these contributions are

$$A^B(s,t,u) = (4/3, 1/3)\, g_{N^*}^2\, (a_1 - a_2 s)/(\Delta^2 - u) \ ,$$

$$B^B(s,t,u) = -(4/3, 1/3)\, g_{N^*}^2\, (b_1 - b_2 s)/(\Delta^2 - u) + (1, -2)\, g_N^2/(M^2 - u) \ , \tag{5}$$

where in units in which $m_\pi = c = \hbar = 1$, the N^* mass Δ is 8.9, $g_{N^*}^2/8\pi = 0.06$, $g_N^2/4\pi = 14.4$, $a_1 = 842$, $a_2 = 23.4$, $b_1 = -157$, and $b_2 = 1.5$. Projecting the partial waves of definite exchange parity from (5), we have

$$A_\ell^{B\pm}(s) = \pm (4/3,\ 1/3)\ g_{N^*}^2 \left[(a_1 - a_2 s)/q^2\right] Q_\ell(1 + t_{N^*}/2q^2) ,$$

$$B_\ell^{B\pm}(s) = \mp (4/3,\ 1/3)\ g_{N^*}^2 \left[(b_1 - b_2 s)/q^2\right] Q_\ell(1 + t_{N^*}/2q^2)$$

$$\pm (1,-2)\ g_N^2 \left[1/q^2\right] Q_\ell(1 + t_N/2q^2) , \qquad (6)$$

where, because of the unequal mass kinematics, the pole positions t_N and t_{N^*} depend on s :

$$t_N = M^2 - (M^2-1)^2/s ,$$

$$t_{N^*} = \Delta^2 - (M^2-1)^2/s . \qquad (7)$$

I now drop the subscript $\pm$ from $F_{t\pm}^B$, since I can work only with F_{t+}^B, and adopt a superscript $\pm$ to indicate exchange parity, so that F_t^{B+} is physical for $J - \frac{1}{2}$ even, and F_t^{B-} for $J - \frac{1}{2}$ odd. Then from Eqs. (6) and (A13),

$$F_t^{B\pm}(s,t) = \pm \Big\{ (1,-2)\ g_N^2 \Big[(C_{12} - C_{22}/\beta_R)\ 2\pi\ \delta(t - t_N)$$

$$- (C_{22}/q^2)\ \mathrm{Im}\ \hat{S}(1 + t/2q^2,\ 1 + t_N/2q^2)\ \theta\ (t - t_N) \Big]$$

$$+ (4/3,\ 1/3)\ g_{N^*}^2 \Big[((a_1 - a_2 s)\ (C_{11} - C_{21}/\beta_R) + (b_1 - b_2 s)$$

$$\times\ (-C_{12} + C_{22}/\beta_R))\ 2\pi\ \delta(t - t_{N^*})$$

(Equation 8 cont'd)

$$+ (1/q^2)\,(-(a_1-a_2 s)\,C_{21} + (b_1-b_2 s)\,C_{22})$$

$$\mathrm{Im}\,\hat{S}(1 + t/2q^2,\ 1 + t_{N^*}/2q^2)\,\theta\,(t - t_{N^*})\Big\}\,, \qquad (8)$$

where $\beta_R = 1 + t/2q^2 + \left[(2 + t/2q^2)\,(t/2q^2)\right]^{1/2}$, the C's are given by Eq. (A3), and $\hat{S}$ is given by Eq. (A12). The Schrödinger equation potential is given by

$$V^{\pm}(r,s) = \frac{-1}{2\pi\mu_R}\int_{t_N}^{\infty} dt\, F_t^{B^{\pm}}(s,t)\, e^{-r\sqrt{t}}/r\,, \qquad (1')$$

the $2s^{-1/2}$ normalization factor in Eq. (1) being already contained in $F_t^{B^{\pm}}$. The numerical values of V depend only on the experimentally determined masses M and Δ and coupling constants g_N^2 and $g_{N^*}^2$.

IV. RESULTS AND CONCLUSIONS

As mentioned above, only the phase shifts for the states with $J = \ell + \frac{1}{2}$ could be calculated. Below 1500 MeV total energy, the calculations show one resonance, in the P_{33} state at 1100 MeV, just above threshold. Its width Γ is 1.5 MeV; this corresponds to a reduced width $\lambda = m^2 \Gamma/q^3$ of 6.2, to be compared with the experimental value of 13.5. The nonresonant phase shifts also agree in a general way with the results of phase-shift analysis, even though there were no parameters that were adjusted to make them agree. In Figs. 1 and 2 the nonresonant phase shifts with $\ell \leq 2$ are compared with the 0- to 700-MeV phase-shift analysis by Roper.[10]

At higher energies the results were not consistent with experiment. Between 1500 and 2500 MeV, the only resonances to appear were a second P_{33} at 1600, and an S_{11} at 2140, and this is clearly wrong. In particular, the Regge recurrences of the $N^*(1238)$ never appear; the trajectory rises only to $\ell = 2.1$ at 1920 MeV. However, the slope of the trajectory at the resonant energy is 0.9 $(\text{GeV})^{-2}$, which is the same slope as that obtained from a straight-line fit to the N^* and its observed recurrences.[11] This result is not surprising if we believe that, while at low energies the Δ trajectory is primarily coupled to the πN channel, at higher energies channels with higher thresholds (and probably higher external spin) are important. In general, my method could not be expected to be correct at high energies, for at least three reasons: at high energies (i) the simple form of the generalized

potential is not justified, (ii) the assumption of elastic unitarity is wrong, and (iii) the differences between the present calculation and the full Mandelstam iteration becomes more acute. Therefore it is understandable that the calculation should fail above 1500 MeV and still produce reasonable results at lower energies, where these three faults seem not to be so important.

The reported results were all obtained from the potential given by Eq. (8). If the cut in F^B has been neglected ($\hat{S}$ set equal to zero), none of the qualitative features would have been changed.

It is interesting to try to understand the relation between the calculation reported here and elastic N/D calculations. For this purpose let us suppose that the assumptions common to both methods, such as elastic unitarity and the particular choice of the generalized potential, were correct. Under this supposition, the approximation involved in N/D is to neglect contributions to the left-hand cut of all but the lowest Mandelstam unitarity iteration. As mentioned in Section I, solving the Schrödinger equation means including all the terms of the unitarity iteration. Only the lowest term is included exactly (with relativistic kinematics)--to include them all would require an infinite iteration just to construct the potential--but it seems reasonable to hope that this is better than neglecting them altogether. If this be correct, then a given (attractive) generalized potential should produce more

scattering in the present method than in N/D , which neglects the attraction produced by iteration of the potential. For a repulsive potential, the iterations alternate in sign, and hence tend to cancel; taking only the lowest-order contribution to the left-hand cut means using too much repulsion. We would expect, then, that in the calculation described in this paper attractive forces would appear stronger, and repulsive forces weaker, than in N/D calculations.

Unfortunately, this comparison is made difficult by the fact that with an adjustable cutoff, any force can be made as strong as one pleases. We then have to push the argument further: if N/D neglects important attractive contributions to the left-hand cut, then in order to obtain a resonance or bound state at the correct mass, it is necessary to make the cutoff higher than if the extra attraction were included. This means that the D function would change more slowly with energy, and residues would thus be greater. So if the method used in this paper is a reliable approximation to the unitarity iteration, we would expect it to produce smaller residues than N/D calculations. Indeed, this is the case. The residue of the N^* reported here is only one half of the physical value, while N/D calculations predict it to be too large.[9,12] Also, in a previous study of $\pi\pi$ scattering using the same methods,[3] the residues, although larger than the experimental values, came out smaller (and the trajectories came out steeper) than in analogous N/D calculations.

However, a comparison of the effective strengths of forces in my method and in N/D can be made in a more direct way. Consider the case of two forces, of opposite signs but comparable magnitudes. If the forces are of the same range, then they will cancel within the Born term. If they are of different ranges, the iteration will make the attractive one stronger, and the repulsive one weaker, although changing the cutoff might not affect the relative strengths of the two forces.

An example of such a case can be found in the πN S_{11} partial wave, where N^* exchange is repulsive and N is attractive. Abers and Zemach[8] estimate the magnitude of the N^* force to be 1.1 times that of the N force; the important point is that they be comparable. Also, the ranges are quite different: because of the unequal π and N masses, the ratio of the ranges of the two forces is not $M/\Delta \approx 0.75$ but rather is t_N/t_{N^*} which at threshold is about 0.44. The facts that the energy dependence of the two forces is different, and that the coupled P_{11} amplitude, although far away, is strong, make difficult the application of the above reasoning to the N/D calculation of the S_{11}. Nevertheless, N/D calculations do produce a negative scattering length (i.e., a net repulsive force),[9,12] although it is known that the scattering length is positive.[10,13] Coulter and Shaw[12] obtained a negative scattering length even when they took account of inelasticity.

As can be seen from Fig. 1a, the scattering length predicted by my calculations is positive. Its value turns out to be

$0.29\ m_\pi^{-1}$; Hamilton and Woolcock[13] give a value of $0.17\ m_\pi^{-1}$. The potential I used differed from that used in Refs. 9 and 12 in that I did not include the force due to ρ exchange. However, since the ρ force is attractive in the S_{11} state, including it would not have decreased the attraction.

One might suspect that the failure of N/D calculations for the S_{11} state indicates a failure of the assumptions, in particular that unknown short-range forces are very important, at least for the s wave. The results presented here suggest the opposite: that when iterations of the potential are taken into account, simple N and N^* exchange is adequate to obtain a reasonable fit to low-energy πN scattering.

ACKNOWLEDGMENTS

I would like to thank Professor Geoffrey F. Chew for many helpful conversations about this problem.

APPENDIX

We need to know the contribution to $F_{t\pm}$ of a particle pole in the crossed reaction. The invariant amplitudes A and B satisfy simple crossing relations, so this contribution to A and B is a pole in t or u. The partial waves will then be given by expressions of the form

$$A_\ell(s) = K_1(s)\, Q_\ell(z_0) ,$$

$$B_\ell(s) = K_2(s)\, Q_\ell(z_0) . \qquad \text{(A1)}$$

Comparison with Eq. (6) shows, for example, for nucleon exchange, $K_1(s) = 0$, $K_2(s) = \pm\,(1,-2)\, g_N^2/q^2$, $z_0 = 1 + t_N(s)/2q^2$. In this appendix we determine $F_\pm$ when A_ℓ and B_ℓ are given by (A1).

From Ref. 7,

$$F_{\ell\pm}(s) = C_{11}\, A_\ell(s) + C_{12}\, B_\ell(s) + C_{21}\, A_{\ell\pm1}(s) + C_{22}\, B_{\ell\pm1}(s) , \qquad \text{(A2)}$$

the matrix C being given by

$$C = \frac{1}{32\pi W^2} \begin{pmatrix} (W+M)^2 - \mu^2 , & \left[(W+M)^2 - \mu^2\right]\left[W - M\right] \\ -(W-M)^2 + \mu^2 , & \left[(W-M)^2 - \mu^2\right]\left[W + M\right] \end{pmatrix}, \qquad \text{(A3)}$$

with $W^2 = s$ and μ = pion mass. Substituting (A1) and (A2) into the definition

$$F_{\pm}(s,t) = \sum (2\ell + 1)\, P_\ell(1 + t/2q^2)\, f_{\ell\pm}(s)\,, \qquad (A4)$$

we get

$$F_{\pm}(s,t) = (K_1C_{11} + K_2C_{12}) \sum (2\ell + 1)\, P_\ell(1 + t/2q^2)\, Q_\ell(z_0) + (K_1C_{21} + K_2C_{22}) \times \sum (2\ell + 1)\, P_\ell(1 + t/2q^2)\, Q_{\ell\pm1}(z_0)\,. \qquad (A5)$$

The first sum is easy:

$$\sum_{\ell=0}^{\infty} (2\ell + 1)\, P_\ell(z)\, Q_\ell(z_0) = 1/(z_0 - z)\,. \qquad (A6)$$

The second sum is not so easy. Define

$$S_{\pm}(z, z_0) = \sum (2\ell + 1)\, P_\ell(z)\, Q_{\ell\pm1}(z_0)\,. \qquad (A7)$$

For s fixed in the physical region, z_0 will be fixed and greater than 1. We shall need to evaluate the discontinuities and the asymptotic behavior of $S_{\pm}$ in the z plane for fixed z_0. The sums in (A7) converge only in an ellipse passing through z_0, so it is necessary to do the sums where they converge, and then continue in z [this continuation is implied in writing (A6)].

Let us first sum S_+ for $1 < z < z_0$. We can use Laplace's integral representations for P_ℓ and Q_ℓ:

$$P_\ell(z) = (1/\pi)\int_0^\pi d\theta \left[\alpha(z,\theta)\right]^\ell , \quad \alpha \equiv z + (z^2-1)^{1/2}\cos\theta ,$$

$$Q_\ell(z_0) = \int_1^\infty dx\, (x^2-1)^{-1/2}\left[\beta(z_0,x)\right]^{-\ell-1} , \quad \beta \equiv z_0 + (z_0^2-1)^{1/2} x . \tag{A8}$$

Although P_ℓ is an entire function, let us choose to stay on the sheet of $(z^2 - 1)^{1/2}$ in which $(z^2 - 1)^{1/2} \to z$ at large $|z|$. From (A6) and (A7),

$$\hat{S}(z,z_0) \equiv S_+(z,z_0) - \frac{1}{\beta_R}\,\frac{1}{z_0 - z} = \sum (2\ell + 1)\, P_\ell(z)\, Q_{\ell+1}(z_0)$$

$$- \frac{1}{\beta_R} \sum (2\ell + 1)\, P_\ell(z)\, Q_\ell(z_0) . \tag{A9}$$

The choice $\beta_R = z + (z^2-1)^{1/2}$ will mean that S has no pole. Now substitute (A8) into (A9):

$$\hat{S}(z,z_0) = \frac{1}{\pi}\sum (2\ell + 1) \int_1^\infty \frac{dx}{(x^2-1)^{1/2}} \int_0^\pi d\theta \left(\frac{1}{\beta} - \frac{1}{\beta_R}\right)\alpha^\ell\, \beta^{-\ell-1} . \tag{A10}$$

The sum can be done inside the integrals,

$$\sum (2\ell + 1)\, \alpha^\ell\, \beta^{-\ell-1} = \frac{1}{\beta - \alpha} + \frac{2\alpha}{(\beta-\alpha)^2} , \tag{A11}$$

since $|\alpha/\beta| < 1$ throughout the region of double integration. We can now do the integral over θ to get

$$\hat{S}(z,z_0) = \frac{-1}{\beta_R}\int_1^\infty \frac{dx}{(x^2-1)^{1/2}} \frac{\beta^2-1}{\beta(\beta^2-2\beta z+1)^{1/2}} \frac{1}{\beta - z + (z^2-1)^{1/2}} \quad . \tag{A12}$$

It is straightforward to show that this integral exists for all z except for $z = z_0$ and $z \in [-1,+1]$, that $|S| \to \text{const} \times |z|^{-3/2}$ at large $|z|$, and that S has a cut from -1 to $+1$ (which does not appear in S_+) and another from z_0 to $+\infty$. Combining (A5), (A6), and (A9), we have

$$F_+(s,t) = \left[(K_1C_{11} + K_2C_{12}) + (K_1C_{12} + K_2C_{22})/\beta_R\right]/(z_0-z) + (K_1C_{21} + K_2C_{22})\,\hat{S}(z,z_0) \quad , \tag{A13}$$

with $z = 1 + t/2q^2$ and $\beta_R = z + (z^2-1)^{1/2}$. $\hat{S}$ can be evaluated numerically from (A12). F_t^B is the imaginary part of (A13), which is 0 if $z < z_0$.

We can sum $S_-(z,z_0)$ in a similar way, and obtain

$$S_-(z,z_0) = \frac{\beta_R}{z_0 - z} + 4\int_1^\infty \frac{dx}{(x^2-1)^{1/2}} \cdot \frac{1}{(\beta^2-2\beta z+1)^{1/2}} \times \left[\frac{\beta z - 1}{\beta - z + (z^2-1)^{1/2}} - \frac{(z^2-1)^{1/2}\,(\beta - z - (z^2-1)^{1/2})}{\beta - z - (z^2-1)^{1/2} + (\beta^2-2\beta z+1)^{1/2}}\right], \tag{A14}$$

but the imaginary part of this integral is $\geq \pi/4$ at large z.

FOOTNOTES AND REFERENCES

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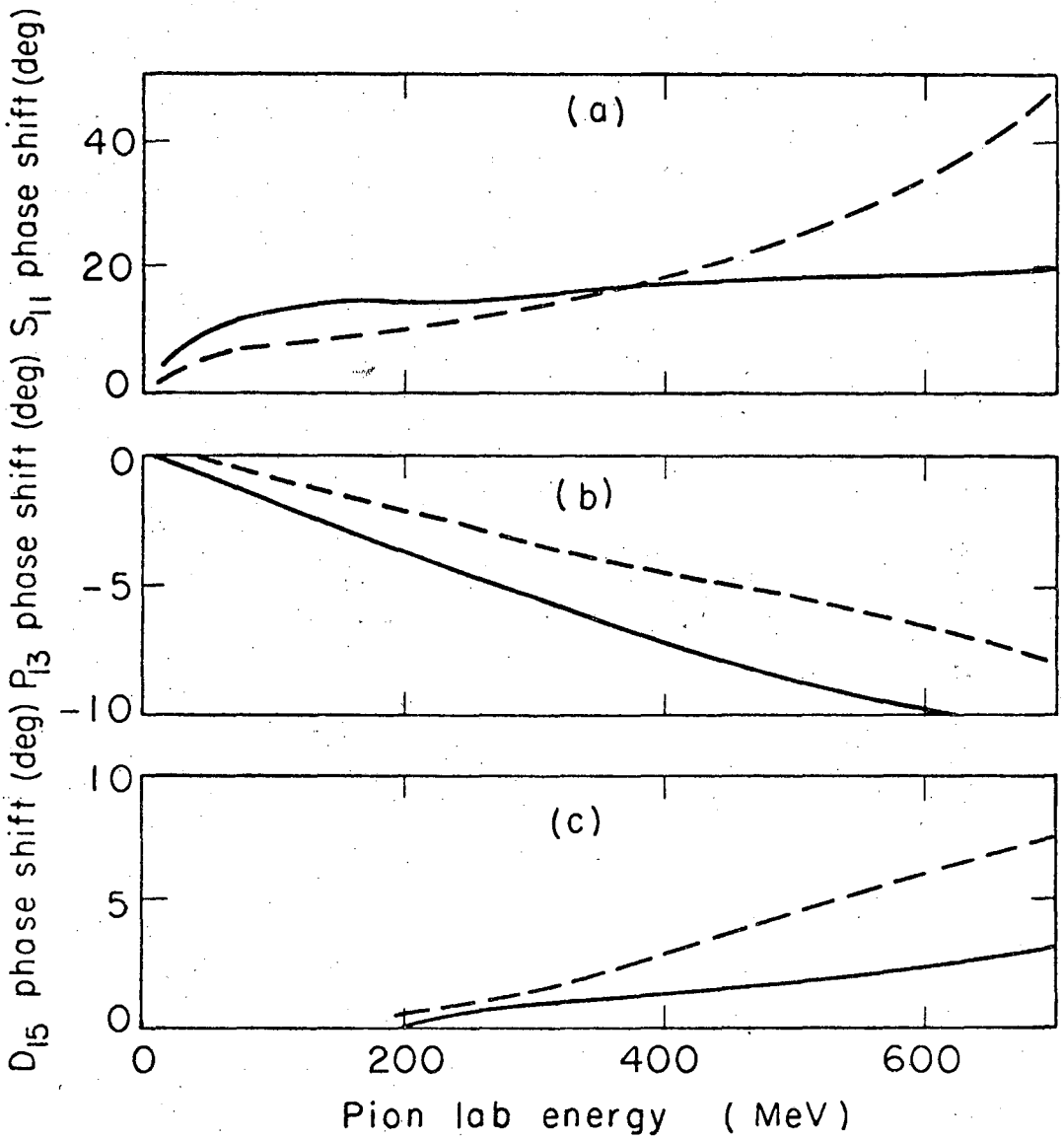
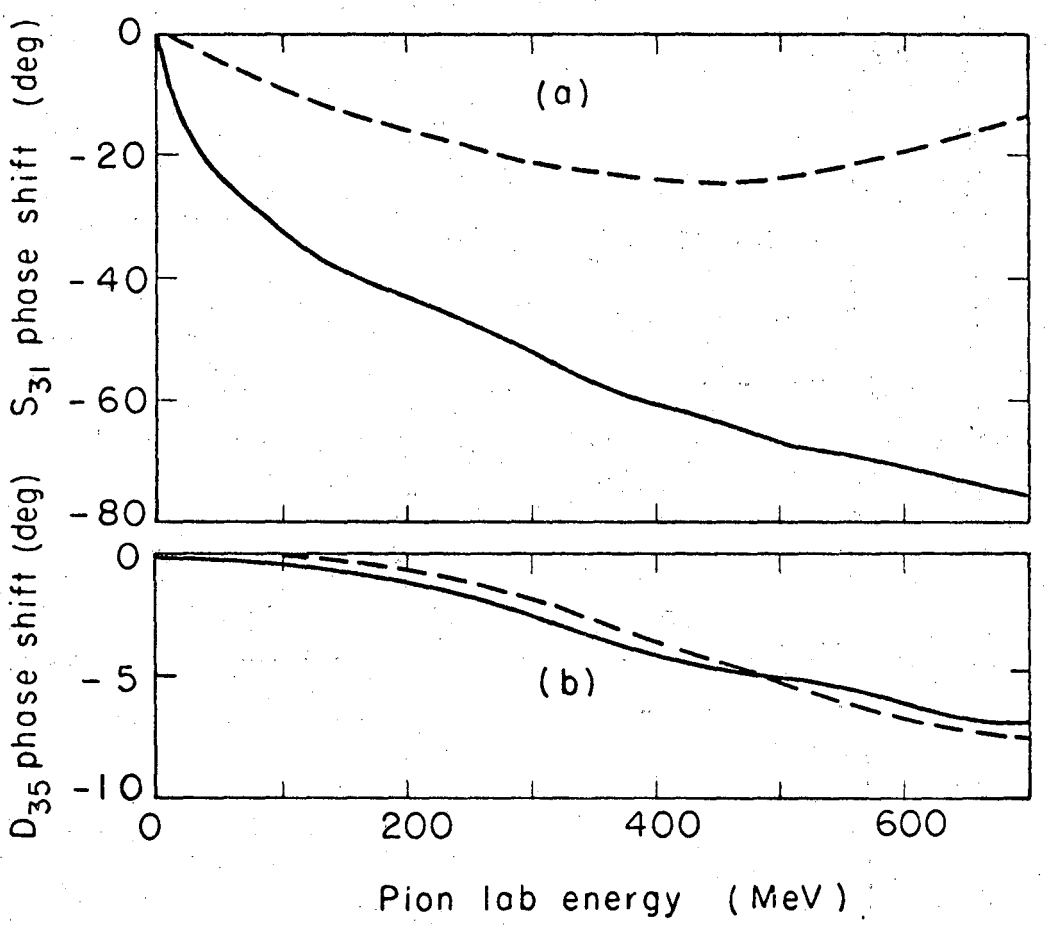
FIGURE CAPTIONS

Fig. 1. $I = \frac{1}{2}$ phase shifts for $\ell \leq 2$ and $J = \ell + \frac{1}{2}$. Solid lines are the phase shifts calculated in this paper; dashed lines the results of Roper, Wright, and Feld (Ref. 10). (a) S_{11} phase shift; (b) P_{13} phase shift; (c) D_{15} phase shift.

Fig. 2. Same as Fig. 1 for nonresonant $I = \frac{3}{2}$ phase shifts. (a) S_{31} phase shift; (b) D_{35} phase shift.


(a)
(b)
(c)
40
20
0
0
-5
-10
10
5
0
0
200
400
600
Pion lab energy (MeV)
D_{15} phase shift (deg) P_{13} phase shift (deg) S_{11} phase shift (deg)

MUB-12719


0
-20
-40
-60
-80
(a)
S_{31} phase shift (deg)
0
-5
-10
(b)
D_{35} phase shift (deg)
0
200
400
600
Pion lab energy (MeV)

MUB 12718

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