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STRING THEORY AND HOLOMORPHIC LINE BUNDLES^{*,†}

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ABSTRACT

Some elementary facts about holomorphic line bundles are discussed along with some applications to string theory.

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1 INTRODUCTION

The current interest in string theory has forced many of us to learn new subjects in mathematics. This is not an easy task. The intuition used in physics and in mathematics is quite different. Things which are obvious to us are completely obscure to mathematicians and *vice versa*. Often one can relate the abstract mathematical theorems to more familiar constructs. In this talk I wish to show how a theorem about line bundles is closely related to material in the physicist's repertoire. This theorem plays an important role in the algebraic geometry approach to string theory. Many of the ideas in algebraic geometry are just souped up complex analysis.

The current era of the applications of complex analysis to string theory begins with the work of Polyakov¹⁾. Many groups²⁾ have subsequently developed the subject in many directions. This is not an exhaustive list. A good introduction to the mathematics may be found in the book by Chern³⁾.

Complex analysis enters into string theory because of several elementary facts. We are interested in studying orientable closed strings. If τ is the time evolution parameter and σ is the position along the string then the string satisfies the wave equation

$$\partial_{\pm} X^{\mu} = 0, \quad (1)$$

where \pm refer to the variables $\tau \pm \sigma$ and X^{μ} is the position of the string in spacetime. The general solution to the wave equations is a linear superposition of left and right moving waves. A right moving solution ϕ_R may be characterized by

$$\phi_R = \phi_R(\tau - \sigma) \implies \partial_{+} \phi_R = 0. \quad (2)$$

Likewise, a left moving solution is characterized by

$$\phi_R = \phi_R(\tau + \sigma) \implies \partial_{-} \phi_R = 0. \quad (3)$$

If one analytically continues to Euclidean time then the wave equation becomes the Laplace equation and the variables $\tau \pm \sigma$ become

$$\tau + \sigma = \bar{z} = \tau - i\sigma, \quad (4)$$

$$\tau - \sigma = z = \tau + i\sigma. \quad (5)$$

The statement that one has a right mover becomes the statement that one has ϕ_R only depends on z , *i.e.*, $\partial_{\bar{z}} \phi_R = 0$. One is thus brought to the theory of analytic functions.

From a more geometrical point of view one has that as a string evolves in time it sweeps out a surface. In fact, it sweeps out a cylinder. The interactions of two strings is geometrically very simple. The two strings fuse at a point and turn into a single string. In the evolution picture one has that the two legs of a pair of pants join to form the waist. Therefore, multiple interactions build up complicated two dimensional surfaces. A classic theorem states that any oriented two dimensional manifold is a Riemann surface. We are again brought back to the theory of analytic functions.

In the nineteenth century, Weirstrass tried to classify analytic functions by their zeroes and poles. This led to the detailed study of meromorphic functions. We will see how the study of such objects plays an important role in string theory. The equations of motion for some string theories wind up being questions about analyticity. For example, the equation of motion for a Weyl fermion is a statement of analyticity. The conservation equation for the energy momentum tensor in a conformally invariant theory is also a statement of analyticity. Pedagogy demands that we begin our study with a discussion of the fundamental theorem of algebra.

2 THE FUNDAMENTAL THEOREM OF ALGEBRA

Gauss gave several distinct proofs of the fundamental theorem of algebra. One of the proofs emphasized the interplay between topology and complex analysis. It may have been the first introduction of the topological concept of the *winding number*. It also used the notion of an *invariance*. If one alters the lower order terms of the polynomial then the zeroes migrate but the total number of zeroes remains constant. One should keep in mind that Gauss did not publish all his results. He was reluctant to publish his discoveries in non-euclidean geometry. He also discovered many of the properties of analytic functions and did not publish them.

Consider a polynomial of degree $n > 0$. P may be viewed as a map $P : \mathbb{C} \rightarrow \mathbb{C}$. Consider a circle S_R centered at the origin and of radius $R \gg 1$. If $z = R \exp(i\theta)$ then a standard estimate shows that

$$P(z) = R^n \exp(in\theta) \left(1 + O\left(\frac{1}{R}\right) \right). \quad (6)$$

Since $P(z)$ is well defined on \mathbb{C} it follows that on the very large circle S_R , $P(z)$

winds around n times as z winds once around S_R . Gauss observed that

$$n = \frac{1}{2\pi i} \int_{S_R} \frac{P'(z)}{P(z)} dz. \quad (7)$$

If P has no zeroes then $P'(z)/P(z)$ is an entire function and by Cauchy's theorem the above integral would vanish. This contradiction requires that P must have at least one root. One can straightforwardly show that P must have n -zeroes.

The fundamental theorem of algebra has a generalization to meromorphic functions. If $f(z)$ is a meromorphic function and if γ is a simply closed positively oriented curve which does not pass through any of the zeroes or poles of f then the number of zeroes inside the curve minus the number of poles inside the curve is given by

$$\# \text{ zeroes} - \# \text{ poles} = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz. \quad (8)$$

It is easy to see why the theorem is true. If z_0 is a zero or pole then near z_0 the function may be written as

$$f(z) = (z - z_0)^n \times [b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \dots], \quad (9)$$

where n is the order of the zero or pole and $b_0 \neq 0$. Consequently one has

$$\frac{f'(z)}{f(z)} = \frac{n}{z - z_0} + \text{analytic near } z_0. \quad (10)$$

The integral over the contour γ may be deformed into integrals over small contours around each of the zeroes and poles.

3 LINE BUNDLES

Whenever one has a field in a quantum field theory one asks "What is it?". Usually, the first question concerns the spin of the field. In a string theory one can ask a more refined question, "the field is a section of which line bundle?" In this section we address the nature of a line bundle.

We will begin our study by considering the motion of a particle in three dimensional space \mathbb{R}^3 in quantum mechanics. We will see that a line bundle is the space where the Schrödinger wavefunction resides. Given a point x in \mathbb{R}^3 one gives a complex number $\psi(x)$. In the Cartesian product $\mathbb{R}^3 \times \mathbb{C}$, the Schrödinger wavefunction

may be represented by its graph. The line bundle is $\mathbb{R}^3 \times \mathbb{C}$, \mathbb{R}^3 is called the base, \mathbb{C} is called the fiber and ψ is called a section of the bundle.

The problem becomes mathematically more interesting if one restricts the motion of the particle to a sphere S^2 . Assume that the particle is electrically charged and that there is a magnetic monopole at the center of the sphere. Dirac showed many years ago that a globally defined wavefunction was impossible. The wavefunction had to have a singularity somewhere.

We will now construct the space where the wavefunction resides. This space will locally look like the Cartesian product $S^2 \times \mathbb{C}$ but it has a non-trivial global structure. Over the northern hemisphere one can choose a non-singular wavefunction ψ_+ . Over the southern hemisphere one can choose a non-singular wavefunction ψ_- . If these two wavefunctions are to define the same physics then there must be a gauge transformations $\exp(i\lambda)$ near the equator which relates them:

$$\psi_+ = \exp(i\lambda) \psi_- . \quad (11)$$

There is a constraint on the function λ . If the wavefunction is going to be single valued then as one goes around the equator the change in λ must be such that $\Delta\lambda = 2\pi n$. Notice that the gauge transformation cannot be globally defined!

Gauge invariance requires that the vector potential in the upper and lower hemispheres to be related by

$$A_+ - A_- = d\lambda . \quad (12)$$

There is a connection between the gauge transformation and the total magnetic flux through the sphere. If N and S respectively denote the northern and southern hemispheres then

$$\int_{S^2} F = \int_N F + \int_S F , \quad (13)$$

$$= \int_N dA_+ + \int_S dA_- , \quad (14)$$

$$= \int_{\partial N} A_+ - \int_{\partial S} A_- , \quad (15)$$

$$= \int_{\partial N} (A_+ - A_-) , \quad (16)$$

$$= \int_{\partial N} d\lambda , \quad (17)$$

$$= \Delta\lambda , \quad (18)$$

$$= 2\pi n . \quad (19)$$

There are several remarks that one can make. The topology of the sphere imposes some constraints. The gauge transformations has to satisfy some restrictions and this imposes a restriction on the flux.

The generalization of these ideas is the theory of complex line bundles. Consider a compact manifold X with an open cover $\{U_a\}$. Locally the line bundle is given by $U_a \times \mathbb{C}$. The local Cartesian products are put together by giving a transition function (gauge transformation) g_{ab} on each non-empty overlap $U_a \cap U_b$. The transition function is a complex valued non-vanishing function on the overlap. One does not care about its behavior outside of the overlap. On a non-empty triple overlap $U_a \cap U_b \cap U_c$ they must satisfy the consistency condition $g_{ab}g_{bc}g_{ca} = 1$. The object that one constructs by putting together this collection of Cartesian products is called a complex line bundle. The manifold X is called the base. A section of the line bundle is given by a collection $\{\psi_a\}$ of locally defined complex valued functions such that on the overlaps they are related by the appropriate transition function:

$$\psi_a = g_{ab}\psi_b . \quad (20)$$

If one defines a covariant derivative on the line bundle L by $D = d + A$, then the curvature (field strength) is $F = dA$. The consistency condition on the triple overlaps forces an analogue of the flux quantization condition. The quantity $c_1(L)$ defined by

$$c_1(L) = \frac{i}{2\pi} \int_X F \quad (21)$$

must be an integer. $c_1(L)$ is called the first Chern class of the line bundle L .

It is easy to verify that the set of all line bundles over X forms an abelian group. If L and L' are line bundles with transition functions $\{g_{ab}\}$ and $\{g'_{ab}\}$ respectively then the product bundle LL' is defined to be the line bundle with transition functions $\{g_{ab} \cdot g'_{ab}\}$. One can show that $c_1(LL') = c_1(L) + c_1(L')$. The identity element of this group I is the line bundle with 1 for its transition functions. This bundle is just the complex valued functions over X .

The inverse bundle L^{-1} is just the bundle with the reciprocal transition functions.

4 HOLOMORPHIC LINE BUNDLES

A holomorphic line bundle is a line bundle where the transition functions are required to be analytic. Such a strong constraint leads to some very powerful and

useful theorems. We discuss one of these theorems in this section. The terms analytic and holomorphic will be used interchangeably. The gauge transformation is given by $\psi'(z, \bar{z}) = g(z)\psi(z, \bar{z})$. Since the transition function is independent of \bar{z} , it is clear that one can define covariant derivatives which do not involve a gauge potential A_z :

$$\nabla_z \psi = \partial_z \psi + A_z \psi, \quad (22)$$

$$\nabla_{\bar{z}} \psi = \partial_{\bar{z}} \psi. \quad (23)$$

Notice that the last line is covariant with respect to holomorphic gauge transformations. The field strength F is given by

$$F_{z\bar{z}} = \partial_{\bar{z}} A_z - \partial_z A_{\bar{z}} = \partial_{\bar{z}} A_z. \quad (24)$$

This is gauge invariant with respect to holomorphic gauge transformations. Normally one thinks that F and polynomials in F are the only gauge invariant quantities available. This is not true in the case of a line bundle. One can define a gauge invariant one-form almost everywhere. This one-form has components given by:

$$B_z = \frac{\nabla_z \psi}{\psi} = \partial_z \log \psi + A_z, \quad (25)$$

$$B_{\bar{z}} = \frac{\nabla_{\bar{z}} \psi}{\psi} = \partial_{\bar{z}} \log \psi. \quad (26)$$

It is easy to verify that the above is gauge invariant with respect to holomorphic transition functions. It is a well defined quantity where ψ is non-vanishing. Note that the curl of B is given by $\partial_{\bar{z}} B_z - \partial_z B_{\bar{z}} = F_{z\bar{z}}$.

We would like to now restrict ourselves to meromorphic sections, $\partial_{\bar{z}} \psi = 0$. For such a section $B_{\bar{z}} = 0$. Let us study the behavior of a section near a point z_0 where one has a 'zero' of order n_0 . If $n_0 > 0$ then one has a legitimate zero of order n_0 . If $n_0 < 0$ then one has a pole of order $|n_0|$. Consider a gauge transformation $\psi'(z) = g(z)\psi(z)$ near z_0 . Since $g(z_0)$ is non-vanishing one must have

$$\psi(z) = (z - z_0)^{n_0} \left[a_0^0 + a_0^1(z - z_0) + a_0^2(z - z_0)^2 + \dots \right], \quad (27)$$

$$\psi'(z) = (z - z_0)^{n_0} \left[a_0'^0 + a_0'^1(z - z_0) + a_0'^2(z - z_0)^2 + \dots \right], \quad (28)$$

where $a_0^0 \neq 0$ and $a_0'^0 \neq 0$. Note that under the gauge transformation there are only two invariants: z_0 and n_0 . A gauge transformation cannot change the location of

the zero or pole, nor can it change its order. This introduces one to the notion of a divisor. The divisor associated with the section ψ is defined to be the formal sum

$$(\psi) = \sum_{\alpha} n_{\alpha} z_{\alpha}. \quad (29)$$

This is just a formal object that keeps track of the zeroes and poles. The order of the divisor is simply $\sum n_{\alpha}$. Note that in Section 2 we computed the 'order' of a meromorphic function in a simply connected domain. We now turn to the computation of the order of a meromorphic section. This will be given by integral formulas defined over the whole Riemann surface X .

Theorem 1 *If L is a holomorphic line bundle then the number of zeroes minus the number of poles of a meromorphic section is given by the first Chern class $c_1(L)$ of the line bundle:*

$$\frac{i}{2\pi} \int_X F.$$

Near each z_0 consider a small open set R_{α} containing z_0 . In the domain $X - \cup R_{\alpha}$, the section ψ is everywhere non-vanishing. Consider the following integral

$$\int_{X - \cup R_{\alpha}} F = - \sum_{\alpha} \int_{\partial R_{\alpha}} B, \quad (30)$$

$$= - \sum_{\alpha} \int_{\partial R_{\alpha}} \frac{\nabla_z \psi}{\psi} dz, \quad (31)$$

$$= - \sum_{\alpha} \int_{\partial R_{\alpha}} \left(\frac{\partial_z \psi}{\psi} + A_z \right) dz. \quad (32)$$

Let us now ask about what happens to the above as one shrinks the size of the R_{α} . Notice that the left hand side approaches

$$\int_X F = (-2\pi i) c_1(L). \quad (33)$$

Since the vector potentials are smooth on each R_{α} it follows that as we shrink the R_{α} , the right hand side approaches $-2\pi i n_{\alpha}$ on each open set. This follows from our discussion of the fundamental theorem of algebra. In conclusion we have

$$c_1(L) = \frac{i}{2\pi} \int_X F = \sum_{\alpha} n_{\alpha}. \quad (34)$$

Note that since each n_{α} is gauge invariant it does not matter which gauge we choose on each R_{α} to do the calculation.

This theorem has several important consequences. The first Chern class $c_1(L)$ depends on the line bundle and not on the section chosen. It is a topological invariant of a line bundle. Thus one learns that all meromorphic sections of a line bundle have the same order. The 'magnetic flux' through the surface determines the 'net number' of zeroes and poles of a section.

There is an immediate corollary of this theorem. *If a line bundle L has a negative Chern class then it has no holomorphic sections.* In other words, any meromorphic section must have at least one pole.

The most important line bundle on a Riemann surface X is the *canonical bundle* K . This is just the holomorphic line bundle whose sections are forms of type $(1,0)$. A form is said to be of type (p,q) if when written in a local complex coordinate system it is of degree p in dz and of degree q in $d\bar{z}$. For example, the curvature is of type $(1,1)$. The first Chern class $c_1(K)$ is essentially the genus of the surface:

$$c_1(K) = 2(g - 1). \quad (35)$$

A compact two dimensional manifold without boundary is topologically equivalent to a sphere with g handles. Note that on a sphere ($g = 0$), there are no holomorphic forms of type $(1,0)$.

Consider the bundle K^{-1} . These are just the vector fields which in a local complex coordinate system can be written as $V^{\alpha} \partial/\partial z^{\alpha}$. Since $c_1(K^{-1}) = -c_1(K)$ it follows that if $g > 1$ then there are no holomorphic vector fields on surfaces with genus greater than one.

On a sphere one has that $c_1(K^{-1}) = 2$ therefore a holomorphic vector field must have two zeroes. If one stereographically projects the sphere onto the plane then the vector field $V^{\alpha} = \alpha + \beta z + \gamma z^2$ is globally analytic. Remember that the point at infinity is a point on the sphere. The coordinates which relate the neighborhood of a point at infinity to a neighborhood of the origin is $z = 1/w$ where w are the local coordinates for a neighborhood of infinity. Under such a change of coordinates one has $V^w = -w^2 V^z$. One thus has that V is analytic everywhere. The α term corresponds to a double zero at infinity, the β term to a simple zero at the origin and a simple zero at infinity, and the γ term to a double zero at the origin. Note that there are no other possibilities for holomorphic vector fields. For example $V^z = z^3$ has a simple pole at infinity. Yet it does define a meromorphic section with a triple

zero at the origin and a simple pole at infinity. Notice that the order of such a section is two as required by the theorem.

The theorem becomes more powerful if one combines it with the following:

Theorem 2 (Riemann-Roch) *If $\gamma(L)$ denotes the number of linearly independent holomorphic sections of a line bundle L then*

$$\gamma(L) - \gamma(L^{-1}K) = (1 - g) + c_1(L).$$

The Riemann-Roch theorem has a multitude of applications. It is most powerful when one has a bundle such that one of the two terms on the left hand side of the above vanishes. In such a situation one can count the number of holomorphic sections of the other bundle.

The theorem can be used to prove that there exist line bundles with holomorphic sections. We prove this theorem in the case $g > 1$. Firstly, notice that if L is any bundle such that $c_1(L) > 0$ then there exists a positive integer r such that $c_1(L^r) > c_1(K) > 0$. Consequently, one has that $L^{-r}K$ has no holomorphic sections since its Chern class is negative. Inserting this into the Riemann-Roch theorem shows that $\gamma(L^r) > g - 1$. L^r has at least $g - 1$ holomorphic sections.

Consider the bundle K^3 for $g > 0$. This bundle has positive Chern class and the remarks of the previous paragraph apply. Inserting into Riemann-Roch yields that K^3 has exactly $3(g - 1)$ linearly independent holomorphic sections. The square of the canonical bundle is called the bundle of quadratic differentials. The number of holomorphic quadratic differentials counts the number of independent complex structures on a Riemann surface.

If one applies Riemann-Roch to the canonical bundle of an arbitrary Riemann surface one has

$$\gamma(K) - \gamma(I) = g - 1. \quad (36)$$

The global analytic functions on a Riemann surface are the constants therefore $\gamma(I) = 1$. We conclude that the number of holomorphic differentials is g , the genus of the surface. This is the analytic way of defining the genus of a Riemann surface.

Applying Riemann-Roch to K^{-1} for a sphere one learns that there are exactly three holomorphic vector fields. Verifying our explicit calculation.

The Riemann-Roch theorem is least powerful when one applies it to spin bundles. A spin bundle σ is a square root of the canonical bundle, $\sigma^2 = K$. On a Riemann

surface of genus g there are 4^g inequivalent spin bundles. Note that $\sigma^{-1}K = (\sigma)^{-2}K\sigma = \sigma$. The left hand side of the Riemann-Roch theorem is identically zero and one cannot extract any information.

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