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Higher Auslander-Reiten Theory

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Jacob Richard West

March 2015

Dissertation Committee:

Professor Julia E. Bergner, Co-Chairperson
Professor Wee Liang Gan, Co-Chairperson
Professor Jacob Greenstein

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ABSTRACT OF THE DISSERTATION

Higher Auslander-Reiten Theory

by

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Doctor of Philosophy, Graduate Program in Mathematics

University of California, Riverside, March 2015

Professor Julia E. Bergner, Co-Chairperson

Professor Wee Liang Gan, Co-Chairperson

A development of Auslander-Reiten theory in the language of stable ∞ -categories is presented. An ∞ -category is a special kind of simplicial set which provides a common generalization of ordinary categories and nice topological spaces. Higher Auslander-Reiten theory can therefore be understood as a homotopy-theoretic analogue of the classical theory, which has proved to be an indispensable tool in many areas of representation theory.

We begin by introducing almost-split and irreducible morphisms in ∞ -categories and establish their basic properties in direct analogy with the classical notions. We go on to describe morphisms determined by objects, a generalization of almost-split morphisms originally formulated by Auslander that until recently received little attention. We prove, using Brown representability, that every collection of maps with domain a compact object C uniquely determines, up to homotopy, a minimal C -determined morphism. From this result, the existence of almost-split and irreducible morphisms can be deduced.

We next describe the analogues of almost-split sequences in stable ∞ -categories and prove that they exist in any compactly generated stable ∞ -category with sufficiently small

objects. Building on our earlier study of morphisms determined by objects, we then introduce Auslander functors on stable ∞ -categories and show that they always exist on compactly generated stable ∞ -categories. In good circumstances, Auslander functors specialize to Serre functors. This observation leads to a general duality formula which specializes to the classical Auslander-Reiten formula on the homotopy category.

Finally, we focus on an important class of examples of compactly generated stable ∞ -categories associated to any Noetherian algebra over a complete local Noetherian ring. In this situation, we give a construction of an Auslander-Reiten translation functor and explain how it recovers the classical Auslander-Reiten translation on the associated (triangulated) homotopy category.

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Chapter 1

Introduction

Higher Auslander-Reiten theory is a reformulation and generalization of classical Auslander-Reiten theory in the language of higher categories. Broadly speaking, this endeavor lies somewhere in the intersection of representation theory and homotopy theory. This work is a contribution to the ongoing efforts of many authors over the last 40 years to develop and understand Auslander-Reiten theory in the broadest generality which supports it. Auslander began these investigations in [5] working with generalized module categories. Happel subsequently introduced Auslander-Reiten theory in triangulated categories in his groundbreaking work on representations of finite dimensional algebras [24, 25, 26], which was later extended to arbitrary compactly generated triangulated categories by Krause [36]. Here we develop Auslander-Reiten theory in stable ∞ -categories, which may be viewed as a higher categorical refinement of triangulated categories [46]. In particular, the theory developed here recovers the Auslander-Reiten theory in the triangulated category setting. In this chapter, we briefly review the history and early development of Auslander-Reiten

theory as an important tool in representation theory. We then describe some of the ways that Auslander-Reiten theory has been used and how it might be regarded as inherently homotopical, motivating higher categories as an appropriate setting for this theory. We conclude with an overview of the material covered in this work, highlighting the main results from each chapter.

1.1 Background

Auslander-Reiten theory originated in the early 1970s as a systematic approach to the study of certain structural features common to categories of representations of Artin algebras. A centerpiece of the theory is a special kind of short exact sequence called an *almost-split sequence*, which may be thought of as the next simplest kind of short exact sequence after split exact sequences. Let \mathcal{A} be an abelian category. A short exact sequence $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ in \mathcal{A} is called *almost-split* if the following conditions are satisfied:

- (1) The sequence $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ is not split exact.
- (2) Any morphism $X \rightarrow X'$ in \mathcal{A} which is not a split monomorphism factors through f .
- (3) Any morphism $Z' \rightarrow Z$ in \mathcal{A} which is not a split epimorphism factors through g .

We summarize the situation with the following diagram:

$$\begin{array}{ccccccc}
 & & & & & & Z' \\
 & & & & & & \downarrow \\
 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \longrightarrow 0. \\
 & & \downarrow & & \swarrow & & \\
 & & X' & & & &
 \end{array}$$

Observe that conditions (2) and (3) are always satisfied by split exact sequences. Auslander introduced almost-split sequences in [2, 3] to facilitate his proof (and generalization) of the first Brauer-Thrall Conjecture, which asserts that a finite dimensional algebra is either of finite representation type (i.e., there are only finitely many isomorphism classes of indecomposable modules over this algebra) or else there are indecomposable modules of arbitrarily large dimension. Auslander and Reiten subsequently established an important existence theorem and many basic properties of almost-split sequences in their joint work on the representation theory of Artin algebras [10, 11, 12, 13], thereby marking the beginning of Auslander-Reiten theory. The classical theory of almost-split sequences found many applications in the representation theory of finite dimensional algebras, as summarized for instance by Reiten in [50] or Auslander in [7], and culminated in the books [21, 14].

The early success of the theory encouraged many authors to explore whether Auslander-Reiten theory could be formulated in contexts broader than the original setting of finitely generated modules over Artin algebras. This exploration began already with Auslander and Reiten in their study of dualizing R -varieties, where R is a commutative Artinian ring [9]. Auslander also established the existence of almost-split sequences in various categories of modules over Noetherian k -algebras, with k a commutative Noetherian ring which is complete and local [6]. Lenzing and Zuazua subsequently characterized the existence of almost-split sequences in any Ext-finite abelian k -category, where k is a commutative Artinian ring [43]. Generalizing in a different direction, Happel introduced *Auslander-Reiten triangles* in triangulated categories, as the correct replacement for almost-split sequences in abelian categories, and proved their existence in the bounded derived category of a finite

dimensional algebra over a field [24]. Krause later showed that Auslander-Reiten triangles exist in any compactly generated triangulated category containing sufficiently small objects [36]. In efforts to unify the abelian and triangulated settings, Beligiannis introduced Auslander-Reiten theory in abstract homotopy categories [16] and Liu developed the theory in Krull-Schmidt categories [44], both works emphasizing the homotopy-theoretic nature of the concepts and tools arising in the theory.

1.2 Motivation

The existence of almost-split sequences (or their generalization in Auslander-Reiten triangles) in a category provides significant structural and finiteness information useful in classification theorems. As already mentioned, the earliest example of this was in Auslander’s work on the representation type of rings. Auslander showed in [3] that if an Artinian ring R has finite representation type, then the category of R -modules has almost-split sequences. Conversely, Auslander proved in [4] that if the category of modules over a connected Artinian ring R has almost-split sequences and the associated Auslander-Reiten quiver has a component of bounded length modules, then R has finite representation type. Another example of the finiteness exhibited by the existence of an Auslander-Reiten theory arises in Happel’s work [24, 26] showing that the bounded derived category of a finite dimensional algebra A has Auslander-Reiten triangles if and only if A has finite global dimension. Continuing with this theme, a recent preprint [1] argues that for a virtually Gorenstein algebra A , the bounded Gorenstein derived category of A has Auslander-Reiten triangles if and only if A is Gorenstein. Exercising the existence of Auslander-Reiten triangles in

a different way, Jorgensen shows in [29, 31] that for a simply connected topological space X with $\dim_k H^*(X; k) < \infty$ and k a field, the bounded derived category associated to the differential graded algebra of singular cochains on X over k has Auslander-Reiten triangles if and only if X admits Poincaré duality over k . More generally, Reiten and van den Bergh proved that for a k -linear triangulated category \mathcal{C} which is Hom-finite and Krull-Schmidt, \mathcal{C} has Auslander-Reiten triangles if and only if \mathcal{C} satisfies Serre duality [49]. While there are still more examples, the above illustrate the kinds of finiteness and duality phenomena that Auslander-Reiten theory makes available and how this information is used in classification.

An essential construction in Auslander and Reiten's original work on almost-split sequences was their translation functor (the dual of the Auslander-Bridger transpose [8]), from which the existence of almost-split sequences was deduced. Crucially, this construction is only well-defined on the stable module category; that is, the Auslander-Reiten translation functor is well-defined up to stable equivalence, a notion of equivalence weaker than isomorphism. Moreover, Auslander's early work in functor categories and Happel's introduction of Auslander-Reiten triangles reflected the understanding that short exact sequences are a structural feature that is stronger than necessary to support Auslander-Reiten theory. Indeed, Beligiannis employs homotopy (co)limits to build an Auslander-Reiten theory in abstract homotopy categories [16].

Here we make the case that $(\infty, 1)$ -categories are the most natural setting for Auslander-Reiten theory, encompassing all previous constructions. Roughly, an (∞, n) -category should be thought of as a category equipped with k -morphisms for all $k > 0$ with the property that a k -morphism is an equivalence when $k > n$, for an appropriate notion of equivalence.

In the $n = 1$ case, there are at least five different models of $(\infty, 1)$ -categories which are all known to be equivalent. For a survey of these various models, see [17]. A particularly flexible implementation of $(\infty, 1)$ -categories is Joyal’s theory of quasi-categories [32, 33, 34, 35], which are simplicial sets satisfying the weak Kan condition of Boardman and Vogt [18]. This theory has been extensively developed by Lurie under the name ∞ -categories [45], and subsequently used to encode a great deal of homotopical algebra in [46]. By construction, higher categories naturally encode weaker notions of equivalence, and it turns out that limits and colimits in $(\infty, 1)$ -categories generalize homotopy limits and colimits. While a high tolerance for abstraction is often necessary when working with stable ∞ -categories, the upshot is that this setting often allows for more intuitive arguments and intrinsic characterizations. In this way, the present work helps provide a unifying conceptual framework for understanding the important ideas and tools of Auslander-Reiten theory.

1.3 Overview

We now discuss the organization and main results of this dissertation. Throughout this work, we rely heavily on the ideas and results found in the books [45] and [46] by Lurie. We will adopt the notation and conventions employed in those books.

In Chapter 2, we begin by reviewing some of the ideas and machinery of ∞ -categories [23, 34, 45] and stable ∞ -categories [46], introducing the preliminary material necessary for developing Auslander-Reiten theory in this setting. A common generalization of ordinary categories and (nice) topological spaces, ∞ -categories are simplicial sets satisfying the weak Kan condition of Boardman and Vogt [18]. Stable ∞ -categories enjoy an exact structure

that is reminiscent of classical abelian categories. An important property of every stable ∞ -category \mathcal{C} is that its (canonically associated) homotopy category $\mathrm{h}\mathcal{C}$ is triangulated (see [46, Theorem 1.1.2.15]). However, unlike its homotopy category, a stable ∞ -category remembers *why* morphisms are homotopic and should therefore be regarded as a refinement of (topological) triangulated categories. A significant feature of this refinement is that the construction of fibers and cofibers in a stable ∞ -category is functorial (in contrast with triangulated categories). Moreover, the fibers and cofibers of a stable ∞ -category provide the exact structure needed to support the construction of an Auslander-Reiten theory. While nearly all the material in this chapter can be found elsewhere in the literature, especially [45, 46], including it here is necessary to ensure that the content in this thesis is reasonably self-contained.

Classical Auslander-Reiten theory identifies several inter-related classes of morphisms which play a distinguished role. In Chapter 3, we introduce these various morphisms into the ∞ -categorical setting and establish their basic properties. An important theme of this chapter is that definitions are always made relative to the homotopy category. Indeed, this is a general principle in $(\infty, 1)$ -category theory because it ensures that the ideas and results are homotopy invariant. After introducing almost-split morphisms in ∞ -categories, we give in Proposition ?? a characterization of right almost-split morphisms in stable ∞ -categories in terms of their associated cofiber sequences. If a right almost-split morphism is also minimal, then Proposition 3.4.12 provides a characterization in terms of the associated fiber sequences. There is a close relationship between almost-split morphisms and irreducible morphisms, as clarified by Theorem 3.5.5 which characterizes irreducible morphisms in sta-

ble ∞ -categories. Auslander recognized very early on that almost-split sequences arose as a special case of a much more general notion, which he called *morphisms determined by objects* [5]. We study morphisms determined by objects in Section 3.6, proving a useful characterization in Theorem 3.6.5 and discussing some closure properties of this class of morphisms. The main result of this chapter is Theorem 3.7.4 which uses Brown representability to prove that there exists morphisms in any compactly generated stable ∞ -category which are right determined by compact objects. From this result, we deduce the existence of right almost-split and irreducible morphisms.

In Chapter 4, we define *Auslander-Reiten sequences* in stable ∞ -categories, the higher categorical analogues of almost-split sequences. After establishing their basic properties, we give a characterization in Proposition 4.1.2 of Auslander-Reiten sequences in terms of minimal almost-split morphisms. Our definition has the expected property that Auslander-Reiten sequences in a stable ∞ -category induce Auslander-Reiten triangles in its associated triangulated homotopy category and all Auslander-Reiten triangles arise in this way. Inspired by Krause [36], we use a version of Brown representability to show in Theorem 4.1.5 that Auslander-Reiten sequences exist in any compactly generated stable ∞ -category with strongly indecomposable objects. In Section 4.2, we investigate the functorial relationship between the end terms of an Auslander-Reiten sequence. To this end, we introduce the notion of an *Auslander functor* on a compactly generated stable ∞ -category and establish in Theorem 4.2.12 that such functors always exist and are unique up to homotopy. Using this result, we formulate an *Auslander-Reiten duality* formula and give a construction of an *Auslander-Reiten translation* functor in Corollary 4.2.16.

In Chapter 5, we specialize to an important class of examples of compactly generated stable ∞ -categories called *derived ∞ -categories*, arising from ordinary abelian categories. Specifically, to any abelian category \mathcal{A} satisfying some mild conditions one can associate a stable ∞ -category $\mathcal{D}(\mathcal{A})$ whose homotopy category is canonically equivalent to the classical derived category of \mathcal{A} (see [46, 1.3]). In this case, Theorem 5.2.3 gives another construction of an Auslander-Reiten translation functor, which is a Serre functor that (in particular) relates the end terms of an Auslander-Reiten sequence. In general such a functor need not exist, but whenever it does the existence of Auslander-Reiten sequences follows as a consequence. Our construction is modeled on the work of Krause and Le [40].

Chapter 2

Preliminaries

In this chapter, we present the preliminary material necessary to develop Auslander-Reiten theory in a higher categorical setting. Specifically, we work with a particular model of $(\infty, 1)$ -categories originally called *weak Kan complexes* by Boardman and Vogt [18], and later *quasi-categories* by Joyal [34]. This thesis relies heavily on the substantial development of this model presented by Lurie in [45] and we will adopt the notation and conventions there, using the term ∞ -categories to refer to the objects of this model. There are however several other models for an $(\infty, 1)$ -category which could be employed, all known to be equivalent [17]. We chose the ∞ -categorical model because its development is the most mature presentation of $(\infty, 1)$ -categories to date. In Section 2.1, we review the definitions and basic machinery we need for working with this model. The presentation here is extremely terse in an effort to introduce the necessary ideas and material as rapidly as possible. For a more detailed introduction to the rich theory of ∞ -categories, see [23, 20] and [45, Chapter 1].

One of the primary structures in Auslander-Reiten theory is a special kind of exact

sequence. In Section 2.3, we discuss the features of a particularly nice class of ∞ -categories, called *stable ∞ -categories*, which are equipped with the necessary exact structure for studying Auslander-Reiten sequences. Nearly all the results in this chapter can be found in [45, 46], as indicated throughout.

2.1 Simplicial sets and ∞ -categories

Definition 2.1.1. Let Δ denote the *simplex category* whose objects are the linearly ordered sets $[n] = \{0, \dots, n\}$ for all integers $n \geq 0$ and morphisms are nondecreasing functions.

Remark 2.1.2. We can and frequently will replace Δ with the much larger (but equivalent) category of *all* finite linearly ordered sets and nondecreasing functions, while still referring to it as Δ by abuse of notation.

Definition 2.1.3. A *simplicial set* is a functor $X: \Delta^{\text{op}} \rightarrow \text{Set}$. A morphism of simplicial sets is a natural transformation of functors. Let Set_Δ denote the category of simplicial sets.

Example 2.1.4. An important example of a simplicial set is the (combinatorial) n -simplex $\Delta^n = \text{Hom}_\Delta(-, [n])$. Observe that this defines an embedding $\Delta \hookrightarrow \text{Set}_\Delta$, $[n] \mapsto \Delta^n$. By Yoneda's lemma, the n -simplices $K[n]$ of a simplicial set K are in bijection with the set of natural transformations $\text{Hom}_{\text{Set}_\Delta}(\Delta^n, K)$. We write K_n to denote either of these sets.

Remark 2.1.5. Let J be a nonempty finite linearly ordered set. Regarding J as an object of Δ as in Remark 2.1.2, we write Δ^J for the simplicial set $\Delta^J = \text{Hom}_\Delta(-, J)$. We frequently use this notation when J is a subset of $[n]$, in which case $\Delta^J \subseteq \Delta^n$ as simplicial sets.

Remark 2.1.6. As with any presheaf category, Set_Δ is complete and cocomplete with limits and colimits computed pointwise. In particular, Set_Δ has an initial object given by the empty simplicial set Δ^{-1} and a final object given by Δ^0 . Moreover, again as with any presheaf category, Set_Δ is a cartesian closed monoidal category. More precisely, if X and Y are simplicial sets, then $X \times Y$ is the simplicial set with $(X \times Y)_n = X_n \times Y_n$ and Y^X is the simplicial set given by $(Y^X)_n = \text{Hom}_{\text{Set}_\Delta}(\Delta^n \times X, Y)$. Then for all simplicial sets X , Y , and Z , we have a functorial bijection $\text{Hom}_{\text{Set}_\Delta}(Z \times X, Y) \cong \text{Hom}_{\text{Set}_\Delta}(Z, Y^X)$. To see this, first observe that if $Z = \Delta^n$, then the bijection is a consequence of the Yoneda Lemma. Using that any simplicial set can be written as a colimit of representables, the result follows by noting that both sides are compatible with the formation of colimits in Z . By setting $Z = Y^X$, we have an evaluation map $e: Y^X \times X \rightarrow Y$ where $e_n: (f, \sigma) \mapsto f(-, \sigma): \Delta^n \rightarrow Y$.

Remark 2.1.7. Let $\delta_i: [n-1] \rightarrow [n]$ be the injective function defined by

$$\delta_i(j) = \begin{cases} j & \text{if } j < i, \\ j+1 & \text{if } j \geq i, \end{cases}$$

and let $\sigma_i: [n+1] \rightarrow [n]$ be the surjective function defined by

$$\sigma_i(j) = \begin{cases} j & \text{if } j \leq i, \\ j-1 & \text{if } j > i. \end{cases}$$

It is tedious but not difficult to check that these functions generate the category Δ (in that every morphism decomposes uniquely as a composition of these). For any simplicial set X , we let $d_i = X(\delta_i)$ and $s_i = X(\sigma_i)$ and refer to these as the *face* and *degeneracy* maps of X , respectively.

Example 2.1.8. The functor $\text{ev}_0: \text{Set}_\Delta \rightarrow \text{Set}$ given by $X \mapsto X_0$, sending a simplicial set X to its set of vertices X_0 , has a left adjoint $c: \text{Set} \rightarrow \text{Set}_\Delta$, which associates to any set S the *constant simplicial set* cS determined by $cS_n = S$ for all $n \geq 0$. The constant functor is fully faithful and determines a monomorphism $cX_0 \rightarrow X$ for any simplicial set X . We say that X is *discrete* if $cX_0 \rightarrow X$ is an isomorphism. The functor $c: \text{Set} \rightarrow \text{Set}_\Delta$ has a left adjoint $\pi_0: \text{Set}_\Delta \rightarrow \text{Set}$, where $\pi_0 X$ is the set of *connected components* of a simplicial set X . More explicitly, $\pi_0 X$ is the coequalizer (in Set) of the diagram

$$X_1 \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} X_0$$

so that two vertices of X_0 are identified in $\pi_0 X$ if there exists an edge between them. We say that X is *connected* if $\pi_0 X$ is a one-point set. One feature of the functor π_0 is that it preserves products.

Example 2.1.9. Let Λ_j^n denote the simplicial set $\Lambda_j^n: \Delta^{\text{op}} \rightarrow \text{Set}$ which sends $[m]$ to the subset of $\text{Hom}_\Delta([m], [n])$ consisting of all those functions $\alpha: [m] \rightarrow [n]$ with the property $\{j\} \cup \alpha([m]) \neq [n]$. By definition, we have $\Lambda_j^n \subseteq \Delta^n$ as simplicial sets. We refer to Λ_j^n as the (n, j) -*horn* or the j -*horn* of Δ^n . If $j \in \{0, n\}$, we call Λ_j^n an *outer horn*, otherwise Λ_j^n is called an *inner horn*. Geometrically, Λ_j^n is the simplicial set obtained from Δ^n by removing the interior and the face opposite vertex j (also called the j th face of Δ^n).

Definition 2.1.10. A simplicial set \mathcal{C} is called an ∞ -*category* if for all $n \geq 2$ and $0 < j < n$, any map $\Lambda_j^n \rightarrow \mathcal{C}$ can be extended to an n -simplex $\Delta^n \rightarrow \mathcal{C}$, as indicated in the diagram

below

$$\begin{array}{ccc} \Lambda_j^n & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow \text{dotted} & \\ \Delta^n & & \end{array}$$

That is, a simplicial set \mathcal{C} is an ∞ -category if every *inner horn* in \mathcal{C} can be lifted to a simplex. Notice that this lift is not required to be unique.

Remark 2.1.11. Definition 2.1.10 was first formulated by Boardman and Vogt [18] under the name *weak Kan complexes* (see Definition 2.1.19 below for the origin of this terminology).

Example 2.1.12. Any ordinary (small) category \mathcal{C} gives rise to an ∞ -category $N(\mathcal{C})$ determined by the equation $N(\mathcal{C})_n = \text{Hom}_{\text{Cat}}([n], \mathcal{C})$, where on the right hand side we regard $[n]$ as a category so that $\text{Hom}_{\text{Cat}}([n], \mathcal{C})$ denotes the set of all functors $[n] \rightarrow \mathcal{C}$. The functor $N: \text{Cat} \rightarrow \text{Set}_\Delta$ is called the (ordinary) *nerve*. It is not difficult to check that the nerve functor is fully faithful. In this way, the theory of ∞ -categories is a generalization of ordinary category theory. Indeed, this generalization is remarkably robust, as shown in [45].

Remark 2.1.13. Motivated by Example 2.1.12, we use the following category-theoretic language when working with ∞ -categories. Let \mathcal{C} be an ∞ -category. We refer to the vertices $\Delta^0 \rightarrow \mathcal{C}$ as *objects* of \mathcal{C} , and to the edges $\Delta^1 \rightarrow \mathcal{C}$ as *morphisms* of \mathcal{C} . We write $X \in \mathcal{C}$ whenever $X: \Delta^0 \rightarrow \mathcal{C}$ is an object of \mathcal{C} . We write $f: X \rightarrow Y$ whenever $f: \Delta^1 \rightarrow \mathcal{C}$ is a morphism of \mathcal{C} with *source* $X = d_1(f)$ and *target* $Y = d_0(f)$. If $X \in \mathcal{C}$ is an object, we write $\text{id}_X = s_0(X): X \rightarrow X$ and refer to this edge as the *identity morphism* on X . We also use the notation $1_X: X \rightarrow X$ for the identity morphism on X .

Example 2.1.14. Any topological space X gives rise to an ∞ -category $\text{Sing}(X)$ determined by $\text{Sing}(X)_n = \text{Hom}_{\mathcal{T}\text{op}}(|\Delta^n|, X)$, where $\text{Hom}_{\mathcal{T}\text{op}}(|\Delta^n|, X)$ is the set of all continuous maps

from the (topological) n -simplex $|\Delta^n| = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n x_i = 1, x_i \geq 0\}$ to X .

The functor $\text{Sing}: \mathcal{Top} \rightarrow \text{Set}_\Delta$ is called the *singular simplicial complex*.

Remark 2.1.15. An important result (due to Quillen) is that $\text{Sing}: \mathcal{Top} \rightarrow \text{Set}_\Delta$ is the right Quillen functor in a Quillen equivalence which identifies simplicial sets with topological spaces (up to weak homotopy equivalence). The left adjoint $|-|: \text{Set}_\Delta \rightarrow \mathcal{Top}$ is called *geometric realization*, computed explicitly as the coequalizer (in \mathcal{Top}) of the diagram

$$\coprod_{[k] \rightarrow [m]} X_k \times \Delta^m \rightrightarrows \coprod_{[n]} X_n \times \Delta^n$$

where each X_n is given the discrete topology and the parallel arrows are determined by the canonical maps $X_m \times \Delta^k \rightarrow X_k \times \Delta^k$ and $X_m \times \Delta^k \rightarrow X_m \times \Delta^m$ for each $[k] \rightarrow [m]$ in Δ .

Definition 2.1.16. A morphism of simplicial sets $X \rightarrow Y$ is a *weak homotopy equivalence* if the induced map $|X| \rightarrow |Y|$ of topological spaces is a weak homotopy equivalence.

Definition 2.1.17. Let K be a simplicial set. The *opposite simplicial set* K^{op} is the functor $K^{\text{op}}: \Delta^{\text{op}} \rightarrow \text{Set}$ determined by setting $K^{\text{op}}([n]) = K([n]^{\text{op}})$, where $[n]^{\text{op}}$ denotes the same set as $[n]$ endowed with the opposite ordering (see Remark 2.1.2).

Remark 2.1.18. A simplicial set \mathcal{C} is an ∞ -category if and only if \mathcal{C}^{op} is an ∞ -category: indeed, \mathcal{C} has the extension property with respect to the horn inclusion $\Lambda_j^n \subseteq \Delta^n$ if and only if \mathcal{C}^{op} has the extension property with respect to the horn inclusion $\Lambda_{n-j}^n \subseteq \Delta^n$.

Definition 2.1.19. A simplicial set K is called a *Kan complex* if for all $n \geq 2$ and $0 \leq j \leq n$, any map $\Lambda_j^n \rightarrow K$ can be extended to an n -simplex $\Delta^n \rightarrow K$, as indicated in the diagram

below

$$\begin{array}{ccc} \Lambda_j^n & \longrightarrow & K \\ \downarrow & \nearrow \text{dotted} & \\ \Delta^n & & \end{array}$$

Let $\mathcal{K}an$ denote the full subcategory of \mathbf{Set}_Δ spanned by the Kan complexes.

Remark 2.1.20. By definition, every Kan complex is in particular an ∞ -category.

Example 2.1.21. Let X be a topological space. Since $|\Delta^n|$ deformation retracts onto each of its horns $|\Lambda_j^n|$, the simplicial set $\mathbf{Sing}(X)$ is a Kan complex. For this reason, we often refer to Kan complexes as *spaces*. Indeed, the Quillen equivalence of Remark 2.1.15 asserts that the unit and counit morphisms $K \rightarrow \mathbf{Sing} |K|$ and $|\mathbf{Sing}(X)| \rightarrow X$ are both weak homotopy equivalences.

Example 2.1.22. Let \mathcal{C} be an ∞ -category. The collection of morphisms between any two objects of an ∞ -category can be organized into a Kan complex (see [45, 1.2.2]). More precisely, for any objects $X, Y \in \mathcal{C}$, let $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \{X\} \times_{\mathcal{C}} \mathcal{C}^{\Delta^1} \times_{\mathcal{C}} \{Y\}$, so that an n -simplex of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a morphism $\sigma: \Delta^n \times \Delta^1 \rightarrow \mathcal{C}$ such that $\sigma|_{\Delta^n \times \{0\}}$ is constant at X and $\sigma|_{\Delta^n \times \{1\}}$ is constant at Y . We use the notation $\mathrm{Map}_{\mathcal{C}}(X, Y)$ to denote any Kan complex with the same weak homotopy type as $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, and refer to $\mathrm{Map}_{\mathcal{C}}(X, Y)$ as the *mapping space* of morphisms from X to Y .

Definition 2.1.23. Let \mathcal{C} be an ∞ -category. We say that two morphisms $f: X \rightarrow Y$ and $g: X \rightarrow Y$ are *homotopic* in \mathcal{C} if there exists a 2-simplex $\sigma: \Delta^2 \rightarrow \mathcal{C}$ of the form

$$\begin{array}{ccc} & X & \\ 1_X \nearrow & & \searrow g \\ X & \xrightarrow{f} & Y. \end{array}$$

In this case, we write $f \simeq g$ and refer to σ as a *homotopy* from f to g .

More generally, we write $h \simeq gf$ if there exists a 2-simplex $\sigma: \Delta^2 \rightarrow \mathcal{C}$ of the form

$$\begin{array}{ccc} & Y & \\ f \nearrow & & \searrow g \\ X & \xrightarrow{h} & Z. \end{array}$$

A morphism $f: X \rightarrow Y$ in \mathcal{C} is called an *equivalence* if there exists a morphism $f': Y \rightarrow X$ such that $ff' \simeq 1_Y$ and $f'f \simeq 1_X$. We say that X and Y are *equivalent* if there exists an equivalence between them. We will generally understand that all meaningful properties of objects are invariant under equivalence. Similarly, all meaningful properties of morphisms are invariant under homotopy and under composition with equivalences.

Proposition 2.1.24 ([45, 1.2.3.5]). *Let \mathcal{C} be an ∞ -category containing objects X and Y . The relation of homotopy is an equivalence relation on the set of edges joining X and Y .*

Proposition 2.1.25 ([45, 1.2.3.8]). *Let \mathcal{C} be an ∞ -category. There exists an ordinary category $\mathrm{h}\mathcal{C}$, called the *homotopy category* of \mathcal{C} , whose objects are the vertices of \mathcal{C} and whose morphisms are homotopy equivalence classes of edges of \mathcal{C} .*

Remark 2.1.26. Let \mathcal{C} be an ∞ -category. A morphism $f: X \rightarrow Y$ in \mathcal{C} is an equivalence if and only if the homotopy equivalence class $[f]$ is an isomorphism in $\mathrm{h}\mathcal{C}$. More generally, we have $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) = \pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y)$.

Definition 2.1.27. Let \mathcal{C} be an ∞ -category. For any simplicial set K , we write $\mathrm{Fun}(K, \mathcal{C})$ to denote the simplicial set \mathcal{C}^K of Remark 2.1.6.

Proposition 2.1.28 ([45, 1.2.7.3]). *Let \mathcal{C} be an ∞ -category. For any simplicial set K , $\mathrm{Fun}(K, \mathcal{C})$ is again an ∞ -category.*

Definition 2.1.29 ([35]). A map of simplicial sets $X \rightarrow Y$ is a *weak categorical equivalence* if for every ∞ -category \mathcal{C} , the induced map $\mathrm{hFun}(Y, \mathcal{C}) \rightarrow \mathrm{hFun}(X, \mathcal{C})$ is an equivalence (of ordinary categories).

Definition 2.1.30. A map of simplicial sets $F: \mathcal{C} \rightarrow \mathcal{D}$ is called a *functor* whenever both \mathcal{C} and \mathcal{D} are ∞ -categories.

Remark 2.1.31. Any functor $F: \mathcal{C} \rightarrow \mathcal{D}$ induces an ordinary functor $\mathrm{h}F: \mathrm{h}\mathcal{C} \rightarrow \mathrm{h}\mathcal{D}$. Moreover, by construction of the mapping space in Example 2.1.22, any functor $F: \mathcal{C} \rightarrow \mathcal{D}$ induces a map of simplicial sets $\mathrm{Map}_{\mathcal{C}}(X, Y) \rightarrow \mathrm{Map}_{\mathcal{D}}(FX, FY)$.

Definition 2.1.32. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor of ∞ -categories. We say that F is *fully faithful* if the induced map $\mathrm{Map}_{\mathcal{C}}(X, Y) \rightarrow \mathrm{Map}_{\mathcal{D}}(FX, FY)$ is a weak homotopy equivalence. We say that F is *essentially surjective* if the induced functor $\mathrm{h}F: \mathrm{h}\mathcal{C} \rightarrow \mathrm{h}\mathcal{D}$ is essentially surjective.

Proposition 2.1.33 ([45, 2.2.5.8]). *A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ of ∞ -categories is a weak categorical equivalence if and only if it is fully faithful and essentially surjective.*

Definition 2.1.34. A simplicial set X is called *weakly contractible* if the geometric realization $|X|$ is weakly contractible, that is, if there exists a weak homotopy equivalence from $|X|$ to the one-point space.

Definition 2.1.35. Let \mathcal{C} be an ∞ -category. An object $X \in \mathcal{C}$ is called *initial* if for every object $Y \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is weakly contractible. Dually, an object $Z \in \mathcal{C}$ is called *final* if for every object $Y \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(Y, Z)$ is weakly contractible.

Initial and final objects are examples of colimits and limits (of empty diagrams). General limits and colimits can be defined in the ∞ -categorical setting (see [45, Chapter 4]), and these possess many of the properties familiar from ordinary category theory. For instance, the following useful result will be used frequently in the sequel.

Lemma 2.1.36 ([45, 4.4.2.1]). *Let \mathcal{C} be an ∞ -category and let $\Delta^2 \times \Delta^1 \rightarrow \mathcal{C}$ be a diagram in \mathcal{C} , depicted as*

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}.$$

Suppose that the left square is a pushout in \mathcal{C} . Then the right square is a pushout if and only if the outer square is a pushout.

Definition 2.1.37 ([45, 5.2.7.6]). Let \mathcal{C} be an ∞ -category and $\mathcal{D} \subseteq \mathcal{C}$ a full subcategory. We will say that a morphism $f: C \rightarrow D$ in \mathcal{C} *exhibits D as a \mathcal{D} -localization of C* if $D \in \mathcal{D}$ and composition with f induces a weak homotopy equivalence

$$\mathrm{Map}_{\mathcal{D}}(D, E) \rightarrow \mathrm{Map}_{\mathcal{C}}(C, E)$$

for every object $E \in \mathcal{D}$. In this situation, we will also say that $f: C \rightarrow D$ is a *localization of C relative to \mathcal{D}* .

Proposition 2.1.38 ([45, 5.2.7.8]). *Let \mathcal{C} be an ∞ -category and $\mathcal{D} \subseteq \mathcal{C}$ a full subcategory.*

The following conditions are equivalent:

- (1) *For every object $C \in \mathcal{C}$, there exists a localization $f: C \rightarrow D$ relative to \mathcal{D} .*
- (2) *The inclusion $\mathcal{D} \subseteq \mathcal{C}$ admits a left adjoint.*

2.2 Additive ∞ -categories

Definition 2.2.1. Let \mathcal{C} be an ∞ -category. A *zero object* of \mathcal{C} is an object which is both initial and final. We say that \mathcal{C} is *pointed* if it contains a zero object.

Lemma 2.2.2. *Let \mathcal{C} be an ∞ -category. Then \mathcal{C} is pointed if and only if the following conditions are satisfied:*

- (1) *The ∞ -category \mathcal{C} has an initial object \emptyset .*
- (2) *The ∞ -category \mathcal{C} has a final object $*$.*
- (3) *There exists a morphism $f: * \rightarrow \emptyset$ in \mathcal{C} .*

Proof. The “only if” direction follows by taking the morphism f to be the identity. Conversely, we have a morphism $g: \emptyset \rightarrow *$ because \emptyset is initial, and moreover $f \circ g \simeq \text{id}_{\emptyset}$. Using that $*$ is final, it must also be that $g \circ f \simeq \text{id}_*$. This shows that f is an equivalence, and hence \mathcal{C} is pointed. \square

Remark 2.2.3. Let \mathcal{C} be an ∞ -category with a zero object 0 . For any $X, Y \in \mathcal{C}$, the natural map

$$\text{Map}_{\mathcal{C}}(0, Y) \times \text{Map}_{\mathcal{C}}(X, 0) \rightarrow \text{Map}_{\mathcal{C}}(X, Y)$$

has contractible domain. We therefore obtain a well-defined morphism $X \rightarrow Y$ in the homotopy category $\text{h}\mathcal{C}$, which we will refer to as the *zero morphism* and also denote by 0 . We say that a morphism is *nonzero* if it is not homotopic to a zero morphism.

Definition 2.2.4. An ∞ -category \mathcal{C} is *additive* if it satisfies the following conditions:

- (1) The ∞ -category \mathcal{C} is pointed.

- (2) The ∞ -category \mathcal{C} admits finite products.
- (3) The ∞ -category \mathcal{C} admits finite coproducts.
- (4) For every pair of objects $X, Y \in \mathcal{C}$, the canonical map $X \amalg Y \rightarrow X \times Y$ classified by the matrix
$$\begin{bmatrix} \text{id}_X & 0 \\ 0 & \text{id}_Y \end{bmatrix}$$
is an equivalence.
- (5) The homotopy category $\text{h}\mathcal{C}$ is enriched in abelian groups.

Remark 2.2.5. The conditions of Definition 2.2.4 are overdetermined. That is, conditions (1), (2), and (5) together imply that $\text{h}\mathcal{C}$ is additive, which implies condition (4); similarly, conditions (1), (3), and (5) together also imply that $\text{h}\mathcal{C}$ is additive. Compare this with [46, Definition 2.4.5.3]. We have chosen to present Definition 2.2.4 in this way because various weaker notions appear in the literature by relaxing some of the above conditions. For instance, in [22, 2.1], the authors say that an ∞ -category is *preadditive* if it satisfies conditions (1), (2), (3), and (4); in [27, 4.4.13], an ∞ -category satisfying conditions (1), (3), and (4) is called *0-semiadditive*.

Remark 2.2.6. Let \mathcal{C} be an ∞ -category satisfying conditions (1), (2), (3), and (4) of Definition 2.2.4. Then $\text{h}\mathcal{C}$ is canonically enriched in commutative monoids. To see this, let $f, g: X \rightarrow Y$ be two morphisms in \mathcal{C} and define $f + g: X \rightarrow Y$ as the composition

$$X \xrightarrow{\delta} X \times X \xrightarrow{f, g} Y \times Y \simeq Y \amalg Y \xrightarrow{\delta'} Y$$

where δ and δ' are the diagonal and codiagonal maps, respectively. This endows $\text{Hom}_{\text{h}\mathcal{C}}(X, Y)$

with the structure of a commutative monoid with identity given by the unique zero morphism from X to Y . Consequently, condition (5) should be understood as requiring the existence of additive inverses. See also [22, Definition 2.6].

Remark 2.2.7. Let \mathcal{D} be an ordinary category that admits finite coproducts. A *cogroup object* of \mathcal{D} is an object $X \in \mathcal{D}$ equipped with a comultiplication map $X \rightarrow X \amalg X$ with the following property: for every object Y , the induced multiplication

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) \times \mathrm{Hom}_{\mathcal{D}}(X, Y) \cong \mathrm{Hom}_{\mathcal{D}}(X \amalg X, Y) \rightarrow \mathrm{Hom}_{\mathcal{D}}(X, Y)$$

determines a group structure on the set $\mathrm{Hom}_{\mathcal{D}}(X, Y)$. By Remark 2.2.6, if \mathcal{C} is an additive ∞ -category, then every $X \in \mathcal{C}$ is a cogroup object of $\mathrm{h}\mathcal{C}$.

Proposition 2.2.8. *Let \mathcal{C} be a pointed ∞ -category which admits finite products and finite coproducts. Then \mathcal{C} is additive if and only if $\mathrm{h}\mathcal{C}$ is additive.*

Proof. The “only if” statement follows from Remark 2.2.5. If $\mathrm{h}\mathcal{C}$ is additive, then in particular $\mathrm{h}\mathcal{C}$ is enriched in abelian groups. Moreover, for any $X, Y \in \mathrm{h}\mathcal{C}$, the canonical map $X \amalg Y \rightarrow X \times Y$ in \mathcal{C} induces an isomorphism in $\mathrm{h}\mathcal{C}$ (because $\mathrm{h}\mathcal{C}$ is additive) and hence determines an equivalence in \mathcal{C} . □

Remark 2.2.9. Let \mathcal{C} be an ∞ -category containing objects X and Y , and satisfying conditions (2), (3), and (4) of Definition 2.2.4. To emphasize the equivalence of condition (4), it is customary to denote both the product and the coproduct by $X \oplus Y$. We refer to $X \oplus Y$ as the *biproduct* or *direct sum* of X and Y .

Definition 2.2.10. Let \mathcal{C} be an additive ∞ -category. An object $X \in \mathcal{C}$ is *decomposable*

if there exist nonzero objects $X_1, X_2 \in \mathcal{C}$ together with an equivalence $X \simeq X_1 \oplus X_2$. An object of \mathcal{C} is called *indecomposable* if it is not decomposable.

Remark 2.2.11. Let \mathcal{C} be an additive ∞ -category. For any object $X \in \mathcal{C}$, it is possible to extract an \mathbb{E}_1 -ring spectrum $\mathrm{End}_{\mathcal{C}}(X)$ with the property that $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, X) = \pi_0 \mathrm{End}_{\mathcal{C}}(X)$ is an (ordinary) associative ring with multiplication given by composition of endomorphisms (see [46, 7.1.2.2]). We will say that an object $X \in \mathcal{C}$ is *strongly indecomposable* if $\pi_0 \mathrm{End}_{\mathcal{C}}(X)$ is a local ring.

The next result establishes the expected relationship. It is an immediate consequence of Lemma 3.1.11 proved in the next chapter.

Proposition 2.2.12. *Let \mathcal{C} be an additive ∞ -category. If $X \in \mathcal{C}$ is strongly indecomposable, then X is indecomposable.*

Definition 2.2.13. Let \mathcal{C} be an additive ∞ -category. An object $X \in \mathcal{C}$ is called *Krull-Schmidt* if X is equivalent to a finite direct sum of strongly indecomposable objects. If every nonzero object of \mathcal{C} is Krull-Schmidt, we will say that \mathcal{C} is a *Krull-Schmidt ∞ -category*.

Proposition 2.2.14. *Let \mathcal{C} be an additive ∞ -category. Then \mathcal{C} is Krull-Schmidt if and only if $\mathrm{h}\mathcal{C}$ is Krull-Schmidt.*

Proof. This follows from the definition together with Proposition 2.2.8. □

2.3 Stable ∞ -categories

Definition 2.3.1. Let \mathcal{C} be a pointed ∞ -category. A *triangle* in \mathcal{C} is a diagram $\Delta^1 \times \Delta^1 \rightarrow \mathcal{C}$, depicted as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

where 0 is a zero object of \mathcal{C} . We say that a triangle in \mathcal{C} is a *fiber sequence* if it is a pullback square, and a *cofiber sequence* if it is a pushout square. We will generally abuse terminology and say that f is a *fiber* of g (respectively, g is a *cofiber* of f) whenever a triangle as above is a fiber sequence (respectively, cofiber sequence). In this situation, we engage in further abuse by simply referring to $X = \text{cofib}(g)$ and $Z = \text{fib}(f)$ as the fiber of g and cofiber of f , respectively. (See Proposition 2.3.4 below.)

Remark 2.3.2. More explicitly, a triangle consists of the following data:

- (1) A pair of morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$.
- (2) A 2-simplex of the form

$$\begin{array}{ccc} & Y & \\ f \nearrow & & \searrow g \\ X & \xrightarrow{h} & Z, \end{array}$$

which identifies h with the composition $g \circ f$.

- (3) A 2-simplex

$$\begin{array}{ccc} & 0 & \\ \nearrow & & \searrow \\ X & \xrightarrow{h} & Z, \end{array}$$

which we view as a *nullhomotopy* of h .

We will generally indicate a triangle by specifying only the pair of maps,

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

with the data of (2) and (3) implicitly assumed.

Definition 2.3.3 ([46, 1.1.1.9]). An ∞ -category \mathcal{C} is *stable* if it satisfies the following conditions:

- (1) There exists a zero object $0 \in \mathcal{C}$.
- (2) Every morphism in \mathcal{C} admits a fiber and a cofiber.
- (3) A triangle in \mathcal{C} is a fiber sequence if and only if it is a cofiber sequence.

The definition of a stable ∞ -category is an axiomatization of the essential features of stable homotopy theory, with axioms analogous to those defining abelian categories. In fact, a central example of a stable ∞ -category is an ∞ -category Sp of spectra, whose homotopy category hSp can be identified with the classical stable homotopy category (see [46, 1.4.3] for more details). An important property of every stable ∞ -category \mathcal{C} is that its homotopy category $\mathrm{h}\mathcal{C}$ is triangulated (see Theorem 2.3.16 below). However, unlike its homotopy category, a stable ∞ -category remembers *why* morphisms are homotopic and should therefore be regarded as a refinement of (topological) triangulated categories. A significant feature of this refinement is that the construction of fibers and cofibers in a stable ∞ -category is functorial (in contrast with triangulated categories).

Proposition 2.3.4 ([46, 1.1.1.7]). *Let \mathcal{C} be a stable ∞ -category. There exist functors*

$$\mathrm{fib}: \mathrm{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}) \text{ and } \mathrm{cofib}: \mathrm{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}),$$

which associate to any morphism in \mathcal{C} a fiber and cofiber sequence, respectively.

Proof Sketch. Let $\mathcal{E} \subseteq \text{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ denote the full subcategory spanned by the cofiber sequences. Let $\theta: \mathcal{E} \rightarrow \text{Fun}(\Delta^1, \mathcal{C})$ be the forgetful functor, written informally as,

$$(X \xrightarrow{f} Y \xrightarrow{g} Z) \mapsto (X \xrightarrow{f} Y).$$

The goal is to prove that θ is a trivial Kan fibration, and therefore admits a section

$$\text{cofib}: \text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \text{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}),$$

which is well-defined up to a contractible space of choices. More explicitly, we decompose this forgetful functor as the composition $\theta = \theta_0 \theta_1$, written informally as,

$$\begin{array}{ccc} X \xrightarrow{f} Y & & X \xrightarrow{f} Y \\ \downarrow & \xrightarrow{\theta_1} & \downarrow \\ 0 \longrightarrow Z & & 0 \end{array} \quad \xrightarrow{\theta_0} \quad X \xrightarrow{f} Y$$

and show that each of the forgetful functors θ_0 and θ_1 are trivial Kan fibrations. The result follows by observing that each domain in the above diagram is an appropriate Kan extension of the corresponding codomain, then applying [45, Proposition 4.3.2.15]. \square

Remark 2.3.5. Composing the fiber functor of Proposition 2.3.4 with evaluation at initial vertex then gives a functor $\text{fib}: \text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C}$, which we refer to using the same name, by abuse of terminology. Similarly, composing the cofiber functor with evaluation at the final vertex gives $\text{cofib}: \text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C}$.

Lemma 2.3.6. *Let \mathcal{C} be a stable ∞ -category. Suppose the following diagram*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \end{array}$$

is a morphism of cofiber sequences. Then

(1) If α and β are equivalences, then γ is also an equivalence.

(2) If β and γ are equivalences, then α is also an equivalence.

Proof. We prove (1), the proof of (2) is similar. Let $\mathcal{E} \subseteq \text{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ be the full subcategory spanned by the cofiber sequences, and let $\theta: \mathcal{E} \rightarrow \text{Fun}(\Delta^1, \mathcal{C})$ denote the forgetful functor, written informally as $\theta: (X \rightarrow Y \rightarrow Z) \mapsto (X \rightarrow Y)$. Statement (1) is that the functor θ is conservative. The construction of the cofiber functor in Proposition 2.3.4 amounts to showing that θ is a trivial Kan fibration (and therefore admits a section which we call the cofiber functor). As a trivial Kan fibration, θ is in particular a left fibration and left fibrations are conservative by [45, 2.1.1.5]. \square

Lemma 2.3.7. *Let \mathcal{C} be a stable ∞ -category. Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a fiber sequence and $\beta: Z' \rightarrow Z$ is any morphism in \mathcal{C} . Then there exists a fiber sequence $X \xrightarrow{f'} Y' \xrightarrow{g'} Z'$ and a morphism $\alpha: Y' \rightarrow Y$ in \mathcal{C} such that the following diagram is a morphism of fiber sequences*

$$\begin{array}{ccccc} X & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \\ \downarrow \text{id}_X & & \downarrow \alpha & & \downarrow \beta \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z. \end{array}$$

Proof. Consider the expanded diagram

$$\begin{array}{ccccc} X & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \\ \downarrow \text{id}_X & & \downarrow \alpha & & \downarrow \beta \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z, \end{array}$$

where the right square is a pullback and the left square (in particular, the map $f': X \rightarrow Y'$) is a consequence of this pullback, so that $g'f' \simeq 0$. To prove that this diagram is a morphism

of fiber sequences, it suffices to show that

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow g' \\ 0 & \longrightarrow & Z' \end{array}$$

is a pullback. For this, consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{f'} & Y' & \xrightarrow{\alpha} & Y \\ \downarrow & & \downarrow g' & & \downarrow g \\ 0 & \longrightarrow & Z' & \xrightarrow{\beta} & Z. \end{array}$$

The outer rectangle is a pullback since $f \simeq \alpha f'$ and $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a fiber sequence. By construction, the right square is a pullback. Hence, by Lemma 2.1.36, we conclude that the left square is also a pullback. \square

By a dual argument, we also have:

Lemma 2.3.8. *Let \mathcal{C} be a stable ∞ -category. Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence and $\alpha: X \rightarrow X'$ is any morphism in \mathcal{C} . Then there exists a cofiber sequence $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z$ and a morphism $\beta: Y \rightarrow Y'$ in \mathcal{C} such that the following diagram is a morphism of cofiber sequences*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \text{id}_Z \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z. \end{array}$$

The next proposition gives a nice characterization of stable ∞ -categories.

Proposition 2.3.9 ([46, 1.1.3.4]). *Let \mathcal{C} be a pointed ∞ -category. Then \mathcal{C} is stable if and only if the following conditions are satisfied:*

- (1) *The ∞ -category \mathcal{C} admits finite limits and finite colimits.*

(2) A square $\Delta^1 \times \Delta^1 \rightarrow \mathcal{C}$ depicted as

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

is a pushout in \mathcal{C} if and only if it is a pullback in \mathcal{C} .

Remark 2.3.10. Another important consequence of Proposition 2.3.4 is the existence of a *suspension functor* $\Sigma: \mathcal{C} \rightarrow \mathcal{C}$ on a stable ∞ -category \mathcal{C} , given as

$$\Sigma: \mathcal{C} \cong \mathrm{Fun}(\Delta^0, \mathcal{C}) \rightarrow \mathrm{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C},$$

where $\mathrm{Fun}(\Delta^0, \mathcal{C}) \rightarrow \mathrm{Fun}(\Delta^1, \mathcal{C})$ associates to any object X a morphism $X \rightarrow 0$, and the second functor is the cofiber. Analogously, the *loop functor* $\Omega: \mathcal{C} \rightarrow \mathcal{C}$ is given by

$$\Omega: \mathcal{C} \cong \mathrm{Fun}(\Delta^0, \mathcal{C}) \rightarrow \mathrm{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C},$$

where the first functor sends X to $0 \rightarrow X$ and the second is the fiber of this morphism. When \mathcal{C} is a stable ∞ -category, these functors are mutually inverse equivalences.

Remark 2.3.11. Let \mathcal{C} be a stable ∞ -category. The suspension functor $\Sigma: \mathcal{C} \rightarrow \mathcal{C}$ is essentially characterized by the existence of natural homotopy equivalences

$$\mathrm{Map}_{\mathcal{C}}(\Sigma X, Y) \xrightarrow{\sim} \Omega \mathrm{Map}_{\mathcal{C}}(X, Y).$$

This assertion follows from the fact that a square

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X \end{array}$$

is a pushout in \mathcal{C} if and only if, for every $Y \in \mathcal{C}$, the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(\Sigma X, Y) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(0, Y) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{C}}(0, Y) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(X, Y) \end{array}$$

is a homotopy pullback of Kan complexes. Since 0 is initial, $\mathrm{Map}_{\mathcal{C}}(0, Y)$ is contractible, and consequently the homotopy pullback determines an equivalence $\mathrm{Map}_{\mathcal{C}}(\Sigma X, Y) \xrightarrow{\sim} \Omega \mathrm{Map}_{\mathcal{C}}(X, Y)$.

In particular, $\pi_0 \mathrm{Map}_{\mathcal{C}}(\Sigma^2 X, Y) \simeq \pi_2 \mathrm{Map}_{\mathcal{C}}(X, Y)$ is an abelian group. Since Σ is an equivalence of ∞ -categories, for every $Z \in \mathcal{C}$ we can choose $X \in \mathcal{C}$ such that $\Sigma^2 X \simeq Z$. Hence, $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Z, Y) = \pi_0 \mathrm{Map}_{\mathcal{C}}(Z, Y)$ is an abelian group and moreover this group structure depends functorially on $Z, Y \in \mathrm{h}\mathcal{C}$.

Proposition 2.3.12. *Every stable ∞ -category is an additive ∞ -category.*

Proof. Let \mathcal{C} be a stable ∞ -category. By definition, \mathcal{C} is pointed. By Proposition 2.3.9, \mathcal{C} admits finite products and finite coproducts. Remark 2.3.11 shows that $\mathrm{h}\mathcal{C}$ is enriched in abelian groups. Thus, \mathcal{C} is an additive ∞ -category (see Remark 2.2.5). \square

Corollary 2.3.13. *Let \mathcal{C} be a stable ∞ -category. Then $\mathrm{h}\mathcal{C}$ is additive.*

Proof. Combine Proposition 2.3.12 and Proposition 2.2.8. \square

Notation 2.3.14. Let \mathcal{C} be a stable ∞ -category and n be an integer. If n is nonnegative, we let $X \mapsto X[n]$ denote the n th power of the suspension functor $\Sigma: \mathcal{C} \rightarrow \mathcal{C}$. If n is nonpositive, we let $X \mapsto X[n]$ denote the $(-n)$ th power of the loop functor $\Omega: \mathcal{C} \rightarrow \mathcal{C}$. We will use the same notation to indicate the induced functors on the homotopy category $\mathrm{h}\mathcal{C}$.

Definition 2.3.15. Let \mathcal{C} be a stable ∞ -category. Suppose we are given a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in the homotopy category $\mathrm{h}\mathcal{C}$. We will say that this diagram is a *distinguished triangle* if there exists a diagram $\Delta^2 \times \Delta^1 \rightarrow \mathcal{C}$, depicted as

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{f}} & Y & \longrightarrow & 0' \\ \downarrow & & \downarrow \tilde{g} & & \downarrow \\ 0 & \longrightarrow & Z & \xrightarrow{\tilde{h}} & W \end{array}$$

where $0, 0' \in \mathcal{C}$ are zero objects, both squares are pushouts, \tilde{f} and \tilde{g} are representatives of the homotopy classes f and g , respectively, and $h: Z \rightarrow X[1]$ is the composition of the homotopy class of \tilde{h} with the equivalence $W \simeq X[1]$ determined by the outer rectangle (Lemma 2.1.36).

Theorem 2.3.16 ([46, 1.1.2.15]). *Let \mathcal{C} be a stable ∞ -category. The suspension functor $\Sigma: \mathcal{C} \rightarrow \mathcal{C}$ together with the collection of distinguished triangles described above endows the homotopy category $\mathrm{h}\mathcal{C}$ with the structure of a triangulated category.*

Definition 2.3.17. Let \mathcal{C} be a stable ∞ -category. For any objects X and Y in \mathcal{C} and any integer n , we define the (Yoneda) *Ext-groups* $\mathrm{Ext}_{\mathcal{C}}^n(X, Y)$ to be the abelian groups $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X[-n], Y) \cong \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y[n])$.

Remark 2.3.18. Let \mathcal{C} be a stable ∞ -category. Any cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathcal{C} gives rise to a long exact sequences

$$\cdots \rightarrow \mathrm{Ext}_{\mathcal{C}}^{n-1}(W, Z) \rightarrow \mathrm{Ext}_{\mathcal{C}}^n(W, X) \rightarrow \mathrm{Ext}_{\mathcal{C}}^n(W, Y) \rightarrow \mathrm{Ext}_{\mathcal{C}}^n(W, Z) \rightarrow \mathrm{Ext}_{\mathcal{C}}^{n+1}(W, X) \rightarrow \cdots$$

and

$$\cdots \rightarrow \mathrm{Ext}_{\mathcal{C}}^{n-1}(X, W) \rightarrow \mathrm{Ext}_{\mathcal{C}}^n(Z, W) \rightarrow \mathrm{Ext}_{\mathcal{C}}^n(Y, W) \rightarrow \mathrm{Ext}_{\mathcal{C}}^n(X, W) \rightarrow \mathrm{Ext}_{\mathcal{C}}^{n+1}(Z, W) \rightarrow \cdots$$

for any $W \in \mathcal{C}$ and all $n \in \mathbb{Z}$. To see this, fix an integer n . We will prove the exactness of the first sequence; the argument for the second sequence is similar. Using that $[n]$ is an auto-equivalence, we are free to replace W by $W[n]$, so we are reduced to proving that

$$\mathrm{Ext}_{\mathcal{C}}^{-1}(W, Z) \rightarrow \mathrm{Ext}_{\mathcal{C}}^0(W, X) \rightarrow \mathrm{Ext}_{\mathcal{C}}^0(W, Y) \rightarrow \mathrm{Ext}_{\mathcal{C}}^0(W, Z) \rightarrow \mathrm{Ext}_{\mathcal{C}}^1(W, X)$$

is exact. Extending the cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ to a diagram

$$\begin{array}{ccccc} \Omega Z & \longrightarrow & 0 & & \\ \downarrow e & & \downarrow & & \\ X & \xrightarrow{f} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow g & & \downarrow \\ 0 & \longrightarrow & Z & \xrightarrow{h} & \Sigma X \end{array}$$

where all squares are pushouts and pullbacks (by Proposition 2.3.9) shows that the composition of any two successive maps in the sequence

$$\mathrm{Ext}_{\mathcal{C}}^{-1}(W, Z) \rightarrow \mathrm{Ext}_{\mathcal{C}}^0(W, X) \rightarrow \mathrm{Ext}_{\mathcal{C}}^0(W, Y) \rightarrow \mathrm{Ext}_{\mathcal{C}}^0(W, Z) \rightarrow \mathrm{Ext}_{\mathcal{C}}^1(W, X)$$

is zero. Next, observe that any map $w: W \rightarrow Y$ such that $gw \simeq 0$ can be extended to a map of fiber sequences as indicated:

$$\begin{array}{ccccc} W & \xrightarrow{\mathrm{id}_W} & W & \longrightarrow & 0 \\ \vdots & & \downarrow w & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z. \end{array}$$

This proves that the sequence is exact at $\mathrm{Ext}_{\mathcal{C}}^0(W, Y)$. The same argument, applied to the fiber sequences $\Omega Z \xrightarrow{e} X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$, respectively, establishes exactness at $\mathrm{Ext}_{\mathcal{C}}^0(W, X)$ and $\mathrm{Ext}_{\mathcal{C}}^0(W, Z)$, as desired.

We will need an additional hypothesis on the stable ∞ -categories under consideration, namely we will primarily work with *compactly generated* stable ∞ -categories.

Definition 2.3.19. Let \mathcal{C} be an ∞ -category.

- (1) An object C of \mathcal{C} is called *compact* if \mathcal{C} admits filtered colimits and the functor $\mathrm{Map}_{\mathcal{C}}(C, -): \mathcal{C} \rightarrow \mathcal{S}$ commutes with filtered colimits (see [45, 5.3.1] and [45, 5.3.4.5]).
- (2) \mathcal{C} is called *compactly generated* if there exists a (small) ∞ -category \mathcal{D} which admits small colimits such that \mathcal{C} is generated under filtered colimits by \mathcal{D} (which can be identified with the full subcategory of compact objects of \mathcal{C}) (see [45, 5.5.7.1] and [45, 5.5.1.1]).

In the case of stable ∞ -categories, compactness and compact generation take a particularly nice form.

Theorem 2.3.20 ([46, 1.4.4.1]).

- (1) A stable ∞ -category \mathcal{C} admits small colimits if and only if \mathcal{C} admits small coproducts.
- (2) Suppose $F: \mathcal{C} \rightarrow \mathcal{D}$ is an exact functor between stable ∞ -categories which admits small colimits. Then F preserves small colimits if and only if F preserves small coproducts.
- (3) Let X be an object of a stable ∞ -category \mathcal{C} . Then X is compact if and only if the following condition is satisfied: For every morphism $f: X \rightarrow \coprod_{\alpha \in A} Y_{\alpha}$ in \mathcal{C} , there exists a finite subset $A_0 \subseteq A$ such that f factors (up to homotopy) through $\coprod_{\alpha \in A_0} Y_{\alpha}$.

Corollary 2.3.21. Let \mathcal{C} be a compactly generated stable ∞ -category. The full subcategory $\mathcal{C}_c \subseteq \mathcal{C}$ spanned by the compact objects is again stable.

Proof. Note that the zero object is compact. We must show that \mathcal{C}_c is stable under the formation of fibers and cofibers. Since translations are equivalences, \mathcal{C}_c is stable under

translations. It therefore suffices by [46, Lemma 1.1.3.3] to show that \mathcal{C}_c is stable under cofibers. Suppose $f: X \rightarrow Y$ is a morphism of compact objects and let $g: Y \rightarrow Z \simeq \text{cofib}(f)$ be a cofiber. We must show that Z is compact. By Theorem 2.3.20, we must show that for every collection of objects $\{W_\alpha\}_{\alpha \in A}$ with coproduct W , the canonical morphism

$$\bigoplus_{\alpha \in A} \text{Ext}_{\mathcal{C}}^0(Z, W_\alpha) \rightarrow \text{Ext}_{\mathcal{C}}^0(Z, W)$$

is an isomorphism. Using the cofiber sequence $X \rightarrow Y \rightarrow Z$, we have a long exact sequence (see Remark 2.3.18)

$$\begin{array}{ccc} \bigoplus_{\alpha} \text{Ext}_{\mathcal{C}}^{-1}(Y, W_{\alpha}) & \xrightarrow{h_0} & \text{Ext}_{\mathcal{C}}^{-1}(Y, W) \\ \downarrow & & \downarrow \\ \bigoplus_{\alpha} \text{Ext}_{\mathcal{C}}^{-1}(X, W_{\alpha}) & \xrightarrow{h_1} & \text{Ext}_{\mathcal{C}}^{-1}(X, W) \\ \downarrow & & \downarrow \\ \bigoplus_{\alpha} \text{Ext}_{\mathcal{C}}^0(Z, W_{\alpha}) & \xrightarrow{h_2} & \text{Ext}_{\mathcal{C}}^0(Z, W) \\ \downarrow & & \downarrow \\ \bigoplus_{\alpha} \text{Ext}_{\mathcal{C}}^0(Y, W_{\alpha}) & \xrightarrow{h_3} & \text{Ext}_{\mathcal{C}}^0(Y, W) \\ \downarrow & & \downarrow \\ \bigoplus_{\alpha} \text{Ext}_{\mathcal{C}}^0(X, W_{\alpha}) & \xrightarrow{h_4} & \text{Ext}_{\mathcal{C}}^0(X, W). \end{array}$$

By the compactness of X and Y , the morphisms h_0, h_1, h_3 and h_4 are all isomorphisms. It now follows from the Five Lemma that h_2 is an isomorphism. \square

Lemma 2.3.22. *Let \mathcal{C} be a compactly generated stable ∞ -category. Suppose $\{S_\alpha : \alpha \in A\}$ is a set of compact generators of \mathcal{C} . If $f: C \rightarrow D$ is a morphism in \mathcal{C} such that the induced map $\text{Hom}_{\text{h}\mathcal{C}}(S_\alpha, C) \rightarrow \text{Hom}_{\text{h}\mathcal{C}}(S_\alpha, D)$ is an isomorphism for every index $\alpha \in A$, then f is an equivalence in \mathcal{C} .*

Proof. Enlarging if necessary, we may assume without loss of generality that the collection $\{S_\alpha : \alpha \in A\}$ is stable under the formation of suspensions. To establish the desired result,

it suffices to show that for every object $X \in \mathcal{C}$, the map $f: C \rightarrow D$ induces a homotopy equivalence $\phi_X: \text{Map}_{\mathcal{C}}(X, C) \rightarrow \text{Map}_{\mathcal{C}}(X, D)$. Let \mathcal{C}' denote the full subcategory of \mathcal{C} spanned by those objects for which ϕ_X is an equivalence. We wish to prove that $\mathcal{C}' = \mathcal{C}$. Since \mathcal{C}' is stable under colimits, it suffices to show that each S_α belongs to \mathcal{C}' . Since each S_α is a cogroup object, each ϕ_{S_α} is a map of group objects of the homotopy category \mathcal{H} of spaces. Therefore, ϕ_{S_α} is a homotopy equivalence if and only if it induces an isomorphism of groups $\pi_n \text{Map}_{\mathcal{C}}(S_\alpha, C) \rightarrow \pi_n \text{Map}_{\mathcal{C}}(S_\alpha, D)$ for every $n \geq 0$ (here the homotopy groups are taken with respect to the base points given by the group structures, i.e. the zero morphisms). Replacing S_α with $\Sigma^n S_\alpha$, we are reduced to the case $n = 0$: that is, to the bijectivity of the maps $\text{Hom}_{\text{h}\mathcal{C}}(S_\alpha, C) \rightarrow \text{Hom}_{\text{h}\mathcal{C}}(S_\alpha, D)$, which completes the proof. \square

Chapter 3

Distinguished morphisms

Auslander-Reiten theory isolates several classes of morphisms which play a distinguished structural role in the categories in which they exist. In this chapter, we introduce these various morphisms and study their properties. By construction, the results of this chapter recover the classical analogues after passing to the homotopy category. In particular, we show that there is a close relationship between almost-split and irreducible morphisms mirroring the classical theory. Auslander recognized in [5, 4] that almost-split morphisms are instances of a more general notion he called morphisms determined by objects. While little attention has been given to morphisms determined by objects in the literature, Ringel recently made a strong case for why Auslander’s insight deserves further investigation [51]. We introduce almost-split morphisms in Section 3.2 and discuss their properties in any additive ∞ -category. In Section 3.4, we show that *minimal* almost-split morphisms have particularly nice features. Section 3.5 investigates the relationships between almost-split morphisms and irreducible morphisms. In Section 3.6, we study how morphisms are de-

terminated by objects, giving a useful characterization in Theorem 3.6.5. Finally, in the last section of this chapter, we discuss existence of these various distinguished morphisms. Theorem 3.7.4 is the main result of this chapter, proving the existence of morphisms determined by compact objects in any compactly generated stable ∞ -category. The existence of almost-split morphisms and irreducible morphisms then follows as consequences of the earlier work in this chapter.

3.1 Retractions and idempotents

Definition 3.1.1. Let \mathcal{C} be an ∞ -category. A morphism $r: X \rightarrow Y$ in \mathcal{C} is called a *retraction* (of X) if there exists a 2-simplex $\Delta^2 \rightarrow \mathcal{C}$ corresponding to a diagram

$$\begin{array}{ccc} & X & \\ i \nearrow & & \searrow r \\ Y & \xrightarrow{\text{id}_Y} & Y. \end{array}$$

In this case, we will also say that i is a *section* (of X) and Y is a *retract* of X . Equivalently, $r: X \rightarrow Y$ is a retraction in \mathcal{C} if it is a retraction in the homotopy category $\text{h}\mathcal{C}$.

If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a triangle in a pointed ∞ -category \mathcal{C} , and f is a retraction, then since f has a right inverse (up to homotopy) and $gf \simeq 0$, it follows that $g \simeq 0$. Similarly, if g is a section, then $f \simeq 0$. Under certain conditions, we also have the converse statements:

Lemma 3.1.2. *Let \mathcal{C} be a pointed ∞ -category.*

- (1) *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a fiber sequence in \mathcal{C} and $g \simeq 0$, then f is a retraction.*
- (2) *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence in \mathcal{C} and $f \simeq 0$, then g is a section.*

Proof.

- (1) If $g \simeq 0$, then, using that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a fiber sequence, the triangle

$$\begin{array}{ccc} Y & \xrightarrow{1} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

induces a map $Y \rightarrow X$ together with a 2-simplex $\Delta^2 \rightarrow \mathcal{C}$,

$$\begin{array}{ccc} & X & \\ \nearrow & & \searrow f \\ Y & \xrightarrow{1} & Y \end{array}$$

exhibiting f as a retraction.

- (2) If $f \simeq 0$, then an analogous argument using that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence shows that g is a section.

□

Remark 3.1.3. Let \mathcal{C} be a stable ∞ -category, and suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a (co)fiber sequence in \mathcal{C} . By Lemma 3.1.2, $f \simeq 0$ if and only if g is a section; similarly, $g \simeq 0$ if and only if f is a retraction.

Remark 3.1.4. Observe that a morphism is both a section and a retraction if and only if it is an equivalence. In particular, if \mathcal{C} is a pointed ∞ -category with a section $s: X \rightarrow Y$ such that $X \xrightarrow{s} Y \rightarrow 0$ is a fiber sequence, then Lemma 3.1.2 implies that s is an equivalence. Similarly, if $r: X \rightarrow Y$ is a retraction and $0 \rightarrow X \xrightarrow{r} Y$ is a cofiber sequence in \mathcal{C} , then r is an equivalence.

Lemma 3.1.5. *Let \mathcal{C} be a stable ∞ -category and suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a fiber sequence in \mathcal{C} . Then f is a section if and only if g is a retraction.*

Proof. Consider the expanded diagram,

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & Y & \longrightarrow & 0 \\
 \downarrow & & \downarrow g & & \downarrow \\
 0 & \longrightarrow & Z & \xrightarrow{h} & \Sigma X \\
 & & \downarrow & & \downarrow \Sigma f \\
 & & 0 & \longrightarrow & \Sigma Y
 \end{array}$$

where the squares with zero objects in the upper right and lower left corners are pushouts.

Using Lemma 2.1.36 (twice), we deduce that all squares are pushouts. Then f is a section if and only if Σf is a section if and only if $h \simeq 0$ if and only if g is a retraction, where the last two equivalences are a consequence of Lemma 3.1.2. \square

Definition 3.1.6. Let Idem denote any simplicial set which has exactly one nondegenerate n -simplex σ_n for each $n \geq 0$ with the property that $d_i \sigma_n = \sigma_{n-1}$ for all $n \geq 1$ and $0 \leq i \leq n$. An *idempotent* in an ∞ -category \mathcal{C} is a map of simplicial sets $\text{Idem} \rightarrow \mathcal{C}$.

Remark 3.1.7. Definition 3.1.6 is a simply rephrasing of [45, Definitions 4.4.5.2 and 4.4.5.4], as those definitions involve additional data necessary for discussing the splitting of coherent idempotents in greater generality than we require here.

Remark 3.1.8. Let \mathcal{C} be an ∞ -category. Giving an idempotent $F: \text{Idem} \rightarrow \mathcal{C}$ is equivalent to giving, for each $n \geq 0$, an n -simplex $\Delta^n \rightarrow \mathcal{C}$ of \mathcal{C} satisfying conditions which exhibit a “coherent idempotent” by explicitly specifying the homotopy (associativity) data between iterated compositions of an idempotent of $\text{h}\mathcal{C}$. More concretely, F_0 determines an object X

of \mathcal{C} , F_1 gives a morphism $e: X \rightarrow X$ of \mathcal{C} , F_2 selects a 2-simplex $\sigma: \Delta^2 \rightarrow \mathcal{C}$ of the form

$$\begin{array}{ccc} & X & \\ e \nearrow & & \searrow e \\ X & \xrightarrow{e} & X \end{array}$$

exhibiting a homotopy $e \simeq e^2$ which witnesses that e is an idempotent of $\mathrm{h}\mathcal{C}$, and for $n > 2$ the n -simplex specified by F_n records the various homotopy relations between e and e^n .

Remark 3.1.9. Let \mathcal{C} be an ∞ -category. Any retraction $r: X \rightarrow Y$ in \mathcal{C} gives rise to an idempotent $e \simeq i \circ r$ in the homotopy category $\mathrm{h}\mathcal{C}$. In general, this idempotent may not lift to a (coherent) idempotent of \mathcal{C} ; for a counter-example, see [46, 1.2.4.8]. However, in the case that \mathcal{C} is a stable ∞ -category, this distinction vanishes; that is, every idempotent of $\mathrm{h}\mathcal{C}$ can be lifted to an idempotent of \mathcal{C} by [46, 1.2.4.6]. We refer to idempotents of the form $e \simeq i \circ r$ arising from retractions as *split idempotents*. If every idempotent in \mathcal{C} is a split idempotent, we say that \mathcal{C} is *idempotent complete*.

Proposition 3.1.10 ([45, 4.4.5.16]). *Let \mathcal{C} be an ∞ -category. If \mathcal{C} admits filtered colimits, then \mathcal{C} is idempotent complete.*

Lemma 3.1.11. *Let \mathcal{C} be an additive ∞ -category containing an object X .*

- (1) *If 0_X and id_X are the only idempotents of X , then X is indecomposable.*
- (2) *If 0_X and id_X are the only idempotents of X , then every retraction $r: X \rightarrow X$ and every section $s: X \rightarrow X$ is an equivalence.*
- (3) *If $\mathrm{End}_{\mathrm{h}\mathcal{C}}(X)$ is a local ring, then the only idempotents are 0_X and id_X .*
- (4) *Assume idempotents in $\mathrm{End}_{\mathrm{h}\mathcal{C}}(X)$ split. If X is indecomposable, then the endomorphism ring $\mathrm{End}_{\mathrm{h}\mathcal{C}}(X)$ is local.*

Proof. (1) Suppose $X \simeq X_1 \oplus X_2$. Let $i_1: X_1 \rightarrow X$ and $p_1: X \rightarrow X_1$ be the canonical biproduct inclusion and projection of X_1 . Then $p_1 i_1 \simeq \text{id}_{X_1}$ and $i_1 p_1$ is an idempotent of X . By assumption, either $i_1 p_1 \simeq 0_X$ which implies $X \simeq X_2$ or $i_1 p_1 \simeq \text{id}_X$ which implies $X \simeq X_1$.

(2) Assume $X \not\simeq 0$. Suppose there exist morphisms $r: X \rightarrow X$ and $i: X \rightarrow X$ such that $ri \simeq \text{id}_X$. If the idempotent $ir \not\simeq \text{id}_X$, then by assumption $ir \simeq 0_X$. This implies $\text{id}_X \simeq ri \simeq (ri)(ri) \simeq r(ir)i \simeq 0_X$, a contradiction.

(3) Assume $\text{End}_{\text{hc}}(X)$ is local and let $e \in \text{End}_{\text{hc}}(X)$ be an idempotent. Since $\text{End}_{\text{hc}}(X)$ is local, either e or $(\text{id}_X - e)$ is an equivalence. In the first case, $e^2 \simeq e$ implies $e \simeq \text{id}_X$; in the second case, $(\text{id}_X - e)^2 \simeq (\text{id}_X - e)$ implies $e \simeq 0_X$.

(4) We prove the contrapositive. Suppose $e: X \rightarrow X$ is a nontrivial idempotent. Then $(\text{id}_X - e)$ is also a nontrivial idempotent and, by assumption, both e and $(\text{id}_X - e)$ are split idempotents. That is, there exist (nontrivial) objects Y and Z together with morphisms $r: X \rightarrow Y$, $i: Y \rightarrow X$, $p: X \rightarrow Z$, and $q: Z \rightarrow X$ such that $ri \simeq \text{id}_Y$, $e \simeq ir$, $pq \simeq \text{id}_Z$, and $(\text{id}_X - e) \simeq qp$. The morphisms r and p induce a unique map $\alpha: X \rightarrow Y \oplus Z$ such that $r \simeq \pi_Y \alpha$ and $p \simeq \pi_Z \alpha$, where π_Y and π_Z are the canonical biproduct projections. Similarly, the morphisms i and q induce a unique map $\beta: Y \oplus Z \rightarrow X$ such that $i \simeq \beta \iota_Y$ and $q \simeq \beta \iota_Z$, where ι_Y and ι_Z are the canonical biproduct inclusions. We claim that the morphisms α and β exhibit an isomorphism $X \cong Y \oplus Z$. First, observe that

$$\beta \alpha \simeq \beta(\iota_Y \pi_Y + \iota_Z \pi_Z) \alpha \simeq ir + qp \simeq e + (\text{id}_X - e) \simeq \text{id}_X.$$

It remains to show that $\alpha \beta \simeq \text{id}_{X \oplus Y}$. Note that $rqp \simeq 0$ implies $rq \simeq 0$ because p is a

retraction and $qpi \simeq 0$ implies $pi \simeq 0$ because q is a section. Then

$$\begin{aligned}
\alpha\beta &\simeq (\iota_Y\pi_Y + \iota_Z\pi_Z)\alpha\beta(\iota_Y\pi_Y + \iota_Z\pi_Z) \\
&\simeq \iota_Y\pi_Y(\alpha\beta)\iota_Y\pi_Y + \iota_Y\pi_Y(\alpha\beta)\iota_Z\pi_Z + \iota_Z\pi_Z(\alpha\beta)\iota_Y\pi_Y + \iota_Z\pi_Z(\alpha\beta)\iota_Z\pi_Z \\
&\simeq \iota_Y(ri)\pi_Y + \iota_Y(rq)\pi_Z + \iota_Z(pi)\pi_Y + \iota_Z(pq)\pi_Z \\
&\simeq \iota_Y\pi_Y + \iota_Z\pi_Z = \text{id}_{Y \oplus Z}.
\end{aligned}$$

This argument proves that X is decomposable. \square

Corollary 3.1.12. *Let \mathcal{C} be a compactly generated stable ∞ -category. An object $X \in \mathcal{C}$ is indecomposable if and only if X is strongly indecomposable.*

Proof. Assume X is strongly indecomposable, that is, $\text{End}_{\text{hc}}(X)$ is a local ring. Then Lemma 3.1.11(3) combined with Lemma 3.1.11(1) shows that X is indecomposable.

Conversely, suppose that X is indecomposable. By Proposition 3.1.10, \mathcal{C} is idempotent complete. Hence, Lemma 3.1.11(4) implies that X is strongly indecomposable. \square

3.2 Almost-split morphisms

Definition 3.2.1. Let \mathcal{C} be an ∞ -category.

- (1) A morphism $f: X \rightarrow Y$ in \mathcal{C} is called *left almost-split* if f is not a section and for any morphism $f': X \rightarrow Y'$ in \mathcal{C} which is not a section, there exists a 2-simplex $\Delta^2 \rightarrow \mathcal{C}$ of the form

$$\begin{array}{ccc}
& & Y \\
& \nearrow f & \\
X & \xrightarrow{\quad f' \quad} & Y'
\end{array}$$

- (2) A morphism $g: Y \rightarrow Z$ in \mathcal{C} is called *right almost-split* if g is not a retraction and for any morphism $g': Y' \rightarrow Z$ in \mathcal{C} which is not a retraction, there exists a 2-simplex $\Delta^2 \rightarrow \mathcal{C}$ of the form

$$\begin{array}{ccc} & Y & \\ \nearrow & & \searrow g \\ Y' & \xrightarrow{g'} & Z. \end{array}$$

Equivalently, a morphism $f: X \rightarrow Y$ is left (right) almost-split in \mathcal{C} if it is left (right) almost-split in the homotopy category $\mathrm{h}\mathcal{C}$.

Remark 3.2.2. A morphism is left almost-split in \mathcal{C} if and only if it is right almost-split in $\mathcal{C}^{\mathrm{op}}$ (see Definition 2.1.17). Consequently, we will mainly focus our attention on right almost-split morphisms. The theory of left almost-split morphisms follows by duality.

Remark 3.2.3. By Definition 3.2.1, if $g: Y \rightarrow Z$ is right almost-split, then any morphism $g': Y \rightarrow Z$ such that $g' \simeq g$ is again right almost-split. Similarly, the collection of right almost-split morphisms is closed under composition with equivalences. Another important feature is that the basic properties of almost-split morphisms only depend on the structure of the homotopy category.

Lemma 3.2.4. *Let \mathcal{C} be an additive ∞ -category. If $g: Y \rightarrow Z$ is a right almost-split morphism in \mathcal{C} , then the endomorphism ring $\mathrm{End}_{\mathrm{h}\mathcal{C}}(Z)$ is local with unique maximal ideal given by $\mathrm{Im} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Z, g) \subseteq \mathrm{End}_{\mathrm{h}\mathcal{C}}(Z)$.*

Proof. Let $J = \mathrm{Im} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Z, g) = [g] \circ \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Z, Y)$. By Lemma A.1.3, it suffices to show that J is the unique maximal right ideal of $\mathrm{End}_{\mathrm{h}\mathcal{C}}(Z)$. The set J is an abelian group because $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Z, Y)$ is an abelian group and composition is linear. The abelian

group J is moreover a right ideal because J is closed under composition on the right by an endomorphism of Z , which is the product structure on $\text{End}_{\text{hc}}(Z)$. To see that J is maximal, suppose $I \subset \text{End}_{\text{hc}}(Z)$ is a proper right ideal. If $[\phi] \in I$, then ϕ cannot be a retraction, otherwise $I = \text{End}_{\text{hc}}(Z)$. Using that $g: Y \rightarrow Z$ is right almost-split, it follows that $\phi: Z \rightarrow Z$ factors through g and thus $I \subseteq J$. \square

Corollary 3.2.5. *Let \mathcal{C} be an additive ∞ -category. If $g: Y \rightarrow Z$ is a right almost-split morphism in \mathcal{C} , then Z is indecomposable.*

Proof. Combine Lemma 3.2.4 and Lemma 3.1.11. \square

Remark 3.2.6. There is an alternative proof of Corollary 3.2.5 which does not rely on knowing that $\text{End}_{\text{hc}}(Z)$ is a local ring. Suppose $Z = Z_1 \oplus Z_2$ with both Z_1 and Z_2 nontrivial, and let $i_1: Z_1 \rightarrow Z$ and $i_2: Z_2 \rightarrow Z$ be the canonical inclusions. Because i_1 and i_2 are not retractions, they each factor through g , so that $i_1 \simeq gh_1$ and $i_2 \simeq gh_2$ for some $h_1: Z_1 \rightarrow Y$ and $h_2: Z_2 \rightarrow Y$. If $p_1: Z \rightarrow Z_1$ and $p_2: Z \rightarrow Z_2$ are the canonical projections, then this implies $g(h_1p_1 + h_2p_2) \simeq i_1p_1 + i_2p_2 \simeq \text{id}_Z$, contradicting that g is not a retraction.

Corollary 3.2.7. *Let \mathcal{C} be an additive ∞ -category. If $g: Y \rightarrow Z$ is a right almost-split morphism in \mathcal{C} with $Y \not\simeq 0$, then g is not a section.*

Proof. Assume $g: Y \rightarrow Z$ is a section with retraction $r: Z \rightarrow Y$. Then $gr: Z \rightarrow Z$ is an idempotent. Combining Lemma 3.2.4 and Lemma 3.1.11(3), either $gr \simeq \text{id}_Z$ or $gr \simeq 0_Z$. The first case implies that g is a retraction, a contradiction. The second case implies $r \simeq 0$, again a contradiction because $Y \not\simeq 0$. \square

Remark 3.2.8. Since any proper ideal of $\text{End}_{\text{hc}}(Z)$ must consist of nonisomorphisms, Lemma 3.2.4 implies that every endomorphism of Z which is not an equivalence factors through $g: Y \rightarrow Z$.

In fact, Remark 3.2.8 admits a slight generalization.

Lemma 3.2.9. *Let \mathcal{C} be an additive ∞ -category containing an object a nonzero object W such that $\text{End}_{\text{hc}}(W)$ is a local ring. If $g: Y \rightarrow Z$ is a right almost-split morphism in \mathcal{C} , then any morphism $r: W \rightarrow Z$ which is not an equivalence factors through g .*

Proof. We must show that $r: W \rightarrow Z$ is not a retraction. Suppose to the contrary that there exists $i: Z \rightarrow W$ such that $ri \simeq \text{id}_Z$. Then $ir: W \rightarrow W$ is an idempotent in hc with $ir \neq \text{id}_W$ because r is not an equivalence. Using that $\text{End}_{\text{hc}}(W)$ is local, Lemma 3.1.11(3) gives that $ir \simeq 0_W$. But this implies that W is a zero object, a contradiction. \square

With some additional hypotheses, Lemma 3.2.4 admits a converse.

Lemma 3.2.10. *Let \mathcal{C} be an additive ∞ -category. Assume $g: Y \rightarrow Z$ is a morphism in \mathcal{C} satisfying the following properties:*

- (1) *The endomorphism ring $\text{End}_{\text{hc}}(Z)$ is local with $\text{rad } \text{End}_{\text{hc}}(Z) = \text{Im } \text{Hom}_{\text{hc}}(Z, g)$.*
- (2) *Every morphism $g': Y' \rightarrow Z$ in \mathcal{C} such that $\text{Im } \text{Hom}_{\text{hc}}(Z, g') \subseteq \text{Im } \text{Hom}_{\text{hc}}(Z, g)$ factors through g .*

Then $g: Y \rightarrow Z$ is right almost-split.

Proof. If $g': Y' \rightarrow Z$ is not a retraction, then $\text{Im } \text{Hom}_{\text{hc}}(Z, g') \subseteq \text{rad } \text{End}_{\text{hc}}(Z)$. Since $\text{rad } \text{End}_{\text{hc}}(Z) = \text{Im } \text{Hom}_{\text{hc}}(Z, g)$, we have that g' factors through g , as desired. \square

Lemma 3.2.11. *Let \mathcal{C} be an additive ∞ -category. Suppose that $g: Y \rightarrow Z$ is a right almost-split morphism in \mathcal{C} . If $h: W \rightarrow Z$ is not a retraction and has the property that g factors through h , then h is right almost-split.*

Proof. Suppose $g': Y' \rightarrow Z$ is not a retraction. Since g is right almost-split, there exists $k: Y' \rightarrow Y$ such that $g' \simeq gk$. Since g factors through h , there exists $k': Y \rightarrow W$ such that $g \simeq hk'$. Hence, $g' \simeq gk \simeq h(k'k)$ shows that g' factors through h . \square

Definition 3.2.12. Let \mathcal{C} be a pointed ∞ -category. A nonzero object Z in \mathcal{C} is called *almost-simple* if the morphism $0 \rightarrow Z$ is right almost-split.

Proposition 3.2.13. *Let \mathcal{C} be a stable ∞ -category. A nonzero object $Z \in \mathcal{C}$ is almost-simple if and only if the cofiber of any morphism $Y \rightarrow Z$ is equivalent to either a zero morphism or a section.*

Proof. Let $g: Y \rightarrow Z$ be any morphism and consider the cofiber sequence $Y \xrightarrow{g} Z \xrightarrow{h} W$. Suppose $0 \rightarrow Z$ is right almost-split. If $g: Y \rightarrow Z$ is a retraction, then $hg \simeq 0$ implies $h \simeq 0$. If $g: Y \rightarrow Z$ is not a retraction, then $g \simeq 0$ because $0 \rightarrow Z$ is right almost-split. Using Lemma 3.1.2, we conclude that h is a section.

Conversely, suppose g is not a retraction. Then, using Lemma 3.1.2 together with the assumption that \mathcal{C} is stable, it must be that h is not equivalent to a zero morphism. Consequently, h is a section, which implies that $g \simeq 0$. Thus, $0 \rightarrow Z$ is right almost-split. \square

3.3 Divisible morphisms

Definition 3.3.1. Let \mathcal{C} be an ∞ -category.

- (1) A nonzero morphism $h: Z \rightarrow W$ is called *left divisible* if it factors through every nonzero morphism $\alpha: W' \rightarrow W$, that is, there exists $\beta: Z \rightarrow W'$ such that $h \simeq \alpha\beta$.
- (2) A nonzero morphism $h: Z \rightarrow W$ is called *right divisible* if it factors through every nonzero morphism $\beta: Z \rightarrow Z'$, that is, there exists $\alpha: Z' \rightarrow W$ such that $h \simeq \alpha\beta$.
- (3) A morphism $h: Z \rightarrow W$ is called *divisible* if it is both left divisible and right divisible.

Remark 3.3.2. A morphism is left divisible in \mathcal{C} if and only if it is right divisible in \mathcal{C}^{op} (see Definition 2.1.17). Consequently, we will mainly focus our attention on right divisible morphisms. The theory of left divisible morphisms follows by duality.

There is a close relationship between almost-split morphisms and divisible morphisms.

Proposition 3.3.3. *Let \mathcal{C} be a stable ∞ -category. Suppose $Y \xrightarrow{g} Z \xrightarrow{h} W$ is a cofiber sequence in \mathcal{C} . Then the morphism $g: Y \rightarrow Z$ is right almost-split if and only if the morphism $h: Z \rightarrow W$ is right divisible.*

Proof. We will make repeated use of Lemma 3.1.2 in this argument.

First, assume that $g: Y \rightarrow Z$ is right almost split and let $\beta: Z \rightarrow Z'$ be any nonzero morphism. Since g is not a retraction, h is nonzero. If $Z'' \xrightarrow{\alpha} Z \xrightarrow{\beta} Z'$ is a fiber sequence, then α is not a retraction and thus factors through g . As h is a cofiber of g , it follows that $h\alpha \simeq 0$. But β is a cofiber of α (because \mathcal{C} is stable), therefore h factors through β .

Conversely, suppose $\alpha: Z'' \rightarrow Z$ is not a retraction. Note that g is not a retraction because h is nonzero, and any cofiber $\beta: Z \rightarrow Z'$ of α is nonzero. Hence, by assumption, h factors through β and consequently $h\alpha \simeq 0$. It now follows that α factors through g , as a fiber of h (because \mathcal{C} is stable). □

Remark 3.3.4. Proposition 3.3.3 might be summarized by the following diagram:

$$\begin{array}{ccccc}
 & & Z'' & & \\
 & \swarrow & \downarrow \alpha & & \\
 Y & \xrightarrow{g} & Z & \xrightarrow{h} & W \\
 & & \downarrow \beta & \searrow & \\
 & & Z' & &
 \end{array}$$

where $\beta: Z \rightarrow Z'$ is any nonzero morphism.

Corollary 3.3.5. *Let \mathcal{C} be a stable ∞ -category. If $h: Z \rightarrow W$ is a right divisible morphism in \mathcal{C} , then Z is strongly indecomposable.*

Proof. Suppose $h: Z \rightarrow W$ is right divisible and consider the fiber sequence $Y \xrightarrow{g} Z \xrightarrow{h} W$.

By Proposition 3.3.3, the morphism $g: Y \rightarrow Z$ is right almost-split. It now follows from Lemma 3.2.4 that Z is strongly indecomposable. \square

Proposition 3.3.6. *Let \mathcal{C} be an additive ∞ -category. If $h: Z \rightarrow W$ is a right divisible morphism in \mathcal{C} , then the left $\text{End}_{\text{hc}}(W)$ -module $\text{Hom}_{\text{hc}}(Z, W)$ has an essential simple socle generated by the homotopy equivalence class $[h] \in \text{Hom}_{\text{hc}}(Z, W)$.*

Proof. Let H be the $\text{End}_{\text{hc}}(W)$ -submodule of $\text{Hom}_{\text{hc}}(Z, W)$ generated by the morphism $[h]: Z \rightarrow W$. If M is any nontrivial submodule of $\text{Hom}_{\text{hc}}(Z, W)$ and $m: Z \rightarrow W$ is a nonzero morphism representing an element of M , then by right divisibility there exists a morphism $w: W \rightarrow W$ such that $h \simeq wm$. Hence, $H \subseteq M$, which shows (in particular) that H is an essential submodule. The socle S of $\text{Hom}_{\text{hc}}(Z, W)$ is the intersection of all essential submodules, so $S \subseteq H \subseteq S$. \square

Theorem 3.3.7. *Let \mathcal{C} be an additive ∞ -category. If $h: Z \rightarrow W$ is a left divisible morphism in \mathcal{C} , then the natural map*

$$\theta: \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(-, W) \rightarrow \mathrm{Hom}_R(\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Z, -), \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Z, W)),$$

where $R = \mathrm{End}_{\mathrm{h}\mathcal{C}}(Z)$, is a monomorphism.

Proof. We will show that θ_X is a monomorphism for any $X \in \mathcal{C}$. It suffices to show that if $\alpha: X \rightarrow W$ is nonzero, then $\theta_X[\alpha]$ is nonzero; that is, we must show that there exists some $\beta: Z \rightarrow X$ such that $\theta_X[\alpha](\beta) = [\alpha\beta]$ is not zero. However, since $\alpha: X \rightarrow W$ is nonzero and $h: Z \rightarrow W$ is left divisible, there exists $\beta: Z \rightarrow X$ such that $h \simeq \alpha\beta$, as desired. \square

3.4 Minimal morphisms

Definition 3.4.1. Let \mathcal{C} be an ∞ -category.

- (1) A morphism $f: X \rightarrow Y$ in \mathcal{C} is called *left minimal* if every $\psi: Y \rightarrow Y$ fitting into a diagram $\Delta^2 \rightarrow \mathcal{C}$ of the form

$$\begin{array}{ccc} & Y & \\ f \nearrow & & \searrow \psi \\ X & \xrightarrow{f} & Y \end{array}$$

is an equivalence.

- (2) A morphism $g: Y \rightarrow Z$ in \mathcal{C} is called *right minimal* if every $\psi: Y \rightarrow Y$ fitting into a diagram $\Delta^2 \rightarrow \mathcal{C}$ of the form

$$\begin{array}{ccc} & Y & \\ \psi \nearrow & & \searrow g \\ Y & \xrightarrow{g} & Z \end{array}$$

is an equivalence.

(3) We say that a morphism is *minimal* if it is both left minimal and right minimal.

Equivalently, a morphism $f: X \rightarrow Y$ is left (right) minimal in \mathcal{C} if it is left (right) minimal in the homotopy category $\mathrm{h}\mathcal{C}$.

Example 3.4.2. Let \mathcal{C} be an ∞ -category. Any section in \mathcal{C} is right minimal. Similarly, any retraction in \mathcal{C} is left minimal. Every equivalence is minimal.

Remark 3.4.3. Let \mathcal{C} be an ∞ -category. A morphism $g: Y \rightarrow Z$ is right minimal in \mathcal{C} if and only if every morphism $\psi: g \rightarrow g$ in the slice category $\mathcal{C}_{/Z}$ over Z is an equivalence; that is, $\pi_0 \mathrm{Map}_{\mathcal{C}_{/Z}}(g, g)$ is a group.

Remark 3.4.4. Let \mathcal{C} be an ∞ -category. Assume $g: Y \rightarrow Z$ in \mathcal{C} is nullhomotopic. Then g is right minimal if and only if every endomorphism $\psi: Y \rightarrow Y$ is an equivalence; that is, $\pi_0 \mathrm{Map}_{\mathcal{C}}(Y, Y)$ is a group.

Lemma 3.4.5. *Let \mathcal{C} be an additive ∞ -category. If $g: Y \rightarrow Z$ is a nonzero morphism in \mathcal{C} such that $\mathrm{End}_{\mathrm{h}\mathcal{C}}(Z)$ is local, then g is left minimal.*

Proof. Suppose $g \simeq \psi g$ for some $\psi: Z \rightarrow Z$. Since $g \not\simeq 0$ and $(1 - \psi)g \simeq 0$, it follows that $(1 - \psi)$ cannot be an equivalence. Since $\mathrm{End}_{\mathrm{h}\mathcal{C}}(Z)$ is local, we must conclude that ψ is an equivalence. □

Corollary 3.4.6. *Let \mathcal{C} be an additive ∞ -category. If $g: Y \rightarrow Z$ is a right almost-split morphism in \mathcal{C} , then g is left minimal.*

Proof. Combine Lemma 3.4.5 and Lemma 3.2.4. □

Corollary 3.4.7. *Let \mathcal{C} be a stable ∞ -category. If $h: Z \rightarrow W$ is a right divisible morphism in \mathcal{C} , then h is right minimal.*

Proof. Combine the dual of Lemma 3.4.5 and Lemma 3.3.5. \square

We say that a morphism is *minimal left almost-split* if it is both left minimal and left almost-split, and similarly for *minimal right almost-split* morphisms. These notions are dual, so as above we restrict our attention to right minimal morphisms.

Lemma 3.4.8. *Suppose $g: Y \rightarrow Z$ and $g': Y' \rightarrow Z'$ are both minimal right almost-split morphisms in an ∞ -category. Then for any equivalence $\psi: Z \rightarrow Z'$, there exists an equivalence $\phi': Y \rightarrow Y'$ such that $\psi g \simeq g' \phi'$.*

Proof. Assume $\psi: Z \rightarrow Z'$ is an equivalence with homotopy inverse $\psi': Z' \rightarrow Z$. Then the composition $\psi' g'$ is not a retraction because g' is not a retraction and consequently, using that g is right almost-split, there exists $\phi: Y' \rightarrow Y$ such that $g \phi \simeq \psi' g'$. Similarly, there exists $\phi': Y \rightarrow Y'$ such that $g' \phi' \simeq \psi g$. Since g is right minimal, $g \simeq \psi' g' \phi' \simeq (\psi' \psi) g$ implies that $\psi' \psi$ is an equivalence; and because g' is right minimal, $g' \simeq \psi g \phi \simeq (\psi \psi') g'$ implies that $(\psi \psi')$ is an equivalence. It now follows that ϕ' is an equivalence. \square

We now turn our attention to the interplay between minimal almost-split morphisms in cofiber sequences of a stable ∞ -category.

Lemma 3.4.9. *Let \mathcal{C} be a stable ∞ -category and suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence in \mathcal{C} . Then f is right minimal if and only if g is left minimal.*

Proof. Suppose f is right minimal and that the following diagram commutes (up to homo-

copy):

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \downarrow \text{id}_Y & & \downarrow \phi \\ Y & \xrightarrow{g} & Z. \end{array}$$

Completing this diagram to a map of cofiber sequences gives a morphism $\psi: X \rightarrow X$, as follows,

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow \psi & & \downarrow \text{id}_Y & & \downarrow \phi \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z. \end{array}$$

Since f is right minimal, ψ is an equivalence. Therefore, by Lemma 2.3.6, ϕ is an equivalence.

The converse is proved similarly. \square

Lemma 3.4.10. *Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence in a stable ∞ -category \mathcal{C} . If $g: Y \rightarrow Z$ is not a retraction and $\text{End}_{\text{hc}}(Z)$ is local, then $f: X \rightarrow Y$ is left minimal.*

Proof. By Lemma 3.1.5, f is not a section because g is not a retraction. Suppose $\psi: Y \rightarrow Y$ is such that $f \simeq \psi f$. This assumption induces a map of cofiber sequences as follows

$$\begin{array}{ccccc} \Omega Z & \xrightarrow{e} & X & \xrightarrow{f} & Y \\ \downarrow \phi & & \downarrow \text{id}_X & & \downarrow \psi \\ \Omega Z & \xrightarrow{e} & X & \xrightarrow{f} & Y. \end{array}$$

Since \mathcal{C} is a stable ∞ -category, $\Omega: \mathcal{C} \rightarrow \mathcal{C}$ is an equivalence. It follows that $\text{End}_{\text{hc}}(\Omega Z)$ is local because $\text{End}_{\text{hc}}(Z)$ is local. Hence, either ϕ or $(1 - \phi)$ is an equivalence. If $(1 - \phi)$ is an equivalence, then $e(1 - \phi) \simeq 0$ implies that $e \simeq 0$. By Lemma 3.1.2, this implies that f is a section, a contradiction. Hence, ϕ is an equivalence and therefore ψ is an equivalence by Lemma 2.3.6. \square

Proposition 3.4.11. *Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence in a stable ∞ -category \mathcal{C} .*

If $g: Y \rightarrow Z$ minimal right almost-split, then $f: X \rightarrow Y$ is left almost-split.

Proof. Since g is not a retraction, f is not a section, by Lemma 3.1.5. Suppose $\alpha: X \rightarrow X'$ is not a section. By Lemma 2.3.8, we have an induced morphism of cofiber sequences

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \text{id}_Z \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z, \end{array}$$

where the left square is a pushout. By way of contradiction, suppose that f' is not a section. Then g' is not a retraction and hence factors through g , as g is right almost-split. Consequently, there exists $\beta': Y' \rightarrow Y$ such that $g' \simeq g\beta'$. The morphism in $\text{Fun}(\Delta^1, \mathcal{C})$ given by

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Z \\ \downarrow \beta' & & \downarrow \text{id}_Z \\ Y & \xrightarrow{g} & Z \end{array}$$

then induces a morphism of cofiber sequences (up to a contractible space of choices)

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z \\ \downarrow \alpha' & & \downarrow \beta' & & \downarrow \text{id}_Z \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z. \end{array}$$

Composition then gives the following morphism of cofiber sequences,

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow \alpha'\alpha & & \downarrow \beta'\beta & & \downarrow \text{id}_Z \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z. \end{array}$$

Since g is right minimal, $g \simeq g(\beta'\beta)$ implies that $(\beta'\beta)$ is an equivalence. It follows from Lemma 2.3.6 that $(\alpha'\alpha)$ is an equivalence, but this contradicts our assumption that α is not a section. Hence, f' is a section and thus α factors through f , which proves that f is left almost-split. \square

Proposition 3.4.12. *Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence in a stable ∞ -category \mathcal{C} .*

If $g: Y \rightarrow Z$ is minimal right almost-split, then $f: X \rightarrow Y$ is minimal left almost-split.

Proof. Combine Proposition 3.4.11 and Lemma 3.4.10. □

3.5 Irreducible morphisms

Definition 3.5.1. Let \mathcal{C} be an ∞ -category. A morphism $f: X \rightarrow Y$ in \mathcal{C} is called *irreducible* if f is neither a section nor a retraction and whenever there is a diagram $\Delta^2 \rightarrow \mathcal{C}$ of the form

$$\begin{array}{ccc} & W & \\ i \nearrow & & \searrow r \\ X & \xrightarrow{f} & Y \end{array}$$

either r is a retraction or i is a section. Equivalently, a morphism $f: X \rightarrow Y$ is irreducible in \mathcal{C} if it is irreducible in the homotopy category $\mathrm{h}\mathcal{C}$.

Irreducible morphisms are closely related to minimal almost-split morphisms.

Proposition 3.5.2. *Let \mathcal{C} be an additive ∞ -category. If $g: Y \rightarrow Z$ be a minimal right almost-split morphism in \mathcal{C} with $Y \neq 0$, then g is irreducible.*

Proof. Assume $g \simeq g_1 g_2$ for some morphisms $g_1: Y' \rightarrow Z$ and $g_2: Y \rightarrow Y'$. As a right almost-split morphism, g is not a retraction. By Corollary 3.2.7, g is also not a section. If g_1 is not a retraction, then since g is right almost-split there exists $k: Y' \rightarrow Y$ such that $g_1 \simeq gk$. Using that g is right minimal, $g \simeq g_1 g_2 \simeq g(kg_2)$ implies that $kg_2: Y \rightarrow Y$ is an equivalence. We conclude that g_2 is a section, proving that g is irreducible. □

Proposition 3.5.2 admits a slight refinement in stable ∞ -categories.

Proposition 3.5.3. *Let $g: Y \rightarrow Z$ be a right almost-split morphism in a stable ∞ -category.*

If $h: X \rightarrow Z$ is an irreducible morphism, then there exists a morphism $i: W \rightarrow Z$ such that $g \simeq (h, i): X \oplus W \rightarrow Z$ (as objects of the slice category over Z).

Proof. As an irreducible morphism, h is not a retraction. Since g is right almost-split, there exists $k: X \rightarrow Y$ such that $h \simeq gk$. Since h is irreducible and g is not a retraction, it must be that k is a section. Let $j: Y \rightarrow X$ be such that $jk \simeq \text{id}_X$. Consider the cofiber sequence

$$\begin{array}{ccc} X & \xrightarrow{k} & Y \\ \downarrow & & \downarrow c \\ 0 & \longrightarrow & W. \end{array}$$

By Lemma 3.1.5, $c: Y \rightarrow W$ is a retraction. Using this pushout, $(\text{id}_Y - kj)k \simeq 0$ implies that there exists $d: W \rightarrow Y$ be such that $dc \simeq \text{id}_Y - kj$. Because c is a retraction and $cdc \simeq c$, we also have that $cd \simeq \text{id}_W$. Now, $j: Y \rightarrow X$ and $c: Y \rightarrow W$ determine a morphism $\alpha: Y \rightarrow X \oplus W$; similarly, $k: X \rightarrow Y$ and $d: W \rightarrow Y$ determine a morphism $\beta: X \oplus W \rightarrow Y$. It is not difficult to check that α and β are mutually inverse equivalences. Let $i \simeq gd: W \rightarrow Z$, and let $\gamma = (h, i): X \oplus W \rightarrow Z$ be the induced morphism. Writing $\iota_X: X \rightarrow X \oplus W$ and $\iota_W: W \rightarrow X \oplus W$ for the canonical inclusions, $h \simeq \gamma\iota_X \simeq gk \simeq g\beta\iota_X$ and $i \simeq \gamma\iota_W \simeq gd \simeq g\beta\iota_W$ implies $\gamma \simeq g\beta$, by uniqueness. \square

Proposition 3.5.3 admits a converse.

Proposition 3.5.4. *Let \mathcal{C} be an additive ∞ -category. If $g = (h, i): X \oplus W \rightarrow Z$ is a minimal right almost-split morphism in \mathcal{C} such that $h: X \rightarrow Z$ is neither a section nor a retraction, then h is irreducible.*

Proof. Assume $h \simeq pq$ for $q: X \rightarrow Y$ and $p: Y \rightarrow Z$ with p not a retraction. We must show

that q is a section. Since g is right almost-split, there exists $\begin{pmatrix} s \\ t \end{pmatrix} : Y \rightarrow X \oplus W$ such that $p \simeq (h \ i) \begin{pmatrix} s \\ t \end{pmatrix} \simeq hs + it$. Therefore,

$$(h, i) \begin{pmatrix} sq & 0 \\ tq & 1_W \end{pmatrix} \simeq (hsq + itq, i) \simeq (h, i).$$

Since $g = (h, i)$ is right minimal, we have that $\begin{pmatrix} sq & 0 \\ tq & 1_W \end{pmatrix}$ is an equivalence. It follows that q is a section, as desired. \square

Theorem 3.5.5. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a cofiber sequence in a stable ∞ -category. Then the following statements are equivalent:*

- (1) *The morphism $f : X \rightarrow Y$ is irreducible.*
- (2) *For any $g' : Y' \rightarrow Z$ there exists either a morphism $h : Y' \rightarrow Y$ such that $g' \simeq gh$ or a morphism $h' : Y \rightarrow Y'$ such that $g \simeq g'h'$.*

Proof. First, assume that $f : X \rightarrow Y$ is an irreducible morphism. Any $g' : Y' \rightarrow Z$ gives rise to a map of cofiber sequences

$$\begin{array}{ccccc} X & \xrightarrow{i'} & E & \xrightarrow{j} & Y' \\ \downarrow \text{id}_X & & \downarrow r & & \downarrow g' \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

where the right square is a pullback, by Lemma 2.3.7. Since $f \simeq ri'$ is irreducible, either r is a retraction or i' is a section. In the first case, there exists $i : Y \rightarrow E$ such that $ri \simeq \text{id}_Y$ and consequently $g \simeq g'(ji)$. In the second case, Lemma 3.1.5 implies that j is a retraction. Hence, there exists $k : Y' \rightarrow E$ such that $jk \simeq \text{id}_{Y'}$ and consequently $g' \simeq g(rk)$.

Now, assume statement (2) holds. Assume that $f \simeq ri$ and consider the induced map of cofiber sequences

$$\begin{array}{ccccc} X & \xrightarrow{i} & E & \xrightarrow{j} & Y' \\ \downarrow \text{id}_X & & \downarrow r & & \downarrow g' \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

so that $Y' \simeq \text{cofib}(i)$. Observe that the right square is a pushout by Lemma 2.1.36, and hence also a pullback. If there exists a morphism $h: Y' \rightarrow Y$ such that $g' \simeq gh$, then using this pullback there exists $k: Y' \rightarrow E$ such that $h \simeq rk$ and $jk \simeq \text{id}_{Y'}$. This implies that j is a retraction and thus i is a section, by Lemma 3.1.5. On the other hand, if there exists $h': Y \rightarrow Y'$ such that $g \simeq g'h'$, then again using that the right square is a pullback gives a morphism $k': Y \rightarrow E$ such that $h' \simeq jk'$ and $rk' \simeq \text{id}_Y$. Hence, r is a retraction. \square

Corollary 3.5.6. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a cofiber sequence in a stable ∞ -category. If $g: Y \rightarrow Z$ is right almost-split and nonzero, then $f: X \rightarrow Y$ is irreducible.*

Proof. By Theorem 3.5.5, it suffices to show that for every $g': Y' \rightarrow Z$ there exists either $h: Y' \rightarrow Y$ such that $g' \simeq gh$ or $h': Y \rightarrow Y'$ such that $g \simeq g'h'$. If g' is not a retraction, then there exists $h: Y' \rightarrow Y$ such that $g' \simeq gh$ because g is right almost-split. On the other hand, if g' is a retraction, then there exists $j: Z \rightarrow Y'$ such that $g'j \simeq \text{id}_Z$. Consequently, $h' \simeq jg$ is such that $g \simeq g'h'$, as desired. \square

3.6 Morphisms determined by objects

Definition 3.6.1 (Auslander [5]). Let \mathcal{C} be an ∞ -category containing an object C . A morphism $g: Y \rightarrow Z$ is called *right C -determined* (or *right determined by C*) if for any

morphism $g': Y' \rightarrow Z$, there exists a 2-simplex $\Delta^2 \rightarrow \mathcal{C}$ of the form

$$\begin{array}{ccc} & Y & \\ \nearrow & & \searrow g \\ Y' & \xrightarrow{g'} & Z \end{array}$$

whenever $\text{Im Hom}_{\text{h}\mathcal{C}}(C, g') \subseteq \text{Im Hom}_{\text{h}\mathcal{C}}(C, g) \subseteq \text{Hom}_{\text{h}\mathcal{C}}(C, Z)$. Equivalently, a morphism $g: Y \rightarrow Z$ is right C -determined if it is right C -determined in $\text{h}\mathcal{C}$.

Example 3.6.2. Let \mathcal{C} be an ∞ -category. Every equivalence of \mathcal{C} is right C -determined for every object $C \in \mathcal{C}$.

Remark 3.6.3. It is not difficult to check that the collection of right determined morphisms is closed under homotopy and composition with equivalences.

Another important class of examples comes from right almost-split morphisms.

Proposition 3.6.4. *Let \mathcal{C} be an additive ∞ -category. A morphism $g: Y \rightarrow Z$ is right almost-split if and only if the following conditions are satisfied:*

- (1) *The endomorphism ring $\text{End}_{\text{h}\mathcal{C}}(Z)$ is local with $\text{rad End}_{\text{h}\mathcal{C}}(Z) = \text{Im Hom}_{\text{h}\mathcal{C}}(Z, g)$.*
- (2) *The morphism $g: Y \rightarrow Z$ is right Z -determined.*

Proof. The “if” statement is Lemma 3.2.10. Conversely, assume that $g: Y \rightarrow Z$ is right almost-split. By Lemma 3.2.4, we know that $\text{End}_{\text{h}\mathcal{C}}(Z)$ is a local ring with unique maximal ideal $\text{rad End}_{\text{h}\mathcal{C}}(Z) = \text{Im Hom}_{\text{h}\mathcal{C}}(Z, g)$. Now, suppose $g': Y' \rightarrow Z$ be such that $\text{Im Hom}_{\text{h}\mathcal{C}}(Z, g') \subseteq \text{Im Hom}_{\text{h}\mathcal{C}}(Z, g)$. It follows that g' is not a retraction and hence factors through g . Thus, $g: Y \rightarrow Z$ is right Z -determined. \square

We now come to our main result of this section.

Theorem 3.6.5. *Let \mathcal{C} be an additive ∞ -category. Suppose $C \in \mathcal{C}$ and set $R = \text{End}_{\text{hc}}(C)$. Fix a morphism $g: Y \rightarrow Z$ and suppose $\eta: \text{Hom}_{\text{hc}}(-, Z) \rightarrow \text{Hom}_R(\text{Hom}_{\text{hc}}(C, -), E)$ is a natural transformation for some right R -module E . Then the sequence*

$$\text{Hom}_{\text{hc}}(-, Y) \xrightarrow{[g] \circ -} \text{Hom}_{\text{hc}}(-, Z) \xrightarrow{\eta} \text{Hom}_R(\text{Hom}_{\text{hc}}(C, -), E) \quad (*)$$

is exact if and only if the sequence

$$\text{Hom}_{\text{hc}}(C, Y) \xrightarrow{[g] \circ -} \text{Hom}_{\text{hc}}(C, Z) \xrightarrow{\eta_C} \text{Hom}_R(\text{Hom}_{\text{hc}}(C, C), E) \quad (*')$$

is exact and $g: Y \rightarrow Z$ is right C -determined.

Proof. Using that η is a natural transformation, we first observe that for any $\alpha: X \rightarrow Z$, the following diagram commutes

$$\begin{array}{ccc} \text{Hom}_{\text{hc}}(Z, Z) & \xrightarrow{[\alpha]^*} & \text{Hom}_{\text{hc}}(X, Z) \\ \downarrow \eta_Z & & \downarrow \eta_X \\ \text{Hom}_R(\text{Hom}_{\text{hc}}(C, Z), E) & \xrightarrow{\beta_\alpha} & \text{Hom}_R(\text{Hom}_{\text{hc}}(C, X), E) \end{array}$$

where $\beta_\alpha(\gamma) = \gamma \circ [\alpha]_*$. Consequently,

$$\eta_X([\alpha]) = (\eta_X \circ [\alpha]^*)([\text{id}_Z]) = (\beta_\alpha \circ \eta_Z)([\text{id}_Z]) = \eta_Z([\text{id}_Z]) \circ [\alpha]_*$$

shows that $\eta_X([\alpha]) = 0$ if and only if $\text{Im } \text{Hom}_{\text{hc}}(C, \alpha) \subseteq \text{Ker } \eta_Z([\text{id}_Z])$. In particular, setting $X = C$, we have $\eta_Z([\text{id}_Z]) \circ [\alpha]_* = \eta_C([\alpha]): \text{End}_{\text{hc}}(C) \rightarrow E$ for any $\alpha: C \rightarrow Z$. Evaluating this morphism on $[\text{id}_C]$ shows that $\eta_Z([\text{id}_Z])([\alpha]) = \eta_C([\alpha])([\text{id}_C])$. Consequently, we have $\text{Ker } \eta_C \subseteq \text{Ker } \eta_Z([\text{id}_Z])$. Using that $\eta_C([\alpha])$ is an $R = \text{End}_{\text{hc}}(C)$ module homomorphism and hence completely determined by its value on $[\text{id}_C]$, we also have $\text{Ker } \eta_Z([\text{id}_Z]) \subseteq \text{Ker } \eta_C$.

Now, assume that the sequence $(*)$ is exact. In particular, the induced sequence $(*)'$ is exact. It remains to show that $g: Y \rightarrow Z$ is right C -determined. Suppose $g': Y' \rightarrow Z$

is a morphism in \mathcal{C} such that $\text{Im Hom}_{\text{hc}}(C, g') \subseteq \text{Im Hom}_{\text{hc}}(C, g)$. We will show that $\eta \circ [g']_* = 0$. Assuming this momentarily, exactness of $(*)$ implies that there exists a unique map $\text{Hom}_{\text{hc}}(-, Y') \rightarrow \text{Hom}_{\text{hc}}(-, Y)$ which factors $[g']_*$ through $[g]_*$. By the Yoneda embedding, this unique map arises from a morphism $[h]: Y' \rightarrow Y$ such that $[g'] = [g] \circ [h]$.

To show that $\eta \circ [g']_* = 0$, we must show that for all $X \in \mathcal{C}$, $\text{Im Hom}_{\text{hc}}(X, g') \subseteq \text{Ker } \eta_X$, that is, for all $\psi: X \rightarrow Y'$ we must show that $\eta_X([g'\psi]) = 0$. By our naturality calculation above, $\eta_X([g'\psi]) = 0$ is equivalent to the condition $\text{Im Hom}_{\text{hc}}(C, g'\psi) \subseteq \text{Ker } \eta_Z([\text{id}_Z])$.

By exactness of $(*)'$, we have that $\text{Ker } \eta_C = \text{Im Hom}_{\text{hc}}(C, g)$. By our assumptions on $g': Y' \rightarrow Z$, we have $\text{Im Hom}_{\text{hc}}(C, g'\psi) \subseteq \text{Im Hom}_{\text{hc}}(C, g') \subseteq \text{Im Hom}_{\text{hc}}(C, g) = \text{Ker } \eta_C$. Noting that $\text{Ker } \eta_C \subseteq \text{Ker } \eta_Z([\text{id}_Z])$ (as shown above) completes the proof.

Conversely, assume $(*)'$ is exact and that $g: Y \rightarrow Z$ is right C -determined. We must show that for every X , $\text{Im Hom}_{\text{hc}}(X, g) = \text{Ker } \eta_X$. By our calculation above, for any $\delta: X \rightarrow Y$, we have $\eta_X([g] \circ [\delta]) = 0$ if and only if $\text{Im Hom}_{\text{hc}}(C, g\delta) \subseteq \text{Ker } \eta_Z([\text{id}_Z])$. Using exactness of $(*)'$, we have $\text{Im Hom}_{\text{hc}}(C, g\delta) \subseteq \text{Im Hom}_{\text{hc}}(C, g) = \text{Ker } \eta_C \subseteq \text{Ker } \eta_Z([\text{id}_Z])$. Hence, $\text{Im Hom}_{\text{hc}}(X, g) \subseteq \text{Ker } \eta_X$. If $\alpha: X \rightarrow Z$ is such that $\eta_X([\alpha]) = 0$, then we know $\text{Im Hom}_{\text{hc}}(C, \alpha) \subseteq \text{Ker } \eta_Z([\text{id}_Z])$. We must show α factors through g . Exactness of $(*)'$ implies $\text{Ker } \eta_C = \text{Im Hom}_{\text{hc}}(C, g)$, and the above calculation shows $\text{Ker } \eta_Z([\text{id}_Z]) = \text{Ker } \eta_C$. Hence, $\text{Im Hom}_{\text{hc}}(C, \alpha) \subseteq \text{Im Hom}_{\text{hc}}(C, g)$. Using that g is right C -determined, the proof is complete. \square

Remark 3.6.6. Let \mathcal{C} be a stable ∞ -category. Fix a morphism $g: Y \rightarrow Z$ and consider

the cofiber sequence

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \downarrow & & \downarrow h \\ 0 & \longrightarrow & W. \end{array}$$

By Remark 2.3.18, this cofiber sequence induces, for every $X \in \mathcal{C}$, an exact sequence

$$\mathrm{Hom}_{\mathrm{hc}}(X, Y) \rightarrow \mathrm{Hom}_{\mathrm{hc}}(X, Z) \rightarrow \mathrm{Hom}_{\mathrm{hc}}(X, W).$$

Now, suppose $\eta: \mathrm{Hom}_{\mathrm{hc}}(-, Z) \rightarrow \mathrm{Hom}_R(\mathrm{Hom}_{\mathrm{hc}}(C, -), E)$ is a natural transformation, where $C \in \mathcal{C}$, $R = \mathrm{End}_{\mathrm{hc}}(C)$, and E is some right R -module. Observe that if there exists a *monomorphism* $\delta: \mathrm{Hom}_{\mathrm{hc}}(-, W) \rightarrow \mathrm{Hom}_R(\mathrm{Hom}_{\mathrm{hc}}(C, -), E)$ such that $\eta = \delta \circ [h]_*$, then the corresponding sequence

$$\mathrm{Hom}_{\mathrm{hc}}(-, Y) \xrightarrow{[g] \circ -} \mathrm{Hom}_{\mathrm{hc}}(-, Z) \xrightarrow{\eta} \mathrm{Hom}_R(\mathrm{Hom}_{\mathrm{hc}}(C, -), E)$$

is exact. To see this, let $\alpha: X \rightarrow Z$ be any morphism in \mathcal{C} . Then $\eta_X([\alpha]) = 0$ if and only if $\delta_X([h\alpha]) = 0$. Since δ_X is a monomorphism, this implies $h\alpha \simeq 0$. As $g \simeq \mathrm{fib}(h)$, we conclude that α factors through g , proving $\mathrm{Ker} \eta \subseteq \mathrm{Im}([g]_*)$. Conversely, for any $\psi: X \rightarrow Y$, $\eta_X([g] \circ [\psi]) = \delta_X([h] \circ [g] \circ [\psi]) = 0$ because $hg \simeq 0$. Hence, $\mathrm{Ker} \eta = \mathrm{Im}([g]_*)$.

Corollary 3.6.7. *Let \mathcal{C} be a stable ∞ category and suppose $Y \xrightarrow{g} Z \xrightarrow{h} W$ is a fiber sequence. If there exists a left divisible morphism $k: C \rightarrow W$ for some $C \in \mathcal{C}$, then the morphism $g: Y \rightarrow Z$ is right C -determined. In particular, if $h: Z \rightarrow W$ is left divisible, then $g: Y \rightarrow Z$ is right Z -determined.*

Proof. Combine Theorem 3.3.7 and Remark 3.6.6. □

Remark 3.6.8. Let \mathcal{C} be a stable ∞ -category. Suppose $C \in \mathcal{C}$ and set $R = \mathrm{End}_{\mathrm{hc}}(C)$.

Theorem 3.6.5 implies that for any $W \in \mathcal{C}$, the morphism $0 \rightarrow W$ is right C -determined

if and only if there exists a monomorphism $\delta: \text{Hom}_{\text{hc}}(-, W) \rightarrow \text{Hom}_R(\text{Hom}_{\text{hc}}(C, -), E)$, for some right R -module E . Consequently, using Remark 3.6.6, a morphism $g: Y \rightarrow Z$ is right C -determined if the canonical morphism $0 \rightarrow W \simeq \text{cofib}(g)$ is right C -determined. Moreover, by Theorem 3.3.7, if $k: C \rightarrow W$ is a left divisible morphism, then $0 \rightarrow W$ is right C -determined and (using Lemma 3.6.13 below) any pullback of this morphism is again right C -determined.

Remark 3.6.9. Let \mathcal{C} be an additive ∞ -category. Suppose $C \in \mathcal{C}$ and set $R = \text{End}_{\text{hc}}(C)$. Suppose $\delta: \text{Hom}_{\text{hc}}(-, W) \rightarrow \text{Hom}_R(\text{Hom}_{\text{hc}}(C, -), E)$ is a natural transformation. The proof of Theorem 3.6.5 showed that, for any $\alpha: X \rightarrow W$, $\delta_X([\alpha]) = \delta_W([\text{id}_W]) \circ [\alpha]_*$. Consequently, $\text{Ker } \delta_X([\alpha]) = \{\varepsilon: C \rightarrow X : [\alpha\varepsilon] \in \text{Ker } \delta_W([\text{id}_W])\}$. The proof of Theorem 3.6.5 also showed that $\text{Ker } \delta_W([\text{id}_W]) = \text{Ker } \delta_C$ (by observing that $\delta_W([\text{id}_W])([\alpha]) = \delta_C([\alpha])([\text{id}_C])$ and evaluation at the identity determines an isomorphism $\text{Hom}_R(R, E) \rightarrow E$). Therefore, if $\text{Ker } \delta_C = 0$, then $\text{Ker } \delta_X([\alpha]) = \text{Ker } \text{Hom}_{\text{hc}}(C, \alpha)$; and if δ_C is an isomorphism, then $\delta_W([\text{id}_W])$ is an isomorphism (as the composition of δ_C with evaluation at $[\text{id}_C]$).

Proposition 3.6.10. *Let \mathcal{C} be an ∞ -category containing an object C . Suppose $g: Y \rightarrow Z$ and $g': Y' \rightarrow Z$ are both right C -determined and right minimal. Then there exists an equivalence $\alpha: Y' \rightarrow Y$ such that $g' \simeq g\alpha$ if and only if $\text{Im } \text{Hom}_{\text{hc}}(C, g) \cong \text{Im } \text{Hom}_{\text{hc}}(C, g')$.*

Proof. Assume $\alpha: Y' \rightarrow Y$ is an equivalence in \mathcal{C} such that $g' \simeq g\alpha$. Then α induces an isomorphism $\text{Hom}_{\text{hc}}(C, Y') \rightarrow \text{Hom}_{\text{hc}}(C, Y)$ which exhibits the desired isomorphism $\text{Im } \text{Hom}_{\text{hc}}(C, g') \cong \text{Im } \text{Hom}_{\text{hc}}(C, g)$.

Conversely, assume that $\text{Im } \text{Hom}_{\text{hc}}(C, g') = \text{Im } \text{Hom}_{\text{hc}}(C, g)$. Since g is right C -determined and $\text{Im } \text{Hom}_{\text{hc}}(C, g') \subseteq \text{Im } \text{Hom}_{\text{hc}}(C, g)$, there exists $\alpha: Y' \rightarrow Y$ such that $g' \simeq$

$g\alpha$. Similarly, using that g' is right C -determined and $\text{Im Hom}_{\text{hc}}(C, g) \subseteq \text{Im Hom}_{\text{hc}}(C, g')$, there exists $\alpha': Y \rightarrow Y'$ such that $g \simeq g'\alpha'$. Hence, $g \simeq g'\alpha' \simeq g(\alpha\alpha')$ implies $(\alpha\alpha')$ is an equivalence because g is right minimal. Likewise, using that g' is right minimal, we have that $(\alpha'\alpha)$ is also an equivalence. These equivalences together imply that $\alpha: Y' \rightarrow Y$ is an equivalence, as desired. \square

We now investigate some of the closure properties of right C -determined morphisms.

Lemma 3.6.11. *Let \mathcal{C} be an ∞ -category containing an object C . The collection of right C -determined morphisms in \mathcal{C} is closed under the formation of retracts. That is, given a diagram*

$$\begin{array}{ccccc} Y & \xrightarrow{q} & W & \xrightarrow{p} & Y \\ \downarrow g & & \downarrow f & & \downarrow g \\ Z & \xrightarrow{i} & X & \xrightarrow{r} & Z \end{array}$$

in \mathcal{C} such that $pq \simeq \text{id}_Y$ and $ri \simeq \text{id}_Z$, if $f: W \rightarrow X$ is right C -determined, then $g: Y \rightarrow Z$ is right C -determined.

Proof. Suppose $g': Y' \rightarrow Z$ is such that $\text{Im Hom}_{\text{hc}}(C, g') \subseteq \text{Im Hom}_{\text{hc}}(C, g)$. Then

$$\text{Im Hom}_{\text{hc}}(C, ig') \subseteq \text{Im Hom}_{\text{hc}}(C, ig) = \text{Im Hom}_{\text{hc}}(C, fq) \subseteq \text{Im Hom}_{\text{hc}}(C, f).$$

Since $f: W \rightarrow X$ is right C -determined, there exists $j: Y' \rightarrow W$ such that $ig' \simeq fj$, which implies $g' \simeq g(pj)$, as desired. \square

Remark 3.6.12. Note that the assumption $pq \simeq \text{id}_Y$ was not used in the above proof.

Lemma 3.6.13. *Let \mathcal{C} be an ∞ -category containing an object C . The collection of right C -determined morphisms in \mathcal{C} is closed under the formation of pullbacks. That is, given a*

pullback diagram

$$\begin{array}{ccc} W & \xrightarrow{p} & Y \\ \downarrow f & & \downarrow g \\ X & \xrightarrow{q} & Z \end{array}$$

in \mathcal{C} , if $g: Y \rightarrow Z$ is right C -determined, then $f: W \rightarrow X$ is right C -determined. Moreover,

if $q: X \rightarrow Z$ is right C -determined, then so is the composition $k \simeq qf \simeq gp: W \rightarrow Z$.

Proof. Suppose $f': W' \rightarrow X$ is a morphism in \mathcal{C} such that $\text{Im Hom}_{\text{hc}}(C, f') \subseteq \text{Im Hom}_{\text{hc}}(C, f)$.

Then $qf \simeq gp$ implies

$$\text{Im Hom}_{\text{hc}}(C, qf') \subseteq \text{Im Hom}_{\text{hc}}(C, qf) = \text{Im Hom}_{\text{hc}}(C, gp) \subseteq \text{Im Hom}_{\text{hc}}(C, g).$$

Since $g: Y \rightarrow Z$ is right C -determined, there exists $j: W' \rightarrow Y$ such that $qf' \simeq gj$. Using that the above diagram is a pullback, there exists $a: W' \rightarrow W$ such that $f' \simeq fa$. Hence, $f: W \rightarrow X$ is right C -determined.

Now, suppose that $q: X \rightarrow Z$ is also right C -determined and that $k': W' \rightarrow Z$ is such that $\text{Im Hom}_{\text{hc}}(C, k') \subseteq \text{Im Hom}_{\text{hc}}(C, k)$. Then $\text{Im Hom}_{\text{hc}}(C, k') \subseteq \text{Im Hom}_{\text{hc}}(C, q)$ and $\text{Im Hom}_{\text{hc}}(C, k') \subseteq \text{Im Hom}_{\text{hc}}(C, g)$. Using that q is right C -determined, there exists $r: W' \rightarrow X$ such that $k' \simeq qr$; likewise, there exists $h: W' \rightarrow Y$ such that $k' \simeq gh$ because g is right C -determined. Using the pullback, there exists $b: W' \rightarrow W$ such that $pb \simeq h$. Hence, $k' \simeq gh \simeq g(pb) \simeq kb$, as desired. \square

Lemma 3.6.14. *Let \mathcal{C} be an ∞ -category which admits coproducts and contains an object C . Assume D is an object of \mathcal{C} equipped with a morphism $D \rightarrow Y'$ for all $Y' \in \mathcal{C}$. If $g: Y \rightarrow Z$ is right C -determined, then $g: Y \rightarrow Z$ is right $(C \amalg D)$ -determined.*

Proof. Suppose $g': Y' \rightarrow Z$ is such that $\text{Im Hom}_{\text{hc}}(C \amalg D, g') \subseteq \text{Im Hom}_{\text{hc}}(C \amalg D, g)$ and let $d: D \rightarrow Y'$ be the given morphism. Then for any $\phi: C \rightarrow Y'$, there is an induced

morphism $\gamma = (\phi, d): C \amalg D \rightarrow Y'$. Since $\operatorname{Im} \operatorname{Hom}_{\mathbf{h}\mathcal{C}}(C \amalg D, g') \subseteq \operatorname{Im} \operatorname{Hom}_{\mathbf{h}\mathcal{C}}(C \amalg D, g)$, there exists $\psi: C \amalg D \rightarrow Y$ such that $g\psi = g'\gamma$. Consequently, if $i: C \rightarrow C \amalg D$ is the canonical morphism, then $g'\phi = g'\gamma i = g\psi i$ shows that $\operatorname{Im} \operatorname{Hom}_{\mathbf{h}\mathcal{C}}(C, g') \subseteq \operatorname{Im} \operatorname{Hom}_{\mathbf{h}\mathcal{C}}(C, g)$. Since g is right C -determined, there exists $j: Y' \rightarrow Y$ such that $g' \simeq gj$, as desired. \square

Remark 3.6.15. If \mathcal{C} is a pointed ∞ -category, then for any object $D \in \mathcal{C}$ there is a canonical morphism $D \rightarrow Y'$ (the zero morphism) for every $Y' \in \mathcal{C}$. Thus, if \mathcal{C} admits coproducts, then Lemma 3.6.14 implies that a right C -determined morphism is right $(C \amalg D)$ -determined for every object D of \mathcal{C} .

3.7 Existence theorems

Following ideas of Krause [36], we use a version of Brown representability to prove the existence of morphisms determined by objects in compactly generated stable ∞ -categories. From this, we derive the existence of minimal right almost-split morphisms.

The following version of Brown's theorem is due to Jacob Lurie, see [46, 1.4.1.2].

Theorem 3.7.1 (Brown representability). *Let \mathcal{C} be a presentable ∞ -category containing a set of objects $\{C_\alpha\}_{\alpha \in A}$ with the following properties:*

- (1) *Each object C_α is a cogroup object of the homotopy category $\mathbf{h}\mathcal{C}$.*
- (2) *Each object $C_\alpha \in \mathcal{C}$ is compact.*
- (3) *The ∞ -category \mathcal{C} is generated by the objects C_α under small colimits.*

Then a functor $F: \mathbf{h}\mathcal{C}^{\mathrm{op}} \rightarrow \mathbf{Set}$ is representable if and only if it satisfies the following conditions:

(a) For every collection of objects X_β in \mathcal{C} , the map $F(\coprod_\beta X_\beta) \rightarrow \prod_\beta F(X_\beta)$ is a bijection.

(b) For every pushout square

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

in \mathcal{C} , the induced map $F(Y') \rightarrow F(X') \times_{F(X)} F(Y)$ is surjective.

Remark 3.7.2. If \mathcal{C} is a stable ∞ -category, then every object of \mathcal{C} is a cogroup object of $\mathbf{h}\mathcal{C}$. Consequently, if \mathcal{C} is a compactly generated stable ∞ -category, then \mathcal{C} satisfies the hypotheses of Theorem 3.7.1.

Lemma 3.7.3. Let \mathcal{C} be a compactly generated stable ∞ -category. Let C be a compact object of \mathcal{C} and set $R = \mathrm{End}_{\mathbf{h}\mathcal{C}}(C)$. For any (ordinary) right R -module Q , the functor

$$F = \mathrm{Hom}_R(\mathrm{Hom}_{\mathbf{h}\mathcal{C}}(C, -), Q) : (\mathbf{h}\mathcal{C})^{\mathrm{op}} \rightarrow \mathcal{A}\mathbf{b}$$

is representable.

Proof. By Theorem 3.7.1, it suffices to show that

(a) For every collection of objects X_β in \mathcal{C} , the map $F(\coprod_\beta X_\beta) \rightarrow \prod_\beta F(X_\beta)$ is a bijection.

(b) For every pushout square

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

in \mathcal{C} , the induced map $F(Y') \rightarrow F(X') \times_{F(X)} F(Y)$ is surjective.

For every collection of objects $\{X_\beta\}$ in \mathcal{C} , the coproduct $\coprod_\beta X_\beta$ in \mathcal{C} agrees with the coproduct in $\mathbf{h}\mathcal{C}$. Moreover, an arbitrary coproduct $\coprod_\beta X_\beta$ can be obtained as a filtered

colimit of finite coproducts. Since C is compact, the functor corepresented by C commutes with filtered colimits; since the hom-functor commutes with finite colimits, it follows that

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, -) : \mathrm{h}\mathcal{C} \rightarrow \mathrm{RMod}_R$$

sends coproducts to coproducts. For any object Q in RMod_R , the hom-functor

$$\mathrm{Hom}_R(-, Q) : (\mathrm{RMod}_R)^{\mathrm{op}} \rightarrow \mathcal{A}\mathbf{b}$$

sends colimits in RMod_R to limits in $\mathcal{A}\mathbf{b}$, and hence $F\left(\coprod_{\beta} X_{\beta}\right) \cong \prod_{\beta} F(X_{\beta})$. This establishes condition (a).

Next, observe that for any cofiber sequence $U \xrightarrow{\mu} V \xrightarrow{\nu} W$ in \mathcal{C} , the functor F gives rise to a sequence $F(W) \xrightarrow{F(\nu)} F(V) \xrightarrow{F(\mu)} F(U)$ in $\mathcal{A}\mathbf{b}$ which is exact at $F(V)$. To see this, note that $F(\mu) \circ F(\nu) = F(\nu \circ \mu) = 0$; and if $\varphi : C \rightarrow V$ is any map such that $\nu\varphi \simeq 0$, then the following diagram can always be completed to a map of cofiber sequences, as indicated,

$$\begin{array}{ccccc} C & \xrightarrow{\mathrm{id}_C} & C & \longrightarrow & 0 \\ \vdots & & \downarrow \varphi & & \downarrow \\ \check{U} & \xrightarrow{\mu} & \check{V} & \xrightarrow{\nu} & \check{W}. \end{array}$$

From this, it follows that the sequence

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, U) \xrightarrow{[\mu] \circ -} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, V) \xrightarrow{[\nu] \circ -} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, W)$$

is exact at $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, V)$. For any object $Q \in \mathrm{RMod}_R$, the hom-functor $\mathrm{Hom}_R(-, Q)$ carries colimits in RMod_R to limits in $\mathcal{A}\mathbf{b}$; consequently, the sequence $F(W) \xrightarrow{F(\nu)} F(V) \xrightarrow{F(\mu)} F(U)$ in $\mathcal{A}\mathbf{b}$ is exact at $F(V)$, as desired.

Now, suppose that

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{g} & Y' \end{array}$$

is a pushout diagram in \mathcal{C} . We must show that the induced map $F(Y') \rightarrow F(X') \times_{F(X)} F(Y)$ is surjective. By Lemma 2.3.8, we can extend this diagram to a map of cofiber sequences

$$\begin{array}{ccccc} X & \xrightarrow{f} & X' & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \text{id}_Z \\ Y & \xrightarrow{g} & Y' & \longrightarrow & Z \end{array}$$

where here $Z = \text{cofib}(f) \simeq \text{cofib}(g)$. Since \mathcal{C} is stable, $\Sigma: \mathcal{C} \rightarrow \mathcal{C}$ is an equivalence with (up to weak homotopy) inverse $\Omega: \mathcal{C} \rightarrow \mathcal{C}$. The induced maps $Z \rightarrow \Sigma X$ and $Z \rightarrow \Sigma Y$ then give rise to boundary maps $F(X) \xrightarrow{\sim} F(\Omega \Sigma X) \rightarrow F(\Omega Z)$ and $F(Y) \xrightarrow{\sim} F(\Omega \Sigma Y) \rightarrow F(\Omega Z)$, making the following diagram a map of exact sequences in $\mathcal{A}b$:

$$\begin{array}{ccccccc} F(Z) & \xrightarrow{a} & F(Y') & \xrightarrow{F(g)} & F(Y) & \xrightarrow{c} & F(\Omega Z) \\ \downarrow \phi & & \downarrow s & & \downarrow t & & \downarrow \psi \\ F(Z) & \xrightarrow{b} & F(X') & \xrightarrow{F(f)} & F(X) & \xrightarrow{d} & F(\Omega Z) \end{array}$$

where ϕ and ψ are bijections.

We would like to see that the induced map $q: F(Y') \rightarrow F(X') \times_{F(X)} F(Y)$ is surjective. This statement follows from a diagram chase: suppose that $(x', y) \in F(X') \times_{F(X)} F(Y)$. Then $F(f)(x') = t(y) \in F(X)$ implies that $\psi c(y) = dt(y) = 0$. Since ψ is injective, we have $c(y) = 0$. So, there exists $y' \in F(Y')$ such that $F(g)(y') = y$. Now, since $t(y) = t(F(g)(y')) = F(f)(s(y')) = F(f)(x')$, we have that $F(f)(s(y') - x') = 0$. So, there exists $w \in F(Z)$ such that $b(w) = s(y') - x'$. Since ϕ is surjective, there exists $w' \in F(Z)$ such that $\phi(w') = w$. Then $a(w') \in F(Y')$ is such that $s(y' - a(w')) = s(y') - b(\phi(w')) = s(y') - b(w) = s(y') - (s(y') - x') = x'$ and $F(g)(y' - a(w')) = y - F(g)(a(w')) = y$. This implies $q(y' - a(w')) = (x', y)$, as desired.

This argument proves that condition (b) holds. Hence, the functor F satisfies the conditions of Theorem 3.7.1 and is therefore representable. \square

Using Brown representability, we now establish the existence of morphisms determined by objects in compactly generated stable ∞ -categories.

Theorem 3.7.4. *Let \mathcal{C} be a compactly generated stable ∞ -category. Suppose $C \in \mathcal{C}$ is compact and set $R = \text{End}_{\text{h}\mathcal{C}}(C)$. For any arbitrary object Z and any (right) R -submodule $H \subseteq \text{Hom}_{\text{h}\mathcal{C}}(C, Z)$, there exists a right minimal map $g: Y \rightarrow Z$ which is right C -determined and satisfies $\text{Im Hom}_{\text{h}\mathcal{C}}(C, g) = H$.*

Proof. Let $\mu: \text{Hom}_{\text{h}\mathcal{C}}(C, Z)/H \rightarrow E$ be an injective envelope in the category of right R -modules (which always exists). Since C is compact, the functor

$$\text{Hom}_R(\text{Hom}_{\text{h}\mathcal{C}}(C, -), E): (\text{h}\mathcal{C})^{\text{op}} \rightarrow \mathcal{A}\mathcal{B}$$

is representable, by Lemma 3.7.3. Let $T(C, E)$ be a representing object, so that

$$\theta: \text{Hom}_{\text{h}\mathcal{C}}(-, T(C, E)) \xrightarrow{\sim} \text{Hom}_R(\text{Hom}_{\text{h}\mathcal{C}}(C, -), E).$$

Applying this isomorphism to the composition

$$\text{Hom}_{\text{h}\mathcal{C}}(C, Z) \xrightarrow{\pi} \text{Hom}_{\text{h}\mathcal{C}}(C, Z)/H \xrightarrow{\mu} E$$

and choosing a representative gives a morphism $\gamma: Z \rightarrow T(C, E)$ (unique up to homotopy).

Suppose the diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \downarrow & & \downarrow \gamma \\ 0 & \longrightarrow & T(C, E) \end{array}$$

is a cofiber sequence in \mathcal{C} . We claim that the morphism $g: Y \rightarrow Z$ satisfies the desired properties. The cofiber sequence above yields an exact sequence (see Remark 2.3.18)

$$\text{Hom}_{\text{h}\mathcal{C}}(-, Y) \xrightarrow{[g]^{\circ-}} \text{Hom}_{\text{h}\mathcal{C}}(-, Z) \xrightarrow{[\gamma]^{\circ-}} \text{Hom}_{\text{h}\mathcal{C}}(-, T(C, E)).$$

Therefore, setting $\eta = \theta \circ [\gamma]_*$ gives an exact sequence

$$\mathrm{Hom}_{\mathrm{hc}}(-, Y) \xrightarrow{[g] \circ -} \mathrm{Hom}_{\mathrm{hc}}(-, Z) \xrightarrow{\eta} \mathrm{Hom}_R(\mathrm{Hom}_{\mathrm{hc}}(C, -), E).$$

Theorem 3.6.5 now implies that g is right C -determined. Using that θ is a monomorphism, Remark 3.6.9 shows that $\mathrm{Ker} \mathrm{Hom}_{\mathrm{hc}}(C, \gamma) = \mathrm{Ker} \theta_Z([\gamma]) = H$. Hence, by exactness, $\mathrm{Im} \mathrm{Hom}_{\mathrm{hc}}(C, g) = H$.

It remains to show that g is right minimal. By Proposition 3.4.9, g is right minimal if and only if γ is left minimal. Suppose that $\varphi: T(C, E) \rightarrow T(C, E)$ is such that $[\varphi] \circ [\gamma] = [\gamma]$. Using that θ_C is an isomorphism, Remark 3.6.9 shows that $\theta_{T(C, E)}([\mathrm{id}_{T(C, E)}])$ is again an isomorphism. Set $\alpha = \theta_{T(C, E)}([\mathrm{id}_{T(C, E)}])$ and let $\psi = \alpha \circ [\varphi]_* \circ \alpha^{-1}: E \rightarrow E$. Then we calculate

$$\mu\pi = \theta_Z([\gamma]) = \alpha \circ [\gamma]_* = \alpha \circ [\varphi]_* \circ [\gamma]_* = \psi \circ \alpha \circ [\gamma]_* = \psi \circ \theta_Z([\gamma]) = \psi\mu\pi.$$

This gives $\mu = \psi\mu$ because π is epic. Since μ is an injective envelope, it is left minimal. It follows that ψ is an isomorphism, and thus φ is an equivalence. Therefore, γ is left minimal and we conclude that g is right minimal. \square

Corollary 3.7.5. *Let \mathcal{C} be a compactly generated stable ∞ -category. If $Z \in \mathcal{C}$ is a strongly indecomposable compact object, then there exists a minimal right almost-split morphism $g: Y \rightarrow Z$ in \mathcal{C} .*

Proof. By Theorem 3.7.4, there exists a right minimal morphism $g: Y \rightarrow Z$ which is right Z -determined and satisfies $\mathrm{Im} \mathrm{Hom}_{\mathrm{hc}}(Z, g) = \mathrm{rad} \mathrm{End}_{\mathrm{hc}}(Z)$. By Proposition 3.6.4, we have that $g: Y \rightarrow Z$ is also right almost-split. \square

Corollary 3.7.6. *Let \mathcal{C} be a compactly generated stable ∞ -category. If $Z \in \mathcal{C}$ is a strongly indecomposable compact object, then there exists a minimal right divisible morphism $h: Z \rightarrow W$ in \mathcal{C} .*

Proof. By Corollary 3.7.5, there exists a minimal right almost-split morphism $g: Y \rightarrow Z$. By Proposition 3.3.3, taking a fiber sequence $Y \xrightarrow{g} Z \xrightarrow{h} W$ gives a morphism $h: Z \rightarrow W$ which is right divisible. By Corollary 3.4.7, the morphism h is right minimal. By Lemma 3.4.9, the morphism h is also left minimal because g is right minimal. \square

Corollary 3.7.7. *Let \mathcal{C} be a compactly generated stable ∞ -category. If $Z \in \mathcal{C}$ is a strongly indecomposable compact object, then there exists an irreducible morphism $f: X \rightarrow Y$ in \mathcal{C} such that $Z \simeq \text{cofib}(f)$.*

Proof. By Corollary 3.7.5, there exists a right almost-split morphism $g: Y \rightarrow Z$. Taking a cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ gives a morphism $f: X \rightarrow Y$. By Corollary 3.5.6, the morphism $f: X \rightarrow Y$ is irreducible. \square

Chapter 4

Auslander-Reiten theory

One of the main structural tools in classical Auslander-Reiten theory is the almost-split sequence, a nonsplit short exact sequence whose existence has significant consequences for the objects appearing at the ends of the sequence. In particular, almost-split sequences are invariants of their end-terms. In this chapter, we introduce Auslander-Reiten sequences in stable ∞ -categories as the higher categorical analogues of almost-split sequences. We prove in Proposition 4.1.7 that Auslander-Reiten sequences are homotopy invariants of the end-terms, and in Theorem 4.1.5 that Auslander-Reiten sequences exist in any compactly generated stable ∞ -categories with strongly indecomposable compact objects. As in the abelian setting, the characterization of Auslander-Reiten sequences in Proposition 4.1.2 shows that they are uniquely determined, up to homotopy, by either one of the morphisms occurring in the sequence.

In their original work, Auslander and Reiten established the existence of almost-split sequences using an isomorphism between injectively and projectively stable morphisms, now

called the Auslander-Reiten formula. In Section 4.2, we study a more general duality phenomenon building on our earlier work on morphisms determined by objects. We introduce Auslander functors on additive ∞ -categories and show in Theorem 4.2.12 that compactly generated stable ∞ -categories always admit such functors. In good circumstances, this construction yields an Auslander-Reiten translation functor as shown in Corollary 4.2.16. An analogue of the Auslander-Reiten formula on the homotopy category then follows as a consequence.

4.1 Auslander-Reiten sequences

Definition 4.1.1. Let \mathcal{C} be a stable ∞ -category. An *Auslander-Reiten sequence* in \mathcal{C} is a cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

such that f is left almost-split and g is right almost-split.

A consequence of the above definition is that any Auslander-Reiten sequence in a stable ∞ -category \mathcal{C} induces an Auslander-Reiten triangle in $\mathrm{h}\mathcal{C}$ (see Definition 2.3.15), and conversely for every Auslander-Reiten triangle in $\mathrm{h}\mathcal{C}$ there is a homotopy equivalence class of Auslander-Reiten sequences over it in \mathcal{C} . We next establish a useful characterization of the Auslander-Reiten sequences in \mathcal{C} .

Proposition 4.1.2. *Let \mathcal{C} be a stable ∞ -category and suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a fiber sequence. Then the following are equivalent:*

- (1) *The fiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an Auslander-Reiten sequence.*

(2) The morphism f is left almost-split and $\text{End}_{\text{hc}}(Z)$ is local.

(3) The morphism f is minimal left almost-split.

(4) The morphism g is right almost-split and $\text{End}_{\text{hc}}(X)$ is local.

(5) The morphism g is minimal right almost-split.

Proof. We prove that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$. The proof that $(1) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$ is dual to our arguments here. The implication $(1) \Rightarrow (2)$ is a consequence of the definition combined with Lemma 3.2.4. Lemma 3.4.10 shows that $(2) \Rightarrow (3)$. The implication $(3) \Rightarrow (1)$ follows from the dual of Proposition 3.4.12. \square

Proposition 4.1.3. *Let \mathcal{C} be a stable ∞ -category. Suppose $\Delta^2 \times \Delta^1 \rightarrow \mathcal{C}$ is a diagram in \mathcal{C} , depicted as*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow g & & \downarrow \\ 0 & \longrightarrow & Z & \xrightarrow{h} & W \end{array}$$

where both squares are pullbacks. Then the fiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an Auslander-Reiten sequence in \mathcal{C} if and only if the morphism $h: Z \rightarrow W$ is divisible.

Proof. By Proposition 3.3.3, the morphism $h: Z \rightarrow W$ is right divisible if and only if $g: Y \rightarrow Z$ is right almost-split. Consider the extended diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & 0 & & \\ \downarrow & & \downarrow g & & \downarrow & & \\ 0 & \longrightarrow & Z & \xrightarrow{h} & W & & \\ & & \downarrow & & \downarrow k & & \\ & & 0 & \longrightarrow & V & & \end{array}$$

where all squares are pushouts. The dual of Proposition 3.3.3 implies that $h: Z \rightarrow W$ is left divisible if and only if $k: W \rightarrow V$ is left almost-split if and only if $f: X \rightarrow Y$ is left almost-split. It now follows that $h: Z \rightarrow W$ is divisible if and only if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an Auslander-Reiten sequence. \square

Proposition 4.1.4. *Let \mathcal{C} be a stable ∞ -category. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an Auslander-Reiten sequence in \mathcal{C} such that f and g are both nonzero, then f and g are irreducible.*

Proof. Apply Corollary 3.5.6 and its dual. \square

The equivalences of Proposition 4.1.2 imply that the study of Auslander-Reiten sequences is equivalent to the study of minimal almost-split morphisms. In particular, our work in Section 3.7 now bears fruit with the following existence result.

Theorem 4.1.5. *Let \mathcal{C} be a compactly generated stable ∞ -category. If $Z \in \mathcal{C}$ is a strongly indecomposable compact object, then there exists an Auslander-Reiten sequence $X \rightarrow Y \rightarrow Z$ in \mathcal{C} .*

Proof. By Corollary 3.7.5, there exists a minimal right almost-split morphism $g: Y \rightarrow Z$ in \mathcal{C} . It now follows from Proposition 4.1.2 that any cofiber sequence $X \rightarrow Y \xrightarrow{g} Z$ is an Auslander-Reiten sequence. \square

Remark 4.1.6. Let \mathcal{C} be a stable ∞ -category and suppose $g: Y \rightarrow Z$ is a minimal right almost-split morphism. Consider the following diagram in which every square is a pushout

and 0 denotes a zero object in \mathcal{C} ,

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & 0 & & \\
\downarrow & & \downarrow g & & \downarrow & & \\
0 & \longrightarrow & Z & \longrightarrow & U & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & \longrightarrow & V & \longrightarrow & W.
\end{array}$$

By Lemma 2.1.36, all rectangles are pushouts, so that there exists an equivalence of cofiber sequences

$$\begin{array}{ccccc}
\Sigma X & \xrightarrow{\Sigma f} & \Sigma Y & \xrightarrow{\Sigma g} & \Sigma Z \\
\downarrow & & \downarrow & & \downarrow \\
U & \longrightarrow & V & \longrightarrow & W.
\end{array}$$

It follows that $U \rightarrow V \rightarrow W$ is again an Auslander-Reiten sequence.

In view of Theorem 4.1.5, we can construct Auslander-Reiten sequences $X \rightarrow Y \rightarrow Z$ in a compactly generated stable ∞ -category \mathcal{C} whenever Z is a strongly indecomposable compact object. However, the construction of this sequence (via Brown representability) is such that the objects X and Y are generally *not* compact. This observation leads to the following natural question: can we construct Auslander-Reiten sequences in $\mathcal{C}_c \subseteq \mathcal{C}$, the full (stable) subcategory of \mathcal{C} spanned by the compact objects? This question has a subtlety: an Auslander-Reiten sequence in \mathcal{C}_c may *not* be an Auslander-Reiten sequence in \mathcal{C} because the almost-split condition depends on the ambient category under consideration. Nevertheless, we can establish a precise relationship between Auslander-Reiten sequences in \mathcal{C}_c and in \mathcal{C} using the notion of *purity*, first introduced in the setting of triangulated categories by Krause [37].

We begin by proving a general uniqueness statement for Auslander-Reiten sequences in a stable ∞ -category. Observe that the uniqueness statement below can be read as saying

that Auslander-Reiten sequences are *homotopy invariants* of their end-terms.

Proposition 4.1.7. *Let \mathcal{C} be a stable ∞ -category. Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z'$ are both Auslander-Reiten sequences in \mathcal{C} . Then the following are equivalent:*

- (1) *There is an equivalence $X \simeq X'$.*
- (2) *There is an equivalence $Z \simeq Z'$.*
- (3) *The cofiber sequences are equivalent.*

Proof. Equivalences in functor categories are computed pointwise, so (3) \Rightarrow [(1) \wedge (2)].

We will show that (1) \Rightarrow (3). The argument for (2) \Rightarrow (3) is then completely analogous.

Assuming (1), suppose $\alpha': X \rightarrow X'$ and $\alpha: X' \rightarrow X$ exhibit the equivalence $X \simeq X'$, that is, $\alpha\alpha' \simeq 1_X$ and $\alpha'\alpha \simeq 1_{X'}$. Since f and f' are both minimal left almost-split, they are in particular not sections. Consequently, using the left almost-split property, there exist maps $\beta: Y \rightarrow Y'$ and $\beta': Y' \rightarrow Y$ such that $f\alpha \simeq \beta'f'$ and $f'\alpha' \simeq \beta f$. These equivalences together give $f \simeq (\beta'\beta)f$ and $f' \simeq (\beta\beta')f'$. Since f and f' are both left minimal, we have that both $(\beta'\beta): Y \rightarrow Y$ and $(\beta\beta'): Y' \rightarrow Y'$ are equivalences. These equivalences together imply that β is an equivalence. Now, the cofiber functor induces a map $\gamma: Z \rightarrow Z'$, which is an equivalence by Lemma 2.3.6. □

We now introduce the requisite notions of purity (see [37]).

Definition 4.1.8. Let \mathcal{C} be an ∞ -category.

- (1) A morphism $f: X \rightarrow Y$ in \mathcal{C} is a *pure-monomorphism* if the induced map

$$\mathrm{Hom}_{\mathrm{hc}}(C, X) \rightarrow \mathrm{Hom}_{\mathrm{hc}}(C, Y)$$

is a monomorphism for every compact object C of \mathcal{C} .

- (2) An object I of \mathcal{C} is *pure-injective* if every pure-monomorphism $I \rightarrow Y$ is a section.
- (3) A map $j: X \rightarrow I$ is a *pure-injective envelope* if I is pure-injective and for every 2-simplex $\Delta^2 \rightarrow \mathcal{C}$ of the form

$$\begin{array}{ccc} & I & \\ j \nearrow & & \searrow i \\ X & \xrightarrow{k} & Y \end{array}$$

$k: X \rightarrow Y$ is a pure-monomorphism if and only if $i: I \rightarrow Y$ is a pure-monomorphism.

Following the work of Krause in [38], our next result establishes a relationship between the Auslander-Reiten sequences of a compactly generated stable ∞ -category \mathcal{C} and those in the full (stable) subcategory $\mathcal{C}_c \subseteq \mathcal{C}$ spanned by the compact objects of \mathcal{C} .

Theorem 4.1.9. *Let \mathcal{C} be a compactly generated stable ∞ -category, let $\mathcal{C}_c \subseteq \mathcal{C}$ be the full subcategory of \mathcal{C} spanned by the compact objects, and suppose we have the following morphism of sequences in \mathcal{C}*

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z \\ \downarrow \varphi & & \downarrow \psi & & \downarrow 1_Z \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z. \end{array}$$

Assume that X' , Y' , and Z are all compact objects.

- (1) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an Auslander-Reiten sequence in \mathcal{C} and $\varphi: X' \rightarrow X$ is a pure-monomorphism, then $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z$ is an Auslander-Reiten sequence in \mathcal{C}_c .
- (2) If $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z$ is an Auslander-Reiten sequence in \mathcal{C}_c and $\varphi: X' \rightarrow X$ is a pure-injective envelope, then $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an Auslander-Reiten sequence in \mathcal{C} .

Proof. We first prove (1). Since $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an Auslander-Reiten sequence, g is right almost-split and hence Z has a local endomorphism ring. Consequently, to show that $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z$ is an Auslander-Reiten sequence in \mathcal{C}_c , it suffices to show that f' is left almost-split in \mathcal{C}_c , by Proposition 4.1.2. Let $\beta: X' \rightarrow C'$ be any morphism in \mathcal{C}_c which is not a section, and let $\alpha: C \rightarrow X'$ be a fiber of β . We have that α is nonzero by Lemma 3.1.2, and because φ is a pure-monomorphism it follows that $\varphi\alpha$ is again nonzero. If $\gamma: \Omega Z \rightarrow X$ is a fiber of f , then γ factors through $\varphi\alpha$, by the dual of Proposition 3.3.3. The induced map $\gamma': \Omega Z \rightarrow X'$ is such that $\gamma \simeq \varphi\gamma'$. Therefore, using again that φ is a pure-monomorphism, the fact that γ factors through $\varphi\alpha$ now implies that γ' factors through α . Invoking the dual of Proposition 3.3.3 once more, this implies that f' is left almost-split.

We now prove (2). As above, it suffices to show that f is left almost-split. Suppose that $\beta: X \rightarrow X''$ is not a section. Since X is a pure-injective object, the map β cannot be a pure-monomorphism. Consequently, $\beta\varphi$ is again not a pure-monomorphism, as φ is a pure-injective envelope. Hence, there exists a compact object C and a nonzero map $\alpha: C \rightarrow X'$ such that $(\beta\varphi)\alpha \simeq 0$. But $\gamma': \Omega Z \rightarrow X'$ factors through α , by the dual of Proposition 3.3.3, for γ' a fiber of f' . If $\gamma: \Omega Z \rightarrow X$ is a fiber of f , then $\gamma \simeq \varphi\gamma'$ and hence $\beta\gamma \simeq 0$. Thus, because f is a cofiber of γ , we have that β factors through f , as desired. \square

Corollary 4.1.10. *Let \mathcal{C} be a compactly generated stable ∞ -category, let $\mathcal{C}_c \subseteq \mathcal{C}$ be the full subcategory of \mathcal{C} spanned by the compact objects, and suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an Auslander-Reiten sequence in \mathcal{C} with Z compact. Then there exists an Auslander-Reiten sequence $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z$ in \mathcal{C}_c if and only if there exists a pure-monomorphism $\varphi: X' \rightarrow X$ in \mathcal{C} .*

Proof. Suppose that $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z$ is an Auslander-Reiten sequence in \mathcal{C}_c . Let $\sigma: X' \rightarrow X''$ be a pure-injective envelope in \mathcal{C} (which always exists). By Lemma 2.3.8, there exists a morphism of cofiber sequences

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z \\ \downarrow \sigma & & \downarrow & & \downarrow 1_Z \\ X'' & \xrightarrow{f''} & Y'' & \xrightarrow{g''} & Z. \end{array}$$

It now follows from Theorem 4.1.9(2) that $X'' \xrightarrow{f''} Y'' \xrightarrow{g''} Z$ is an Auslander-Reiten sequence in \mathcal{C} . By Proposition 4.1.7, we conclude that $X \simeq X''$.

Conversely, suppose $\varphi: X' \rightarrow X$ is a given (nonzero) pure-monomorphism with X' a compact object. If $\gamma: \Omega Z \rightarrow X$ is a fiber of f , then the dual of Proposition 3.3.3 implies that γ factors through φ because f is left almost-split; that is, there exists $\gamma': \Omega Z \rightarrow X'$ such that $\gamma \simeq \varphi\gamma'$. Extend this morphism to a cofiber sequence $\Omega Z \xrightarrow{\gamma'} X' \xrightarrow{f'} Y' \xrightarrow{g'} Z$. Finally, using that f' is a cofiber of γ' , these data extend to a map of cofiber sequences

$$\begin{array}{ccccccc} \Omega Z & \xrightarrow{\gamma'} & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z \\ \downarrow 1_{\Omega Z} & & \downarrow \varphi & & \downarrow \psi & & \downarrow 1_Z \\ \Omega Z & \xrightarrow{\gamma} & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z. \end{array}$$

Therefore, by Theorem 4.1.9(1), we conclude that $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z$ is an Auslander-Reiten sequence in \mathcal{C}_c . □

4.2 Auslander-Reiten duality

Let \mathcal{C} be a compactly generated stable ∞ -category. In Theorem 3.7.4, we established a general existence result for right determined morphisms in \mathcal{C} . The proof of this theorem

relied on a natural isomorphism

$$\theta: \operatorname{Hom}_{\mathbf{h}\mathcal{C}}(-, T(C, E)) \xrightarrow{\sim} \operatorname{Hom}_R(\operatorname{Hom}_{\mathbf{h}\mathcal{C}}(C, -), E)$$

constructed in Lemma 3.7.3, using Brown representability. Our goal in this section is to study the functoriality of this isomorphism in C and E , as well as the interplay between this construction and the translation functor $[n]: \mathcal{C} \rightarrow \mathcal{C}$ (see Notation 2.3.14). We begin by first making the observation that the natural transformation θ is completely determined by the functor $\operatorname{Hom}_{\mathbf{h}\mathcal{C}}(C, -)$ together with a single additional morphism. This observation has essentially already been made in the proof of Theorem 3.6.5 and in Remark 3.6.9, but we make it explicit here.

Lemma 4.2.1. *Let \mathcal{C} be an additive ∞ -category. Suppose $C \in \mathcal{C}$ and set $R = \operatorname{End}_{\mathbf{h}\mathcal{C}}(C)$. Write $\psi = \operatorname{Hom}_{\mathbf{h}\mathcal{C}}(C, -): \mathbf{h}\mathcal{C} \rightarrow \mathbf{Ab}$ for the functor corepresented by C . Then any natural transformation*

$$\theta: \operatorname{Hom}_{\mathbf{h}\mathcal{C}}(-, T) \rightarrow \operatorname{Hom}_R(\operatorname{Hom}_{\mathbf{h}\mathcal{C}}(C, -), E)$$

is completely determined by ψ and the R -module morphism $\theta_T([\operatorname{id}_T]): \operatorname{Hom}_{\mathbf{h}\mathcal{C}}(C, T) \rightarrow E$.

Moreover, $\theta_T([\operatorname{id}_T]) = \operatorname{ev}_{[\operatorname{id}_C]} \circ \theta_C: \operatorname{Hom}_{\mathbf{h}\mathcal{C}}(C, T) \rightarrow \operatorname{Hom}_R(\operatorname{End}_{\mathbf{h}\mathcal{C}}(C), E) \rightarrow E$.

Proof. Assume θ is a natural transformation as above. Then the morphism $\theta_T([\operatorname{id}_T])$ exists and for any $\alpha: Z \rightarrow T$, naturality gives a commuting diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathbf{h}\mathcal{C}}(T, T) & \xrightarrow{[\alpha]^*} & \operatorname{Hom}_{\mathbf{h}\mathcal{C}}(Z, T) \\ \downarrow \theta_T & & \downarrow \theta_Z \\ \operatorname{Hom}_R(\operatorname{Hom}_{\mathbf{h}\mathcal{C}}(C, T), E) & \xrightarrow{\beta_\alpha} & \operatorname{Hom}_R(\operatorname{Hom}_{\mathbf{h}\mathcal{C}}(C, Z), E), \end{array}$$

where $\beta_\alpha(\mu) = \mu \circ \psi([\alpha])$. In particular, we have $\theta_Z([\alpha]) = \theta_T([\operatorname{id}_T]) \circ \psi([\alpha])$.

Conversely, suppose $\mu_T: \text{Hom}_{\text{h}\mathcal{C}}(C, T) \rightarrow E$ is given. Set $\theta_T([\text{id}_T]) = \mu_T$ and for every morphism $\alpha: Z \rightarrow T$ in \mathcal{C} , define $\theta_Z([\alpha]) = \mu_T \circ \psi([\alpha])$. To see that these definitions determine a natural transformation θ as above, let $f: Z' \rightarrow Z$ be any morphism in \mathcal{C} and observe that

$$(\theta_{Z'} \circ [f]^*)([\alpha]) = \theta_{Z'}([\alpha] \circ [f]) = \mu_T \circ \psi([\alpha] \circ [f]) = (\mu_T \circ \psi([\alpha])) \circ \psi([f]) = (\beta_f \circ \theta_Z)([\alpha])$$

ensures the requisite diagrams commute.

Finally, we note that if $\alpha: C \rightarrow T$, then $\theta_C([\alpha])([\text{id}_C]) = \theta_T([\text{id}_T])(\alpha)$. \square

Remark 4.2.2. Let \mathcal{C} be an additive ∞ -category. Suppose R is a ring equipped with a (unital) ring homomorphism $\rho: R \rightarrow \text{End}_{\text{h}\mathcal{C}}(C)$ for some $C \in \mathcal{C}$, and let $\psi = \text{Hom}_{\text{h}\mathcal{C}}(C, -)$ denote the functor corepresented by C . For every object $Z \in \mathcal{C}$, write ψ_Z for the following morphism, induced by the functor ψ ,

$$\psi_Z: \text{Hom}_{\text{h}\mathcal{C}}(Z, T) \rightarrow \text{Hom}_R(\text{Hom}_{\text{h}\mathcal{C}}(C, Z), \text{Hom}_{\text{h}\mathcal{C}}(C, T)).$$

For any natural transformation

$$\theta: \text{Hom}_{\text{h}\mathcal{C}}(-, T) \rightarrow \text{Hom}_R(\text{Hom}_{\text{h}\mathcal{C}}(C, -), E),$$

let $\theta_T([\text{id}_T])_Z$ denote the morphism

$$\theta_T([\text{id}_T])_Z: \text{Hom}_R(\text{Hom}_{\text{h}\mathcal{C}}(C, Z), \text{Hom}_{\text{h}\mathcal{C}}(C, T)) \rightarrow \text{Hom}_R(\text{Hom}_{\text{h}\mathcal{C}}(C, Z), E)$$

induced by composition with $\theta_T([\text{id}_T]): \text{Hom}_{\text{h}\mathcal{C}}(C, T) \rightarrow E$.

By Lemma 4.2.1, we have $\theta_Z = \theta_T([\text{id}_T])_Z \circ \psi_Z$. If any two of these morphisms are isomorphisms, then so is the third. In the case that θ is a natural isomorphism, every θ_Z is

an isomorphism. Consequently, every ψ_Z is a section and every $\theta_T([\text{id}_T])_Z$ is a retraction. Moreover, $\theta_T([\text{id}_T]) = \text{ev}_{[\text{id}_C]} \circ \theta_C$ is an isomorphism if and only if $\text{ev}_{[\text{id}_C]}$ is an isomorphism, and in this case every $\theta_T([\text{id}_T])_Z$ is an isomorphism, which implies every ψ_Z is an isomorphism. However, it need not be the case that $\theta_T([\text{id}_T])$ is an isomorphism. On the other hand, if $E = \text{Hom}_{\text{hc}}(C, T)$, then $\theta = \text{Hom}_{\text{hc}}(C, -)$ which is generally not an isomorphism without additional finiteness assumptions.

Proposition 4.2.3. *Let \mathcal{C} be an additive ∞ -category. Suppose there exists an object C of \mathcal{C} equipped with a unital ring homomorphism $\rho: R \rightarrow \text{End}_{\text{hc}}(C)$ together with a natural isomorphism*

$$\theta: \text{Hom}_{\text{hc}}(-, TE) \xrightarrow{\sim} \text{Hom}_R(\text{Hom}_{\text{hc}}(C, -), E)$$

for every right R -module $E \in \text{RMod}_R$. Then there exists a functor $T(C, -): \text{RMod}_R \rightarrow \text{hc}$, unique up to isomorphism, verifying the isomorphisms, natural in Z and E ,

$$\text{Hom}_{\text{hc}}(Z, T(C, E)) \cong \text{Hom}_R(\text{Hom}_{\text{hc}}(C, Z), E).$$

Proof. For each $E \in \text{RMod}_R$, define $T(C, E) = TE$. We must show that this definition is functorial. Let $e: E \rightarrow E'$ be any morphism of R -modules. Writing e_Z for $\text{Hom}_R(\text{Hom}_{\text{hc}}(C, Z), e)$, we have a commuting diagram

$$\begin{array}{ccc} \text{Hom}_{\text{hc}}(Z, TE) & \xrightarrow{\tilde{e}_Z} & \text{Hom}_{\text{hc}}(Z, TE') \\ \downarrow \theta_Z & & \downarrow \theta'_Z \\ \text{Hom}_R(\text{Hom}_{\text{hc}}(C, Z), E) & \xrightarrow{e_Z} & \text{Hom}_R(\text{Hom}_{\text{hc}}(C, Z), E') \end{array}$$

where $\tilde{e}_Z = (\theta'_Z)^{-1} \circ e_Z \circ \theta_Z$. Setting $Z = TE$ and evaluating at the identity produces a morphism $[Te] = \tilde{e}_{TE}([\text{id}_{TE}]): TE \rightarrow TE'$. We claim that $\tilde{e}_Z = \text{Hom}_{\text{hc}}(Z, [Te])$; that

is, for any $\alpha: Z \rightarrow TE$, we must show $\tilde{e}_Z(\alpha) = [Te] \circ \alpha$. Using Lemma 4.2.1 and writing $\psi = \text{Hom}_{\text{h}\mathcal{C}}(C, -)$, we calculate

$$\tilde{e}_Z(\alpha) = (\theta'_Z)^{-1}(e \circ \theta_Z(\alpha)) = (\theta'_Z)^{-1}(e \circ \theta_{TE}([\text{id}_{TE}]) \circ \psi(\alpha))$$

and $[Te] = \tilde{e}_{TE}([\text{id}_{TE}]) = (\theta'_{TE})^{-1}(e \circ \theta_{TE}([\text{id}_{TE}]))$ implies

$$\begin{aligned} \tilde{e}_Z(\alpha) &= (\theta'_Z)^{-1}(\theta'_{TE}([Te]) \circ \psi(\alpha)) \\ &= (\theta'_Z)^{-1}(\theta'_{TE'}([\text{id}_{TE'}]) \circ \psi([Te]) \circ \psi(\alpha)) \\ &= (\theta'_Z)^{-1}(\theta'_Z([Te] \circ \alpha)) \\ &= [Te] \circ \alpha. \end{aligned}$$

Hence, the natural isomorphisms $\theta(C, E)$ vary functorially in E and thereby determine a functor $T(C, -): \text{RMod}_R \rightarrow \text{h}\mathcal{C}$ satisfying the stated properties. \square

Remark 4.2.4. Note that the ring homomorphism $\rho: R \rightarrow \text{End}_{\text{h}\mathcal{C}}(C)$ of Proposition 4.2.3 was inconsequential in the proof.

Our next goal is fix a right R -module E and study the functorality in C (whenever this makes sense) of the natural isomorphism

$$\theta(C, E): \text{Hom}_{\text{h}\mathcal{C}}(-, T(C, E)) \xrightarrow{\sim} \text{Hom}_R(\text{Hom}_{\text{h}\mathcal{C}}(C, -), E).$$

Let $f: C \rightarrow C'$ be any morphism. To establish a morphism between $\theta(C, E)$ and $\theta(C', E)$, we must have that f induces an R -module homomorphism $\text{Hom}_{\text{h}\mathcal{C}}(C', Z) \rightarrow \text{Hom}_{\text{h}\mathcal{C}}(C, Z)$ for every Z . Consequently, the ring R cannot depend on any fixed object C and the R -module structure maps $\rho_C: R \rightarrow \text{End}_{\text{h}\mathcal{C}}(C)$ must have the property that $f \circ \rho_C(r) = \rho_{C'}(r) \circ f$ for all $r \in R$. It is convenient to repackage this information in a more sophisticated form.

Definition 4.2.5. Let \mathcal{C} be an additive ∞ -category. The *center* of \mathcal{C} , denoted $\mathfrak{Z}(\mathcal{C})$, is the ring of natural transformations of the identity functor on the homotopy category $\mathrm{h}\mathcal{C}$, that is, $\mathfrak{Z}(\mathcal{C}) = \mathrm{Nat}(\mathrm{Id}_{\mathrm{h}\mathcal{C}}, \mathrm{Id}_{\mathrm{h}\mathcal{C}})$.

The functoriality we are seeking in the above discussion is now equivalent to giving a unital ring homomorphism $R \xrightarrow{\rho} \mathfrak{Z}(\mathcal{B})$, where $\mathcal{B} \subseteq \mathcal{C}$ is the full additive subcategory spanned by those objects C for which a natural isomorphism $\theta(C, E)$ exists. Observe that for any $C \in \mathcal{B}$, the evaluation map $\mathrm{ev}_C: \mathfrak{Z}(\mathcal{B}) \rightarrow \mathrm{End}_{\mathrm{h}\mathcal{C}}(C)$ is a ring homomorphism with image in the center of $\mathrm{End}_{\mathrm{h}\mathcal{C}}(C)$. That is, the requisite R -module structure above is given by the composition $\rho_C = \mathrm{ev}_C \circ \rho$. To summarize, Proposition 4.2.3 shows that the natural transformations $\theta(C, E)$ are automatically functorial in E , but functoriality in C requires the *additional structure* of a ring homomorphism $R \rightarrow \mathfrak{Z}(\mathcal{B})$ as a prerequisite. It turns out that this structure is also sufficient.

Proposition 4.2.6. *Let \mathcal{C} be an additive ∞ -category. Suppose $\mathcal{B} \subseteq \mathcal{C}$ is a full additive subcategory equipped with a unital ring homomorphism $\rho: R \rightarrow \mathfrak{Z}(\mathcal{B})$ such that for some fixed right R -module E , there exists a natural isomorphism*

$$\theta: \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(-, TC) \xrightarrow{\sim} \mathrm{Hom}_R(\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, -), E)$$

associated to every object $C \in \mathcal{B}$. Then there exists a functor $T(-, E): \mathrm{h}\mathcal{B} \rightarrow \mathrm{h}\mathcal{C}$, unique up to isomorphism, verifying the isomorphisms, natural in Z and C ,

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Z, T(C, E)) \cong \mathrm{Hom}_R(\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, Z), E).$$

Proof. For each $C \in \mathcal{B}$, define $T(C, E) = TC$. We must show that this definition is functorial. Let $f: C \rightarrow C'$ be any morphism in \mathcal{B} . Writing f_Z for $\mathrm{Hom}_R(\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(f, Z), E)$,

we have a commuting diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Z, TC) & \xrightarrow{\tilde{f}_Z} & \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Z, TC') \\
\downarrow \theta_Z & & \downarrow \theta'_Z \\
\mathrm{Hom}_R(\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, Z), E) & \xrightarrow{f_Z} & \mathrm{Hom}_R(\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C', Z), E)
\end{array}$$

where $\tilde{f}_Z = (\theta'_Z)^{-1} \circ f_Z \circ \theta_Z$. Setting $Z = TC$ and evaluating at the identity produces a morphism $[Tf] = \tilde{f}_{TC}([\mathrm{id}_{TC}]): TC \rightarrow TC'$. We claim that $\tilde{f}_Z = \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Z, [Tf])$; that is, for any $\alpha: Z \rightarrow TC$, we must show $\tilde{f}_Z(\alpha) = [Tf] \circ \alpha$. Using Lemma 4.2.1 and the notation $\psi = \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, -)$ and $f^Z = \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(f, Z)$, we calculate

$$\tilde{f}_Z(\alpha) = (\theta'_Z)^{-1}(\theta_Z(\alpha) \circ f^Z) = (\theta'_Z)^{-1}(\theta_{TC}([\mathrm{id}_{TC}]) \circ \psi(\alpha) \circ f^Z).$$

Noting that $\psi(\alpha) \circ f^Z = f^{TC} \circ \psi'(\alpha)$ and $[Tf] = \tilde{f}_{TC}([\mathrm{id}_{TC}]) = (\theta'_{TC})^{-1}(\theta_{TC}([\mathrm{id}_{TC}]) \circ f^{TC})$, the above calculation implies

$$\begin{aligned}
\tilde{e}_Z(\alpha) &= (\theta'_Z)^{-1}(\theta'_{TC}([Tf]) \circ \psi'(\alpha)) \\
&= (\theta'_Z)^{-1}(\theta'_{TC'}([\mathrm{id}_{TC'}]) \circ \psi'([Tf]) \circ \psi'(\alpha)) \\
&= (\theta'_Z)^{-1}(\theta'_Z([Tf] \circ \alpha)) \\
&= [Tf] \circ \alpha.
\end{aligned}$$

Hence, the natural isomorphisms $\theta(C, E)$ vary functorially in C and thereby determine a functor $T(-, E): \mathrm{h}\mathcal{B} \rightarrow \mathrm{h}\mathcal{C}$ satisfying the stated properties. \square

Remark 4.2.7. The proof of Proposition 4.2.6 is nearly identical to that of Proposition 4.2.3, except that we must exercise the R -module structure precisely when observing that the R -module homomorphisms $\psi(\alpha) = \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, \alpha)$ and $f^Z = \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(f, Z)$ commute.

Remark 4.2.8. In the situation of Proposition 4.2.6, the functor $T(-, E): \mathbf{hB} \rightarrow \mathbf{hC}$ commutes with any auto-equivalence $F: \mathbf{hC} \rightarrow \mathbf{hC}$ which restricts to a functor $\mathbf{hB} \rightarrow \mathbf{hB}$. To see this, it suffices to show that $\mathrm{Hom}_{\mathbf{hC}}(Z, T(FC)) \cong \mathrm{Hom}_{\mathbf{hC}}(Z, F(TC))$ for all $Z \in \mathbf{C}$. Since F is an auto-equivalence, for any $Z \in \mathbf{C}$ we can find W such that $Z \simeq FW$. Now, using that F is fully faithful, we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{hC}}(Z, T(FC)) &\cong \mathrm{Hom}_{\mathbf{hC}}(FW, T(FC)) \cong \mathrm{Hom}_R(\mathrm{Hom}_{\mathbf{hC}}(FC, FW), E) \\ &\cong \mathrm{Hom}_R(\mathrm{Hom}_{\mathbf{hC}}(C, W), E) \cong \mathrm{Hom}_{\mathbf{hC}}(W, TC) \\ &\cong \mathrm{Hom}_{\mathbf{hC}}(FW, F(TC)) \cong \mathrm{Hom}_{\mathbf{hC}}(Z, F(TC)). \end{aligned}$$

This yields the desired natural isomorphism $TF \cong FT$.

Remark 4.2.9. In the situation of Proposition 4.2.6, the functor $T(-, E): \mathbf{hB} \rightarrow \mathbf{hC}$ may not be full nor faithful. To see this, observe that (suppressing E in the notation) the induced map

$$\mathrm{Hom}_{\mathbf{hB}}(C, D) \rightarrow \mathrm{Hom}_{\mathbf{hC}}(TC, TD)$$

coincides with the composition

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{hB}}(C, D) & \xrightarrow{\quad} & \mathrm{Hom}_R(\mathrm{Hom}_R(\mathrm{Hom}_{\mathbf{hC}}(C, D), E), E) \\ & \searrow & \downarrow \\ & & \mathrm{Hom}_R(\mathrm{Hom}_{\mathbf{hC}}(D, TC), E) \\ & & \downarrow \\ & \searrow & \mathrm{Hom}_{\mathbf{hC}}(TC, TD) \end{array}$$

where the horizontal arrow is the canonical R -module (evaluation) morphism and the vertical arrows are the structural isomorphisms arising in the construction of $T(-, E)$. It follows

that $T(-, E)$ is fully faithful if and only if the canonical R -module morphism

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, D) \rightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, D), E), E)$$

is an isomorphism.

Remark 4.2.10. Let \mathcal{C} be an additive ∞ -category. Suppose $\mathcal{B} \subseteq \mathcal{C}$ is a full additive subcategory equipped with a unital ring homomorphism $\rho: R \rightarrow \mathfrak{Z}(\mathcal{B})$ such that for *every* right R -module E , there exists a natural isomorphism

$$\theta(C, E): \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(-, T(C, E)) \xrightarrow{\sim} \mathrm{Hom}_R(\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, -), E)$$

associated to every object $C \in \mathcal{B}$. Then by Propositions 4.2.3 and 4.2.6, for each $C \in \mathcal{B}$ and each $E \in \mathrm{RMod}_R$, we have functors $T(C, -): \mathrm{RMod}_R \rightarrow \mathrm{h}\mathcal{C}$ and $T(-, E): \mathrm{h}\mathcal{B} \rightarrow \mathrm{h}\mathcal{C}$, unique up to isomorphism. It is not difficult to check that for any $f: C \rightarrow C'$ and any $e: E \rightarrow E'$, the following diagram commutes

$$\begin{array}{ccc} T(C, E) & \xrightarrow{T(f, E)} & T(C', E) \\ T(C, e) \downarrow & & \downarrow T(C', e) \\ T(C, E') & \xrightarrow{T(f, E')} & T(C', E'). \end{array}$$

Hence, in this situation, we have a functor $T: \mathrm{h}\mathcal{B} \times \mathrm{RMod}_R \rightarrow \mathrm{h}\mathcal{C}$.

Definition 4.2.11. Let \mathcal{C} be an additive ∞ -category. Suppose $\mathcal{B} \subseteq \mathcal{C}$ is a full additive subcategory equipped with a unital ring homomorphism $\rho: R \rightarrow \mathfrak{Z}(\mathcal{B})$. A *right Auslander functor* on \mathcal{C} is any functor

$$T^\rho: \mathrm{h}\mathcal{B} \times \mathrm{RMod}_R \rightarrow \mathrm{h}\mathcal{C}$$

verifying the isomorphisms, natural in every variable,

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Z, T(C, E)) \cong \mathrm{Hom}_R(\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, Z), E).$$

We often suppress ρ in the notation, writing T in place of T^ρ and leaving ρ implicit.

Theorem 4.2.12. *Let \mathcal{C} be a compactly generated stable ∞ -category. Then \mathcal{C} admits a canonical right Auslander functor.*

Proof. Let $\mathcal{C}_c \subseteq \mathcal{C}$ denote the full subcategory spanned by the compact objects of \mathcal{C} . Set $R = \mathfrak{Z}(\mathcal{C}_c)$, so that $\rho: R \rightarrow \mathfrak{Z}(\mathcal{C}_c)$ is just the identity. For any pair $(C, E) \in \mathbf{h}\mathcal{C}_c \times \mathbf{RMod}_R$, Lemma 3.7.3 yields a natural isomorphism

$$\theta(C, E): \mathrm{Hom}_{\mathbf{h}\mathcal{C}}(-, T(C, E)) \xrightarrow{\sim} \mathrm{Hom}_R(\mathrm{Hom}_{\mathbf{h}\mathcal{C}}(C, -), E).$$

Combining Propositions 4.2.3 and 4.2.6 together with Remark 4.2.10, there exists a functor $T: \mathbf{h}\mathcal{C}_c \times \mathbf{RMod}_R \rightarrow \mathbf{h}\mathcal{C}$, unique up to isomorphism, verifying the required natural isomorphisms. \square

Corollary 4.2.13. *Let \mathcal{C} be a compactly generated stable ∞ -category. Let $\mathcal{C}_c \subseteq \mathcal{C}$ denote the full subcategory spanned by the compact objects and suppose $\rho: R \rightarrow \mathfrak{Z}(\mathcal{C}_c)$ is a unital ring homomorphism. Then for every integer $n \in \mathbb{Z}$, there exist an isomorphism, natural in every variable,*

$$\mathrm{Ext}_{\mathcal{C}}^n(Z, T(C, E)) \cong \mathrm{Hom}_R(\mathrm{Ext}_{\mathcal{C}}^{-n}(C, Z), E).$$

Proof. By Theorem 4.2.12, there exists a right Auslander functor $T: \mathbf{h}\mathcal{C}_c \times \mathbf{RMod}_R \rightarrow \mathbf{h}\mathcal{C}$ verifying the isomorphisms, natural in every variable,

$$\mathrm{Hom}_{\mathbf{h}\mathcal{C}}(Z, T(C, E)) \cong \mathrm{Hom}_R(\mathrm{Hom}_{\mathbf{h}\mathcal{C}}(C, Z), E).$$

As a stable ∞ -category, the functor $[n]: \mathbf{h}\mathcal{C} \rightarrow \mathbf{h}\mathcal{C}$ induced by the loop and suspension

functors on \mathcal{C} is an auto-equivalence. Thus, for any $n \in \mathbb{Z}$, we have canonical isomorphisms

$$\begin{aligned} \mathrm{Ext}_{\mathcal{C}}^n(Z, T(C, E)) &\cong \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Z[-n], T(C, E)) \\ &\cong \mathrm{Hom}_R(\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, Z[-n]), E) \\ &\cong \mathrm{Hom}_R(\mathrm{Ext}_{\mathcal{C}}^{-n}(C, Z), E), \end{aligned}$$

which completes the proof. \square

Remark 4.2.14. Let \mathcal{C} be a compactly generated stable ∞ -category. In the situation of Corollary 4.2.13, we will refer to the isomorphisms

$$\mathrm{Ext}_{\mathcal{C}}^n(Z, T(C, E)) \cong \mathrm{Hom}_R(\mathrm{Ext}_{\mathcal{C}}^{-n}(C, Z), E)$$

collectively as the *Auslander-Reiten duality* on \mathcal{C} .

Remark 4.2.15. Let \mathcal{C} be a compactly generated stable ∞ -category. By Theorem 4.2.12, we have a right Auslander functor $T: \mathrm{h}\mathcal{C}_c \times \mathrm{RMod}_R \rightarrow \mathrm{h}\mathcal{C}$. Let $\mathrm{h}\mathcal{C}_0 \subseteq \mathrm{h}\mathcal{C}$ denote the essential image of T . Then any morphism $g: Y \rightarrow Z$ in \mathcal{C} with cofiber $W \simeq \mathrm{cofib}(g)$ in $\mathrm{h}\mathcal{C}_0$ is right C -determined for some compact object C of $\mathrm{h}\mathcal{C}_c$. To see this, observe that the defining isomorphisms of T together with Theorem 3.6.5 imply that $0 \rightarrow T(C, E)$ is right C -determined for any $E \in \mathrm{RMod}_R$. If $Y \xrightarrow{g} Z \rightarrow T(C, E)$ is a cofiber sequence, then it is also a fiber sequence because \mathcal{C} is stable. Using Lemma 3.6.13, the pullback of $0 \rightarrow T(C, E)$ along any morphism is again right C -determined. It follows that g is right C -determined.

Corollary 4.2.16. *Let \mathcal{C} be a compactly generated stable ∞ -category. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ denote the full subcategory spanned by the strongly indecomposable compact objects of \mathcal{C} . Then there exists a functor $\tau: \mathrm{h}\mathcal{C}_0 \rightarrow \mathrm{h}\mathcal{C}$ satisfying the following property: if $X \rightarrow Y \rightarrow Z$ is any Auslander-Reiten sequence in \mathcal{C} with $Z \in \mathcal{C}_0$, then $X \simeq \tau Z$.*

Proof. By Theorem 4.2.12, there exists a right Auslander functor $T: \mathbf{h}\mathcal{C}_c \times \mathbf{RMod}_R \rightarrow \mathbf{h}\mathcal{C}$, where $R = \mathfrak{Z}(\mathcal{C}_c)$. Let $E \in \mathbf{RMod}_R$ be an injective envelope of R , regarded as a right R -module (note that because \mathcal{C} is presentable, R is a set without changing universe). Then because Z is compact, we have an isomorphism

$$\mathrm{Hom}_{\mathbf{h}\mathcal{C}}(Z, T(Z, E)) \cong \mathrm{Hom}_R(\mathrm{Hom}_{\mathbf{h}\mathcal{C}}(Z, Z), E).$$

By the proof of Theorem 3.7.4, the morphism $\gamma: Z \rightarrow T(Z, E)$ corresponding under the above isomorphism to the morphism $\mathrm{End}_{\mathbf{h}\mathcal{C}}(Z) \rightarrow \mathrm{End}_{\mathbf{h}\mathcal{C}}(Z)/\mathrm{rad}\,\mathrm{End}_{\mathbf{h}\mathcal{C}}(Z) \rightarrow E$ is left minimal and any fiber of γ is right Z -determined. That is, suppose that in the following diagram,

$$\begin{array}{ccccc} \Omega T(Z, E) & \xrightarrow{f'} & Y' & \longrightarrow & 0 \\ \downarrow & & \downarrow g' & & \downarrow \\ 0 & \longrightarrow & Z & \xrightarrow{\gamma} & T(Z, E) \end{array}$$

every square is a pullback in \mathcal{C} . Then g' is right minimal by Lemma 3.4.9 and right almost-split by Proposition 3.6.4, and thus $\Omega T(Z, E) \xrightarrow{f'} Y' \xrightarrow{g'} Z$ is an Auslander-Reiten sequence by Proposition 4.1.2. It now follows from Proposition 4.1.7 that $X \simeq \Omega T(Z, E)$. Hence, defining $\tau: \mathbf{h}\mathcal{C}_0 \rightarrow \mathbf{h}\mathcal{C}$ as the composition

$$\tau: \mathbf{h}\mathcal{C}_0 \cong \mathbf{h}\mathcal{C}_0 \times \{E\} \longrightarrow \mathbf{h}\mathcal{C}_c \times \mathbf{RMod}_R \xrightarrow{T} \mathbf{h}\mathcal{C} \xrightarrow{[-1]} \mathbf{h}\mathcal{C}$$

completes the proof. □

Remark 4.2.17. Let \mathcal{C} be a compactly generated stable ∞ -category. In the situation of Corollary 4.2.16, we will refer to the functor τ as an *Auslander-Reiten translation* functor. Moreover, we say that $X \simeq \tau Z$ is the *Auslander-Reiten translate* of Z .

Remark 4.2.18. Let \mathcal{C} be a compactly generated stable ∞ -category. The Auslander-Reiten duality of Remark 4.2.14 restricted to the full subcategory of strongly indecomposable compact objects reduces to the following *Auslander-Reiten formula*, with $(-)^* = \mathrm{Hom}_R(-, E)$,

$$\mathrm{Ext}_{\mathcal{C}}^{n+1}(W, \tau Z) \cong \mathrm{Ext}_{\mathcal{C}}^{-n}(Z, W)^*.$$

Chapter 5

Derived ∞ -categories

To any abelian category \mathcal{A} with enough projectives or enough injectives, one can associate a stable ∞ -category $\mathcal{D}(\mathcal{A})$ whose homotopy category is canonically equivalent to the classical derived category of \mathcal{A} (see [46, Section 1.3]). Generalizing an earlier result of Krause [39], Lurie showed in [46, Theorem 1.3.6.7] that when \mathcal{A} is a locally Noetherian abelian category, the stable ∞ -category $\mathcal{D}(\mathcal{A})$ is compactly generated.

In this chapter, we first review the construction of these algebraic stable ∞ -categories associated to ordinary abelian categories. To this end, we study the differential graded category of chain complexes with values in an additive category and recall Lurie’s construction of a differential graded nerve functor. Focusing on an important example, we then show how to explicitly construct an Auslander functor

$$T: \mathcal{K}(\mathcal{A}) \rightarrow \mathcal{K}(\mathcal{A})$$

on a compactly generated stable ∞ -category $\mathcal{K}(\mathcal{A})$ arising from the locally Noetherian abelian category \mathcal{A} of modules over a Noetherian k -algebra, where k is a complete local

Noetherian ring. Our construction closely follows Krause and Le [40], which generalizes a similar construction going back to Happel [24, 26]. The functor T has the property that it recovers the translation functor of Krause and Le after passing to the homotopy category. In contrast with the Auslander-Reiten translation functor of Corollary 4.2.16, the construction in this chapter occurs entirely at the ∞ -categorical level (see Theorem 5.2.3).

5.1 Differential graded categories

Definition 5.1.1. Let \mathcal{A} be an additive category. A *chain complex* $A = (A_*, d_*)$ with values in \mathcal{A} is a composable sequence of morphisms in \mathcal{A}

$$\cdots \longrightarrow A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A_{-1} \longrightarrow \cdots$$

such that $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$. The maps d_n are called the *differentials* of the chain complex A . We denote by $\text{Ch}(\mathcal{A})$ the collection of all chain complexes with values in \mathcal{A} .

The collection of chain complexes with values in an additive category can be organized into a differential graded category. Before explaining this construction, we first recall the definition of a differential graded category.

Definition 5.1.2. Let k be a commutative ring. A *differential graded category* \mathcal{C} over k consists of the following data:

- A collection of objects $\text{Ob } \mathcal{C}$.
- For every pair of objects X and Y of \mathcal{C} , a chain complex of k -modules

$$\cdots \rightarrow \text{Map}_{\mathcal{C}}(X, Y)_1 \rightarrow \text{Map}_{\mathcal{C}}(X, Y)_0 \rightarrow \text{Map}_{\mathcal{C}}(X, Y)_{-1} \rightarrow \cdots$$

denoted $\text{Map}_{\mathcal{C}}(X, Y)$.

- For every triple of objects X , Y , and Z , a composition map

$$\text{Map}_{\mathcal{C}}(Y, Z) \otimes_k \text{Map}_{\mathcal{C}}(X, Y) \rightarrow \text{Map}_{\mathcal{C}}(X, Z)$$

which we can identify with a collection of k -bilinear maps

$$\circ: \text{Map}_{\mathcal{C}}(Y, Z)_p \times \text{Map}_{\mathcal{C}}(X, Y)_q \rightarrow \text{Map}_{\mathcal{C}}(X, Z)_{p+q}$$

satisfying the Leibniz rule: $d(g \circ f) = dg \circ f + (-1)^p g \circ df$.

- For every object X , an *identity morphism* $\text{id}_X \in \text{Map}_{\mathcal{C}}(X, X)_0$ satisfying

$$g \circ \text{id}_X = g \quad \text{id}_X \circ f = f$$

for all $g \in \text{Map}_{\mathcal{C}}(X, Y)_p$ and $f \in \text{Map}_{\mathcal{C}}(Y, X)_q$.

The composition law is required to be associative in the following sense: for every triple

$f \in \text{Map}_{\mathcal{C}}(W, X)_p$, $g \in \text{Map}_{\mathcal{C}}(X, Y)_q$, and $h \in \text{Map}_{\mathcal{C}}(Y, Z)_r$, we have

$$(h \circ g) \circ f = h \circ (g \circ f).$$

In the special case where $k = \mathbb{Z}$ is the ring of integers, we refer to a differential graded category over k simply as a *differential graded category* or *dg-category*.

Example 5.1.3. A dg-category with a single object is a differential graded ring.

Example 5.1.4. Let \mathcal{A} be an additive category. We regard $\text{Ch}(\mathcal{A})$ as a differential graded category as follows:

- The objects of $\text{Ch}(\mathcal{A})$ are the chain complexes with values in \mathcal{A} .

- For every pair of chain complexes (A, d^A) and (B, d^B) , we construct a chain complex of abelian groups $\text{Map}_{\text{Ch}(\mathcal{A})}(A, B)_*$ by setting, for every integer p ,

$$\text{Map}_{\text{Ch}(\mathcal{A})}(A, B)_p = \prod_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{A}}(A_n, B_{n+p})$$

with differentials $d_p: \text{Map}_{\text{Ch}(\mathcal{A})}(A, B)_p \rightarrow \text{Map}_{\text{Ch}(\mathcal{A})}(A, B)_{p-1}$ given by the formula

$$(d_p f)_n = d_{n+p}^B \circ f_n - (-1)^p f_{n-1} \circ d_n^A.$$

Observe that for any triple of chain complexes A , B , and C in $\text{Ch}(\mathcal{A})$, composition in \mathcal{A} gives a bilinear map

$$\text{Map}_{\text{Ch}(\mathcal{A})}(B, C)_p \times \text{Map}_{\text{Ch}(\mathcal{A})}(A, B)_q \rightarrow \text{Map}_{\text{Ch}(\mathcal{A})}(A, C)_{p+q}$$

satisfying the Leibniz rule: $d(g \circ f) = dg \circ f + (-1)^p g \circ df$. Consequently, the above construction endows $\text{Ch}(\mathcal{A})$ with the structure of a differential graded category (over \mathbb{Z}). Moreover, if k is a commutative ring and \mathcal{A} is an additive k -category, then $\text{Ch}(\mathcal{A})$ is a differential graded category over k .

Construction 5.1.5 ([46, 1.3.1.6]). Let \mathcal{C} be a differential graded category over a commutative ring k . We associate to \mathcal{C} a simplicial set $N_{\text{dg}}(\mathcal{C})$, called the *differential graded nerve* of \mathcal{C} , as follows: For each $n \geq 0$, we define $N_{\text{dg}}(\mathcal{C})_n$ to be the set of all ordered pairs

$$(\{X_i : i \in [n]\}, \{f_I : I \subseteq [n]\})$$

where:

- For $0 \leq i \leq n$, the X_i are objects of \mathcal{C} .

- For every subset $I = \{i_- < i_1 < \cdots < i_m < i_+\}$ with $m \geq 0$, f_I is an element of the k -module $\text{Map}_{\mathcal{C}}(X_{i_-}, X_{i_+})_m$ satisfying the equation

$$df_I = \sum_{1 \leq j \leq m} (-1)^j \left(f_{I - \{i_j\}} - f_{\{i_j < \cdots < i_m < i_+\}} \circ f_{\{i_- < i_1 < \cdots < i_j\}} \right).$$

If $\alpha: [m] \rightarrow [n]$ is a nondecreasing function, then the induced map $N_{\text{dg}}(\mathcal{C})_n \rightarrow N_{\text{dg}}(\mathcal{C})_m$ is given by

$$(\{X_i : i \in [n]\}, \{f_I : I \subseteq [n]\}) \mapsto (\{X_{\alpha(j)} : j \in [m]\}, \{g_J : J \subseteq [m]\})$$

where

$$g_J = \begin{cases} f_{\alpha(J)} & \text{if } \alpha|_J \text{ is injective} \\ \text{id}_{X_i} & \text{if } J = \{j, j'\} \text{ and } \alpha(j) = \alpha(j') = i \\ 0 & \text{otherwise.} \end{cases}$$

Remark 5.1.6. Let \mathcal{C} be a differential graded category over a commutative ring k . The 0-simplices $N_{\text{dg}}(\mathcal{C})_0$ of the simplicial set $N_{\text{dg}}(\mathcal{C})$ can be identified with the objects of \mathcal{C} . The 1-simplices of $N_{\text{dg}}(\mathcal{C})$ are pairs of objects X and Y of \mathcal{C} together with a degree 0 map $f \in \text{Map}_{\mathcal{C}}(X, Y)_0$ such that $df = 0$. A 2-simplex of $N_{\text{dg}}(\mathcal{C})$ consists of a triple of objects X, Y, Z , a triple of maps $f \in \text{Map}_{\mathcal{C}}(X, Y)_0$, $g \in \text{Map}_{\mathcal{C}}(Y, Z)_0$, $h \in \text{Map}_{\mathcal{C}}(X, Z)_0$ satisfying $df = dg = dh = 0$, together with a degree 1 map $p \in \text{Map}_{\mathcal{C}}(X, Z)_1$ satisfying $dp = (g \circ f) - h$.

Proposition 5.1.7 ([46, 1.3.1.10]). *Let \mathcal{C} be a differential graded category over a commutative ring k . Then the simplicial set $N_{\text{dg}}(\mathcal{C})$ is an ∞ -category.*

Proof. Fix $n \geq 2$ and $0 < j < n$, and let $\phi_0: \Lambda_j^n \rightarrow N_{\text{dg}}(\mathcal{C})$. We must show that ϕ_0 can be extended to an n -simplex $\phi: \Delta^n \rightarrow N_{\text{dg}}(\mathcal{C})$. The image of ϕ_0 consists of the data of

a pair $(\{X_i : 0 \leq i \leq n\}, \{f_I : I \subseteq [n]\})$ where the X_i are objects of \mathcal{C} and the morphisms $f_I \in \text{Map}_{\mathcal{C}}(X_{i_-}, X_{i_+})_m$ are defined for every subset $I = \{i_- < i_1 < \dots < i_m < i_+\} \subseteq [n]$ such that $I \neq [n], [n] - \{j\}$, and satisfy the equation

$$df_I = \sum_{1 \leq k \leq m} (-1)^k (f_{I - \{i_k\}} - f_{\{i_k < \dots < i_m < i_+\}} \circ f_{\{i_- < i_1 < \dots < i_k\}}).$$

To extend these data to an n -simplex, we must produce morphisms $f_{[n] - \{j\}}$ and $f_{[n]}$ such that the above equation still holds. Setting $f_{[n]} = 0$, there is a unique solution for $f_{[n] - \{j\}}$ satisfying our constraints:

$$f_{[n] - \{j\}} = \sum_{0 < k < n} (-1)^{j+k} f_{\{k < \dots < n\}} \circ f_{\{0 < \dots < k\}} - \sum_{\substack{0 < k < n \\ k \neq j}} (-1)^{j+k} f_{I - \{k\}}.$$

□

Definition 5.1.8. Let \mathcal{A} be an additive category. We denote by $\mathcal{K}(\mathcal{A})$ the ∞ -category $\text{N}_{\text{dg}}(\text{Ch}(\mathcal{A}))$ and refer to $\mathcal{K}(\mathcal{A})$ as the ∞ -category of chain complexes with values in \mathcal{A} .

The importance of the differential graded nerve to our present discussion is the following result.

Proposition 5.1.9 ([46, 1.3.2.10]). *Let \mathcal{A} be an additive category. Then the ∞ -category $\mathcal{K}(\mathcal{A})$ is stable.*

Remark 5.1.10. Let \mathcal{A} be an additive category. The cofiber of a morphism $f : A \rightarrow B$ in $\mathcal{K}(\mathcal{A})$ can be identified with the *mapping cone* $M(f)$ of f in $\text{Ch}(\mathcal{A})$, where in each degree $M(f)_n = A_{n-1} \oplus B_n$.

As justification for our notation, we next observe that the homotopy category $\text{h}\mathcal{K}(\mathcal{A})$ can be canonically identified with $\text{K}(\mathcal{A})$, the classical category of chain complexes modulo

homotopy (see Remark 5.1.13 below). To see this, we first observe that two other categories can be extracted from the data of a differential graded category.

Remark 5.1.11. Let \mathcal{C} be a differential graded category over a commutative ring k .

There is a category $\mathbf{u}\mathcal{C}$ associated to \mathcal{C} with the same objects and with morphisms given by, for each pair of objects X and Y ,

$$\mathrm{Hom}_{\mathbf{u}\mathcal{C}}(X, Y) = Z_0(\mathrm{Map}_{\mathcal{C}}(X, Y)_*) = \{f \in \mathrm{Map}_{\mathcal{C}}(X, Y)_0 : df = 0\}.$$

We refer to $\mathbf{u}\mathcal{C}$ as the category *underlying* the differential graded category \mathcal{C} .

There is another category associated to \mathcal{C} , called the *homotopy category* of \mathcal{C} and denoted $\mathbf{h}\mathcal{C}$, which consists of the following data:

- The objects of $\mathbf{h}\mathcal{C}$ are the objects of \mathcal{C} .
- For every pair of objects X and Y in $\mathbf{h}\mathcal{C}$, we define

$$\mathrm{Hom}_{\mathbf{h}\mathcal{C}}(X, Y) = H_0(\mathrm{Map}_{\mathcal{C}}(X, Y)_*) = \mathrm{coker}(\mathrm{Map}_{\mathcal{C}}(X, Y)_1 \xrightarrow{d} \mathrm{Hom}_{\mathbf{u}\mathcal{C}}(X, Y)).$$

More explicitly, a morphism $[f] \in \mathrm{Hom}_{\mathbf{h}\mathcal{C}}(X, Y)$ is an equivalence class of morphisms $f \in \mathrm{Hom}_{\mathbf{u}\mathcal{C}}(X, Y)$, where two maps $f, g \in \mathrm{Hom}_{\mathbf{u}\mathcal{C}}(X, Y)$ are equivalent if there exists a map $h \in \mathrm{Map}_{\mathcal{C}}(X, Y)_1$ such that $dh = f - g$.

- Composition in $\mathbf{h}\mathcal{C}$ is determined by the formula $[g] \circ [f] = [g \circ f]$.

Proposition 5.1.12 ([46, 1.3.1.11]). *Let \mathcal{C} be a differential graded category over a commutative ring k . The homotopy category $\mathbf{h}\mathcal{C}$ of \mathcal{C} is canonically isomorphic to the homotopy category $\mathbf{hN}_{\mathrm{dg}}(\mathcal{C})$ of the ∞ -category $\mathbf{N}_{\mathrm{dg}}(\mathcal{C})$.*

Proof. Using the identification of $N_{\text{dg}}(\mathcal{C})_0$ with the objects of \mathcal{C} together with description of the morphisms $N_{\text{dg}}(\mathcal{C})_1$ supplied by Remark 5.1.6, we have a functor $u\mathcal{C} \rightarrow hN_{\text{dg}}(\mathcal{C})$ which is bijective on objects and surjective on morphisms. To complete the proof, observe that the homotopy relation of Definition 2.1.23 applied to $N_{\text{dg}}(\mathcal{C})$ (via the description of $N_{\text{dg}}(\mathcal{C})_2$ in Remark 5.1.6) agrees with the (homology) relation defining $h\mathcal{C}$ in Remark 5.1.11. \square

We can now describe the homotopy category $h\mathcal{K}(\mathcal{A})$ explicitly.

Remark 5.1.13. Let \mathcal{A} be an additive category. The differential graded category $\text{Ch}(\mathcal{A})$ (see Example 5.1.4) has two other categories canonically associated to it (see Remark 5.1.11).

The category *underlying* the differential graded category $\text{Ch}(\mathcal{A})$ has morphisms given by

$$\text{Hom}_{\text{Ch}(\mathcal{A})}(A_*, B_*) = \left\{ f \in \text{Map}_{\text{Ch}(\mathcal{A})}(A_*, B_*)_0 : df = 0 \right\}.$$

More explicitly, $f \in \text{Hom}_{\text{Ch}(\mathcal{A})}(A_*, B_*)$ consists of a sequence of maps $\{f_n : A_n \rightarrow B_n : n \in \mathbb{Z}\}$ such that $(d_0 f)_n = d_n^B \circ f_n - f_{n-1} \circ d_n^A = 0$. These maps are precisely the morphisms of chain complexes in the additive category (Definition 5.1.1). Succinctly stated, the category underlying the differential graded category $\text{Ch}(\mathcal{A})$ is the additive category of chain complexes usually also denoted $\text{Ch}(\mathcal{A})$.

The homotopy category $h\text{Ch}(\mathcal{A})$ has morphisms given by

$$\text{Hom}_{h\text{Ch}(\mathcal{A})}(A_*, B_*) = H_0(\text{Map}_{\text{Ch}(\mathcal{A})}(A_*, B_*)_*) .$$

Here $[f] \in \text{Hom}_{h\text{Ch}(\mathcal{A})}(A_*, B_*)$ is an equivalence class of morphisms in $\text{Hom}_{\text{Ch}(\mathcal{A})}(A_*, B_*)$,

where two morphisms $f, g \in \text{Hom}_{\text{Ch}(\mathcal{A})}(A_*, B_*)$ are equivalent if there exists $h \in \text{Map}_{\text{Ch}(\mathcal{A})}(A_*, B_*)_1$

such that $dh = f - g$. More explicitly, we have

$$(d_1h)_n = d_{n+1}^B \circ h_n + h_{n-1} \circ d_n^A = f_n - g_n.$$

In other words, f and g are *chain morphisms* which are considered equivalent if they are *chain homotopic*, that is, if their difference is *nullhomotopic*. This homotopy category is often denoted $K(\mathcal{A}) = \mathrm{hCh}(\mathcal{A})$.

Combining the above observation with Proposition 5.1.12, we see that $\mathrm{hK}(\mathcal{A}) = \mathrm{hN}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$ can be canonically identified with the category $K(\mathcal{A})$ of chain complexes modulo homotopy.

Remark 5.1.14. A consequence of Theorem 2.3.20 is that a stable ∞ -category \mathcal{C} is compactly generated if and only if the triangulated category $\mathrm{h}\mathcal{C}$ is compactly generated (see [46, 1.4.4.3] and [47]). In other words, we can find examples of compactly generated stable ∞ -categories \mathcal{C} by finding compactly generated triangulated categories which can be realized as the homotopy category of \mathcal{C} . One such example is the stable ∞ -category of spectra which is freely generated by a single compact object (the sphere spectrum) under small colimits (see [46, 1.4.4] for more details). By Remark 5.1.13, another source of examples arises from homotopy categories of chain complexes $K(\mathcal{A}) = \mathrm{hK}(\mathcal{A})$ which are compactly generated for appropriate abelian categories \mathcal{A} . Krause showed in [39] that if $\mathcal{A}_{\mathrm{inj}}$ is the full subcategory of injective right R -modules over a Noetherian ring R , then $K(\mathcal{A}_{\mathrm{inj}})$ is compactly generated. A result of Neeman [48], generalizing an earlier result of Jorgensen [30], established that if $\mathcal{A}_{\mathrm{proj}}$ is the full subcategory of projective R -modules over a left coherent ring R (this holds in particular for any Noetherian ring), then $K(\mathcal{A}_{\mathrm{proj}})$ is compactly generated. We will return to the compactly generated stable ∞ -category $\mathcal{K}(\mathcal{A}_{\mathrm{inj}})$ in Theorem 5.1.22 below.

Definition 5.1.15. Let \mathcal{C} and \mathcal{D} be differential graded categories over a commutative ring k . A *differential graded functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of the following data:

- For every object X of \mathcal{C} , an object $F(X)$ of \mathcal{D} .
- For every pair of objects X and Y of \mathcal{C} , a morphism of chain complexes of k -modules

$$F_{X,Y}: \text{Map}_{\mathcal{C}}(X, Y) \rightarrow \text{Map}_{\mathcal{D}}(F(X), F(Y)).$$

These data are required to satisfy the following conditions:

- For every object X of \mathcal{C} , we have $F_{X,X}(\text{id}_X) = \text{id}_{F(X)}$.
- For every triple of objects X, Y, Z of \mathcal{C} and every pair of morphisms $f \in \text{Map}_{\mathcal{C}}(X, Y)_p$ and $g \in \text{Map}_{\mathcal{C}}(Y, Z)_q$, we have $F_{X,Z}(g \circ f) = F_{Y,Z}(g) \circ F_{X,Y}(f)$.

Remark 5.1.16. By [46, Proposition 1.3.1.20], the differential graded nerve (Construction 5.1.5) determines a right Quillen functor $\mathcal{C} \mapsto \text{N}_{\text{dg}}(\mathcal{C})$ from $\text{Cat}_{\text{dg}} k$, differential graded categories over k endowed with the Dwyer-Kan-Tabuada model structure [46, Proposition 1.3.1.19], to the category of simplicial sets Set_{Δ} , endowed with the Joyal model structure [45, Theorem 2.2.5.1]. In fact, the proof of [46, Proposition 1.3.1.20] shows that N_{dg} preserves weak equivalences. Explicitly, if $F: \mathcal{C} \rightarrow \mathcal{D}$ is a differential graded functor such that for every pair $X, Y \in \mathcal{C}$, the morphism of chain complexes of k -modules $F_{X,Y}: \text{Map}_{\mathcal{C}}(X, Y) \rightarrow \text{Map}_{\mathcal{D}}(FX, FY)$ is a quasi-isomorphism and the induced functor $\text{h}\mathcal{C} \rightarrow \text{h}\mathcal{D}$ is an equivalence of categories, then the functor $\text{N}_{\text{dg}}(F): \text{N}_{\text{dg}}(\mathcal{C}) \rightarrow \text{N}_{\text{dg}}(\mathcal{D})$ is an equivalence of ∞ -categories.

In particular, a functor $F: \mathcal{A} \rightarrow \mathcal{B}$ of additive categories gives rise (canonically) to a differential graded functor $\tilde{F}: \text{Ch}(\mathcal{A}) \rightarrow \text{Ch}(\mathcal{B})$ obtained by applying F componentwise,

and hence a functor of ∞ -categories $N_{\text{dg}}(\tilde{F}): \mathcal{K}(\mathcal{A}) \rightarrow \mathcal{K}(\mathcal{B})$. By abuse of notation, we often simply write F in place of $N_{\text{dg}}(\tilde{F})$. Observe that if $F: \mathcal{A} \rightarrow \mathcal{B}$ is an equivalence of categories, then the induced functor $\tilde{F}: \text{Ch}(\mathcal{A}) \rightarrow \text{Ch}(\mathcal{B})$ is again an equivalence of differential graded categories. Consequently, by [46, Proposition 1.3.1.20], the corresponding functor $N_{\text{dg}}(\tilde{F}): \mathcal{K}(\mathcal{A}) \rightarrow \mathcal{K}(\mathcal{B})$ is an equivalence of ∞ -categories.

We next discuss resolutions of objects.

Definition 5.1.17. Let \mathcal{A} be an abelian category.

- (1) A chain complex $Q \in \text{Ch}(\mathcal{A})$ is called *dg-injective* if every $Q_n \in \mathcal{A}$ is injective and for every exact complex $E \in \text{Ch}(\mathcal{A})$, the chain complex $\text{Map}_{\text{Ch}(\mathcal{A})}(E, Q)$ is again exact.
- (2) A chain complex $P \in \text{Ch}(\mathcal{A})$ is called *dg-projective* if every $P_n \in \mathcal{A}$ is projective and for every exact complex $E \in \text{Ch}(\mathcal{A})$, the chain complex $\text{Map}_{\text{Ch}(\mathcal{A})}(P, E)$ is again exact.

The importance of dg-injective and dg-projective objects in $\text{Ch}(\mathcal{A})$ is that these objects serve as the appropriate generalizations (from a homological point of view) of injective and projective objects in \mathcal{A} . Indeed, the projective objects in $\text{Ch}(\mathcal{A})$ are nothing but contractible (split) complexes of projective objects, and consequently not very useful. On the other hand, dg-projective objects of $\text{Ch}(\mathcal{A})$ are those complexes P for which the functor $\text{Map}_{\text{Ch}(\mathcal{A})}(P, -)$ preserves quasi-isomorphisms. We record this well-known fact in the next Lemma.

Lemma 5.1.18. *Let \mathcal{A} be an abelian category and assume $f: X \rightarrow Y$ is a quasi-isomorphism in $\text{Ch}(\mathcal{A})$.*

(1) If Q is dg-injective, then the induced map $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(Y, Q) \rightarrow \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(X, Q)$ is a quasi-isomorphism.

(2) If P is dg-projective, then the induced map $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(P, X) \rightarrow \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(P, Y)$ is a quasi-isomorphism.

Proof. We prove (1), the proof of (2) is similar. Recall that a morphism of chain complexes $f: X \rightarrow Y$ is a quasi-isomorphism if and only if the mapping cone $M(f)$ is exact (see Remark 5.1.10). Since Q is dg-injective, we have that $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M(f), Q)$ is again exact. It remains to observe that $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M(f), Q)$ is isomorphic to the mapping cone of the induced morphism $\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(Y, Q) \rightarrow \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(X, Q)$, which completes the proof. \square

Another reason to consider dg-injective and dg-projective objects is that they arise naturally in resolutions of unbounded complexes. For more details, see for instance [55, 15, 19, 28, 53].

Lemma 5.1.19 ([55]). *Let R be a ring. For every complex X of R -modules, there exist quasi-isomorphisms $X \rightarrow Q$ and $P \rightarrow X$ where Q is dg-injective and P is dg-projective.*

Remark 5.1.20. Lemma 5.1.19 can be formulated and proved in the more general situation of Grothendieck abelian categories, but we do not need that level of generality here.

Let \mathcal{A} be an abelian category. Let $\mathrm{Ch}_{\mathrm{dgp}}(\mathcal{A}) \subseteq \mathrm{Ch}(\mathcal{A})$ denote the full subcategory of chain complexes spanned by the dg-projectives and set $\mathcal{K}_{\mathrm{dgp}}(\mathcal{A}) = \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}_{\mathrm{dgp}}(\mathcal{A}))$. Similarly, let $\mathrm{Ch}_{\mathrm{dgi}}(\mathcal{A}) \subseteq \mathrm{Ch}(\mathcal{A})$ denote the full subcategory of chain complexes spanned by the dg-injectives and set $\mathcal{K}_{\mathrm{dgi}}(\mathcal{A}) = \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}_{\mathrm{dgi}}(\mathcal{A}))$. By Proposition 5.1.9, $\mathcal{K}_{\mathrm{dgp}}(\mathcal{A})$ and $\mathcal{K}_{\mathrm{dgi}}(\mathcal{A})$ are both stable ∞ -categories.

Proposition 5.1.21. *Let R be a ring. Let \mathcal{A} denote the category of right R -modules.*

- (1) *The inclusion of ∞ -categories $\mathcal{K}_{\text{dgp}}(\mathcal{A}) \hookrightarrow \mathcal{K}(\mathcal{A})$ admits a right adjoint G .*
- (2) *Let α be a morphism in $\mathcal{K}(\mathcal{A})$. Then $G(\alpha)$ is an equivalence in $\mathcal{K}_{\text{dgp}}(\mathcal{A})$ if and only if α is a quasi-isomorphism of chain complexes.*
- (3) *The composite $F: \mathcal{K}_{\text{dgi}}(\mathcal{A}) \hookrightarrow \mathcal{K}(\mathcal{A}) \xrightarrow{G} \mathcal{K}_{\text{dgp}}(\mathcal{A})$ is fully faithful.*

Proof. To show that the inclusion $\mathcal{K}_{\text{dgp}}(\mathcal{A}) \hookrightarrow \mathcal{K}(\mathcal{A})$ admits a right adjoint, it suffices by Proposition 2.1.38 (applied to opposite categories) to show that for every object X of $\mathcal{K}(\mathcal{A})$ there exists a morphism $f: X' \rightarrow X$ with $X' \in \mathcal{K}_{\text{dgp}}(\mathcal{A})$ such that for every $P \in \mathcal{K}_{\text{dgp}}(\mathcal{A})$, the induced map

$$\text{Map}_{\text{Ch}(\mathcal{A})}(P, X') \rightarrow \text{Map}_{\text{Ch}(\mathcal{A})}(P, X)$$

is an isomorphism in the homotopy category, that is, a quasi-isomorphism. By Lemma 5.1.18, it suffices to find a quasi-isomorphism $f: X' \rightarrow X$. By Lemma 5.1.19, there exists $X' \in \mathcal{K}_{\text{dgp}}(\mathcal{A})$ and a quasi-isomorphism $f: X' \rightarrow X$, completing the proof of the first statement.

Let $\alpha: X \rightarrow Y$ be a morphism in $\mathcal{K}(\mathcal{A})$. By Lemma 5.1.19, α gives rise to a commutative diagram

$$\begin{array}{ccc} G(X) & \xrightarrow{G(\alpha)} & G(Y) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\alpha} & Y \end{array}$$

in which the vertical arrows are quasi-isomorphisms. It follows that $G(\alpha)$ is a quasi-isomorphism if and only if α is a quasi-isomorphism. As a morphism between projective

complexes, $G(\alpha)$ is a quasi-isomorphism if and only if $G(\alpha)$ is a chain homotopy equivalence, which proves the second statement.

To prove the last assertion, let $X, Y \in \mathcal{K}(\mathcal{A})$ and consider the diagram

$$\mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(X, Y) \rightarrow \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(GX, Y) \leftarrow \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(GX, GY)$$

induced by the quasi-isomorphisms $GX \rightarrow X$ and $GY \rightarrow Y$ of Lemma 5.1.19. If Y is dg-injective, then the first map in the above diagram is a quasi-isomorphism by Lemma 5.1.18(1). For any $X \in \mathcal{K}(\mathcal{A})$, we have that $GX \in \mathcal{K}_{\mathrm{dgp}}(\mathcal{A})$ and so the second map in the above diagram is also quasi-isomorphism by Lemma 5.1.18(2). This argument proves that F is fully faithful. \square

In view of Theorem 4.1.5, we are most interested in those stable ∞ -categories $\mathcal{K}(\mathcal{A})$ which are compactly generated. Following Krause [39], Lurie proved the following result:

Theorem 5.1.22 ([46, 1.3.6.7]). *Let \mathcal{A} be a locally Noetherian abelian category, and let $\mathcal{A}_{\mathrm{inj}} \subseteq \mathcal{A}$ denote the full subcategory spanned by the injective objects. Then $\mathcal{K}(\mathcal{A}_{\mathrm{inj}})$ is a compactly generated stable ∞ -category. Moreover, an object $Q \in \mathcal{K}(\mathcal{A}_{\mathrm{inj}})$ is compact if and only if it satisfies the following conditions:*

- (1) *Q is (equivalent to) a left-bounded complex of injectives; in particular, $H_n(Q) \cong 0$ for $n \gg 0$.*
- (2) *The homology objects $H_n(Q)$ vanish for $n \ll 0$.*
- (3) *For all $n \in \mathbb{Z}$, $H_n(Q)$ is a Noetherian object of \mathcal{A} .*

If \mathcal{A} is a category of unitary right modules over a right Noetherian ring, then is a locally Noetherian abelian category. More generally, any closed subcategory of \mathcal{A} is locally Noetherian. See [52] for more examples and details.

5.2 An Auslander-Reiten translation functor

For the remainder of this section, we fix a commutative Noetherian ring k which is complete and local, and a Noetherian k -algebra R . Let \mathcal{M} denote the category of right R -modules. Our assumptions on R ensure that \mathcal{M} is a locally Noetherian abelian k -category (see [52]). By Theorem 5.1.22, the stable ∞ -category $\mathcal{K}(\mathcal{M}_{\text{inj}})$ is compactly generated and compact objects are quasi-isomorphic to bounded complexes of finitely generated right R -modules. Moreover, since any left-bounded complex of injectives is dg-injective, we see that every compact $Q \in \mathcal{K}(\mathcal{M}_{\text{inj}})$ is dg-injective.

Let E be an injective cogenerator in the category of k -modules (explicitly, we may take an injective envelope $E = E(k/m)$ where m denotes the unique maximal ideal of k). The (contravariant) functor $D = \text{Hom}_k(-, E)$ determines a self-duality on the full subcategory of finite length k -modules. In general the functor D does not implement a Morita duality, however, it does for instance when k is a commutative Noetherian ring which is complete and local, by a result of Matlis (see [42, 19.55]). Our construction of an Auslander-Reiten translation functor will make use of the so-called *Nakayama functors* given by

$$\nu = D \text{Hom}_R(-, R): \mathcal{M} \rightarrow \mathcal{M}$$

$$\eta = \text{Hom}_R(DR, -): \mathcal{M} \rightarrow \mathcal{M}.$$

It is well known that the Nakayama functors form an adjoint pair (ν, η) which induce a

mutually inverse equivalence of categories

$$\mathcal{M}_{\text{proj}}^{\text{fg}} \xrightleftharpoons[\eta]{\nu} \mathcal{M}_{\text{inj}}^{\text{fg}}$$

between the full subcategory $\mathcal{M}_{\text{proj}}^{\text{fg}} \subseteq \mathcal{M}$ spanned by the finitely generated projective right R -modules and the full subcategory $\mathcal{M}_{\text{inj}}^{\text{fg}} \subseteq \mathcal{M}$ spanned by the finitely generated injective right R -modules (e.g., [54, Lemma 5.1]). Using Remark 5.1.16 and [45, 5.2.2.8], the Nakayama functors $(\nu, \eta): \mathcal{K}(\mathcal{M}) \rightarrow \mathcal{K}(\mathcal{M})$ are adjoint and induce an equivalence of ∞ -categories

$$\mathcal{K}(\mathcal{M}_{\text{proj}}^{\text{fg}}) \xrightleftharpoons[\eta]{\nu} \mathcal{K}(\mathcal{M}_{\text{inj}}^{\text{fg}}). \quad (5.2.0.1)$$

Our goal is show that the Nakayama functor ν implements the functorial representability essential for constructing an Auslander-Reiten translation functor. We begin by recalling that $\text{Ch}(\mathcal{M})$ carries a closed monoidal structure. If X is any chain complex of right R -modules and Y is any chain complex of left R -modules, then we define a chain complex of k -modules $X \otimes_R Y$ via the equation

$$(X \otimes_R Y)_n = \coprod_{k \in \mathbb{Z}} X_k \otimes_R Y_{n-k}$$

with differential $d_n: (X \otimes Y)_n \rightarrow (X \otimes Y)_{n-1}$ determined by $d_k^X \otimes 1 + (-1)^k \otimes d_{n-k}^Y$ on each component. An important property of this construction is the adjoint isomorphism

$$\text{Map}_k(X \otimes_R Y, Z) \cong \text{Map}_R(X, \text{Map}_k(Y, Z))$$

where Z is any complex of k -modules. (For ease of notation here, we have written Map_k and Map_R in place of the appropriate $\text{Map}_{\text{Ch}(\mathcal{A})}$ mapping complex (see Example 5.1.4)).

Let $\text{Ch}^-(\mathcal{M}) \subseteq \text{Ch}(\mathcal{M})$ denote the full subcategory of $\text{Ch}(\mathcal{M})$ spanned by those chain complexes X such that $X_n = 0$ for $n \ll 0$; that is, $\text{Ch}^-(\mathcal{M})$ consists of right-bounded

complexes. Similarly, let $\text{Ch}^+(\mathcal{M})$ denote the full subcategory of $\text{Ch}(\mathcal{M})$ spanned by those chain complexes Y such that $Y_n = 0$ for $n \gg 0$; that is, $\text{Ch}^+(\mathcal{M})$ consists of left-bounded complexes.

Lemma 5.2.1. *For any $X \in \text{Ch}^-(\mathcal{M})$ and $Y \in \text{Ch}^+(\mathcal{M})$, there is a natural morphism of k -module complexes*

$$Y \otimes_R \text{Map}_R(X, R) \rightarrow \text{Map}_R(X, Y)$$

which is an isomorphism if every X_n is a finitely generated projective right R -module.

Proof. For any right R -modules M and N , the homomorphism $\varphi: N \otimes_R \text{Hom}_R(M, R) \rightarrow \text{Hom}_R(M, N)$ given by $\varphi(n \otimes f)(m) = nf(m)$ is an isomorphism provided M is finitely generated and projective.

Viewing R as concentrated in degree 0, we have

$$\begin{aligned} (Y \otimes_R \text{Map}_R(X, R))_n &= \coprod_{j \in \mathbb{Z}} Y_j \otimes_R \text{Map}_R(X, R)_{n-j} \\ &= \coprod_{j \in \mathbb{Z}} \left(Y_j \otimes_R \prod_{i \in \mathbb{Z}} \text{Hom}_R(X_i, R_{i+n-j}) \right) \\ &= \prod_{j \in \mathbb{Z}} (Y_j \otimes_R \text{Hom}_R(X_{j-n}, R)) \\ &\xrightarrow{\varphi} \prod_{j \in \mathbb{Z}} \text{Hom}_R(X_{j-n}, Y_j) \cong \prod_{i \in \mathbb{Z}} \text{Hom}_R(X_i, Y_{i+n}) \\ &= \text{Map}_R(X, Y)_n, \end{aligned}$$

where the isomorphism between the coproduct and product follows from the boundedness assumptions on X and Y , which imply only finitely many nonzero terms in the coproduct.

□

Lemma 5.2.2. *For any $X \in \text{Ch}^-(\mathcal{M})$ and $Y \in \text{Ch}^+(\mathcal{M})$, there is a natural morphism of k -module complexes*

$$\beta_{X,Y}: D \text{Map}_R(X, Y) \rightarrow \text{Map}_R(Y, \nu X), \quad (5.2.2.1)$$

which is an isomorphism if every X_n is a finitely generated projective right R -module.

Proof. Let $\alpha: Y \otimes_R \text{Map}_R(X, R) \rightarrow \text{Map}_R(X, Y)$ be the morphism of k -module complexes constructed in Lemma 5.2.1, which is an isomorphism if every X_n is finitely generated and projective. Then

$$\begin{aligned} D \text{Map}_R(X, Y) &= \text{Map}_k(\text{Map}_R(X, Y), E) \\ &\xrightarrow{\alpha^*} \text{Map}_k(Y \otimes_R \text{Map}_R(X, R), E) \\ &\cong \text{Map}_R(Y, \text{Map}_k(\text{Map}_R(X, R), E)) \\ &= \text{Map}_R(Y, \nu X), \end{aligned}$$

where the last isomorphism is the usual adjunction. □

Theorem 5.2.3. *Let $G: \mathcal{K}(\mathcal{M}) \rightarrow \mathcal{K}_{\text{dgp}}(\mathcal{M})$ be the (unbounded) projective resolution functor of Proposition 5.1.21. Then the endofunctor*

$$T: \mathcal{K}(\mathcal{M}_{\text{inj}}) \hookrightarrow \mathcal{K}(\mathcal{M}) \xrightarrow{G} \mathcal{K}_{\text{dgp}}(\mathcal{M}) \hookrightarrow \mathcal{K}(\mathcal{M}_{\text{proj}}) \xrightarrow{\nu} \mathcal{K}(\mathcal{M}_{\text{inj}})$$

has the following properties:

- (1) *The functor T is exact and preserves all coproducts.*
- (2) *The functor T is fully faithful when restricted to the full subcategory of compact objects*

$$\mathcal{K}_c(\mathcal{M}_{\text{inj}}) \subseteq \mathcal{K}(\mathcal{M}_{\text{inj}}).$$

(3) The functor T satisfies the Auslander-Reiten formula, that is, for any $X, Q \in \mathcal{K}(\mathcal{M}_{\text{inj}})$ such that Q is compact, there is an equivalence

$$D \operatorname{Map}_{\mathcal{K}(\mathcal{M}_{\text{inj}})}(Q, X) \simeq \operatorname{Map}_{\mathcal{K}(\mathcal{M}_{\text{inj}})}(X, TQ). \quad (5.2.3.1)$$

(4) The functor T admits a right adjoint.

Proof. As a right adjoint, G commutes with limits. As a left adjoint, the functor ν commutes with colimits, and hence finite limits because $\mathcal{K}(\mathcal{M}_{\text{proj}})$ is a stable ∞ -category. It follows that T is exact by [46, 1.1.4.1]. To see that G commutes with arbitrary coproducts, consider the diagram

$$\begin{array}{ccc} \coprod_{\alpha} GX_{\alpha} & \xrightarrow{\quad} & G \coprod_{\alpha} X_{\alpha} \\ & \searrow \quad \swarrow & \\ & \coprod_{\alpha} X_{\alpha} & \end{array}$$

The right vertical arrow is a quasi-isomorphism by Lemma 5.1.19. The left vertical arrow is a quasi-isomorphism because homology commutes with arbitrary coproducts (in this setting). It follows that the horizontal arrow is a quasi-isomorphism. This proves the first statement.

To prove the second statement, first note that every compact object is dg-injective. By Proposition 5.1.21, we have that G is fully faithful on compact objects. Next, observe that G sends compact objects to bounded complexes of finitely generated projectives. Using 5.2.0.1, this observation implies that ν is also fully faithful on compact objects. It now follows that T is fully faithful on compact objects.

To establish the Auslander-Reiten formula, it suffices to assume that X is also compact because $\mathcal{K}(\mathcal{M}_{\text{inj}})$ is compactly generated. By Lemma 5.1.19 and the properties of Q as a compact object, there exists a quasi-isomorphism $GQ \rightarrow Q$ where GQ is a bounded complex

of finitely generated projective right R -modules. Since X is compact, it is dg-injective, and hence by Lemma 5.1.18 induces a quasi-isomorphism

$$\mathrm{Map}_{\mathrm{Ch}(\mathcal{M}_{\mathrm{inj}})}(Q, X) \rightarrow \mathrm{Map}_{\mathrm{Ch}(\mathcal{M}_{\mathrm{inj}})}(GQ, X).$$

Using that the functor D preserves quasi-isomorphisms, we have a quasi-isomorphism

$$D \mathrm{Map}_{\mathrm{Ch}(\mathcal{M}_{\mathrm{inj}})}(GQ, X) \rightarrow D \mathrm{Map}_{\mathrm{Ch}(\mathcal{M}_{\mathrm{inj}})}(Q, X).$$

Finally, the properties of GQ allow us to apply the isomorphism $\beta_{GQ, X}$ of (5.2.2.1) giving

$$\mathrm{Map}_{\mathrm{Ch}(\mathcal{M}_{\mathrm{inj}})}(X, \nu GQ) \xleftarrow{\cong} D \mathrm{Map}_{\mathrm{Ch}(\mathcal{M}_{\mathrm{inj}})}(GQ, X) \xrightarrow{\sim} D \mathrm{Map}_{\mathrm{Ch}(\mathcal{M}_{\mathrm{inj}})}(Q, X).$$

This establishes the Auslander-Reiten formula (5.2.3.1).

To prove the last statement, it suffices by [45, 5.5.2.9] to show that T preserves small colimits. Since T preserves all coproducts and $\mathcal{K}(\mathcal{M}_{\mathrm{inj}})$ is a presentable stable ∞ -category, it follows from Theorem 2.3.20 that T preserves small colimits. \square

Remark 5.2.4. In the situation of Theorem 5.2.3, the right adjoint to T can be described explicitly as the Nakayama functor η followed by an injective resolution.

The existence of Auslander-Reiten sequences now follows as a consequence of the existence of an Auslander-Reiten translation functor, which determines a functorial relationship between the end terms.

Corollary 5.2.5. *If $Z \in \mathcal{K}(\mathcal{M}_{\mathrm{inj}})$ is compact and strongly indecomposable, then there exists an Auslander-Reiten sequence $X \rightarrow Y \rightarrow Z$ in $\mathcal{K}(\mathcal{M}_{\mathrm{inj}})$. Moreover, $X \simeq \Omega TZ$.*

Proof. Observe that $D = \mathrm{Hom}_k(-, E)$ is exact because E is injective. Therefore, D commutes with the formation of homology. Combining this observation with Proposition 5.1.12

and Remark 5.1.13, the Auslander-Reiten formula (5.2.3.1) of Theorem 5.2.3 induces an isomorphism (functorial in Q and Z) on the homotopy category

$$\mathrm{Hom}_{K(\mathcal{M}_{\mathrm{inj}})}(Q, TZ) \cong \mathrm{Hom}_k(\mathrm{Hom}_{K(\mathcal{M}_{\mathrm{inj}})}(Z, Q), E).$$

The proof of Theorem 4.1.5 now establishes the desired result. □

Appendix A

Some algebra

A.1 Local rings

The material here is standard, but included for completeness. We used notes by E.L. Lady as a reference for some of the material in this section, available online [41].

Definition A.1.1. A unital ring is called *local* if it has a unique maximal left ideal.

Lemma A.1.2. *Let R be a nontrivial unital ring. If $r, s \in R$ are such that $rs = 1$ and $sr \neq 1$, then neither sr nor $1 - sr$ is left or right invertible.*

Proof. We have $sr(1 - sr) = 0 = (1 - sr)sr$. If sr is left or right invertible, then $(1 - sr) = 0$ implies $sr = 1$, a contradiction. Likewise, if $(1 - sr)$ is left or right invertible, then $sr = 0$ implies $1 = rs = (rs)^2 = r(sr)s = 0$, contradicting the nontriviality of R . \square

Theorem A.1.3. *Let R be a unital ring. The following are equivalent:*

- (a) *R is local.*

(b) R has a unique maximal right ideal.

(c) The Jacobson radical $J(R)$ is the unique maximal left ideal and the unique maximal right ideal of R .

(d) $1 \neq 0$ in R and the sum of any two non-units in R is a non-unit.

(e) $1 \neq 0$ in R and if $r \in R$, then r or $1 - r$ is a unit.

Proof. (a) \Rightarrow (b): Let $\mathfrak{m} \subset R$ be the unique maximal left ideal of R . We first show that \mathfrak{m} is a right ideal. For any $r \in R$, $\mathfrak{m}r$ is a left ideal. If $\mathfrak{m}r = R$, then $mr = 1$ for some $m \in \mathfrak{m}$. Since $rm \in \mathfrak{m}$, we know $rm \neq 1$. Thus, by Lemma A.1.2, $(1 - rm)$ is not left invertible. Consequently, $R(1 - rm) \neq R$ is a proper left ideal and so $(1 - rm) \in R(1 - rm) \subset \mathfrak{m}$, since \mathfrak{m} is the unique maximal left ideal of R . But then $1 = (1 - rm) + rm \in \mathfrak{m}$, a contradiction. Therefore, $\mathfrak{m}r \neq R$ implies $\mathfrak{m}r \subset \mathfrak{m}$, which shows that \mathfrak{m} is a right ideal.

Now, suppose $I \subset R$ is a right ideal and $i \in I$. If $RI = R$, then $ri = 1$ for some $r \in R$. But as above, $ir \in I$ implies $ir \neq 1$. Again, by Lemma A.1.2, this implies that both ir and $1 - ir$ do not have left inverses. Hence, $R(ir) \neq R$ and $R(1 - ir) \neq R$ are both proper left ideals and therefore contained in \mathfrak{m} . But this implies $1 \in \mathfrak{m}$, a contradiction. So, $RI \neq R$ is a proper left ideal and $I \subset RI \subset \mathfrak{m}$, which shows that \mathfrak{m} is also the unique maximal right ideal of R .

(b) \Rightarrow (a): This follows by an argument completely analogous to the one above.

(a) \wedge (b) \Leftrightarrow (c): Since $J(R)$ is the intersection of all maximal left ideals of R , the equivalence is clear.

(a) \wedge (b) \Rightarrow (d): Let $\mathfrak{m} \subset R$ be the unique maximal left ideal of R . If $r \in R$ is left invertible, then $Rr = R$, and so $r \notin \mathfrak{m}$. By (b), \mathfrak{m} is also the unique maximal right ideal, it

follows that $rR = R$. Hence, r is also right invertible and thus invertible. Now, if $r, s \in R$ are both non-units, then they are not left invertible, which implies that $Rr \neq R$ and $Rs \neq R$ from which we conclude $r, s \in \mathfrak{m}$, since \mathfrak{m} is the unique maximal left ideal. Hence, $r + s \in \mathfrak{m}$, which shows that $r + s$ is a non-unit.

(d) \Rightarrow (a): Let I be the collection of all non-units in R . By assumption, I is an abelian group. If $r, s \in R$ are such that $rs = 1$ and $sr \neq 1$, then by Lemma A.1.2 we have that sr and $(1 - sr)$ are elements of I . But this implies 1 is a non-unit, which is absurd. Therefore, left invertible implies invertible under these assumptions, and I consists of all elements of R which are not left invertible. Consequently, for any $r \in R$ and $i \in I$, ri cannot be left invertible since i is not left invertible. So, $ri \in I$ and I is a left ideal. Moreover, if J is a proper left ideal, then no element of J is left invertible, that is, $J \subset I$. Hence, I is the unique maximal left ideal of R .

(d) \Rightarrow (e): Let I be the unique maximal left ideal of R consisting of all elements of R which are not left invertible. If $r \in R$ is not invertible, then r is not left invertible, by the argument above. Therefore, $Rr \subset I$. If $R(1 - r) \subset I$, then $R = Rr + R(1 - r) \subset I$, which is absurd. Therefore, $R(1 - r) \not\subset I$ is a left ideal not contained in I . Since I is the unique maximal left ideal of R , it must be that $R(1 - r) = R$. So, $(1 - r)$ is left invertible, which in this case implies invertible.

(e) \Rightarrow (d): Suppose $r, s \in R$ are both non-units. If $r + s$ is a unit, then $v(r + s) = 1 = (r + s)v$, for some $v \in R$. So, $rv = 1 - sv$ and $vr = 1 - vs$. If rv is not a unit, then $sv = 1 - rv$ is a unit, which implies that s has a right inverse. If vr is not a unit, then $vs = 1 - vr$ is a unit, which implies that s also has a left inverse. Since this cannot be, we

must have that vr is a unit, and so r has a left inverse. Let $u \in R$ be such that $ur = 1$. Since r is a non-unit, it must be that $ru \neq 1$. Therefore, by Lemma A.1.2, ru and $1 - ru$ are both non-units, contradicting our assumptions. We conclude that $r + s$ must also be a non-unit. \square

Remark A.1.4. The statement and arguments above (particularly in (d)) shows that the set of non-units of R forms a unique maximal left ideal, equal to $J(R)$, which is equivalent to the complement of all left invertible elements. The same argument shows that this ideal is also equal to the complement of all right invertible elements.

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