Title

The Local Langlands Correspondence, Rapoport-Zink Spaces, and Shimura Varieties

Author

Bertoloni Meli, Alexander

Publication Date

2020

Peer reviewed|Thesis/dissertation

The Local Langlands Correspondence, Rapoport-Zink Spaces, and Shimura Varieties

by

Alexander Bertoloni Meli

A dissertation submitted in partial satisfaction of
the requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committe in charge

Professor Sug Woo Shin, Chair
Professor Denis Auroux
Professor Xinyi Yuan
Professor Nicholas Jewell

Spring 2020

Abstract

The Local Langlands Correspondence, Rapoport-Zink Spaces, and Shimura Varieties

by

Alexander Bertoloni Meli

Doctor in Philosophy in Mathematics

University of California Berkeley

Professor Sug Woo Shin, Chair

The connection between the Langlands correspondence and the cohomology of Rapoport-Zink spaces and Shimura varieties has been the subject of extensive mathematical research over the past few decades. In this thesis, we extend the existing theory in two key ways. Firstly, we give an explicit combinatorial description of the cohomology of Rapoport-Zink spaces of EL-type, building off of earlier work by Harris–Taylor and Shin ([HT01], [Shi12b], [HT01]). Secondly, joint with Alex Youcis, we state a list of axioms for the supercuspidal local Langlands correspondence and prove that they characterize the correspondence in certain cases. The most important of our axioms arises naturally in the study of the cohomology of Shimura varieties and was first stated in work of Scholze and Scholze–Shin ([Sch13b], [SS13]). We verify these axioms in the case of unramified unitary groups.

This thesis is dedicated to:

Domenico Bertoloni Meli
Rebecca Bertoloni Meli
Sofia Bertoloni Meli
Lisa Minh Nguyen

Contents

Introduction

This dissertation focuses on the local Langlands correspondence and its relation to the cohomology of certain moduli spaces studied in arithmetic geometry: Rapoport-Zink spaces and Shimura varieties. Our study of this relationship provides new results in two directions. On the one hand, we use representation theory to shed light on the structure of the etale cohomology of these moduli spaces. On the other, we use the geometry of these spaces to prove new properties satisfied by the Langlands correspondence. We begin with a general discussion of the Langlands correspondence and its relation to Rapoport-Zink spaces and Shimura varieties. We show how this relationship leads to some of the key questions answered in this thesis and give a broad outline of our main results. We leave detailed introductions and theorem statements to the introductory sections of Parts 1, 2, and 3.

The key object of study in this thesis is the local Langlands correspondence for a p-adic field F. For the group GL_n, the correspondence gives a bijection between certain irreducible $\mathrm{GL}_n(F)$ representations and n-dimensional Galois representations. More generally, the correspondence states that there exists a natural finite-to-one map from the set of isomorphism classes of irreducible admissible representations of $\mathrm{G}(F)$, for G a reductive group, to the set of equivalence classes of homomorphisms $\mathcal{L}_F \to {}^L G$, where $\mathcal{L}_F$ is the Langlands group of F (given by the product $W_F \times \mathrm{SL}_2(\mathbb{C})$ where W_F is the Weil group of F) and ${}^L G := \widehat{G} \rtimes W_F$ is the L-group of G.

One approach to the study of such a correspondence of representations is to find a natural representation of $W_F \times \mathrm{G}(F)$ whose decomposition into irreducible representations is governed by the Langlands correspondence. In some sense, this approach is analogous to the theory of Schur-Weyl duality where one relates the representation theory of the symmetric group S_k and the group $\mathrm{GL}_n(\mathbb{C})$ via the decomposition of the natural action of $\mathrm{GL}_n(\mathbb{C}) \times S_k$ on the k-fold tensor product $\mathbb{C}^n \otimes ... \otimes \mathbb{C}^n$.

In our case, the ℓ-adic cohomology of Rapoport-Zink spaces provides a natural choice of a $W_F \times \mathrm{G}(F)$-representation. For a certain class of representations of $\mathrm{G}(F)$ known as *supercuspidal* representations, the Kottwitz conjecture [RV14, Conjecture 7.3] gives a precise and relatively simple description of the relationship between the "supercuspidal part" of the cohomology of Rapoport-Zink spaces and the local Langlands correspondence. On the other hand, for general representations this relationship is known to be quite complicated and remains mysterious.

In Part 1 we restrict our attention to Rapport-Zink spaces that are of unramified EL-type. Building on work of Shin [Shi12b], we are able to give an explicit combinatorial description of the ℓ-adic cohomology of these spaces in terms of the local Langlands correspondence, the Jacquet-Langlands correspondence, and parabolic induction functors. Our description generalizes the Kottwitz conjecture in this case. We then use this explicit description to verify a conjecture of Harris on these cohomology spaces.

Parts 2 and 3 use formulas arising from geometry to prove new results on the local Langlands correspondence. The key geometric objects of Parts 2 and 3 are Shimura varieties, which are the global analogues of Rapoport-Zink spaces. Parts 2 and 3 stem from work that was completed jointly with Alex Youcis.

In Part 2 we prove that the local Langlands correspondence for unramified unitary groups satisfies certain trace identities which we refer to as the *Scholze–Shin equations*. These equations were first studied by Scholze in [Sch13b] and generalized by Scholze–Shin in [SS13]. In our setting, these equations relate the trace of a discrete series L-parameter for an unramified unitary group U with the sum of the trace distributions of the representations in its L-packet. Involved in this trace identity are certain functions $f_{\tau,h}$ that occur naturally in the study of p-part of the cohomology of Shimura varieties and were first defined in our case in the thesis of Alex Youcis ([You19]). We prove the *Scholze-Shin equations* via a careful study of the p-part of the cohomology of Shimura varieties following works of Langlands, Kottwitz, and Scholze.

In Part 3, we study the problem of the characterization of the local Langlands correspondence for supercuspidal representations. Our approach is inspired by Scholze's proof of the Langlands correspondence in the GL_n case and in particular, crucially uses the Scholze–Shin equations. We give a list of axioms that we show uniquely characterize the local Langlands correspondence for certain groups. In particular, combining parts 2 and 3, we deduce a new characterization of the local Langlands correspondence in the case of unramified unitary groups.

Acknowledgements

I would like to thank my advisor Sug Woo Shin for his patience and kindness and generosity with ideas. I am supremely grateful for his mentorship.

I thank my great friend and collaborator Alex Youcis. Parts 2 and 3 of this thesis are the fruits, thus far, of our collaborations. He has helped me to become a better mathematician, and more importantly, a better person.

I thank Kaoru Hiraga for sketching to us the proof of Proposition 3.4.2 in Part 3. This allowed us to avoid having an axiom requiring the atomic stability property.

I thank Tasho Kaletha for several discussions relating to the contents of Part 3 of this thesis. Our exposition and results benefited substantially from his feedback.

I kindly thank the many mathematicians with whom I have had helpful discussions on this project including Hiraku Atobe, Brian Conrad, Laurent Fargues, Paul Hamacher, Michael Harris, Naoki Imai, Wansu Kim, Yoichi Mieda, Kieu Hieu Nguyen, Peter Scholze, and Eva Viehmann. I particularly want to thank Masao Oi who has carefully answered a great number of my questions and is always a pleasure to discuss mathematics with.

Finally, I want to thank my mathematical brothers and sister: Rahul Dalal, Ian Gleason, Zixin Jiang, DongGyu Lim, and Sander Mack-Crane. I have learned so much from countless conversations and seminars with them.

Part 1. The l-adic Cohomology of Unramified Rapoport-Zink Spaces and Harris's Conjecture

1.1. Introduction

Our goal in this Part is to give a description of the l-adic cohomology of unramified Rapoport-Zink spaces of EL-type. These spaces are moduli spaces of p-divisible groups associated to unramified Weil-restrictions of general linear groups and can be thought of as generalizations of Lubin-Tate spaces.

This generalizes, for these particular spaces, the Kottwitz conjecture stated in [RV14, Conj 7.3]. The Kottwitz conjecture describes the supercuspidal part of the l-adic cohomology of Rapoport-Zink spaces, and is known in the cases we consider by work of Shin [Shi12b, Cor 1.3]. We prove our description of this cohomology is compatible with a conjecture of Harris [Har01, Conj 5.4], generalizing the Kottwitz conjecture to parabolic inductions of supercuspidal representations.

Our main result describes the cohomology of these Rapoport-Zink spaces as a formal alternating sum (indexed by certain root theoretic data) of representation-theoretic constructions including the local Langlands correspondence, parabolic inductions, and Jacquet modules.

We prove our result inductively using two formulas from the literature. The first of these is Shin's averaging formula [Shi12b, Thm 7.5] which is proven using Mantovan's formula [Man05, Thm 22]. Mantovan's formula connects the cohomology of Rapoport-Zink spaces, Igusa varieties and Shimura varieties. The second formula is the Harris-Viehmann conjecture of [RV14, Conj 8.4] which relates the cohomology of so-called non-basic Rapoport-Zink spaces to a product of Rapoport-Zink spaces of lower dimension. A proof of this conjecture is expected to appear in a forthcoming paper of Scholze.

To carry out our induction, we prove combinatorial analogues of the above formulas phrased purely in terms of root-theoretic data. Interestingly, we are able to prove these analogues for general quasisplit reductive groups, though at present we can only connect them to the cohomology of Rapoport-Zink spaces of unramified EL-type. To do so in other cases, one would need to generalize Shin's averaging formula.

We now describe the main results of this Part more precisely. We fix an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. We study Rapoport-Zink spaces of unramified EL-type which we denote $\mathbb{M}_{b,\mu}$. These are moduli spaces of p-divisible groups coming from an unramified EL-datum consisting of

(1) a finite unramified extension $F \subset \overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$,

(2) a finite dimensional F vector space V which defines the group $G = \mathrm{Res}_{F/\mathbb{Q}_p}\mathrm{GL}(V)$,
(3) a $G_{\overline{\mathbb{Q}_p}}$-conjugacy class of cocharacters $\{\mu\}$, with $\mu : \mathbb{G}_m \to G_{\overline{\mathbb{Q}_p}}$, and such that the weights of μ are elements of $\{0, 1\}$.
(4) an element b of a finite set $\mathbf{B}(G, \mu)$ which defines a group J_b that is an inner twist of a Levi subgroup M_b of G.

Roughly one can think of b, μ as specifying the Newton and Hodge polygons of a p-divisible group and J_b as the automorphism group of the isocrystal b.

Let $\mathbb{Q}_p^{ur}$ denote the maximal unramified extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}_p}$, and let $\widehat{\mathbb{Q}_p^{ur}}$ denote its completion. Then the spaces $\mathbb{M}_{b,\mu}$ are formal schemes over $\widehat{\mathbb{Q}_p^{ur}}$. One constructs a tower of rigid spaces $\mathbb{M}_{U,b,\mu}^{rig}$ over the generic fiber $\mathbb{M}_{b,\mu}^{rig}$ of $\mathbb{M}_{b,\mu}$, where the index U runs over compact open subgroups of $G(\mathbb{Q}_p)$. Associated to such a tower we have a cohomology space $[H^\bullet(G, b, \mu)]$ which is an element of the Grothendieck group $\mathrm{Groth}(G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_{E_{\{\mu\}_G}})$ of admissible representations of $G(\mathbb{Q}_p)$, $J_b(\mathbb{Q}_p)$ and $W_{E_{\{\mu\}_G}}$, where the latter group is the Weil group of the reflex field, $E_{\{\mu\}_G}$, of $\{\mu\}$. This construction can be thought of as an alternating sum of a direct limit over $U \subset G$ of l-adic cohomology groups with the actions of $G(\mathbb{Q}_p)$ and $J_b(\mathbb{Q}_p)$ arising from Hecke correspondences and isogenies of p-divisible groups, respectively. We refer to §1.3.1 for a precise definition.

The cohomology object $[H^\bullet(G, b, \mu)]$ gives rise to a map of Grothendieck groups

$$\mathrm{Mant}_{G,b,\mu} : \mathrm{Groth}(J_b(\mathbb{Q}_p)) \to \mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_{\{\mu\}_G}})$$

which maps a representation ρ to the alternating sum of the $J_b(\mathbb{Q}_p)$-linear Ext groups of $[H^\bullet(G, b, \mu)]$ and ρ.

The map $\mathrm{Mant}_{G,b,\mu}$ has been studied by many authors. Harris and Taylor [HT01] used this construction to prove the local Langlands correspondence for general linear groups. It also appears naturally in Mantovan's work relating the cohomology of Shimura varieties, Igusa varieties, and Rapoport-Zink spaces [Man05]. Fargues studied $\mathrm{Mant}_{G,b,\mu}$ for basic b in some EL and PEL-cases in [Far04]. Shin combined Mantovan's formula with his trace formula description of the cohomology of Igusa varieties to prove instances of local-global Langlands compatibilities [Shi11].

In [Shi12b], Shin proved an averaging formula for $\mathrm{Mant}_{G,b,\mu}$ which is key to our work. He defined a map

$$\mathrm{Red}_b : \mathrm{Groth}(G(\mathbb{Q}_p)) \to \mathrm{Groth}(J_b(\mathbb{Q}_p))$$

which up to a character twist is given by composing the un-normalized Jacquet module

$$\mathrm{Jac}^G_{P_b^{op}} : \mathrm{Groth}(G(\mathbb{Q}_p)) \to \mathrm{Groth}(M_b(\mathbb{Q}_p))$$

with the Jacquet-Langlands map of Badulescu [Bad1]

$$\mathrm{LJ} : \mathrm{Groth}(M_b(\mathbb{Q}_p)) \to \mathrm{Groth}(J_b(\mathbb{Q}_p)).$$

Shin uses global methods and so necessarily works with a large but inexplicit class of representations which he denotes *accessible*. This set loosely consists of those representations isomorphic to the p-component of an automorphic representation appearing in the cohomology of a certain unitary similitude group Shimura variety. In particular, the essentially square integrable representations in $\mathrm{Groth}(G(\mathbb{Q}_p))$ are accessible.

In what follows $r_{-\mu}$ is a finite dimensional representation of $\widehat{G} \rtimes W_{E_{\{\mu\}_G}}$ which restricts to the representation of highest weight $-\mu$ on $\widehat{G}$, and LL is the semisimplifed local Langlands correspondence from [HT01]. Shin shows the following result.

Theorem 1.1.1 (Shin's Averaging Formula)**.** *Assume π is an accessible representation of $G(\mathbb{Q}_p)$. Then*

$$\sum_{b \in \mathbf{B}(G,\mu)} \mathrm{Mant}_{G,b,\mu}(\mathrm{Red}_b(\pi)) = [\pi][r_{-\mu} \circ \mathrm{LL}(\pi)|_{W_{E_{\{\mu\}_G}}}],$$

where the above formula is correct up to a Tate twist which we omit for clarity and $[\pi][\rho]$ is our notation for an element $\pi \boxtimes \rho \in \mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_{\{\mu\}_G}})$.

Additionally we have the conjecture of Harris and Viehmann which allows us to write $\mathrm{Mant}_{G,b,\mu}$ for non-basic b (b is basic when it corresponds to an isocrystal with a single slope) in terms of $\mathrm{Mant}_{G',b',\mu'}$ such that G' is a general linear group of smaller rank than G. This conjecture was formulated in work of [Har01] and [RV14] and is expected to be proven in forthcoming work of Scholze. In what follows, Ind is the un-normalized parabolic induction functor.

Conjecture 1.1.2 (Harris-Viehmann)*.*

$$\mathrm{Mant}_{G,b,\mu} = \sum_{(M_b,\mu') \in \mathcal{I}^{G,\mu}_{M_b,b'}} \mathrm{Ind}^G_{P_b}(\otimes_{i=1}^k \mathrm{Mant}_{M_{b'_i},b'_i,\mu'_i}),$$

where we omit a Tate twist which we discuss at length in §1.3.2. The finite set $\mathcal{I}^{G,\mu}_{M_b,b'}$ is described in Proposition 1.2.25.

Shin's averaging formula and the Harris Viehmann conjecture allow one to compute $\mathrm{Mant}_{G,b,\mu} \circ \mathrm{Red}_b$ recursively. The latter lets us compute $\mathrm{Mant}_{G,b,\mu}$ for non-basic b given that we know $\mathrm{Mant}_{G',b',\mu'}$ for G' of lower rank and the former lets us compute $\mathrm{Mant}_{G,b,\mu}$ for the unique basic $b \in \mathbf{B}(G,\mu)$ if we know it for all non-basic $b \in \mathbf{B}(G,\mu)$. One of our main results is to give a non-recursive description of $\mathrm{Mant}_{G,b,\mu} \circ \mathrm{Red}_b$ which we now describe.

Let $G = \mathrm{Res}_{F/\mathbb{Q}_p}\mathrm{GL}(V)$ as before, choose a rational Borel subgroup B of G, and a rational maximal torus $T \subset B \subset G$. Then we consider pairs (M_S, μ_S) where $M_S \subset T$ is a Levi subgroup of a parabolic subgroup P_S containing B, and $\mu_S \in X_*(T)$ is dominant as a cocharacter of M_S. We call a pair of the above form a *cocharacter pair* for G.

We associate to a cocharacter pair (M_S, μ_S) the map of representations $[M_S, \mu_S] : \mathrm{Groth}(G(\mathbb{Q}_p)) \to \mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_{\{\mu_S\}_{M_S}}})$, which up to a character twist is given by

$$\pi \mapsto [(\mathrm{Ind}_{P_S}^{G} \circ [\mu_S] \circ \mathrm{Jac}_{P_S^{op}}^{G})(\pi)]$$

and

$$[\mu_S] : \mathrm{Groth}(M_S(\mathbb{Q}_p)) \to \mathrm{Groth}(M_S(\mathbb{Q}_p) \times W_{E_{\{\mu_S\}_{M_S}}})$$

given by

$$\pi \mapsto [\pi][r_{-\mu_S} \circ LL(\pi)]$$

Then our main result, which follows from Theorem 1.3.12 is

Theorem 1.1.3. *Suppose* $\mathrm{Mant}_{\mathrm{G,b},\mu}$ *corresponds to a tower of unramified Rapoport-Zink spaces of EL-type. We assume that the Harris-Viehmann conjecture is true. Then if* $\rho \in \mathrm{Groth}(G(\mathbb{Q}_p))$ *is essentially square-integrable, we have*

$$\mathrm{Mant}_{G,b,\mu}(\mathrm{Red}_b(\rho)) = \sum_{(M_S,\mu_S) \in \mathcal{R}_{G,b,\mu}} (-1)^{L_{M_S,M_b}} [M_S, \mu_S](\rho),$$

where $\mathcal{R}_{G,b,\mu}$ *is a collection of cocharacter pairs with a combinatorial definition and* $(-1)^{L_{M_S,M_b}}$ *is an easily determined sign.*

Shin conjectures ([Shi12b, Conj 8.1]) that the averaging formula holds for all admissible representations of $G(\mathbb{Q}_p)$. If this were indeed the case, then our result would also immediately hold for all admissible representations of $G(\mathbb{Q}_p)$.

A crucial part of the proof of the above theorem is the following unconditional result, which is perhaps interesting in its own right.

Theorem 1.1.4 (Imprecise version of Theorem 1.2.24 and Corollary 1.2.28)**.** *For general quasisplit* G *and a cochcaracter* μ *(not necessarily* *minuscule), combinatorial analogues of Shin's formula and the Harris-Viehmann conjecture hold true.*

This result suggests that perhaps the combinatorics of cocharacter pairs is related to $\mathrm{Mant}_{G,b,\mu}$ in cases more general than Rapoport-Zink spaces of unramified EL-type. However, we caution the reader that the existence of nontrivial L-packets and nontrivial endoscopy in more general groups will likely complicate the situation.

In §1.4, we use our combinatorial formula to prove the EL-type cases of a conjecture of Harris ([Har01, Conj 5.4]). This conjecture describes $\mathrm{Mant}_{G,b,\mu}(I_M^G(\rho))$ for ρ a supercuspidal representation of $M(\mathbb{Q}_p)$ for M a Levi subgoup of G. In this case, I_M^G denotes normalized parabolic induction. In particular, we show the following result, which is Conjecture 1.4.4.

Theorem 1.1.5 (Harris conjecture)**.** *We assume that Shin's averaging formula holds for all admissible representations of $G(\mathbb{Q}_p)$ and that the Harris-Viehmann conjecture is true. Let ρ be a supercuspidal representation of $M(\mathbb{Q}_p)$. Then up to a precise character twist and sign which we omit for clarity,*

$$\mathrm{Mant}_{G,b,\mu}(LJ(I_M^{M_b}(\rho))) = [I_M^G(\rho)]\left[\bigoplus_{(M,\mu')\in \mathrm{Rel}_{M,b}^{G,\mu}} r_{-\mu'} \circ LL(\rho)\right]$$

for an explicit set of cocharacter pairs $\mathrm{Rel}_{M,b}^{G,\mu}$.

We prove our result for $I_M^G(\rho)$ not necessarily irreducible and b not necessarily basic, which is a generalization of what Harris conjectured for the G we consider.

Finally, in Appendix 1.5.1 we give an example to show that for general representations ρ, one cannot hope for an expression as simple as that in Harris's conjecture.

1.2. Cocharacter Formalism

In this section we define and study the notion of a *cocharacter pair*. This notation will be used in the third and fourth sections of this part, where we describe the cohomology of certain Rapoport-Zink spaces in terms of cocharacter pairs. We endeavor to use a similar notation to [Kot97].

This section is divided into five subsections. These are structured so that the first contains the basic definitions and the fourth and fifth subsections contain the most important results. The second and third subsections prove a number of technical lemmas that the reader may want to skip at first and refer to as necessary.

1.2.1. **Notation and Preliminary Definitions.** For the remainder of this section, we fix G a connected quasisplit reductive group defined over $\mathbb{Q}_p$. This is a significantly more general setting than we will need for applications in this part. However, we choose to work in this generality because doing so is both conceptually clearer and potentially useful for future applications. The ideas in §5 of [Kot97] might allow one to remove the quasisplit assumption, but we do not attempt this here as it is unnecessary for the applications. Moreover, Kottwitz's study of the set $\mathbf{B}(G)$ in that section relies on understanding the quasisplit case first.

Remark. The reader will notice that most of this section makes sense over an arbitrary field. The assumption that we work over $\mathbb{Q}_p$ is used in section 1.2.4 when we connect cocharacter pairs to the set $\mathbf{B}(G)$ defined by Kottwitz. However, in §5.1 of [Kot97], Kottwitz shows that over $\mathbb{Q}_p$, the set $\mathbf{B}(G)$ is parametrized by a disjoint union of sets of the form $X^*(Z(\widehat{M_S})^\Gamma)^+$ for M_S a standard Levi subgroup of G. These latter sets make sense over general fields and one could make sense generally of all the results of this section by replacing $\mathbf{B}(G)$ with the sets parametrizing it.

Since G is quasisplit, we can pick a Borel subgroup $B \subset G$ defined over $\mathbb{Q}_p$ and a maximal split torus $A \subset B$ of G. We choose T to be a maximal torus defined over $\mathbb{Q}_p$ satisfying $A \subset T \subset B$. We define $X^*(A)$ and $X_*(A)$ respectively to be the character and cocharacter groups of $A_{\overline{\mathbb{Q}_p}}$.

The group G has a relative root datum $(X^*(A), \Phi^*(G,A), X_*(A), \Phi_*(G,A))$, where $\Phi^*(G,A)$ and $\Phi_*(G,A)$ respectively denote the set of relative roots and relative coroots of G and the torus A. Our choice of Borel subgroup B determines a decomposition $\Phi^*(G,A) = \Phi^*(G,A)^+ \coprod \Phi^*(G,A)^-$ of positive and negative roots and a subset $\Delta \subset \Phi^*(G,A)^+$ of simple roots. Analogous statements are also true for the coroots. The set of parabolic subgroups $P \supset B$ defined over $\mathbb{Q}_p$ are called *standard parabolic subgroups*. We define P_S to be the unique standard parabolic subgroup such that $\Phi^*(P_S, A) = \Phi^*(G,A)^+ \cup (\Phi_*(G,A)^- \cap \mathrm{Span}_{\mathbb{Z}}(S))$. There is an inclusion *preserving* bijection between the set of standard parabolic subgroups and subsets of Δ given by $S \mapsto P_S$.

We let N_S be the unipotent radical of the standard parabolic subgroup P_S. It is a standard result that there exists a connected reductive subgroup $M \subset P_S$ so that the natural map $M \to P_S/N_S$ is an isomorphism. In particular, this gives us a Levi decomposition $P_S = MN_S$ and the subgroup M is called a Levi subgroup of P_S. The subgroup M is not unique but any two Levi subgroups of P_S are conjugate by an element of N_S. However, we have fixed a maximal torus T and there is a unique Levi subgroup M_S containing T. The subgroup M_S is constructed explicitly as the centralizer $C_G(Z)$, where $Z \subset T$ is the connected component of the intersection of the kernels of the roots in S. We refer to the Levi subgroups M_S that we produce in this way as *standard Levi subgroups.*

Define

$$\mathfrak{A} := X_*(A).$$

We have the closed rational Weyl chamber

$$\overline{C}_{\mathbb{Q}} = \{x \in \mathfrak{A}_{\mathbb{Q}} : \langle x, \alpha \rangle \geqslant 0, \alpha \in \Delta\}.$$

We define for each standard Levi subgroup,

$$\mathfrak{A}_{M_S,\mathbb{Q}} := \{x \in \mathfrak{A}_{\mathbb{Q}} : \langle x, \alpha \rangle = 0, \alpha \in S\},$$

and denote the *strictly dominant* elements of $\mathfrak{A}_{M_S,\mathbb{Q}}$ by

$$\mathfrak{A}^+_{M_S,\mathbb{Q}} = \{x \in \mathfrak{A}_{\mathbb{Q}} : \langle x, \alpha \rangle = 0, \alpha \in S, \langle x, \alpha \rangle > 0, \alpha \in \Delta \backslash S\},$$

and we have

$$\coprod_{M_S} \mathfrak{A}^+_{M_S,\mathbb{Q}} = \overline{C}_{\mathbb{Q}}.$$

There is a partial ordering of $\mathfrak{A}_{\mathbb{Q}}$ given by $\mu \preceq \mu'$ if $\mu' - \mu$ is a non-negative rational combination of simple roots.

Definition 1.2.1. *We define a* cocharacter pair *for a group G (relative to some fixed choice of T and B defined over $\mathbb{Q}_p$) to be a pair (M_S, μ_S) such that $M_S \subset G$ is a standard Levi subgroup and $\mu_S \in X_*(T)$ satisfies $\langle \mu_S, \alpha \rangle \geqslant 0$ for each positive* absolute *root α of T in the Lie algebra of $M_{S,\overline{\mathbb{Q}_p}}$. Positivity for absolute roots is determined by the Borel subgroup B which we have fixed.*

We denote the set of cocharacter pairs for G by $\mathcal{C}_G$.

Remark. We caution the reader that the cocharacter μ_S need not be an element of $X_*(A)$, even though M_S is defined over $\mathbb{Q}_p$.

We could define cocharacter pairs more canonically as the set of equivalence classes of pairs (M, μ) such that M is a Levi subgroup of G defined over $\mathbb{Q}_p$ and μ is a cocharacter of M. Two pairs $(M, \mu), (M', \mu')$ are equivalent if M, M' are conjugate in $G_{\mathbb{Q}_p}$ and μ, μ' are conjugate in $M_{\overline{\mathbb{Q}_p}}$. We choose not to do this as in practice we will often need to work with the unique dominant cocharacter in a conjugacy class relative to a fixed based root datum.

Let $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. Since we have assumed T and B are defined over $\mathbb{Q}_p$, Γ acts on $T_{\overline{\mathbb{Q}_p}}$ and $B_{\overline{\mathbb{Q}_p}}$. This gives us a natural left action of Γ on $X_*(T)$ given explicitly by $(\gamma \cdot \mu)(g) = \gamma(\mu(\gamma^{-1}(g))$ for $\mu \in X_*(T)$ and $\gamma \in \Gamma$. We get an analogous left action on $X^*(T)$ and one can easily check that the pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$ is Γ invariant under these actions.

We have

$$X_*(T)^{\Gamma} = \mathfrak{A}.$$

Indeed, a Γ-invariant cocharacter μ factors through the identity component of T^{Γ}, where T^{Γ} is the subscheme defined by $T^{\Gamma}(\overline{\mathbb{Q}_p}) = T(\overline{\mathbb{Q}_p})^{\Gamma}$. But the identity component of T^{Γ} is the torus A. Conversely any cocharacter of A induces a Γ-invariant cocharacter via the natural inclusion $A \hookrightarrow T$.

Given $\mu \in X_*(T)$, we construct an element μ^{Γ} of $\mathfrak{A}_{\mathbb{Q}}$ as follows:

$$\mu^{\Gamma} = \frac{1}{[\Gamma : \Gamma_{\mu}]} \sum_{\gamma \in \Gamma/\Gamma_{\mu}} \gamma(\mu)$$

where Γ_{μ} is the stabilizer of μ in Γ. Then $\mu^{\Gamma} \in X_*(T)^{\Gamma}_{\mathbb{Q}} = \mathfrak{A}_{\mathbb{Q}}$.

Given a standard Levi subgroup M_S, we let $W^{\mathrm{rel}}_{M_S}$ denote the relative Weyl group of M_S. The group $W^{\mathrm{rel}}_{M_S}$ is defined to be the subgroup of the relative Weyl group, W^{rel}, that is generated by the reflections corresponding to simple roots in S.

Definition 1.2.2. *We define a map*

$$\theta_{M_S} : X_*(T) \to \mathfrak{A}_{\mathbb{Q}},$$

given by

$$\theta_{M_S}(\mu) = \frac{1}{|W^{\mathrm{rel}}_{M_S}|} \sum_{\sigma \in W^{\mathrm{rel}}_{M_S}} \sigma(\mu^{\Gamma}).$$

We are now ready to describe a formalism that will prove useful in studying the cohomology of certain Rapoport-Zink spaces. Crucial to everything that follows is a partial ordering on the set $\mathcal{C}_G$ of cocharacter pairs for G.

Definition 1.2.3. *We define a partial ordering on $\mathcal{C}_G$ which we denote by the symbol $\leqslant$. Unfortunately, our definition is somewhat indirect: we first define when $(M_{S_2}, \mu_{S_2}) \leqslant (M_{S_1}, \mu_{S_1})$ for $M_{S_2} \subset M_{S_1}$ (equivalently $S_2 \subset S_1$) and $S_1 \backslash S_2$ contains a single element (in other words, M_{S_2} is* *a maximal proper Levi subgroup of M_{S_1}). We then extend the relation to all cocharacter pairs by taking the transitive closure.*

Let M_{S_2}, M_{S_1} be standard Levi subgroups of G such that $M_{S_2} \subset M_{S_1}$ and $S_1 \backslash S_2$ is a singleton. For cocharacter pairs $(M_{S_2}, \mu_{S_2}), (M_{S_1}, \mu_{S_1}) \in \mathcal{C}_G$, we write $(M_{S_2}, \mu_{S_2}) \leqslant (M_{S_1}, \mu_{S_1})$ if μ_{S_2} is conjugate to μ_{S_1} in $M_{S_1 \overline{\mathbb{Q}_p}}$ and $\theta_{M_{S_2}}(\mu_{S_2}) \succ \theta_{M_{S_1}}(\mu_{S_1})$. We then take the transitive closure to extend to a partial ordering on $\mathcal{C}_G$.

The following example shows that the above definition depends on the assumption that $S_1 \backslash S_2$ is a singleton.

Example 1.2.4. Consider $G = \mathrm{GL}_4$ with T the diagonal torus and B the upper triangular matrices. We can pick a basis for $X_*(T)$ of cocharacters $\widehat{e_i}$ defined so that $\widehat{e_i}(g)$ is the diagonal matrix with 1 in every position except for the ith, which equals g. Then we can identify an element of $X_*(T)$ with its coordinate vector in this basis. Finally, we use additional parenthesis to indicate the product structure of the standard Levi subgroup M_S. Using this notation, the set of cocharacter pairs that are less than or equal to $(\mathrm{GL}_4, (1^2, 0^2))$ is given in the diagram at the start of Appendix 1.5.1.

In particular, we see that $(\mathrm{GL}_1^4, (1)(1)(0)(0)) \leqslant (\mathrm{GL}_4, (1^2, 0^2))$ since we have a chain of cocharacter pairs where each Levi subgroup is maximal in the next:

$$\begin{aligned} &(\mathrm{GL}_1^4, (1)(1)(0)(0)) \leqslant (\mathrm{GL}_1 \times \mathrm{GL}_2 \times \mathrm{GL}_1, (1)(1,0)(0)) \\ &\leqslant (\mathrm{GL}_3 \times \mathrm{GL}_1, (1^2, 0)(0)) \leqslant (\mathrm{GL}_4, (1^2, 0^2)). \end{aligned}$$

However, it is not the case that $(\mathrm{GL}_1^4, (1)(0)(1)(0)) \leqslant (\mathrm{GL}_4, (1^2, 0^2))$ even though $\theta_{\mathrm{GL}_1^4}((1,0,1,0)) \succ \theta_{\mathrm{GL}_4}((1,1,0,0))$ and the cocharacters are conjugate in G.

Finally, we remark that the fact that all the related cocharacter pairs in the above example have equal (as opposed to just conjugate) cocharacters is very much a result of us choosing a fairly small group G. Even for $G = \mathrm{GL}_5$, this is not the case.

Definition 1.2.5. *We define a cocharacter pair (M_S, μ_S) for G to be* strictly decreasing *if $\theta_{M_S}(\mu_S) \in \mathfrak{A}^+_{M_S,\mathbb{Q}}$. We denote by $\mathcal{SD} \subset \mathcal{C}_G$ the strictly decreasing elements of $\mathcal{C}_G$ and by $\mathcal{SD}_\mu$ (for dominant $\mu \in X_*(T)$) the strictly decreasing elements $(M_S, \mu_S) \in \mathcal{C}_G$ such that $(M_S, \mu_S) \leqslant (G, \mu)$.*

Remark. The θ_{M_S} map can be thought of as associating a tuple of slopes to a cocharacter pair. Then the strictly decreasing cocharacter pairs with Levi subgroup M_S are the ones whose slope tuple lies in the image of the Newton map $\nu : \mathbf{B}(G)_{M_S} \to \mathfrak{A}_{M_S,\mathbb{Q}}$. The above statement is made precise by Proposition 1.2.21.

1.2.2. **An Alternate Characterization of the Averaging Map.** The following two subsections consist of a collection of lemmas developing the theory of the map θ_{M_S} and the set of strictly decreasing elements $\mathcal{SD}$ of $\mathcal{C}_G$.

In this section, we give an alternate description of the map θ_{M_S}. To do so, we will need several properties of cocharacters and root data which we record in the following lemma. For this lemma only, we consider T and G defined over a more general class of fields so that these results also apply to the complex dual groups $\widehat{T}$ and $\widehat{G}$.

Lemma 1.2.6. *Let $F \supset \mathbb{Q}$ be a field and $\overline{F}$ an algebraic closure. Let G be a connected quasisplit reductive group defined over F. Suppose that $T \subset G$ is a maximal torus defined over F and that the group scheme $T_{\overline{F}}$ admits an action defined over $\overline{F}$ by a finite group Λ. Let $X^*(T^\Lambda)$ denote the characters of the subgroup scheme of Λ-fixed points of $T_{\overline{F}}$. The anti-equivalence of categories between tori and finitely generated free Abelian groups given by $T_{\overline{F}} \mapsto X^*(T)$ induces an action of Λ on $X^*(T)$. We then have the following.*

(1) *There is a unique isomorphism $X^*(T^\Lambda) \cong X^*(T)_\Lambda$ such that the following diagram commutes.*

$$\begin{tikzcd} X^*(T) \arrow[r, "res"] \arrow[dr, two heads, "proj"'] & X^*(T^\Lambda) \arrow[d, leftrightarrow] \\ & X^*(T)_\Lambda \end{tikzcd}$$

(2) *Let $M_S \subset G$ be a standard Levi subgroup. Let $W^{\text{abs}}_{M_S}, W^{\text{rel}}_{M_S}$ denote the absolute and relative Weyl groups of M_S and let $\Gamma = \mathrm{Gal}(\overline{F}/F)$. Then $W_{M_S,\text{rel}}$ acts on $X_*(T)^\Gamma$ via its natural identification with $\mathfrak{A}$ and Γ acts on $X_*(T)^{W_{M_S,\text{abs}}}$ since for $w \in W_{M_S,\text{abs}}$, and $\gamma \in \Gamma$, and $\mu \in X_*(T)^{W_{M_S,\text{abs}}}$, we have $w(\gamma(\mu)) = \gamma(\gamma^{-1}(w)(\mu)) = \gamma(\mu)$. Then the identity map on $X_*(T)$ induces an isomorphism of groups*

$$(X_*(T)^{W_{M_S,\text{abs}}})^\Gamma \cong (X_*(T)^\Gamma)^{W_{M_S,\text{rel}}}$$

(3) *The natural map $X_*(T)^\Lambda_{\mathbb{Q}} \hookrightarrow X_*(T)_{\mathbb{Q}} \twoheadrightarrow X_*(T)_{\mathbb{Q},\Lambda}$ induces an isomorphism $X_*(T)^\Lambda_{\mathbb{Q}} \cong X_*(T)_{\Lambda,\mathbb{Q}}$.*

Proof. The functor $T \mapsto X^*(T)$ is an anti-equivalence between the categories of diagonalizable groups over $\overline{F}$ and finitely generated Abelian groups. The diagram for the universal property for Λ-invariants is that of Λ-coinvariants but with all the arrows reversed. Thus, there must exist a unique isomorphism between $X^*(T^\Lambda)$ and $X^*(T)_\Lambda$ that makes the diagram

$$\begin{tikzcd} X^*(T) \arrow[r, "res"] \arrow[dr, two heads, "proj"'] & X^*(T^\Lambda) \arrow[d, leftrightarrow] \\ & X^*(T)_\Lambda \end{tikzcd}$$

commute. This proves (1).

In [Kot84a, Lem 1.1.3], Kottwitz proves that the identity map on $X_*(T)$ induces an isomorphism

$$(X_*(T)^\Gamma)/W_{M_S}^{\text{rel}} \cong (X_*(T)/W_{M_S}^{\text{abs}})^\Gamma.$$

Thus, to prove (2), we need only show that this isomorphism gives a bijection of the singleton orbits. This will give an isomorphism of groups (not just sets) between $(X_*(T)^{W_{M_S,\text{abs}}})^\Gamma$ and $(X_*(T)^\Gamma)^{W_{M_S,\text{rel}}}$ that is induced from the identity map on $X_*(T)$.

Kottwitz's isomorphism maps the $W_{M_S}^{\text{rel}}$-orbit of $\mu \in X_*(T)^\Gamma$ to its $W_{M_S}^{\text{abs}}$ orbit in $X_*(T)$. Thus, it suffices to show that if $\mu \in X_*(T)^\Gamma$ is invariant by $W_{M_S}^{\text{rel}}$ then it is also invariant by $W_{M_S}^{\text{abs}}$. If μ is invariant by $W_{M_S}^{\text{rel}}$, then the pairing of μ with each relative root of M_S is 0. Thus the image of μ lies in the intersection of the kernels of the relative roots of M_S which is $Z(M_S) \cap A$. Therefore, μ is invariant under the action of $W_{M_S}^{\text{abs}}$.

Finally, we note that the proof of Kottwitz uses the fact that the intersection of the absolute Weyl chamber $\overline{C}_{\mathbb{Q}}^{\text{abs}}$ with the image of $X_*(A)$ in $X_*(T)$ gives the relative Weyl chamber $\overline{C}_{\mathbb{Q}}$. Indeed, this follows easily from the fact that the restriction of the set of absolute simple roots Δ^{abs} relative to our choice of B and T equals the set of relative simple roots Δ (see Proposition 1.5.2). An analogous fact is known for the Weyl chambers in the character group $X^*(T)$ (see Proposition 1.5.4) but this seems to be much more subtle.

For (3), we need to construct an inverse to the map

$$X_*(T)_{\mathbb{Q}}^\Lambda \hookrightarrow X_*(T)_{\mathbb{Q}} \twoheadrightarrow X_*(T)_{\mathbb{Q},\Lambda}.$$

Take $[\mu] \in X_*(T)_{\mathbb{Q},\Lambda}$ for $\mu \in X_*(T)_{\mathbb{Q}}$. Then

$$\frac{1}{\Lambda}\sum_{\lambda\in\Lambda}\lambda(\mu) \in X_*(T)_{\mathbb{Q}}^\Lambda$$

is independent of the choice of lift of $[\mu]$ to $X_*(T)_{\mathbb{Q}}$ and gives an inverse to the map above. $\square$

Let A_{M_S} be the maximal split torus in the center of M_S. Then

$$X_*(A_{M_S})_{\mathbb{Q}} \cong \mathfrak{A}_{M_S,\mathbb{Q}}.$$

We now prove a lemma that we will need to use to describe the alternate characterization of θ_{M_S}.

Lemma 1.2.7. *(1) There is a natural isomorphism $X^*(Z(\widehat{M_S})^\Gamma)_{\mathbb{Q}} \cong \mathfrak{A}_{M_S,\mathbb{Q}}$ defined via a series of canonical identifications.*

(2) The isomorphism in (1) *coincides with the one constructed in §4.4.3 of [Kot97].*

Proof. We prove (1) first. By Lemma 1.2.6, we have the following isomorphisms.

$$\begin{aligned} X^*(\widehat{T}^{W^{\text{abs}}_{M_S},\Gamma})_{\mathbb{Q}} &\cong X^*(\widehat{T})_{\mathbb{Q},W^{\text{abs}}_{M_S},\Gamma} = X_*(T)_{\mathbb{Q},W^{\text{abs}}_{M_S},\Gamma} \\ &\cong X_*(T)_{\mathbb{Q}}^{W^{\text{abs}}_{M_S},\Gamma} \quad \cong X_*(T)_{\mathbb{Q}}^{\Gamma,W^{\text{rel}}_{M_S}} \\ &\cong X_*(A_{M_S})_{\mathbb{Q}} \quad\quad \cong \mathfrak{A}_{M_S,\mathbb{Q}}. \end{aligned}$$

We explicate the isomorphism $X_*(T)_{\mathbb{Q}}^{\Gamma,W^{\text{rel}}_{M_S}} \cong X_*(A_{M_S})_{\mathbb{Q}}$. This follows from the isomorphism $X_*(A)^{W^{\text{rel}}_{M_S}} \cong X_*(A_{M_S})$ which we now describe. Suppose we have $\mu \in X_*(A)^{W^{\text{rel}}_{M_S}}$. Equivalently, for each relative root α of $\text{Lie}(M_S)$, we have $\sigma_\alpha(\mu) = \mu$ (where σ_α is the reflection in the Weyl group corresponding to α). Since $\sigma_\alpha(\mu) = \mu - \langle \mu, \alpha \rangle \check{\alpha}$, this is equivalent to $\langle \mu, \alpha \rangle = 0$ for all relative roots α of $\text{Lie}(M_S)$, which in turn is equivalent to the statement that $\text{im}(\mu) \subset \bigcap_\alpha \ker\alpha$. Finally, this is equivalent to $\text{im}(\mu) \subset Z(M_S) \cap A$. Since the image of a cocharacter is connected, we in fact have that $\mu \in X_*(A_{M_S})$.

To finish the argument, we need to construct an isomorphism

$$X^*(Z(\widehat{M_S})^\Gamma)_{\mathbb{Q}} \cong X^*(\widehat{T}^{W^{\text{abs}}_{M_S},\Gamma})_{\mathbb{Q}}.$$

Note that it is necessary to take the tensor product with $\mathbb{Q}$ here as $Z(\widehat{M_S})$ and $\widehat{T}^{W^{\text{abs}}_{M_S}}$ need not be isomorphic.

It suffices to show that

$$X^*(Z(\widehat{M_S}))_{\mathbb{Q}} \cong X^*(\widehat{T}^{W^{\text{abs}}_{M_S}})_{\mathbb{Q}}.$$

The group $Z(\widehat{M_S})$ is equal to the intersection of the kernels of the roots of $\widehat{M_S}$ and so $X^*(Z(\widehat{M_S}))$ is identified with $X^*(\widehat{T})/R$ where R is the $\mathbb{Z}$-module spanned by the roots of $\widehat{M_S}$. By Lemma 1.2.6, $X^*(\widehat{T}^{W^{\text{abs}}_{M_S}}) \cong$

$X^*(\widehat{T})_{W^{\text{abs}}_{M_S}} = X^*(\widehat{T})/D$ where D is the $\mathbb{Z}$ module spanned by $w(\mu) - \mu$ for every $w \in W^{\text{abs}}_{M_S}$ and $\mu \in X^*(\widehat{T})$. Since $Z(\widehat{M_S}) \subset \widehat{T}^{W^{\text{abs}}_{M_S}}$, we have a natural surjection

$$X^*(\widehat{T}^{W^{\text{abs}}_{M_S}}) \twoheadrightarrow X^*(Z(\widehat{M_S})).$$

By our previous discussion, the kernel of this map is R/D. Thus, to prove our claim, it suffices to show that R/D is finite. But if α is a root of $\widehat{M_S}$, then $\sigma_\alpha(\alpha) - \alpha = -2\alpha$. Thus $2R \subset D$ and so we have the desired result.

We now show (2). The map in [Kot97, §4.4.3] is defined as follows:

$$\mathfrak{A}_{M_S,\mathbb{Q}} \to X_*(T)_{\mathbb{Q}} = X^*(\widehat{T})_{\mathbb{Q}} \xrightarrow{res} X^*(Z(\widehat{M_S})^\Gamma)_{\mathbb{Q}},$$

where the final map is restriction of characters. By Lemma 1.2.6 (1), this last map is the same as the composition

$$X^*(\widehat{T})_{\mathbb{Q}} \to X^*(\widehat{T})_{\mathbb{Q}, W^{\text{abs}}_{M_S}, \Gamma} \cong X^*(\widehat{T}^{W^{\text{abs}}_{M_S}, \Gamma})_{\mathbb{Q}} \cong X^*(Z(\widehat{M_S})^\Gamma)_{\mathbb{Q}},$$

Thus, by applying Lemma 1.2.6 and the proof of Lemma 1.2.7, we get that the entire map is given by

$$\mathfrak{A}_{M_S,\mathbb{Q}} \cong X_*(T)^{\Gamma, W^{\text{rel}}_{M_S}}_{\mathbb{Q}} \cong X_*(T)^{W^{\text{abs}}_{M_S}, \Gamma}_{\mathbb{Q}} \cong X_*(T)_{\mathbb{Q}, W^{\text{abs}}_{M_S}, \Gamma},$$

$$\cong X^*(\widehat{T}^{W^{\text{abs}}_{M_S}, \Gamma})_{\mathbb{Q}} \cong X^*(Z(\widehat{M_S})^\Gamma)_{\mathbb{Q}}.$$

We observe that this is the inverse of what we wrote down above. $\square$

We are now ready to give our alternate characterization of the map θ_{M_S}.

Proposition 1.2.8. *[Alternate Characterization of θ_{M_S}] The map θ_{M_S} that was introduced in Definition 1.2.2 is equal to the composition*

$$X_*(T) = X^*(\widehat{T}) \xrightarrow{res} X^*(Z(\widehat{M_S})^\Gamma) \to X^*(Z(\widehat{M_S})^\Gamma)_{\mathbb{Q}} \cong \mathfrak{A}_{M_S,\mathbb{Q}} \subset \mathfrak{A}_{\mathbb{Q}},$$

where the final isomorphism is the one described in Lemma 1.2.7.

Proof. We recall Definition 1.2.2 where θ_{M_S} is defined to be the composition

$$X_*(T) \to X_*(T)^\Gamma_{\mathbb{Q}} \to X_*(T)^{\Gamma, W^{\text{rel}}_{M_S}}_{\mathbb{Q}} \subset \mathfrak{A}_{\mathbb{Q}},$$

where both maps are averages over the relevant group. As we now show, this is the same as the composition

$$X_*(T) \to X_*(T)^{W^{\text{abs}}_{M_S}}_{\mathbb{Q}} \to X_*(T)^{W^{\text{abs}}_{M_S}, \Gamma}_{\mathbb{Q}} \cong X_*(T)^{\Gamma, W^{\text{rel}}_{M_S}}_{\mathbb{Q}} \subset \mathfrak{A}_{\mathbb{Q}},$$

where the first two maps are averages and the third is as in Lemma 1.2.6 (2). Indeed for $\mu \in X_*(T)$,

$$\frac{1}{|W^{\mathrm{rel}}_{M_S}|}\sum_{w \in W^{\mathrm{rel}}_{M_S}}\sum_{\gamma\in\Gamma} w(\gamma(\mu)),$$

is invariant by $W^{\mathrm{abs}}_{M_S}$ by Lemma 1.2.6 (2) and so equals (keeping in mind that $W^{\mathrm{rel}}_{M_S} \subset W^{\mathrm{abs}}_{M_S}$ by Corollary 1.5.3)

$$\frac{1}{|W^{\mathrm{abs}}_{M_S}|}\sum_{w \in W^{\mathrm{abs}}_{M_S}}\sum_{\gamma\in\Gamma} w(\gamma(\mu)) = \frac{1}{|W^{\mathrm{abs}}_{M_S}|}\sum_{w \in W^{\mathrm{abs}}_{M_S}}\sum_{\gamma\in\Gamma} \gamma(w)(\gamma(\mu))$$

$$= \frac{1}{|W^{\mathrm{abs}}_{M_S}|}\sum_{w \in W^{\mathrm{abs}}_{M_S}}\sum_{\gamma\in\Gamma} \gamma(w(\mu)) = \frac{1}{|W^{\mathrm{abs}}_{M_S}|}\sum_{\gamma\in\Gamma}\sum_{w \in W^{\mathrm{abs}}_{M_S}} \gamma(w(\mu)).$$

Now, we consider the following commutative diagram.

$$\begin{tikzcd}
X_*(T)_{\mathbb{Q}} \arrow[rr, "avg"] \arrow[dr] & & X_*(T)_{\mathbb{Q}}^{W^{\mathrm{abs}}_{M_S}} \arrow[rr, "avg"] \arrow[dr] & & X_*(T)_{\mathbb{Q}}^{W^{\mathrm{abs}}_{M_S},\Gamma} \\
& X_*(T)_{\mathbb{Q},W^{\mathrm{abs}}_{M_S}} \arrow[ur, "avg"] \arrow[dr] & & X_*(T)_{\mathbb{Q},\Gamma}^{W^{\mathrm{abs}}_{M_S}} \arrow[ur, "avg"] & \\
& & X_*(T)_{\mathbb{Q},W^{\mathrm{abs}}_{M_S},\Gamma} \arrow[ur, "avg"] & &
\end{tikzcd}$$

The commutativity essentially follows from the definition of the averaging maps. The benefit of this is that now we can write θ_{M_S} as the composition of

$$X_*(T) \to X_*(T)_{W^{\mathrm{abs}}_{M_S}} \to X_*(T)_{W^{\mathrm{abs}}_{M_S},\Gamma} \to X_*(T)_{\mathbb{Q},W^{\mathrm{abs}}_{M_S},\Gamma}$$
$$\to X^*(T)_{\mathbb{Q},\Gamma}^{W^{\mathrm{abs}}_{M_S}} \to X_*(T)_{\mathbb{Q}}^{W^{\mathrm{abs}}_{M_S},\Gamma} \cong X_*(T)^{\Gamma,W^{\mathrm{rel}}_{M_S}} \subset \mathfrak{A}_{\mathbb{Q}}$$

where we no longer need to base change the first three spaces to $\mathbb{Q}$ because denominators are not introduced in the maps until later.

Using the equality between cocharacters of T and characters of $\widehat{T}$, we rewrite this as

$$X_*(T) = X^*(\widehat{T}) \to X^*(\widehat{T})_{W^{\mathrm{abs}}_{M_S}} \to X^*(\widehat{T})_{W^{\mathrm{abs}}_{M_S},\Gamma} \to X^*(\widehat{T})_{\mathbb{Q},W^{\mathrm{abs}}_{M_S},\Gamma}$$
$$\to X^*(\widehat{T})_{\mathbb{Q},\Gamma}^{W^{\mathrm{abs}}_{M_S}} \to X^*(\widehat{T})_{\mathbb{Q}}^{W^{\mathrm{abs}}_{M_S},\Gamma} = X_*(T)_{\mathbb{Q}}^{W^{\mathrm{abs}}_{M_S},\Gamma} \cong X_*(T)^{\Gamma,W^{\mathrm{rel}}_{M_S}} \subset \mathfrak{A}_{\mathbb{Q}}.$$

Now we invoke Lemma 1.2.6 (1) to get that the above composition is equal to

$$X_*(T) = X^*(\widehat{T}) \xrightarrow{res} X^*(\widehat{T}^{W^{\mathrm{abs}}_{M_S},\Gamma}) \to X^*(\widehat{T}^{W^{\mathrm{abs}}_{M_S},\Gamma})_{\mathbb{Q}} \cong X^*(\widehat{T})_{\mathbb{Q},W^{\mathrm{abs}}_{M_S},\Gamma}$$
$$\to X^*(\widehat{T})^{W^{\mathrm{abs}}_{M_S}}_{\mathbb{Q},\Gamma} \to X^*(\widehat{T})^{W^{\mathrm{abs}}_{M_S},\Gamma}_{\mathbb{Q}} = X_*(T)^{W^{\mathrm{abs}}_{M_S},\Gamma}_{\mathbb{Q}} \cong X_*(T)^{\Gamma,W^{\mathrm{rel}}_{M_S}} \subset \mathfrak{A}_{\mathbb{Q}}.$$

The final step is to observe that we have a commutative diagram

$$\begin{tikzcd} X^*(\widehat{T}^{W^{\mathrm{abs}}_{M_S},\Gamma}) \arrow[r] \arrow[d, "res"] & X^*(\widehat{T}^{W^{\mathrm{abs}}_{M_S},\Gamma})_{\mathbb{Q}} \arrow[d, "\sim"] \\ X^*(Z(\widehat{M_S})^{\Gamma}) \arrow[r] & X^*(Z(\widehat{M_S})^{\Gamma})_{\mathbb{Q}}. \end{tikzcd}$$

Thus, the previous expression equals

$$X_*(T) = X^*(\widehat{T}) \xrightarrow{res} X^*(\widehat{T}^{W^{\mathrm{abs}}_{M_S},\Gamma}) \xrightarrow{res} X^*(Z(\widehat{M_S})^{\Gamma}) \to X^*(Z(\widehat{M_S})^{\Gamma})_{\mathbb{Q}}$$
$$\cong X^*(\widehat{T}^{W^{\mathrm{abs}}_{M_S},\Gamma})_{\mathbb{Q}} \cong X^*(\widehat{T})_{\mathbb{Q},W^{\mathrm{abs}}_{M_S},\Gamma} \to X^*(\widehat{T})^{W^{\mathrm{abs}}_{M_S}}_{\mathbb{Q},\Gamma}$$
$$\to X^*(\widehat{T})^{W^{\mathrm{abs}}_{M_S},\Gamma}_{\mathbb{Q}} = X_*(T)^{W^{\mathrm{abs}}_{M_S},\Gamma}_{\mathbb{Q}} \cong X_*(T)^{\Gamma,W^{\mathrm{rel}}_{M_S}} \subset \mathfrak{A}_{\mathbb{Q}}.$$

comparing with Lemma 1.2.7, we can rewrite θ_{M_S} as

$$X_*(T) = X^*(\widehat{T}) \xrightarrow{res} X^*(Z(\widehat{M_S})^{\Gamma}) \to X^*(Z(\widehat{M_S})^{\Gamma})_{\mathbb{Q}} \cong \mathfrak{A}_{M_S,\mathbb{Q}} \subset \mathfrak{A}_{\mathbb{Q}}$$

as desired.

□

We record the following useful corollary of the ideas discussed in the above argument.

Corollary 1.2.9. *Suppose that $\mu, \mu' \in X_*(T)$ are conjugate in $M_{S,\overline{\mathbb{Q}_p}}$. Then $\theta_{M_S}(\mu) = \theta_{M_S}(\mu')$.*

Proof. By the observation at the start of Proposition 1.2.8, θ_{M_S} is equivalently defined as the composition

$$X_*(T) \to X_*(T)^{W^{\mathrm{abs}}_{M_S}}_{\mathbb{Q}} \to X_*(T)^{W^{\mathrm{abs}}_{M_S},\Gamma}_{\mathbb{Q}} \cong X_*(T)^{\Gamma,W^{\mathrm{rel}}_{M_S}}_{\mathbb{Q}} \subset \mathfrak{A}_{\mathbb{Q}}.$$

In particular, μ and μ' are mapped to the same element under the first map in the above composition. □

1.2.3. **Strictly Decreasing Cocharacter Pairs.** In this section, we prove a number of properties of strictly decreasing cocharacter pairs and their relation to the partial order we defined in Definition 1.2.3. As always, we let σ_α denote the reflection in the relative Weyl group corresponding to the relative root α.

Lemma 1.2.10. *If $x \in \mathfrak{A}_{\mathbb{Q}}$ is dominant, then*

$$y = \frac{1}{|W^{\mathrm{rel}}_{M_S}|} \sum_{\sigma \in W^{\mathrm{rel}}_{M_S}} \sigma(x)$$

is also dominant. If in addition, $\langle x, \alpha \rangle > 0$ for some $\alpha \in \Delta \backslash S$, then we also have $\langle y, \alpha \rangle > 0$.

Proof. For the first part of the lemma, we claim that if we can show that $\langle \sigma(x), \alpha \rangle \geqslant 0$ for each $\sigma \in W^{\mathrm{rel}}_{M_S}$ and $\alpha \in \Delta \backslash S$, then we are done. This follows because if a collection of cocharacters pair non-negatively with α, then so will their average. Thus for $\alpha \in \Delta \backslash S$, we get $\langle y, \alpha \rangle \geqslant 0$. For $\alpha \in S$, we automatically have $\langle y, \alpha \rangle = 0$ since $0 = y - \sigma_\alpha(y) = \langle y, \alpha \rangle \check{\alpha}$.

Pick $\alpha \in \Delta \backslash S$. Then the root group of α is contained in the unipotent radical N_S of P_S. The group N_S is normalized by M_S. In particular, for any $\sigma \in W^{\mathrm{rel}}_{M_S}$, the root group of $\sigma^{-1}(\alpha)$ is contained in N_S and hence $\sigma^{-1}(\alpha)$ is also a positive root. Thus $\langle \sigma(x), \alpha \rangle = \langle x, \sigma^{-1}(\alpha) \rangle \geqslant 0$ as desired.

To prove the second part, we notice since $\langle x, \alpha \rangle > 0$, the term in y corresponding to $\sigma = 1$ has positive pairing with α. Since all the other terms have non-negative pairing with α, we must have that $\langle y, \alpha \rangle > 0$. □

Lemma 1.2.11. *If x as in the previous lemma is dominant, then*

$$\frac{1}{|W^{\mathrm{rel}}_{M_S}|} \sum_{\sigma \in W^{\mathrm{rel}}_{M_S}} \sigma(x) \preceq x$$

Proof. It suffices to show that for any $\sigma \in W^{\mathrm{rel}}_{M_S}$, we have $\sigma(x) \preceq x$. This is a standard fact ([Bou68, Ch6 1.6.18, p. 158]). □

Corollary 1.2.12. *Let $(M_S, \mu_S) \in \mathcal{SD}$ be a strictly decreasing cocharacter pair and let $(M_{S'}, \mu_{S'}) \in \mathcal{C}_G$ and suppose that $(M_S, \mu_S) \leqslant (M_{S'}, \mu_{S'})$. Then $(M_{S'}, \mu_{S'}) \in \mathcal{SD}$.*

Proof. We need to show that for each $\beta \in \Delta \backslash S'$, that $\langle \theta_{M_{S'}}(\mu_{S'}), \beta \rangle > 0$. By 1.2.9, $\theta_{M_{S'}}(\mu_{S'}) = \theta_{M_{S'}}(\mu_S)$. Further, we observe that

$$\theta_{M_{S'}}(\mu_S) = \frac{1}{|W^{\mathrm{rel}}_{M_{S'}}|} \sum_{\sigma \in W^{\mathrm{rel}}_{M_{S'}}} \sigma(\theta_{M_S}(\mu_S)). \tag{1}$$

Since $\theta_{M_S}(\mu_S)$ is dominant by assumption and satisfies $\langle \theta_{M_S}(\mu_S), \beta \rangle > 0$, we can apply 1.2.10 to get the desired result. □

The following easy uniqueness result is quite useful.

Lemma 1.2.13. *Let* $(M_{S_1}, \mu_{S_1}), (M_{S_2}, \mu_{S_2}), (M_{S_2'}, \mu_{S_2'}) \in \mathcal{C}_G$. *Suppose further that* $(M_{S_1}, \mu_{S_1}) \leqslant (M_{S_2}, \mu_{S_2})$, *that* $(M_{S_1}, \mu_{S_1}) \leqslant (M_{S_2'}, \mu_{S_2'})$. *If* $M_{S_2} = M_{S_2'}$, *then* $(M_{S_2}, \mu_{S_2}) = (M_{S_2'}, \mu_{S_2'})$.

Proof. By definition, $\mu_{S_1}, \mu_{S_2}, \mu_{S_2'}$ are all conjugate in M_{S_2}. But also, μ_{S_2} and $\mu_{S_2'}$ are dominant in the absolute root system. Thus they are equal. $\square$

We now define the notion of a cocharacter pair being strictly decreasing relative to a Levi subgroup.

Definition 1.2.14. *Let* $M_S \subsetneq M_{S'}$ *be standard Levi subgroups of* G. *We say* (M_S, μ_S) *is strictly decreasing relative to* $M_{S'}$ *if* $\langle \theta_{M_S}(\mu_S), \alpha \rangle > 0$ *for* $\alpha \in S' \backslash S$.

Remark. Recall that by construction, $\langle \theta_{M_S}(\mu_S), \alpha \rangle = 0$ for $\alpha \in S$. Thus, $(M_S, \mu_S) \in \mathcal{SD}$ exactly when it is strictly decreasing relative to G.

Lemma 1.2.15. *Let* $(M_{S_1}, \mu_{S_1}), (M_{S_1'}, \mu_{S_1'}) \in \mathcal{C}_G$ *be cocharacter pairs such that* $(M_{S_1}, \mu_{S_1}) \leqslant (M_{S_1'}, \mu_{S_1'})$. *Let* $M_{S_2} \supset M_{S_1}$ *be a standard Levi subgroup of* G *and suppose* (M_{S_1}, μ_{S_1}) *is strictly decreasing relative to* M_{S_2}. *Then* $(M_{S_1'}, \mu_{S_1'})$ *is strictly decreasing relative to* $M_{S_1' \cup S_2}$.

Proof. We first reduce to the case where M_{S_1} is a maximal Levi subgroup of $M_{S_1'}$ (i.e. $S_1' = S_1 \cup \{\alpha\}$ for some $\alpha \in \Delta \backslash S_1$). To do so, we recognize that the relation $(M_{S_1}, \mu_{S_1}) \leqslant (M_{S_1'}, \mu_{S_1'})$ definitionally implies that there is a finite sequence of cocharacter pairs

$$(M_{S_1}, \mu_{S_1}) = (M_{S^0}, \mu_{S^0}) \leqslant \ldots \leqslant (M_{S^k}, \mu_{S^k}) = (M_{S_1'}, \mu_{S_1'})$$

where each M_{S^i} is a maximal Levi subgroup of $M_{S^{i+1}}$. Thus, if we prove the lemma in the maximal Levi subgroup case, we can inductively prove it in the general case.

We now assume that $M_{S_1} \subset M_{S_1'}$ is a maximal Levi subgroup so that $S_1' = S_1 \cup \{\alpha\}$ for some $\alpha \in \Delta \backslash S_1$. We need to show that $\langle \theta_{M_{S_1'}}(\mu_{S_1'}), \beta \rangle > 0$ for each $\beta \in S_1' \cup S_2 \backslash S_1'$. First note that any such β is an element of $S_2 \backslash S_1$. By Corollary 1.2.9, since μ_{S_1} and $\mu_{S_1'}$ are conjugate in $M_{S_1'}$, we have $\theta_{M_{S_1'}}(\mu_{S_1}) = \theta_{M_{S_1'}}(\mu_{S_1'})$. Thus we are reduced to showing $\langle \theta_{M_{S_1'}}(\mu_{S_1}), \beta \rangle > 0$ for $\beta \in S_2 \backslash S_1$.

Note that since (M_{S_1}, μ_{S_1}) is strictly decreasing relative to M_{S_2}, we have $\theta_{M_{S_1}}(\mu_{S_1})$ is dominant relative to the root datum of M_{S_2} and $\langle \theta_{M_{S_1}}(\mu_{S_1}), \beta \rangle > 0$. Therefore, by Equation (1) and Lemma 1.2.10, $\langle \theta_{M_{S_1'}}(\mu_{S_1}), \beta \rangle > 0$ as desired. $\square$

Proposition 1.2.16. *Let $(M_S, \mu_S) \in \mathcal{C}_G$ and suppose it is strictly decreasing relative to some standard Levi subgroup $M_{S'} \supset M_S$. Then there is a unique $(M_{S'}, \mu_{S'}) \in \mathcal{C}_G$ such that $(M_S, \mu_S) \leqslant (M_{S'}, \mu_{S'})$. We call $(M_{S'}, \mu_{S'})$ the* extension *of (M_S, μ_S) to $M_{S'}$.*

In the case where $S' = S \cup \{\alpha\}$ for $\alpha \in \Delta \backslash S$, the converse is true. Specifically, if $(M_S, \mu_S) \in \mathcal{C}_G$ and there exists $(M_{S'}, \mu_{S'}) \in \mathcal{C}_G$ satisfying $(M_{S'}, \mu_{S'}) \geqslant (M_S, \mu_S)$ with $S' = S \cup \{\alpha\}$, then (M_S, μ_S) is strictly decreasing relative to $M_{S'}$.

Proof. We begin by proving the first statement. Uniqueness follows from Lemma 1.2.13. For existence, we first reduce to the case where M_S is a maximal Levi subgroup of $M_{S'}$. Suppose we have proven the proposition in this reduced case. We might then try to prove the general case by iteratively applying the reduced case of the proposition to a chain of standard Levi subgroups $M_S = M_{S_0} \subset \ldots \subset M_{S_k} = M_{S'}$ such that each is maximal in the next. Such a chain clearly exists, but to apply the reduced case of the proposition we need to show that if we have constructed a cocharacter pair $(M_{S_i}, \mu_{S_i}) \geqslant (M_S, \mu_S)$ then (M_{S_i}, μ_{S_i}) is strictly decreasing relative to $M_{S'}$. This follows from Lemma 1.2.15.

Now, we let $\mu_{S'}$ be the unique conjugate of μ_S which is dominant in $M_{S'}$. If we can show that $\theta_{M_{S'}}(\mu_{S'}) \prec \theta_{M_S}(\mu_S)$, then $(M_{S'}, \mu_{S'})$ will satisfy the conditions of the proposition. By Corollary 1.2.9 and Equation (1),

$$\theta_{M_{S'}}(\mu_{S'}) = \theta_{M_{S'}}(\mu_S) = \frac{1}{|W_{M_{S'}}|} \sum_{\sigma \in W_{M_{S'}}} \sigma(\theta_{M_S}(\mu_S)),$$

so we can reduce to showing that

$$\frac{1}{|W_{M_{S'}}|} \sum_{\sigma \in W_{M_{S'}}} \sigma(y) \prec y,$$

for any y satisfying $\langle y, \alpha \rangle > 0$ for $\alpha \in S' \backslash S$ and $\langle y, \alpha \rangle = 0$ for $\alpha \in S$. Any such y is dominant in the root datum of $M_{S'}$ and so by Lemma 1.2.11,

$$\frac{1}{|W_{M_{S'}}|} \sum_{\sigma \in W_{M_{S'}}} \sigma(y) \preceq y.$$

Further, the above equation cannot be an equality because y has positive pairing with each root of $S' \backslash S$ while $\frac{1}{|W_{M_{S'}}|} \sum_{\sigma \in W_{M_{S'}}} \sigma(y)$ has 0 pairing with these roots.

To prove the converse, suppose that $(M_S,\mu_S)\leqslant(M_{S'},\mu_{S'})$ and $S'=S\cup\{\alpha\}$ for some $\alpha\in\Delta\backslash S$. Then by Corollary 1.2.9

$$\theta_{M_{S'}}(\mu_{S'})=\theta_{M_{S'}}(\mu_S)=\frac{\theta_{M_S}(\mu_S)+\sigma_\alpha(\theta_{M_S}(\mu_S))}{2},$$

and so

$$\theta_{M_S}(\mu_S)-\theta_{M_{S'}}(\mu_{S'})=\frac{\theta_{M_S}(\mu_S)-\sigma_\alpha(\theta_{M_S}(\mu_S))}{2}=\frac{1}{2}\langle\theta_{M_S}(\mu_S),\alpha\rangle\check{\alpha}.$$

Since by assumption $\theta_{M_{S'}}(\mu_{S'})\prec\theta_{M_S}(\mu_S)$, it follows that $\langle\theta_{M_S}(\mu_S),\alpha\rangle>0$. □

Remark. Note that the converse of the above proposition is not true in the general case.

Corollary 1.2.17. *Fix a standard Levi subgroup M_S and roots $\alpha_1,\alpha_2\in\Delta\backslash S$. Suppose we have cocharacter pairs $(M_S,\mu_S),(M_{S\cup\{\alpha_1\}},\mu_{S\cup\{\alpha_1\}}),(M_{S\cup\{\alpha_1,\alpha_2\}},\mu_{S\cup\{\alpha_1,\alpha_2\}})\in\mathcal{C}_G$ satisfying*

$$(M_S,\mu_S)\leqslant(M_{S\cup\{\alpha_1\}},\mu_{S\cup\{\alpha_1\}})\leqslant(M_{S\cup\{\alpha_1,\alpha_2\}},\mu_{S\cup\{\alpha_1,\alpha_2\}})$$

and that (M_S,μ_S) is strictly decreasing relative to $M_{S\cup\{\alpha_2\}}$.

Then the extension of (M_S,μ_S) to $M_{S\cup\{\alpha_2\}}$, which we denote $(M_{S\cup\{\alpha_2\}},\mu_{S\cup\{\alpha_2\}})$, satisfies

$$(M_S,\mu_S)\leqslant(M_{S\cup\{\alpha_2\}},\mu_{S\cup\{\alpha_2\}})\leqslant(M_{S\cup\{\alpha_1,\alpha_2\}},\mu_{S\cup\{\alpha_1,\alpha_2\}})$$

Proof. By the second statement of Proposition 1.2.16, we have that (M_S,μ_S) is strictly decreasing relative to $M_{S\cup\{\alpha_1\}}$. Then by Lemma 1.2.15, $(M_{S\cup\{\alpha_2\}},\mu_{S\cup\{\alpha_2\}})$ is strictly decreasing relative to $M_{S\cup\{\alpha_1,\alpha_2\}}$. Thus by Proposition 1.2.16, we have $(M_{S\cup\{\alpha_2\}},\mu_{S\cup\{\alpha_2\}})\leqslant(M_{S\cup\{\alpha_1,\alpha_2\}},\mu_{S\cup\{\alpha_1,\alpha_2\}})$ as desired. □

Proposition 1.2.18. *Let $S\subset S_1\subset S_2$ be subsets of Δ and suppose $(M_S,\mu_S),(M_{S_2},\mu_{S_2})\in\mathcal{C}_G$ with*

$$(M_S,\mu_S)\leqslant(M_{S_2},\mu_{S_2})$$

and (M_S,μ_S) is strictly decreasing relative to M_{S_1}. Then the unique extension (M_{S_1},μ_{S_1}) of (M_S,μ_S) to M_{S_1} satisfies

$$(M_{S_1},\mu_{S_1})\leqslant(M_{S_2},\mu_{S_2}).$$

Proof. Since $(M_S,\mu_S)\leqslant(M_{S_2},\mu_{S_2})$, there is an increasing chain of cocharacter pairs

$$(M_S,\mu_S)=(M_{S^0},\mu_{S^0})\leqslant\ldots\leqslant(M_{S^k},\mu_{S^k})=(M_{S_2},\mu_{S_2})$$

such that each standard Levi subgroup is maximal in the next. The content of this proposition is that we can pick a chain such that (M_{S_1},μ_{S_1})

appears. By Lemma 1.2.15, we can assume that M_S is maximal in M_{S_1}. Let α be the unique element of $S_1 \backslash S$.

Pick a chain of cocharacter pairs $(M_S, \mu_S) = (M_{S^0}, \mu_{S^0}) \leqslant ... \leqslant (M_{S^k}, \mu_{S^k}) = (M_{S_2}, \mu_{S_2})$ as above. Chains of cocharacter pairs are determined by an ordering on the roots in $S_2 \backslash S = \{\alpha_1, ..., \alpha_k\}$, such that the $S^i = S \cup \{\alpha_1, ..., \alpha_i\}$. The root α appears in this chain so $\alpha = \alpha_i$ for some i. If $i = 1$ we are done. Otherwise, we consider $(M_{S^{i-2}}, \mu_{S^{i-2}}) \leqslant (M_{S^{i-1}}, \mu_{S^{i-1}}) \leqslant (M_{S^i}, \mu_{S^i})$. By Lemma 1.2.15, $(M_{S^{i-2}}, \mu_{S^{i-2}})$ is strictly decreasing relative to $M_{S^{i-2} \cup \{\alpha\}}$ and so by Corollary 1.2.17 (applied so that $(M_{S^{i-2}}, \mu_{S^{i-2}})$ takes the place of (M_S, μ_S) in Corollary 1.2.17), we get a new chain of cocharacter pairs between (M_S, μ_S) and (M_{S_2}, μ_{S_2}) where we switch the positions of α, α_{i-1} in the corresponding ordering of $S_2 \backslash S$. By repeating this argument, we can construct a chain where $\alpha = \alpha_1$, which is what we need. $\square$

The preceding propositions give us the following picture. Given a cocharacter pair (M_S, μ_S) we check which simple roots α satisfy $\langle \theta_{M_S}(\mu_S), \alpha \rangle > 0$. Suppose there are n such simple roots. Then we get 2^n standard Levi subgroups containing M_S corresponding to adding different subsets of these simple roots. The cocharacter pair (M_S, μ_S) has a unique extension to each of the Levi subgroups and the poset lattice of these co-character pairs can be thought of as the graph of an n dimensional cube in the following way. The vertices of the cube are the 2^n cocharacter pairs extending (M_S, μ_S) that we have just constructed. For two such pairs $(M_{S_1}, \mu_{S_1}), (M_{S_2}, \mu_{S_2})$, we draw an edge between the two corresponding vertices if either $S_1 \subset S_2$ and $|S_2 \backslash S_1| = 1$, or $S_2 \subset S_1$ and $|S_1 \backslash S_2| = 1$. We can upgrade this graph to a directed graph by stipulating that an edge between (M_{S_1}, μ_{S_1}) and (M_{S_2}, μ_{S_2}) is directed from (M_{S_1}, μ_{S_1}) to (M_{S_2}, μ_{S_2}) if $(M_{S_2}, \mu_{S_2}) < (M_{S_1}, \mu_{S_1})$.

Finally, note that for any two pairs (M_{S_1}, μ_{S_1}) and (M_{S_2}, μ_{S_2}) corresponding to vertices in the above cube, we have $(M_{S_2}, \mu_{S_2}) \leqslant (M_{S_1}, \mu_{S_1})$ if and only if there is a directed path in the cube travelling from the vertex of (M_{S_1}, μ_{S_1}) to that of (M_{S_2}, μ_{S_2}).

1.2.4. **Connection With Isocrystals.** We now investigate the relation between strictly decreasing cocharacter pairs and Kottwitz's theory of isocrystals with additional structure. See [Kot97] for omitted details on the theory of isocrystals.

An isocrystal is a pair (V, Φ) where V is a finite dimensional $\widehat{\mathbb{Q}_p^{ur}}$ vector space and $\Phi : V \to V$ is an additive transformation satisfying $\Phi(av) = \sigma(a)\Phi(v)$ for $a \in \widehat{\mathbb{Q}_p^{ur}}, v \in V$ and σ the arithmetic Frobenius morphism. As before, let G be a connected quasisplit reductive group defined over $\mathbb{Q}_p$ and consider the set of isomorphism classes of exact $\otimes$-functors from $\mathrm{Rep}(G)$ to Isoc, the category of isocrystals. Such isomorphism classes are classified by $H^1(W_{\mathbb{Q}_p}, G(\widehat{\overline{\mathbb{Q}_p^{ur}}}))$ which we denote $\mathbf{B}(G)$ (where $W_{\mathbb{Q}_p}$ is the Weil group of $\mathbb{Q}_p$).

In §4.2 of [Kot97], Kottwitz constructs the Newton map $\nu : \mathbf{B}(G) \to \overline{C}_{\mathbb{Q}}$ and the Kottwitz map $\kappa : \mathbf{B}(G) \to X^*(Z(\widehat{G})^\Gamma)$. An element of $\mathbf{B}(G)$ is uniquely determined by its image under these maps.

We say that the standard Levi subgroup M_S is associated to $b \in \mathbf{B}(G)$ if $\nu(b) \in \mathfrak{A}^+_{M_S,\mathbb{Q}}$. Henceforth, we will often denote the standard Levi subgroup associated to b by M_b. Notice that many elements of $\mathbf{B}(G)$ could be associated to the same Levi subgroup. We call b *basic* if $M_b = G$. We write

$$\mathbf{B}(G) = \coprod_{S \subset \Delta} \mathbf{B}(G)_{M_S}$$

such that $\mathbf{B}(G)_{M_S}$ consists of those $b \in \mathbf{B}(G)$ associated to M_S. We denote by $\mathbf{B}(M_S)^+$ the maximal subset of $\mathbf{B}(M_S)$ such that $\nu(\mathbf{B}(M_S)^+) \subset \overline{C}_{\mathbb{Q}}$. In §5.1 of [Kot97], Kottwitz uses the Kottwitz map for M_S to construct canonical bijections

$$\mathbf{B}(G)_{M_S} \cong \mathbf{B}(M_S)^+_{M_S} \cong X^*(Z(\widehat{M_S})^\Gamma)^+ \tag{2}$$

where Kottwitz constructs a canonical isomorphism

$$X^*(Z(\widehat{M_S})^\Gamma)_{\mathbb{Q}} \cong \mathfrak{A}_{M_S,\mathbb{Q}} \tag{3}$$

and $X^*(Z(\widehat{M_S})^\Gamma)^+$ denotes the subset of $X^*(Z(\widehat{M_S})^\Gamma)$ mapping to $\mathfrak{A}^+_{M_S,\mathbb{Q}}$. In fact, Kottwitz shows that the composition of the above isomorphisms gives the Newton map

$$\mathbf{B}(G)_{M_S} \to \mathfrak{A}^+_{M_S,\mathbb{Q}} \hookrightarrow \overline{C}_{\mathbb{Q}}.$$

For a further discussion of Equation (3), we refer the reader to Lemma 1.2.7.

We now prove an important lemma that will be used to relate the set $\mathbf{B}(G)$ to the strictly decreasing elements of $\mathcal{C}_G$.

Lemma 1.2.19. *Fix a standard Levi subgroup M_S of G and let $(M_S, \mu_S) \in \mathcal{SD}$. Then $\theta_{M_S}(\mu_S) \in \nu(\mathbf{B}(G)_{M_S})$.*

Proof. We first describe the set $\nu(\mathbf{B}(G)_{M_S})$. By Equations (2) and (3), the set $\nu(\mathbf{B}(G)_{M_S})$ is equal to the image of $X^*(Z(\widehat{M_S})^\Gamma)^+$ in $\mathfrak{A}_{M_S,\mathbb{Q}}$. Thus, to prove this lemma, it suffices to show that θ_{M_S} factors through the map $X^*(Z(\widehat{M_S})^\Gamma) \hookrightarrow X^*(Z(\widehat{M_S})^\Gamma)_{\mathbb{Q}} \cong \mathfrak{A}_{M_S,\mathbb{Q}}$ where the isomorphism is as in Equation (3) or Lemma 1.2.7. Then, since (M_S, μ_S) is strictly decreasing, the factoring of θ_{M_S} will map μ_S to an element of

$X^*(Z(\widehat{M_S})^\Gamma)^+$ as desired. That θ_{M_S} factors in this way follows from the alternate characterization of θ_{M_S} given in Proposition 1.2.8. $\square$

Definition 1.2.20. *Fix $\mu \in X_*(T)$. Then we recall the following definition of Kottwitz [Kot97, §6.2]:*

$$\mathbf{B}(G, \mu) := \{b \in \mathbf{B}(G) : \nu(b) \preceq \theta_T(\mu), \kappa(b) = \mu|_{Z(\hat{G})^\Gamma}\}.$$

Now we prove the key result of this section, which permits us to associate an element of $\mathbf{B}(G)$ to each strictly decreasing cocharacter pair.

Proposition 1.2.21. *We have a natural map*

$$\mathcal{T} : \mathcal{SD} \to \mathbf{B}(G)$$

defined as follows. Let $(M_S, \mu_S) \in \mathcal{SD}$. Then there exists a $b \in \mathbf{B}(G)$ so that $\kappa(b) = \mu_S|_{Z(\hat{G})^\Gamma}$ and $\nu(b) = \theta_{M_S}(\mu_S)$. We note that by construction, b is unique. Then we define $\mathcal{T}((M_S, \mu_S)) = b$. Furthermore, we show that

$$\mathcal{T}(\mathcal{SD}_\mu) \subset \mathbf{B}(G, \mu).$$

Proof. We first define b. Note that since (M_S, μ_S) is strictly decreasing, $\theta_{M_S}(\mu_S) \in \mathfrak{A}^+_{M_S,\mathbb{Q}}$. By Proposition 1.2.8, it follows that $\mu_S|_{Z(\widehat{M_S})^\Gamma} \in X^*(Z(\widehat{M_S})^\Gamma)^+$ and so we can define b to be the element of $\mathbf{B}(G)$ corresponding to $\mu_S|_{Z(\widehat{M_S})^\Gamma}$ under the isomorphism $B(G)_{M_S} \cong X^*(Z(\widehat{M_S})^\Gamma)^+$ of Equation (2). Recall that the composition of this isomorphism with Equation (3) induces the Newton map restricted to $\mathbf{B}(G)_{M_S}$. Thus, we have $\theta_{M_S}(\mu_S) = \nu(b)$. Equation (4.9.2) of [Kot97] implies that $\kappa(b) = \mu_S|_{Z(\hat{G})^\Gamma}$.

It remains to show that if $(M_S, \mu_S) \in \mathcal{SD}_\mu$ then the element $b \in \mathbf{B}(G)$ that we have constructed lies in the set $\mathbf{B}(G, \mu)$. For this, we need to show that $\nu(b) = \theta_{M_S}(\mu_S) \preceq \theta_T(\mu)$.

We claim that $\theta_T(\mu) \succeq \theta_T(\mu_S)$. After all, by ([Bou68, Ch6 1.6.18, p. 158]), we have $\mu \succeq \mu_S$. Then the claim follows from Corollary 1.5.5.

Now we claim that $\theta_T(\mu_S)$ is dominant in the relative root system of M_S. To prove the claim, we first observe that μ_S is dominant relative to the absolute root system of M_S. As above, the Galois group Γ preserves the Weyl chamber corresponding to the positive absolute roots given by B. Thus, $\gamma(\mu_S)$ is dominant for each $\gamma \in \Gamma$, and so $\theta_T(\mu_S)$ is dominant relative to the absolute roots of M_S. The intersection of the closed positive Weyl chamber for the absolute root datum of M_S with $\mathfrak{A}_\mathbb{Q}$ is the Weyl chamber for relative root datum of M_S (cf. proof of Lemma 1.2.6 (2)). Thus, $\theta_T(\mu_S)$ is dominant with respect to the relative roots as desired.

Finally, we apply Lemma 1.2.11 and Equation (1) to get

$$\theta_T(\mu_S) \succeq \theta_{M_S}(\mu_S),$$

which finishes the proof. □

QUESTION 1.2.1. Can one describe the image

$$\mathcal{T}(\mathcal{SD}_\mu) \subset \mathbf{B}(G,\mu)?$$

Fix $G = GL_n$ with T and B the diagonal maximal torus and upper triangular Borel subgroup respectively. Suppose μ has weights 1 and 0. Then we claim $\mathcal{T}(\mathcal{SD}_\mu) = \mathbf{B}(G,\mu)$. Indeed, pick any $b \in \mathbf{B}(GL_n,\mu)$. Then without loss of generality, $\nu_b = ((a_1/b_1)^{x_1 b_1}, ..., (a_r/b_r)^{x_r b_r})$ for some $a_i, b_i \in \mathbb{N}$ such that a_i/b_i is written in reduced form. Then let M be the standard Levi subgroup isomorphic to $GL_{x_1 b_1} \times ... \times GL_{x_r b_r}$ and embedded diagonally. Since $b \in \mathbf{B}(GL_n,\mu)$, we must have that $\mu = (1^{\sum_{i=1}^r x_i a_i}, 0^{n - \sum_{i=1}^r x_i a_i})$. Finally, we define $\mu' \in X_*(T)$ by $\mu' = (1^{x_1 a_1}, 0^{x_1 b_1 - x_1 a_1}, ..., 1^{x_r a_r}, 0^{x_r b_r - x_r a_r})$. Then we note that μ' is dominant in the root system of M so that $(M,\mu') \in \mathcal{C}_G$. Moreover, $\theta_M(\mu') = \nu_b$ so that $(M,\mu') \in \mathcal{SD}$. Then since μ' and μ are conjugate in GL_n, it is easy to see that $(M,\mu') \leqslant (GL_n,\mu)$. In conclusion, we have shown that $(M',\mu') \in \mathcal{SD}_\mu$ and $\mathcal{T}((M',\mu')) = b$ as desired.

On the other hand for different choices of μ, we can have $\mathcal{T}(\mathcal{SD}_\mu) \subsetneq \mathbf{B}(G,\mu)$. For instance, let $G = GL_3$, let $\mu = (2,0,0)$, and let $b \in \mathbf{B}(G,\mu)$ be such that $\nu_b = (1, 1/2, 1/2)$. Then it is easy to check that $\mathcal{T}(\mathcal{SD}_\mu)$ does not contain b.

1.2.5. **The Induction and Sum Formulas.** We are now ready to prove our main theorems on cocharacter pairs. We begin by defining some key subsets of $\mathcal{C}_G$, the set of cocharacter pairs for G. In this section we fix a dominant $\mu \in X_*(T)$ and $b \in \mathbf{B}(G,\mu)$.

Definition 1.2.22. *We define the sets $\mathcal{T}_{G,b,\mu}$ and $\mathcal{R}_{G,b,\mu}$ as follows:*

$$\mathcal{T}_{G,b,\mu} := \mathcal{T}^{-1}(b) \cap \mathcal{SD}_\mu$$

and

$$\mathcal{R}_{G,b,\mu} := \{(M_{S_1},\mu_{S_1}) \in \mathcal{C}_G : (M_{S_1},\mu_{S_1}) \leqslant (M_{S_2},\mu_{S_2}) \text{ for a } (M_{S_2},\mu_{S_2}) \in \mathcal{T}_{G,b,\mu}\}.$$

Definition 1.2.23. *Let $\mathbb{Z}\langle \mathcal{C}_G \rangle$ denote the free Abelian group generated by the set of cocharacter pairs for G.*

We define $\mathcal{M}_{G,b,\mu} \in \mathbb{Z}\langle \mathcal{C}_G \rangle$ by

$$\mathcal{M}_{G,b,\mu} = \sum_{(M_S,\mu_S) \in \mathcal{R}_{G,b,\mu}} (-1)^{L_{M_S,M_b}} (M_S,\mu_S)$$

such that for $M_{S_1} \subset M_{S_2}$, $L_{M_{S_1},M_{S_2}}$ is defined to be $|S_2 \backslash S_1|$.

Remark. We observe that for $(M_S, \mu_S) \in \mathcal{SD}$, if $\mathcal{T}((M_S, \mu_S)) = b$, then $M_S = M_b$.

We will show in Theorem 1.3.12 that at least in the case where G is an unramified restriction of scalars of a general linear group, $\mathcal{M}_{G,b,\mu}$ is related to the cohomology of Rapoport-Zink spaces for G. Thus one expects there to be a combinatorial analogue of the Harris-Viehmann conjecture (Conjecture 1.3.3). We call this combinatorial analogue the *induction formula*. Perhaps the more surprising result is that there is also an analogue of Shin's averaging formula (which we call the *sum formula*) [Shi12b, Thm 7.5]. We first prove the sum formula.

Theorem 1.2.24 (Sum Formula)**.** *The following holds in $\mathbb{Z}\langle \mathcal{C}_G \rangle$:*

$$\sum_{b \in B(G,\mu)} \mathcal{M}_{G,b,\mu} = (G, \mu).$$

Proof. We need to show that

$$\sum_{b \in B(G,\mu)} \mathcal{M}_{G,b,\mu} = (G, \mu),$$

or equivalently

$$\sum_{b \in B(G,\mu)} \sum_{(M_S,\mu_S) \in \mathcal{R}_{G,b,\mu}} (-1)^{L_{M_S,M_b}} (M_S, \mu_S) = (G, \mu).$$

We prove this equality by counting how many times a given cocharacter pair shows up on the left-hand side. The pair (G, μ) shows up exactly once in the left-hand sum as an element of $\mathcal{R}_{G,b,\mu}$ for b the unique basic element of $\mathbf{B}(G, \mu)$. Suppose$(M_S, \mu_S) \in \mathcal{C}_G$ is some other cocharacter pair. Then define

$$Y_{(M_S,\mu_S)} = \{b \in \mathbf{B}(G, \mu) : (M_S, \mu_S) \in \mathcal{R}_{G,b,\mu}\}.$$

We are reduced to showing

$$\sum_{b \in Y_{(M_S,\mu_S)}} (-1)^{L_{M_S,M_b}} = 0. \tag{4}$$

Our general strategy will be to show that the left-hand side of equation 4 vanishes for each $(M_S, \mu_S) < (G, \mu)$ by inducting on the size of $\Delta \backslash S$. However, in the case that $(M_S, \mu_S) \in \mathcal{SD}_\mu$, we can prove the vanishing without an inductive argument. We show this first before discussing the induction.

Suppose now that $(M_S, \mu_S) \in \mathcal{SD}_\mu$. By Corollary 1.2.12, every pair $(M_{S'}, \mu_{S'}) \in \mathcal{C}_G$ satisfying $(M_S, \mu_S) \leqslant (M_{S'}, \mu_{S'}) \leqslant (G, \mu)$ is strictly decreasing and thus by Proposition 1.2.21, we have $\mathcal{T}((M_{S'}, \mu_{S'})) \in \mathbf{B}(G, \mu)$. These are precisely the elements $b \in \mathbf{B}(G, \mu)$ so that $(M_S, \mu_S) \in \mathcal{R}_{G,b,\mu}$. By the discussion after Proposition 1.2.18, we can associate the graph of a cube to the set of $(M_{S'}, \mu_{S'})$ such that each cocharacter pair is a vertex. To the vertex associated to $(M_{S'}, \mu_{S'})$ we attach the sign $(-1)^{L_{M_S, M'_S}}$. We note that adjacent vertices in this graph will have opposite signs since if $(M_{S'}, \mu_{S'})$ and $(M_{S''}, \mu_{S''})$ have adjacent vertices, then the cardinality of S' and S'' differs by 1. Now, it is a standard fact that if we associate an element of $\{1, -1\}$ to each vertex of the graph of an n-dimensional cube for $n \geqslant 1$ so that adjacent vertices have opposite signs, then the sum of all the signs is 0. This implies that the left-hand side of Equation (4) vanishes in the strictly decreasing case.

Now we discuss the inductive argument. The base case will be for pairs $(M_S, \mu_S) < (G, \mu)$ satisfying $|\Delta \backslash S| = 1$. The second statement of Proposition 1.2.16 implies that in this case (M_S, μ_S) is strictly decreasing relative to G, which means that $(M_S, \mu_S) \in \mathcal{SD}_\mu$. Thus, the base case is proven by the previous paragraph.

We now discuss the inductive step. Suppose $(M_S, \mu_S) < (G, \mu)$. If (M_S, μ_S) is strictly decreasing, then we are done by the above. Suppose now that (M_S, μ_S) is not strictly decreasing. We claim that (M_S, μ_S) must be strictly decreasing with respect to at least some standard Levi subgroup of G that properly contains M_S. After all, since $(M_S, \mu_S) < (G, \mu)$, there must exist at least some $\alpha \in \Delta \backslash S$ and $(M_{S \cup \{\alpha\}}, \mu_{S \cup \{\alpha\}}) \in \mathcal{C}_G$ so that $(M_S, \mu_S) \leqslant (M_{S \cup \{\alpha\}}, \mu_{S \cup \{\alpha\}})$. Then by Proposition 1.2.16, this implies that (M_S, μ_S) is strictly decreasing relative to $M_{S \cup \{\alpha\}}$.

Thus, let $M_{S'}$ be the maximal standard Levi subgroup of G such that (M_S, μ_S) is strictly decreasing relative to $M_{S'}$. We can write $S' = S \cup \{\alpha_1, ..., \alpha_n\}$ where $\alpha_i \neq \alpha_j$ for $i \neq j$ and each $\alpha_i \in \Delta \backslash S$. We denote by X the n-cube of cocharacter pairs above (M_S, μ_S) as in the discussion after Proposition 1.2.18.

We claim that

$$\sum_{b \in Y_{(M_S, \mu_S)}} (-1)^{L_{M_S, M_b}}$$
$$= - \sum_{(M_{S'}, \mu_{S'}) \in X \backslash \{(M_S, \mu_S)\}} \sum_{b \in Y_{(M_{S'}, \mu_{S'})}} (-1)^{L_{M_{S'}, M_b}}.$$

Given this claim, we see that to finish the proof, it suffices to show that the right-hand side is identically 0. However, the right-hand side consists of a sum of a number of terms similar to the left-hand side but for pairs $(M_{S'}, \mu_{S'})$ in place of (M_S, μ_S). Note that each S' is strictly larger than S and thus we are done by induction.

We now prove the claim. Moving all the terms to one side, we need only show that

$$\sum_{(M_{S'},\mu_{S'})\in X} \sum_{b\in Y_{(M_{S'},\mu_{S'})}} (-1)^{L_{M_{S'},M_b}} = 0.$$

Fix $b \in \mathbf{B}(G,\mu)$. Then it suffices to show the contribution from b in the above formula vanishes. Thus, we must show

$$\sum_{(M_{S'},\mu_{S'})\in X\cap\mathcal{R}_{G,b,\mu}} (-1)^{L_{M_{S'},M_b}} = 0. \tag{5}$$

We examine the structure of $X \cap \mathcal{R}_{G,b,\mu}$ when it is nonempty. If we can show that the cocharacter pairs in this set form a sub-cube of X of positive dimension, then we will be done by the standard fact that if we place alternating signs on the vertices of a cube and add up all the signs we get 0.

Clearly, any $(M_{S'},\mu_{S'}) \in X \cap \mathcal{R}_{G,b,\mu}$ must satisfy $M_S \subset M_{S'} \subset M_b$. The subset of X satisfying this latter property forms a sub-cube of X since its elements are indexed by subsets of $S_b \backslash S$, where S_b is the subset of Δ corresponding to M_b in the standard way (note that by Lemma 1.2.13, there is at most one element of $X \cap \mathcal{R}_{G,b,\mu}$ for each standard Levi $M_{S'}$). Moreover, this latter set cannot form a cube of dimension 0 for then we would have $M_S = M_b$ and so $X \cap \mathcal{R}_{G,b,\mu} = \{(M_S,\mu_S)\}$ which would imply that (M_S,μ_S) is strictly decreasing contrary to assumption.

Thus to finish the proof, we need only show that every $(M_{S'},\mu_{S'})$ such that

(1) $M_S \subset M_{S'} \subset M_b$,

(2) $(M_S,\mu_S) \leqslant (M_{S'},\mu_{S'})$,

(3) (M_S,μ_S) is strictly decreasing relative to $M_{S'}$,

satisfies $(M_{S'},\mu_{S'}) \leqslant (M_b,\mu_b)$ for some $(M_b,\mu_b) \in \mathcal{T}_{G,b,\mu}$. Since we assumed that $X \cap \mathcal{R}_{G,b,\mu} \neq \varnothing$, then in fact there is an $(M_b,\mu_b) \in \mathcal{T}_{G,b,\mu}$ with $(M_S,\mu_S) \leqslant (M_b,\mu_b)$. Then the desired result follows from Proposition 1.2.18. $\square$

We now turn to the induction formula. Fix a standard Levi subgroup M_S of G. Then our choice of maximal torus T and Borel subgroup B of G provides us with natural choices $B \cap M_S$ and T of a Borel subgroup and maximal torus of M_S. This allows us to define the set $\mathcal{C}_{M_S}$ of cocharacter pairs for M_S. There is a natural inclusion

$$i^G_{M_S} : \mathcal{C}_{M_S} \hookrightarrow \mathcal{C}_G. \tag{6}$$

The image of this inclusion is precisely the set of cocharacter pairs $(M_{S'}, \mu_{S'})$ where $S' \subset S$. This inclusion preserves the partial ordering of cocharacter pairs. The strictly decreasing elements of $\mathcal{C}_{M_S}$ map to the elements of $\mathcal{C}_G$ which are strictly decreasing relative to M_S.

Now choose a $b \in \mathbf{B}(G, \mu)$ and rational Levi M_S such that $M_b \subset M_S \subset G$. We have a unique $b' \in \mathbf{B}(M_b)^+_{M_b}$ corresponding to b under the isomorphism given by Equation (2). The inclusion $M_b \subset M_S$ induces a map

$$\mathbf{B}(M_b) \to \mathbf{B}(M_S).$$

Let b_S be the image of b' under this map.

The following definition will be important in relating cocharacter pairs of a group G to those of a standard Levi. Compare with [RV14, Equation (8.1)].

Definition 1.2.25. *Let M_S be a standard Levi subgroup of G, let $\mu \in X_*(T)$ be a dominant cocharacter and choose $b \in \mathbf{B}(G, \mu)$. We take $b_S \in \mathbf{B}(M_S)$ as constructed in the previous paragraph and define the set* $\mathcal{I}^{G,\mu}_{M_S, b_S} = \{(M_S, \mu_S) \in \mathcal{C}_{M_S} : b_S \in \mathbf{B}(M_S, \mu_S), \mu_S$ *is conjugate to* μ *in* $G\}$.

We first check the following transitivity property of $\mathcal{I}^{G,\mu}_{M_S, b_S}$.

Proposition 1.2.26. *Fix $(G, \mu) \in \mathcal{C}_G$ and $b \in \mathbf{B}(G, \mu)$. Suppose M_{S_2} and M_{S_1} are standard Levi subgroups of G such that $M_b \subset M_{S_2} \subset M_{S_1}$. Then*

$$\mathcal{I}^{G,\mu}_{M_{S_2}, b_{S_2}} =$$

$$\{(M_{S_2}, \mu_{S_2}) \in \mathcal{C}_{M_{S_2}} : (M_{S_2}, \mu_{S_2}) \in \mathcal{I}^{M_{S_1}, \mu_{S_1}}_{M_{S_2}, b_{S_2}} \text{ for a } (M_{S_1}, \mu_{S_1}) \in \mathcal{I}^{G,\mu}_{M_{S_1}, b_{S_1}}\}.$$

Proof. We show each set is a subset of the other. Take $(M_{S_2}, \mu_{S_2}) \in \mathcal{I}^{G,\mu}_{M_{S_2}, b_{S_2}}$. Let μ_{S_1} be the unique dominant cocharacter conjugate to μ_{S_2} in M_{S_1}. Then we consider (M_{S_1}, μ_{S_1}) as an element of $\mathcal{C}_{M_{S_1}}$ and just need to show that $b_{S_1} \in \mathbf{B}(M_{S_1}, \mu_{S_1})$ since we already know that $b_{S_2} \in \mathbf{B}(M_{S_2}, \mu_{S_2})$ by assumption. Thus, we need only show that $\nu(b_{S_1}) \leqslant \theta_T(\mu_{S_1})$ and $\kappa(b_{S_1}) = \mu_{S_1}|_{Z(\widehat{M_{S_1}})^\Gamma}$.

We prove the inequality first. By assumption, $\nu(b_{S_2}) \preceq \theta_T(\mu_{S_2})$ and by Equations (2) and (3), $\nu(b_{S_1}) = \nu(b) = \nu(b_{S_2})$. Since μ_{S_1} and μ_{S_2} are conjugate in M_{S_1} and μ_{S_1} is dominant, it follows from [Bou68, Ch6 1.6.18, p. 158] that $\mu_{S_2} \preceq \mu_{S_1}$. Then, by Corollary 1.5.5 it follows that

$\theta_T(\mu_{S_2}) \preceq \theta_T(\mu_{S_1})$ in the relative root system. Combining all this data, we get

$$\nu(b_{S_1}) = \nu(b_{S_2}) \preceq \theta_T(\mu_{S_2}) \preceq \theta_T(\mu_{S_1}),$$

as desired.

To prove $\kappa(b_{S_1}) = \mu_{S_1}|_{Z(\widehat{M_{S_1}})^\Gamma}$, we note that by Equation (4.9.2) of [Kot97] and the fact that $b_{S_2} \in \mathbf{B}(M_{S_2}, \mu_{S_2})$, we have

$$\kappa(b_{S_1}) = \mu_{S_2}|_{Z(\widehat{M_{S_1}})^\Gamma}.$$

Then μ_{S_1} and μ_{S_2} are conjugate in M_{S_1} so there exists a $w \in W^{\text{abs}}_{M_{S_1}}$ so that $w(\mu_1) = \mu_2$. This implies that μ_1 and μ_2 are conjugate in $\widehat{M_{S_1}}$ and in particular equal when restricted to $Z(\widehat{M_{S_1}})$. This implies the desired equality.

To show the converse inclusion, we start with $(M_{S_2}, \mu_{S_2}) \in \mathcal{I}^{M_{S_1}, \mu_{S_1}}_{M_{S_2}, b_{S_2}}$ for some $(M_{S_1}, \mu_{S_1}) \in \mathcal{I}^{G,\mu}_{M_{S_1}, b_{S_1}}$ and need to show that $b_{S_2} \in \mathbf{B}(M_{S_2}, \mu_{S_2})$ and that μ_{S_2} is conjugate to μ in G. But $(M_{S_2}, \mu_{S_2}) \in \mathcal{I}^{M_{S_1}, \mu_{S_1}}_{M_{S_2}, b_{S_2}}$ implies that $b_{S_2} \in \mathbf{B}(M_{S_2}, \mu_{S_2})$ and also that μ_{S_2} is conjugate to μ_{S_1} in M_{S_1}. Further, $(M_{S_1}, \mu_{S_1}) \in \mathcal{I}^{G,\mu}_{M_{S_1}, b_{S_1}}$ implies that μ_{S_1} is conjugate to μ in G. Thus, μ_{S_2} is conjugate to μ in G as desired. $\square$

The set $\mathcal{I}^{G,\mu}_{M_S, b_S}$ will primarily be useful because it allows us to relate the set $\mathcal{T}_{G,b,\mu}$ to analogous constructions in M_S. This is encapsulated in the following proposition.

Proposition 1.2.27. *Fix M_S, μ and b as in Definition 1.2.25. The natural inclusion $i^G_{M_S} : \mathcal{C}_{M_S} \hookrightarrow \mathcal{C}_G$ of Equation (6) induces a bijection*

$$\coprod_{(M_S, \mu_S) \in \mathcal{I}^{G,\mu}_{M_S, b_S}} \mathcal{T}_{M_S, b_S, \mu_S} \cong \mathcal{T}_{G,b,\mu}$$

Proof. We first show that

$$i^G_{M_S}\Big(\coprod_{(M_S, \mu_S) \in \mathcal{I}^{G,\mu}_{M_S, b_S}} \mathcal{T}_{M_S, b_S, \mu_S}\Big) \supset \mathcal{T}_{G,b,\mu}.$$

Since $M_b \subset M_S$, it follows from the discussion after Equation (6) that

$$\mathcal{T}_{G,b,\mu} \subset i^G_{M_S}(\mathcal{C}_{M_S}).$$

Thus, pick an arbitrary element of $\mathcal{T}_{G,b,\mu}$ of the form $i^G_{M_S}(M_b, \mu_b)$ for $(M_b, \mu_b) \in \mathcal{C}_{M_S}$. The cocharacter pair $i^G_{M_S}(M_b, \mu_b)$ is strictly decreasing, and therefore so is $(M_b, \mu_b) \in \mathcal{C}_{M_S}$. By Proposition 1.2.16 we can find $(M_S, \mu_S) \in \mathcal{C}_{M_S}$ such that $(M_b, \mu_b) \leqslant (M_S, \mu_S)$. Observe that since $i^G_{M_S}(M_b, \mu_b) \leqslant (G, \mu)$, the cocharacter μ_b is conjugate to μ in G and therefore μ_S must be as well by construction. If we can show that $\mathcal{T}((M_b, \mu_b)) = b_S$, then we will be done because by Proposition 1.2.21, this implies that $b_S \in \mathbf{B}(M_S, \mu_S)$ and so therefore that $(M_S, \mu_S) \in \mathcal{I}^{G,\mu}_{M_S,b_S}$ and $(M_b, \mu_b) \in \mathcal{T}_{M_S,b_S,\mu_S}$.

By assumption, $\mathcal{T}(i^G_{M_S}(M_b, \mu_b)) = b \in \mathbf{B}(G, \mu)$. Recall that the map $\mathcal{T}$ is defined so that a strictly decreasing $(M_b, \mu_b) \in \mathcal{C}_G$ which satisfies $(M_b, \mu_b) \leqslant (G, \mu)$ is mapped first to the element $\mu_b|_{Z(\widehat{M_b})^\Gamma} \in X^*(Z(\widehat{M_b})^\Gamma)^+$. Then, this element is identified with an element of $\mathbf{B}(G)$ via the isomorphisms of Equation (2):

$$X^*(Z(\widehat{M_b}))^\Gamma)^+ \cong \mathbf{B}(M_b)^+_{M_b} \cong \mathbf{B}(G)_{M_b},$$

where the second isomorphism above is induced by the inclusion $M_b \hookrightarrow G$. We have the commutative diagram

$$\begin{tikzcd} \mathbf{B}(M_b) \arrow[r] \arrow[dr] & \mathbf{B}(M_S) \arrow[d] \\ & \mathbf{B}(G) \end{tikzcd}$$

where each map is induced from the inclusion of groups. By definition, the element $b' \in \mathbf{B}(M_b)^+$ maps to $b \in \mathbf{B}(G)$ and $b_S \in \mathbf{B}(M_S)$ respectively. Thus, we see that by construction, $\mathcal{T}((M_b, \mu_b)) = b_S$.

Conversely, suppose $(M_b, \mu_b) \in \mathcal{T}_{M_S,b_S,\mu_S}$ for some $(M_S, \mu_S) \in \mathcal{I}^{G,\mu}_{M_S,b_S}$. Since $b' \in \mathbf{B}(M_b)^+_{M_b}$, it follows from the definition of b_S and $\mathcal{T}_{M_S,b_S,\mu_S}$ that $\mu_b|_{Z(\widehat{M_b})^\Gamma}$ is an element of $X^*(Z(\widehat{M_b})^\Gamma)^+$. This implies that $i^G_{M_S}(M_b, \mu_b) \in \mathcal{SD}$. By Proposition 1.2.16, we have an extension of $i^G_{M_S}(M_b, \mu_b)$ to G, and since μ_b and μ are conjugate in G by assumption, it follows that this extension is (G, μ) so that $i^G_{M_S}(M_b, \mu_b) \leqslant (G, \mu)$. It follows from these facts that $i^G_{M_S}(M_b, \mu_b) \in \mathcal{T}_{G,b,\mu}$.

Finally, we remark that for distinct $(M_S, \mu_S), (M_S, \mu'_S) \in \mathcal{I}^{G,\mu}_{M_S,b_S}$ the sets $\mathcal{T}_{M_S,b_S,\mu_S}$ and $\mathcal{T}_{M_S,b_S,\mu'_S}$ are indeed disjoint by Lemma 1.2.13. □

As a corollary of this result, we have the induction formula.

Corollary 1.2.28 (Induction Formula)**.** *We continue using the notation of the previous proposition. The natural map*

$$i^G_{M_S} : \mathcal{C}_{M_S} \hookrightarrow \mathcal{C}_G,$$

induces a map

$$i^G_{M_S} : \mathbb{Z}\langle \mathcal{C}_{M_S} \rangle \hookrightarrow \mathbb{Z}\langle \mathcal{C}_G \rangle,$$

which gives an equality

$$\sum_{(M_S,\mu_S)\in\mathcal{I}^{G,\mu}_{M_S,b_S}} i^G_{M_S}(\mathcal{M}_{M_S,b_S,\mu_S}) = \mathcal{M}_{G,b,\mu}.$$

Proof. It follows from Proposition 1.2.27 that the map $i^G_{M_S}$ induces a bijection

$$\coprod_{(M_S,\mu_S)\in\mathcal{I}^{G,\mu}_{M_S,b_S}} \mathcal{R}_{M_S,b_S,\mu_S} \cong \mathcal{R}_{G,b,\mu}.$$

We remark that for distinct $(M_S,\mu_S),(M_S,\mu'_S)\in\mathcal{I}^{G,\mu}_{M_S,b_S}$ we have $\mathcal{R}_{M_S,b_S,\mu_S}\cap\mathcal{R}_{M_S,b_S,\mu'_S}=\varnothing$ by Lemma 1.2.13.

The corollary then follows from the definition of $\mathcal{M}_{G,b,\mu}$. □

This result can be thought of as an analogue of the *Harris-Viehmann* conjecture which we discuss in the next section.

In the cases we are interested in, we will also need a description of how cocharacter pairs behave with respect to products.

Suppose $G = G_1\times \ldots \times G_k$ and $T = T_1 \times \ldots \times T_k$ such that T_i is a maximal torus for G_i. Then

$$X_*(T)\cong X_*(T_1)\oplus \ldots \oplus X_*(T_k),$$

and any standard Levi subgroup admits a product decomposition

$$M_S \cong M_{S_1}\times \ldots \times M_{S_k},$$

such that $T_i \subset M_{S_i}\subset G_i$. Then any cocharacter pair (M_S,μ_S) of G corresponds to a tuple of cocharacter pairs

$$((M_{S_1},\mu_{S_1}),\ldots,(M_{S_k},\mu_{S_k}))\in\mathcal{C}_{G_1}\times\ldots\times\mathcal{C}_{G_k},$$

in the obvious way. The pair $(M_S,\mu_S)\in\mathcal{C}_G$ is strictly decreasing if and only if each pair $(M_{S_i},\mu_{S_i})\in\mathcal{C}_{G_i}$ is, and if $\mathcal{T}((M_S,\mu_S)) = b\in\mathbf{B}(G,\mu)$, then we also have $\mathcal{T}_i((M_{S_i},\mu_{S_i})) = b_i\in\mathbf{B}(G_i,\mu_i)$ where $\mathcal{T}_i$ is the map $\mathcal{T}$ defined for the group G_i. Thus, $b\mapsto(b_1,\ldots,b_k)$ under the natural bijection

$$\mathbf{B}(G)\cong\mathbf{B}(G_1)\times\ldots\times\mathbf{B}(G_k).$$

We record the following proposition

Proposition 1.2.29. *We use the notation of the previous two paragraphs.*

The natural bijection

$$\mathcal{C}_G\cong\mathcal{C}_{G_1}\times\ldots\times\mathcal{C}_{G_k},$$

induces bijections

$$\mathcal{T}_{G,b,\mu}\cong\mathcal{T}_{G_1,b_1,\mu_1}\times\ldots\times\mathcal{T}_{G_k,b_k,\mu_k},$$

and
$$\mathcal{R}_{G,b,\mu} \cong \mathcal{R}_{G_1,b_1,\mu_1} \times \ldots \times \mathcal{R}_{G_k,b_k,\mu_k}.$$
Further, under the natural isomorphism $\mathbb{Z}\langle \mathcal{C}_G \rangle \cong \mathbb{Z}\langle \mathcal{C}_{G_1} \rangle \otimes \ldots \otimes \mathbb{Z}\langle \mathcal{C}_{G_k} \rangle$ *we have*
$$\mathcal{M}_{G,b,\mu} = \mathcal{M}_{G_1,b_1,\mu_1} \otimes \ldots \otimes \mathcal{M}_{G_k,b_k,\mu_k}.$$

1.3. Cohomology of Rapoport-Zink spaces and the Harris-Viehmann Conjecture

In this section, we define the Rapoport-Zink spaces we will work with and show how we can describe their cohomology using the language developed in the previous section. We also give a statement of the Harris-Viehmann conjecture, and explain the necessity of a small correction to the conjecture. We follow [Far04], [Shi12b], and [RV14].

The theory necessarily involves several choices of signs. This is often a point of confusion, so we describe our conventions here. We choose the cocharacter μ appearing in the definition of Rapoport-Zink spaces to have non-negative weights, in agreement with most authors. In this part, we use the contravariant Dieudonne functor, which means that our p-divisible groups will have isocrystals in the set $\mathbf{B}(G,\mu)$ (as opposed to $\mathbf{B}(G,-\mu)$ for the covariant theory). This convention agrees with that of [Far04] and [RV14], but [Shi12b] uses the opposite convention. We use the local Langlands correspondence for $\mathrm{GL}_n(\mathbb{Q}_p)$ as in [HT01, pg. 2]. In particular, we normalize the local Artin map so that uniformizers correspond to geometric Frobenius elements.

1.3.1. **Rapoport-Zink Spaces of EL-Type.** We fix the following notation. Suppose G is a reductive group defined over a field k and $\mu \in X_*(G)$. Then if H is a subgroup of G such that μ factors through the inclusion $X_*(H) \hookrightarrow X_*(G)$, we denote by $\{\mu\}_H$ the $H(\overline{k})$ conjugacy class of μ and by $E_{\{\mu\}_H}$ the field of definition of $\{\mu\}_H$ (i.e the smallest extension of k so that each element of $\mathrm{Gal}(\overline{k}/E_{\{\mu\}_H})$ stabilizes $\{\mu\}_H$).

Now we define the Rapoport-Zink data we consider.

Definition 1.3.1. *An* unramified Rapoport-Zink datum of EL type *is a tuple*
$(F, V, \{\mu\}_G, b)$ *where*

(1) F *is a finite unramified extension of* $\mathbb{Q}_p$*,*
(2) V *is a finite dimensional* F *vector space,*
(3) $G := \mathrm{Res}_{F/\mathbb{Q}_p}(\mathrm{GL}_F(V))$*,*
(4) $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}_p}} \to G_{\overline{\mathbb{Q}_p}}$ *is a cocharacter inducing a weight decomposition* $V \otimes \widehat{\mathbb{Q}_p^{ur}} \cong V_0 \oplus V_1$ *where* $\mu(z)$ *acts by* z^i *on* V_i*,*

(5) $b \in \mathbf{B}(G,\mu)$.

We fix a Borel subgroup $B \subset G$ defined over $\mathbb{Q}_p$, a $\mathbb{Q}_p$-split torus $A \subset G$ of maximal rank in G and such that $A \subset B$, and a maximal torus $T \subset B$ containing A and defined over $\mathbb{Q}_p$. We can choose μ in the above definition so that it is dominant relative to B.

Let $\mathbb{X}$ be a p-divisible group defined over $\overline{\mathbb{F}_p}$ with an action of $\mathcal{O}_F$ and such that the isocrystal attached to $\mathbb{X}$ by the contravariant Dieudonne functor is isomorphic to $(V_F, b\sigma)$. We consider the moduli functor $\mathbb{M}_{b,\mu}$ such that for S a scheme over $\mathcal{O}_{\widehat{\mathbb{Q}_p^{ur}}}$ with p locally nilpotent, $\mathbb{M}_{b,\mu}(S) = \{(X,i,\rho)\}/\sim$. Where X is a p-divisible group defined over S, $i : \mathcal{O}_F \to \mathrm{End}_F(X)$, and $\rho : \mathbb{X} \times_{\overline{\mathbb{F}_p}} \overline{S} \to \overline{X}$ is a quasi-isogeny ($\overline{S}, \overline{X}$ are the reductions modulo p).

By work of Rapoport and Zink [RZ96, Thm 3.25], the above moduli problem is represented by a formal scheme over $\mathcal{O}_{\widehat{\mathbb{Q}_p^{ur}}}$ which we also denote by $\mathbb{M}_{b,\mu}$. We have the generic fiber $\mathbb{M}_{b,\mu}^{rig}$ which is a rigid analytic space over $\widehat{\mathbb{Q}_p^{ur}}$. Further, we get a tower of coverings $\mathbb{M}_{b,\mu,U}^{rig}$ of $\mathbb{M}_{b,\mu}^{rig}$ for each compact open subgroup $U \subset G(\mathbb{Q}_p)$.

For a fixed prime $l \neq p$, we denote by $H_c^j(\mathbb{M}_{b,\mu,U}^{rig} \times \overline{\widehat{\mathbb{Q}_p^{ur}}}, \overline{\mathbb{Q}_l})$ the etale cohomology with compact supports. This is a $\overline{\mathbb{Q}_l}$ vector space which is a smooth representation of $J_b(\mathbb{Q}_p) \times W_{E_{\{\mu\}_G}}$, where J_b is the inner form of the standard Levi subgroup M_b associated to b (as constructed in §3.3 of [Kot97]) and $W_{E_{\{\mu\}_G}}$ is the Weil group of $E_{\{\mu\}_G}$ (for example see [RV14, Prop 6.1]).

We use the notation $\mathrm{Groth}(\cdot)$ for the Grothendieck group of admissible representations of topological groups. See §$I.2$ of [HT01] for the precise definition of these Grothendieck groups.

Let P_b be the standard parabolic subgroup with Levi factor M_b and denote the opposite parabolic by P_b^{op}. We define J_P^G, Jac_P^G to be the normalized and un-normalized Jacquet module functors, and we define I_P^G, Ind_P^G to be the normalized and un-normalized parabolic induction functors. Often, if $M \subset P$ is the standard Levi subgroup of P and we are taking I_P^G or $I_{P^{op}}^G$ to be a map of Grothendieck groups, we will write I_M^G to remind the reader that these maps do not depend on choice of P, P^{op} when considered as maps of Grothedieck groups.

In [Man05], Mantovan considers the following construction (see also [Shi12b]). We define a map

$$\mathrm{Mant}_{G,b,\mu} : \mathrm{Groth}(J_b(\mathbb{Q}_p)) \to \mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_{\{\mu\}_G}}),$$

by

$$\mathrm{Mant}_{G,b,\mu}(\rho) =$$

$$\sum_{i,j\geqslant 0}(-1)^{i+j}\varinjlim_{U\subset G(\mathbb{Q}_p)}\mathrm{Ext}^i_{J_b(\mathbb{Q}_l)}(H^j_c(\mathbb{M}^{rig}_{b,\mu,U}\times\overline{\hat{\mathbb{Q}}^{ur}_p},\overline{\mathbb{Q}_l}),\rho)(-\mathrm{dim}\mathbb{M}^{rig}_{b,\mu,U}).$$

In §6.2 of [Shi12b] and §2.4 of [Shi11], Shin considers a map

$$\mathrm{Red}_b : \mathrm{Groth}(G(\mathbb{Q}_p)) \to \mathrm{Groth}(J_b(\mathbb{Q}_p)).$$

We follow the construction given in [Shi11][1]. We define Red_b by

$$\pi \mapsto e(J_b)(\mathrm{LJ}\circ \mathrm{J}^G_{P_b^{op}}(\pi)\otimes\delta^{\frac{1}{2}}_{P_b}),$$

where

$$LJ : \mathrm{Groth}(M_b(\mathbb{Q}_p)) \to \mathrm{Groth}(J_b(\mathbb{Q}_p)),$$

is the map defined by Badulescu extending the inverse Jacquet - Langlands correspondence (see [Bad07, Prop 3.2]) and $e(J_b)$ is the Kottwitz sign as defined in [Kot83].

We now describe the main result of [Shi12b]. The cocharacter μ of G is a map $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}_p}} \to \prod_{\tau\in\mathrm{Hom}(F,\overline{\mathbb{Q}_p})} GL_{n,\overline{\mathbb{Q}_p}}$ such that the weights in each GL_n factor are 1s or 0s. Thus we let p_τ, q_τ denote the number of 1 and 0 weights respectively in the factor corresponding to τ.

The following formula is the main theorem in [Shi12b, Thm 7.5].

Theorem 1.3.2 (Shin)**.** *We have the following equality for accessible representations in* $\mathrm{Groth}(G(\mathbb{Q}_p)\times W_{E_{\{\mu\}_G}})$.

$$\sum_{b\in B(G,\mu)}\mathrm{Mant}_{b,\mu}(\mathrm{Red}_b(\pi)) = [\pi][r_{-\mu}\circ \mathrm{LL}(\pi)|_{W_{E_{\{\mu\}_G}}}\otimes|\cdot|^{-\sum_\tau \mathrm{p}_\tau\mathrm{q}_\tau/2}].$$

Loosely speaking, accessible representations in Shin's paper are character twists of the local components of global representations that can be found within the cohomology of Shimura varieties. Shin shows that all essentially square-integrable representations are accessible.

In this case LL is the semisimplified local Langlands correspondence (known by the work of [HT01] for instance). The map $r_{-\mu}$ is the algebraic representation of $\widehat{G}\rtimes W_{E_{\{\mu\}_G}}\subset {}^LG$ defined by Kottwitz ([Kot84a, Lem 2.1.2]). It is characterized by the fact that $r_{-\mu}|_{\widehat{G}}$ is the irreducible representation of extreme weight $-\mu$ and if we take a Γ-invariant splitting of $\widehat{G}$, then the subgroup $W_{E_{\{\mu\}_G}}$ of ${}^LG$ acts trivially on the highest weight vector of $r_{-\mu}$ associated with this splitting.

[1]We believe the construction given before Lemma 6.2 of [Shi12b] has a slight typo. There, Red_b is defined by $\pi\mapsto e(J_b)(\mathrm{LJ}\circ\mathrm{Jac}^G_{P_b^{op}}(\pi))$. As maps of Grothendieck groups, $\mathrm{Jac}^G_{P_b^{op}} = \mathrm{J}^G_{P_b^{op}}\otimes\delta^{\frac{1}{2}}_{P_b^{op}} = \mathrm{J}^G_{P_b^{op}}\otimes\delta^{-\frac{1}{2}}_{P_b}$. But this is not equal to $\mathrm{J}^G_{P_b^{op}}(\pi)\otimes\delta^{\frac{1}{2}}_{P_b}$, which is the construction given in [Shi11] that is compatible with [HT01].

Remark. The Tate twist appearing on the right-hand side of the above formula comes from the dimension formula for Shimura varieties and is equal to $-\langle \rho_G, \mu \rangle$ where ρ_G is the half sum of the positive roots in G.

The above theorem is analogous to the sum formula for cocharacter pairs (Theorem 1.2.24). The induction formula (Corollary 1.2.28) is related to the Harris-Viehmann conjecture (Conjecture 1.3.3 in this document). A proof of this conjecture is expected to appear in forthcoming work of Scholze.

1.3.2. **Harris-Viehmann Conjecture.** We now state the Harris - Viehmann conjecture following Rappoport and Viehmann in [RV14]. In this subsection, we return to the notation of §1.2 so that in particular, G is a connected, quasisplit reductive group defined over $\mathbb{Q}_p$.

Choose a dominant minuscule $\mu \in X_*(T)$ (where we can consider μ as a cocharacter of G since $T \subset G$) and a $b \in \mathbf{B}(G, \mu)$. Associated to b, we have the standard Levi subgroup M_b. Suppose we have a standard rational Levi subgroup M_S so that $M_b \subset M_S \subset G$. We define b', b_S as we did before Definition 1.2.25.

In [RV14, Equation (6.2)], the authors associate a cohomological construction to the triple (G, b, μ) which they denote $H^\bullet((G, [b], \{\mu\}))$. This construction is a map of Grothendieck groups: $H^\bullet((G, [b], \{\mu\}))$: $\mathrm{Groth}(J_b(\mathbb{Q}_p)) \to \mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_{\{\mu\}}})$ and agrees with $\mathrm{Mant}_{\mathrm{G,b,\mu}}$ in the case above. We will denote this construction $H^\bullet(G, b, \mu)$ since we deal with dominant cocharacters instead of conjugacy classes. Then they have the following conjecture.

Conjecture 1.3.3 (Harris-Viehmann). For $\rho \in \mathrm{Groth}(J_b(\mathbb{Q}_p))$, we have the equality

$$H^\bullet(G, b, \mu)[\rho] =$$
$$\sum_{(M_S, \mu_S) \in \mathcal{I}^{G,\mu}_{M_S, b_S}} (\mathrm{Ind}^G_{P_S} H^\bullet(M_S, b_S, \mu_S)[\rho]) \otimes [1][|\cdot|^{\langle \rho_G, \mu_S \rangle - \langle \rho_G, \mu \rangle}],$$

in $\mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_{\{\mu\}_G}})$. The parabolic induction only modifies the $\mathrm{Groth}(G(\mathbb{Q}_p))$ parts of these representations.

Remark. We need to explain several things in the above conjecture. First we explain why the right-hand side is a representation of $W_{E_{\{\mu\}_G}}$, second we check that the conjecture satisfies a transitivity property, and third we give an example justifying the extra character twist appearing in our formulation. This twist is not present in the original formulation of the conjecture.

We first explain why the right-hand side is a representation of $W_{E_{\{\mu\}_G}}$. We start with a general lemma.

Lemma 1.3.4. *Suppose a group Λ acts on a finite set S. Suppose further that for each $s \in V$, we attach a vector space V_s and for each $\lambda \in \Lambda$ and $s \in S$ we have an isomorphism*

$$i(s, \lambda) : V_s \to V_{\lambda(s)}.$$

We suppose further that $i(s,1)$ is the identity map and that $i(\lambda_1(s), \lambda_2) \circ i(s, \lambda_1) = i(s, \lambda_2\lambda_1)$. Then $\bigoplus_{s \in S} V_s$ is naturally a representation of Λ.

Let $\{s_1, ..., s_k\} \subset S$ be a set of one representative from each Λ-orbit in S. Then

$$\bigoplus_{s \in S} V_s \cong \bigoplus_{i=1}^{k} \mathrm{Ind}_{\mathrm{stab}(s_i)}^{\Lambda} V_{s_i},$$

where Ind *refers to the induced representation (*not *parabolic induction).*

Proof. The proof is clear from the definition of induced representation. ∎

Moreover, we record the following transitivity property for later use.

Lemma 1.3.5. *Suppose that Λ acts on S as before. Let $S_1 \coprod ... \coprod S_k = S$ be a partition of S so that Λ acts on $\{S_1, ..., S_k\}$. Suppose we have for each $s \in S$ a vector space V_s and isomorphisms $i(s, \lambda)$ as above. Then by Lemma 1.3.4 we can consider the* $\mathrm{stab}(S_i) \subset \Lambda$ *representation $V_{S_i} = \bigoplus_{s \in S_i}$. For each $\lambda \in \Lambda$, we get isomorphisms $i(S_i, \lambda) : V_{S_i} \to V_{\lambda(S_i)}$. Thus, again by Lemma 1.3.4, we get a Λ representation $\bigoplus_i V_{S_i}$. This representation is isomorphic to the Λ representation $\bigoplus_{s \in S} V_s$ we get from applying Lemma 1.3.4 to S.*

Now we discuss the $W_{E_{\{\mu\}_G}}$-action in the Harris-Viehmann conjecture. Observe that for $\mu \in X_*(G)$, if $\gamma \in W_{E_{\{\mu\}_G}}$ stabilizes $\{\mu\}_{M_S}$ then it also stabilizes $\{\mu\}_G$ so that $W_{E_{\{\mu\}_{M_S}}} \subset W_{E_{\{\mu\}_G}}$.

Now we claim that $W_{E_{\{\mu\}_G}}$ acts on $\mathcal{I}^{G,\mu}_{M_S, b_S}$ and that the stabilizer of (M_S, μ_S) under this action is $W_{E_{\{\mu\}_{M_S}}}$. To prove the first part of the claim, we pick $\gamma \in W_{E_{\{\mu\}_G}}$ and observe that since M_S and P_S are defined over $\mathbb{Q}_p$, we have $\gamma(M_S) = M_S$ and $\gamma(\mu_S)$ is dominant in M_S. Thus $(M_S, \gamma(\mu_S)) \in \mathcal{C}_{M_S}$ so we need only check that $b_S \in \mathbf{B}(M_S, \gamma(\mu_S))$ and $\gamma(\mu_S) \sim_G \mu$. The first check follows from the fact that

$$\theta_T(\mu_S) = \theta_T(\gamma(\mu_S)),$$

and

$$\mu_S|_{Z(\widehat{M_S})^\Gamma} = \gamma(\mu_S)|_{Z(\widehat{M_S})^\Gamma}.$$

The second check follows because γ stabilizes $\{\mu\}_G$.

To prove the second part of the claim, we note that if $\mu_S = \gamma(\mu_S)$ then γ stabilizes $\{\mu_S\}_{M_S}$. Conversely, if γ stabilizes $\{\mu_S\}_{M_S}$ then since it maps dominant elements relative to M_S to dominant elements, we must have $\gamma(\mu_S) = \mu_S$.

We observe that we have now shown that $W_{E_{\{\mu\}_G}}$ acts on the collection of Rapoport-Zink data (M_S, b_S, μ_S) for $(M_S, \mu_S) \in \mathcal{I}^{G,\mu}_{M_S,b_S}$. By [RV14, Proposition 5.3.iv], these actions induce morphisms of the corresponding towers of rigid spaces and therefore the spaces $H^\bullet(M_S, b_S, \mu_S)[\rho]$ for $\rho \in \mathrm{Groth}(J_b(\mathbb{Q}_p))$. Thus by Lemma 1.3.4 we get an action of $W_{E_{\{\mu\}_G}}$ on the sum of vector spaces

$$\sum_{(M_S,\mu_S)\in\mathcal{I}^{G,\mu}_{M_S,b_S}} H^\bullet(M_S, b_S, \mu_S)[\rho],$$

and therefore on

$$\sum_{(M_S,\mu_S)\in\mathcal{I}^{G,\mu}_{M_S,b_S}} \mathrm{Ind}^G_{P_S}(H^\bullet(M_S, b_S, \mu_S)[\rho]).$$

We remark that the character twist by $-\mathrm{dim}\mathcal{M}^{rig}_{b,\mu,U}$ in the definition of $H^\bullet(M_S, b_S, \mu_S)$ is not an obstacle to defining the $W_{E_{\{\mu\}_G}}$-action as the dimensions of the spaces associated to (M_S, b_S, μ_s) and $(M_S, b_S, \gamma(\mu_S))$ are the same (for $\gamma \in W_{E_{\{\mu\}_G}}$). Also we observe that the twist by $[1][|\cdot|^{\langle\rho_G,\mu_S\rangle - \langle\rho_G,\mu\rangle}]$ is harmless as it is constant over orbits of $W_{E_{\{\mu_S\}_G}}$. This concludes our discussion of the $W_{E_{\{\mu\}_G}}$ action.

We now check that the Harris-Viehmann conjecture is transitive. By this, we mean that if we have standard Levi subgroups M_{S_1} and M_{S_2} of G such that $M_b \subset M_{S_2} \subset M_{S_1} \subset G$, then first applying the conjecture to (G, b, μ) and the inclusion $M_{S_1} \subset G$ and then applying the conjecture to each resulting $(M_{S_1}, b_{S_1}, \mu_{S_1})$ for $(M_{S_1}, \mu_{S_1}) \in \mathcal{I}^{G,\mu}_{M_{S_1},b_{S_1}}$ and the inclusion $M_{S_2} \subset M_{S_1}$ should be the same as applying the conjecture to (G, b, μ) and the inclusion $M_{S_2} \subset G$.

We need to check that the character twists match, that

$$\mathcal{I}^{G,\mu}_{M_{S_2},b_{S_2}} =$$

$$\{(M_{S_2}, \mu_{S_2}) \in \mathcal{C}_{M_{S_2}} : (M_{S_2}, \mu_{S_2}) \in \mathcal{I}^{M_{S_1},\mu_{S_1}}_{M_{S_2},b_{S_2}} \text{ for a } (M_{S_1}, \mu_{S_1}) \in \mathcal{I}^{G,\mu}_{M_{S_1},b_{S_1}}\}.$$

and that the $W_{E_{\{\mu\}_G}}$ actions are the same.

To check the characters match, it suffices to check that for $(M_{S_1}, \mu_{S_1}), (M_{S_2}, \mu_{S_2}) \in \mathcal{C}_G$ such that $(M_{S_2}, \mu_{S_2}) \leqslant (M_{S_1}, \mu_{S_1}) \leqslant (G, \mu)$, we have

$$\langle \rho_G, \mu_{S_2} \rangle - \langle \rho_G, \mu \rangle = (\langle \rho_G, \mu_{S_1} \rangle - \langle \rho_G, \mu \rangle) + (\langle \rho_{M_{S_1}}, \mu_{S_2} \rangle - \langle \rho_{M_{S_1}}, \mu_{S_1} \rangle).$$

This reduces to showing the equality

$$\langle \rho_{G \backslash M_{S_1}}, \mu_{S_1} \rangle = \langle \rho_{G \backslash M_{S_1}}, \mu_{S_2} \rangle, \tag{7}$$

where $\rho_{G \backslash M_{S_1}}$ is the half-sum of the absolute roots of G that are not roots of M_{S_1}. Since μ_{S_2} and μ_{S_1} are conjugate in M_{S_1}, there exists a $w \in W^{\mathrm{abs}}_{M_{S_1}}$ so that $w(\mu_1) = \mu_2$. Then the desired equality follows from the fact that the pairing $\langle \cdot, \cdot \rangle$ is $W^{\mathrm{abs}}_{M_{S_1}}$-invariant and that $W^{\mathrm{abs}}_{M_{S_1}}$ stabilizes the set of positive absolute roots in G but not M_{S_1}. To prove this second fact, note that M_{S_1} normalizes the unipotent radical U_{S_1} of P_{S_1} and that the roots of $\mathrm{Lie}(U_{S_1})$ are precisely the positive absolute roots of G that are not contained in M_{S_1}.

The second check is precisely Proposition 1.2.26, and the third check follows from Proposition 1.2.26 and Lemma 1.3.5.

Now we compute an example to illustrate the necessity of the extra Tate twist in our statement of Conjecture 1.3.3. The following example is also discussed in [Shi12b, §8.3]

Example 1.3.6. Let $n_1 < n_2$ be coprime positive integers and let $G = \mathrm{GL}_{n_1+n_2}$. Fix T the standard maximal torus of diagonal matrices and B the Borel subgoup of upper triangular matrices. Let μ be the minuscule cocharacter with weight vector $(1^2, 0^{n_1+n_2-2})$ and $b \in \mathbf{B}(G, \mu)$ satisfying $\nu_b = ((1/n_1)^{n_1}, (1/n_2)^{n_2})$. Let ρ_1, ρ_2 be supercuspidal representations of $\mathrm{GL}_{n_1}(\mathbb{Q}_p), \mathrm{GL}_{n_2}(\mathbb{Q}_p)$ respectively. Define the standard Levi subgroup $M_b = \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}$, and consider the representation $\pi = I^G_{M_b}(\rho_1 \boxtimes \rho_2)$. We will be interested in computing $\mathrm{Mant}_{G,b,\mu}(\mathrm{Red}_b(\pi))$.

The key point is that we can use Shin's formula (Theorem 1.3.2 in this document) and known cases of the Harris-Viehmann conjecture due to Mantovan ([Man08]) to do this computation, even though the Harris-Viehmann conjecture is not known to be true in the case of M_b since b is not of Hodge-Newton type.

We observe that there are only 3 elements $b' \in \mathbf{B}(G, \mu)$ that satisfy

$$\mathrm{Mant}_{G,b',\mu}(\mathrm{Red}_{b'}(\pi)) \neq 0.$$

After all, the fact that ρ_1, ρ_2 are supercuspidal and the geometric lemma of Bernstein-Zelevinski (§2.11 of [BZ77]) forces $M_{b'}$ to be one of $G, \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}, \mathrm{GL}_{n_2} \times \mathrm{GL}_{n_1}$. In the case where $M_{b'} = G$, we also get 0 since $LJ(\pi) = 0$. Thus, if we write out Shin's formula for our π, the only elements of $\mathbf{B}(G,\mu)$ whose terms contribute to the left-hand side are b, b_1, b_2 where b is as before and b_1, b_2 are defined by

$$\nu_{b_1} = ((2/n_1)^{n_1}, 0^{n_2}), \nu_{b_2} = ((2/n_2)^{n_2}, 0^{n_1}).$$

Thus, we have $M_{b_1} = M_b = \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}$ and $M_{b_2} = \mathrm{GL}_{n_2} \times \mathrm{GL}_{\mathrm{n}_1}$. Note that b_1, b_2 are both of Hodge-Newton type so that we can apply the results of Mantovan.

We have

$$\mathrm{Mant}_{G,b_1,\mu}(\mathrm{Red}_{b_1}(\pi)) = \mathrm{Mant}_{G,b_1,\mu}(LJ(\delta_{P_{b_1}}^{\frac{1}{2}} \otimes J^G_{P^{op}_{b_1}} I^G_{M_{b_1}}(\rho_1 \boxtimes \rho_2))).$$

By the geometric lemma of Bernstein-Zelevinski (§2.11 of [BZ77]) we have that the above equals

$$\mathrm{Mant}_{G,b_1,\mu_1}(LJ((\rho_1 \boxtimes \rho_2) \otimes \delta_{P_{b_1}}^{\frac{1}{2}})).$$

We recall that $\delta_{P_{b_1}} = (|\cdot|^{n_2} \circ \det) \boxtimes (|\cdot|^{-n_1} \circ \det)$ and henceforth use the notation $\rho(n)$ to mean $(|\cdot|^n \circ \det) \otimes \rho$. Thus, we can rewrite the above formula as

$$\mathrm{Mant}_{G,b_1,\mu_1}(LJ(\rho_1(n_2/2)) \boxtimes LJ(\rho_2(-n_1/2))).$$

Then applying the Harris-Viehmann formula we get that the above equals

(8)

$$\mathrm{Ind}^G_{M_b} \mathrm{Mant}_{\mathrm{GL}_{n_1},(1^2,0^{n_1-2})}(LJ(\rho_1(\frac{n_2}{2}))) \boxtimes \mathrm{Mant}_{\mathrm{GL}_{n_2},(0^{n_2})}(LJ(\rho_2(\frac{-n_1}{2}))).$$

Since ρ_1 and ρ_2 are supercuspidal, we can compute (by an easy application of Shin's formula for instance) that

$$\mathrm{Mant}_{\mathrm{GL}_{n_1},(1^2,0^{n_1-2})}(LJ(\rho_1(n_2/2)))$$
$$= [\rho_1(n_2/2)][r_{(-1^2,0^{n_1-2})} \circ LL(\rho_1(n_2/2)) \otimes |\cdot|^{2-n_1}],$$

and so Equation (8) becomes equal to

$$[\pi][r_{(-1^2,0^{n_1-2})} \circ LL(\rho_1(n_2/2)) \otimes |\cdot|^{2-n_1} \otimes r_{(0^{n_2})} \circ LL(\rho_2(-n_1/2))].$$

Pulling the twists through the $r_{-\mu}$ maps, we get

$$[\pi][(r_{(-1^2,0^{n_1-2})} \boxtimes r_{(0^{n_2})}) \circ (LL(\rho_1) \oplus LL(\rho_2)) \otimes |\cdot|^{2-n_1-n_2}].$$

Repeating this computation for the b_2 term, we get

$$\mathrm{Mant}_{G,b_2,\mu}(\mathrm{Red}_{b_2}(\pi))$$
$$= [\pi][(r_{(-1^2,0^{n_2-2})} \boxtimes r_{(0^{n_1})}) \circ (LL(\rho_2) \oplus LL(\rho_1)) \otimes |\cdot|^{2-n_1-n_2}].$$

We now compare these terms to the righthand side of Shin's formula. There the term is

$$[\pi][r_{-\mu} \circ LL(\pi) \otimes |\cdot|^{2-n_1-n_2}].$$

Now $LL(\pi) = LL(\rho_1) \oplus LL(\rho_2)$. Thus, we can restrict $r_{-\mu}$ to $\widehat{M}_b \subset \widehat{G}$ (we have been ignoring the Galois part of ${}^LG$ in this example since G is a split group). Using the theory of weights, we get

$$r_{-\mu}|_{\widehat{M}}$$

$$= [r_{(-1^2,0^{n_1-2})} \boxtimes r_{(0^{n_2})}] \oplus [r_{(-1,0^{n_1-1})} \boxtimes r_{(-1,0^{n_2-1})}] \oplus [r_{(0^{n_1})} \boxtimes r_{(-1^2,0^{n_2-2})}],$$

and so we see that the contributions for b_1, b_2 which we computed above will cancel terms on the righthand side of Shin's formula leaving us with

$$\mathrm{Mant}_{\mathrm{G,b},\mu}(\mathrm{Red}_b(\pi))$$

$$= [\pi][(r_{(-1,0^{n_1-1})} \boxtimes r_{(-1,0^{n_2-1})}) \circ (LL(\rho_1) + LL(\rho_2)) \otimes |\cdot|^{2-n_1-n_2}].$$

However, if the Harris-Viehmann conjecture without the extra Tate twist were to hold for b, we would get

$$\mathrm{Mant}_{G,b,\mu}(\mathrm{Red}_b(\pi)) = \mathrm{Mant}_{G,b,\mu}(LJ(\rho_1(n_2/2)) \boxtimes LJ(\rho_2(-n_1/2)))$$

$$= [\pi][r_{(-1,0^{n_1-1})} \boxtimes r_{(-1,0^{n_2-1})} \circ (LL(\rho_1) + LL(\rho_2))|\cdot|^{1-n_2}].$$

Thus, we see the Tate twists do not agree.

In general, the righthand side of Shin's formula has a twist of $-\langle \rho_G, \mu \rangle$ where ρ_G is the half sum of the positive roots of G. Suppose now that $b \in \mathbf{B}(G,\mu)$ and $b' \in \mathbf{B}(M_b)^+$ corresponds to b under Equation (2). Then for any $(M_b, \mu') \in \mathcal{I}^{G,\mu}_{M_b,b'}$, we would expect the Galois part of $\mathrm{Mant}_{M_b,b',\mu'}(\rho)$ for $\rho \in \mathrm{Groth}(J_b(\mathbb{Q}_p))$ to come with a twist of $-\langle \rho_{M_b}, \mu' \rangle$. Then the Galois part of $\mathrm{Mant}_{G,b,\mu}(\mathrm{Red}_b(\pi))$ for $\pi \in \mathrm{Groth}(G(\mathbb{Q}_p))$ would carry an extra twist of $-\langle \frac{\det(Ad_{N_b}(M_b))|_T}{2}, \mu' \rangle$ corresponding to twisting $J^G_{P_b^{op}}(\pi)$ by $\delta^{\frac{1}{2}}_{P_b}$ in the definition of Red_b . We note that

$$\langle \rho_{M_b}, \mu' \rangle + \langle \frac{\det(Ad_{N_b}(M_b))|_T}{2}, \mu' \rangle = \langle \rho_G, \mu' \rangle,$$

Thus, we see that the difference between these Tate twists is

$$\langle \rho_G, \mu' \rangle - \langle \rho_G, \mu \rangle.$$

which is the twist in Conjecture 1.3.3

Remark. We note that in the Hodge-Newton case studied by Mantovan, $\mu = \mu'$ (as in the notation of the previous paragraph) so that this extra twist vanishes, agreeing with Mantovan's results ([Man08, Corollary 5], cf. [RV14, Theorem 8.8]).

We now give an alternate version of the Harris-Viehmann conjecture that we will use in numerous arguments in this document. Suppose that G, b, μ are as in Theorem 1.3.2. The standard Levi subgroup M_b has a natural product decomposition

$$M_b = M_1 \times ... \times M_k$$

so that under the natural isomorphism

$$\mathbf{B}(M_b) \cong \mathbf{B}(M_1) \times \mathbf{B}(M_k), b' \mapsto (b'_1, ..., b'_k),$$

each $\nu(b_i)$ has a single slope. Now pick $(M_b, \mu_b) \in \mathcal{I}^{G,\mu}_{M_b,b'}$. Then the local Shimura variety datum (M_b, b', μ_b) decomposes into a collection $(M_1, b'_1, \mu_{b,1}), ..., (M_k, b'_k, \mu_{b,k})$. In §5.2.$(ii)$ of [RV14], the authors show that the local Shimura variety associated to (M_b, b', μ_b) is the product of those associated to $(M_i, b'_i, \mu_{b,i})$. Furthermore using the Kunneth formula (as in [Man08, p. 15]), we get that for $\rho_1 \boxtimes ... \boxtimes \rho_k \in \mathrm{Groth}(M_1(\mathbb{Q}_p) \times ... \times M_k(\mathbb{Q}_p))$,

$$\mathrm{Mant}_{M_b,b',\mu_b}(\rho_1 \boxtimes ... \boxtimes \rho_k) = \boxtimes_{i=1}^k \mathrm{Mant}_{G_i,b'_i,\mu_{b,i}}(\rho_i),$$

as a representation of $M_b \times W_{E_{\{\mu_b\}_{M_b}}}$ (the group $W_{E_{\{\mu_b\}_{M_b}}}$ acts diagonally through the product $W_{E_{\{\mu_{b,1}\}_{M_1}}} \times ... \times W_{E_{\{\mu_{b,k}\}_{M_k}}}$).

Thus, we have the following alternate form of the Harris-Viehmann conjecture for the Rapoport-Zink spaces we consider.

Conjecture 1.3.7 (Alternate Form of Harris-Viehmann Conjecture). We use the notation of the previous paragraphs so that in particular, (G, b, μ) comes from an unramified Rapoport-Zink space of EL-type as in Definition 1.3.1. Then for any $\rho \in \mathrm{Groth}(J_b(\mathbb{Q}_p))$, we have the following equality in $\mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_{\{\mu\}_G}})$:

$$\mathrm{Mant}_{G,b,\mu}(\rho) =$$

$$\sum_{(M_b,\mu_b) \in \mathcal{I}^{G,\mu}_{M_b,b'}} \mathrm{Ind}^G_{P_b}(\boxtimes_{i=1}^k \mathrm{Mant}_{M_b,b'_i,\mu_{b,i}}(\rho_i)) \otimes [1][|\cdot|^{\langle \rho_G, \mu_b \rangle - \langle \rho_G, \mu \rangle}].$$

1.3.3. **Proof of Theorem 1.1.3.** The combination of the Harris - Viehmann conjecture and sum formula allows us to relate the cohomology of Rapoport-Zink spaces to the cocharacter pairs studied in §2. To do so, we attach a map of Grothendieck groups to each cocharacter pair. We return to the notation of §3.1.

Fix a cocharacter pair $(G, \mu) \in \mathcal{C}_G$. Suppose $(M_S, \mu_S) \in \mathcal{C}_G$ and satisfies $\mu_S \sim_G \mu$. We associate (M_S, μ_S) to a map of representations

$$[M_S, \mu_S] : \mathrm{Groth}(G(\mathbb{Q}_p)) \to \mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_{\{\mu_S\}_{M_S}}}),$$

given by

$$\pi \mapsto (\mathrm{Ind}_{P_S}^{G} \circ [\mu_S] \circ (\delta_{P_S} \otimes \mathrm{Jac}_{P_S^{op}}^{G}))(\pi) \otimes [1][|\cdot|^{\langle \rho_G, \mu_S\rangle - \langle \rho_G, \mu\rangle}],$$

with

$$[\mu_S] : \mathrm{Groth}(M_S(\mathbb{Q}_p)) \to \mathrm{Groth}(M_S(\mathbb{Q}_p) \times W_{E_{\{\mu\}_{M_S}}}),$$

given by

$$\pi \mapsto [\pi][r_{-\mu_S} \circ LL(\pi)|_{W_{E_{\{\mu_S\}_{M_S}}}} \otimes |\cdot|^{-\langle \rho_{M_S}, \mu_S\rangle}].$$

Remark. We note that the map $[M_S, \mu_S]$ is only defined relative to a cocharacter pair (G, μ).

Remark. We observe an interesting property of the maps $[M_S, \mu_S]$. Fix (G, μ) and consider (M_S, μ_S) such that $\mu_S \sim_G \mu$. Since the normalized Jacquet module and parabolic induction functors behave better with respect to the local Langlands correspondence, it makes sense to rewrite $[M_S, \mu_S]$ in terms of these maps. We get

$$[M_S, \mu_S] = (I_{M_S}^{G} \otimes \delta_{P_S}^{-\frac{1}{2}} \circ [\mu_S] \circ (\delta_{P_S}^{\frac{1}{2}} \otimes J_{P_S^{op}}^{G})) \otimes [1][|\cdot|^{\langle \rho_G, \mu_S - \mu\rangle}].$$

Note that the twists by the modular character cancel in the admissible part but do not cancel in the Galois part. Thus, the total Tate twist of the Galois part is

$$\begin{aligned}\langle \rho_G, \mu_S - \mu\rangle - \langle \rho_{M_S}, \mu_S\rangle - \langle \frac{\det(Ad_{N_S}(M_S))|_T}{2}, \mu_S\rangle \\ = -\langle \rho_G, \mu\rangle.\end{aligned}$$

This twist does not depend on (M_S, μ_S) but rather only on (G, μ). Thus, as we will see in the computations of the next section, it is possible for large cancellations to occur in computations of $\mathrm{Mant}_{G,b,\mu}(\rho)$ for various ρ.

We now prove some lemmas relating to these maps before tackling the main theorem.

Lemma 1.3.8. *Let M_{S_1}, M_{S_2} be standard Levi subgroups of G satisfying $M_{S_2} \subset M_{S_1}$. Consider the natural map*

$$i_{M_{S_1}}^{G} : \mathcal{C}_{M_{S_1}} \to \mathcal{C}_G,$$

as defined in Equation (6). Let $(M_{S_2}, \mu_{S_2}) \in \mathcal{C}_{M_{S_1}}$. Suppose further that we have fixed pairs $(M_{S_1}, \mu_{S_1}) \in \mathcal{C}_{M_{S_1}}$ and $(G, \mu) \in \mathcal{C}_G$ so that $\mu_{S_2} \sim_{M_{S_1}} \mu_{S_1}$ and $\mu_{S_2} \sim_G \mu$. Then for $\pi \in \mathrm{Groth}(G_{\mathbb{Q}_p})$,

$$i_{M_{S_1}}^{G}([M_{S_2}, \mu_{S_2}])(\pi) =$$

$$(\mathrm{Ind}_{P_{S_1}}^G \circ [M_{S_2}, \mu_{S_2}] \circ (\delta_{P_{S_1}} \otimes \mathrm{Jac}_{P_{S_1}^{op}}^G))(\pi) \otimes [1][|\cdot|^{\langle \rho_G, \mu_{S_1}\rangle - \langle \rho_G, \mu\rangle}],$$

where we write

$$i_{M_{S_1}}^G([M_{S_2}, \mu_{S_2}]) : \mathrm{Groth}(G(\mathbb{Q}_p)) \to \mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_{\{\mu_{S_2}\}M_{S_2}}}),$$

to denote the map associated to $i_{M_{S_1}}^G((M_{S_2}, \mu_{S_2}))$ in the manner above.

Proof. We first note that by transitivity of the Jacquet module and modulus character constructions, we have

$$\delta_{P_{S_2}} \otimes \mathrm{Jac}_{P_{S_2}^{op}}^G = (\delta_{P_{S_2} \cap M_1} \otimes \mathrm{Jac}_{P_{S_2}^{op}}^{M_{S_1}}) \circ (\delta_{P_{S_1}} \otimes \mathrm{Jac}_{P_{S_1}^{op}}^G).$$

Hence, we just need to check that the twists on the Galois parts of both sides match. By Remark 1.3.3, both twists are by $-\langle \rho_G, \mu \rangle$ □

Lemma 1.3.9. *Suppose we are in the situation of Proposition 1.2.29 so that $G = G_1 \times ... \times G_k$ is a connected reductive group with standard Levi subgroup $M_S = M_{S_1} \times ... \times M_{S_k}$. Fix cocharacter pairs $(M_S, \mu_S), (G, \mu) \in \mathcal{C}_G$ with $\mu_S \sim_G \mu$. The bijection $\mathcal{C}_G \cong \mathcal{C}_{G_1} \times ... \mathcal{C}_{G_k}$ takes (M_S, μ_S) to $((M_{S_1}, \mu_{S_1}), ..., (M_{S_k}, \mu_{S_k}))$ and (G, μ) to $((G_1, \mu_1), ..., (G_k, \mu_k))$ and we have $\mu_{S_i} \sim_{G_i} \mu_i$. Then we define*

$$\boxtimes_{i=1}^k [M_{S_i}, \mu_{S_i}] : \mathrm{Groth}(G(\mathbb{Q}_p)) \to \mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_{\{\mu_S\}M_S}})$$

by

$$\pi_1 \boxtimes ... \boxtimes \pi_k \mapsto [M_{S_1}, \mu_{S_1}](\pi_1) \boxtimes ... \boxtimes [M_{S_k}, \mu_{S_k}](\pi_k).$$

Then we have the following equality of homomorphisms of Grothendieck groups:

$$\boxtimes_{i=1}^k [M_{S_i}, \mu_{S_i}] = [M_S, \mu_S]$$

Proof. We have $\boxtimes_{i=1}^k [M_{S_i}, \mu_{S_i}]$ equals

$$\begin{aligned}
&= \boxtimes_{i=1}^k \mathrm{Ind}_{P_{S_i}}^{G_i} \circ [\mu_{S_i}] \circ (\delta_{P_{S_i}} \otimes \mathrm{Jac}_{P_{S_i}^{op}}^{G_i}) \otimes [1][|\cdot|^{\langle \rho_{G_i}, \mu_{S_i} - \mu_i \rangle}] \\
&= \mathrm{Ind}_{P_S}^G \circ [\mu] \circ (\delta_{P_S} \otimes \mathrm{Jac}_{P_S^{op}}^G) \otimes [1][|\cdot|^{\sum_{i=1}^k \langle \rho_{G_i}, \mu_{S_i} - \mu_i \rangle}] \\
&= \mathrm{Ind}_{P_S}^G \circ [\mu] \circ (\delta_{P_S} \otimes \mathrm{Jac}_{P_S^{op}}^G) \otimes [1][|\cdot|^{\langle \rho_G, \mu_S - \mu \rangle}] \\
&= [M_S, \mu_S].
\end{aligned}$$

□

For some finite subset $C \subset \mathcal{C}_G$, such that each $(M_S, \mu_S) \in C$ satisfies $\mu_S \sim_G \mu$, we would like to make sense of a sum

$$\sum_{(M_S, \mu_S) \in C} [M_S, \mu_S].$$

This makes sense as a map $\mathrm{Groth}(G(\mathbb{Q}_p)) \to \mathrm{Groth}(G(\mathbb{Q}_p) \times W_E)$ where $W_E = \bigcap_{(M_S,\mu_S)\in C} W_{E_{\{\mu_S\}_{M_S}}}$. However, for our purposes, we would like to understand when we can extend the image of this map to a representation in $\mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_{\{\mu\}_G}})$.

Lemma 1.3.10. *Fix a pair $(G,\mu) \in \mathcal{C}_G$. Consider a finite subset $C \subset \mathcal{C}_G$ such that if $(M_S,\mu_S) \in C$ then $\mu_S \sim_G \mu$. Furthermore, suppose that for each $\gamma \in W_{E_{\{\mu\}_G}}$ and element $(M_S,\mu_S) \in C$, we have $(M_S,\gamma(\mu_S)) \in C$. Then*

$$\sum_{(M_S,\mu_S)\in C} [M_S,\mu_S],$$

is a map

$$\mathrm{Groth}(G(\mathbb{Q}_p)) \to \mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_{\{\mu\}_G}})$$

in a natural way.

Proof. Our construction is analogous to that of Lemma 1.3.4. We fix $\rho \in \mathrm{Groth}(G(\mathbb{Q}_p))$ and give

$$V_C = \bigoplus_{(M_S,\mu_S)\in C} [M_S,\mu_S](\rho),$$

the structure of a $G(\mathbb{Q}_p) \times W_{E_{\{\mu\}_G}}$ representation. Suppose that $C = C_1 \coprod ... \coprod C_n$ where each C_i is a single $W_{E_{\{\mu\}_G}}$-orbit. Then for each i, we give

$$V_{C_i} = \bigoplus_{(M_S,\mu_S)\in C_i} [M_S,\mu_S](\rho),$$

the structure of a $G(\mathbb{Q}_p) \times W_{E_{\{\mu\}_G}}$-representation and then define the $G(\mathbb{Q}_p) \times W_{E_{\{\mu\}_G}}$-structure on V_C to be the direct sum of the V_{C_i}.

Suppose now that C contains a single $W_{E_{\{\mu\}_G}}$ orbit. In this case, we will show that

$$\bigoplus_{(M_S,\mu_S)\in C} [M_S,\mu_S](\rho),$$

can be given the structure of a $\mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_{\{\mu\}_G}})$ representation equal to

$$[\mathrm{Ind}^G_{P_S}(\delta_{P_S} \otimes \mathrm{Jac}^G_{P_S^{op}}(\rho))]$$
$$\boxtimes [r \circ LL(\delta_{P_S} \otimes \mathrm{Jac}^G_{P_S^{op}}(\rho))|_{W_{E_{\{\mu\}_G}}} \otimes |\cdot|^{-\langle \rho_G, \mu_S - \mu\rangle - \langle \rho_{M_S}, \mu_S\rangle}],$$

where r is the induced representation (*not* parabolic induction) given by

$$\mathrm{Ind}^{\widehat{M_S} \rtimes W_{E_{\{\mu\}_G}}}_{\widehat{M_S} \rtimes W_{E_{\{\mu_S\}_{M_S}}}} (r_{-\mu_S}),$$

for a fixed choice of $(M_S, \mu_S) \in C$. The isomorphism class of r will not depend on this choice.

We study the representation r. Fix representatives $\gamma_1, ..., \gamma_k \in W_{E_{\{\mu\}_G}}/W_{E_{\{\mu_S\}_{M_S}}}$ so that $\gamma_1 = 1$. Then r is defined to be the sum of k copies of $r_{-\mu_S}$ indexed by the γ_i and acted on by $W_{E_{\{\mu\}_G}}$ in the standard way. We check that the ith copy of $r_{-\mu_S}$ is a representation of $\widehat{M_S} \rtimes W_{E_{\{\gamma_i(\mu_S)\}_{M_S}}}$ and isomorphic to $r_{-\gamma_i(\mu_S)}$. Let V_i be the underlying vector space of the ith copy of $r_{-\mu_S}$. Then V_i is naturally a representation of $\widehat{M_S} \rtimes \gamma_i W_{E_{\{\mu_S\}_{M_S}}} \gamma_i^{-1} = \widehat{M_S} \rtimes W_{E_{\{\gamma_i(\mu_S)\}_{M_S}}}$.

Now suppose $v \in V_1$ is a weight vector of $\widehat{T} \subset \widehat{M_S}$ of weight μ'. Then we show that $(1, \gamma_i)v \in V_i$ has weight $\gamma_i(\mu')$. After all, for $t \in \widehat{T}$, we have

$$
\begin{aligned}
r((t,1))((1,\gamma_i)v) &= (t, \gamma_i)v \\
&= (1, \gamma_i)(\gamma_i^{-1}(t), 1)v \\
&= (1, \gamma_i) r_{-\mu_S}((\gamma_i^{-1}(t), 1))(v) \\
&= (1, \gamma_i)\mu'(\gamma_i^{-1}(t))v \\
&= \gamma_i(\mu')(t)(1, \gamma_i)v.
\end{aligned}
$$

In particular, we have shown that V_i is irreducible of extreme weight $-\gamma_i(\mu_S)$ as an $\widehat{M_S}$-representation (since $r_{-\mu_S}$ is irreducible of extreme weight $-\mu_S$ as an $\widehat{M_S}$-representation). It is a simple check similar to the above that $W_{E_{\{\gamma_i(\mu_S)\}_{M_S}}}$ acts trivially on the highest weight space of V_i. This proves that V_i is isomorphic to $r_{-\gamma_i(\mu_S)}$.

In particular, this shows that we can give

$$
\bigoplus_{\gamma_i \in W_{E_{\{\mu\}_G}}/W_{E_{\{\mu_S\}_{M_S}}}} r_{-\gamma_i(\mu_S)} \circ LL(\delta_{P_S} \otimes \mathrm{Jac}^G_{P_S^{op}}(\rho))|_{W_{E_{\{\gamma_i(\mu_S)\}_{M_S}}}},
$$

the structure of a $W_{E_{\{\mu\}_G}}$ representation isomorphic to

$$
r \circ LL(\delta_{P_S} \otimes \mathrm{Jac}^G_{P_S^{op}}(\rho))|_{W_{E_{\{\mu\}_G}}}.
$$

To conclude the proof, we just need to check that the $|\cdot|$ twists on each $[M_S, \gamma_i(\mu_S)]$-term are the same. This follows because ρ_G and ρ_{M_S} are both invariant by $W_{E_{\{\mu\}_G}}$. $\square$

We would like to check the following:

Lemma 1.3.11. *The sum $\mathcal{M}_{G,b,\mu}$ as in Definition 1.2.23 gives a map*

$$
[\mathcal{M}_{G,b,\mu}] : \mathrm{Groth}(G(\mathbb{Q}_p)) \to \mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_{\{\mu\}_G}}),
$$

where

$$[\mathcal{M}_{G,b,\mu}] := \sum_{(M_S,\mu_S)\in\mathcal{R}_{G,b,\mu}} (-1)^{L_{M_S,M_b}}[M_S,\mu_S].$$

Proof. By Lemma 1.3.10, it suffices to show that $\mathcal{M}_{G,b,\mu}$ is invariant under the natural action of $W_{E_{\{\mu\}_G}}$ on $\mathbb{Z}\langle\mathcal{C}_G\rangle$. Pick $\gamma \in W_{E_{\{\mu\}_G}}$. Since the action of γ on a cocharacter pair fixes the standard Levi subgroup in the first factor, signs will not be an issue and we will be done if we can check that $\mathcal{R}_{G,b,\mu}$ is γ-invariant. But if $(M_b,\mu_b) \in \mathcal{T}_{G,b,\mu}$ then it is a simple consequence of the definition of $\mathcal{T}$ that so is $(M_b,\gamma(\mu_b))$. Furthermore if $(M_S,\mu_S) \leqslant (M_b,\mu_b)$ then $(M_S,\gamma(\mu_S)) \leqslant (M_b,\gamma(\mu_b))$ by definition of the partial order relation (remarking that $\theta_{M_S}(\mu_S) = \theta_{M_S}(\gamma(\mu_S))$). This shows that $\mathcal{R}_{G,b,\mu}$ is γ-invariant as desired. ☐

If we combine the previous lemma with Proposition 1.2.29, and Lemma 1.3.9 we get

$$\boxtimes_{i=1}^{k} [\mathcal{M}_{G_i,b_i,\mu_i}] = [\mathcal{M}_{G,b,\mu}]. \tag{9}$$

We now prove the key result of this section which provides the connection between Mant and cocharacter pairs.

Theorem 1.3.12. *Assume that the Harris-Viehmann conjecture is true for the general linear groups we consider.*

(1) *We have the following equality of morphisms* $\mathrm{Groth}^2(G(\mathbb{Q}_p)) \to \mathrm{Groth}^2(G(\mathbb{Q}_p) \times W_{E_{\{\mu\}_G}})$*:*

$$\mathrm{Mant}_{G,b,\mu} \circ \mathrm{Red}_b = [\mathcal{M}_{G,b,\mu}].$$

where $\mathrm{Groth}^2(G(\mathbb{Q}_p))$ *is defined to be the span of the essentially square integrable representations in* $\mathrm{Groth}(G(\mathbb{Q}_p))$*.*

(2) *Now assume further that Theorem 1.3.2 holds for all admissible representations of* $\mathrm{Groth}(G(\mathbb{Q}_p))$*. Then the above equality holds as morphisms* $\mathrm{Groth}(G(\mathbb{Q}_p)) \to \mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_{\{\mu\}_G}})$*.*

Proof. We prove the second statement first. We prove this result by induction on the rank of $X_*(T)$.

If the rank of $X_*(T)$ is 1, then $\mathbf{B}(G,\mu)$ is a singleton and so the result follows from Theorem 1.3.2.

Suppose the result holds for all non-basic $b \in \mathbf{B}(G,\mu)$ with $\mathrm{Rk}(X_*(T)) \leqslant r$. Then by Theorem 1.3.2 and Theorem 1.2.24, the result holds for all $b \in \mathbf{B}(G,\mu)$ with $\mathrm{Rk}(X_*(T)) \leqslant r$.

Finally, suppose the result holds for all $b \in \mathbf{B}(G,\mu)$ with $\mathrm{Rk}(X_*(T)) \leqslant r$. Then suppose $X_*(T)$ has rank $r+1$ and choose

$b \in \mathbf{B}(G,\mu)$ such that b is not basic. We write $M_b = M_{b_1} \times \ldots \times M_{b_k}$. By the Harris-Viehmann formula,

$$\mathrm{Mant}_{G,b,\mu} \circ \mathrm{Red}_b$$

$$= \sum_{(M_b,\mu_b)\in \mathcal{I}^{G,\mu}_{M_b,b'}} (\mathrm{Ind}^G_{P_b} \circ \otimes_{i=1}^k \mathrm{Mant}_{M_{b_i},b'_i,\mu_{bi}} \circ \mathrm{Red}_b) \otimes [1][|\cdot|^{\langle \rho_G,\mu_b\rangle - \langle \rho_G,\mu\rangle}]$$

$$= \sum_{(M_b,\mu_b)\in \mathcal{I}^{G,\mu}_{M_b,b'}} (\mathrm{Ind}^G_{P_b} \circ \otimes_{i=1}^k (\mathrm{Mant}_{M_{b_i},b'_i,\mu_{bi}} \circ \mathrm{Red}_{b'_i}) \circ (\delta_{P_b} \otimes \mathrm{Jac}^G_{P_b^{op}}))$$

$$\otimes[1][|\cdot|^{\langle \rho_G,\mu_b\rangle - \langle \rho_G,\mu\rangle}].$$

By inductive assumption we get that this equals

$$\sum_{(M_b,\mu_b)\in \mathcal{I}^{G,\mu}_{M_b,b'}} (\mathrm{Ind}^G_{P_b} \circ \otimes_{i=1}^k [\mathcal{M}_{M_{b_i},b'_i,\mu_{bi}}] \circ (\delta_{P_b} \otimes \mathrm{Jac}^G_{P_b^{op}})) \otimes [1][|\cdot|^{\langle \rho_G,\mu_b\rangle - \langle \rho_G,\mu\rangle}]$$

and now by Equation (9)

$$= \sum_{(M_b,\mu_b)\in \mathcal{I}^{G,\mu}_{M_b,b'}} (\mathrm{Ind}^G_{P_b} \circ [\mathcal{M}_{M_b,b',\mu_b}] \circ (\delta_{P_b} \otimes \mathrm{Jac}^G_{P_b^{op}})) \otimes [1][|\cdot|^{\langle \rho_G,\mu_b\rangle - \langle \rho_G,\mu\rangle}].$$

Finally, by Corollary 1.2.28 and Lemma 1.3.8

$$= [\mathcal{M}_{G,b,\mu}].$$

We must check that the $W_{E_{\{\mu\}_G}}$ structure coming from Remark 1.3.2 is compatible with that of Lemma 1.3.10. Pick $\rho \in \mathrm{Groth}(G(\mathbb{Q}_p))$. By inductive assumption and Lemma 1.3.8, for each $(M_b,\mu_b) \in \mathcal{I}^{G,\mu}_{M_b,b'}$, the $W_{E_{\{\mu_b\}_{M_b}}}$-structures on

$$(\mathrm{Ind}^G_{P_b} \circ \mathrm{Mant}_{M_b,b',\mu_b} \circ \mathrm{Red}_{b'} \circ (\delta_{P_b} \otimes \mathrm{Jac}^G_{P_b^{op}}))(\rho) \otimes [1][|\cdot|^{\langle \rho_G,\mu_b\rangle - \langle \rho_G,\mu\rangle}],$$

and

$$i^G_{M_b}([\mathcal{M}_{M_b,b',\mu_b}])(\rho),$$

are the same. Thus by Lemma 1.3.4, the $W_{E_{\{\mu\}_G}}$-structure on $\mathrm{Mant}_{G,b,\mu}(\mathrm{Red}_b(\rho))$ is a direct sum over the $W_{E_{\{\mu\}_G}}$-orbits of $\mathcal{I}^{G,\mu}_{M_b,b'}$ of induced representations of the form

$$\mathrm{Ind}^{W_{E_{\{\mu\}_G}}}_{W_{E_{\{\mu_b\}_{M_b}}}} i^G_{M_b}([\mathcal{M}_{M_b,b',\mu_b}])(\rho).$$

This $W_{E_{\{\mu\}_G}}$-structure matches the one on $[\mathcal{M}_{G,b,\mu}]$ (coming from Lemma 1.3.10) by the transitivity of the induced representation construction (see Lemma 1.3.5 for instance).

We now prove the first statement of the theorem. To do so, we need to show that if we restrict ourselves to the span of the essentially square integrable representations $\mathrm{Groth}^2(G(\mathbb{Q}_p)) \subset \mathrm{Groth}(G(\mathbb{Q}_p))$, then we can remove the first assumption. In particular, these representations are accessible, so we have Theorem 1.3.2 unconditionally. In the above proof we need only observe that the Jacquet module $\mathrm{Jac}^G_{P^{op}}(\rho)$ is a sum of essentially square integrable representations for $\rho \in \mathrm{Irr}^2(G(\mathbb{Q}_p))$. Thus, to get the result for $\mathrm{Groth}^2(G(\mathbb{Q}_p))$ by induction, our inductive assumption need only hold for all $\mathrm{Groth}^2(G'(\mathbb{Q}_p))$ for $rkG' < rkG$. This shows that under the condition that the Harris-Viehmann conjecture is true in the cases we consider, the theorem is true for essentially square integrable representations without any other assumptions. □

1.4. Harris's Generalization of the Kottwitz Conjecture (proof of Theorem 1.5)

In this section, we discuss an explicit computation using the above results. In particular, we prove that Shin's formula for all admissible representations combined with the Harris-Viehmann conjecture proves Harris's conjecture for the general linear groups considered in §3. This conjecture is distinct from the Harris-Viehmann conjecture and is [Har01, Conj 5.4].

We begin by discussing the Kottwitz conjecture, which appears as [Shi12b, Cor 7.7] in the cases we consider, and more generally as [RV14, Conj 7.3]. Fix G as in section 3 of this document and a cocharacter pair (G, μ) such that μ is minuscule. Let $b \in \mathbf{B}(G, \mu)$ be the unique basic element. Now, consider ρ a representation of $J_b(\mathbb{Q}_p)$ such that $JL(\rho)$ is a supercuspidal representation of $G(\mathbb{Q}_p)$. Then

$$\mathrm{Mant}_{G,b,\mu}(\mathrm{Red}_b(JL(\rho))) = \mathrm{Mant}_{G,b,\mu}(\rho),$$

but by Theorem 1.3.12, the lefthand side equals

$$[\mathcal{M}_{G,b,\mu}](JL(\rho)).$$

Now we see that since $JL(\rho)$ is supercuspidal, each term of the form $[M_S, \mu_S](JL(\rho))$ is 0 when M_S is a proper Levi subgroup of G. Thus,

$$\mathrm{Mant}_{G,b,\mu}(\rho) = [\mathcal{M}_{G,b,\mu}](JL(\rho)) = [JL(\rho)][r_{-\mu} \circ LL(\rho)| \cdot |^{-\langle \rho_G, \mu \rangle}].$$

This result is the Kottwitz conjecture for G. Alternatively, if $b \in \mathbf{B}(G, \mu)$ is not basic, then no cocharacter pairs with G as the Levi subgroup will appear in $\mathcal{M}_{G,b,\mu}$ and so

$$\mathrm{Mant}_{G,b,\mu}(\rho) = 0.$$

Of course, these results are already known by [Shi12b], but we review them as motivation for Harris's conjecture.

We begin with the following useful definition.

Definition 1.4.1. *Fix $(G,\mu) \in \mathcal{C}_G$ and $b \in \mathbf{B}(G,\mu)$. Let M_S be a standard Levi subgroup such that $M_S \subset M_b$. We define the subset $\mathrm{Rel}^{G,\mu}_{M_S,b} \subset \mathcal{C}_G$ as the set*

$$\{(M_S,\mu_S) \in \mathcal{C}_G : \exists (M_b,\mu_b) \in \mathcal{T}_{G,b,\mu},\ \ \theta_{M_b}(\mu_b) = \theta_{M_S}(\mu_S),\ \ \mu_b \sim_{M_b} \mu_S\}.$$

The notation $\mu_S \sim_{M_b} \mu_b$ is defined to mean that μ_S and μ_b are conjugate in M_b. Note that we do not require $(M_S,\mu_S) \leqslant (G,\mu)$ or $(M_S,\mu_S) \leqslant (M_b,\mu_b)$.

We record the following useful properties of $\mathrm{Rel}^{G,\mu}_{M_S,b}$.

Lemma 1.4.2. *We use the same notation as in the previous definition. Then*

$$\mathrm{Rel}^{G,\mu}_{M_S,b} = \coprod_{(M_b,\mu_b) \in \mathcal{I}^{G,\mu}_{M_b,b'}} \mathrm{Rel}^{M_b,\mu_b}_{M_S,b'}.$$

Proof. If $(M_S,\mu_S) \in \mathrm{Rel}^{G,\mu}_{M_S,b}$, then there is an $(M_b,\mu_b) \in \mathcal{T}_{G,b,\mu}$ such that $\theta_{M_b}(\mu_b) = \theta_M(\mu_S)$ and $\mu_S \sim_{M_b} \mu_b$. Then by Proposition 1.2.27, there is a unique $(M_b,\mu') \in \mathcal{I}^{G,\mu}_{M_b,b'}$ such that $(M_b,\mu_b) \in \mathcal{T}_{M_b,b',\mu'}$ and so $(M_S,\mu_S) \in \mathrm{Rel}^{M_b,\mu_b}_{M_S,b'}$. The reverse inclusion is analogous. □

Lemma 1.4.3. *The set $\mathrm{Rel}^{G,\mu}_{M_S,b}$ is invariant under the action of $W_{E_{\{\mu\}_G}}$.*

Proof. If $(M_S,\mu_S) \in \mathrm{Rel}^{G,\mu}_{M_S,b}$ then we can find $(M_b,\mu_b) \in \mathcal{T}_{G,b,\mu}$ with $\theta_{M_b}(\mu_b) = \theta_{M_S}(\mu_S)$ and $\mu_b \sim_{M_b} \mu_S$. By a similar argument to Lemma 1.3.11, we show that for each $\gamma \in W_{E_{\{\mu\}_G}}$, we have $(M_b,\gamma(\mu_b)) \in \mathcal{T}_{G,b,\mu}$ and $\theta_{M_S}(\gamma(\mu_S)) = \theta_{M_b}(\gamma(\mu_b))$ and $\gamma(\mu_S) \sim_{M_b} \gamma(\mu_b)$. This finishes the proof. □

Equipped with the above definition, we can now make the following restatement and slight generalization of [Har01, Conj 5.4] for the G that we consider. Our statement is a generalization because we consider non-basic b and do not assume the representation $I^G_{M_S}(\rho)$ is irreducible.

Conjecture 1.4.4 (Harris). Fix a $b \in \mathbf{B}(G,\mu)$ and a standard Levi subgroup $M_S \subset M_b$. Then for $\rho \in \mathrm{Groth}(M_S(\mathbb{Q}_p))$ a supercuspidal representation, the following representations are equal in $\mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_{\{\mu\}_G}})$:

$$\mathrm{Mant}_{G,b,\mu}(e(J_b)LJ(\delta^{\frac{1}{2}}_{G,P_b} \otimes I^{M_b}_{M_S}(\rho)))$$

and

$$[I^G_{M_S}(\rho)]\left[\bigoplus_{(M_S,\mu_S) \in \mathrm{Rel}^{G,\mu}_{M_S,b}} r_{-\mu_S} \circ LL(\rho)|_{W_{E_{\{\mu_S\}_{M_S}}}} |\cdot|^{-\langle \rho_G,\mu\rangle}\right].$$

Here $r_{-\mu_S}$ is a representation of $\widehat{M_S} \rtimes W_{E_{\{\mu_S\}_{M_S}}}$ but the righthand side naturally acquires the structure of a $G(\mathbb{Q}_p) \times W_{E_{\{\mu\}_G}}$ representation from Lemma 1.4.3 and the proof of Lemma 1.3.10.

In particular, for b basic, this says that

$$\mathrm{Mant}_{G,b,\mu}(\mathrm{Red}_b(I^G_{M_S}(\rho)))$$

$$= [I^G_{M_S}(\rho)] \left[\bigoplus_{(M_S,\mu_S)\in \mathrm{Rel}^{G,\mu}_{M_S,b}} r_{-\mu_S} \circ LL(\rho)|_{W_{\{\mu_S\}_{M_S}}} |\cdot|^{-\langle \rho_G, \mu\rangle} \right].$$

We will prove this conjecture assuming that Shin's formula (Theorem 1.3.2 of this Part) holds for all admissible representations.

We proceed by induction on the rank of T. The key observation will be that Harris's conjecture is compatible with the Harris-Viehmann conjecture and Shin's formula. We will first assume that $I^G_{M_S}(\rho)$ is irreducible and later remove this assumption.

The following proposition shows that Conjecture 1.4.4 is compatible with the Harris-Viehmann conjecture (Conjecture 1.3.3).

Proposition 1.4.5. *Fix $b \in \mathbf{B}(G,\mu)$ non-basic and fix a standard Levi subgroup M_S of G satisfying $M_S \subset M_b$. Pick $\rho \in \mathrm{Groth}(M_S(\mathbb{Q}_p))$ and suppose that $I^G_{M_S}(\rho)$ is irreducible. Suppose that Conjecture 1.4.4 for ρ holds for $\mathrm{Mant}_{M_b,b',\mu_b}$ for each $(M_b,\mu_b) \in \mathcal{I}^{G,\mu}_{M_b,b'}$. Then Conjecture 1.4.4 holds for $\mathrm{Mant}_{G,b,\mu}$.*

Proof. We compute

$$\mathrm{Mant}_{G,b,\mu}(e(J_b)LJ(\delta^{\frac{1}{2}}_{G,P_b} \otimes I^{M_b}_{M_S}(\rho))) =$$

$$\sum_{\substack{(M_b,\mu_b)\\ \in \mathcal{I}^{G,\mu}_{M_b,b'}}} \mathrm{Ind}^G_{P_b}(\mathrm{Mant}_{M_b,b',\mu_b}(e(J_b)LJ(\delta^{\frac{1}{2}}_{G,P_b} \otimes I^{M_b}_{M_S}(\rho)))) \otimes [1][|\cdot|^{\langle \rho_G, \mu_b - \mu\rangle}]$$

so by assumption this equals

$$\sum_{(M_b,\mu_b)\in \mathcal{I}^{G,\mu}_{M_b,b'}} [\mathrm{Ind}^G_{P_b}(\delta^{\frac{1}{2}}_{G,P_b} \otimes I^{M_b}_{M_S}(\rho))]$$

$$\boxtimes \left[\bigoplus_{(M_S,\mu_S)\in \mathrm{Rel}^{M_b,\mu_b}_{M_S,b'}} r_{-\mu_S} \circ LL(I^{M_b}_{M_S}(\rho))|_{W_{E_{\{\mu_S\}_{M_S}}}} |\cdot|^{S} \right],$$

where $S = -\langle \rho_{M_b}, \mu_b \rangle + \langle \rho_G, \mu_b - \mu \rangle - \langle \frac{\det(Ad_{N_n}(M_b))|_T}{2}, \mu_b \rangle = -\langle \rho_G, \mu \rangle$ (following the discussion in Remark 1.3.3). Now simplifying the above expression, we get

$$\sum_{(M_b,\mu_b)\in \mathcal{I}^{G,\mu}_{M_b,b'}} [I^G_{M_S}(\rho)] \left[\bigoplus_{\substack{(M_S,\mu_S) \\ \in \mathrm{Rel}^{M_b,\mu_b}_{M_S,b'}}} r_{-\mu_S} \circ LL(I^G_M(\rho))|_{W_{E_{\{\mu_S\}_{M_S}}}} |\cdot|^{-\langle \rho_G, \mu \rangle} \right].$$

Thus, we are reduced to showing that

$$\mathrm{Rel}^{G,\mu}_{M_S,b} = \coprod_{(M_b,\mu_b)\in \mathcal{I}^{G,\mu}_{M_b,b'}} \mathrm{Rel}^{M_b,\mu_b}_{M_S,b'}.$$

This is just Lemma 1.4.2. □

With Proposition 1.4.5 in hand, it remains to show that if Conjecture 1.4.4 holds for all non-basic $b \in \mathbf{B}(G,\mu)$ then it holds for the basic b. The key to proving this is Theorem 1.3.2.

We begin by making some observations about $r_{-\mu}$. Since we assumed $I^G_{M_S}(\rho)$ is irreducible, we have $LL(I^G_{M_S}(\rho)) = LL(\rho)$ and the image of this representation lies inside ${}^L M_S \subset {}^L G$. Thus, the term $[r_{-\mu} \circ LL(I^G_{M_S}(\rho))|_{W_{E_{\{\mu\}_G}}}]$ depends only on the restriction $r_{-\mu}|_{\widehat{M_S} \rtimes W_{E_{\{\mu\}_G}}}$. Since μ is assumed to be minuscule, we have the following equality of $\widehat{M_S}$ representations.

$$r_{-\mu}|_{\widehat{M_S}} = \bigoplus_{(M_S,\mu_S)\in \mathcal{C}_G, \mu_S \sim_G \mu} r_{-\mu_S}|_{\widehat{M_S}}. \tag{10}$$

We further note that each $r_{-\mu_S}$ is a representation of $\widehat{M_S} \rtimes W_{E_{\{\mu_S\}_{M_S}}}$. Since $\{(M_S,\mu_S) \in \mathcal{C}_G : \mu_S \sim_G \mu\}$ is invariant under the natural action of $W_{E_{\{\mu\}_G}}$, it follows from the proof of 1.3.10 that the right-hand side of the above equation can be promoted to a representation of $\widehat{M_S} \rtimes W_{E_{\{\mu\}_G}}$ so that 10 is an equality of $W_{E_{\{\mu\}_G}}$ representations.

Now we recall the following subsets of W^{rel} defined in §2.11 of [BZ77].

Definition 1.4.6. *Let M_S, N_S be standard Levi subgroups of G. We define*

$$W^{M_S} = \{w \in W^{\mathrm{rel}} : w(M_S \cap B) \subset B\},$$

$$W^{M_S,N_S} = \{w \in W^{\mathrm{rel}} : w(M_S \cap B) \subset B, w^{-1}(N_S \cap B) \subset B\}$$

We record the following lemma:

Lemma 1.4.7. *[BZ77, Lem 2.11] Suppose M_S, N_S are standard Levi subgroups of G and $w \in W^{M_S,N_S}$. Then $w(M_S) \cap N_S$ and $w^{-1}(N_S) \cap M_S$ are standard Levi subgroups.*

Lemma 1.4.8. *Suppose M_S is a standard Levi subgroup of G. Then W^{M_S} contains a unique representative of each left coset of $W^{\mathrm{rel}}_{M_S}$. Equivalently, $(W^{M_S})^{-1}$ contains a unique representative of each right coset of $W^{\mathrm{rel}}_{M_S}$.*

Proof. Suppose $w \in W^{\mathrm{rel}}$. Then $B' = w^{-1}(B)$ is a Borel subgroup of G containing the maximal torus T. Since B' contains exactly one of each root and its negative, $B' \cap M_S$ is a Borel subgroup of M_S. In particular, since $B' \cap M_S, B \cap M_S$ are both Borel subgroups of M_S containing T, there exists a $w_m \in W^{\mathrm{rel}}_{M_S}$ so that

$$w_m(B \cap M_S) = B' \cap M_S.$$

Then $ww_m(B \cap M_S) = B \cap M_S \subset B$, so that $ww_m \in W^{M_S}$. Thus the coset $wW^{\mathrm{rel}}_{M_S}$ contains at least one element of W^{M_S}.

Suppose $ww_m, ww'_m \in wW^{\mathrm{rel}}_{M_S} \cap W^{M_S}$. In particular, $ww'_m = (ww_m)(w_m^{-1}w'_m)$. But ww_m takes all positive roots of M_S to positive roots of G, and equivalently, negative roots of M_S to negative roots of G. Thus, if $w_m^{-1}w'_m$ takes any positive root of M_S to a negative root of M_S, then ww'_m cannot be an element of W^{M_S}. In particular, this implies that $w_m^{-1}w'_m = 1$ which shows uniqueness. □

Lemma 1.4.9. *Suppose M_S is a standard Levi subgroup of G and $x \in \mathfrak{A}^+_{\mathbb{Q},M_S}$ and $w \in W^{rel}$. Then $w(x) = x$ if and only if $w \in W^{\mathrm{rel}}_{M_S}$.*

Proof. Recall that by assumption, G is quasi-split over $\mathbb{Q}_p$ and A is a split torus of G of maximal rank. Pick $g \in N_G(A)(\overline{\mathbb{Q}_p})$ so that g projects to $w \in W^{\mathrm{rel}} = N_G(A)(\overline{\mathbb{Q}_p})/Z_G(A)(\overline{\mathbb{Q}_p})$. Then the equation $w(x) = x$ implies that $g \in Z_G(x)(\overline{\mathbb{Q}_p})$. The centralizer of a cocharacter is a Levi subgroup, and since $x \in \mathfrak{A}^+_{\mathbb{Q},M_S}$, we have $Z_G(x) = M_S$. In particular, $g \in N_{M_S}(A)(\overline{\mathbb{Q}_p})$ and so $w \in W^{\mathrm{rel}}_{M_S}$.

We remark that x is not a cocharacter, but that $Z_G(x)$ still makes sense as there is an induced action of G on $X_*(A)_{\mathbb{Q}}$. □

We can now prove the following key proposition.

Proposition 1.4.10. *Fix $(G, \mu) \in \mathcal{C}_G$ and suppose $(M_S, \mu_S) \in \mathcal{C}_G$ satisfies $\mu_S \sim_G \mu$. Then there exists a unique $b \in \mathbf{B}(G, \mu)$ and a unique $w \in W^{M_S,M_b}$ so that $(w(M_S), w(\mu_S)) \in \mathrm{Rel}^{G,\mu}_{w(M_S),b}$.*

Proof. We first discuss uniqueness. By assumption, $w(M_S)$ is a standard Levi subgroup. Then w induces an equality $wW^{\mathrm{rel}}_{M_S}w^{-1} = W^{\mathrm{rel}}_{w(M_S)}$.

In particular, W^{rel} acts on $X_*(T)$ through Corollary 1.5.3 and it follows that

$$w(\theta_{M_S}(\mu_S)) = \theta_{w(M_S)}(w(\mu_S)).$$

Since $(w(M_S), w(\mu_S)) \in \mathrm{Rel}^{G,\mu}_{w(M_S),b}$, it follows that $\theta_{w(M_S)}(w(\mu_S))$ is dominant in the relative root system. In particular, $\theta_{w(M_S)}(w(\mu_S))$ must be equal to the unique element x in the W^{rel} orbit of $\theta_{M_S}(\mu_S)$ which is dominant in $\mathfrak{A}_{\mathbb{Q}}$. Now $x \in \mathfrak{A}^+_{M_{S'},\mathbb{Q}}$ for a unique $M_{S'}$. Since any $(M_b, \mu_b) \in \mathcal{T}_{G,b,\mu}$ is definitionally strictly decreasing, it follows that even though we can't yet conclude the uniqueness of b, we have shown that any other b_1 must satisfy $M_{b_1} = M_b = M_{S'}$.

Now, suppose we had $w, w' \in W^{M_S,M_b}$ such that

$$w(\theta_{M_S}(\mu_S)) = x = w'(\theta_{M_S}(\mu_S)).$$

Then in particular, $w'w^{-1}$ stabilizes x and so by Lemma 1.4.9, $w'w^{-1} \in W^{\mathrm{rel}}_{M_b}$. So w and w' are in the same right coset $W^{\mathrm{rel}}_{M_b}w$. However, $W^{M_S,M_b} \subset (W^{M_b})^{-1}$. By Lemma 1.4.8, $(W^{M_b})^{-1}$ contains a unique representative of each right coset of $(W^{M_b})^{-1}$ and so there is a unique $w \in (W^{M_b})^{-1}$ satisfying $w(\theta_{M_S}(\mu_S)) = x$. In particular, this implies that $w = w'$. Thus, we have shown that w is unique, if it exists. There is exactly one $\mu' \in X_*(T)$ such that $\mu' \sim_{M_b} w(\mu)$ and μ' is dominant in M_b. Then $(M_b, \mu') \in \mathcal{T}_{G,b,\mu}$ for at most one $b \in \mathbf{B}(G,\mu)$. This shows uniqueness.

To prove existence, we again define x to be the unique dominant element in the W^{rel}-orbit of $\theta_{M_S}(\mu_S)$. Define $M_{S'} = Z_G(x)$ and take the unique $w \in (W^{M_{S'}})^{-1}$ such that $w(\theta_{M_S}(\mu_{M_S})) = x$. We would like to show that $w \in W^{M_S,M_{S'}}$.

By definition,

$$w(M_S) \subset w(Z_G(\theta_{M_S}(\mu_S))) = Z_G(x) = M_{S'}.$$

Suppose it is not the case that $w(M_S \cap B) \subset B$. In particular, w maps a positive root r of M_S to a root $w(r)$ of $M_{S'}$ which is not positive. In particular, $-w(r)$ is positive and so $w^{-1}(-w(r)) = -r$ is positive (since $w \in (W^{M_{S'}})^{-1}$). But this is clearly a contradiction. Thus, in fact $w \in W^{M_S,M_{S'}}$.

By Lemma 1.4.7, $w(M_S) \cap M_{S'} = w(M_S)$ is a standard Levi. It remains to show that $(w(M_S), w(\mu_S))$ is a cocharacter pair and an element of $\mathrm{Rel}^{G,\mu}_{w(M_S),b}$. Now if r is a positive root in the absolute root system of $w(M_S)$, then $\langle r, w(\mu_S)\rangle = \langle w^{-1}(r), \mu_S\rangle \geqslant 0$ (since (M_S, μ_S) is a cocharacter pair and $w^{-1}(r)$ is a positive root of M_S). Thus, $(w(M_S), w(\mu_S))$ is a cocharacter pair. By construction, $x = \theta_{w(M_S)}(w(\mu_S)) = \theta_{M_{S'}}(w(\mu_S))$. Suppose $\mu' \in X_*(T)$ is the unique cocharacter conjugate to $w(\mu_S)$ in $M_{S'}$ and dominant in $M_{S'}$. Then by Corollary 1.2.9, $(M_{S'}, \mu')$ is strictly decreasing and therefore $(M_{S'}, \mu') \in \mathcal{T}_{G,b,\mu}$ for some b and so $(w(M_S), w(\mu_S)) \in \mathrm{Rel}^{G,\mu}_{w(M_S),b}$. $\square$

Corollary 1.4.11. *Fix a cocharacter pair $(G, \mu) \in \mathcal{C}_G$ and a standard Levi subgroup M_S of G. For $b \in \mathbf{B}(G, \mu)$, define W_b by $\{w \in W^{M_S,M_b} : w(M_S) \subset M_b\}$. Then the previous lemma gives a bijection*

$$\{(M_S, \mu_S) \in \mathcal{C}_G : \mu_S \sim_G \mu\} \cong \coprod_{b\in\mathbf{B}(G,\mu)} \coprod_{w\in W_b} \mathrm{Rel}^{G,\mu}_{w(M_S),b}.$$

Proof. By the construction in the previous proposition, it is clear that given an $(M_S, \mu_S) \in \mathcal{C}_G$ we get an element of the right-hand side of the above equation. Conversely, an element $(w(M_S), \mu')$ of the right-hand side comes with a fixed $w \in W_b$ and so we can recover $(M_S, w^{-1}(\mu'))$ on the left-hand side. $\square$

We are now ready to finish the proof of Conjecture 1.4.4. By inductive assumption we assume we've shown Conjecture 1.4.4 for G with maximal torus of rank less than n. Then Proposition 1.4.5 implies that Conjecture 1.4.4 holds for G with maximal torus of rank n in the case where b is not basic. It remains to prove the basic case, for which it suffices to show that Theorem 1.3.2 is compatible with Conjecture 1.4.4. We have

$$\sum_{b\in\mathbf{B}(G,\mu)} \mathrm{Mant}_{G,b,\mu}(\mathrm{Red}_b(I^G_{M_S}(\rho)))$$

$$= \sum_{b\in\mathbf{B}(G,\mu)} \mathrm{Mant}_{G,b,\mu}(e(J_b)LJ(\delta^{\frac{1}{2}}_{P_b} \otimes J^G_{P^{op}_b} I^G_{M_S}(\rho))).$$

By the geometric lemma of [BZ77] and noting that W^{M_S,M_b} defined with respect to B is equal to the analogous set defined with respect to B^{op}, we have

$$J^G_{P^{op}_b} I^G_{M_S}(\rho) = \sum_{w\in W^{M_S,M_b}} I^{M_b}_{M'_b}(w(J^{M_S}_{P'^{op}_S}(\rho))),$$

where $M'_S = M_S \cap w^{-1}(M_b), M'_b = w(M_S) \cap M_b$. By the assumption that ρ is supercuspidal we must have $M'_S = M_S$ and $M'_b = w(M_S)$. In this case, we have from the geometric lemma that $w(M_S)$ is a standard Levi subgroup. Thus we get that the previous expression is equal to

$$\sum_{b\in\mathbf{B}(G,\mu)} \mathrm{Mant}_{G,b,\mu}(e(J_b) \sum_{w\in W_b} LJ(\delta^{\frac{1}{2}}_{P_b} \otimes I^{M_b}_{w(M_S)}(w(\rho))),$$

where $W_b \subset W^{M_S,M_b}$ is the subset of w such that $w(M_S) \subset M_b$. We now apply Corollary 1.4.4 by inductive assumption to get

$$\sum_{b \in \mathbf{B}(G,\mu)} \sum_{w \in W_b} [I^G_{w(M_S)}(w(\rho))]$$

$$\boxtimes \left[\bigoplus_{(w(M_S),\mu') \in \mathrm{Rel}^{G,\mu}_{w(M_S),b}} r_{-\mu'} \circ LL(I^G_{w(M_S)}(w(\rho)))|_{W_{E_{\{\mu'\}_{w(M_S)}}}} |\cdot|^{-\langle \rho_G, \mu \rangle} \right].$$

By [BZ77, Thm 2.9], we have that

$$[I^G_{w(M_S)}(w(\rho))] = [I^G_{M_S}(\rho)],$$

and since $I^G_{M_S}(\rho)$ is assumed to be irreducible, we have

$$LL(I^G_{M_S}(\rho)) = LL(\rho).$$

Finally, we note that $W_{E_{\{w^{-1}(\mu')\}_{M_S}}} = W_{E_{\{\mu'\}_{w(M_S)}}}$ and we have an equality

$$[r_{-\mu'} \circ LL(w(\rho))|_{W_{E_{\{\mu'\}_{w(M_S)}}}}] = [r_{-w^{-1}(\mu')} \circ LL(\rho)|_{W_{E_{\{w^{-1}(\mu')\}_{M_S}}}}].$$

Thus the above expression becomes

$$\sum_{b \in \mathbf{B}(G,\mu)} \sum_{w \in W_b} [I^G_{M_S}(\rho)]$$

$$\boxtimes \left[\bigoplus_{(w(M_S),\mu') \in \mathrm{Rel}^{G,\mu}_{w(M_S),b}} r_{-w^{-1}(\mu')} \circ LL(\rho)|_{W_{E_{\{w^{-1}(\mu')\}_{M_S}}}} |\cdot|^{-\langle \rho_G, \mu \rangle} \right].$$

By Corollary 1.4.11 this equals

$$[I^G_{M_S}(\rho)][\bigoplus_{(M_S,\mu_S):\mu_S \sim_G \mu} r_{-\mu_S} \circ LL(\rho)|_{W_{E_{\{\mu_S\}_{M_S}}}} |\cdot|^{-\langle \rho_G, \mu \rangle}].$$

Finally, we apply the decomposition given by Equation (10) to get

$$[I^G_{M_S}(\rho)][r_{-\mu}|_{\widehat{M_S} \rtimes W_{E_{\{\mu\}_G}}} \circ LL(\rho)|_{W_{E_{\{\mu\}_G}}} |\cdot|^{-\langle \rho_G, \mu \rangle}],$$

which is the desired result.

Finally, we show that Conjecture 1.4.4 holds even if $I^G_{M_S}(\rho)$ is not irreducible. Our verification that Conjecture 1.4.4 is compatible with the Harris-Viehmann conjecture did not rely on the irreducibility of $I^G_{M_S}(\rho)$. Thus in the case where we do not assume $I^G_{M_S}(\rho)$ is irreducible, it would suffice to show that Conjecture 1.4.4 is true in the case where b is basic. If b is basic, then $M_b = G$ so we have

$$\mathrm{Mant}_{G,b,\mu}(e(J_b)LJ(\delta^{\frac{1}{2}}_{G,P_b} I^{M_b}_{M_S}(\rho))) = \mathrm{Mant}_{G,b,\mu}(\mathrm{Red}_b(I^G_{M_S}(\rho))).$$

This can now be computed by cocharacter pairs using the results of §3. If $I_{M_S}^G(\rho)$ is assumed to be irreducible, then for each cocharacter pair $(M_{S'}, \mu_{S'})$ of G, we have

$$[M_{S'}, \mu_{S'}](I_{M_S}^G(\rho)) = (\mathrm{Ind}_{P_{S'}}^G \circ [\mu_{S'}])(\delta_{P_S}^{\frac{1}{2}} \otimes J_{P_{S'}^{op}}^G I_{M_S}^G(\rho)) \otimes [1][|\cdot|^{\langle \rho_G, \mu_{S'} - \mu\rangle}]$$

$$= (\mathrm{Ind}_{P_{S'}}^G \circ [\mu_{S'}])(\bigoplus_{w \in W_\rho} \delta_{P_{S'}}^{\frac{1}{2}} \otimes I_{w(M_S)}^{M_{S'}}(w(\rho))) \otimes [1][|\cdot|^{\langle \rho_G, \mu_{S'} - \mu\rangle}],$$

where W_ρ is the subset of $w \in W^{M_S, M_{S'}}$ such that $w(M_S) \subset M_{S'}$. Then the above equals

$$[I_{M_S}^G(\rho)] \left[\bigoplus_{w \in W_\rho} r_{-\mu_{S'}} \circ LL(w(\rho)) |\cdot|^{-\langle \rho_G, \mu\rangle} \right].$$

Thus we see that applying various $[M_{S'}, \mu_{S'}]$ to $I_{M_S}^G(\rho)$ in the irreducible case will always yield the same term of $\mathrm{Groth}(G(\mathbb{Q}_p))$ (namely $[I_{M_S}^G(\rho)]$) and so when evaluating $\mathrm{Mant}_{G,b,\mu}(\mathrm{Red}_b(I_{M_S}^G(\rho))$ as a sum of cocharacter pairs, the different Galois terms must cancel to give Conjecture 1.4.4. Thus, if we can show that in the reducible case, the $\mathrm{Groth}(G(\mathbb{Q}_p))$ part of each $[M_{S'}, \mu_{S'}](I_{M_S}^G(\rho))$ is fixed and the Galois part is identical to the irreducible case, then Conjecture 1.4.4 must hold for this case as well.

The first part of our previous computation did not depend on the irreducibility of $I_{M_S}^G(\rho)$ so we still have

$$[M_{S'}, \mu_{S'}](I_{M_S}^G(\rho))$$

$$= (\mathrm{Ind}_{P_{S'}}^G \circ [\mu_{S'}])(\bigoplus_{w \in W_\rho} \delta_{P_{S'}}^{\frac{1}{2}} \otimes I_{w(M_S)}^{M_{S'}}(w(\rho))) \otimes [1][|\cdot|^{\langle \rho_G, \mu_{S'} - \mu\rangle}].$$

Suppose now that $I_{w(M_S)}^{M_{S'}}(w(\rho)) = \pi_1 \oplus ... \oplus \pi_k$. Then using that for all i, we have $LL(\pi_i) = LL(w(\rho))$,

$$[\mu_{S'}](I_{w(M_S)}^{M_{S'}}(w(\rho))) = \oplus_{i=1}^k [\pi_i][r_{-\mu_{S'}} \circ LL(\pi_i) \otimes |\cdot|^{-\langle \rho_{M_{S'}}, \mu_{S'}\rangle}]$$

$$= \oplus_{i=1}^k [\pi_i][r_{-\mu_{S'}} \circ LL(w(\rho)) \otimes |\cdot|^{-\langle \rho_{M_{S'}}, \mu_{S'}\rangle}]$$

$$= [I_{w(M_S)}^{M_{S'}}(w(\rho))][r_{-\mu_{S'}} \circ LL(w(\rho)) \otimes |\cdot|^{-\langle \rho_{M_{S'}}, \mu_{S'}\rangle}]$$

Thus, the expression for $[M_{S'}, \mu_{S'}](I_{M_S}^G(\rho))$ becomes

$$[I_{M_S}^G(\rho)] \left[\bigoplus_{w \in W^{M_S, M_{S'}}} r_{-\mu_{S'}} \circ LL(w(\rho)) |\cdot|^{-\langle \rho_G, \mu\rangle} \right],$$

as desired.

1.5.1. **Examples.** In this section, we give an example to show that even in the unramified EL-type case, we do not get an expression as simple as Harris's conjecture for $\mathrm{Mant}_{G,b,\mu}(\rho)$ for general ρ. We generally use the same notation as in the computation in Example 1.3.6.

Let $G = \mathrm{GL}_4$, suppose μ has weights $(1^2, 0^2)$, and take b basic. Let T be the diagonal maximal torus and B be the Borel subgroup of upper triangular matrices. Then the set of cocharacter pairs less than or equal to (G, μ) is as follows.

$$\begin{tikzcd}
 & & (\mathrm{GL}_4, (1^2, 0^2)) \arrow[dll] \arrow[d] \arrow[drr] & & \\
(\mathrm{GL}_3 \times \mathrm{GL}_1, (1^2, 0)(0)) \arrow[d] \arrow[drr] & & (\mathrm{GL}_2^2, (1^2)(0^2)) & & (\mathrm{GL}_1 \times \mathrm{GL}_3, (1)(1, 0^2)) \arrow[dll] \arrow[d] \\
(\mathrm{GL}_2 \times \mathrm{GL}_1^2, (1^2)(0)(0)) & & (\mathrm{GL}_1 \times \mathrm{GL}_2 \times \mathrm{GL}_1, (1)(1, 0)(0)) \arrow[d] & & (\mathrm{GL}_1^2 \times \mathrm{GL}_2, (1)(1)(0^2)) \\
 & & (\mathrm{GL}_1^4, (1)(1)(0)(0)) & &
\end{tikzcd}$$

Let $\rho \in \mathrm{Groth}(\mathrm{GL}_1(\mathbb{Q}_p))$ and consider π the unique essentially square integrable quotient of $I^G_{\mathrm{GL}_1^4}(\rho \boxtimes \rho(1) \boxtimes \rho(2) \boxtimes \rho(3))$. We want to compute $\mathrm{Mant}_{G,b,\mu}(\mathrm{Red}_b(\pi))$.

We introduce some notation which will allow us to describe the answer to this question. The results of §2 of [Zel80] show that $I^G_{\mathrm{GL}_1^4}(\rho \boxtimes \rho(1) \boxtimes \rho(2) \boxtimes \rho(3))$ has exactly 8 irreducible subquotients. If π' is one such subquotient, then $J^G_{B^{op}}(\pi')$ will be a finite sum of representations of the form $\rho(\lambda(0)) \boxtimes \rho(\lambda(1)) \boxtimes \rho(\lambda(2)) \boxtimes \rho(\lambda(3))$ where λ is a permutation of $\{0, 1, 2, 3\}$. In particular, if Ω denotes the set of all such permutations of $\rho \boxtimes \rho(1) \boxtimes \rho(2) \boxtimes \rho(3)$, then each permutation lies in the Jacquet module of exactly one irreducible subquotient of $I^G_{\mathrm{GL}_1^4}(\rho \boxtimes \rho(1) \boxtimes \rho(2) \boxtimes \rho(3))$ so that the irreducible subquotients correspond to a partition of Ω. We use the following shorthand: we define the notation (0123) to refer to the representation $\rho(0) \boxtimes \rho(1) \boxtimes \rho(2) \boxtimes \rho(3)$. Following Zelevinsky, our 8 irreducible subquotients naturally correspond to vertices of a 3-dimensional cube, and so we denote them by binary strings of length 3. Then if we denote the subset of Ω corresponding to some subquotient

π' by $\Omega(\pi')$,we have

$$\begin{aligned}
\Omega([000]) &= \{(3210)\} \\
\Omega([100]) &= \{(2310), (2130), (2103)\} \\
\Omega([010]) &= \{(3120), (1320), (1302), (3102), (1032)\} \\
\Omega([001]) &= \{(3201), (3021), (0321)\} \\
\Omega([110]) &= \{(1203), (1023), (1230)\} \\
\Omega([101]) &= \{(2013), (2031), (0213), (0231), (2301)\} \\
\Omega([011]) &= \{(3012), (0312), (0132)\} \\
\Omega([111]) &= \{(0123)\}
\end{aligned}$$

In particular, our representation π corresponds to $[111]$ under the above notation. A tedious computation using Theorem 1.3.12 yields the following

Proposition 1.5.1.

$$\begin{aligned}
\mathrm{Mant}_{G,b,\mu}(\mathrm{Red}_b(\pi)) = {} & [111][\widetilde{LL(\rho)}^2(-7) + \widetilde{LL(\rho)}^2(-6)] \\
& - ([110][\widetilde{LL(\rho)}^2(-5)] + [011][\widetilde{LL(\rho)}^2(-5)]) \\
& + [010][\widetilde{LL(\rho)}^2(-4)] \\
& - [000][\widetilde{LL(\rho)}^2(-3)]
\end{aligned}$$

We finish by remarking that the set of cocharacter pairs less than or equal to (G, μ) has some special properties in the above case that make the general case more complicated.

For instance, each $\mathcal{T}_{G,b,\mu}$ has at most a single element. However, if G has a nontrivial action by Γ, this need not be the case.

In the case we consider, we have a single cocharacter pair for each Levi subgroup. In general, this need not be the case. For instance, if $G = \mathrm{GL}_5, \mu = (1^3, 0^2)$, then $(\mathrm{GL}_3 \times \mathrm{GL}_2, (1^3)(0^2))$, $(\mathrm{GL}_3 \times \mathrm{GL}_2, (1^2, 0)(1, 0))$ are both less than (G, μ).

Further, in the above example, each cocharacter pair (M_S, μ_S) had the property that μ_S was dominant as a cocharacter of G relative to B. In general this need not be the case. In fact, $(\mathrm{GL}_1^5, (1)(1)(0)(1)(0)) \leqslant (\mathrm{GL}_5, (1^3, 0^2))$.

1.5.2. **Relative Root Systems and Weyl Chambers.** In this section we prove a fact about root systems that is needed in the text (for instance in the proof of Proposition 1.2.21). We assume that G is a quasisplit group over a field k of characteristic 0 and pick a separable closure k^{sep}. We fix a split k-torus A of maximal rank in G and choose a maximal torus T and Borel subgroup B both defined over k and such that $A \subset T \subset B$. Associated to this data, we have an absolute root datum

$$(X^*(T), \Phi^*(G,T), X_*(T), \Phi_*(G,T)),$$

and a relative root datum

$$(X^*(A), \Phi^*(G,A), X_*(A), \Phi_*(G,A)).$$

Our choice of B also gives sets Δ of absolute simple roots and ${}_k\Delta$ of relative simple roots. Note that we also have a natural restriction map

$$\mathrm{res} : X^*(T) \to X^*(A),$$

and that by definition an absolute root in $\Phi^*(G,T)$ restricts to an element of $\Phi^*(G,A) \cup \{0\}$.

We record two standard consequences of our assumption that G is quasisplit.

Proposition 1.5.2. *Let G be quasisplit and use the notations as above. Then,*

(1) The centralizer $Z_G(A) = T$,
(2) We have $\mathrm{res}(\Delta) = {}_k\Delta$. The key point being that no absolute simple root restricts to the trivial character.

We have the following easy consequence on the structure of the Weyl group of the relative root system. Recall that the absolute Weyl group W equals

$$N_G(T)(k^{sep})/Z_G(T)(k^{sep}),$$

and the relative Weyl group W^{rel} is $N_G(A)(k)/Z_G(A)(k)$.

Corollary 1.5.3. *We have the following equality: $W^{\mathrm{rel}} = W^{\Gamma}$, where $\Gamma = \mathrm{Gal}(k^{sep}/k)$.*

Proof. It suffices to show that $Z_G(A) = Z_G(T)$ and that $N_G(A)(k) = N_G(T)(k)$. For the first equality, we note that by the quasisplit assumption, $Z_G(A) = T = Z_G(T)$. For the second equality, we note that any $g \in N_G(A)(k)$ must also normalize the centralizer of A which is T. Conversely, if $g \in N_G(T)(k)$ then g normalizes the unique maximal k-split sub-torus of T which is A. $\square$

Define the absolute Weyl chamber $\overline{C}^*_{\mathbb{Q}} \subset X^*(T)_{\mathbb{Q}}$ by $\{x \in X^*(T)_{\mathbb{Q}} : \langle \check{\alpha}, x \rangle \geqslant 0, \alpha \in \Delta\}$ and define the relative Weyl chamber ${}_k\overline{C}^*_{\mathbb{Q}} \subset X^*(A)_{\mathbb{Q}}$ analogously. The key result of this section is that

$$\mathrm{res}(\overline{C}^*_{\mathbb{Q}}) \;=\; {}_k\overline{C}^*_{\mathbb{Q}}.$$

Despite its simple statement, the author has been unable to locate a convenient reference of this fact. For $x \in X^*(T)_{\mathbb{Q}}$ and $\alpha \in \Delta$, we need to relate $\langle \breve{\alpha}, x\rangle$ and $\langle \widetilde{\mathrm{res}(\alpha)}, \mathrm{res}(x)\rangle$. If we let $\sigma_\alpha \in W$ be the reflection corresponding to the root α, then we have

$$x - \sigma_\alpha(x) = \langle \breve{\alpha}, x\rangle \alpha. \tag{11}$$

and analogously for $\widetilde{\mathrm{res}(\alpha)}$. Thus it will suffice to relate σ_α and $\sigma_{\mathrm{res}(\alpha)}$.

Note that since B is defined over k, we have $\gamma(\Delta) = \Delta$ for every $\gamma \in \Gamma$. Moreover, for each $\alpha \in \Delta$, we have $\mathrm{res}(\gamma(\alpha)) = \mathrm{res}(\alpha)$. After all, Γ acts trivially on $X^*(A)_{\mathbb{Q}}$ and the restriction map is Γ-equivariant.

Now fix $\alpha \in \Delta$ and let W_α be the subgroup of W generated by the elements $\sigma_{\gamma(\alpha)}$ for each $\gamma \in \Gamma$. We claim that if we can find a nontrivial Γ-invariant element of W_α, then it must equal $\sigma_{\mathrm{res}(\alpha)}$. To prove this, we first recall the construction of σ_α and $\sigma_{\mathrm{res}(\alpha)}$ (see [Bor91, pg 230]) for instance). Given a root $\alpha \in \Phi^*(G, T)$ we can define a group $G_\alpha = Z_G(T_\alpha)$ where $T_\alpha = \ker(\alpha)^0 \subset T$. Then $N_{G_\alpha}(T)(k^{sep})/Z_{G_\alpha}(T)(k^{sep})$ embeds into W and has a unique nontrivial element which is σ_α. Analogously, we define $A_{\mathrm{res}(\alpha)}$ and $G_{\mathrm{res}(\alpha)} = Z_G(A_{\mathrm{res}(\alpha)})$. Then $N_{G_{\mathrm{res}(\alpha)}}(A)(k)/Z_{G_{\mathrm{res}(\alpha)}}(A)(k)$ embeds into W^{rel} and has a unique nontrivial element that is identified with $\sigma_{\mathrm{res}(\alpha)}$.

Now, by Corollary 1.5.3 we have

$$N_{G_{\mathrm{res}(\alpha)}}(A)(k)/Z_{G_{\mathrm{res}(\alpha)}}(A)(k) = N_{G_{\mathrm{res}(\alpha)}}(T)(k)/Z_{G_{\mathrm{res}(\alpha)}}(T)(k).$$

Thus to complete the proof of the claim, we need to show that

$$N_{G_\alpha}(T)(k^{sep})/Z_{G_\alpha}(T)(k^{sep}) \hookrightarrow N_{G_{\mathrm{res}(\alpha)}}(T)(k^{sep})/Z_{G_{\mathrm{res}(\alpha)}}(T)(k^{sep}). \tag{12}$$

After all, the unique nontrivial Γ-invariant element of the group on the right is $\sigma_{\mathrm{res}(\alpha)}$ and the group on the left contains σ_α. Since we get the same equation if we replace α everywhere with $\gamma(\alpha)$, this will imply that

$$W_\alpha \subset N_{G_{\mathrm{res}(\alpha)}}(T)(k^{sep})/Z_{G_{\mathrm{res}}}(T)(k^{sep}).$$

Now, Equation (12) follows from the fact that

$$Z_{G_\alpha}(T) = Z_{G_{\mathrm{res}(\alpha)}}(T) = T$$

and

$$N_{G_\alpha}(T) \subset N_{G_{\mathrm{res}(\alpha)}}(T).$$

We are now interested in finding a nontrivial Γ-invariant element of the group W_α defined above. In fact, W_α will be a finite Coxeter group and the element we seek is the unique element of longest length. We need to compute this element explicitly, which we now do. We treat two cases. Suppose first that the elements of the Γ-orbit of σ_α commute pairwise. Then clearly the product $\prod_{\gamma \in \Gamma/\mathrm{stab}(\sigma_\alpha)} \sigma_{\gamma(\alpha)}$ is Γ-invariant.

In the second case, suppose that the Γ-orbit of σ_α has precisely two elements which we denote X and Y. Then we have $(XY)^k = 1$ for some $k \geqslant 2$ which we assume to be minimal. If k is even, then $(XY)^{k/2}$ is invariant and nontrivial and if k is odd, then $Y(XY)^{(k-1)/2}$ is invariant and nontrivial.

We now prove that any Γ action on the simple roots Δ of G is a combination of these cases. The action of Γ on Δ induces an action on the associated (not necessarily connected) Dynkin diagram D. Each $\gamma \in \Gamma$ maps connected components of D to connected components and so there is an induced action of Γ on the set of connected components $\pi_0(D)$.

Now fix an $\alpha \in \Delta$ and consider the Γ-orbit $\Gamma\alpha$ of α. Suppose D^i is a connected component of D such that $D^i \cap \Gamma\alpha \neq \varnothing$. Then via the classification of connected Dynkin diagrams, we see that $\Gamma\alpha \cap D^i$ contains either a single node, 2 non-adjacent nodes, 2 adjacent nodes, or 3 nodes where no two are adjacent. In particular, these are all covered by the cases we considered above, so we can find an element w_i of W_α that is invariant by the action of $\mathrm{stab}(D^i) \subset \Gamma$. Then $\Gamma\alpha$ consists of finitely many disjoint copies of one of the above possibilities and so we see that $\prod_i w_i$ is Γ-invariant and an element of W_α and therefore equal to $\sigma_{\mathrm{res}(\alpha)}$. Equipped with this description, we now give a proof of the main result of this section.

Proposition 1.5.4. *We continue to observe the assumptions made above. In particular, G is a quasisplit group over k. Then the map* $\mathrm{res} : X^*(T) \twoheadrightarrow X^*(A)$ *induces an equality*

$$\mathrm{res}(\overline{C}^*_{\mathbb{Q}}) = {}_k\overline{C}^*_{\mathbb{Q}}.$$

Proof. We first show that $\mathrm{res}(\overline{C}^*_{\mathbb{Q}}) \subset {}_k\overline{C}^*_{\mathbb{Q}}$. Pick $x \in \overline{C}^*_{\mathbb{Q}}$ and $\alpha \in \Delta$. Then we need to show that

$$\langle \widetilde{\mathrm{res}(\alpha)}, \mathrm{res}(x) \rangle \geqslant 0$$

or equivalently, that

$$\mathrm{res}(x) - \sigma_{\mathrm{res}(\alpha)}(\mathrm{res}(x))$$

is a non-negative multiple of $\mathrm{res}(\alpha)$. Note that res is W^Γ-equivariant (where W^Γ acts as W^{res} on $X^*(A)$). Thus, it suffices to show that

$$\mathrm{res}(x - \sigma_{\mathrm{res}(\alpha)}(x))$$

is a non-negative multiple of $\mathrm{res}(\alpha)$. Thus, we need to compute $x - \sigma_{\mathrm{res}(\alpha)}(x)$. We do so using our description of $\sigma_{\mathrm{res}(\alpha)}$.

We first consider the case where the Γ-orbit of σ_α consists of pairwise commuting elements. Equivalently, the elements of $\Gamma\alpha$ are pairwise orthogonal. Then

$$\sigma_{\mathrm{res}(\alpha)} = \sigma_{\alpha_n} \circ ... \circ \sigma_{\alpha_1}$$

for $\{\alpha_1, ..., \alpha_n\} = \Gamma\alpha$. Since x is dominant in the absolute root system, we have

$$x - \sigma_{\alpha_i}(x) = a_i\alpha_i$$

for some $a_i \geqslant 0$. Then since α_i is orthogonal to α_j for $i \neq j$, we have $\sigma_{\alpha_i}(\alpha_j) = \alpha_j$. Thus,

$$\begin{aligned} x - \sigma_{\mathrm{res}(\alpha)}(x) &= \sum_{i=1}^{n} (\sigma_{\alpha_1} \circ ... \circ \sigma_{\alpha_{i-1}})(x) - (\sigma_{\alpha_1} \circ ... \circ \sigma_{\alpha_i})(x) \\ &= \sum_{i=1}^{n} (\sigma_{\alpha_1} \circ ... \circ \sigma_{\alpha_{i-1}})(x - \sigma_{\alpha_i}(x)) \\ &= \sum_{i=1}^{n} (\sigma_{\alpha_1} \circ ... \circ \sigma_{\alpha_{i-1}})(a_i\alpha_i) \\ &= \sum_{i=1}^{n} a_i\alpha_i. \end{aligned}$$

Thus in this case,

$$\mathrm{res}(x - \sigma_{\mathrm{res}(\alpha)}(x)) = (a_1 + ... + a_n)\mathrm{res}(\alpha)$$

and $a_1 + ... + a_n \geqslant 0$ as desired.

Now we consider the case where $\Gamma\alpha = \{\alpha, \beta\}$ and α and β are adjacent in D and connected by a single edge. Then $\sigma_\alpha(\beta) = \alpha + \beta = \sigma_\beta(\alpha)$. In this case, $\sigma_{\mathrm{res}(\alpha)} = \sigma_\beta \circ \sigma_\alpha \circ \sigma_\beta$. By assumption, we have that $x - \sigma_\alpha(x) = a\alpha$ and $x - \sigma_\beta(x) = b\beta$ for a and b non-negative. Thus,

$$\begin{aligned} x - \sigma_{\mathrm{res}(\alpha)}(x) &= (x - \sigma_\beta(x)) + \sigma_\beta(x - \sigma_\alpha(x)) + (\sigma_\beta \circ \sigma_\alpha)(x - \sigma_\beta(x)) \\ &= b\beta + a(\alpha + \beta) + b\alpha \\ &= (a + b)(\alpha + \beta), \end{aligned}$$

which projects to $2(a + b)\mathrm{res}(\alpha)$ and $2(a + b) \geqslant 0$ as desired.

Finally, we must consider the case where $\Gamma\alpha$ equals $\{\alpha_1, \beta_1, ..., \alpha_n, \beta_n\}$ such that α_i and β_i are connected by a single edge in D but for $i \neq j$, neither α_i nor β_i are connected to either α_j or β_j. We compute $x - (\sigma_{\beta_i} \circ \sigma_{\alpha_i} \circ \sigma_{\beta_i})(x)$ as in the previous paragraph. Then if we let $w_i = \sigma_{\beta_i} \circ \sigma_{\alpha_i} \circ \sigma_{\beta_i}$, we have

$$\sigma_{\mathrm{res}(\alpha)} = w_1 \circ .. \circ w_n.$$

Now we can compute $x - \sigma_{\mathrm{res}(\alpha)}(x)$ as in the commuting case, substituting w_i for σ_{α_i}. We see in this case that

$$\mathrm{res}(x - \sigma_{\mathrm{res}(\alpha)}(x)) = 2(a_1 + b_1 + ... + a_n + b_n)\mathrm{res}(\alpha).$$

This concludes the proof that $\mathrm{res}(\overline{C}^*_{\mathbb{Q}}) \subset {}_k\overline{C}^*_{\mathbb{Q}}$.

It remains to show that we actually have equality. We claim it suffices to show that the fundamental weight $\delta_{\mathrm{res}(\alpha)}$ is an element of $\mathrm{res}(\overline{C}^*_{\mathbb{Q}})$. Recall that $\delta_{\mathrm{res}(\alpha)}$ is the element in the $\mathbb{Q}$-span of the relative roots defined so that the pairing with $\breve{\mathrm{res}(\alpha)}$ is 1 and the pairing is 0 with all the other relative simple coroots. To show the claim proves our result, we note there is a natural isomorphism $X^*(A)_{\mathbb{Q}} \cong X^*(A_0)_{\mathbb{Q}} \times X^*(A')_{\mathbb{Q}}$ where A_0 is the maximal k-split central torus and A' is the identity component of the intersection of A with the derived subgroup of G. Then ${}_k\overline{C}^*_{\mathbb{Q}}$ corresponds under this identification to the product of $X^*(A_0)_{\mathbb{Q}}$ with the projection of ${}_k\overline{C}^*_{\mathbb{Q}}$ to $X^*(A')$. Then we have a natural map $X^*(Z(G)^0)_{\mathbb{Q}} \twoheadrightarrow X^*(A_0)_{\mathbb{Q}}$ where $Z(G)^0$ is the identity component of the center of G and $X^*(Z(G)^0)_{\mathbb{Q}} \subset \overline{C}^*_{\mathbb{Q}}$. Thus it suffices to show that $\mathrm{res}(\overline{C}^*_{\mathbb{Q}})$ surjects onto the projection of ${}_k\overline{C}^*_{\mathbb{Q}}$ to $X^*(A')$. This latter space is identified with the set of non-negative linear combinations of the fundamental relative weights, thus proving the claim.

To prove that $\delta_{\mathrm{res}(\alpha)}$ is an element of $\mathrm{res}(\overline{C}^*_{\mathbb{Q}})$, we make use of an equivalent description of $\delta_{\mathrm{res}(\alpha)}$. It is the unique element in the $\mathbb{Q}$-span of the relative roots so that $\sigma_{\mathrm{res}(\beta)}(\delta_{\mathrm{res}(\alpha)}) = \delta_{\mathrm{res}(\alpha)}$ for $\mathrm{res}(\alpha)$ and $\mathrm{res}(\beta)$ distinct simple roots and $\sigma_{\mathrm{res}(\beta)}(\delta_{\mathrm{res}(\alpha)}) = \delta_{\mathrm{res}(\alpha)} - \mathrm{res}(\beta)$ when $\mathrm{res}(\alpha) = \mathrm{res}(\beta)$.

In the case where the elements of $\Gamma\alpha$ are mutually orthogonal, we have by the above characterization of fundamental weights that the absolute fundamental weight δ_α restricts to $\delta_{\mathrm{res}(\alpha)}$. In the case where $\Gamma\alpha$ has two elements that are connected in D, then δ_α restricts to $2\delta_{\mathrm{res}(\alpha)}$. In the final case, δ_α restricts to $2\delta_{\mathrm{res}(\alpha)}$. Thus, in all cases, we can find an element of $X^*(T)_{\mathbb{Q}}$ that restricts to $\delta_{\mathrm{res}(\alpha)}$. This completes the proof. ☐

We record an important corollary of this proposition.

Corollary 1.5.5. *Suppose $\mu, \mu' \in X_*(T)_{\mathbb{Q}}$ and $\mu \geq \mu'$. Let μ^Γ be the average of μ over its Γ orbit. Then $\mu^\Gamma \geq \mu'^\Gamma$ in $X_*(A)_{\mathbb{Q}}$. We caution that the first inequality means that $\mu - \mu'$ is a non-negative combination of absolute simple coroots, while the second means that $\mu^\Gamma - \mu'^\Gamma$ is a non-negative combination of relative simple coroots.*

Proof. Recall that the action of Γ stabilizes $\check{\Delta}$. Thus for each $\gamma \in \Gamma$, we have $\gamma(\mu) \geq \gamma(\mu')$ and so also $\mu^\Gamma \geq \mu'^\Gamma$ in the absolute root system. Thus, we are reduced to showing that if $x \in X_*(T)^\Gamma_{\mathbb{Q}}$ is a non-negative combination of simple absolute coroots, then it is also a non-negative combination of simple relative coroots (under the identification $X_*(A)_{\mathbb{Q}} = X_*(T)^\Gamma_{\mathbb{Q}}$).

Equivalently, we need to show that if x has non-negative pairing with every element of $\overline{C}^*_{\mathbb{Q}}$, then x has non-negative pairing with every element of ${}_k\overline{C}^*_{\mathbb{Q}}$. This is indeed equivalent because x has non-negative pairing with each element of $\overline{C}^*_{\mathbb{Q}}$ if and only if it has non-negative pairing with each fundamental weight δ_α and this is the case if and only if x is a non-negative combination of simple roots.

Finally, this equivalent statement is an immediate consequence of the proposition. $\square$

Part 2. The Scholze-Shin Conjecture for Unramified Unitary Groups (with Alex Youcis)

2.1. Introduction and Notation

The goal of this Part is to explore the extent to which the results of [Sch13b] can be generalized to unitary groups.

More explicitly, in [Sch13b] Scholze is able to to show that the local Langlands conjecture for $\mathrm{GL}_n(F)$, where F is a finite extension of $\mathbb{Q}_p$, can be characterized by explicitly constructed 'test functions'. Less cryptically, he shows that for every cutoff function $h \in C_c^\infty(\mathrm{GL}_n(\mathcal{O}_F), \mathbb{Q})$ and every element $\tau \in W_F$, there is an explicitly defined function $f_{\tau,h} \in \mathscr{H}(\mathrm{GL}_n(F))$ with the property that for any irreducible smooth representation π_p of $\mathrm{GL}_n(F)$ that

$$\mathrm{tr}(f_{\tau,h} \mid \pi_p) = \mathrm{tr}(h \mid \pi_p)\,\mathrm{tr}(\tau \mid \mathsf{LL}(\pi_p)), \tag{13}$$

where LL is the local Langlands correspondence for $\mathrm{GL}_n(F)$ as in [HT01]. Moreover, Scholze shows that (13) uniquely characterizes the correspondence LL.

The function $f_{\tau,h}$ was constructed by Scholze in the earlier work [Sch13a] and can be defined in terms of the cohomology of certain tubular neighborhoods inside of Rapoport-Zink spaces associated to $\mathrm{GL}_n(F)$. Note that, in particular, $f_{\tau,h}$ implicitly depends on the choice of a dominant cocharacter of $\mathrm{GL}_{n,F}$ which, in the above, is the cocharacter corresponding to the standard representation.

Scholze's function theoretic characterization of the local Langlands conjecture for $\mathrm{GL}_n(F)$ has many applications, examples of which we now list. Philosophically it suggests that the deep and somewhat abstract Langlands correspondence can be understood, in some sense, in terms of explicit functions which one might be able to algorithmically or combinatorially describe. A function theoretic characterization of the Langlands correspondence allows for a more concrete study of the endoscopic case of the Langlands functoriality principle, by studying the transfer of these characterizing functions between endoscopic groups. Finally, the function theoretic characterization of the local Langlands conjecture lends itself to be used to study the Langlands correspondence in more fluid situations (for example to study the local Langlands correspondence in families as in [JNS17]).

Given the above, especially in any attempt to study functoriality using these 'test functions', one desires to generalize this result of Scholze to an arbitrary reductive group G over $\mathbb{Q}_p$. In [SS13] Scholze and S.W. Shin study the cohomology groups $H^*(\mathsf{Sh}, \mathcal{F}_\xi)$ where Sh is the Shimura variety attached to certain compact unitary similitude groups $\mathbf{G}$ (those with no endoscopy as in §2.2.5). In particular, they describe the decomposition of the $\mathbf{G}(\mathbb{A}_f) \times W_{E_{\mathfrak{p}}}$ $H^*(\mathsf{Sh}, \mathcal{F}_\xi)$, where E is the reflex field for Sh and $\mathfrak{p}$ is a prime of E lying over a *split place* p of $\mathbb{Q}$ (see loc. cit. for the definition of split, which is slightly less restrictive than the usual notion of split), in terms of the local Langlands conjecture of $\mathbf{G}(\mathbb{Q}_p)$ which is (a product of terms of the form) $\mathrm{GL}_n(F)$.

They also formulate generalizations of the formula (13) to groups G over $\mathbb{Q}_p$ other than $\mathsf{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_{n,F}$. In particular, they state the following:

Conjecture 1 (Scholze-Shin)**.** *Let G be an unramified group over $\mathbb{Q}_p$ with $\mathbb{Z}_p$-model $\mathcal{G}$ and let μ be a dominant cocharacter of $G_{\overline{\mathbb{Q}_p}}$ with reflex field E. Let $\tau \in W_{\mathbb{Q}_p}$ and let $h \in C_c^\infty(\mathcal{G}(\mathbb{Z}_p), \mathbb{Q})$. Let (H, s, η) be an endoscopic group for G and let h^H be the transfer of h. Then, for every tempered L-parameter φ with associated semi-simple parameter λ we have*

$$S\Theta_\varphi(f^H_{\tau,h}) = \mathrm{tr}\left(s^{-1}\tau \mid (r_{-\mu} \circ \eta \circ \lambda\,|_{W_E} \mid \cdot \mid_E^{-\langle \rho,\mu\rangle}\right) S\Theta_\varphi(h). \tag{14}$$

We refer the reader to [SS13, §7] for a detailed explanation of the notation but we note that $S\Theta_\varphi$ is the stable distribution of φ which associates to a function $f \in \mathscr{H}(H(\mathbb{Q}_p))$ the quantity

$$S\Theta_\varphi(f) := \sum_{\pi_p \in \Pi(\varphi)} r_\pi \,\mathrm{tr}(f \mid \pi_p), \tag{15}$$

where $\Pi(\pi_p)$ is the L-packet of φ and r_π is a natural number associated to π (see [SS13, §6]).

Remark. As remarked before, the function $f_{\tau,h}$ depends on the choice of μ, but we suppress this dependency throughout this article since it will always be clear from context.

Note that to make sense of Conjecture 1 one must have the analogue of the functions $f_{\tau,h}$ for G as well as the knowledge of the local Langlands conjecture for H. In this conjecture we are concerned with the case where $H = G$. In this case, the existence of the functions $f_{\tau,h}$ follows from the results of [You19] and the local Langlands conjecture for H follows from the results of [Mok15].

The desire for the presence of endoscopic groups in Conjecture 1 is related to the fact that to characterize the local Langlands conjecture for groups G different from $\mathsf{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_{n,F}$, for which non-trivial L-packets appear, one expects the need to relate any association with endoscopic transfer, which the necessitates a formula like Equation (14) for an arbitrary endoscopic group H.

The result of the methods in this Part is the following (stated as Theorem 2.4.15):

Theorem 1. *The Scholze-Shin conjecture holds with the following assumptions:*

(1) $G = \mathsf{Res}_{F/\mathbb{Q}_p} U$ where U is an inner form of $U_{E/F}(n)^*$ and $E/\mathbb{Q}_p$ is unramified.
(2) The parameter ψ is tempered.
(3) The L-packet of ψ contains a square integrable representation.
(4) (H, s, η) is the trivial endoscopic triple, and μ is miniscule

Remark. In fact, we prove the above result for all local A-parameters ψ containing a representation π_p appearing as a local constituent of a representation π appearing in the cohomology of the unitary Shimura varieties we consider and such that π_∞ is discrete series.

We now describe the contents of this Part, pointing out interesting results which are incidental to the proof of Theorem 1.

In Section 1, we explore the notion of *relevant endoscopy.* Informally speaking, the relevant endoscopy of a global group $\mathbf{G}$ is the set of endoscopic triples showing up in the stabilization of the trace formula for $\mathbf{G}$. More rigorously, we define an endosopic triple $(\mathbf{H}, s, \eta)$ to be *relevant* if it can be completed to an endoscopic quadruple $(\mathbf{H}, s, \eta, \gamma_{\mathbf{H}})$ (as in Definition 2.2.4). We show that this notion of relevance is intimately related to an *a priori* different notion of relevance for (H, s, η) which means that it can be upgraded to a quadruple $(H, s, {}^L\eta, \psi^H)$ where ψ^H is an A-parameter for H and ${}^L\eta \circ \psi^H$ is relevant for G.

Remark. Here our notion of A-parameter is somewhat loose. In Section 1 we develop a method to analyze the above when the A-parameters of an algebraic group G over a local or global field F is taken to mean certain homomorphisms $\psi : \mathcal{L}_\psi \to {}^LG$ where $\mathcal{L}_\psi$ is some extension of W_F by a pro-reductive connected algebraic group. In particular, we shall apply this in the cases when F is local (in which case these are the usual notion of A-parameters) and when G is a global unitary group in which case they are the A-parameters in [Kal+14, §1.3.4].

This then allows one to get a good understanding of the explicit relationship between a unitary group $\mathbf{G}$ having no relevant endoscopy and certain global parameters ψ of $\mathbf{G}$ (as in [Kal+14]) having trivial reduced global centralizer group $\overline{S_\psi}$. Namely, we show the following (labeled as Proposition 2.2.31 in the main body of the paper):

Theorem 2. *Let $\mathbf{G} = \mathsf{Res}_{F/\mathbb{Q}}\mathbf{U}$ be a global unitary group and let ψ be a relevant A-parameter of $\mathbf{G}$ such that ψ_∞ is elliptic for some infinite place ∞ of F. Then, if $\mathbf{G}$ has no relevant endoscopy then $\overline{S_\psi} = 1$.*

As a corollary of this, using the deep work of [Kal+14], we obtain, using the notation of Theorem 2, the following (labeled as Lemma 2.2.32 in the main body of the paper):

Corollary 1. *Let π be an automorphic representation for $\mathbf{G}$ which is discrete at infinity. Then, if $\mathbf{G}$ has no relevant endoscopy the following equality holds*

$$L^2_{\mathrm{disc}}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))[\pi^p] = \bigoplus_{\pi'_p \in \Pi_{\psi_p}(\mathbf{G}(\mathbb{Q}_p),\omega_p)} \pi'_p \tag{16}$$

where ψ is the A-parameter associated to π.

For a precise description of notation see the discussion surrounding Lemma 2.2.32. In words, this lemma says that under suitable conditions on $\mathbf{G}$ and π the away-from-p isotypic component of $L^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$ associated to π consists of precisely representations with local p-component lying in the packet of ψ_p and, moreover, that these appear with multiplicity one.

In Section 2 of this Part we show a decomposition of the cohomology of a compact Shimura variety with no endoscopy. More precisely, we have the following (labeled as Theorem 2.3.2):

Theorem 3. *Let $\mathbf{G}$ be a reductive group over $\mathbb{Q}$ which has no relevant endoscopy and for which $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic. Suppose that Sh is a Shimura variety associated $\mathbf{G}$ with reflex field $\mathbf{E}_{\boldsymbol{\mu}}$. Then, for any algebraic $\overline{\mathbb{Q}_\ell}$-representation ξ of $\mathbf{G}$ and any prime $\mathfrak{p}$ of $\mathbf{E}_{\boldsymbol{\mu}}$ there is a decomposition of virtual $\overline{\mathbb{Q}_\ell}$-representations of $\mathbf{G}(\mathbb{A}_f) \times W_{\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}}$*

$$H^*(\mathsf{Sh}, \mathcal{F}_\xi) = \bigoplus_{\pi_f} \pi_f \boxtimes \sigma(\pi_f), \tag{17}$$

where π_f ranges over admissible $\overline{\mathbb{Q}_\ell}$-representations of $\mathbf{G}(\mathbb{A}_f)$ such that there exists an automorphic representation π of $\mathbf{G}(\mathbb{A})$ such that;

(1) $\pi_f \cong (\pi)_f$ (using our identification $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$)
(2) $\pi_\infty \in \Pi_\infty(\xi)$.

Moreover, for each π_f there exists a cofinite set $S(\pi_f) \subseteq S^{\mathrm{ur}}(\pi_f)$ of primes p such that for each prime $\mathfrak{p}$ over $\mathbf{E}_{\boldsymbol{\mu}}$ lying over p and each $\tau \in W_{\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}}$ the following equality holds:

$$\mathrm{tr}(\tau \mid \sigma(\pi_f)) = a(\pi_f)\,\mathrm{tr}(\tau \mid r_{-\boldsymbol{\mu}} \circ \varphi_{\pi_p}) p^{\frac{1}{2}v(\tau)[\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}:\mathbb{Q}_p]\dim \mathsf{Sh}}, \tag{18}$$

for some integer $a(\pi_f)$ (see Definition 2.3.6).

Besides the singling out of the notion of relevance of endoscopy this theorem has minimal original content, essentially being a technical exercise in showing that the results of [Kot92a] are applicable to the general situation with the results of [KSZ] as a replacement for the results of [Kot92b]. We have included the work here mostly for the convenience of the reader, and to help fix ideas and notation that occur in Section 3 of this Part.

In Section 3, we combine the results of the last two sections, together with the work of [Shi11] and [You19], to deduce Theorem 1.

To begin, we show that one can make explicit improvements to Theorem 3 in the case that $\mathbf{G} = \mathsf{Res}_{F/\mathbb{Q}}\mathbf{U}$ for a unitary group $\mathbf{U}$. Namely, we show the following (see the contents of §2.4.2):

Theorem 4. *Let $E/\mathbb{Q}$ be a CM field with F its totally real subfield. Let $\mathbf{U}$ be an inner form of $U_{E/F}(n)^*$ and set $\mathbf{G} := \mathsf{Res}_{F/\mathbb{Q}}\mathbf{U}$. Assume that $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic and has no relevant endoscopy. Let Sh be a Shimura variety associated to $\mathbf{G}$. Then, for any algebraic $\overline{\mathbb{Q}_\ell}$-representation ξ and* any prime $\mathfrak{p}$ *of E there is a decomposition of virtual $\overline{\mathbb{Q}_\ell}[\mathbf{G}(\mathbb{A}_f) \times W_{\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}}]$-modules*

$$H^*(\mathsf{Sh}, \mathcal{F}_\xi)(\chi) = \bigoplus_{\pi_f} \pi_f \boxtimes a(\pi_f)\left(r_{-\boldsymbol{\mu}} \circ \mathsf{LL}(\pi_p)\right), \tag{19}$$

where π_f ranges over admissible $\overline{\mathbb{Q}_\ell}$-representations of $\mathbf{G}(\mathbb{A}_f)$ such that there exists an automorphic representation π of $\mathbf{G}(\mathbb{A})$ such that;

(1) $\pi_f \cong (\pi)_f$ (using our identification $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$)
(2) $\pi_\infty \in \Pi_\infty(\xi)$.

and χ is some global character and $a(\pi_f)$ is an integer (see Definition 2.3.6).

We also obtain, using Theorem 4 and Corollary 1, the further refinement:

Corollary 2. *Let π be be an automorphic representation of $\mathbf{G}$ such that π_∞ is discrete series. Then, for any prime $\mathfrak{p}$ of E and any algebraic $\overline{\mathbb{Q}_\ell}$-representation ξ we have a decomposition of virtual $\mathbb{Q}_\ell[\mathbf{G}(\mathbb{Q}_p) \times W_{\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}}]$-modules*

$$H^*(\mathsf{Sh}, \mathcal{F}_\xi)[\pi_f^p] = \bigoplus_{\pi'_p \in \Pi_{\psi_p}(\mathbf{G}(\mathbb{Q}_p), \omega_p)} \pi'_p \boxtimes \sigma(\pi_f^p \otimes \pi'_p). \tag{20}$$

We then use the trace formula in [You19] together with Theorem 4 and Corollary 2 to deduce Theorem 1. To do this though, one must first lift local representations at p to global representations of some unitary group, and some care must be chosen in the conditions necssary to do this. We appeal to the results of [Shi12a] which is where the square-integrability conditions enter into the equation.

Remark. We note that while much of this part is written with the specific focus on unramified unitary groups, the rough strategy to prove the Scholze-Shin conjecture seems applicable to a much wider class of groups. The main impediments to generalizing is the lack of results like [Kal+14] and [Shi11] to apply to non-unitary groups.

Notations and conventions.

2.1.0.1. *General.*

- Unless stated otherwise p is a prime and ℓ is a prime different from p.
- We will (sometimes implicitly) fix an isomorphism $\iota : \overline{\mathbb{Q}_\ell} \xrightarrow{\approx} \mathbb{C}$.
- Unless stated otherwise all fields are assumed of characteristic 0.
- For a number field F and a finite place v of F we shall denote by F_v the completion of F at v, $\mathcal{O}_v$ its integer ring, and k_v its residue field.
- For a number field F we denote by $\mathbb{A}_F$ the topological ring of F-adeles and by $\mathbb{A}_{F,f}$ the topological subring of finite F-adeles. We shall shorten $\mathbb{A}_\mathbb{Q}$ to $\mathbb{A}$ and $\mathbb{A}_{\mathbb{Q},f}$ to $\mathbb{A}_f$.

2.1.0.2. *Galois theory.*

- For a field F and an algebraic extension F'/F we shall use $\mathrm{Gal}(F'/F)$ to denote the Galois group of F' over F. We shall shorten $\mathrm{Gal}(\overline{F}/F)$ to Γ_F.
- For a local or global field F we shall denote by W_F the Weil group of F (as in [Tat79, §1]) with its implicit continuous map with dense image $W_F \to \Gamma_F$. For every finite Galois extension F' of F we shall use this map to canonically, and implicitly, define an isomorphism $W_F/W_{F'} \cong \Gamma_F/\Gamma_{F'}$ and shall thus use $\mathrm{Gal}(F'/F)$ to denote the common group.
- For a non-archimedean local field F with residue field k we shall shall denote by $I_F \subseteq W_F \subseteq \Gamma_F$ the inertia subgroup of F.
- For a finite field F we shall denote by Frob_F, or just Frob if F is clear from context, the geometric Frobenius element in Γ_F.
- For a non-archimedean local field F with residue field k we shall denote by Frob_F a lift of Frob_k along the canonical surjection $W_F \to \Gamma_k$.

- For a local field F we shall denote by v_F, or just v when F is clear from context, the valuation map $v : W_F \to \mathbb{Z}$ where we have normalized so that $v(\mathrm{Frob}_F) = 1$.

2.1.0.3. *Reductive groups.*

- All reductive groups are assumed connected.
- In contexts revolving arbitrary fields F we shall denote algebraic groups over F with non-boldfaced letters like G. In the context where F is a global field we will often denote a group over F in the boldface font (e.g. $\mathbf{G}$). For a place v of F we shall denote shorten $\mathbf{G}_{\mathbb{Q}_v}$ to $\mathbf{G}_v$. If there is some distinguished place v_0 of F of interest to us we shall often use the non-boldfaced notation G to denote $\mathbf{G}_{v_0}$.
- For an algebraic group G over a field F we denote by G° the connected component of G and by $\pi_0(G)$ the component group G/G°.
- For an algebraic group G over a field F we denote by $Z(G)$ the center of G and by $Z_G(\gamma)$ the centralizer of an element $\gamma \in G(F)$.
- For an algebraic group G over a field F and an element $\gamma \in G(F)$ we denote by I_γ the group $Z_G(\gamma)^\circ$.
- For an algebraic group G we denote $G/Z(G)$ by G^{ad} and the derived subgroup by G^{der}.
- For a reductive group G over a field F we denote by A_G the maximal F-split torus in $Z(G)$.
- For a reductive group G over a field F we shall denote by $X_*(G)$ the Γ_F-set of homomorphisms $\mathbb{G}_{m,\overline{F}} \to G_{\overline{F}}$ and by $X^*(G)$ the Γ_F-module of homomorphisms $G_{\overline{F}} \to \mathbb{G}_{m,\overline{F}}$. Note that if G is a torus then $X_*(G_{\overline{F}})$ is also a Γ_F-module. We denote by $X^*_F(G)$ the group of homomorphisms $\mathbb{G}_{m,F} \to G$ and identify it implicitly with the subgroup $X^*(G)^{\Gamma_F}$ of $X^*(G)$.
- For a reductive group G over a field F we denote by $\{G\}$ the set of conjugacy classes in $G(F)$, by $\{G\}_s$ the set of stable conjugacy classses in $G(F)$, and by $\{G\}^{\text{s.s.}}$ and $\{G\}_s^{\text{s.s.}}$ the analogues with $G(F)$ replaced by the set $G(F)^{\text{s.s.}}$ of semisimple elements of $G(F)$. For an element $\gamma \in G(F)$ we denote by $\{\gamma\}$ (resp. $\{\gamma\}_s$) its image in $\{G\}$ (resp. $\{G\}_s$).
- For a reductive group G over a field F and two elements γ and γ' in $G(F)$ we use the notation $\gamma \sim \gamma'$ to indicate that γ and γ' are conjugate, and the notation $\gamma \sim_{\mathrm{st}} \gamma'$ to denote that γ and γ' are stably conjugate.

- For a reductive group G over a field F and a semi-simple element $\gamma \in G(F)$ we denote by $S(\gamma)$ the collection of conjugacy classes contained in the stable conjugacy class $\{\gamma\}_s$.
- For a reductive group G over a local field F and a semi-simple element $\gamma \in G(F)$ we denote by $a(\gamma)$ the cardinality of the kernel of the natural map

$$H^1(F, I_\gamma) \to H^1(F, Z_G(\gamma)) \tag{21}$$

  which is finite by the assumption that F is local. Note that if G^{der} is simply connected then $a(\gamma) = 1$ and so this term will often times not factor in to our work (despite its presence in many references).
- For a reductive group G over a field F we denote by $G(F)^{\mathrm{ell}}$ the set of elliptic elements of $G(F)$ (see §2.5.1.1 for a discussion of ellipticity).
- If G is an algebraic group over a characteristic 0 local field we will topologize $G(F)$ in the standard way (e.g. as in [Con12b]). We shall then denote the connected component of $G(F)$ with this topology by $G(F)^0$.
- If F is a global field and $\mathbf{G}$ a reductive group over F we shall topologize $G(\mathbb{A}_F)$ and $\mathbf{G}(\mathbb{A}_{F,f})$ in the standard ways (again see [Con12b]) .
- For a number field F and a reductive group $\mathbf{G}$ over F we denote by $S(\mathbf{G})$ the set of finite places v of F for which $\mathbf{G}_v$ is unramified (i.e. which admits a reductive model over $\mathrm{Spec}(\mathcal{O}_v)$ in the sense of [Con14, Definition 3.1.1]).
- For a number field F and a reductive group $\mathbf{G}$ over F we will often implicitly choose a reductive model $\mathcal{G}_v$ of $\mathbf{G}_v$ over $\mathrm{Spec}(\mathcal{O}_v)$ for all $v \in S(\mathbf{G})$.
- We shall denote by $K_{0,v}$ the hyperspecial subgroup $\mathcal{G}_v(\mathcal{O}_v) \subseteq \mathbf{G}(F_v)$ for all $v \in S(\mathbf{G})$. For finite $v \notin S(\mathbf{G})$ or infinite v we shall define $K_{0,v}$ to be $\mathbf{G}(F_v)$.
- We will implicitly make the identification of topological groups

$$G(\mathbb{A}_F) \cong \prod_v{}' (\mathbf{G}(F_v), K_{0,v}) \tag{22}$$

  and the identification

$$G(\mathbb{A}_{F,f}) \cong \prod_{v \text{ finite}}{}' (\mathbf{G}(F_v), K_{0,v}) \tag{23}$$

  obtained by (passing to the colimit) in [Con12b, Theorem 3.6].

- For a reductive group over a number field F we denote by $\mathbf{G}(\mathbb{A}_F)^1$ the subgroup of $\mathbf{G}(\mathbb{A}_F)$ defined as follows

$$\mathbf{G}(\mathbb{A})^1 := \{g \in \mathbf{G}(\mathbb{A}) : |\nu(g)| = 1 \text{ for all } \nu \in X^*(\mathbf{G})^{\Gamma_F}\} \tag{24}$$

where $\mathbb{A}_F^\times$ is given the usual norm.
- For a reductive group $\mathbf{G}$ over the number field F we note that evidently (by the product rule) that $\mathbf{G}(F) \subseteq \mathbf{G}(\mathbb{A}_F)^1$ we define the *adelic quotient* of $\mathbf{G}$, denoted $[\mathbf{G}]$, to be the topological space $\mathbf{G}(\mathbb{A})^1/\mathbf{G}(\mathbb{Q})$ which is a measure space whenever $G(\mathbb{A})$ is given a measure.
- For F a global field and $\mathbf{G}$ a reductive group over F we denote by $\tau(\mathbf{G})$ the *Tamagawa number* of $\mathbf{G}$ defined to be $\mathrm{vol}([G])$ when $\mathbf{G}(\mathbb{A})$ is endowed with the Tamagawa measure (as in [Wei12, Chapter II]). See [PS92, Theorem 5.6] for a proof that such a volume is finite.
- For $\mathbf{G}$ a reductive group over $\mathbb{Q}$ and K a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ we denote by $Z(\mathbb{Q})_K$ the group $Z(\mathbf{G})(\mathbb{Q}) \cap K$ and by Z_K the group $Z(\mathbf{G})(\mathbb{A}_f) \cap K$.
- Let F be a local field and G a reductive group over F. We denote by $e(G)$ the *Kottwitz sign* as in [Kot83].

2.1.0.4. *Harmonic analysis.*

- Let F be a number field and $\mathbf{G}$ a reductive group over F. Let C be an algebraically closed field and let π_f be an irreducible admissible C-representation of $\mathbf{G}(A_{F,f})$. Then, we shall denote by

$$\pi_f = \sideset{}{'}\bigotimes_v \pi_{f,v} \tag{25}$$

the Flath decomposition with respect to the set $\{K_{0,v}\}$ as in [Fla79]. We then denote by $S^{\mathrm{ur}}(\pi_f)$ the set of $v \in S(\mathbf{G})$ such that $\pi_{f,v}$ is $K_{0,v}$ unramified (i.e. for which $\pi_{f,v}^{K_{0,v}} \neq 0$) and call a place v in $S^{\mathrm{ur}}(\pi_f)$ *unramified*. Again, we will make it clear when things fundamentally change with different choices of $K_{0,v}$.
- If $v \in S^{\mathrm{ur}}(\pi_f)$ let us denote by $\varphi_{\pi_{f,v}}$ the associated unramified local Langlands parameter $W_{F_v} \to {}^L\mathbf{G}_v$ as in [Bor79, Chapter II].
- Let F be a non-archimedean local field and let G be a reductive group over F. For a characteristic 0 field C We denote by $\mathscr{H}_C(G(F))$, or just $\mathscr{H}(G(F))$ when C is clear the *Hecke algebra* as in [Car+79, §1.3] where we have implicitly (often times clear from context) fixed a $\mathbb{Q}$-valued Haar measure dg on $G(F)$.

For a compact open subgroup K of $G(F)$ we shall denote by $\mathscr{H}_C(G(F),K)$, or just $\mathscr{H}(G(F),K)$ when C is clear from context, as in loc. cit.

- Let F be a local field and G a reductive group over F. Let us suppose that $\phi\in\mathscr{H}_C(G(F))$ and that $\gamma\in G(F)$ is semi-simple. Then, we define the *orbital integral* of ϕ, denoted $O_\gamma(\phi)$, to be the quantity

$$O_\gamma(\phi):=\int_{I_\gamma(F)\backslash G(F)}\phi(g\gamma g^{-1})\,dg \tag{26}$$

We define the *stable orbital integral* of ϕ, denoted $SO_\gamma(\phi)$, to be the quantity

$$SO_\gamma(\phi)=\sum_{\gamma'\sim_{\mathrm{st}}\gamma}e(I_{\gamma'})a(\gamma')O_\gamma(\phi) \tag{27}$$

- Let F be a global field and let $\mathbf{G}$ be a reductive group over F. Let ϕ be an element of $\mathscr{H}_C(\mathbf{G}(\mathbb{A}_F))$ and $\gamma\in\mathbf{G}(\mathbb{A}_F)$ semi-simple (i.e. that each of its local factors is semi-simple). We then define the *orbital integral* of ϕ, denoted $O_\gamma(\phi)$, to be the quantity

$$O_\gamma(\phi)=\int_{\mathbf{I}_\gamma(\mathbb{A}_F)\backslash\mathbf{G}(\mathbb{A}_F)}\phi(g\gamma g^{-1})\,dg \tag{28}$$

Assume now that $\gamma\in\mathbf{G}(F)$. We define the *stable orbital integral* of ϕ, denoted $SO_\gamma(\phi)$, to be the quantity

$$\sum_i e(\mathbf{I}_{\gamma_i})O_{\gamma_i}(\phi) \tag{29}$$

Here i ranges over the set

$$\ker(F,\mathbf{I}(\overline{\mathbb{A}_F}))\to H^1(F,\mathbf{G}(\overline{\mathbb{A}_F})) \tag{30}$$

The element $\gamma_i\in\mathbf{G}(\mathbb{A}_F)$ is the one associated to i by applying [Kot86b, §4.1] place by place. Note, in particular, that for all places v of F the v^{th}-component of γ_i is stably conjugate to γ.

- Suppose that $\mathbf{G}$ is a reductive group over $\mathbb{Q}$ and $\xi_\mathbb{C}$ is an algebraic representation of $\mathbf{G}_\mathbb{C}$. Let $\Pi_\infty(\xi_\mathbb{C})$ be the set of isomorphism classes of all irreducible $\mathbf{G}(\mathbb{R})$-representations having the same central and infinitesimal character as the contragredient representation and let $\Pi^0_\infty(\xi_\mathbb{C})$ be the subset of discrete series representations in $\Pi_\infty(\xi_\mathbb{C})$. If ξ is an algebraic $\overline{\mathbb{Q}_\ell}$-representation of $\mathbf{G}$ we use our identification of $\overline{\mathbb{Q}_\ell}$ and $\mathbb{C}$ to obtain a corresponding $\mathbb{C}$-representation $\xi_\mathbb{C}$ and we set $\Pi_\infty(\xi):=\Pi_\infty(\xi_\mathbb{C})$ and $\Pi^0_\infty(\xi):=\Pi^0_\infty(\xi_\mathbb{C})$

- Let $\mathbf{G}$ be a reductive group over $\mathbb{Q}$. Let π be a $\mathbb{C}$-representation (or $\overline{\mathbb{Q}_\ell}$-representation using our identification of $\overline{\mathbb{Q}_\ell}$ and $\mathbb{C}$). We set $m(\pi)$ to be the multiplicity of π in $L^2_{\mathrm{disc}}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$.

2.1.0.5. *Algebraic geometry.*

- For a variety X over a field k and a lisse $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ on X with $\mathrm{char}(k)\neq\ell$ we then denote by $H^*(X,\mathcal{F})$ the virtual $\overline{\mathbb{Q}_\ell}$-space

$$\sum_{i=0}^{2\dim(X)}(-1)^i H^i(X_{\overline{k}},\mathcal{F}_{\overline{k}}). \tag{31}$$

2.1.0.6. *Shimura varieties.*

- We shall denote Shimura data as $(\mathbf{G},X)$ as in [Mil04, Definition 5.5].
- We shall assume that all of our Shimura data are of abelian type.
- We shall assume only that our Shimura data satisfy axioms **SV**1, **SV**2, and **SV**3 as in [Mil04], but will often assume that our Shimura data also satisfies axiom of **SV**5.
- If $(\mathbf{G},X)$ is a Shimura datum, we shall denote its associated reflex field (as in [Mil04, Definition 12.2]) by $E(\mathbf{G},X)$ or, when $(\mathbf{G},X)$ is clear from context, just E.
- For every neat (as on [Mil04, Page 34]) compact open subgroup K of $\mathbf{G}(\mathbb{A}_f)$ we denote by $\mathsf{Sh}_K(\mathbf{G},X)$, or Sh_K when $(\mathbf{G},X)$ is clear from context, the canonical model (in the sense of [Mil04, Definition 12.8]) of the complex variety $\mathsf{Sh}_K(G,X)_{\mathbb{C}}$ (as in [Mil04, Definition 5.14]) over its reflex field E.
- We denote by Sh the E-scheme $\varprojlim_K \mathsf{Sh}_K$ as K runs over the neat compact open subgroups of $\mathbf{G}(\mathbb{A}_f)$. Note that this exists by [Stacks, Tag 01YX] since the transition maps for the system $\{\mathsf{Sh}_K\}$ have finite (and thus affine) transition maps.
- Let ℓ be a prime and let ξ be an algebraic $\overline{\mathbb{Q}_\ell}$-representation of G (i.e. an algebraic representation $\xi : G_{\overline{\mathbb{Q}_\ell}}\to\mathrm{GL}_{\overline{\mathbb{Q}_\ell}}(V)$ for some $\overline{\mathbb{Q}_\ell}$-space V) such that for the induced map

$$G(\mathbb{A}_f)\xrightarrow{\text{proj.}}G(\mathbb{Q}_\ell)\hookrightarrow G(\overline{\mathbb{Q}_\ell})\to\mathrm{GL}_{\overline{\mathbb{Q}_\ell}}(V)$$

has the property that $Z(\mathbb{Q})_K\subseteq\ker\xi$ for all sufficiently small compact open subgroups $K\subseteq G(\mathbb{A}_f)$.

2.2. Relevant Global Endoscopy

2.2.1. **Introduction.** In this section, we discuss the notion of relevant global endoscopy. Loosely, for a group $\mathbf{G}$ defined over a number field F, we say that an elliptic endoscopic datum $(\mathbf{H}, s, \eta)$ is *relevant* if it appears in the stable trace formula for the group $\mathbf{G}$. We then prove some applications of our discussion which will be necessary for our main results.

2.2.2. **Definitions and statements.** We assume for convenience in this entire part that G^{der} is simply connected. We begin by recalling the definition of endoscopic datum as in [Shi10, §2.1].

Definition 2.2.1. *An endoscopic datum for a reductive group G over a field F consists of a triple (H, s, η) where H is a quasisplit reductive group, $\eta : \widehat{H} \to \widehat{G}$ is an embedding and $s \in \widehat{H}$ such that*

- *We have an equality $\eta(\widehat{H}) = Z_{\widehat{G}}(s)^0$,*
- *The $\widehat{G}$-conjugacy class of η is fixed by Γ_F,*
- *The image of s in $Z(\widehat{H})/Z(\widehat{G})$ lies in $(Z(\widehat{H})/Z(\widehat{G}))^{\Gamma_F}$,*
- *The image of $s \in H^1(F, Z(\widehat{G}))$ is trivial if F is local and locally trivial if F is global.*

An endoscopic datum is defined to be elliptic *if $(Z(\widehat{H})^{\Gamma})^{\circ} \subset Z(\widehat{G})$.*

We record now our definition of isomorphism between endoscopic data:

Definition 2.2.2. *An isomorphism between endoscopic data (H_1, s_1, η_1) and (H_2, s_2, η_2) is an isomorphism $\alpha : H_2 \to H_1$ such that there exists $g \in \widehat{G}$ such that $\widehat{\alpha}(s_1) = s_2 \mod Z(\widehat{G})$ and the following diagram commutes:*

$$\begin{tikzcd} \widehat{H_1} \arrow[r, "\eta_1"] \arrow[d, "\widehat{\alpha}"'] & \widehat{G} \arrow[d, "\mathrm{Int}(g)"] \\ \widehat{H_2} \arrow[r, "\eta_2"] & \widehat{G}. \end{tikzcd} \tag{32}$$

We denote the set of isomorphism classes of endoscopic data for G by $\mathcal{E}(G)$ and we denote the set of isomorphism classes of elliptic endoscopic data by $\mathcal{E}^{\mathrm{ell}}(G)$.

Note that the map $\widehat{\alpha}$ is Γ_F-invariant and only well-defined up to a choice of splittings (see [Kot84b, §1.8]) and hence up to $\widehat{H_1}^{\Gamma_F}$-conjugacy but that the above diagram makes sense for any choice of $\widehat{\alpha}$ in this class.

Note also that we will often confuse $\widehat{H}$ for $\eta(\widehat{H})$ and so, in particular, will often confuse s and $\eta(s)$.

Now, since we assume G^{der} is simply connected, for each endoscopic datum (H, s, η), there exists a lift of η to an L-map ${}^L\eta : {}^LH \to {}^LG$ (see [Lan79, Prop 1]). The following lemma will be useful to us.

Lemma 2.2.3. *Suppose that (H_1, s_1, η_1) and (H_2, s_2, η_2) are endoscopic data and fix lifts ${}^L\eta_1$ and ${}^L\eta_2$ of η_1 and η_2 respectively. Suppose further that $\alpha : H_2 \to H_1$ gives an isomorphism of endoscopic data and $g \in \widehat{G}$ is as in 2.2.2. Then for each choice of $\widehat{\alpha}$, there exists a lift ${}^L\alpha$ of α such that the following diagram commutes:*

$$\begin{tikzcd} {}^LH_1 \arrow[r, "{}^L\eta_1"] \arrow[d, "{}^L\alpha"'] & {}^LG \arrow[d, "\mathrm{Int}(g)"] \\ {}^LH_2 \arrow[r, "{}^L\eta_2"] & {}^LG. \end{tikzcd} \tag{33}$$

Moreover, the $\widehat{H_1}$-conjugacy class of ${}^L\alpha$ does not depend on the choice of $\widehat{\alpha}$ or g.

Proof. We want to define ${}^L\alpha$ to equal ${}^L\eta_2^{-1} \circ \mathrm{Int}(g) \circ {}^L\eta_1$. For this to make sense, we need to show that the image of $\mathrm{Int}(g) \circ {}^L\eta_1$ is contained in the image of ${}^L\eta_2$.

Now there exists for each $w \in W_F$ and $i \in \{1, 2\}$, elements $g(w)_i \in \widehat{G}$ so that ${}^L\eta_i(1, w) = (g(w)_i, w)$. We observe that for any $h_i \in \widehat{H_i}$, we have

$$(g(w)_i (w \cdot \eta_i)(h_i), w) = {}^L\eta_i(1, w)\, {}^L\eta_i(w^{-1}(h_i), 1) \tag{34}$$
$$= {}^L\eta_i(h_i, w) \tag{35}$$
$$= {}^L\eta_i(h_i, 1)\, {}^L\eta_i(1, w) \tag{36}$$
$$= (\eta_i(h_i) g(w)_i, w), \tag{37}$$

so that

$$\mathrm{Int}(g(w)_i^{-1})(\eta_i(h_i)) = (w \cdot \eta_i)(h_i). \tag{38}$$

Now, it suffices to check that for each $(1, w) \in {}^LH_1$ there exists an $(h_2, w) \in {}^LH_2$ such that

$$(g g(w)_1 w(g^{-1}), w) = (\eta_2(h_2) g(w)_2, w). \tag{39}$$

Hence we need to check that $g g(w)_1 w(g^{-1}) g(w)_2^{-1} \in \eta_2(\widehat{H_2})$. It suffices to show that this element lies in $Z_{\widehat{G}}(\eta_2(\widehat{H_2}))$ since for any maximal torus T of $\widehat{H_2}$, we have $\eta_2(T)$ is a maximal torus of $\widehat{G}$ and so

$$Z_{\widehat{G}}(\eta_2(\widehat{H_2})) \subset Z_{\widehat{G}}(\eta_2(T)) = \eta_2(T) \subset \eta_2(\widehat{H_2}). \tag{40}$$

Now pick $h_2 \in \widehat{H_2}$. We observe that using equation (38), we have

$$\begin{aligned}
&\mathrm{Int}(gg(w)_1 w(g^{-1}) g(w)_2^{-1})(\eta_2(h_2)) = \mathrm{Int}(gg(w)_1 w(g^{-1}))((w \cdot \eta_2)(h_2)) && (41)\\
&= \mathrm{Int}(gg(w)_1)(w(g^{-1}\eta_2(w^{-1}(h_2))g)) && (42)\\
&= \mathrm{Int}(gg(w)_1)(w(\eta_1(\widehat{\alpha}^{-1}(w^{-1}(h_2))))) && (43)\\
&= \mathrm{Int}(gg(w)_1)((w \cdot \eta_1)(\widehat{\alpha}^{-1}(h_2))) && (44)\\
&= \mathrm{Int}(g)(\eta_1(\widehat{\alpha}^{-1}(h_2))) && (45)\\
&= \eta_2(h_2), && (46)
\end{aligned}$$

as desired.

Now we show the second statement of the lemma. As above, we have that the map $\widehat{\alpha}$ is unique up to $\widehat{H_1}^{\Gamma_F}$-conjugacy. For a fixed choice of $\widehat{\alpha}$ if we have pick two different $g, g' \in \widehat{G}$ such that the requisite diagram commutes, then $\mathrm{Int}(g^{-1}g')$ fixes $\eta_1(\widehat{H_1})$ pointwise and so $g^{-1}g' \in \eta_1(Z(\widehat{H_1}))$. Hence any two ${}^L\alpha$ will differ at most up to conjugacy by an element of $\widehat{H_1}$. ☐

We are now ready to define the notion of relevant endoscopy. We begin with some definitions following [Shi10, §2.3].

The first definition is that of the set of so-called *endoscopic quadruples* for the group G:

Definition 2.2.4. *For F a local or global field define $\mathcal{EQ}_F(G)$ to be the set of equivalence classes of tuples (H, s, η, γ_H) such that (H, s, η) is an endoscopic triple and $\gamma_H \in H(F)$ transfers to $G(F)$ and is (G, H)-regular and semisimple. The tuples (H, s, η, γ_H) and $(H', s', \eta', \gamma'_H)$ are equivalent if there exists an isomorphism $\alpha : H' \to H$ inducing an isomorphism of endoscopic data and such that $\alpha(\gamma'_H)$ is stably conjugate to γ_H. We define the subset $\mathcal{EQ}_F^{\mathrm{ell}}(G) \subset \mathcal{EQ}_F(G)$ to consist of those equivalence classes such that (H, s, η) is elliptic.*

We now define a set of pairs associated to G consisting, essentially, of a semi-simple element γ of $G(F)$ and an element of its Kottwitz group $\mathfrak{K}(I_\gamma/F)$ (see 2.5.1.5 for a recollection of the Kottwitz group). More precisely:

Definition 2.2.5. *For F a local or global field define $\mathcal{SS}_F(G)$ to be the set of equivalence classes of pairs (γ, κ) such that $\gamma \in G(F)$ is semisimple and $\kappa \in \mathfrak{K}(I_\gamma/F)$. Two pairs (γ, κ) and (γ', κ') are equivalent if γ and γ' are stably conjugate in G and κ and κ' are equal under the canonical isomorphism $\mathfrak{K}(I_\gamma/F) \cong \mathfrak{K}(I_{\gamma'}/F)$. We define the subset*

$\mathcal{SS}_F^{\mathrm{ell}}(G) \subset \mathcal{SS}_F(G)$ *to be the equivalence classes of pairs where* γ *is elliptic.*

Now we have the following key bijection due to Kottwitz:

Proposition 2.2.6. *The natural map*

$$\mathcal{EQ}_F(\mathbf{G}) \to \mathcal{SS}_F(\mathbf{G}), \tag{47}$$

given by

$$(H, s, \eta, \gamma_H) \mapsto (\gamma, \eta(s)), \tag{48}$$

(where γ *is some transfer of* γ_H *to* $\mathbf{G}(F)$*) is well-defined and a bijection. Moreover this map restricts to give a bijection*

$$\mathcal{EQ}_F^{\mathrm{ell}}(\mathbf{G}) \to \mathcal{SS}_F^{\mathrm{ell}}(\mathbf{G}). \tag{49}$$

Proof. See [Shi10, Lemma 2.8] as well as [Kot86b, Lemma 9.7]. □

We are now ready to define the notion of relevant endoscopy.

Definition 2.2.7. *Let* F *be a number field and* $\mathbf{G}$ *a reductive group over* F*. We have a natural projection map*

$$\mathcal{EQ}_F(\mathbf{G}) \to \mathcal{E}(\mathbf{G}). \tag{50}$$

which restricts to a map

$$\mathcal{EQ}_F^{\mathrm{ell}}(\mathbf{G}) \to \mathcal{E}^{\mathrm{ell}}(\mathbf{G}). \tag{51}$$

We define the subsets $\mathcal{RE}(\mathbf{G}) \subset \mathcal{E}(\mathbf{G})$ *and* $\mathcal{RE}^{\mathrm{ell}}(\mathbf{G}) \subset \mathcal{E}^{\mathrm{ell}}(\mathbf{G})$ *to be the images of the first and second maps respectively. We say that the set* $\mathcal{RE}(\mathbf{G})$ *is the set of* relevant global endoscopy *of* $\mathbf{G}$ *and that* $\mathcal{RE}^{\mathrm{ell}}(\mathbf{G})$ *is the set of* relevant elliptic global endoscopy*.*

We now state the representation-theoretic analogue of 2.2.6, part of a general web of analogies between representation theory and conjugacy classes. Such constructions appear for instance in works of Kottwitz (see the proof of [Kot84b, Prop 11.3.2]) and Shelstad ([She83, §4.2]). We choose to provide the details in this work.

For the remainder of this subsection, let us fix F to be a local or global field and G a reductive group over F.

We shall use the notion of A-parameters which we now recall. To do this we will be using the notion of the Langlands group $\mathcal{L}_F$ as in the introduction of [Art02]. When F is a local field such a group is $W_F \times \mathrm{SL}_2(\mathbb{C})$ but when F is a number field the existence of such a Langlands group (for which we use Langlands original pro-algebraic formalism) is conjectural. We shall then only use its basic properties assumed for such a group as in loc. cit.

We shall denote by K the kernel of the projection map $\mathcal{L}_F \to W_F$ which is a connected pro-algebraic group over $\mathbb{C}$ (which we often tacitly identify with its $\mathbb{C}$-points).

We begin with the definition of an L-parameter since this will make the definition of an A-parameter easier to parse:

Definition 2.2.8. *Let $\mathcal{L}_F$ be the Langlands group. Then, an L -* parameter *for G is a continuous map $\phi : \mathcal{L}_F \to {}^L G$ such that the following conditions hold:*

(1) The restriction of the map $\phi_{|K}$ has image in $\widehat{G} \subseteq {}^L G$ and is algebraic as a map $K \to \widehat{G}$.

(2) The diagram

$$\begin{tikzcd} \mathcal{L}_F \arrow[r, "\phi"] \arrow[dr] & {}^L G \arrow[d] \\ & W_F \end{tikzcd} \tag{52}$$

is commutative.

(3) For all $w \in \mathcal{L}_F$ the element $\phi(w) \in {}^L G$ is semisimple or, in other words, that under any representation ${}^L G \to \mathrm{GL}_n(\mathbb{C})$ (in the sense of [Bor79, §2.6]) the image of $\phi(w)$ is semi-simple.

Two L parameters ϕ_1 and ϕ_2 for G are said to be equivalent *if there exists $g \in \widehat{G}$ such that*

$$w \mapsto g^{-1}\phi_2(w) g \phi_1(w)^{-1} \tag{53}$$

is a (locally) trivial 1 cocycle of $\mathcal{L}_F$ taking values in $Z(\widehat{G})$.

In the case that F is local, we say that the L-parameter ϕ is relevant *if whenever $\phi(\mathcal{L}_F) \subset \mathcal{P}$ for $\mathcal{P}$ a parabolic subgroup of ${}^L G$ (in the sense of [Bor79, §3]), then $\mathcal{P}$ is conjugate in ${}^L G$ to ${}^L P$ for some parabolic subgroup $P \subseteq G$. In the case that F is global, we say that ϕ is relevant if for each place v of F, we have $\phi_v := \psi|_{\mathcal{L}_{F_v}}$ is relevant.*

We then move on to the slight variant of L-parameters known as A-parameters:

Definition 2.2.9. *Let $\mathcal{L}_F$ be the Langlands group. Then, an A -* parameter *for G is a continuous map $\psi : \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ such that the following conditions hold:*

(1) The restriction $\psi_{|\mathcal{L}_F}$ is an L-parameter.

(2) The restriction $\psi_{|\mathrm{SL}_2(\mathbb{C})}$ takes image in $\widehat{G}$ and the resulting map of complex Lie groups is holomorphic.

(3) The diagram

$$
\begin{tikzcd}
\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \arrow[r, "\psi"] \arrow[dr] & {}^L G \arrow[d] \\
 & W_F
\end{tikzcd}
\tag{54}
$$

is commutative.

(4) The image of $\psi(\mathcal{L}_F)$ in ${}^L G$ is bounded (i.e. relatively compact).

Two A parameters ψ_1 and ψ_2 for G are said to be equivalent *if there exists $g \in \widehat{G}$ such that*

$$w \mapsto g^{-1}\psi_2(w) g \psi_1(w)^{-1} \tag{55}$$

is a (locally) trivial 1 cocycle of $\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C})$ taking values in $Z(\widehat{G})$.

In the case that F is local, we say that the A-parameter ψ is relevant *if whenever $\psi(\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C})) \subset \mathcal{P}$ for $\mathcal{P} \subset {}^L G$ a parabolic subgroup, then $\mathcal{P}$ is conjugate in ${}^L G$ to ${}^L P$ for some parabolic subgroup $P \subseteq G$. In the case that F is global, we say that ψ is relevant if for each place v of F, we have $\psi_v := \psi|_{\mathcal{L}_{F_v} \times \mathrm{SL}_2(\mathbb{C})}$ is relevant.*

We also need the notion of when, for (H, s, η) an endoscopic triple for G, two A-parameters ψ_1^H and ψ_2^H of H are $Z(\widehat{G})$-equivalent. This definition is as follows:

Definition 2.2.10. *Let (H, s, η) and endoscopic group of G. Then, two A-parameters ψ_1^H and ψ_2^H of H are said to be $Z(\widehat{G})$-*equivalent *if there exists an element $h \in \widehat{H}$ such that the map*

$$w \mapsto h^{-1}\psi_2^H(w) h \psi_1^H(w)^{-1}, \tag{56}$$

is a (locally) trivial 1-cocycle of $\mathcal{L}_F \times SL_2(\mathbb{C})$ valued in $Z(\widehat{G})$.

We need the following definitions as in [Kot84b, §10].

Definition 2.2.11. *Let G be a reductive group over F and let ψ be an A parameter for G. Then we define C_ψ to be the set of $g \in \widehat{G}$ such that g commutes with the image of ψ. We also define S_ψ as the set of $g \in \widehat{G}$ such that*

$$w \mapsto g^{-1}\psi(w) g \psi(w)^{-1}, \tag{57}$$

is a (locally) trivial 1-cocycle of $\mathcal{L}_F \times SL_2(\mathbb{C})$ valued in $Z(\widehat{G})$. Note that evidently $Z(\widehat{G}) \subseteq S_\psi$ and we define $\overline{S_\psi}$ to be $S_\psi / Z(\widehat{G})$.

We define an A-parameter ψ to be *elliptic* if ψ factors through no proper Levi subgroup of ${}^L G$ and we have the following lemma of Kottwitz

Lemma 2.2.12. *The following are equivalent.*

(1) The parameter ψ is elliptic,

(2) $C_\psi^\circ \subset Z(\widehat{G})$,

(3) $S_\psi^\circ \subset Z(\widehat{G})$.

Proof. See [Kot84b, Lemma 10.3.1]. □

We now move towards stating our desired bijection. We begin first by defining the set on one side of the bijection. Roughly, this consists of A-parameters for endoscopic groups for G. More precisely:

Definition 2.2.13. *Define the set $\mathcal{EP}_F(G)$ to be equivalences classes of quadruples $(H, s, {}^L\eta, \psi^H)$ where ${}^L\eta : {}^LH \to {}^LG$ is an L-map, $(H, s, {}^L\eta|_{\widehat{H}})$ is an endoscopic datum, and ψ^H is an A-parameter of H such that ${}^L\eta \circ \psi^H$ is relevant.*

Two quadruples $(H_1, s_1, {}^L\eta_1, \psi_1^H)$ and $(H_2, s_2, {}^L\eta_2, \psi_2^H)$ are equivalent if there is an isomorphism $\alpha : H_2 \to H_1$ of endoscopic data such that ${}^L\alpha \circ \psi_1^H$ is $Z(\widehat{G})$-equivalent to ψ_2^H. By 2.2.3, note that the choice of ${}^L\alpha$ is unique up to $\widehat{H_1}$-conjugacy and that the notion of $Z(\widehat{G})$ equivalence does not depend on this choice.

We define $\mathcal{EP}_F^{\mathrm{ell}}(G) \subset \mathcal{EP}_F(G)$ to be the subset consisting of those tuples such that (H, s, η) is an elliptic endoscopic datum and ${}^L\eta \circ \psi^H$ is elliptic.

We then have the following definition of the other set in our desired bijection:

Definition 2.2.14. *Define the set $\mathcal{SP}_F(G)$ of equivalence classes of pairs $(\psi, \overline{s})$ such that ψ is a relevant Arthur parameter of G and $\overline{s} \in \overline{S_\psi}$. Two pairs $(\psi_1, \overline{s}_1)$ and $(\psi_2, \overline{s}_2)$ are equivalent if ψ_1 and ψ_2 are equivalent by some $g \in \widehat{G}$ such that $\mathrm{Int}(g)(\overline{s}_1)$ and $\overline{s}_2$ are conjugate in $\overline{S_{\psi_2}}$.*

We define $\mathcal{SP}_F^{\mathrm{ell}}(G) \subset \mathcal{SP}_F(G)$ to consist of those pairs such that ψ is elliptic.

We can now finally state our desired bijection:

Proposition 2.2.15. *The map*

$$[H, s, {}^L\eta, \psi^H] \mapsto [{}^L\eta \circ \psi^H, \overline{\eta(s)}] \tag{58}$$

gives a well-defined bijection $\mathcal{EP}_F(G) \to \mathcal{SP}_F(G)$. Moreover, this map restricts to a bijection

$$\mathcal{EP}_F^{\mathrm{ell}}(G) \to \mathcal{SP}_F^{\mathrm{ell}}(G). \tag{59}$$

We now consider the case where F is a global field and $\mathbf{G}$ is a reductive group over F. We have another construction analogous to that of

$\mathcal{RE}(\mathbf{G})$ and $\mathcal{RE}^{\text{ell}}(\mathbf{G})$. Namely we define $\mathcal{REP}(\mathbf{G})$ to be the image of the projection

$$\mathcal{EP}_F(\mathbf{G}) \to \mathcal{E}(\mathbf{G}), \tag{60}$$

and $\mathcal{REP}^{\text{ell}}(\mathbf{G})$ to be the image of the projection

$$\mathcal{EP}_F^{\text{ell}}(\mathbf{G}) \to \mathcal{E}(\mathbf{G}). \tag{61}$$

This suggests the following

Question 2.2.16. Is it true that

$$\mathcal{REP}(\mathbf{G}) = \mathcal{RE}(\mathbf{G}), \tag{62}$$

and

$$\mathcal{REP}^{\text{ell}}(\mathbf{G}) = \mathcal{RE}^{\text{ell}}(\mathbf{G})? \tag{63}$$

An important remark to make is that the previous discussion as well as the statement of 2.2.15 for global F are contingent on the definition of the global Langlands group $\mathcal{L}_F$. In fact, our proof of 2.2.15 uses this group in a somewhat nontrivial way, as we need to use ψ to construct a Galois action on $\widehat{H}$. We instead we prove the following result, which can be seen as evidence of the conjectured inclusion $\mathcal{REP}^{\text{ell}}(\mathbf{G}) \subset \mathcal{RE}^{\text{ell}}(\mathbf{G})$. This result carries no hidden conjectures on the Langlands correspondence. In particular, we will use it in the proof of our main result on the Scholze-Shin conjecture.

Theorem 2.2.17. *Suppose that F is a totally real number field. Suppose that we have a triple $(\mathbf{H}, s, {}^L\eta)$ such that $(\mathbf{H}, s, \eta)$ is an endoscopic group for $\mathbf{G}$ and ${}^L\eta$ is an extension of η to ${}^LH$. In particular, for each place v of F we get an endoscopic datum $(\mathbf{H}_v, s, {}^L\eta_v)$ of $\mathbf{G}_v$. Suppose further that for each place v, we have an A-parameter $\psi_v^{\mathbf{H}}$ of $\mathbf{H}_v$ such that ${}^L\eta_v \circ \psi_v^{\mathbf{H}}$ is relevant. We assume further that at each real place v_∞, $(\mathbf{H}_{v_\infty}, s, \eta)$ is elliptic and that $\mathbf{H}_{v_\infty}$ has an elliptic maximal torus. Then in fact $(\mathbf{H}, s, \eta) \in \mathcal{RE}(\mathbf{G})$.*

Remark 2.2.18. The restriction that F is totally real is not really a strong condition since it is almost implied by the later assumptions. In particular, to have that $\mathbf{H}_{V_\infty}$ has an elliptic maximal torus for all infinite places v_∞ implies, unless $\mathbf{H}$ is itself a torus, that F is totally real.

2.2.3. **Proof of 2.2.15.** We now give the proof of the key bijection 2.2.15. Before we begin the proof in earnest, it will be helpful to establish two useful general lemmata.

The first is the following:

Lemma 2.2.19. *Let X be a complex reductive group. Let $s \in X(\mathbb{C})$ be semisimple and set $Y := Z_X(s)^\circ$. Then, the map $N_X(Y) \to \mathrm{Out}(Y)$ given on $\mathbb{C}$-points by sending $x \in N_X(Y)(\mathbb{C})$ to $\mathrm{Int}(x)_{|Y}$ has finite image.*

Proof. Let us note that $Z_X(Z(Y))^\circ$ is contained in the kernel of the map $N_X(Y) \to \mathrm{Out}(Y)$. Indeed, it suffices to show that $Z_X(Z(Y))^\circ \subseteq Y$. We first observe that $s \in Z(Y)$. Evidently $s \in Z(Z_X(s)) \subseteq Z_X(s)$ so the only non-trivial statement is that s is actually in $Z_X(s)^\circ = Y$. But, note that since s is semisimple, we have $s \in T(\mathbb{C})$ for T a maximal torus of X. Hence $s \in T(\mathbb{C}) \subset Y$ and so $s \in Y$ and thus $s \in Z(Y)$. Therefore, $Z_X(Z(Y)) \subseteq Z_X(s)$ and thus $Z_X(Z(Y))^\circ \subseteq Z_X(s)^\circ = Y$.

To finish the proof, it suffices to show that $N_X(Y)/Z_X(Z(Y))^\circ$ is finite. But, since $Z_X(Z(Y))^\circ$ is finite index in $Z_X(Z(Y))$ it suffices to show that $N_X(Y)/Z_X(Z(Y))$ is finite. Note though that $N_X(Y) \subseteq N_X(Z(Y))$ since $Z(Y)$ is a characteristic subgroup of Y. Thus, we get an inclusion

$$N_X(Y)/Z_X(Z(Y)) \hookrightarrow N_X(Z(Y))/Z_X(Z(Y)) \tag{64}$$

and thus it suffices to show this latter group is finite. Of course, this is equivalent to showing that $N_X(Z(Y))^\circ$ and $Z_X(Z(Y))^\circ$ coincide. Since $Z(Y)$ is multiplicative (since Y is reductive by [Hum95, §2.2]) this claim follows from [Hum75, Corollary, §16.3]. ☐

The second lemma is the following:

Lemma 2.2.20. *Let F be a field of characteristic 0. Let X be reductive group over $\overline{F}$ and let $\mathcal{S}$ be a splitting of X. Then, given a finite Galois extension F'/F and a homomorphism $\xi : \mathrm{Gal}(F'/F) \to \mathrm{Out}(X)$, there exists a unique quasi-split group H over F such that there is an isomorphism $\widehat{H} \xrightarrow{\approx} X$ equivariant (up to inner automorphisms).*

Proof. Let Ψ be the based root datum associated to the triple (X, B, T) and let (X', B', T') be the dual triple with associated root datum $\Psi^\vee$. Let X'_0 be the unique split model of X' over F. Note then that we have natural isomorphisms of (constant) group (schemes)

$$\mathrm{Out}((X'_0)_{\overline{F}}) \cong \mathrm{Out}(X') \cong \mathrm{Aut}(\Psi^\vee) \cong \mathrm{Aut}(\Psi) \cong \mathrm{Out}(X) \tag{65}$$

Note then associated to ξ is a homomorphism $\xi^\vee : \mathrm{Gal}(F'/F) \to \mathrm{Out}((X'_0)_{\overline{F}})$. Then, by Proposition 2.5.68 we get a unique associated quasi-split inner form H of X'_0. Moreover, it's clear from construction that the natural map $\Gamma_F \to \mathrm{Out}(H_{\overline{F}})$ coincides with $\xi^\vee$. It is then not hard to see that we have a natural isomorphism $\overline{H} \xrightarrow{\approx} X$ as desired. ☐

We now return to the proof of Proposition 2.2.15:

Proof. (Proposition 2.2.15) We first define a map $\mathcal{EP}_F(G) \to \mathcal{SP}_F(G)$. Pick a representative $(H, s, {}^L\eta, \psi^H)$ of $[H, s, {}^L\eta, \psi^H] \in \mathcal{EP}_F(G)$. We then get a parameter ψ of G given by $\psi^H \circ {}^L\eta$.

Now, by definition of endoscopic triple we have that $w \mapsto s^{-1}w(s)$ is a (locally) trivial 1-cocycle of W_F with values in $Z(\widehat{G})$ and this induces a (locally) trivial 1-cocycle of $\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C})$ via the projection $\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to W_F$. But then we have for all $w \in \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C})$

$$s^{-1}\psi^H(w)s\psi^H(w)^{-1} = s^{-1}w(s) \tag{66}$$

so that $\eta(s) \in S_\psi$. Conversely, pick an equivalence class $[\psi, \overline{s}] \in \mathcal{SP}_F(G)$ and pick a representative $(\psi, \overline{s})$. Let $s \in S_\psi$ be a lift of $\overline{s}$. Define $\widehat{H} := Z_{\widehat{G}}(s)^0$ and define η to be the natural embedding $\widehat{H} \hookrightarrow \widehat{G}$. Now, for any $g \in \mathrm{im}(\psi) \subset {}^LG$, the map $\mathrm{Int}(g) : \widehat{G} \to \widehat{G}$ stabilizes $\widehat{H}$ and hence gives a continuous homomorphism

$$\overline{\psi} : \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Out}(\widehat{H}). \tag{67}$$

given by sending an element $(w, x) \in \mathcal{L}_F$ to the image of $\mathrm{Int}(\psi(w,x))_{|\widehat{H}}$ under the map $\mathrm{Aut}(\widehat{H}) \to \mathrm{Out}(\widehat{H})$. To see the continuity note that the map $\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$ is definitionally continuous. The map ${}^LG \to \mathrm{Aut}({}^LG)$ is also clearly continuous. The map $\mathrm{Aut}({}^LG) \to \mathrm{Out}(\widehat{H})$ is continuous as one can clearly reduce to the split case in which case it reduces to checking the continuity of the map $\mathrm{Aut}(\widehat{G}) \to \mathrm{Out}(\widehat{H})$ but this is clear since this map of groups can be promoted to a functor of the associated group schemes. We claim that $\overline{\psi}$ has finite image. To see this note that it suffices to show that the image of $N_{{}^LG}(\widehat{H}) \to \mathrm{Out}(\widehat{H})$ has finite image. Note though that there is a finite extension E/F such that G_E is split so that ${}^L(G_E)$ is merely $\widehat{G} \times \Gamma_E$. Since ${}^L(G_E)$ is finite index in ${}^LG$ it's not hard to see that we can reduce to the case when G is split. The claim then immediately follows from Lemma 2.2.19. Now note that any continuous finite quotient of $\mathcal{L}_F$ is of the form $\mathrm{Gal}(F'/F)$ for some finite extension F'/F. Indeed, evidently $\mathrm{SL}_2(\mathbb{C})$ has no non-trivial finite continuous quotients. Thus, it suffices to prove the claim for $\mathcal{L}_F$. Now, if K denotes the kernel of $\mathcal{L}_F \to W_F$ then K is a connected pro-reductive complex group. Thus, K also has no non-trivial finite continuous quotients. Thus, we've reduced the claim to W_F for which the claim is obvious. Thus, we have associated to (s, ψ) a homomorphism $\overline{\psi} : \mathrm{Gal}(F'/F) \to \mathrm{Out}(\widehat{H})$ which, by Lemma 2.2.20, allows us to find a quasi-split group H over F whose dual group is naturally isomorphic to $\widehat{H}$ equivariant for the Γ_F actions on both sides.

We now claim that that (H, s, η) is an endoscopic datum for G. It remains to check that the conjugacy class of η is Γ_F-invariant and that the image of $s \in H^1(F, Z(\widehat{G}))$ is (locally) trivial. For the first check, we pick $w \in \Gamma_F$ and need to show that the constructed action of w on $\widehat{H}$ differs from the action of w on $\widehat{G}$ by an inner automorphism of $\widehat{G}$. In other words we need to show that for all $\sigma \in \Gamma_F$ that there exists some $g_\sigma \in \widehat{G}$ such that

$$\sigma_{\widehat{G}} \circ \eta \circ \sigma_{\widehat{H}}^{-1} = \mathrm{Int}(g_\sigma) \circ \eta \tag{68}$$

This is true by construction. For the second property, we note that the image of s in $H^1(F, Z(\widehat{G}))$ is definitionally given by $w \mapsto s^{-1}w(s)$ for $w \in \Gamma_F$. Since Γ_F acts on $\widehat{H}$, and thus $Z(\widehat{G}) \subseteq \widehat{H}$, through $\mathrm{Gal}(F'/F)$ we see that this cocycle is induced from a cocycle in $H^1(\mathrm{Gal}(F'/F), Z(\widehat{G}))$. Now we observe that for any lift $w' \in \mathcal{L}_F \times SL_2(\mathbb{C})$ of w, we have

$$s^{-1}\psi(w')s\psi(w')^{-1} = s^{-1}w(s). \tag{69}$$

Since $s \in S_\psi$, this gives the desired result.

By our assumption that G^{der} is simply connected, we can extend η to a map ${}^L\eta : {}^LH \to {}^LG$. Then we need to check that the parameter ψ factors through ${}^L\eta$. We shall follow techniques discussed in unpublished notes of Kottwitz. Let us begin by defining the subgroup $\mathcal{H}$ of ${}^LG$ as the set of elements $x \in {}^LG$ such that there exists an element $y \in {}^LH$ such that the equality

$$\mathrm{Int}(x) \circ {}^L\eta = {}^L\eta \circ \mathrm{Int}(y), \tag{70}$$

holds. Note that $\mathcal{H}$ depends only on ${}^L\eta\mid_{\widehat{H}}$ and, in particular, only on the endoscopic triple (H, s, η). We then have the following observation of Kottwitz:

Lemma 2.2.21. *The set $\mathcal{H}$ is a subgroup of ${}^LG$ which is a split extension of W_F by $\widehat{H}$.*

Proof. The proof is due to unpublished work of Kottwitz.

There exists a finite extension K/F such that the action of Γ_F on $\widehat{H}$ and $\widehat{G}$ factors through Γ_K. Now pick $\sigma \in \mathrm{Gal}(K/F)$ and $w \in W_F$ such that w projects to $\sigma \in \mathrm{Gal}(K/F)$. Then $(1, w) \in {}^LH$ acts on $\widehat{H}$ by σ. By definition, there exists a $g_\sigma \in \widehat{G}$ such that $\mathrm{Int}(g_\sigma) \circ \eta = \sigma \cdot \eta$. Then

$$\eta \circ (1, w) = \mathrm{Int}(\sigma(g_\sigma), w)) \circ \eta, \tag{71}$$

which implies $\mathcal{H}$ surjects onto W_F.

Now the kernel of $\mathcal{H} \to W_F$ consists of $x \in \widehat{G}$ such that there exists $y \in \widehat{H}$ and $\mathrm{Int}(x) \circ \eta = \eta \circ \mathrm{Int}(y)$. Clearly $\eta(\widehat{H})$ is contained in this set. Conversely, we have that $\mathrm{Int}(x^{-1}\eta(y))$ acts trivially on $\widehat{H}$. In particular, $x^{-1}\eta(y)$ must centralize a maximal torus $\widehat{T}_H$ of $\eta(\widehat{H})$. Then $\widehat{T}_H$ is maximal in $\widehat{G}$ as well so $x^{-1}\eta(y) \in \widehat{T}_H \subset \eta(\widehat{H})$. Hence $x \in \eta(\widehat{H})$.

We now prove that the extension

$$1 \to \eta(\widehat{H}) \to \mathcal{H} \to W_F \to 1 \tag{72}$$

is split. We proceed as follows. Let $\widehat{T} \subset \widehat{B}$ be maximal torus and Borel of $\widehat{H}$ and let $\mathcal{T}$ be the subgroup of $\mathcal{H}$ of elements preserving the pair $(\eta(\widehat{T}), \eta(\widehat{B}))$. Then $\mathcal{T}$ is an extension of W_F by $\eta(\widehat{T})$.

Then [Lan79, Lemma 4] says that if there exists a field K that is a finite Galois extension of F such that the action of W_F on $\widehat{T}$ factors through $\mathrm{Gal}(K/F)$, then $\mathcal{T}$ is split. Since this is the case, $\mathcal{T}$ is split so we can take a splitting $c : W_F \to \mathcal{T}$. Then this is also a splitting of $\mathcal{H}$. □

We then observe that for any ${}^L\eta$, we have ${}^L\eta({}^LH) \subset \mathcal{H}$. In particular, ${}^L\eta$ gives a map of extensions of W_F by $\eta(\widehat{H})$ and hence is an isomorphism onto $\mathcal{H}$.

Thus, to show that ψ factors through ${}^L\eta$, we need only show that $\mathrm{im}(\psi) \subset \mathcal{H}$. We need to show that for each $x \in \mathrm{im}(\psi)$, there exists $y \in {}^LH$ such that the projections of x and y to W_F agree and

$$\mathrm{Int}(x) \circ \eta = \eta \circ \mathrm{Int}(y), \tag{73}$$

on $\widehat{H}$. First pick $w \in \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C})$ and consider $\psi(w)$. Then we check that there exists an element $y \in {}^LH$ such that $\mathrm{Int}(\psi(w)) \circ \eta = \eta \circ \mathrm{Int}(y)$. But indeed this follows immediately from the fact that the L-action of the projection $\overline{w} \in W_F$ on $\widehat{H} \subset {}^LH$ differs from that of $\mathrm{Int}(\psi(w))$ by an element of $\mathrm{Inn}(\widehat{H})$. We then define a parameter ψ^H such that ${}^L\eta \circ \psi^H = \psi$.

We now show the map we have constructed is well-defined. First, one can also easily show that choosing a different lift of $\overline{s}$ gives an isomorphic endoscopic datum. Next, suppose that $(\psi_1, \overline{s_1})$ is equivalent to $(\psi_2, \overline{s_2})$ by some $g \in \widehat{G}$ satisfying $w \mapsto g\psi_1(w)g^{-1}\psi(w)_2^{-1}$ is a (locally) trivial cocycle of $\mathcal{L}_F$ valued in $Z(\widehat{G})$. Then by assumption $g\overline{s_1}g^{-1}$ is conjugate by some $s \in S_{\psi_2}$ to $\overline{s_2}$ and so the groups $\widehat{H}_1$ and $\widehat{H}_2$ are conjugate in $\widehat{G}$ by sg. Moreover, it is easy to check that the map $\mathrm{Int}(sg) : \widehat{H}_1 \to \widehat{H}_2$ will preserve the actions of Γ_F up to an inner automorphism of $\widehat{H}_2$ and hence descends to an isomorphism $\alpha : H_2 \to H_1$ defined over F. The map α then gives an isomorphism of the endoscopic data (H_1, s_1, η_1) and (H_2, s_2, η_2) and ${}^L\mathrm{Int}(sg) \circ \psi_1^H$ is $Z(G)$-equivalent to ψ_2^H. This shows the map is well-defined.

To conclude the proof, we must show that the maps $\mathcal{EP}_F(G) \to \mathcal{SP}_F(G)$ and $\mathcal{SP}_F(G) \to \mathcal{EP}_F(G)$ that we have constructed are inverses of each other. It is clear that the composition $\mathcal{SP}_F(G) \to \mathcal{EP}_F(G) \to \mathcal{SP}_F(G)$ is the identity. Indeed, the first map sends $[\overline{s}, \psi]$ to an element of $\mathcal{EP}_F(G)$ of the form $[H, s, {}^L\eta, \psi^H]$ where s is a lift of $\overline{s}$ to S_ψ and ${}^L\eta \circ \psi^H = \psi$. The second map then takes $[H, s, {}^L\eta, \psi^H]$ to $[\overline{\eta(s)}, {}^L\eta \circ \psi^H]$. But, by definition $\overline{\eta(s)} = \overline{s}$ and ${}^L\eta \circ \psi^H = \psi$ from where the conclusion follows.

We now show that the composition $\mathcal{EP}_F(G) \to \mathcal{SP}_F(G) \to \mathcal{EP}_F(G)$ is the identity. Take a representative $(H, s, {}^L\eta, \psi^H)$ of $[H, s, {}^L\eta, \psi^H] \in \mathcal{EP}_F(G)$. Then we want to show that this is equivalent to the tuple $(H', s', {}^L\eta', \psi^{H'})$ that we get from applying the composition $\mathcal{EP}_F(G) \to \mathcal{SP}_F(G) \to \mathcal{EP}_F(G)$ to $(H, s, {}^L\eta, \psi^H)$. Note that, up to equivalence, we can assume that $s' = s$ and so we have a map of complex Lie groups ${\eta'}^{-1} \circ \eta : \widehat{H} \to \widehat{H'}$.

We claim this map is equivariant for each $w \in \Gamma_F$ up to conjugation by some $h \in \widehat{H}$. There exists some finite extension E/F such that the actions of Γ_F on both groups factor through $\mathrm{Gal}(E/F)$ hence we need only prove the claim for $w \in \mathrm{Gal}(E/F)$. Pick a lift $w' \in \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C})$ of w, the action of w on each group differs by an inner automorphism from the action of conjugation by $\psi^H(w')$ or $\psi^{H'}(w')$ respectively. So then we have (up to conjugation which we denote by $\sim$) for $h \in \widehat{H}$:

$$(w \cdot ({\eta'}^{-1} \circ \eta))(h) = w({\eta'}^{-1}\eta(w^{-1}(h)) \tag{74}$$

$$\sim \mathrm{Int}(\psi^{H'}(w'))({\eta'}^{-1}\eta(\mathrm{Int}(\psi^H(w')^{-1})(h)) \tag{75}$$

$$= ({\eta'}^{-1} \circ \mathrm{Int}(\psi(w')) \circ \mathrm{Int}(\psi(w')^{-1}) \circ \eta)(h) \tag{76}$$

$$= (\eta' \circ \eta)(h). \tag{77}$$

This proves the claim and implies that the isomorphism descends to an isomorphism $\alpha : H' \to H$ defined over F. This satisfies $\widehat{\alpha}(s) = s'$ mod $Z(\widehat{G})$ and hence gives the desired isomorphism of endoscopic data. Moreover, it is clear that we have an equivalence
$(H, s, {}^L\eta, \psi^H), (H', s', {}^L\eta', \psi^{H'})$.

We now check that the bijection restricts to give a bijection

$$\mathcal{EP}_F^{\mathrm{ell}}(G) \to \mathcal{SP}_F^{\mathrm{ell}}(G). \tag{78}$$

We need to check that if $[\psi, \overline{s}] \in \mathcal{SP}_F(G)^{\mathrm{ell}}$, then the tuple
$(H, s, {}^L\eta, \psi^H)$ we construct from $(\psi, \overline{s})$ satisfies that (H, s, η) is elliptic. But we have $\eta((Z(\widehat{H})^{\Gamma_F})^0) \subset \eta(C^0_{\psi^H}) \subset C^0_\psi \subset Z(\widehat{G})$ as desired. Note that the last equality holds by [Kot84b, lemma 10.3.1]. $\square$

2.2.4. **Proof of 2.2.17.** We now prove our main result on relevancy of global endoscopy. We need to construct a $(\mathbf{G},\mathbf{H})$-regular $\gamma_{\mathbf{H}} \in \mathbf{H}(F)$ such that $\gamma_{\mathbf{H}}$ transfers to some elliptic $\gamma \in \mathbf{G}(F)$. To do so, we first need the following proposition.

Proposition 2.2.22 ([Kot90, pg 188])**.** $\mathbf{G}$ *be a group over a totally number field F. Let $(\mathbf{H}, s, \eta)$ be an endoscopic datum of $\mathbf{G}$ such that $(\mathbf{H}_v, s, \eta)$ is elliptic for all infinite places v of F. Let $\gamma_{\mathbf{H}} \in \mathbf{H}(F)$ be a $(\mathbf{G},\mathbf{H})$-regular semisimple element such that $\gamma_{\mathbf{H}}$ transfers to an element of $\mathbf{G}(F_v)$ for each place v of F and $\gamma_{\mathbf{H}}$ is elliptic as an element of $\mathbf{H}(F_v)$ for all infinite places v of F. Then in fact, $\gamma_{\mathbf{H}}$ transfers to a semisimple $\gamma \in \mathbf{G}(F)$.*

Let us note that it suffices to consider the case when $F = \mathbb{Q}$. Indeed, set $\mathbf{G}' := \mathsf{Res}_{F/\mathbb{Q}}\mathbf{G}$ and set $(\mathbf{H}', s', \eta')$ to be so that $\mathbf{H}' = \mathsf{Res}_{F/\mathbb{Q}}\mathbf{H}$, the element $s' := (s, ..., s) \subset \widehat{\mathbf{H}'} = \widehat{\mathbf{H}}^m$ (where $m := [F : \mathbb{Q}]$), and η' is the map $\widehat{\mathbf{H}'} \to \widehat{\mathbf{G}'}$ given by

$$\eta'(h_1, \ldots, h_m) := (\eta(h_1), \ldots, \eta(h_1), \ldots, \eta(h_m)) \tag{79}$$

Then, if we let $\gamma_{\mathbf{H}'}$ be equal to $\gamma_{\mathbf{H}}$ as an element of $\mathbf{H}'(\mathbb{Q}) = \mathbf{H}(F)$ we get the desired result.

Before we begin the proof in earnest, we record here a general fact:

Lemma 2.2.23. *Let X be a reductive group over a field F. Then, there is a short exact sequence of Γ_F-modules*

$$1 \to K \to Z(\widehat{X})^\circ \to \widehat{Z(X)^\circ} \to 1 \tag{80}$$

where K is some finite Γ_F-module. If F is a local field, this in turn induces a natural isogeny of abelian groups

$$(Z(\widehat{X})^\circ)^{\Gamma_F} \to (\widehat{Z(X)^\circ})^{\Gamma_F} \tag{81}$$

Proof. Let us begin by noting that we have a short exact sequence of connected reductive F-groups

$$1 \to Z(X)^\circ \to X \to Q \to 1 \tag{82}$$

where $Q := X/Z(X)^\circ$ is semisimple. We then get a short exact sequence of Γ_F-modules

$$1 \to Z(\widehat{Q}) \to Z(\widehat{X}) \to \widehat{Z(X)^\circ} \to 1 \tag{83}$$

Note that since Q is semisimple, $Z(\widehat{Q})$ is finite (e.g. [Kot84b, (1.8.4)]) from where the first part of the proposition follows.

Let us now consider the associated long exact sequence of Γ_F-modules

$$1 \to Z(\widehat{Q})^{\Gamma_F} \to (Z(\widehat{X})^\circ)^{\Gamma_F} \to (\widehat{Z(X)^\circ})^{\Gamma_F} \to H^1(F, Z(\widehat{Q})) \tag{84}$$

We are then done by observing that since F is a local field that $H^1(F, Z(\widehat{Q}))$ is finite. $\square$

Proof. (Proposition 2.2.22) By assumption there exists a $\gamma \in \mathbf{G}(\mathbb{A})$ such that $\gamma_{\mathbf{H}}$ transfers to γ. Let $\psi : \mathbf{G}^* \to \mathbf{G}$ be a quasisplit inner twist of $\mathbf{G}$. By [Kot82, Theorem 4.1], $\gamma_{\mathbf{H}}$ transfers to some $\gamma^* \in \mathbf{G}^*(\mathbb{Q})$.

Now, as in [Kot86b, §6], the elements γ^*, γ determine an element $\mathrm{obs}(\gamma) \in \mathfrak{K}(I_{\gamma^*}/\mathbb{Q})^D$ such that γ is conjugate in $\mathbf{G}(\mathbb{A})$ to an element of $\mathbf{G}(\mathbb{Q})$ if and only if $\mathrm{obs}(\gamma)$ is trivial.

Lemma 2.2.24. *The element $\gamma^* \in \mathbf{G}(\mathbb{R})$ is $\mathbb{R}$-elliptic.*

Proof. Since $\gamma_{\mathbf{H}}$ is $(\mathbf{G}, \mathbf{H})$-regular and elliptic in $\mathbf{H}(\mathbb{R})$, it follows that γ^* is elliptic in $\mathbf{G}^*(\mathbb{R})$. Indeed, recall first that since $\mathbf{H}$ is an endoscopic group of G that $Z(\mathbf{G}) \subseteq Z(\mathbf{H})$ as $\mathbb{Q}$-groups (e.g. see the second to last paragraph of [Shi10, Page 5]). Note then that since $\gamma_{\mathbf{H}}$ is $(\mathbf{G}, \mathbf{H})$-regular that I_γ and I_{γ^*} are inner forms (e.g. see [Kot86b, §3]). Thus,

$$Z(\mathbf{G}) \subseteq Z(\mathbf{H}) \subseteq Z(I_\gamma) = Z(I_{\gamma^*}) \tag{85}$$

holds and thus

$$Z(\mathbf{G}_{\mathbb{R}}) \subseteq Z(\mathbf{H}_{\mathbb{R}}) \subseteq Z(I_{\gamma,\mathbb{R}}) = Z(I_{\gamma^*,\mathbb{R}}) \tag{86}$$

holds by base change.

To show that γ^* is elliptic we need to show that $Z(I_{\gamma^*,\mathbb{R}})^\circ / Z(\mathbf{G}_{\mathbb{R}})^\circ$ is $\mathbb{R}$-anisotropic. By assumption we have that $Z(I_{\gamma,\mathbb{R}})^\circ / Z(H_{\mathbb{R}})^\circ$ is $\mathbb{R}$-anisotropic. Since (H, s, η) is $\mathbb{R}$-elliptic we have that $Z(\mathbf{H}_{\mathbb{R}})^\circ_s = Z(\mathbf{G}_{\mathbb{R}})^\circ_s$ (e.g. see the second to last paragraph of [Shi10, Page 5]), which implies the desired consequence. $\square$

Lemma 2.2.25. *The containment $(Z(\widehat{I_{\gamma^*}})^{\Gamma_\infty})^\circ \subset Z(\widehat{\mathbf{G}})$ holds.*

Proof. Begin by noting that

$$Z(\widehat{I_{\gamma^*}})^{\Gamma_\infty} = Z(\widehat{I_{\gamma^*,\mathbb{R}}})^{\Gamma_\infty} \tag{87}$$

Now, by assumption we have that $T := Z(I_{\gamma^*,\mathbb{R}})^\circ$ is an elliptic torus in $\mathbf{G}_{\mathbb{R}}$. Then, by lemma 2.5.37 implies that $(\widehat{T}/Z(\widehat{\mathbf{G}}))^{\Gamma_\infty}$ is finite (note that $Z(\widehat{\mathbf{G}}) = Z(\widehat{\mathbf{G}_{\mathbb{R}}})$ so we ignore the difference). Thus, *a foritiori*, we know that $\widehat{T}^{\Gamma_\infty}/Z(\widehat{\mathbf{G}})^{\Gamma_\infty}$ is finite. In particular, since $(Z(\widehat{\mathbf{G}})^{\Gamma_\infty})^\circ$ is finite index in $Z(\widehat{\mathbf{G}})^{\Gamma_\infty}$, we have that $(Z(\widehat{\mathbf{G}})^{\Gamma_\infty})^\circ$ is finite index in $\widehat{T}^{\Gamma_\infty}$.

Now, note that we're trying to show that $((Z(\widehat{I_{\gamma^*,\mathbb{R}}})^{\Gamma_\infty})^\circ \subseteq Z(\widehat{\mathbf{G}})$ so it suffices to show that $(Z(\widehat{I_{\gamma^*,\mathbb{R}}})^{\Gamma_\infty})^\circ = (Z(\widehat{\mathbf{G}})^{\Gamma_\infty})^\circ$. Note that evidently $(Z(\widehat{\mathbf{G}})^{\Gamma_\infty})^\circ$ is contained in $(Z(\widehat{I_{\gamma^*,\mathbb{R}}})^{\Gamma_\infty})^\circ$, and since the latter is connected it suffices to show that the former is finite index in the latter.

Now, we know that $(Z(\widehat{\mathbf{G}})^{\Gamma_\infty})^\circ$ is finite index in $\widehat{T}^{\Gamma_\infty}$. Note though that by Lemma 2.2.23 we have an isogeny of abelian groups

$$(Z(\widehat{I_{\gamma*,\mathbb{R}}})^\circ)^{\Gamma_\infty} \to ((Z(I_{\gamma*,\mathbb{R}})^\circ)\widehat{\ })^{\Gamma_\infty} =: \widehat{T}^{\Gamma_\infty} \tag{88}$$

which is equivariant for the inclusions of $(Z(\widehat{\mathbf{G}})^{\Gamma_\infty})^\circ$ on both sides. In particular, since $(Z(\widehat{\mathbf{G}})^{\Gamma_\infty})^\circ$ is finite index in $\widehat{T}^{\Gamma_F}$ it's also finite index in $(Z(\widehat{I_{\gamma*,\mathbb{R}}})^\circ)^{\Gamma_\infty}$.

Note then that we have the exact sequence of Γ_∞-modules

$$1 \to Z(\widehat{I_{\gamma*,\mathbb{R}}})^\circ \to Z(\widehat{I_{\gamma*,\mathbb{R}}}) \to \pi_0(Z(\widehat{I_{\gamma*,\mathbb{R}}})) \to 1 \tag{89}$$

which gives us the exact sequence

$$1 \to (Z(\widehat{I_{\gamma*,\mathbb{R}}})^\circ)^{\Gamma_\infty} \to Z(\widehat{I_{\gamma*,\mathbb{R}}})^{\Gamma_\infty} \to \pi_0(Z(\widehat{I_{\gamma*,\mathbb{R}}}))^{\Gamma_\infty} \tag{90}$$

which shows that, since $\pi_0(Z(\widehat{I_{\gamma*,\mathbb{R}}}))$ is finite, that $(Z(\widehat{I_{\gamma*,\mathbb{R}}})^\circ)^{\Gamma_\infty}$ is finite index in $Z(\widehat{I_{\gamma*,\mathbb{R}}})^{\Gamma_\infty}$. Since $(Z(\widehat{G})^{\Gamma_\infty})^\circ$ is finite index in $(Z(\widehat{I_{\gamma*,\mathbb{R}}})^\circ)^{\Gamma_\infty}$ it follows that it's also finite index in $Z(\widehat{I_{\gamma*,\mathbb{R}}})^{\Gamma_\infty}$. It follows that $(Z(\widehat{G})^{\Gamma_\infty})^\circ$ must be finite index in $(Z(\widehat{I_{\gamma*,\mathbb{R}}})^{\Gamma_\infty})^\circ$ from where the conclusion follows. $\square$

Now, the action of Γ on $Z(\widehat{I_{\gamma*}})$ factors through some finite quotient Γ_K let σ be the nontrivial element of $\Gamma_\mathbb{R}$. This gives a conjugacy class $\{\sigma\} \subset \Gamma_K$. Then by Cebotarev Density, we can find some finite place v of $\mathbb{Q}$ such that the conjugacy class of Frob_v equals $\{\sigma\}$. In particular, for such a v, we have

$$(Z(\widehat{I_{\gamma*}})^{\Gamma_v})^0 \subset Z(\widehat{I_{\gamma*}})^{\Gamma_\infty} \subset Z(\widehat{\mathbf{G}}). \tag{91}$$

Now, recall that the set of $\mathbf{G}(\mathbb{Q}_v)$ conjugacy classes in the stable conjugacy class of γ^* is in bijection with $\ker[H^1(\mathbb{Q}_v, I_{\gamma*}) \to H^1(\mathbb{Q}_v, \mathbf{G})]$. Then by the Kottwitz isomorphism we have the bijection

$$\ker[H^1(\mathbb{Q}_v, I_{\gamma*}) \to H^1(\mathbb{Q}_v, \mathbf{G})] \cong \ker[\pi_0(Z(\widehat{I_{\gamma*}})^{\Gamma_v})^D \to \pi_0(Z(\widehat{\mathbf{G}})^{\Gamma_v})^D]. \tag{92}$$

Now, $\mathfrak{K}(I_{\gamma*}/\mathbb{Q}_v)$ equals the image of $Z(\widehat{I_{\gamma*}})^\Gamma_v$ under the map

$$Z(\widehat{I_{\gamma*}})^\Gamma_v \to [Z(\widehat{I_{\gamma*}})/Z(\widehat{\mathbf{G}})]^{\Gamma_v}. \tag{93}$$

Since the kernel of this map is $Z(\mathbf{G})^{\Gamma_v}$ and we have shown that in our case

$$(Z(\widehat{I_{\gamma*}})^\Gamma)^0 \subset Z(\widehat{\mathbf{G}}), \tag{94}$$

it follows that in fact, the map

$$Z(\widehat{I_{\gamma*}})^\Gamma_v \to [Z(\widehat{I_{\gamma*}})/Z(\widehat{\mathbf{G}})]^{\Gamma_v} \tag{95}$$

factors through $\pi_0(Z(\widehat{I_{\gamma *}})^{\Gamma_v})$ and hence, we have an exact sequence

$$\pi_0(Z(\widehat{\mathbf{G}})^{\Gamma_v}) \to \pi_0(Z(\widehat{I_{\gamma *}})^{\Gamma_v} \to \mathfrak{K}(I_{\gamma *}/\mathbb{Q}_v) \to 1. \tag{96}$$

Dualizing gives

$$\mathfrak{K}(I_{\gamma *}/\mathbb{Q}_v)^D = \ker[\pi_0(Z(\widehat{I_{\gamma *}})^{\Gamma_v})^D \to \pi_0(Z(\widehat{\mathbf{G}})^{\Gamma_v})^D], \tag{97}$$

and so in conclusion, we have a bijection

$$\ker[H^1(\mathbb{Q}_v, I_{\gamma *}) \to H^1(\mathbb{Q}_v, \mathbf{G})] \twoheadrightarrow \mathfrak{K}(I_{\gamma *}/\mathbb{Q}_v)^D. \tag{98}$$

By definition, we have a surjection

$$\mathfrak{K}(I_{\gamma *}/\mathbb{Q}_v)^D \twoheadrightarrow \mathfrak{K}(I_{\gamma *}/\mathbb{Q})^D. \tag{99}$$

Finally, we observe that $\mathfrak{K}(I_{\gamma *}/\mathbb{Q}_v) \cong \mathfrak{K}(I_{\gamma}/\mathbb{Q}_v)$ so that we in fact have a surjection

$$\ker[H^1(\mathbb{Q}_v, I_{\gamma}) \to H^1(\mathbb{Q}_v, \mathbf{G})] \twoheadrightarrow \mathfrak{K}(I_{\gamma *}/\mathbb{Q})^D. \tag{100}$$

In particular, it follows that we can modify γ at the place v by some stable conjugate such that $\mathrm{obs}(\gamma)$ vanishes. This then implies the desired result. ☐

We now return to the proof of 2.2.17. By 2.2.22, we just need to find a semisimple $(\mathbf{G}, \mathbf{H})$-regular $\gamma_{\mathbf{H}} \in \mathbf{H}(F)$ that transfers to each $\mathbf{G}(F_v)$ and is elliptic at each real place.

We now reduce the question of transferring $\gamma_{\mathbf{H}}$ to that of transferring a torus T of $\mathbf{H}$. More precisely, we record the following lemma

Lemma 2.2.26. *Let (H, s, η) be an endoscopic group for G such that H and G are defined over a local field F. Suppose $T \subset H$ is a maximal torus defined over F and that T transfers to G in the sense of [Shi10] after remark 2.6. Then for any semisimple $\gamma \in \mathbf{T}(F)$, we have that γ transfers to $G(F)$ in the sense of [Shi10, §2.3].*

Proof. This is clear from definition. ☐

Hence, to prove 2.2.17, it suffices to find a maximal torus $\mathbf{T} \subset \mathbf{H}$ defined over F that transfers to $\mathbf{G}$ since the $(\mathbf{G}, \mathbf{H})$-regular elements are dense in $\mathbf{T}$. By 2.5.12, there exists a $\mathbf{T}$ defined over F and such that for each place v of F that $\mathbf{G}_v$ is not quasisplit, we have $\mathbf{T}_v$ is elliptic. In the quasisplit cases, it is clear that $\mathbf{T}_v$ transfers. Hence it suffices to show that if $(\mathbf{H}_v, s, {}^L\eta_v, \psi_v^{\mathbf{H}}, \mathbf{T}_v)$ is such that $(\mathbf{H}_v, s, \eta_v)$ is an endoscopic datum, $\psi_v^{\mathbf{H}}$ is an A-parameter of $\mathbf{H}_v$ such that ${}^L\eta \circ \psi_v^{\mathbf{H}}$ is a relevant parameter of $\mathbf{G}_v$, and $\mathbf{T}_v$ is an elliptic maximal torus of $\mathbf{H}_v$ defined over F_v, then $\mathbf{T}_v$ transfers to $\mathbf{G}_v$.

Now consider the torus $\eta_v((Z(\widehat{\mathbf{H}_v})^{\Gamma_{F_v}})^\circ) \subset \widehat{\mathbf{G}_v}$. Then the centralizer of this torus in ${}^L\mathbf{G}_v$ surjects onto W_{F_v} since it contains ${}^L\eta({}^L\mathbf{H}_v)$. In particular, we have that $Z_{{}^L\mathbf{G}_v}(\eta_v((Z(\widehat{\mathbf{H}}_v)^{\Gamma_{F_v}})^\circ))$ is a Levi subgroup of ${}^L\mathbf{G}_v$ by [Bor79, Lemma 3.5]. To simplify notation, we denote this subgroup $\mathcal{M}$. By assumption, since clearly ${}^L\eta_v$ factors through $\mathcal{M}$, we have that $\mathcal{M}$ is relevant. Hence $\mathcal{M}$ in conjugate by an element of $\widehat{\mathbf{G}_v}$ to a subgroup ${}^LM \subset {}^L\mathbf{G}_v$ such that $M \subset \mathbf{G}_v$ is a standard Levi subgroup. Since we are only concerned with the endoscopic datum $(\mathbf{H}_v, s, \eta_v)$ up to isomorphism, we can replace it with any isomorphic datum $(\mathbf{H}_v, s, \eta_v \circ \mathrm{Int}(g))$. In particular, we can and do assume without loss of generality that $\mathcal{M} = {}^LM$.

We claim that $(\mathbf{H}_v, s, \eta_v)$ is an elliptic endoscopic datum for M. We first check that $(\mathbf{H}_v, s, \eta_v)$ is an endoscopic datum for M. To see that the conjugacy class of η_v is Γ_{F_v}-invariant, we note that ${}^L\eta({}^L\mathbf{H}_v) \subset \mathcal{M}$. Since W_{F_v} and Γ_{F_v} act through some finite quotient $\mathrm{Gal}(K/F_v)$ on $\widehat{\mathbf{H}}_v$ and $\widehat{\mathbf{G}}_v$, it suffices to show that the conjugacy class of η is invariant under the action of some arbitrary $\sigma \in \mathrm{Gal}(K/F_v)$. Let $w \in W_{F_v}$ be a lift of σ. Then ${}^L\eta(1, w) = (m, w) \in {}^LM$ and we have

$$\sigma \cdot \eta = \sigma_{\widehat{\mathbf{G}_v}} \circ \eta \circ \sigma^{-1}_{\widehat{\mathbf{H}_v}} \tag{101}$$

$$= \mathrm{Int}((1, w)) \circ \eta \circ \mathrm{Int}((1, w^{-1})) \tag{102}$$

$$= \mathrm{Int}((1, w)(w^{-1}(m^{-1}), w^{-1})) \circ \eta \tag{103}$$

$$= \mathrm{Int}(m^{-1}) \circ \eta, \tag{104}$$

as desired. The only remaining check to show that $(\mathbf{H}_v, s, \eta_v)$ is an endoscopic datum is that the image of s in $H^1(F_v, Z(\widehat{M})^{\Gamma_{F_v}})$ is trivial, but this follows immediately from the functoriality of these cohomology groups. Finally, to prove that the datum is elliptic, we observe that by assumption, $\eta_v((Z(\widehat{\mathbf{H}}_v)^{\Gamma_{F_v}})^\circ) \subset Z(\widehat{M})$.

Now, we transfer $\mathbf{T}_v$ to M^* and observe that since the endoscopic datum is elliptic, $\mathbf{T}_v$ must be elliptic in M^*. In particular, it follows that $\mathbf{T}_v$ transfers to M and therefore $\mathbf{G}_v$. This completes the proof.

2.2.5. **No relevant global endoscopy.** Our goal in this subsection is to discuss the case where a group $\mathbf{G}$ possesses no relevant endoscopic groups other than the trivial one.

Namely, let us make the following definition:

Definition 2.2.27. *Let $\mathbf{G}$ be a reductive group over a number field F. We say that $\mathbf{G}$ has* no relevant global endoscopy *if $\mathcal{RE}(\mathbf{G})$ consists (up to equivalence) only of the trivial endoscopic triple $(\mathbf{G}, e, \mathrm{id})$. We say that $\mathbf{G}$ has* no relevant global elliptic endoscopy *if $\mathcal{RE}^{\mathrm{ell}}(\mathbf{G})$ consists (up to equivalence) only of the trivial endoscopic triple $(\mathbf{G}, e, \mathrm{id})$.*

We make the following useful observation:

Lemma 2.2.28. *Let* $\mathbf{G}$ *be a reductive group over a number field* F*. Then,* $\mathbf{G}$ *has no relevant global endoscopy if and only if for all semi-simple* $\gamma \in \mathbf{G}(F)$ *we have that* $\mathfrak{K}(I_\gamma/F) = 0$*. Similarly,* $\mathbf{G}$ *has no relevant global elliptic endoscopy if for all semi-simple and elliptic* $\gamma \in \mathbf{G}(F)$ *we have that* $\mathfrak{K}(I_\gamma/F) = 0$*.*

Proof. Suppose first that $\mathbf{G}$ has no relevant global endoscopy. Pick $(\gamma, \kappa) \in \mathcal{SS}_F(\mathbf{G})$. Note then that by Proposition 2.2.6, we get an element $(\mathbf{H}, s, \eta, \gamma_{\mathbf{H}}) \in \mathcal{EQ}_F(\mathbf{G})$ associated to (γ, κ). By assumption, we then know that $(\mathbf{H}, s, \eta) \sim (\mathbf{G}, e, \mathrm{id})$ and so in particular, $\eta(s) \in Z(\widehat{G})$, which implies κ is trivial.

Conversely, suppose that $\mathfrak{K}(I_\gamma/F)$ is trivial for all semi-simple $\gamma \in \mathbf{G}(F)$. Let $(\mathbf{H}, s, \eta)$ be an element of $\mathcal{RE}(\mathbf{G})$. Choose some semi-simple $\gamma_{\mathbf{H}} \in \mathbf{H}(F)$ such that $(\mathbf{H}, s, \eta, \gamma_{\mathbf{H}})$ is an element of $\mathcal{EQ}_F(\mathbf{G})$. Note that by Proposition 2.2.6 we get associated to this quadruple a pair $(\gamma, \kappa) \in \mathcal{SS}_F(\mathbf{G})$. By our assumption we have that $\kappa = 0$. Pick a transfer γ^* of γ to $\mathbf{G}^*(F)$. Then $(\mathbf{G}^*, e, \mathrm{id}, \gamma)$ is an element of $\mathcal{EQ}_F(\mathbf{G})$ which maps to $(\gamma, 0)$ under Proposition 2.2.6. Thus, we deduce that $(\mathbf{H}, s, \eta, \gamma_{\mathbf{H}}) \sim (\mathbf{G}^*, e, \mathrm{id}, \gamma)$ as desired.

The elliptic version is similar. $\square$

We will be mostly interested in reductive groups $\mathbf{G}$ such that $\mathbf{G}^{\mathrm{ad}}$ is F-anisotropic and which satisfy the Hasse principle (i.e. that $\ker^1(F, \mathbf{G}) = 0$), in which case the condition of no relevant global (elliptic) endoscopy takes the following particularly simple form:

Proposition 2.2.29. *Let* F *be a number field and* $\mathbf{G}$ *be a reductive group over* F*. Assume further that* $\mathbf{G}^{\mathrm{ad}}$ *is* F*-anistropic and satisfies the Hasse principle. Then, the following are equivalent:*

(1) $\mathbf{G}$ *has no relevant global endoscopy.*
(2) $\mathbf{G}$ *has no relevant global elliptic endoscopy.*
(3) For all maximal F*-tori* $T \subset \mathbf{G}$ *one has that the containment* $Z(\widehat{\mathbf{G}})^\Gamma \subseteq \widehat{T}^\Gamma$ *is actually an equality.*

Proof. Let us begin by observing that 1. and 2. are equivalent simply because every semi-simple element of $\mathbf{G}(F)$ is elliptic. Thus, it suffices to prove the equivalence of 1. and 3.

Note that since $\mathbf{G}$ satisfies the Hasse principle, we have that $\ker^1(\Gamma, Z(\widehat{\mathbf{G}}))$ vanishes (e.g. see [Kot84b, Remark 4.4]). Thus, it's fairly easy to see that for any semi-simple γ in $\mathbf{G}(F)$ we have that

$$\mathfrak{K}(I_\gamma/F) = Z(\widehat{I}_\gamma)^\Gamma / Z(\widehat{\mathbf{G}})^\Gamma \tag{105}$$

and thus the implication of 3. implies 1. follows immediately from Lemma 2.5.36. The implication that 1. implies 3. would follow quite simply if every maximal torus T in G were of the form I_γ for some semi-simple $\gamma \in \mathbf{G}(F)$. But, this follows immediately from Theorem 2.5.20.

□

2.2.6. **An application to the representation theory of unitary groups.** In this subsection, we derive some results on the representation theory of global unitary groups with no relevant global endoscopy. In particular, we show that the relevant elliptic A-parameters of such groups satisfy $\overline{S_\psi} = 1$. While one could prove this in enough cases using special assumptions to prove our main result, we prefer the present, more systematic, approach.

Let $F/\mathbb{Q}$ be a total real extension of number fields and E/F be a quadratic imaginary extension. Let n be an odd natural number and $(U_{E/F}(n), \omega)$ be an inner twist of $U_{E/F}(n)^*$ having no relevant endoscopy. Such a group exists by 2.4.2.

In the course of our proof, we need to appeal to the bijection 2.2.15 in the global case. To avoid making assumptions about the global Langlands group $\mathcal{L}_{\mathbb{Q}}$, we work with "automorphic A-parameters" in the sense of [Kal+14, §1.3.4]. This notion is originally due to Arthur [Art13]. We note that an automorphic parameter yields at each place v of F, a localization ψ_v which is an A-parameter of $\mathbf{U}_v$ [Kal+14, §1.3.5]. Moreover, one can make sense of the groups C_ψ and S_ψ for such parameters [Kal+14, §1.3.4]. In particular, we note that the words elliptic and relevant make sense for automorphic parameters. Thus, a first step is to prove a version of 2.2.15 for automorphic parameters.

Proposition 2.2.30. *Let E/F be a quadratic extension of number fields. Let $\mathbf{U}$ be an inner form of $U_{E/F}(N)^*$. Let us make the following notational definitions*

- *Set $\mathcal{AEP}_F(\mathbf{U})$ to be the set of all quadruples $(H, s, {}^L\eta, \psi^{\mathfrak{e}})$ where $(H, s, {}^L\eta)$ is an extended endoscopic datum of $\mathbf{U}$ And $\psi^{\mathfrak{e}} = (\psi^n, \tilde{\psi}^{\mathfrak{e}}) \in \Psi(H, {}^L\eta)$ (as in [Kal+14, §1.3.6]).*
- *Set $\mathcal{ASP}_F(\mathbf{U})$ to be the set of all pairs $(\overline{s}, \psi)$ where $\psi = (\psi^n, \tilde{\psi}) \in \Psi(\mathbf{U}, \eta_{\chi_k})$ and $\overline{s} \in \overline{S_\psi}$.*

We then have a bijection $\mathcal{AEP}_F(\mathbf{U}) \to \mathcal{ASP}_F(\mathbf{U})$ given by

$$[H, s, {}^L\eta, \psi^H] \mapsto [{}^L\eta \circ \tilde{\psi}^{\mathfrak{e}}, \overline{\eta(s)}], \tag{106}$$

Moreover, this bijection is compatible via localization with the local version of 2.2.15 using the localization map in [Kal+14, §1.3.5].

Proof. The bijection is constructed analogously to the proof of 2.2.15. We first define the inverse map. Given $[\overline{s}, \psi] \in \mathcal{ASP}_F(\mathbf{U})$ we need to construct an element of $\mathcal{AEP}_F(\mathbf{U})$, In particular, $\mathcal{L}_\psi$ is an extension of W_F by a pro-reductive group just as $\mathcal{L}_F$ was. Since this was the key property of $\mathcal{L}_F$ that we used, we can construct the datum $(H, s, {}^L\eta)$ using a lift of $\overline{s}$ and $\tilde{\psi} : \mathcal{L}_\psi \to {}^L\mathbf{U}$ as in the proof of 2.2.15. Then we can conclude as before that $\tilde{\psi}$ factors through the image of ${}^L\eta$ and hence gives rise to a parameter $\psi^{\mathfrak{e}}$ such that $\eta_\chi \circ {}^L\eta \circ \tilde{\psi}^{\mathfrak{e}} = \tilde{\psi}^n$ as desired. As in 2.2.15 we conclude that this map is the desired inverse.

Now we prove compatibility with the local version of 2.2.15. We need to show that if v is a place of F, then the bijection in 2.2.15 identifies $[H_v, s_v, {}^L\eta_v, \psi_v^{\mathfrak{e}}]$ with $[\overline{s}, \psi_v]$. This follows from the commutative diagram after Proposition 1.3.3 in [Kal+14]. □

2.2.6.1. *The Triviality of $\overline{S_\psi}$.* In this subsection, we prove that relevant elliptic parameters of the group $\mathbf{U} := U_{E/F}(n)$ satisfy $\overline{S_\psi} = 1$.

Proposition 2.2.31. *Let ψ be a relevant elliptic automorphic A - parameter of $\mathbf{U}$ such that for some infinite place v_∞ of F, we have ψ_{v_∞} is elliptic. Then we have $\overline{S_\psi} = 1$.*

Proof. Suppose for contradiction that $\overline{S_\psi}$ has a nontrivial element $\overline{s}$ and pick a lift $s \in S_\psi$.

Then for each place v of F, we see that identifying $\widehat{\mathbf{U}} \subset {}^L\mathbf{U}$ with $\widehat{\mathbf{U}_v} \subset {}^L\mathbf{U}_v$, we get that $s \in S_{\psi_v}$ so that $(\psi_v, \overline{s}_v) \in \mathcal{SP}_{F_v}(\mathbf{G}_v)$ and hence by 2.2.15 we get an endoscopic datum (H_v, s_v, η_v) of $\mathbf{G}_v$. Under our identifications, $\widehat{H_v} \subset {}^L\mathbf{G}_v$ and η_v is the inclusion map. Moreover $\eta_v(s_v) = s$. In particular, we have for all v that $\eta_v(\widehat{H_v}) = Z_{\widehat{\mathbf{G}}}(s)^0$.

By 2.2.30, we get a datum $[\mathbf{H}, s, {}^L\eta, \psi^{\mathfrak{e}}] \in \mathcal{AEP}_F(\mathbf{U})$. In particular, we have a global endoscopic datum $(\mathbf{H}, s, \eta)$ that localizes at each place v to (H_v, s_v, η_v). Now, v_∞ ramifies over E since E/F is imaginary and hence $\mathbf{U}_{v_\infty}$ is an inner form of $U_{E_{v_\infty}/F_{v_\infty}}(n)$. Since we assumed ψ_{v_∞} is elliptic, it follows from 2.2.15 that $(H_{v_\infty}, s_{v_\infty}, \eta_{v_\infty})$ is an elliptic endoscopic datum.

We now pick a lift ${}^L\eta$ of η and note that for each place v, we get a map ${}^L\eta_v$. Now, we recall that the choice of the lift ${}^L\eta_v$ in the construction of the map $\mathcal{SP}_F(\mathbf{G}_v) \to \mathcal{EP}_F(\mathbf{G}_v)$ is arbitrary and picking a different lift does not change $(\mathbf{H}_v, s, \eta_v)$. In particular, we could have picked at each place v, the lift ${}^L\eta_v$ of η_v that we got from localizing ${}^L\eta$. Note however that doing so does change the parameters $\psi^{\mathbf{H}_v}$.

In particular, we now have, without loss of generality, a tuple $(\mathbf{H}, s, {}^L\eta)$ and for each $v \in F$, a parameter $\psi^{\mathbf{H}_v}$ of $\mathbf{H}_v$ such that ${}^L\eta_v \circ$

$\psi^{\mathbf{H}_v}$ is relevant. Furthermore, since ψ_∞ was assumed to be elliptic, $(\mathbf{H}_\infty, s, \eta_v)$ is elliptic. Furthermore, $\mathbf{H}$ is a product of unitary groups and so has an elliptic maximal torus. In particular, we are now in the situation to apply 2.2.17. We get that there exists a semisimple $\gamma_{\mathbf{H}} \in \mathbf{H}(F)$ such that $(\mathbf{H}, s, \eta, \gamma_{\mathbf{H}}) \in \mathcal{RE}(\mathbf{U})$. Now by 2.2.6 we get an element $(\gamma, \kappa) \in \mathcal{SS}_F^{\mathrm{ell}}(\mathbf{G})$. Since s is nontrivial in $\overline{S_\psi}$, it follows that κ is nontrivial. This contradicts that for $\mathbf{U}$, all $\mathfrak{K}(I_\gamma/F)$ are trivial. $\square$

2.2.6.2. *Isotypic Components.* Now, let $\mathbf{G} = \mathsf{Res}_{F/\mathbb{Q}}\mathbf{U}$ and choose χ_κ, Ξ for $\mathbf{U}$ as in [Kal+14, Thm. 1.7.1]. Then it follows from that theorem that we have a decomposition

$$L^2_{\mathrm{disc}}(\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A}_F)) = \bigoplus_{\psi \in \Psi_2(\mathbf{U}^*, \eta_{\chi_\kappa})} \bigoplus_{\pi \in \Pi_\psi(\mathbf{U}, \omega, \epsilon_\psi)} \pi. \tag{107}$$

Now we fix a representation $\boldsymbol{\pi}$ of $\mathbf{G}(\mathbb{A}_\mathbb{Q})$ that is discrete at ∞. Since $\mathbf{G}(\mathbb{A}_\mathbb{Q}) = \mathbf{U}(\mathbb{A}_F)$, we can equivalently consider $\boldsymbol{\pi}$ to be a representation of $\mathbf{U}(\mathbb{A}_F)$. We call this representation $\boldsymbol{\pi}'$ so as to avoid confusion. Now, at any place p of $\mathbb{Q}$, we have

$$\boldsymbol{\pi}_p = \bigotimes_{v|p} \boldsymbol{\pi}'_v. \tag{108}$$

Then the Satake parameters of $\boldsymbol{\pi}'$ determine a unique parameter $\psi_{\boldsymbol{\pi}'}$ of $\mathbf{U}$ such that $\boldsymbol{\pi}' \in \Pi_{\psi_{\boldsymbol{\pi}'}}(\mathbf{U}, \xi)$. Since $\boldsymbol{\pi}'$ is discrete at each infinite place, it follows that $\psi_{\boldsymbol{\pi}'}$ has trivial Arthur SL_2-factor and hence is generic. Hence by the comment after equation [Kal+14, (1.2.4)], we have that each element of $\Pi_{\psi_{\boldsymbol{\pi}'}}(\mathbf{U}, \omega)$ is irreducible. Moreover each element of the packet appears with multiplicity 1 by the global multiplicity formula.

Now by 2.2.31, it follows that $\Pi_{\psi_{\boldsymbol{\pi}'}}(\mathbf{U}, \omega, \epsilon_{\psi_{\boldsymbol{\pi}'}}) = \Pi_{\psi_{\boldsymbol{\pi}'}}(\mathbf{U}, \omega)$ or, in other words, the condition involving $\epsilon_{\psi_{\boldsymbol{\pi}'}}$ is vacuous. In particular, if we let $\boldsymbol{\pi}'^p$ denote the factor of $\boldsymbol{\pi}$ that is the complement of $\bigotimes_{v|p} \boldsymbol{\pi}'_v$, then we have

$$L^2_{\mathrm{disc}}(\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A}_F))[\boldsymbol{\pi}'^p] = \bigotimes_{v|p} \bigoplus_{\pi'_v \in \Pi_{\psi_{\boldsymbol{\pi}'_v}}(\mathbf{U}_v, \omega)} \pi'_v \tag{109}$$

We can define a parameter $\psi_{\boldsymbol{\pi}}$ of the group $\mathbf{G}$. Since $\mathbf{G}_p = \prod_{v|p} \mathbf{U}_v$, it follows that

$$\bigoplus_{\pi_p \in \Pi_{\psi_{\boldsymbol{\pi}_p}}(\boldsymbol{G}(\mathbb{Q}_p), \omega)} \pi_p = \bigotimes_{v|p} \bigoplus_{\pi'_v \in \Pi_{\psi_{\boldsymbol{\pi}'_v}}(\mathbf{U}_v, \omega)} \pi'_v. \tag{110}$$

In particular, we record the following result.

Lemma 2.2.32. *We have the following decomposition.*

$$L^2_{\mathrm{disc}}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))[\boldsymbol{\pi}^p] = \bigoplus_{\psi_{\boldsymbol{\pi}_p}\in\Pi_{\psi_p}(\boldsymbol{G}(\mathbb{Q}_p),\omega)} \pi_p. \tag{111}$$

2.3. The ℓ-adic Cohomology of Compact Shimura Varieties with No Endoscopy

2.3.1. **Introduction.** We state in this section a result on the decomposition of the cohomology of certain compact Shimura varieties $\mathsf{Sh}(\mathbf{G}, X)$ in the case when $(\mathbf{G}, X)$ has no relevant global endoscopy (in the sense of §2.2.5). The results here are largely a technical generalization of the results in [Kot92a] using the newly proven results of [KSZ] checking, in all cases, that the methods of [Kot92a] work in this more general setting under the umbrella assumption of no endoscopy.

This decomposition will be key to understanding the Scholze–Shin conjecture at a given bad place in terms of the already established Scholze–Shin conjecture at a good place which, at least in the case of the trivial endoscopic triple, is just a rephrasing of the results of [Kot84a].

2.3.2. **Statement of the decomposition result.** Let us now state the decomposition result of interest to us. To do this, we begin by detailing the necessary setup.

We start with a Shimura datum $(\mathbf{G}, X)$ which we assume to be of abelian type. We assume further that our group satisfies Axiom **SV**5 of [Mil04]. By [Mil04, Theorem 5.26] (and the succeeding discussion) this is equivalent to assuming that $(A_{\mathbf{G}})_{\mathbb{R}} = A_{\mathbf{G}_{\mathbb{R}}}$. We assume further that $\mathbf{G}/Z(\mathbf{G})$ is $\mathbb{Q}$-anisotropic. Note that this implies that if $\mathbf{T}$ is a maximal torus in $\mathbf{G}$ then $\mathbf{T}_{\mathbb{R}}$ is an elliptic maximal torus in $\mathbf{G}_{\mathbb{R}}$. Thus, in particular, we see that $\mathbf{G}(\mathbb{R})$ has discrete series (see [Kna01, Theorem 12.20]). We also assume that $\mathbf{G}^{\mathrm{der}}$ is simply connected.

Most importantly, we assume that the group $\mathbf{G}$ has no relevant global endoscopy (in the sense of §2.2.5). This is the key assumption which makes the proof of Theorem 2.3.1 below possible.

Let us fix a prime ℓ and let ξ be an algebraic $\overline{\mathbb{Q}_\ell}$-representation of $\mathbf{G}$ (i.e. an algebraic representation $\xi : \mathbf{G}_{\overline{\mathbb{Q}_\ell}} \to \mathrm{GL}_{\overline{\mathbb{Q}_\ell}}(V)$ for some $\overline{\mathbb{Q}_\ell}$-space V) which induces a representation

$$\mathbf{G}(\mathbb{A}_f) \xrightarrow{\text{proj.}} \mathbf{G}(\mathbb{Q}_\ell) \hookrightarrow \mathbf{G}(\overline{\mathbb{Q}_\ell}) \to \mathrm{GL}_{\overline{\mathbb{Q}_\ell}}(V)$$

which we also denote ξ.

Let us also note that from the conjugacy class X one obtains a conjugacy class of cocharacters $\boldsymbol{\mu}$ of $\mathbf{G}_{\mathbb{C}}$ as on [Mil04, Page 111] which (as in loc. cit.) induces a unique conjugacy class of cocharacters, also denoted $\boldsymbol{\mu}$, over $\overline{\mathbb{Q}}$. Moreover, by definition, the reflex field $E(\boldsymbol{G}, X)$ is precisely the reflex field of $\boldsymbol{\mu}$ as in §2.5.1.4. We denote this field by $\mathbf{E}_{\boldsymbol{\mu}}$. Then, by the contents of §2.5.1.4 we obtain a representation $r_{\boldsymbol{\mu}} : \widehat{G} \rtimes W_{\mathbf{E}_{\boldsymbol{\mu}}} \to \mathrm{GL}(V(\boldsymbol{\mu}))$.

Finally, fix an isomorphism $\iota_l : \overline{\mathbb{Q}_\ell} \cong \mathbb{C}$ which we implicitly use throughout the sequel. In particular, via ι_ℓ we get an algebraic representation $\xi_{\mathbb{C}}$ over $\mathbb{C}$.

With these assumptions, and in the notation as above the following holds:

Theorem 2.3.1. *There is a decomposition of virtual* $\overline{\mathbb{Q}_\ell}[\mathbf{G}(\mathbb{A}_f) \times W_{\mathbf{E}_{\boldsymbol{\mu}}}]$*-representations*

$$H^*(\mathsf{Sh}, \mathcal{F}_\xi) = \bigoplus_{\pi_f} \pi_f \boxtimes \sigma(\pi_f), \tag{112}$$

where π_f *ranges over admissible* $\overline{\mathbb{Q}_\ell}$*-representations of* $\mathbf{G}(\mathbb{A}_f)$ *such that there exists an automorphic representation* π *of* $\mathbf{G}(\mathbb{A})$ *such that;*

(1) $\pi_f \cong (\pi)_f$ *(using our identification* $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$*)*
(2) $\pi_\infty \in \Pi_\infty(\xi)$*.*

Moreover, for each π_f *there exists a cofinite set* $S(\pi_f) \subseteq S^{\mathrm{ur}}(\pi_f)$ *of primes* p *such that for each prime* $\mathfrak{p}$ *over* $\mathbf{E}_{\boldsymbol{\mu}}$ *lying over* p *and each* $\tau \in W_{\mathbf{E}_{\boldsymbol{\mu}_{\mathfrak{p}}}}$ *the following equality holds:*

$$\mathrm{tr}(\tau \mid \sigma(\pi_f)) = a(\pi_f)\,\mathrm{tr}(\tau \mid r_{-\boldsymbol{\mu}} \circ \varphi_{\pi_p}) p^{\frac{1}{2}v(\tau)[\mathbf{E}_{\boldsymbol{\mu}_{\mathfrak{p}}}:\mathbb{Q}_p]\dim \mathsf{Sh}}, \tag{113}$$

for some integer $a(\pi_f)$ *(see Definition 2.3.6).*

As stated in the introduction, the proof of this result (closely imitating [Kot92a]) is broken up in to three main steps. These, very roughly, go as follows:

- **Step 1**: Construct a function f which projects the cohomology $H^*(\mathsf{Sh}, \mathcal{F}_\xi)$ on to its π_f-isotypic component so that, by construction, the quantity $\mathrm{tr}(f \times \tau \mid H^*(\mathsf{Sh}, \mathcal{F}_\xi))$ agrees with left-hand side of (113).
- **Step 2**: Use results of Kisin-Shin-Zhu in [KSZ] to express the quantity $\mathrm{tr}(f \times \tau \mid H^*(\mathsf{Sh}, \mathcal{F}_\xi))$ in terms of sums of orbital integrals.
- **Step 3**: Pseudo-stabilize the result to obtain the right-hand side of (113).

The rest of Part I will be dedicated to carefully carrying out this proof step-by-step.

2.3.3. **The function f.** In this subsection we construct a smooth function $f : G(\mathbb{A}) \to \mathbb{C}$ alluded to in Step 1 above. This function f, which will admit a factorization $f = f_\infty f^\infty$, is deceptively notated since it really depends on the following data:

- An admissible $\overline{\mathbb{Q}_\ell}$-representation π_f of $\mathbf{G}(\mathbb{A}_f)$.
- A compact open subgroup K of $G(\mathbb{A}_f)$ such that π_f has a non-zero K-invariant vector.
- The set $\Pi^0_\infty(\xi)$.

The function f will be constructed in a highly non-explicit way. This is relevant since the entrance of the cofinite set $S(\pi_f) \subseteq S^{\mathrm{ur}}(\pi_f)$ in Theorem 2.3.1 enters in to the picture via f. Namely hidden in Step 2 of the outline above is the assumption that at p one can decompose f as $f = f^p \mathbb{1}_{K_{0,p}}$. Thus, the inexplicitness of f is part and parcel with the inexplicitness of the cofinite set $S(\pi_f)$.

2.3.3.1. *The construction of f_∞ and basic properties.* Let us begin by recalling the basic setup of the theory of pseudo-coefficients in the context that we need them. Let us fix χ to be a smooth character $A_{\mathbf{G}}(\mathbb{R})^0 \to \mathbb{C}^\times$. We then define the following set:

Definition 2.3.2. *The set $\mathscr{H}(\mathbf{G}(\mathbb{R}), \chi)$ is the set of all smooth functions $f : \mathbf{G}(\mathbb{R}) \to \mathbb{C}$ such that*

(1) $f(ag) = \chi(a) f(g)$ for all $a \in A_{\mathbf{G}}(\mathbb{R})^0$.
(2) The function $f\chi^{-1} : \mathbf{G}(\mathbb{R})/A_{\mathbf{G}}(\mathbb{R})^0 \to \mathbb{C}$ is compactly supported.

Let us now consider the set $\Pi_\infty(\chi)$ of irreducible admissible representations of $\mathbf{G}(\mathbb{R})$ with central character χ and let $\Pi^0_\infty(\chi)$ denote the subset of $\Pi_\infty(\chi)$ consisting of those elements which are discrete series for $\mathbf{G}(\mathbb{R})$. Let us note that for a fixed $\pi^0_\infty \in \Pi^0_\infty(\chi)$ we make the following definition:

Definition 2.3.3. *A* pseudo-coefficient *for π^0_∞ is an element $f_{\pi^0_\infty} \in \mathscr{H}(\mathbf{G}(\mathbb{R}), \chi^{-1})$ such that for all tempered $\pi_\infty \in \Pi_\infty(\chi)$ we have that*

$$\mathrm{tr}(f_{\pi^0_\infty} \mid \pi_\infty) = \begin{cases} 1 & \text{if} \quad \pi_\infty \cong \pi^0_\infty \\ 0 & \text{if} \quad \text{otherwise} \end{cases} \tag{114}$$

Let us be clear about what the above trace means. Namely, for π_∞ in $\Pi_\infty(\chi)$ we set $\mathrm{tr}(f_{\pi^0_\infty} \mid \pi_\infty)$ to be the trace of the operator

$$v \mapsto \int_{\mathbf{G}(\mathbb{R})/A_{\mathbf{G}}(\mathbb{R})^0} f_{\pi^0_\infty}(g) \pi_\infty(g)(v)\, dg \tag{115}$$

which is well-defined since the product of $f_{\pi_\infty^0}$ and π_∞ transform by the identity under $A_{\mathbf{G}}(\mathbb{R})^0$ and since $f_{\pi_\infty^0}$ is compactly supported on $\mathbf{G}(\mathbb{R})/A_{\mathbf{G}}(\mathbb{R})^0$.

The existence of such pseudo-coefficients can be deduced from the research announcement [CD85], with a full proof found in the references of said paper.

Let us now fix an element $\pi_\infty^0 \in \Pi_\infty^0(\xi)$ which, in particular, is an element of $\Pi_\infty^0(\chi_\xi^{-1})$. Let us denote by $f_{\pi_\infty^0} \in \mathscr{H}(\mathbf{G}(\mathbb{R}), \chi_\infty)$ the pseudo-coefficient of π_∞^0 in the sense discussed above.

We record the following equality:

Proposition 2.3.4. *For any $\gamma_\infty \in \mathbf{G}(\mathbb{R})$ semisimple, the following equality holds:*

(116)

$$\mathrm{SO}_{\gamma_\infty}(g) = \begin{cases} \mathrm{tr}(\xi(\gamma_\infty))\mathrm{vol}(A_{\mathbf{G}}(\mathbb{R})^0/I_\infty(\mathbb{R}))^{-1}e(I_\infty) & \text{if} \quad \gamma_\infty \in \mathbf{G}(\mathbb{R})^{\mathrm{ell}} \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

where $g := (-1)^{\dim \mathsf{Sh}} f_{\pi_\infty^0}$ and I_∞ is the unique anisotropic modulo center inner form of I_{γ_∞}.

Remark 2.3.5. Note that the existence of I_∞ follows from Lemma 2.5.11. Indeed, since we are assuming that $\mathbf{G}(\mathbb{R})$ has an elliptic maximal torus we know from Corollary 2.5.9 that for $\gamma_\infty \in \mathbf{G}(\mathbb{R})^{\mathrm{ell}}$ we have that $\gamma_\infty \in T(\mathbb{R})$ for some maximal elliptic torus T of $\mathbf{G}_{\mathbb{R}}$. Note then that $T \subseteq I_{\gamma_\infty}$ and thus I_{γ_∞} has an elliptic maximal torus, which shows that Lemma 2.5.11 applies.

Let us note that in the above formula the quantity $\mathrm{SO}_{\gamma_\infty}(g)$ is sensical (in the sense that the integrals defining this stable orbital integral converge) since $f_{\pi_\infty^0}$ is compactly supported on $\mathbf{G}(\mathbb{R})/A_{\mathbf{G}}(\mathbb{R})^+$ and so, in particular, compactly supported on $I_{\gamma_\infty}(\mathbb{R})\backslash\mathbf{G}(\mathbb{R})$ since $A_{\mathbf{G}}(\mathbb{R})^+ \subseteq I_{\gamma_\infty}(\mathbb{R})$.

Proposition 2.3.4. We follow [Kot92a, §3.1]. Let us first assume that γ_∞ is strongly regular. Note that then that since γ_∞ is elliptic strongly regular, we have that $I_{\gamma_\infty} = I_\infty$. Now we have:

(117)

$$O_\gamma(f_{\pi_\infty^0}) = \begin{cases} \mathrm{vol}(A_{\mathbf{G}}(\mathbb{R})^0\backslash I_{\gamma_\infty}(\mathbb{R}))^{-1}\Theta_{\pi_\infty^0}(\gamma_\infty^{-1}) & \text{if} \quad \gamma_\infty \text{ elliptic} \\ 0 & \text{if} \quad \gamma_\infty \text{ not elliptic} \end{cases}$$

where $\theta_{\pi_\infty^0}$ is the function associated to the Harisha-Chandra character of π_∞^0 by Harish-Chandra's theorem (for a proof of this formula see [Art93, Theorem 5.1]). Suppose now that γ_∞ is strongly regular elliptic.

Then, by Proposition 2.5.21 we deduce that

(118)
$$\begin{aligned}\mathrm{SO}_{\gamma_\infty}(g) &:= \sum_{[\gamma'_\infty]\sim_s[\gamma_\infty]} O_{\gamma'_\infty}(g)\\ &= \sum_{w\in W_{\mathbb{C}}/W_{\mathbb{R}}} O_{w\cdot\gamma_\infty}(g)\\ &= \sum_{w\in W_{\mathbb{C}}/W_{\mathbb{R}}} (-1)^{\dim \mathsf{Sh}}\mathrm{vol}(A_{\mathbf{G}}(\mathbb{R})^0\backslash I_{\gamma_\infty}(\mathbb{R}))^{-1}\Theta_{\pi^0_\infty}(w\cdot\gamma_\infty^{-1}))\end{aligned}$$

Note that in the first line the lack of the terms $a(\gamma'_\infty)$ is due to our assumption that $\mathbf{G}^{\mathrm{der}}$ is simply connected, and the lack of the Kottwitz sign is because $I_{\gamma'_\infty}$, by assumption, is a torus which has trivial Kottwitz sign (since it is quasi-split).

Let us write $\pi^0_\infty := \pi(\varphi, B_0)$ as in [Kot90, Page 185]. Then, this last term is equal, by the Harish-Chandra character formula, to

(119)
$$\sum_{w\in W_{\mathbb{C}}/W_{\mathbb{R}}} (-1)^{\dim \mathsf{Sh}}\mathrm{vol}(\frac{I_{\gamma_\infty}(\mathbb{R})}{A_{\mathbf{G}}(\mathbb{R})^+})^{-1}\sum_{w'\in W_{\mathbb{R}}}\chi_{w'\cdot B_0}(w\cdot\gamma_\infty^{-1})\Delta_{w'\cdot B_0}(w\cdot\gamma_\infty^{-1})$$

But, this is visibly equal to

(120)
$$\sum_{w\in W_{\mathbb{C}}/W_{\mathbb{R}}} (-1)^{\dim \mathsf{Sh}}\mathrm{vol}(\frac{I_{\gamma_\infty}(\mathbb{R})}{A_{\mathbf{G}}(\mathbb{R})^0})^{-1}\sum_{w'\in W_{\mathbb{R}}}\chi_{w'\cdot(w\cdot B_0)}(\gamma_\infty^{-1})\Delta_{w'\cdot(w\cdot B_0)}(\gamma_\infty^{-1})$$

which is equal to

(121)
$$\sum_{w'\in W_{\mathbb{R}}}\sum_{w\in W_{\mathbb{C}}/W_{\mathbb{R}}} (-1)^{\dim \mathsf{Sh}}\mathrm{vol}(\frac{I_{\gamma_\infty}(\mathbb{R})}{A_{\mathbf{G}}(\mathbb{R})^0})^{-1}\chi_{w'\cdot(w\cdot B_0)}(\gamma_\infty^{-1})\Delta_{w'\cdot(w\cdot B_0)}(\gamma_\infty^{-1})$$

which, by concatenation, is equal to

$$(-1)^{\dim \mathsf{Sh}}\sum_{w''\in W_{\mathbb{C}}}\mathrm{vol}(A_{\mathbf{G}}(\mathbb{R})^0\backslash I_{\gamma_\infty}(\mathbb{R}))^{-1}\chi_{w''\cdot B}(\gamma_\infty^{-1})\Delta_{w''\cdot B_0}(\gamma_\infty^{-1}) \tag{122}$$

But, by the Weyl character formula this is equal to

$$\mathrm{vol}(A_{\mathbf{G}}(\mathbb{R})^0\backslash I_{\gamma_\infty}(\mathbb{R}))^{-1}\,\mathrm{tr}\,\xi(\gamma_\infty) \tag{123}$$

as desired.

For the case for general elliptic $\gamma_\infty \in \mathbf{G}(\mathbb{R})$ (not necessarily strongly regular) we proceed as follows. Note that by Corollary 2.5.10 γ_∞ is contained in some elliptic maximal torus of $\mathbf{G}_{\mathbb{R}}$. The result then follows from the above description and [She83, Lemma 2.9.3].

Note that the Kottwitz sign $e(I_\infty)$ enters due to the difference in sign conventions between this article and that of Shelstad (see [She83, Page 2.12]). $\square$

With the above, we are now well-positioned to define f_∞ and observe its basic properties. Namely, let us define f_∞ as follows:

$$f_\infty := \frac{(-1)^{\dim \mathsf{Sh}}}{|\Pi^0_\infty(\xi)|} \sum_{\pi^0_\infty \in \Pi^\infty_0} f_{\pi^0_\infty} \tag{124}$$

Note that this sum is sensical since $\Pi^0_\infty(\xi)$ is a finite set.

Note then that by the definition of pseudo-coefficients we have that for any π_∞ an irreducible tempered representation of $\mathbf{G}(\mathbb{R})$ in $\Pi_\infty(\xi)$, the following equality holds:

$$\operatorname{tr}(f_\infty \mid \pi_\infty) = \begin{cases} \dfrac{(-1)^{\dim \mathsf{Sh}}}{|\Pi^0_\infty(\xi)|} & \text{if} \quad \pi_\infty \in \Pi^0_\infty(\xi) \\ 0 & \text{if} \quad \text{otherwise} \end{cases} \tag{125}$$

with this trace having the same meaning as the discussion succeeding Definition 2.3.3.

The last thing we record is that there is a certain well-defined integer $a(\pi_f)$ associated to any admissible irreducible representation π_f of $\mathbf{G}(\mathbb{A}_f)$. Namely, let us make the following definition.

Definition 2.3.6. *Let notation be as above. We then define $a(\pi_f)$ as follows:*

$$a(\pi_f) := \sum_{\pi_\infty \in \Pi_\infty(\xi)} m(\pi_f \otimes \pi_\infty) \operatorname{tr}(f_\infty \mid \pi_\infty) \tag{126}$$

Let us begin by observing the following:

Lemma 2.3.7. *The equality*

$$a(\pi_f) = \sum_{\pi_\infty \in \Pi_\infty(\xi)} m(\pi_f \otimes \pi_\infty) \operatorname{tr}((-1)^{\dim \mathsf{Sh}} f_{\pi^0_\infty} \mid \pi_\infty) \tag{127}$$

holds for any $\pi^0_\infty \in \Pi^0_\infty(\xi)$.

Proof. Let K be any compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ such that $\pi_f^K \neq 0$. Let us then note that the $\mathbb{C}$-space V of automorphic representations such that $\varpi_\infty \in \Pi_\infty(\xi)$ is an admissible $\mathbf{G}(\mathbb{A}_f)$-representation by Harish-Chandra's theorem (e.g. see [BJ79, Theorem 1.7]). Let us choose any function h as in Proposition 2.3.12 where normalize so that $\operatorname{tr}(h \mid \pi_f) = 1$. Note then that our desired equality is equivalent to

$$\sum_\pi m(\pi) \operatorname{tr}(h f_\infty \mid \pi) = \sum_\pi m(\pi) \operatorname{tr}(h (-1)^{\dim \mathsf{Sh}} f_{\pi^0_\infty} \mid \pi) \tag{128}$$

where π travels over automorphic $\mathbf{G}(\mathbb{A}_f)$-representations with central character agreeing with that of $\xi_{\mathbb{C}}^\vee$. But, by Proposition 2.5.59 this is equivalent to the claim that

(129)
$$\tau(\mathbf{G})\sum_{\{\gamma\}_s\in\{\mathbf{G}\}_s^{\text{s.s.}}} SO_\gamma(hf_\infty)=\tau(\mathbf{G})\sum_{\{\gamma\}_s\in\{\mathbf{G}\}_s^{\text{s.s.}}} SO_\gamma(h(-1)^{\dim \mathsf{Sh}}f_{\pi_\infty^0})$$

But, note that the left-hand side of this equality is equal, by definition of f_∞, to

$$|\Pi_\infty^0(\xi)|\tau(\mathbf{G})\sum_{\{\gamma\}_s\in\{\mathbf{G}\}_s^{\text{s.s.}}}\sum_{\pi_\infty} SO_\gamma(h(-1)^{\dim \mathsf{Sh}}f_{\pi_\infty}) \tag{130}$$

Note though that by Proposition 2.5.59 we have that

$$SO_\gamma(h(-1)^{\dim \mathsf{Sh}}f_{\pi_\infty})=SO_\gamma(h(-1)^{\dim \mathsf{Sh}}f_{\pi_\infty^0}) \tag{131}$$

(because both sides are equal to the expression given in Proposition 2.5.59) from where the conclusion follows. □

The following proposition will be useful shortly:

Proposition 2.3.8. *The complex number $a(\pi_f)$ is an element of $\mathbb{Z}$.*

Proof. It suffices to show that if $f_{\pi_\infty^0}$ is a pseudo-coefficient for an element $\pi_\infty^0\in\Pi_\infty^0(\xi)$ then $\operatorname{tr}(f_{\pi_\infty^0}\mid\pi_\infty)\in\mathbb{Z}$ for every $\pi_\infty\in\Pi_\infty(\xi)$. Suppose that π_∞ has the same central character as π_∞^0. We know that π_∞, as an element of the Grothendieck group of representations of $\mathbf{G}(\mathbb{R})$, is a $\mathbb{Z}$-linear combination of standard representations (e.g. see [ABV12, Lemma 1.20]. We then use the fact (see [CD90, Corollaire Page 213]) that the trace of a pseudocoefficient for π_∞^0 is 0 on all standard representations except π_∞^0.

□

Finally, we record the following alternative description of the integer $a(\pi_f)$:

Proposition 2.3.9. *We have an equality*

$$a(\pi_f)=\sum_{\pi_\infty\in\Pi_\infty^0} m(\pi_f\otimes\pi_\infty)N^{-1}\operatorname{ep}(\pi_\infty\otimes\xi_{\mathbb{C}}) \tag{132}$$

where $N=|\Pi_\infty^0|\cdot|\pi_0(\mathbf{G}(\mathbb{R})/Z(\mathbf{G})(\mathbb{R}))|$ and $\operatorname{ep}(\pi_\infty\otimes\xi_{\mathbb{C}})$ is the Euler-Poincare characteristic of $H^*(\mathfrak{g},K_\infty,\pi_\infty\otimes\xi_{\mathbb{C}})$.

Proof. See [Kot92a, Lemma 3.2] and [Kot92a, Lemma 4.2]. The only assertion thatr used in the proof that requires justifcation is the fact that $K_\infty/Z(\mathbf{G})(\mathbb{R})$ is connected in our situation. But, this follows from the observation that if K'_∞ is a maximal compact subgroup of $\mathbf{G}^{\text{der}}(\mathbb{R})$

(which is connected by [PS92, Theorem 7.6] given our assumption that $\mathbf{G}^{\mathrm{der}}$ is simply connected) then K'_∞ surjects on to $K_\infty/Z(\mathbf{G})(\mathbb{R})$. $\square$

Corollary 2.3.10. *Let K be a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ such that $\pi_f^K \neq 0$. Then, $H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)[\pi_f^K] \neq 0$ if and only if $a(\pi_f) \neq 0$.*

Proof. This follows from [BR94, frm-e.3] as well as [BC+83]. Again, note that by our assumption that $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic, we know that $\mathsf{Sh}_M(\mathbf{G}, X)^{\mathrm{an}}_{\mathbb{C}}$ is proper for all neat $M \subseteq \mathbf{G}(\mathbb{A}_f)$ (by [Pau04, Lemma 3.1.5]) and so L^2-cohomology agrees with singular cohomology, and thus has a comparison with étale cohomology by Artin's comparison theorem. $\square$

Finally, we record the following result of Vogan-Zuckerman. Namely:

Proposition 2.3.11 ([VZ84])**.** *Suppose that ξ is regular. Then, we have the equality $a(\pi_f) = (-1)^{\dim \mathsf{Sh}} m(\pi_f \otimes \pi_\infty^0)$.*

2.3.3.2. *The construction of f^∞.* To construct f^∞ we first start with the following basic observation:

Proposition 2.3.12. *Let $K \subseteq \mathbf{G}(\mathbb{A}_f)$ be compact open and let V be an admissible semisimple $\overline{\mathbb{Q}_\ell}[G(\mathbb{A}_f)]$-representation. Then, there exists some $P \in \mathscr{H}(G(\mathbb{A}_f), K)$ such that the action of P on V is the projector of V onto $V^K[(\pi^\infty)^K]$.*

Proof. This follows immediately from the general version of the Jacobson Density Theorem (e.g. as in [Lor07, F20]). Namely, if we decompose

$$V^K = \bigoplus_i V_i^{e_i} \tag{133}$$

where V_i are the simple components of $\mathscr{H}(G(\mathbb{A}_f), K)$ then by loc. cit. we can find some element $P \in \mathscr{H}(G(\mathbb{A}_f), K)$ such that the image of P in $\mathrm{End}_{\overline{\mathbb{Q}_\ell}}(V)$ is the projector of V^K onto $V^K[(\pi^\infty)^K]$. Noting then that since $P \in \mathscr{H}(G(\mathbb{A}_f), K)$ we have that $P = Pe_K$ and noting that e_K projects V onto V^K, the conclusion follows. $\square$

We can then construct the function f^∞ by taking P to be any element of $\mathscr{H}(\mathbf{G}(\mathbb{A}_f), K)$ from the previous proposition where we take

$$V := \bigoplus_{i=1}^{2\dim(\mathsf{Sh})} H^i(\mathsf{Sh}, \mathcal{F}_\xi) \tag{134}$$

To do this, it suffices to show that V is semisimple and admissible. For the first property note that since $\mathsf{Sh} \to \mathsf{Sh}_K$ is a pro-finite Galois cover, the Leray spectral sequence implies that

$$V^K = H^i(\mathsf{Sh}, \mathcal{F}_\xi)^K = H^i(\mathsf{Sh}_K, \mathcal{F}_\xi) \tag{135}$$

the latter term of which is finite-dimensional by standard algebraic geometry. For the second property we use the following well-known result:

Theorem 2.3.13. *For all $i \geqslant 0$ The admissible $\overline{\mathbb{Q}_\ell}[\mathbf{G}(\mathbb{A}_f)]$ - representation $H^i(\mathsf{Sh}, \mathcal{F}_\xi)$ is semisimple.*

Proof. It suffices to show, by Artin's comparison theorem, that for any embedding of E into $\mathbb{C}$ that the $\overline{\mathbb{Q}_\ell}[\mathbf{G}(\mathbb{A}_f)]$-representation

$$\varinjlim_K H^i_{\text{sing}}(\mathsf{Sh}^{\text{an}}_{K,\mathbb{C}}, \mathcal{F}^{\text{an}}_{\xi,K}) \tag{136}$$

is semisimple. This follows at once from [BR94, §2.3] as well as [BC+83]. Note that since $\mathbf{G}^{\text{ad}}$ is $\mathbb{Q}$-anisotropic that $\mathsf{Sh}^{\text{an}}_{K,\mathbb{C}}$ is compact for all K (see [Pau04, Lemma 3.1.5]), and thus the L^2-cohomology of $\mathsf{Sh}^{\text{an}}_{K,\mathbb{C}}$ agrees with singular cohomology. □

Let us note that for any f^∞ defined as above we can renormalize such that for π'_f any admissible $\overline{\mathbb{Q}_\ell}[\mathbf{G}(\mathbb{A}_f)]$-representation for which the space $H(\mathsf{Sh}_K, \mathcal{F}_\xi)[(\pi'_f)^K]$ is non-zero then

$$\operatorname{tr}(f^\infty \mid \pi'_f) = \begin{cases} 1 & \text{if} \quad \pi_f \cong \pi'_f \\ 0 & \text{if} \quad \text{otherwise} \end{cases} \tag{137}$$

In the sequel we fix such a function f^∞. It is worth noting that we cannot specify the trace of f^∞ on representations whose K-invariants do not appear in $H(\mathsf{Sh}_K, \mathcal{F}_\xi)$. It is also noting that f^∞ is not unique. This non-unicity will be a non-issue for us, and so we have chosen to not notate the non-unicity of f.

2.3.4. **A geometric trace formula in the case of good reduction.** We recall here the statement of the relevant version of the main formula from [KSZ] necessary to prove Theorem 2.3.1. We keep the assumptions from §2.3.2 although the only pivotal assumption for the version of the results of [KSZ] that we use is the assumption that $\mathbf{G}^{\text{der}}$ is simply connected.

Let us fix the notation as in §2.3.2. We also fix the following extra notation. Let us fix a prime $p \in S(\mathbf{G})$. Fix a finite place $\mathfrak{p}$ of $\mathbf{E}_{\boldsymbol{\mu}}$ lying over p. Since $\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}/\mathbb{Q}_p$ is unramified (by Corollary 2.5.30) we know that $\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}} \cong \mathbb{Q}_{p^r}$ for some $r \geqslant 1$. Fix $K^p \subseteq \mathbf{G}(\mathbb{A}^p_f)$ to be a neat compact open subgroup and set $K := K^p K_{0,p}$.

Before we proceed let us make the following observation:

Lemma 2.3.14. *For $K^p \subseteq \mathbf{G}(\mathbb{A}^p_f)$ sufficiently small the group $Z(\mathbb{Q})_K$ is trivial.*

Proof. Let us note that since we are assuming that $(A_{\mathbf{G}})_{\mathbb{R}} = A_{\mathbf{G}_{\mathbb{R}}}$ that for all sufficiently small compact open subgroups K_1 of $G(\mathbb{A}_f)$ we have that $Z(\mathbb{Q})_{K_1}$ is trivial (e.g. see [Mil04, Remark 5.27]). Note then that by possibly shrinking K_1, we may assume that $K_1 = K_1^p K_p$ with $K_p \subseteq K_{0,p}$. Since $K_p \subseteq K_{0,p}$ is of finite index, $Z(\mathbb{Q})_{K_1^p K_{0,p}}$ is finite. Now, since $Z(\mathbb{Q})$ embeds diagonally into $\mathbf{G}(\mathbb{A}_f)$, we can shrink K_1^p to some K^p such that $Z(\mathbb{Q})_{K^p K_{0,p}}$ is trivial as desired. $\square$

Given this lemma we assume, in all future discussion, that K is small enough so that $Z(\mathbb{Q})_K$ is the trivial group.

We continue, as in [KSZ, §5.5], to fix the following extra data/notation:

- Fix $j \geqslant 1$ and set $n := rj$.
- Let $\mathfrak{t} = (\gamma_0, (\gamma_\ell)_{\ell \neq p}, \delta)$ be a (equivalence class of) degree n (punctual) Kottwitz triple(s) as in [KSZ, Definition 2.7.1] or [Kot90, Page 165].
- For such a Kottwitz triple $\mathfrak{t} = (\gamma_0, (\gamma_\ell), \delta)$ set $I_0(\mathfrak{t}) := I_{\gamma_0}$ and for each place v of $\mathbb{Q}$ set $I_v(\mathfrak{t})$ to be the inner form of $(I_0(\mathfrak{t}))_v$ as in [KSZ, §4.7.18] (see also [Kot90, Page 169] and [Kot90, Page 171]).
- Let us denote by $I(\mathfrak{t})$ the unique inner form of I_0 such that $I(\mathfrak{t})_v \cong I_v(\mathfrak{t})$ for all v (e.g. see [KSZ, Proposition 4.7.19] and [Kot90, Page 171]).
- Let $\alpha(\mathfrak{t}) \in \mathfrak{K}(I_0/\mathbb{Q})^D$ as in [Kot90, §2] and [KSZ, p. 4.7.13].
- Set $R := \mathsf{Res}_{\mathbb{Q}_{p^n}/\mathbb{Q}_p} \mathbf{G}_{\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}}$ and let θ be the automorphism of R corresponding to the Frobenius element of $\mathrm{Gal}(\mathbb{Q}_{p^n}/\mathbb{Q}_p)$. Let $R_{\delta \rtimes \theta}$ be as [KSZ, Definition 1.5.1]. Namely, for a $\mathbb{Q}_p$-algebra A we set

$$R_{\delta \rtimes \theta}(A) = \{g \in \mathbf{G}(A \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^n}) : g\delta\sigma(g)^{-1} = \delta\} \tag{138}$$

- Let us a fix a Haar measures dg^p on $\mathbf{G}(\mathbb{A}_f^p)$ arbitrarily and a Haar measure dg_p on $R(\mathbb{Q}_p)$ where we require that the mass of $R(\mathbb{Z}_p)$ is 1.
- Also choose Haar measures on $I_p = I(\mathbb{Q}_p)$ and $I(\mathbb{A}_f^p)$. Note that we have an isomorphism $I_p \cong R_{\delta \rtimes \theta}$ and for all $\ell \neq p$ we also have isomorphisms $I_\ell \cong Z_G(\gamma_\ell)$. Having fixed such isomorphism we can transfer these Haar measures to Haar measures on $R_{\delta \rtimes \theta}(\mathbb{Q}_p)$ and $I_\gamma(\mathbb{A}_f^p)$.
- Let $\mu : \mathbb{G}_{m,\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}} \to \mathbf{G}_{\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}}}$ be any element of $\boldsymbol{\mu}_\mathfrak{p}$.
- Let us denote by ϕ_n denote $\mathbb{1}_{R(\mathbb{Z}_p)\mu(p)^{-1}R(\mathbb{Z}_p)}$.

- We define the twisted orbital integral

$$TO_\delta(\phi_n) := \int_{R_{\delta\rtimes\theta}(\mathbb{Q}_p)\backslash R(\mathbb{Q}_p)} \phi_n(g^{-1}\delta\sigma(g))dg. \tag{139}$$

- Define $c_1(\mathfrak{t}) := \mathrm{vol}(I(\mathbb{Q})Z_K\backslash I(\mathbb{A}_f))$.
- Set $c_2(\mathfrak{t}) = |\ker(\ker^1(\mathbb{Q}, I_0) \to \ker^1(\mathbb{Q}, \mathbf{G}))|$.
- Set $c(\mathfrak{t}) := c_1(\mathfrak{t})c_2(\mathfrak{t})$.

We then state the main result of [KSZ] specialized to our current situation:

Theorem 2.3.15 ([KSZ, Theorem 5.5.2])**.** *For sufficiently small K^p, we have the following. Let $f^p \in \mathscr{H}(\mathbf{G}(\mathbb{A}_f^p), K^p)$. Normalize the action of $f^p dg^p$ on $H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)$ such that $\mathrm{vol}_{dg^p}(K^p)^{-1}1_{K^p}dg^p = 1$. Then the quantity*

$$\mathrm{tr}(\Phi^j \times \mathbb{1}_{K_{0,p}} f^p dg^p \mid H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)) \tag{140}$$

is equal to

$$\sum_{\substack{\mathfrak{t}=(\gamma_0,\gamma,\delta)\\ \alpha(\gamma_0,\gamma,\delta)=1}} c(\mathfrak{t})O_\gamma(f^p)TO_\delta(\phi_n)\,\mathrm{tr}\,\xi(\gamma_0) \tag{141}$$

Proof. In the following we merely justify the simplifications to [KSZ, Theorem 5.5.2] made in the above.

First let us note that since $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic that Sh_K is proper (e.g. [Pau04, Lemma 3.1.5] noting that $\mathbf{G}$ being $\mathbb{Q}$-anisotropic is equivalent to $G(\mathbb{Q})$ containing no unipotent elements by [BT72, §8]). This obviously allows us to replace compactly supported cohomology by normal étale cohomology.

This observation also allows us to take $j = 1$ (or $m = 1$ in the notation of [KSZ]). Indeed, the proof of [KSZ, Theorem 5.5.2] uses the Fujiwara-Varshavsky trace formula which requires that j is sufficiently large. But, in [Var07, Theorem 2.3.2, c)] a bound is given on permissible j that, in particular, implies that j need only be at least 1 if the integral canonical model Sh_K is proper and the Hecke correspondence is étale. The latter is clear (e.g. see [Kis10, Theorem 2.3.8]). The former follows in the Hodge type case by work of Madapusi-Pera (e.g. see [Per12, Corollary 4.1.7]) and follows in the abelian type case by reduction to geometric connected components and using the fact that such components admit finite surjections from components of Hodge type Shimura varieties.

Next note that having shrunk K^p sufficiently small we have, by Lemma 2.3.14, that $Z(\mathbb{Q})_K$ is trivial. This allows us to ignore the stipulations about ξ present in [KSZ, §5.5] as well as replace the set

$(\mathbf{G}(\mathbb{Q}) \cap \mathbf{G}(\mathbb{R})^{\mathrm{ell}})\backslash Z(\mathbb{Q})_K$ (i.e. $\Sigma_{Z(\mathbb{Q})_K,\mathbb{R}\text{-ell}}(\mathbf{G})$ in the notation of [KSZ, §5.5]) with $\mathbf{G}(\mathbb{Q}) \cap \mathbf{G}(\mathbb{R})^{\mathrm{ell}}$ (i.e. $\Sigma_{\mathbb{R}\text{-ell}}$ in the notation of [KSZ, §5.5]). This is what allows us to combine the double sum in [KSZ, Theorem 5.5.2] into a single sum of Kottwitz triples.

The absence of the terms $\iota_{\mathbf{G}}(\gamma_0)$ and $\bar{\iota}_{\mathbf{G}}(\gamma_0)$ is explained by the assumption that $\mathbf{G}^{\mathrm{der}}$ is simply connected. This assumption also explains the lack of connected components on our R-groups. Indeed, note that $R_{\delta \rtimes \theta}$ is connected since it's an inner form of $Z_{\mathbf{G}}(\gamma)_{\mathbb{Q}_p}$ by [KSZ, Lemma 1.5.3].

The last thing to note is the usage of degree n classical (or punctual in the language of [KSZ]) Kottwitz triples instead of p^n-admissible cohomological Kottwitz triples as is written in [KSZ, Theorem 5.5.2]. The reason that this is permissible is that the natural map from such p^n-cohomological Kottwitz triples to degree n classical Kottwitz triples is a bijection (e.g. see [KSZ, p. 4.7.12]) and the fact that the term $O(\gamma_0, \alpha^p, [b])$ (as in loc. cit.) associated to a p^n-admissible cohomological Kottwitz triple $(\gamma_0, \alpha^p, [b])$ is defined in terms of the associated degree n Kottwitz triple. A similar statement holds for the Kottwitz invariant $\alpha(\mathfrak{t})$. □

2.3.5. **Proof of Theorem 2.3.1.** We are now prepared to combine the material from the last two subsections, together with the contents of 2.5.2, to prove our desired claim.

We first prove the following, analogizing the results in [Kot92a, §5]:

Theorem 2.3.16. *For all $j \geqslant 1$ and all $f = f^p \mathbb{1}_{K_{0,p}} f_\infty$ where f^p is an element of $\mathscr{H}(G(\mathbb{A}_f^p), K^p)$ and f_∞ is as in §2.3.3 the following equality holds*

$$\mathrm{tr}(\Phi^j \times (f^p \mathbb{1}_{K_{0,p}}) \mid H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)) = \tau_K(\mathbf{G}) \sum_{\{\gamma\}_s \in \{\mathbf{G}\}_s^{\mathrm{s.s.}}} \mathrm{SO}_\gamma(f^p f_n f_\infty) \tag{142}$$

Here we denote by $\tau_K(\mathbf{G})$ the number

$$\tau_K(\mathbf{G}) := \mathrm{vol}(\mathbf{G}(\mathbb{Q})\backslash \mathbf{G}(\mathbb{A})/Z_K A_{\mathbf{G}}(\mathbb{R})^0) \tag{143}$$

which is sensical since $\mathbf{G}(\mathbb{Q})\backslash \mathbf{G}(\mathbb{A})/Z_K A_{\mathbf{G}}(\mathbb{R})^0$ has finite volume as it is a quotient of $[G]$. Also, f_n denotes the unramified base change of ϕ_n along $G_{\mathbb{Q}_p} \to \mathsf{Res}_{\mathbb{Q}_{p^n}/\mathbb{Q}_p} G_{\mathbb{Q}_{p^n}}$ (see the proof of Theorem 2.3.16 for details of the definition).

Before we begin, it's useful to note the following lemma:

Lemma 2.3.17. *For any classical degree n Kottwitz triple $\mathfrak{t} = (\gamma_0, \gamma, \delta)$ we have that*

$$c(\mathfrak{t}) = \tau_K(\mathbf{G}) \mathrm{vol}(A_{\mathbf{G}}(\mathbb{R})^0 \backslash I_\infty(\mathbb{R}))^{-1} \tag{144}$$

Here I_∞ is as in Lemma 2.3.4.

Proof. This is [KSZ, p. 6.1.1]. □

Proof. (Proof of Theorem 2.3.16) Let us begin by noting that by Theorem 2.3.15 in conjunction with Lemma 2.3.17

$$\mathrm{tr}(\Phi^j \times (f^p \mathbb{1}_{K_{0,p}}) \mid H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)) \tag{145}$$

is equal to

$$\tau_K(\mathbf{G}) \sum_{\substack{\mathfrak{t}=(\gamma_0,\gamma,\delta)\\ \alpha(\gamma_0,\gamma,\delta)=1}} \mathrm{vol}(A_{\mathbf{G}}(\mathbb{R})^0 \backslash I_\infty(\mathbb{R}))^{-1} O_\gamma(f^p) TO_\delta(\phi_n)\, \mathrm{tr}\, \xi(\gamma_0) \tag{146}$$

Note though that since $\alpha(\gamma_0, \gamma, \delta)$ is a character of $\mathfrak{K}(\gamma_0, \mathbf{G}, F)$, which is trivial by our assumption that $\mathbf{G}$ has no relevant global endoscopy, we can rewrite this as

$$\tau_K(\mathbf{G}) \sum_{\mathfrak{t}=(\gamma_0,\gamma,\delta)} \mathrm{vol}(A_{\mathbf{G}}(\mathbb{R})^0 \backslash I_\infty(\mathbb{R}))^{-1} O_\gamma(f^p) TO_\delta(\phi_n)\, \mathrm{tr}\, \xi(\gamma_0) \tag{147}$$

Also, note that since $\alpha(\mathfrak{t}) = \alpha(\gamma_0, \gamma, \delta) = 1$ we know by [KSZ, Proposition 4.7.19] that there exists some reductive group $I(\mathfrak{t})$ over $\mathbb{Q}$ such that we have isomorphisms $I(\mathfrak{t})_v \cong I_v(\mathfrak{t})$ for all $v \neq p, \infty$, $I(\mathfrak{t})_p \cong R_{\delta \rtimes \theta}$, and $I(\mathfrak{t})_\infty \cong I_\infty$ where I_∞ is the inner form of $(I_{\gamma_0})_{\mathbb{R}}$ from Proposition 2.5.59. So then we know that

$$e(I_\delta) \prod_{v \neq p,\infty} e(\gamma_v) e(I_\infty) = e(I) = 1 \tag{148}$$

Thus, we may rewrite this sum as

$$\begin{aligned} &\tau_K(\mathbf{G}) \sum_{\mathfrak{t}=(\gamma_0,\gamma,\delta)} \mathrm{vol}(A_{\mathbf{G}}(\mathbb{R})^0 / I_\infty(\mathbb{R}))^{-1} \\ &\cdot \prod_{v \neq p,\infty} e(\gamma_v) O_\gamma(f^p) e(I_\delta) TO_\delta(\phi_n) e(I_\infty)\, \mathrm{tr}\, \xi(\gamma_0) \end{aligned} \tag{149}$$

Now, by Proposition 2.3.4 we know that

$$\mathrm{tr}(\xi(\gamma_0)) = \mathrm{vol}(A_{\mathbf{G}}(\mathbb{R})^+ / I_\infty(\mathbb{R})) e(I_\infty) SO_{\gamma_0}(f_\infty) \tag{150}$$

So that our sum becomes(noting that the two copies of $e(I_\infty)$ cancel):

$$\tau_K(\mathbf{G}) \sum_{\mathfrak{t}=(\gamma_0,\gamma,\delta)} \prod_{v \neq p,\infty} e(\gamma_v) O_\gamma(f^p) e(I_\delta) TO_\delta(\phi_n) SO_{\gamma_0}(f_\infty) \tag{151}$$

Let us denote by b the base change morphism

$$\mathscr{H}(\mathbf{G}(\mathbb{Q}_{p^n}), \mathcal{G}_p(\mathbb{Z}_{p^n})) \to \mathscr{H}(\mathbf{G}(\mathbb{Q}_p), K_{0,p}) \tag{152}$$

as in the introduction [Kot86a]. One then knows that, by [Lab90, prop 3] (see also [Clo90, thm 1.1]), that

$$\sum_{\substack{\delta\in G(\mathbb{Q}_{p^n})/\sim_\sigma \\ N(\delta)\sim\gamma_0}} e(\delta)TO_\delta(\phi_n)=SO_{\gamma_0}(f_n) \tag{153}$$

Thus, we see that we can rewrite our sum as

$$\tau_K(\mathbf{G})\sum_{(\gamma_0,\gamma)}\prod_{v\neq p,\infty} e(\gamma_v)\mathrm{O}_\gamma(f^p)SO_{\gamma_0}(f_n)SO_{\gamma_0}(f_\infty) \tag{154}$$

But, by the definition of a stable orbital integral on $\mathbb{A}_f^p$, we see that we can rewrite this as

$$\tau_K(\mathbf{G})\sum_{\gamma_0}SO_{\gamma_0}(f^p)SO_{\gamma_0}(f_n)SO_{\gamma_0}(f_\infty)=\tau_K(\mathbf{G})\sum_{\gamma_0}SO_{\gamma_0}(f^pf_nf_\infty) \tag{155}$$

Now, note that while γ_0 a priori only runs over the elements of $\mathbf{G}(\mathbb{Q})$ which are elliptic in $\mathbf{G}(\mathbb{R})$, note that by Proposition 2.3.4 we have that $SO_{\gamma_0}(f_\infty)$ is zero for γ_0 not elliptic in $\mathbf{G}(\mathbb{R})$. Thus, we can actually equate this sum to

$$\tau_K(\mathbf{G})\sum_{\gamma_0\in\{\mathbf{G}\}_s^{\text{s.s.}}}SO_{\gamma_0}(f^pf_nf_\infty) \tag{156}$$

from where the conclusion follows. ☐

We are now in a position to apply Proposition 2.5.59 to the above to obtain (keeping the notation of Theorem 2.3.16) an equality between $\mathrm{tr}(\Phi^j\times(f^p\mathbb{1}_{K_{0,p}}))\mid H^*(\mathsf{Sh}_K,\mathcal{F}_\xi))$ and

$$\begin{aligned}&\tau_K(\mathbf{G})/\tau(\mathbf{G})\sum_{\pi\in\Pi_\chi(\mathbf{G})}m(\pi)\,\mathrm{tr}(f\mid\pi)\\ =&\mathrm{vol}(Z_K/Z(\mathbb{Q})_K)^{-1}\sum_{\pi\in\Pi_\chi(\mathbf{G})}m(\pi)\,\mathrm{tr}(f\mid\pi)\\ =&\mathrm{vol}(Z_K)^{-1}\sum_{\pi\in\Pi_\chi(G)}m(\pi)\,\mathrm{tr}(f\mid\pi),\end{aligned} \tag{157}$$

where $f:=f^pf_nf_\infty$ and the last equality follows from the assumption that K is small enough that $Z(\mathbb{Q})_K$ is trivial. Here we are denoted by χ the restriction to $A_{\mathbf{G}}(\mathbb{R})^0$ the of the central character of $\xi_{\mathbb{C}}^\vee$. Note that, by construction, f_∞ transforms under the center by the central character of $\xi_{\mathbb{C}}$ so, in particular, we see that $f\in\mathscr{H}(\mathbf{G}(\mathbb{A}),\chi^{-1})$.

Let us now begin the proof of the result in earnest. Let us note that since $\mathsf{Sh}_K(\mathbf{G},X)$ is proper for all neat compact open subgroups K of

$\mathbf{G}(\mathbb{A}_f)$ we know from the proper base change theorem that an inclusion $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ gives rise to an isomorphism

$$H^*(\mathsf{Sh}, \mathcal{F}_\xi) \xrightarrow{\approx} H^*(\mathsf{Sh}_{\mathbb{C}}, \mathcal{F}_\xi) \tag{158}$$

Moreover, by Artin's comparison theorem we obtain a natural isomorphism $\overline{\mathbb{Q}}_\ell$-spaces

$$H^*(\mathsf{Sh}_{\mathbb{C}}, \mathcal{F}_\xi) \xrightarrow{\approx} H^*_{\mathrm{sing}}(\mathsf{Sh}^{\mathrm{an}}_{\mathbb{C}}, \mathcal{F}^{\mathrm{an}}_\xi) \tag{159}$$

where we we imprecisely denoting by $H^*_{\mathrm{sing}}(\mathsf{Sh}^{\mathrm{an}}_{\mathbb{C}}, \mathcal{F}^{\mathrm{an}}_\xi)$ the space

$$\varinjlim_K H^*_{\mathrm{sing}}(\mathsf{Sh}_K(G,X)^{\mathrm{an}}_{\mathbb{C}}, \mathcal{F}^{\mathrm{an}}_{\xi,K}) \tag{160}$$

which is in the Grothendieck group of $\overline{\mathbb{Q}_\ell}$-spaces.

Note that by Theorem 2.3.13 this $\overline{\mathbb{Q}_\ell}[\mathbf{G}(\mathbb{A}_f)]$-module is semisimple. Thus, by definition, there exists a decomposition

$$H^*(\mathsf{Sh}^{\mathrm{an}}_{\mathbb{C}}, \mathcal{F}^{\mathrm{an}}_\xi) = \bigoplus_{\pi_f} \pi_f \boxtimes \sigma(\pi_f), \tag{161}$$

where π_f ranges over irreducible admissible $\mathbf{G}(\mathbb{A}_f)$-representations contained in $H^*(\mathsf{Sh}^{\mathrm{an}}_{\mathbb{C}}, \mathcal{F}^{\mathrm{an}}_\xi)$ and $\sigma(\pi_f)$ is a virtual $\overline{\mathbb{Q}_\ell}$-space.

Let us note that since the $\mathbf{G}(\mathbb{A}_f)$-action on the tower Sh is defined $\mathbf{E}_{\boldsymbol{\mu}}$-rationally that the action of $\mathbf{G}(\mathbb{A}_f)$ and $\Gamma_{\mathbf{E}_{\boldsymbol{\mu}}}$ commute. For this reason, we see that the induced action of $\Gamma_{\mathbf{E}_{\boldsymbol{\mu}}}$ on $H^*(\mathsf{Sh}^{\mathrm{an}}_{\mathbb{C}}, \mathcal{F}^{\mathrm{an}}_\xi)$ induced from the above isomorphisms has the property that it preserves $\sigma(\pi_f)$, and thus we see that $\sigma(\pi_f)$ is a virtual $\overline{\mathbb{Q}_\ell}$-representation (recalling our identification of $\overline{\mathbb{Q}_\ell}$ and $\mathbb{C}$) of $W_{\mathbf{E}_{\boldsymbol{\mu}}}$.

Thus, in conclusion, pulling this decomposition back along the above isomorphisms we obtain a decomposition

$$H^*(\mathsf{Sh}, \mathcal{F}_\xi) = \bigoplus_{\pi_f} \pi_f \boxtimes \sigma(\pi_f) \tag{162}$$

where π_f travels over admissible $\overline{\mathbb{Q}_\ell}$-representations of $\mathbf{G}(\mathbb{A}_f)$ contained in $H^*(\mathsf{Sh}, \mathcal{F}_\xi)$ and $\sigma(\pi_f)$ is a virtual $\overline{\mathbb{Q}_\ell}$-representation of $\Gamma_{\mathbf{E}_{\boldsymbol{\mu}}}$.

Remark 2.3.18. Note that, *a priori*, the virtual $\overline{\mathbb{Q}_\ell}$-representation $\sigma(\pi_f)$ of Γ_E depends on the above chosen ambient identifications/data. But, as our description in Theorem 2.3.1 shows, the traces of a dense set of elements of Γ_E are independent of these choices, and thus so is $\sigma(\pi_f)$.

We now fix for once and for all an admissible $\overline{\mathbb{Q}}_l$ representation π_f^0 of $\mathbf{G}(\mathbb{A}_f)$ satisfying the conditions of Theorem 2.3.1. In particular, we assume there exists an automorphic $\overline{\mathbb{Q}}_l$-representation π of $\mathbf{G}(\mathbb{A})$ such that π is isomorphic to $\pi_f^0\pi_\infty$ where $\pi_\infty \in \Pi_\infty(\xi)$.

We now fix a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ satisfying the following properties.

- We assume that K is a neat subgroup,
- that $Z(\mathbb{Q})_K = 1$,
- and that π_f^K is nonempty.

We now fix f^∞ as in section 2.3.3.2. Finally, we need to determine the cofinite set $S(\pi_f^0) \subset S(\mathbf{G})$ of theorem 2.3.1. We define $S(\pi_f^0)$ so that for each $p \in S(\pi_f^0)$,

(1) the group $\mathbf{G}_{\mathbb{Q}_p}$ is unramified,

(2) we have a factorization $K = K^p K_{0,p}$ where $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ and $K_{0,p} \subset \mathbf{G}(\mathbb{Q}_p)$ is a hyperspecial subgroup,

(3) we can factor $f^\infty = f^p \mathbb{1}_{K_{0,p}}$ where $f^p \in \mathscr{H}(\mathbf{G}(\mathbb{A}_f^p))$.

We briefly explain why the factorization in the third item can be made for all but finitely many p. We can write

$$f^\infty = \sum_i c_i \mathbb{1}_{K a_i K}, \tag{163}$$

where $c_i \in \mathbb{C}, a_i \in \mathbf{G}(\mathbb{A}_f)$. Now, for all but finitely many places, we have for all i, $(a_i)_p \in K_p$. Hence if S is the finite set of primes where this does not happen, we can write

$$f^\infty = (\sum_i c_i \mathbb{1}_{K_S (a_i)_S K_S}) \cdot \mathbb{1}_{K^S}, \tag{164}$$

which gives the desired factorization.

Now fix $p \in S(\pi_f^0)$ and a prime $\mathfrak{p}$ of $\mathbf{E}_{\boldsymbol{\mu}}$ lying over p. Now fix a $\tau \in W_{\mathbf{E}_{\mu_\mathfrak{p}}}$. We aim to describe $\mathrm{tr}(\tau \mid \sigma(\pi_f))$ as in theorem 2.3.1. Note that since $H^*(\mathsf{Sh}, \mathcal{F}_\xi)$ is unramified at p (by smooth proper base change given the existence of smooth proper models by combinging [Kis10] and [Per12]) we may as well assume that $\tau = \Phi^j$ for some j where we denote by Φ the geometric Frobenius element of $W_{\mathbf{E}_{\mu_\mathfrak{p}}}$.

On the first hand, let us observe that we have the equality

$$\begin{aligned}\operatorname{tr}(\Phi^j \times f^\infty \mid H^*(\mathsf{Sh}, \mathcal{F}_\xi)) &= \operatorname{tr}(\Phi^j \times f^\infty \mid H^*(\mathsf{Sh}_K(\mathbf{G}, X), \mathcal{F}_{\xi,K}) \\ &= \sum_{\pi_f,\, \pi_f^K \neq 0} \operatorname{tr}(\Phi^j \times f^\infty \mid \pi_f^K \boxtimes \sigma(\pi_f)) \\ &= \sum_{\pi_f,\, \pi_f^K \neq 0} \operatorname{tr}(f^\infty \mid \pi_f^K) \operatorname{tr}(\Phi^j \mid \sigma(\pi_f)) \\ &= \operatorname{tr}(\Phi^j \mid \sigma(\pi_f^0))\end{aligned} \tag{165}$$

where the last equality follows from the definition of f^∞.

On the other hand, by Equation (157), we have

$$\operatorname{tr}(\Phi^j \times (f^p \mathbb{1}_{K_{0,p}})) \mid H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)) = \operatorname{vol}(Z_K)^{-1} \sum_{\pi \in \Pi_\chi(\mathbf{G})} m(\pi) \operatorname{tr}(f \mid \pi), \tag{166}$$

where $f = f^p f_n f_\infty$.

Now by 2.3.6, we can rewrite the right hand side of the above equation as

$$\operatorname{vol}(Z_K)^{-1} \sum_{\pi_f \in \Pi_{f,\chi}(\mathbf{G})} a(\pi_f) \operatorname{tr}(f^p f_n \mid \pi_f), \tag{167}$$

where $\Pi_{f,\chi}(\mathbf{G})$ denotes the set of admissible $\mathbf{G}(\mathbb{A}_f)$-representations π_f such that there exists a π_∞ an admissible $\mathbf{G}(\mathbb{R})$-representation such that $\pi_f \otimes \pi_\infty$ is an element of $\Pi_\chi(\mathbf{G})$.

At this point, we note that for any π_f, we have the equality

$$\operatorname{tr}(f^p f_n \mid \pi_f) = \operatorname{tr}(f^p \mid \pi_f^p) \operatorname{tr}(f_n \mid (\pi_f)_p) \tag{168}$$

$$= \operatorname{tr}(f^p \mathbb{1}_{K_{0,p}} \mid \pi_f) \operatorname{tr}(f_n \mid (\pi_f)_p), \tag{169}$$

where the last step follows because $\operatorname{tr}(\mathbb{1}_{K_{0,p}} \mid (\pi_f)_p)$ equals 1 or 0 based on whether ${\pi_f}_p^{K_{0,p}}$ is nonempty or empty and in the latter case, we would also have $\operatorname{tr}(f_n \mid (\pi_f)_p) = 0$.

Now, by [Kot84a, Theorem 2.1.3], we have

$$\operatorname{tr}(f_n \mid (\pi_f)_p) = \operatorname{vol}(Z_K) \operatorname{tr}(\tau \mid r_{-\mu} \circ \varphi_{(\pi_f)_p}) p^{\frac{1}{2} j [\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}} : \mathbb{Q}_p] \dim \mathsf{Sh}}. \tag{170}$$

Finally, putting all the pieces together and recalling that $f^p \mathbb{1}_{K_{0,p}} = f^\infty$ which projects to the $(\pi_f^0)^K$-isotypic part of $H^*(\mathsf{Sh}_K(\mathbf{G}, X), \mathcal{F}_\xi)$, we get

$$\operatorname{tr}(\Phi^j \times f^\infty \mid H^*(\mathsf{Sh}_K(\mathbf{G}, X), \mathcal{F}_\xi)) \tag{171}$$

is equal to

$$a(\pi_f) \operatorname{tr}(\tau \mid r_{-\mu} \circ \varphi_{(\pi_f)_p}) p^{\frac{1}{2} j [\mathbf{E}_{\boldsymbol{\mu}_\mathfrak{p}} : \mathbb{Q}_p] \dim \mathsf{Sh}} \tag{172}$$

Combining this with Equation (165) proves Theorem 2.3.1.

2.4. The Unramified Scholze-Shin Conjecture: the Trivial Endoscopic Triple

2.4.1. Unramified unitary groups and their representations.

In this subsection, we construct the various groups and representations that we use in the proof of our main result.

2.4.1.1. *Local and global unitary groups.* To begin, we fix a prime p of $\mathbb{Q}$ and a finite unramified extension $F/\mathbb{Q}_p$. We fix an isomorphism $\iota_p : \overline{\mathbb{Q}_p} \to \mathbb{C}$. Let E/F be the unique unramified extension of degree 2 and define $U_{E/F}(n)^*$ to be the unique up to isomorphism quasisplit unitary group of rank n over F for the extension E/F as in 2.5.78. Define G to be the group $\mathsf{Res}_{F/\mathbb{Q}_p} U_{E/F}(n)^*$. Note that G is unramified since $E/\mathbb{Q}_p$ is unramified. Note that $G_{\overline{\mathbb{Q}_p}}$ is isomorphic to a product of GL_n factors. We fix a nontrivial minuscule cocharacter μ of $G_{\overline{\mathbb{Q}_p}}$ by fixing a minuscule cocharacter μ_i of each factor the form

$$\mu_i(z) = \begin{pmatrix} z & & & & & \\ & \ddots & & & & \\ & & z & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}, \tag{173}$$

where the number of z factors and 1 factors in the above expression are a_i and b_i respectively. We assume that for at least one i, we have $a_i \notin \{0, n\}$.

Note that such a μ is minuscule but that not all minuscule μ are of this form. Since E is unramified over $\mathbb{Q}_p$, it is Galois and hence the reflex field of μ is a subfield of E which we denote E_μ.

We now note the following:

Lemma 2.4.1. *There exists an extension of number fields $\mathbf{E}/\mathbf{F}$ satisfying the following properties:*

1. *$\mathbf{E}_{\mathfrak{q}} = E$ and $\mathbf{F}_{\mathfrak{p}} = F$ for some primes $\mathfrak{q}$ of $\mathbf{E}$ and $\mathfrak{p}$ of F such that $\mathfrak{q} \cap \mathbf{F} = \mathfrak{p}$.*
2. *$\mathbf{F}$ is totally real.*
3. *$\mathbf{E}$ is a quadratic imaginary extension of $\mathbf{F}$.*
4. *$\mathbf{F} \neq \mathbb{Q}$.*

Proof. The construction of $\mathbf{F}$ follows from [Art13, Lemma 6.2.1] by taking any $r_0 > 1$. Indeed, the construction of loc. cit. produces $\mathbf{F}$

satisfying the desired conditions of 1. and 2. and the existence of more than one real place on $\mathbf{F}$ implies condition 3.

We argue about the existence of $\mathbf{E}$ similarly. Indeed, the only assumption for which the arguments of loc. cit. don't apply directly to is the assumption that $\mathbf{E}/\mathbf{F}$ is imaginary. But, this follows immediately from the method of loc. cit. since for an embedding of $F^2 \hookrightarrow \mathbb{R}^2$ the monic polynomials with imaginary roots is open since it corresponds to $(b, c) \in \mathbb{R}^2$ such that $b^2 - 4c < 0$. $\square$

We now define $\mathbf{U}^*$ to be the group $U_{\mathbf{E}/\mathbf{F}}(n)^*$ and $\mathbf{G}^*$ to be $\mathsf{Res}_{\mathbf{F}/\mathbb{Q}}\mathbf{U}^*$. The previously defined ι_p induces an isomorphism $\mathbf{G}^*_{\overline{\mathbb{Q}_p}} \cong \mathbf{G}^*_{\mathbb{C}}$. Note that G is a direct factor of $\mathbf{G}^*_{\mathbb{Q}_p}$ and hence again by ι_p, we get that $G_{\mathbb{C}}$ is a direct factor of $\mathbf{G}^*_{\mathbb{C}}$. Define a minuscule cocharacter $\boldsymbol{\mu}$ of $\mathbf{G}^*$ so that $\boldsymbol{\mu}$ restricts to μ on the $G_{\mathbb{C}}$ factor and is trivial elsewhere.

We would now like to record the existence of a certain unitary group over a global field.

Proposition 2.4.2. *There exists an inner form $\mathbf{U}$ of $\mathbf{U}^*$ and hence an inner form $\mathbf{G} := \mathsf{Res}_{\mathbf{F}/\mathbb{Q}}\mathbf{U}$ of $\mathbf{G}^*$ such that:*

1. *The group $\mathbf{G}^{\mathrm{ad}}$ is $\mathbf{F}$-anisotropic.*
2. *The group $\mathbf{G}$ has no relevant global endoscopy.*
3. *The group G is a direct factor of $\mathbf{G}_{\mathbb{Q}_p}$.*
4. *Let $\{v\}$ denote the infinite places of $\mathbf{F}$. Given any set $\{(p_v, q_v)\}$ of pairs of non-negative integers such that $p_v + q_v = n$ we have that $U_v \cong U(p_v, q_v)$.*

Proof. We shall use the terminology as in Lemma 2.5.88. In particular, we shall construct $\mathbf{U}$ by constructing $\mathbf{U}_v \in \mathsf{InnForm}(\mathbf{U}^*_v)$ for all places v of $\mathbf{F}$. Begin by setting $\mathbf{U}_v := U(p_v, q_v)$ for each $v \mid \infty$ as in condition 4. of the proposition. Let us also set $\mathbf{U}_{v_0} := \mathbf{U}^*_{v_0}$ where $v_0 = \mathfrak{p}$ is the prime from Lemma 2.4.1. Choose some finite place v_0' of F different than v_0 and set $\mathbf{U}_{v_0'} := D^{\times}_{\frac{1}{n}}$. Let us set

$$\epsilon := \sum_{v \mid \infty} \epsilon_v(\mathbf{U}_v) + \epsilon_{v_0}(\mathbf{U}_{v_0}) + \epsilon_{v_0'}(\mathbf{U}_{v_0'}) \tag{174}$$

This is an element of $\mathbb{Z}/2\mathbb{Z}$. If $\epsilon = 0$ let us set $\mathbf{U}_v := \mathbf{U}^*_v$ for all $v \nmid \infty$ such that $v \notin \{v_0, v_0'\}$. If $\epsilon \neq 0$ then necessarily n is even. In this case choose some finite split (relative to E) place v_0'' and set $\mathbf{U}_{v_0''} := D^{\times}_{\frac{n-1}{n}}$ and then set $\mathbf{U}_v := \mathbf{U}^*_v$ for $v \nmid \infty$ such that $v \notin \{v_0, v_0', v_0''\}$. By construction we have that $\sum_v \epsilon_v(\mathbf{U}_v) = 0$ and thus by Lemma 2.5.88 there exists a unique $\mathbf{U} \in \mathsf{InnForm}(\mathbf{U}^*)$ such that $\mathbf{U}_{\mathbb{Q}_v} \cong \mathbf{U}_v$.

Note that

$$\mathbf{G}_{\mathbb{Q}_p} \cong \prod_{v|p} \mathsf{Res}_{\mathbf{F}_v/\mathbb{Q}_p} \mathbf{U}_v \tag{175}$$

and thus by construction we see that $\mathbf{G}_{\mathbb{Q}_p}$ contains as a factor $\mathsf{Res}_{\mathbf{F}_{v_0}/\mathbb{Q}_p}\mathbf{U}_v$. But, by construction, $\mathbf{U}_v \cong U_{E/F}(n)^*$ and $\mathbf{F}_{v_0} \cong F$ and thus condition 3. is automatically satisfied. Also, evidently condition 4. is satisfied. Thus, it remains to show that conditions 1. and 2. are satisfied

Now, note that since $\mathbf{U}$ is an element of $\mathsf{InnForm}(U_{\mathbf{E}/\mathbf{F}}(n^*))$ we know by Lemma 2.5.79 that $U \cong U(\Delta, *)$ where Δ is some central simple $\mathbf{E}$-algebra. Combining Lemma 2.5.82 and Lemma 2.5.90 it suffices to show that Δ must be a division algebra. To do this, note that by Lemma 2.5.76 one has an isomorphism $\mathbf{U}_E \cong \Delta^\times$. By 1. of Lemma 2.5.82 it suffices to show that $U(E)$ contains no non-trivial unipotent elements. But, $\mathbf{U}(E) \subseteq U_{v_0'}(E_{v_0'})$. Note though that we have an isomorphism $(U_{v_0'})_{E_{v_0'}} \cong (D^\times_{\frac{1}{n}})^2$ and since $(D^\times_{\frac{1}{n}})^2$ is anisotropic modulo center we see that this contains no non-trivial unipotent elements as desired. $\square$

We now fix global groups $\mathbf{U}$ and $\mathbf{G}$ satisfying the statement of 2.4.2 where we fix the set $\{(p_v, q_v)\}$ so that $p_v = a_v$ and $q_v = b_v$ where we recall that $\{(a_v, b_v)\}$ comes from the definition of $\boldsymbol{\mu}$. We get a conjugacy class of cocharacters of $\mathbf{G}$ associated to $\boldsymbol{\mu}$. We denote the reflex field of this conjugacy class by $\mathbf{E}_{\boldsymbol{\mu}}$. In the present case, $\mathbf{E}$ and $\mathbf{F}$ are not assumed to be Galois. Hence it need not be true that $\mathbf{F} \subset \mathbf{E}_{\boldsymbol{\mu}}$. All we can say is that $\mathbf{E}_{\boldsymbol{\mu}}$ is a subfield of the Galois closure, $c(\mathbf{E})$ of $\mathbf{E}$. Since we have fixed the isomorphism $\iota_p : \overline{\mathbb{Q}_p} \to \mathbb{C}$, we get a cocharacter of $\mathbf{G}_{\mathbb{Q}_p}$ which we also call $\boldsymbol{\mu}$. On the one hand, the reflex field of this $\boldsymbol{\mu}$ is given by the completion of $\mathbf{E}_{\boldsymbol{\mu}}$ at the place $\mathfrak{p}$ over p corresponding to ι_p. On the other hand, by construction, $\mathbf{G}_{\mathbb{Q}_p} = G \times G'$ and hence $\boldsymbol{\mu} = (\mu, \mu')$ where μ is fixed before and μ' is trivial. Hence the reflex field of $\boldsymbol{\mu}$ in $\mathbf{G}_{\mathbb{Q}_p}$ is E_μ. Thus, we have shown that if $\mathfrak{p}$ is the place of $\mathbf{E}_{\boldsymbol{\mu}}$ determined by ι_p, then $\mathbf{E}_{\boldsymbol{\mu}_{\mathfrak{p}}} = E_\mu$.

2.4.1.2. *Shimura data for unitary groups.* In this subsection we will write down the general conditions necessary to have a Shimura datum of the form $(\mathbf{G}, X)$ where $\mathbf{G} = \mathsf{Res}_{\mathbf{F}/\mathbb{Q}}\mathbf{U}$ and where $\mathbf{F}$ is some number field, $\mathbf{E}$ is a quadratic extension, and $\mathbf{U}$ is an inner form of $U_{E/F}(n)^*$ for some n. We will then, in particular, verify that we can find a Shimura datum of abelian type $(\mathbf{G}, X)$ where $\mathbf{G}$ is as in §2.4.1.1. See [RSZ17, §3] for an alternative discussion of the following.

Let us begin by saying that $\mathbf{U}$ (or $\mathbf{G}$) is of *non-compact type* if for some infinite place v of $\mathbf{F}$ we have that $\mathbf{U}_{\mathbf{F}_v}$ is not $\mathbb{R}$-anisotropic. In other words, $\mathbf{G}$ is of compact type if $\mathbf{G}(\mathbb{R})$ is compact, and being of non-compact type just means that it is not of compact type. We then have the following claim:

Lemma 2.4.3. *Suppose that E is a CM field and $\mathbf{G}$ is of non-compact type. Then, there is a Shimura datum $(\mathbf{G}, X)$ of abelian type.*

Proof. So, let us assume that $\mathbf{U}$. Let

$$h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}} \cong \prod_i U(p_i, q_i) \tag{176}$$

(where we have a priori fixed this latter isomorphism) be defined in terms of its projections h_i defined as follows. If $p_i = 0$ or $q_i = 0$ we define h_i to be trivial. Otherwise, define h_i as follows:

$$h_i(z) := \begin{pmatrix} \frac{z}{\bar{z}} & & & & & \\ & \ddots & & & & \\ & & \frac{z}{\bar{z}} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \tag{177}$$

where there are p_i entries of $\frac{z}{\bar{z}}$ and q_i entries of 1. Set X to be the $\mathbf{G}(\mathbb{R})$-conjugacy class of h. We claim that $(\mathbf{G}, X)$ is a Shimura datum of abelian type.

The fact that $(\mathbf{G}, X)$ is a Shimura datum is elementary and left to the reader (the assumption that $\mathbf{U}$ is of non-compact type being used in Axiom SV3 of [Mil04]). To see that it's of abelian type, we must find an associated Hodge type datum. Let $G\mathbf{U}$ denote the associated unitary similitude group associated to $\mathbf{U}$ and set $\mathbf{H} := \mathsf{Res}_{\mathbf{F}/\mathbb{Q}} G\mathbf{U}$. We then define $\mathbf{H}^{\mathbb{Q}}$ to be the fiber product $\mathbf{H} \times_{\mathsf{Res}_{\mathbf{F}/\mathbb{Q}}\mathbb{G}_{m,F}} \mathbb{G}_{m,\mathbb{Q}}$ where the map $\mathbf{H} \to \mathsf{Res}_{\mathbf{F}/\mathbb{Q}}\mathbb{G}_{m,\mathbf{F}}$ is the similitude character and the map $\mathbb{G}_{m,\mathbb{Q}} \to \mathsf{Res}_{\mathbf{F}/\mathbb{Q}}\mathbb{G}_{m,\mathbf{F}}$ is the usual inclusion. We define a morphism

$$h' : \mathbb{S} \to (\mathbf{H}^{\mathbb{Q}})_{\mathbb{R}} \tag{178}$$

as follows. Begin by noting that

$$(\mathbf{H}^{\mathbb{Q}})_{\mathbb{R}} = \tag{179}$$

$$\left\{ (g_i) \in \prod_i GU(p_i, q_i)\rangle\rangle) : c(g_i) = c(g_j) \text{ for all } i, j \text{ and } c(g_i) \in \mathbb{R}^\times \right\}$$

Let us fix one such isomorphism. We then define h', via this fixed isomorphism, by its projections h'_i to each $GU(p_i, q_i)$ by

$$h'_i(z) := \begin{pmatrix} z & & & & & \\ & \ddots & & & & \\ & & z & & & \\ & & & \overline{z} & & \\ & & & & \ddots & \\ & & & & & \overline{z} \end{pmatrix} \tag{180}$$

where there are p_i copies of z, and q_i copies of $\overline{z}$. One can then check that $(\mathbf{H}^{\mathbb{Q}}, h')$ defines a PEL type Shimura datum (e.g. see [Mil04, Chapter 8]).

Note now that $(\mathbf{H}^{\mathbb{Q}})^{\mathrm{der}}$ is naturally isomorphic to $\mathsf{Res}_{\mathbf{F}/\mathbb{Q}}\mathbf{U}^{\mathrm{der}}$ which is, likewise, equal to $\mathbf{G}^{\mathrm{der}}$. Let $(\mathbf{H}^{\mathbb{Q}})^{\mathrm{der}} \to \mathbf{G}^{\mathrm{der}}$ be the identity map. It's not hard to see then that this induces an isomorphism of Shimura datum between $((\mathbf{H}^{\mathbb{Q}})^{\mathrm{ad}}, (h')^{\mathrm{ad}})$ and $(\mathbf{G}^{\mathrm{ad}}, h^{\mathrm{ad}})$. Thus, $(\mathbf{G}, X)$ is of abelian type. □

We now observe that $\mathbf{G}$ as in §2.4.1.1 is of non-compact type since μ and hence $\boldsymbol{\mu}$ is non-trivial. We can define a Shimura datum $(\mathbf{G}, X)$ as in the previous lemma. In particular, we note that by construction, the conjugacy class of cocharacters of $\mathbf{G}_{\mathbb{C}}$ associated to X contains $\boldsymbol{\mu}$ as an element.

2.4.1.3. *Local and global representations.* We now fix a square integrable irreducible admissible representation $\pi_p^0 \in \mathbb{C}[G(\mathbb{Q}_p)]$. We also fix a Shimura datum $(\mathbf{G}, X)$ as in the last subsection, as well as an algebraic $\overline{\mathbb{Q}_\ell}$-representation ξ of $\mathbf{G}$ with regular highest weight. We have by assumption $\mathbf{G}_{\mathbb{Q}_p} = G \times G'$. Fix a square integrable representation π'_p of $G'(\mathbb{Q}_p)$ so that $\pi_p^0 \boxtimes \pi'_p$ is a square-integrable representation of $\mathbf{G}(\mathbb{Q}_p)$.

We need the following proposition

Proposition 2.4.4. *There exists a representation π of $\mathbf{G}(\mathbb{A})$ such that π_f appears $H^*(\mathsf{Sh}(\mathbf{G}, X), \mathcal{F}_\xi)$ and such that $\pi_p \cong \pi_p^0 \boxtimes \pi'_p$.*

Proof. This is an easy consequence of [Shi12a, Theorem 5.7]. We set S to be the places of $\mathbb{Q}$ where $\mathbf{G}$ is ramified plus the place p. Then we fix a square integrable representation π_S of $\mathbf{G}(\mathbb{Q}_S)$ such that $(\pi_S)_p = \pi_p^0 \boxtimes \pi'_p$. We let $\widehat{U}$ be the $\widehat{\mu}^{pl}$-regular set equal to the orbit $\mathcal{O}$ of the unramified unitary characters of $\mathbf{G}(\mathbb{Q}_S)$ acting on π_S as in [Shi12a, Example 5.6]. We note that at p, we have that any $\pi'_S \in \widehat{U}$ satisfies $(\pi'_S)_p = \pi_p^0 \boxtimes \pi'_p$

since $\mathbf{G}(\mathbb{Q}_p)$ has no split torus in its center. We then apply Theorem 5.7 of Shin's paper to get the desired result. Note in particular, that π_f appears in $H^*(\mathsf{Sh}(\mathbf{G}, X), \mathcal{F}_\xi)$ since it is ξ-cohomological at ∞. $\square$

We now fix a global π satisfying the properties of the above theorem. Note that since we have assumed ξ has regular highest weight, it follows from the remark after Theorem 1 of [Kot92a] that π is discrete and hence elliptic at infinity.

2.4.2. **Construction of the global Galois representation.** We continue with the notation fixed as in 2.4.1. In this subsection only, we use the Galois form of L-groups. We do so because we work primarily with Galois representations instead of A-parameters.

2.4.2.1. *Unitary Shimura varieties.* We first define a morphism of L-groups

$$\lambda : {}^L\mathbf{G} \to {}^L\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}_n. \tag{181}$$

As a group, $\widehat{\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}_n}$ is isomorphic to

$$\left(\prod_{\Gamma_{\mathbb{Q}}/\Gamma_{\mathbf{E}}} \mathrm{GL}_n(\mathbb{C})\right) \rtimes \Gamma_{\mathbb{Q}}. \tag{182}$$

We fix a subset $X \subset \Gamma_{\mathbb{Q}}/\Gamma_{\mathbf{E}}$ such that the map

$$\Gamma_{\mathbb{Q}}/\Gamma_{\mathbf{E}} \to \Gamma_{\mathbb{Q}}/\Gamma_{\mathbf{F}}, \tag{183}$$

induces a bijection

$$X \xrightarrow{\approx} \Gamma_{\mathbb{Q}}/\Gamma_{\mathbf{F}}. \tag{184}$$

We define $X^\perp := \Gamma_{\mathbb{Q}}/\Gamma_{\mathbf{E}} \backslash X$. We now construct λ by

$$\lambda(g_1, ..., g_m, w) = (g_1, ..., g_m, J_N(g_1^{-1})^t J_N^{-1}, ..., J_N(g_m^{-1})^t J_N^{-1}, w), \tag{185}$$

where the left hand side is an element of

$$(\prod_{\Gamma_{\mathbb{Q}}/\Gamma_{\mathbf{F}}} \mathrm{GL}_n(\mathbb{C})) \rtimes \Gamma_{\mathbb{Q}} = {}^L\mathbf{G}, \tag{186}$$

and the right hand side is an element of

$$(\prod_{X} \mathrm{GL}_n(\mathbb{C}) \times \prod_{X^\perp} \mathrm{GL}_n(\mathbb{C})) \rtimes \Gamma_{\mathbb{Q}} = {}^L\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}_n(\mathbb{C}). \tag{187}$$

2.4.2.2. *The identification of $\sigma(\pi_f)$.* We continue with notation as in 2.4.1. In particular, $(\mathbf{G}, X)$ is an abelian type Shimura datum, ξ is an irreducible algebraic representation of $\mathbf{G}_{\mathbb{C}}$, and π is an irreducible automorphic representation of $\mathbf{G}(\mathbb{A})$ that is ξ-cohomological at ∞. By 2.5.63, we get an irreducible discrete automorphic representation $\mathrm{BC}(\pi)$ of $GL_n(\mathbb{A}_{\mathbf{E}})$ that is conjugate self-dual with infinitesimal character $(\xi \otimes \xi)^\vee$. Note that since ξ is regular, that $(\xi \otimes \xi)^\vee$ is slightly regular so that we can apply [Shi11, Theorem 1.2].

We now apply [Shi11, Theorem 1.2] to get a representation $\sigma(\mathrm{BC}(\pi))$ of $\Gamma_{\mathbf{E}}$ with coefficients in $\overline{\mathbb{Q}_\ell}$. In this subsection, we identify an explicit relationship of the Galois representation $\sigma(\pi_f)$, as in Theorem 2.3.1, and the representation $\sigma(\mathrm{BC}(\pi))$ of $GL_n(\overline{\mathbb{Q}_\ell})$, as in [Shi11, Theorem 1.2].

Now consider the representation

$$\sigma := \iota_\ell \sigma(\mathrm{BC}(\pi)) : \Gamma_{\mathbf{E}} \to \mathrm{GL}_n(\mathbb{C}). \tag{188}$$

We identify $GL_n(\mathbb{C})$ with $\widehat{\mathrm{GL}_{n\mathbf{E}}} \subset {}^L GL_{n\mathbf{E}}$ and consider the equivalence class $[\sigma]$ up to conjugacy by an element of $\widehat{GL_{n\mathbf{E}}}$. Thus, we have $[\sigma] \in H^1(\Gamma_E, \widehat{GL_{n\mathbf{E}}})$. Now, by a variant of Shapiro's lemma, [Bor79, Lemma 4.5], we get a class of $H^1(\Gamma_{\mathbb{Q}}, \widehat{\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}_{n\mathbf{E}}})$. Pick a representative ρ of this class. Then again by [Bor79, Lemma 4.5], we have that the projection of ρ to the factor corresponding to the trivial coset of $\Gamma_{\mathbf{E}}$ is a representative of $[\sigma]$.

We need a few lemmas.

Lemma 2.4.5. *Let E/F be an unramified extension of p-adic local fields. Let H be an unramified reductive group over E. Fix a hyperspecial subgroup $K = \mathcal{H}(\mathcal{O}_E) \subset H(E)$ and let π be an irreducible admissible representation of $H(E)$ unramified with respect to K. Then since $H(E) = (\mathsf{Res}_{E/F}H)(F)$, we can also naturally consider π to be an unramified representation of $(\mathsf{Res}_{E/F}H)(F)$ with respect to $(\mathsf{Res}_{\mathcal{O}_E/\mathcal{O}_F}\mathcal{H})(\mathcal{O}_F)$. We denote this representation by π'.*

Now, let $\psi_\pi = \mathsf{LL}_E(\pi)$ and $\mathcal{I}\psi_\pi$ be the equivalence class of parameters of $\mathsf{Res}_{E/F}H$ coming from ψ_π by Shapiro's lemma. Then $\mathcal{I}\psi_\pi = LL_F(\pi')$.

Proof. (Sketch) Let us note that since H is unramified it has an unramified maximal torus. Indeed, let $\mathcal{H}$ be a reductive model for H over $\mathcal{O}_E$. Note that the variety of maximal tori X is smooth over $\mathcal{O}_E$ (e.g. see [Con14, Theorem 3.2.6]) we can use Hensel's lemma to lift a maximal torus of $\mathcal{H}_k$ (where k is the residue field) to a maximal torus of $\mathcal{H}$ whose generic fiber is an unramified torus of H. Note then that by the argument in [BR94, §1.12] we can then reduce the argument to that of tori. This is then a well-known result (e.g. see [Lan+97]). $\square$

We now return to the notation before the previous lemma.

Lemma 2.4.6. *For each place p of $\mathbb{Q}$ such that $\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}_n, \mathbf{E}$ and $\mathrm{BC}(\pi)$ are unramified at p, we have $\rho|_{\Gamma_{\mathbb{Q}_p}} = LL_{\mathbb{Q}_p}(\mathrm{BC}(\pi)_p)$.*

Proof. We consider the following diagram

$$\begin{tikzcd} H^1(\mathbf{E}, \widehat{\mathrm{GL}_n, \mathbf{E}}) \arrow[r] \arrow[d, leftrightarrow] & \prod_{\mathfrak{p}|p} H^1(\mathbf{E}_{\mathfrak{p}}, \widehat{\mathrm{GL}_n, \mathbf{E}_{\mathfrak{p}}}) \arrow[d, leftrightarrow] \\ H^1(\mathbb{Q}, \mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\widehat{\mathrm{GL}_n, \mathbf{E}}) \arrow[r] & \prod_{\mathfrak{p}|p} H^1(\mathbb{Q}_p, \mathsf{Res}_{\mathbf{E}_{\mathfrak{p}}/\mathbb{Q}_p}\widehat{\mathrm{GL}_n, \mathbf{E}_{\mathfrak{p}}}), \end{tikzcd} \tag{189}$$

where the vertical arrows are Shapiro isomorphisms, the top horizontal arrow is a product of restriction maps to each $\Gamma_{\mathbf{E}_{\mathfrak{p}}}$, and the bottom horizontal map is the composition of the restriction to $\Gamma_{\mathbb{Q}_p}$ and the isomorphism

$$H^1(\mathbb{Q}_p, (\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\widehat{\mathrm{GL}_n, \mathbf{E}})_{\mathbb{Q}_p}) \cong \prod_{\mathfrak{p}|p} H^1(\mathbb{Q}_p, \mathsf{Res}_{\mathbf{E}_{\mathfrak{p}}/\mathbb{Q}_p}\widehat{\mathrm{GL}_n, \mathbf{E}_{\mathfrak{p}}}). \tag{190}$$

We claim that this diagram commutes. Indeed the vertical maps are just projections onto the identity coset factors and the horizontal maps are products of restrictions.

But now, we have from [Shi11, Thm 1.2] that $\sigma|_{\mathbf{E}_{\mathfrak{p}}} = \mathsf{LL}_{\mathbf{E}_{\mathfrak{p}}}(\mathrm{BC}(\pi)_{\mathfrak{p}})$. Then by commutativity of the above diagram and the previous lemma we get the desired result. $\square$

We now take the dominant cocharacter $\boldsymbol{\mu}$ of $\mathbf{G}_{\mathbb{C}} \cong \prod_{\Gamma_{\mathbb{Q}}/\Gamma_{\mathbf{F}}} (\mathrm{GL}_n)_{\mathbb{C}}$ associated to the Shimura datum $(\mathbf{G}, X)$ and write it as a product of cocharacters $(\boldsymbol{\mu}_{\tau_1}, ..., \boldsymbol{\mu}_{\tau_m})$ where τ ranges over $\Gamma_{\mathbb{Q}}/\Gamma_{\mathbf{F}}$. We then define the cocharacter $(-\boldsymbol{\mu}, 0)$ of

$$(\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}_n)_{\mathbb{C}} = (\prod_{X} \mathrm{GL}_n(\mathbb{C}) \times \prod_{X^{\perp}} \mathrm{GL}_n(\mathbb{C})) \tag{191}$$

so that the character is $-\boldsymbol{\mu} = (-\boldsymbol{\mu}_{\tau_1}, ... - \boldsymbol{\mu}_{\tau_m})$ on the copies of GL_n indexed by X and 0 on the copies of GL_n indexed by $X^{\perp}$. We denote the reflex field of $(\boldsymbol{\mu}, 0)$ by $\mathbf{E}_{(\boldsymbol{\mu},0)}$. Then using ι_p, we consider $(\boldsymbol{\mu}, 0)$ as a cocharacter of $(\mathsf{Res}_{\mathbf{E}/\mathbb{Q}}\mathrm{GL}_n)_{\mathbb{Q}_p}$ and observe that the localization of $\mathbf{E}_{(\boldsymbol{\mu},0)}$ at the place $\mathfrak{p}$ equals $E_{(\mu,0)}$ and moreover we have the following observation:

Lemma 2.4.7. *We have an equality of fields* $(\mathbf{E}_{\boldsymbol{\mu}})_{\mathfrak{p}} = (\mathbf{E}_{(\boldsymbol{\mu},0)})_{\mathfrak{p}}$.

Proof. Let us note that it suffices to show that the reflex fields of the local cocharacters $\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}$ and $(\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}, 0)$ agree. To do this let us note that we have a natural embedding of $\mathbb{Q}$-groups

$$\mathbf{G} \hookrightarrow \mathsf{Res}_{\mathbf{E}/\mathbb{Q}} \mathrm{GL}_{n,\mathbf{E}} \tag{192}$$

Upon base changing this to $\overline{\mathbb{Q}}$ we obtain a Galois invariant embedding

$$\mathbf{G}_{\overline{\mathbb{Q}}} \hookrightarrow (\mathsf{Res}_{E/\mathbb{Q}} \mathrm{GL}_{n,E})_{\overline{\mathbb{Q}}} \cong \prod_{X} \mathrm{GL}_{n,\overline{\mathbb{Q}}} \times \prod_{X^{\perp}} \mathrm{GL}_{n,\overline{\mathbb{Q}}} \tag{193}$$

with notation as above. In particular, we see that we get a natural $\Gamma_{\mathbb{Q}_p}$-equivariant embedding

$$\mathbf{G}_{\overline{\mathbb{Q}_p}} \hookrightarrow \prod_{X} \mathrm{GL}_{n,\overline{\mathbb{Q}_p}} \times \prod_{X^{\perp}} \mathrm{GL}_{n,\overline{\mathbb{Q}_p}} \tag{194}$$

Note that this map sends $\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}$ to $(\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}, J_N \boldsymbol{\mu}_{\overline{\mathbb{Q}_p}} J_N^{-1})$. It is fairly evident then that the reflex fields of $\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}$ and $(\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}, J_N \boldsymbol{\mu}_{\overline{\mathbb{Q}_p}} J_N^{-1})$ are equal. Indeed, only non-trivial factors of $\boldsymbol{\mu}$ correspond to elements of X coming from $\mathsf{Res}_{F_v/\mathbb{Q}_p} U_{E_w/F_v}$, and $E_w/\mathbb{Q}_p$ is Galois and so the only relevant part of the Galois action on the right hand side of Equation (194) act by the transposition interchanging X and $X^{\perp}$ and then by the natural action of $\mathrm{Gal}(F_v/\mathbb{Q}_p)$. Finally, one sees that $(\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}, 0)$ and $(\boldsymbol{\mu}_{\overline{\mathbb{Q}_p}}, J_N \boldsymbol{\mu}_{\overline{\mathbb{Q}_p}} J_N^{-1})$ have the same reflex field since $J_N \boldsymbol{\mu}_{\overline{\mathbb{Q}_p}} J_N^{-1}$ is never conjugate to $\boldsymbol{\mu}$ by our assumption that $\boldsymbol{\mu}$ is non-trivial. The conclusion follows. ☐

Let $\mathbf{E}^*$ be the compositum of $\mathbf{E}_{\boldsymbol{\mu}}$ and $\mathbf{E}_{(\boldsymbol{\mu},0)}$. We have $\mathbf{E}^*_{\mathfrak{p}} = E_{\mu}$. We then get a representation

$$r_{(-\boldsymbol{\mu},0)} : {}^{L}(\mathsf{Res}_{\mathbf{E}/\mathbb{Q}} \mathrm{GL}_n)|_{\Gamma_{\mathbf{E}_{(\boldsymbol{\mu},0)}}} \to \mathrm{GL}_N(\mathbb{C}), \tag{195}$$

as described in the notation at the beginning of this Part. We record the following lemma.

Lemma 2.4.8. *Take* $\lambda : {}^{L}\mathbf{G} \to {}^{L}\mathsf{Res}_{\mathbf{E}/\mathbb{Q}} \mathrm{GL}_n$ *as in* (181). *Then we have have an equality restricted to* $\widehat{\mathbf{G}} \rtimes \Gamma_{\mathbf{E}^*}$.

$$r_{(-\boldsymbol{\mu},0)} \circ \lambda = r_{-\boldsymbol{\mu}}. \tag{196}$$

Proof. This follows more or less immediately from the definition of λ. ☐

We then have the following proposition:

Proposition 2.4.9. *Let q be an element of $S(\pi_f)$ and $\mathfrak{q}$ any place of $\mathbf{E}^*$ lying over q. Then, we have an equality*

$$\mathrm{tr}\left(\Phi_{\mathfrak{q}} \mid r_{(-\boldsymbol{\mu},0)} \circ \rho|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}\right) = \mathrm{tr}(\Phi_{\mathfrak{q}} \mid r_{-\boldsymbol{\mu}} \circ \mathsf{LL}_{\mathbb{Q}_q}(\pi_q)|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}). \tag{197}$$

Before giving the proof of the above proposition, we record the following corollary, which is the key result of the section.

Corollary 2.4.10. *For each $q \in S(\pi_f)$ and each place $\mathfrak{q}$ of $\mathbf{E}^*$ lying over q, we have the following equality*

$$a(\pi_f)\,\mathrm{tr}(\Phi_{\mathfrak{q}}|(r_{(-\boldsymbol{\mu},0)} \circ \rho|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}) \otimes |\cdot|^{\frac{\dim \mathsf{Sh}}{2}}) = \mathrm{tr}(\Phi_{\mathfrak{q}}|\sigma(\pi_f)). \tag{198}$$

In particular, it follows that we have the following equality in the Grothendieck group of $W_{\mathbf{E}^*}$-representations

$$a(\pi_f)[(r_{(-\boldsymbol{\mu},0)} \circ \rho) \otimes |\cdot|^{\frac{\dim \mathsf{Sh}}{2}}] = \sigma(\pi_f), \tag{199}$$

and hence by [Shi11, Thm 1.2], for any (not just unramified) prime q of $\mathbb{Q}$ and each place $\mathfrak{q}$ of $\mathbf{E}^*$ over q, and for $\tau \in W_{\mathbf{E}^*_{\mathfrak{q}}}$,

$$a(\pi_f)\,\mathrm{tr}(\tau|(r_{(-\boldsymbol{\mu},0)} \circ \rho|_{\Gamma_{W_{\mathbf{E}^*_{\mathfrak{q}}}}}) \otimes |\cdot|^{\frac{\dim \mathsf{Sh}}{2}}) = \mathrm{tr}(\tau|\sigma(\pi_f)). \tag{200}$$

In particular, we will want to apply this corollary to the chosen prime p and the place $\mathfrak{p}$ of $\mathbf{E}^*$ coming from ι_p.

Proof. (Proposition 2.4.9) By 2.4.6 and since $\Phi_{\mathfrak{q}} \in \Gamma_{\mathbb{Q}_q}$, we have

$$\mathrm{tr}(\Phi_{\mathfrak{q}} \mid r_{(-\boldsymbol{\mu},0)} \circ \rho|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}) = \mathrm{tr}(\Phi_{\mathfrak{q}} \mid r_{(-\boldsymbol{\mu},0)} \circ \mathsf{LL}_{\mathbb{Q}_q}(\mathrm{BC}(\pi)_q)|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}). \tag{201}$$

Now, by 2.5.63, the above equals

$$\mathrm{tr}(\Phi_{\mathfrak{q}} \mid r_{(-\boldsymbol{\mu},0)} \circ \mathsf{LL}_{\mathbb{Q}_q}(\mathrm{BC}_q(\pi_q))|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}). \tag{202}$$

By the compatibility of local base change with the unramified local Langlands correspondence [Min11, Thm 4.1], we then have

$$\mathrm{tr}(\Phi_{\mathfrak{q}} \mid r_{(-\boldsymbol{\mu},0)} \circ \mathsf{LL}_{\mathbb{Q}_q}(\mathrm{BC}_q(\pi_q))|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}) = \mathrm{tr}(\Phi_{\mathfrak{q}} \mid r_{(-\boldsymbol{\mu},0)} \circ \lambda \circ \mathsf{LL}_{\mathbb{Q}_q}(\pi_q)|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}). \tag{203}$$

Finally, by 2.4.8, we get

$$\mathrm{tr}(\Phi_{\mathfrak{p}} \mid r_{(-\boldsymbol{\mu},0)} \circ \lambda \circ \mathsf{LL}_{\mathbb{Q}_p}(\pi_p)|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}) = \mathrm{tr}(\Phi_{\mathfrak{p}} \mid r_{-\boldsymbol{\mu}} \circ \mathsf{LL}_{\mathbb{Q}_p}(\pi_p)|_{\Gamma_{\mathbf{E}^*_{\mathfrak{q}}}}) \tag{204}$$

$\square$

2.4.3. **Traces at places of bad reduction and pseudo - stabilization.** In this subsection we record an analogue of the trace formula as in §2.3.4, as well as the pseudo-stabilization of that formula as in §2.3.5. In particular, we keep the notation and assumptions the same as in §2.3.4 throughout this subsection with one exception. Namely, we fix a compact open subgroup $K_p \subseteq K_{0,p}$ and then set $K := K^p K_p$.

The first main result is the following:

Theorem 2.4.11 ([You19, Theorem 4.4.1])**.** *Let $h \in \mathscr{H}_{\mathbb{Q}}(\mathcal{G}(\mathbb{Z}_p), K_p)$ and let $\tau \in W_{E_{\mathfrak{p}}}$. Then, there exists a a function $\phi_{\tau,h} \in \mathscr{H}_{\mathbb{Q}}(\mathbf{G}((E_{\mathfrak{p}})_j))$ (independent of the choice of ℓ) such that for any $f^p \in \mathscr{H}_{\overline{\mathbb{Q}_\ell}}(\mathbf{G}(\mathbb{A}_f^p), K^p)$ the following equality holds*

(205)
$$\operatorname{tr}(\tau \times f^p h \mid H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)) = \sum_{\substack{\mathfrak{t}=(\gamma_0,\gamma,\delta) \\ \alpha(\gamma_0,\gamma,\delta)=1}} c(\mathfrak{t}) O_\gamma(f^p) TO_\delta(\phi_{\tau,h}) \operatorname{tr} \xi(\gamma_0)$$

The proof of the above, or rather the simplifications to the formula made in [You19, Theorem 4.4.1], are the same as in the proof of Theorem 2.3.15.

Let us now fix a function $f^p \in \mathscr{H}_{\mathbb{Q}}(\mathbf{G}(\mathbb{A}_f^p), K^p)$ with the property $f^p \mathbb{1}_{K_p}$ is a projector from $H^*(\mathsf{Sh}, \mathcal{F}_\xi)$ on to $H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)[(\pi_f)^{K^p}]$ and let f_∞ be as in §2.3.3.1. Let us also set $f_{\tau,h} \in \mathscr{H}(\mathbf{G}(\mathbb{Q}_p))$ to be a transfer of $\phi_{\tau,h}$ (which exists by the results of [Wal08]).

We then have the following claim:

Proposition 2.4.12. *The following equality holds:*

$$\operatorname{tr}(\tau \times f^p h \mid H^*(\mathsf{Sh}, \mathcal{F}_\xi)) = \tau_K(\mathbf{G}) \sum_{\{\gamma\}_s \in \{\mathbf{G}\}_s^{\mathrm{s.s.}}} SO_\gamma(f^p f_{\tau,h} f_\infty) \tag{206}$$

Proof. The proof of this result is exactly the same as in the proof of Theorem 2.3.16. The only substantive change is that the proof of the analogue of (153) is now by the twisted fundamental lemma (as in [Wal08]). □

We then deduce that

$$\operatorname{tr}(\tau \times f^p h \mid H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)) = \sum_{\pi \in \Pi_\chi(\mathbf{G})} m(\pi) \operatorname{tr}(f^p f_{\tau,h} f_\infty) \tag{207}$$

Note then that we can rewrite the right-hand side of this equation as

(208)
$$\sum_{\substack{\pi_f \\ \in \Pi_{f,\chi}(\mathbf{G})}} a(\pi_f) \operatorname{tr}(f^p f_{\tau,h} \mid \pi_f) = \sum_{\substack{\pi_f \\ \in \Pi_{f,\chi}(\mathbf{G})}} a(\pi_f) \operatorname{tr}(f^p \mathbb{1}_{K_p} \mid \pi_f) \operatorname{tr}(f_{\tau,h} \mid \pi_p)$$

Note though that by construction $a(\pi_f)\,\mathrm{tr}(f^p \mathbb{1}_{K_p})$ will vanish unless $(\pi_f)^K$ has non-trivial isotypic component in $H^*(\mathsf{Sh}_K, \mathcal{F}_\xi)$ and the away-from-p component of π_f agrees with that of $\pi_{0,f}^p$. Let us call this set S.

From this, we see that our sum reduces to

$$\sum_{\pi_f \in S} a(\pi_f)\,\mathrm{tr}(f_{\tau,h} \mid \pi_{f,p}) \tag{209}$$

Note though that we have the following reuslt:

Lemma 2.4.13. *The set S is precisely $\Pi_{\psi_p}(\mathbf{G}(\mathbb{Q}_p), \xi_p)$ where ψ_p is the A-parameter associated to $\pi_{0,p}$.*

Proof. Let us denote by S' the set of $\mathbf{G}(\mathbb{Q}_p)$-components of the irreducible factors of $L^2_{\mathrm{disc}}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))[\pi^p]$. By Matushima's formula it is clear that $S \subseteq S'$. Moreover, by Lemma 2.2.32 we know that S' is precisely $\Pi_{\psi_p}(\mathbf{G}(\mathbb{Q}_p), \omega)$. Thus, it suffices to show that $S = S'$.

Equivalently, by Corollary 2.3.10, for every $\pi_p \in S'$ we need to show that $a(\pi_p \otimes \pi_f^p) \neq 0$. But, since ξ is regular we see by Theorem 2.3.11 that $a(\pi_p \otimes \pi_f^p) \neq 0$ if and only if $m(\pi_p \otimes \pi^p) \neq 0$. This is precisely the claim that $S = S'$. $\square$

From the above, we deduce the following:

Proposition 2.4.14.

$$\mathrm{tr}(\tau \times f^p h \mid H^*(\mathsf{Sh}, \mathcal{F}_\xi)) = a(\pi_f) \sum_{\pi_p \in \Pi_{\psi_p}(\mathbf{G}(\mathbb{Q}_p), \xi_p)} \mathrm{tr}(f_{\tau,h} \mid \pi_p) \tag{210}$$

2.4.4. **The Scholze-Shin conjecture in certain unramified cases.** In this subsection we prove the main result of this Part . Let E/F and G be as in §2.4.1.1 and π_p^0 a square integrable representation of $G(\mathbb{Q}_p)$ and $\pi_p^0 \boxtimes \pi_p'$ an irreducible square integrable representation of $\mathbf{G}(\mathbb{Q}_p)$ as in §2.4.1.3. Let ψ_p and ψ_p' be the Arthur parameters associated to π_p^0 and π_p' respectively as in [Kal+14, Theorem 1.6.1]. In particular, $\pi_p^0 \boxtimes \pi_p'$ has Arthur parameter $\psi_p \oplus \psi_p'$. Since π_p^0 and π_p' are tempered, ψ_p and ψ_p' are also bounded Langlands parameters. Let $(\mathbf{G}, X)$ be as in §2.4.1.2 and let $\boldsymbol{\mu}$ and μ be as in §2.4.1.1.

We now prove the following which is a special case of the Scholze-Shin conjecture [SS13, Conj 7.1].

Theorem 2.4.15. *Pick any natural number $j \geqslant 1$ and $\tau \in \mathrm{Frob}^j I_{E_\mu} \subset W_{E_\mu}$. Pick $h \in \mathscr{H}(G(\mathbb{Z}_p))$. Then*

$$\sum_{\pi_p \in \Pi_{\psi_p}(G)} \mathrm{tr}(f^G_{\tau,h} \mid \pi_p) = \mathrm{tr}(\tau \mid r_{-\mu} \circ \psi_p|_{W_{E_\mu}} \otimes |\cdot|^{\frac{\dim \mathsf{Sh}}{2}}) \sum_{\pi_p \in \Pi_{\psi_p}(G)} \mathrm{tr}(h \mid \pi_p). \tag{211}$$

Proof. This follows from combining the results of the previous sections. We choose π as in 2.4.1.3 and $f^p \in \mathscr{H}(\mathbf{G}(\mathbb{A}_f^p))$ as in 2.4.11 such that f^p projects to the π^p isotypic piece of $H^*(\mathsf{Sh}, \mathcal{F}_\xi))$. Fix any $h^{\mathbf{G}} \in \mathscr{H}(G(\mathbb{Z}_p) \times G'(\mathbb{Z}_p))$. Note that $\tau \in E_\mu = \mathbf{E}_{\mathfrak{p}}^*$ as discussed in the paragraph before 2.4.8.

On the one hand, by 2.4.14, we have

$$\operatorname{tr}(\tau \times f^p h^{\mathbf{G}} \mid H^*(\mathsf{Sh}, \mathcal{F}_\xi)) = a(\pi_f) \sum_{\pi_p \in \Pi_{\psi_p \oplus \psi'_p}(\mathbf{G}_{\mathbb{Q}_p})} \operatorname{tr}(f^{\mathbf{G}}_{\tau, h^{\mathbf{G}}} \mid \pi_p). \tag{212}$$

On the other hand, by 2.3.1, we have

$$\operatorname{tr}(\tau \times f^p h^{\mathbf{G}} \mid H^*(\mathsf{Sh}, \mathcal{F}_\xi)) = \operatorname{tr}(\tau \times f^p h^{\mathbf{G}} \mid \bigoplus_{\pi_f} \pi_f \boxtimes \sigma(\pi_f)), \tag{213}$$

and hence by definition of f^p as well as the argument in 2.4.12 using 2.2.32,

$$\operatorname{tr}(\tau \times f^p h^{\mathbf{G}} \mid H^*(\mathsf{Sh}, \mathcal{F}_\xi)) = \operatorname{tr}(\tau \times h^{\mathbf{G}} \mid \bigoplus_{\pi_p \in \Pi_{\psi_p \oplus \psi'_p}(\mathbf{G})} \pi_p \boxtimes \sigma(\pi_f)). \tag{214}$$

Now, using 2.4.10, the above equals

$$a(\pi_f) \operatorname{tr}(\tau \mid (r_{(-\boldsymbol{\mu},0)} \circ \rho|_{W_{\mathbf{E}_{\mathfrak{p}}^*}}) \otimes |\cdot|^{\frac{\dim \mathsf{Sh}}{2}}) \sum_{\pi'_p \in \Pi_{\psi_p \oplus \psi'_p}(\mathbf{G}_{\mathbb{Q}_p})} \operatorname{tr}(h^{\mathbf{G}} \mid \pi_p). \tag{215}$$

Finally, by 2.4.6, compatibility of the local Langlands correspondence and local base change ([Mok15, Theorem 3.2.1 (a)]), and 2.4.8, we have that

$$\begin{aligned} r_{(-\boldsymbol{\mu},0)} \circ \rho|_{W_{\mathbf{E}_{\mathfrak{p}}^*}} &\cong r_{(-\boldsymbol{\mu},0)} \circ \psi_{BC(\pi)_p}|_{W_{\mathbf{E}_{\mathfrak{p}}^*}} \\ &\cong r_{(-\boldsymbol{\mu},0)} \circ \lambda \circ (\psi_p \oplus \psi'_p)|_{W_{\mathbf{E}_{\mathfrak{p}}^*}} \\ &= r_{-\boldsymbol{\mu}} \circ (\psi_p \oplus \psi'_p)|_{W_{\mathbf{E}_{\mathfrak{p}}^*}} \end{aligned} \tag{216}$$

Hence the righthand side of the previous equality becomes
(217)

$$a(\pi_f) \operatorname{tr}(\tau | (r_{-\boldsymbol{\mu}} \circ (\psi_p \oplus \psi'_p)|_{W_{\mathbf{E}_{\mathfrak{p}}^*}}) \otimes |\cdot|^{\frac{\dim \mathsf{Sh}}{2}}) \sum_{\pi_p \in \Pi_{\psi_p \oplus \psi'_p}(\mathbf{G}_{\mathbb{Q}_p})} \operatorname{tr}(h^{\mathbf{G}} \mid \pi_p).$$

Finally, combining the the two equations for $\operatorname{tr}(\tau \times f^p h^{\mathbf{G}} \mid H^*(\mathsf{Sh}, \mathcal{F}_\xi))$ gives that

$$\sum_{\pi_p \in \Pi_{\psi_p \oplus \psi'_p}(\mathbf{G}_{\mathbb{Q}_p})} \operatorname{tr}(f^{\mathbf{G}}_{\tau, h^{\mathbf{G}}} \mid \pi_p) \tag{218}$$

is equal to

$$\mathrm{tr}(\tau \mid (r_{-\boldsymbol{\mu}} \circ (\psi_p \oplus \psi'_p)|_{W_{\mathbf{E}^*_{\mathfrak{p}}}}) \otimes |\cdot|^{\frac{\dim \mathsf{Sh}}{2}}) \sum_{\pi_p \in \Pi_{\psi_p \oplus \psi'_p}(\mathbf{G}_{\mathbb{Q}_p})} \mathrm{tr}(h^{\mathbf{G}} \mid \pi_p). \tag{219}$$

We now need to translate this equation to one for G instead of $\mathbf{G}_{\mathbb{Q}_p}$. Since our choice of $h^{\mathbf{G}}$ was arbitrary, we pick it so that $h^{\mathbf{G}} = h \times h'$ where h' has trace 1 on a single representation in the packet $\Pi_{\psi'_p}(G')$ and trace 0 on the others. We can do this since local A-packets are finite (e.g. see [HG, Proposition 8.5.2]). Since μ is trivial on G', we have that $f_{\tau,h'} = h'$. Indeed, the triviality of μ' implies that the space $\mathscr{D}_\infty(\mathcal{G}', [b'], \mu')$ (where μ' is the projetion to μ) as in [You19] is the trivial $\mathcal{G}'(\mathbb{Z}_p)$-torsor for any $[b']$ as in loc. cit. In particular, this implies that $H^*(\mathscr{D}_\infty(\mathcal{G}', [b'], \mu'), \mathbb{Q}_\ell)$ is nothing more than $C_c^\infty(\mathcal{G}'(\mathbb{Z}_p))$. Since the action of τ is through right multiplication by b' it's clear that the trace of $\tau \times h$ on $\mathscr{D}_\infty(\mathcal{G}', [b'], \mu')$, which is by definition $f_{\tau,h'}(b')$, is just $h'(b')$. Moreover, we have that

$$f^{\mathbf{G}}_{\tau,h\times h'} = f^{G}_{\tau,h} \times f^{G'}_{\tau,h'} = f_{\tau,h} \times h'. \tag{220}$$

as there is a natural splitting of the space

$$\mathscr{D}_\infty(\mathcal{G} \times \mathcal{G}', [(b, b')]\mu) \cong \mathscr{D}_\infty(\mathcal{G}, [b], \mu) \times \mathscr{D}_\infty(\mathcal{G}'[b'], \mu') \tag{221}$$

which is equivariant for the action of $\mathcal{G}(\mathbb{Z}_p) \times \mathcal{G}'(\mathbb{Z}_p)$.

Then, using that $\Pi_{\psi_p \oplus \psi'_p}(\mathbf{G}_{\mathbb{Q}_p}) = \Pi_{\psi_p}(G) \times \Pi_{\psi'_p}(G')$, we get

$$\sum_{\pi_p \in \Pi_{\psi_p}(G)} \mathrm{tr}(f^G_{\tau,h} \mid \pi_p) = \tag{222}$$

$$\mathrm{tr}(\tau \mid (r_{-\boldsymbol{\mu}} \circ (\psi_p \oplus \psi'_p)|_{W_{\mathbf{E}^*_{\mathfrak{p}}}}) \otimes |\cdot|^{\frac{\dim \mathsf{Sh}}{2}}) \sum_{\pi_p \in \Pi_{\psi_p}(G)} \mathrm{tr}(h \mid \pi_p).$$

Now we denote by μ', the cocharacter of $G'_{\overline{\mathbb{Q}_p}}$ such that under ι_p, (μ, μ') maps to $\boldsymbol{\mu}$. By construction μ' is trivial and hence $r_{\mu'}$ is the trivial representation. In particular, we get

$$\mathrm{tr}(\tau \mid r_{-\boldsymbol{\mu}} \circ (\psi_p \oplus \psi'_p)) = \mathrm{tr}(\tau \mid (r_{-\mu} \circ \psi_p) \otimes (r_{-\mu'} \circ \psi'_p)) = \mathrm{tr}(\tau \mid r_{-\mu} \circ \psi_p). \tag{223}$$

Making this substitution gives

$$\sum_{\pi_p \in \Pi_{\psi_p}(G)} \mathrm{tr}(f^G_{\tau,h} \mid \pi_p) = \mathrm{tr}(\tau \mid (r_{-\mu} \circ \psi_p)|_{W_{E_\mu}} \otimes |\cdot|^{\frac{\dim \mathsf{Sh}}{2}}) \sum_{\pi_p \in \Pi_{\psi_p}(G)} \mathrm{tr}(h \mid \pi_p) \tag{224}$$

as desired. ∎

2.5. Appendices for Part II

2.5.1. **Appendix 1: Some lemmas about reductive groups.** The goal of this appendix is to collect some loosely related facts about reductive groups, especially with a focus on reductive groups over $\mathbb{R}$.

2.5.1.1. *Elliptic elements and tori.* In this subsection we clarify the relationship between several notions of ellipticity for elements of a reductive group.

So, let us fix a field F of characteristic 0 and let G be a reductive group over F. We begin with the following definition which is unambiguous:

Definition 2.5.1. *A torus T in G containing $Z(G)^\circ$ is said to be* elliptic *if the torus $T/Z(G)^\circ$ is F-anisotropic.*

It is often times the case that a torus T contains not only $Z(G)^\circ$ but $Z(G)$ (e.g. maximal tori). In this case, one might wonder whether one obtains a fundamentally different definition by requiring that $T/Z(G)$ is F-anisotropic. As the following lemma shows, by applying it to the obvious isogeny $T/Z(G)^\circ \to T/Z(G)$, the answer is no. For this reason, we will often times not careful between discussions of the F-anitropicity of $T/Z(G)$ for $T/Z(G)^\circ$ for a torus T containing $Z(G)^\circ$ (again, mostly in the case when T is a maximal torus):

Lemma 2.5.2. *Let T_1 and T_2 be isogenous tori over F. Then, T_1 is F-anisotropic if and only if T_2 is.*

Proof. Let $f : T_1 \to T_2$ be an isogeny. Note then that we get an inclusion $X^*(T_2) \hookrightarrow X^*(T_1)$ with finite cokernel. We and thus an inclusion $X^*(T_2)^\Gamma \hookrightarrow X^*(T_1)^\Gamma$ with finite cokernel. Since $X^*(T_i)^\Gamma$ is free we see that $X^*(T_2)^\Gamma$ is trivial if and only if $X^*(T_1)^\Gamma$ is trivial as desired. $\square$

The definition of what it means for a semisimple element γ in $G(F)$ to be 'elliptic' is a little less clear. Namely, we have the following:

Definition 2.5.3. *A semisimple element γ in $G(F)$ is* elliptic *if $Z(Z_G(\gamma))^\circ$ is an elliptic torus. We will say that such an element γ is* strongly elliptic *if γ is contained in $T(F)$ for some elliptic maximal torus T of G.*

Note that evidently strongly elliptic implies elliptic. Indeed, if T is an elliptic maximal torus such that $\gamma \in T(F)$ then T is a maximal torus in $Z_G(\gamma)$ and thus $\subseteq Z(G)^\circ Z(Z_G(\gamma))^\circ$ is a subtorus of T. Since T is elliptic this implies that $Z(Z_G(\gamma))^\circ$ is elliptic.

Of course, it can't be true in general that elliptic implies strongly elliptic since there are reductive groups which contain no elliptic maximal tori but which contain elliptic elements.

Example 2.5.4. For any perfect field F the maximal tori in $\mathrm{GL}_{n,F}$ are of the form $\prod_{i=1}^{k} \mathsf{Res}_{E_i/F}\,\mathbb{G}_{m,E_i}$ where E_i/F are field extensions and $\sum_{i=1}^{k}[E_i : F] = n$. Moreover, one can check that amongst these the elliptic maximal tori are those of the form $\mathsf{Res}_{E/F}\mathbb{G}_{m,E}$ where $[E : F] = n$. Thus, we see that $\mathrm{GL}_{n,F}$ has an elliptic maximal torus if and only if F admits an extension of degree n.

In particular, we see that $\mathrm{GL}_{n,\mathbb{R}}$ admits an elliptic maximal torus if and only if $n = 2$. That said, $\mathrm{GL}_{n,\mathbb{R}}$ has elliptic elements for all $n \geqslant 1$. Indeed, for any group G the identity element $G(F)$ is elliptic.

That said, in most of the cases of interest to us the definitions coincide. For instance, we have the following observation:

Proposition 2.5.5. *Let F be a p-adic local field. Then, a semisimple element γ in $G(F)$ is elliptic if and only if it's strongly elliptic.*

Lemma 2.5.6. *Let F be a p-adic local field and let H be a reductive group over F. Then, H contains an elliptic maximal torus.*

Proof. By [PS92, Theorem 6.21] we know that $H/Z(H)$ contains a maximal anisotropic torus T. Evidently the preimage of T under the projection map $H \to H/Z(H)$ produces the desired elliptic maximal torus. $\square$

Proof. (Proposition 2.5.5) As we've already observed, it suffices to show that if $\gamma \in G(F)$ is elliptic, then it's strongly elliptic. That said, note that $H := I_\gamma$ contains an elliptic maximal torus T which is evidently a maximal torus of G since H contains a maximal torus of G and thus has the same rank as G. By definition, this implies that $T/Z(H)$ is F-anistropic. That said note that by our assumption the split rank of $Z(H)$ and the split rank of $Z(G)$ coincides. Thus, $T/Z(H)$ having split rank 0 implies that $T/Z(G)$ has split rank 0. Since γ is contained in $T(F)$ the claim follows. $\square$

We would like to extend this result to all characteristic 0 local fields and so, in particular, extend this result to $\mathbb{R}$ (note that the only elliptic torus in a group G over $\mathbb{C}$ is $Z(G)^\circ$). But, as we observed in Example 2.5.4 such a result fails for trivial reasons over $\mathbb{R}$ for general groups. That said, one can ask whether the notion of elliptic and strongly elliptic do agree for semisimple elements in $G(\mathbb{R})$ where G is a reductive group over $\mathbb{R}$ that does contain an elliptic maximal torus. The answer is yes.

To see this, we begin with the following well-known result:

Lemma 2.5.7. *Let G be a reductive group over $\mathbb{R}$. Then, for every compact subgroup K contained in $G(\mathbb{R})$ there exists an $\mathbb{R}$-anisotropic group H and a closed embedding $H \hookrightarrow G$ such that $H(\mathbb{R}) = K$.*

Proof. This is [Ser93, §5 Proposition 2]. □

One consequence of this is the following:

Lemma 2.5.8. *Let G be a reductive group over $\mathbb{R}$. Then, all maximal anisotropic tori in G are conjugate. Moreover, all maximal elliptic tori in G are conjugate.*

Proof. Let us begin by showing that the former statement implies the latter. Namely, let T_1 and T_2 be two maximal elliptic tori in G. Note then that by standard theory we have a decomposition $T_i = T_i^s T_i^a$ where T_i^s is the maximal split subtorus of T_i and T_i^a is the maximal anisotropic subtorus. Moreover, we have that $T_i^s \cap T_i^a$ is finite. Note that by our ellipticity assumptions we have that $T_i^s = (Z(G)^\circ)^s$ for $i = 1, 2$.

Let us note that T_i^a are maximal aniostropic tori in G, as we now show. By symmetry we need only consider the case when $i = 1$. Now, suppose that T is an anisotropic torus of G strictly containing T_1^a. Then, evidently $TZ(G)^\circ$ is an elliptic torus of G strictly containing T_1 which contradicts assumptions.

So, assuming that all anisotropic tori in G are conjugate there exists some $g \in G(\mathbb{R})$ such that $gT_1^a g^{-1} = T_2^a$. Note then evidently that since conjugation by g fixes $Z(G)$ pointwise that

$$gT_1 g^{-1} = g(T_1^a Z(G)^\circ)g^{-1} = T_2^a Z(G)^\circ = T_2 \tag{225}$$

as desired.

Suppose now that T_1 and T_2 are maximal anisotropic tori in G. Note then that $T_1(\mathbb{R})$ and $T_2(\mathbb{R})$ are compact subgroups of G and thus contained in maximal compact subgroups K_1 and K_2 of $G(\mathbb{R})$. Now, it is well-known (e.g. see [Con14, Theorem D.2.8]) that K_1 and K_2 are conjugate by an element of $G(\mathbb{R})$. Thus without loss of generality we may assume the equality $K := K_1 = K_2$. Moreover, by Lemma 2.5.7 we know that $K = H(\mathbb{R})$ for H some $\mathbb{R}$-anisotropic subgroup of G.

We claim that both $T_1(\mathbb{R})$ and $T_2(\mathbb{R})$ are maximal tori in K in the sense of the theory of compact Lie groups (i.e. that they are maximal connected compact abelian subgroups). Indeed, suppose not. Then there exists a connected compact abelian subgroup $S \subseteq K = H(\mathbb{R})$

properly containing $T_1(\mathbb{R})$. But, by [Con14, Theorem D.2.4] this implies that there exists some connected $\mathbb{R}$-anisotropic group $S^{\mathrm{alg}} \subseteq H$ such that $S^{\mathrm{alg}}(\mathbb{R}) = S$. Note then that by the Zariski denseness of $\mathbb{R}$-points (e.g. see [Mil17, Theorem 17.9.3]) we have that S^{alg} properly contains T_1. But, since S is dense in S^{alg} we see that S^{alg} is necessarily abelian. Thus, S^{alg} is an anisotropic torus in H properly containing T_1. This contradicts that T_1 is a maximal anisotropic torus in G. By symmetry the claim also applies for T_2.

Thus, since $T_1(\mathbb{R})$ and $T_2(\mathbb{R})$ are maximal tori in K in the sense of the theory of compact Lie groups we know from the theory of such groups that $T_1(\mathbb{R})$ and $T_2(\mathbb{R})$ are conjugate by an element of K. Then, again by density of $T_1(\mathbb{R})$ in T_1, we deduce that T_1 is conjugate to T_2. More rigorously let $g \in K = H(\mathbb{R})$ conjugate $T_1(\mathbb{R})$ to $T_2(\mathbb{R})$. Note then that conjugation map by g sends $T_1(\mathbb{R})$ into $T_2 \subseteq G$ from which density of $T_1(\mathbb{R})$ in T_1 implies that conjugation by g takes T_1 into T_2. This implies that $\dim T_1 \leqslant \dim T_2$. By symmetry we deduce that $\dim T_2 \leqslant \dim T_1$. Then, since $gT_1g^{-1} \subseteq T_2$ and gT_1g^{-1} and T_2 are both connected and smooth we deduce that $gT_1g^{-1} = T_2$ as desired. □

Two important corollaries of this result are the following:

Corollary 2.5.9. *Let G Be a reductive group over $\mathbb{R}$ and suppose that G has an elliptic maximal torus. Then, every maximal elliptic torus in G is an elliptic maximal torus.*

Corollary 2.5.10. *Let G be a reductive group over $\mathbb{R}$ and suppose that G has an elliptic maximal torus T_0. Then, every elliptic element γ in $G(\mathbb{R})$ is strongly elliptic.*

Proof. Note that, by definition, γ is contained in an elliptic torus T_1 of G (namely $T_1 = Z(Z_G(\gamma))^\circ$). Note then that T_1 is contained in some maximal elliptic torus T of G. But, by the previous corollary T is a maximal torus in G. The conclusion follows. □

We finally record the following well-known results concerning the existence of elliptic maximal tori in groups over $\mathbb{R}$. Namely, while it is classical that every reductive group G over $\mathbb{R}$ admits a unique anisotropic form. That said, the existence of an anisotropic modulo center inner form is not guaranteed and is related to the existence of an elliptic maximal torus. Namely:

Lemma 2.5.11. *Let G be a connected reductive group over $\mathbb{R}$. Then, G admits an elliptic maximal torus if and only if G admits an anisotropic modulo center inner form.*

2.5.1.2. *Local-to-global construction of elliptic maximal tori.* In this subsection we would like to verify that if G is a reductive group over a number field F we can construct maximal tori in G which become elliptic over some some finite set of places S of F as long as there are no tautological obstructions (i.e. that G has no elliptic maximal tori at one of the places in S). More rigorously:

Proposition 2.5.12. *Let F be a number field and let G be a connected reductive group over F. Suppose that S is a finite set of places of F such that for all $v \in S$ the group G_{F_v} contains an elliptic maximal torus. Then, there exists a maximal torus T in G such that T_{F_v} is an elliptic maximal torus in G_{F_v} for all $v \in S$.*

To prove this it will be helpful to set up some notation and recall some classical results concerning the moduli of maximal tori in G. For now, let F be any field of characteristic 0 and let G be a connected reductive group over F. To begin, let us define X to be the functor associating to an F-algebra R the set $X(R)$ of maximal tori in G_R (e.g. in the sense of [Con14, Definition 3.2.1]). Then, we have the following result:

Lemma 2.5.13. *The functor X is represented by a smooth, irreducible, and quasi-affine F-scheme (also denoted X). Moreover, for any maximal torus T_0 in G there is a canonical isomorphism $G/N_G(T_0) \to X$.*

Proof. See [Con14, Theorem 3.2.6] for the first statement minus the smoothness and irreducibility and the second statement. Note that the conditions that the maximal tori in $G_{\overline{F}}$ are self-centralizing follows immediately from the reductive hypotheses on G. The smoothness and irreducibility of X then follow *a fortiori* from the second statement given the smoothness and irreducibility of G. □

We shall need the following structural result of Chevalley concerning X:

Theorem 2.5.14 (Chevalley)**.** *The scheme X is F-rational.*

Now, for any field F' containing F let us denote by $X^e(F')$ the subset of $X'(F)$ consisting of F'-elliptic maximal tori in $G_{F'}$. Be careful that, despite the notation, $X^e(F')$ is evidently not functorial in F'.

We then have the following observation:

Lemma 2.5.15. *Suppose that F is a characteristic* 0 *local field. Then, $X^e(F)$ is an open (possibly empty) subset of $X(F)$ where the latter is endowed with the usual topology F-topology.*

Proof. Let us denote by $\mathbb{T}$ the universal maximal torus over X. For a point $x \in X(F)$ we denote by $\mathbb{T}_x$ the corresponding torus of G since split rank is an isogeny invariant (e.g. see Lemma 2.5.2). It then suffices to show that the isogeny class of $\mathbb{T}_x$ is locally constant in x. To do this we proceed as follows. Let us note that X is rational and smooth, so that $\mathbb{T}$ gives rise (by [Con14, Corollary B.3.6]) to a continuous representation $\pi_1(X, \overline{x_0}) \to \mathrm{GL}_n(\mathbb{Z})$ (where n is the rank of $\mathbb{T}$).

Note that this representation must factor through a finite quotient Q of $\pi_1(X, \overline{x_0})$. Note that for $x \in X(F)$ the torus $\mathbb{T}_x$ clearly corresponds to the composition $\Gamma_F \to \pi_1(X, \overline{x_0}) \to \mathrm{GL}_n(\mathbb{Z})$ which we denote ρ_x. Note, in particular that for any $x \in X(F)$ we have that ρ_x has image bounded by $|Q|$ and so Γ_F factors through a quotient of size $|Q|$. Since F has only finitely many extensions of size $|Q|$ we see that there must be some finite extension F'/F such that ρ_x factors through $\mathrm{Gal}(F'/F)$ for all $x \in X(F)$.

Let us denote, for each $x \in X(F)$, the composition of ρ_x with the embedding $\mathrm{GL}_n(\mathbb{Z}) \hookrightarrow \mathrm{GL}_n(\mathbb{Q})$ by $\rho_x^{\mathbb{Q}}$. Then, by the Brauer-Nesbitt theorem we know that $\rho_x^{\mathbb{Q}} \cong \rho_{x'}^{\mathbb{Q}}$ if and only if $\chi_{\rho_x(g)} = \chi_{\rho_{x'}(g)}$ for all $g \in \mathrm{Gal}(F'/F)$ where we have used χ_T to denote the characteristic polynomial for T. But, since the coefficients of ρ_x are roots of unity, we know that $\chi_{\rho_x(g)} = \chi_{\rho_x(g')}$ if and only if they agree modulo N for N sufficiently large. In other words, we see that if $\mathbb{T}_x[N] \cong \mathbb{T}_{x'}[N]$ then $\mathbb{T}_x$ and $\mathbb{T}_{x'}$ are isogenous.

Let us then pick a point $x \in X(F)$ and consider the finite étale cover $\underline{\mathrm{Isom}}(\mathbb{T}[n], \mathbb{T}_{x_0}[N])$ of X. Note then that since the point $x_0 \in X(F)$ has a lift to a point of $\underline{\mathrm{Isom}}(\mathbb{T}[n], \mathbb{T}_{x_0}[N])(F)$ then by standard theory (e.g. see [Poo17, Theorem 3.5.73.(i)]) there exists a neighboorhod U of x_0 in $X(F)$ such that $\underline{\mathrm{Isom}}(\mathbb{T}[N], \mathbb{T}_{x_0}[N])(F) \to X(F)$ admits a section. By the above, this implies that $\mathbb{T}_x$ is isogenous to $\mathbb{T}_{x_0}$ for all $x \in U$, and so the conclusion follows. $\square$

Using the above results we can now prove Proposition 2.5.12:

Proof. (Proposition 2.5.12) Let us denote by F_S the usual F-algebra $\prod_{v \in S} F_v$. Note then that we have a natural diagonal embedding $X(F) \to X(F_S)$. Moreover, since X is F-rational, smooth, and irreducible we know that the image of $X(F)$ in $X(F_S)$ is dense (e.g. see [PS92, Proposition 7.3]). Now, by assumption we have that $X^e(F_v)$ is non-empty for all $v \in S$ and thus combining this with Lemma 2.5.15 we see that $\prod_{v \in S} X^e(F_v)$ is a non-empty open subset of $X(F_S)$. Since $X(F)$ is a dense subset of $X(F_S)$ we thus deduce that $X(F)$ and $\prod_{v\in S} X(F_v)$ must have a point in common. The conclusion follows. $\square$

2.5.1.3. *Stable conjugacy for strongly regular elements over* $\mathbb{R}$. The goal of this subsection is to clarify the nature of stable conjugacy for strongly regular elements in $G(\mathbb{R})$ where G is a reductive group over $\mathbb{R}$.

Before we begin, let us fix some notation that will be used below (as well as the main body of the paper).

Definition 2.5.16. *Let T be a maximal torus in G. For any Levi subgroup M of G containing T we denote by $W(M,T)$ the Weyl group scheme $N_M(T)/T$. We will denote by $W_{\mathbb{C}}(M,T)$ the group*

$$W_{\mathbb{C}}(M,T) := N_G(T)(\mathbb{C})/T(\mathbb{C}) = W(M,T)(\mathbb{C}) \tag{226}$$

We denote by $W_{\mathbb{R}}(M,T)$ the group

$$W_{\mathbb{R}}(M,T) := N_G(T)(\mathbb{R})/T(\mathbb{R}) \subseteq W(M,T)(\mathbb{R}) \tag{227}$$

where this last containment can be strict in general. When $M = G$ we use the shortenings $W_{\mathbb{C}}$ and $W_{\mathbb{R}}$ of the above notation.

Remark 2.5.17. For the sake of notational comparison, let us note that if T is an elliptic maximal torus then $W_{\mathbb{R}}$ is often written (for example in Harish-Chandra's parametrization of discrete series) as W_c and called the *compact Weyl group*. The reason is that in this case $W_{\mathbb{R}}$ is equal to $W(K, T(\mathbb{R}))$ for any maximal compact subgroups of $G(\mathbb{R})$ containing $T(\mathbb{R})$. The reason of course, is that $N_G(T)(\mathbb{R})$, containing $T(\mathbb{R})$ as a finite index subgroup, is itself compact and so contained in a maximal compact subgroup of $G(\mathbb{R})$.

We also recall the following well-known definitions:

Definition 2.5.18. *Let G be a reductive group over a field F. A semsimimple element γ in $G(F)$ is* regular *if I_γ is a (necessarily maximal) torus of G. We say that γ is* strongly regular *if $Z_G(\gamma)$ is a (necessarily maximal) torus of G.*

Recall that if G^{der} is simply connected then these two notions coincide. Indeed, in the case by the following well-known result of Steinberg:

Theorem 2.5.19 (Steinberg)**.** *Let G be a reductive group over a field F and assume that G^{der} is simply connected. Then, for any semisimple $\gamma \in G(F)$ we have that $Z_G(\gamma)$ is connected.*

Proof. To show that $Z_G(\gamma)$ is connected it suffices to show that $Z_G(\gamma)_{\overline{F}}$ is connected, and so it suffices to assume that F is algebraically closed. Note that we have a short exact sequence of groups

$$0 \to G^{\mathrm{der}} \to G \to G^{\mathrm{ab}} \to 0 \tag{228}$$

Note that since G is reductive we have that $G = G^{\mathrm{der}} Z(G)$ and so $Z(G)$ surjects onto G^{ab}. Since $Z_G(\gamma) \supseteq Z(G)$ we deduce that $Z_G(\gamma)$ surjects onto G^{ab}. Thus, the sequence (228) gives rise to the sequence

$$0 \to G^{\mathrm{der}} \cap Z_G(\gamma) \to Z_G(\gamma) \to G^{\mathrm{ab}} \to 0 \tag{229}$$

Thus, since G^{ab} is connected since G is, it suffices to show that $G^{\mathrm{der}} \cap Z_G(\gamma)$ is connected. Note that since $G = G^{\mathrm{der}} Z_G(\gamma)$ that there exists some $z \in Z(G)(F)$ such that $\gamma z \in G^{\mathrm{der}}(F)$. Clearly $Z_G(\gamma) = Z_G(\gamma z)$ and so it suffices to assume that $\gamma \in G^{\mathrm{der}}(F)$. Note then that $G^{\mathrm{der}} \cap Z_G(\gamma) = Z_{G^{\mathrm{der}}}(\gamma)$. Thus, it finally suffices to assume that $G = G^{\mathrm{der}}$. In this setting one can find a proof in [Ste06, §5] or [Hum95, §2.11] $\square$

It will also be helpful to record the following basic observation:

Theorem 2.5.20 (Steinberg)**.** *Let G be a reductive group over a field F. Then, the set U of regular elements of G is an open subset of F. In particular, $U(F)$ is dense in G.*

Proof. The fact that U is open follows from [Ste65, p. 1.3]. Note then that since G is unirational (e.g. see [Mil17, Theroem 17.93]) the same is true for U. Thus, $U(F)$ is Zariski dense in U. But, since U is open in G and G is irreducible (e.g. by [Mil17, Summary 1.36]) we know that U is dense in G so that $U(F)$ is dense in G as desired. $\square$

We now state our target proposition:

Proposition 2.5.21. *Let G be a reductive group over $\mathbb{R}$ and let T be a maximal torus in $\mathbb{R}$. Let S be a maximal split subtorus of T and set $M := Z_G(S)$. Let $\gamma \in T(\mathbb{R})$ be strongly regular. Then:*

$$\{\gamma\}_s = \bigcup_{w \in W_{\mathbb{C}}(M,T)} \{w\gamma w^{-1}\} = \bigcup_{w \in W_{\mathbb{C}}(M,T)/W_{\mathbb{R}}(M,T)} \{w\gamma w^{-1}\} \tag{230}$$

An immediate corollary, the case of most interest to us, is the following:

Corollary 2.5.22. *Let G be a reductive group over $\mathbb{R}$ and suppose that T is a maximal elliptic torus then*

$$\{\gamma\}_s = \bigcup_{w \in W_{\mathbb{C}}} \{w\gamma w^{-1}\} = \bigsqcup_{w \in W_{\mathbb{C}}/W_{\mathbb{R}}} \{w\gamma w^{-1}\} \tag{231}$$

Proof. This follows immediately from the proposition since one can take S to be a maximal split subtorus of $Z(G)$ so that $M = G$. $\square$

Example 2.5.23. Let $G = \mathrm{SL}_{2,\mathbb{R}}$. Then, the classic example of two non-conjugate but stably conjugate elements of $\mathrm{SL}_2(\mathbb{R})$ is $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\gamma' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note though that $\gamma \in T(\mathbb{R})$ where T is the elliptic maximal torus

$$T = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a^2 + b^2 = 1 \right\} \subseteq \mathrm{SL}_{2,\mathbb{R}} \tag{232}$$

Moreover, note that $|W_{\mathbb{C}}| = 2$ with the non-trivial class represented by $w := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Moreover, it's not hard to check that

$$\mathrm{Int}(w) : T \to T \tag{233}$$

is given by

$$\mathrm{Int}(w) : \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{234}$$

Thus, the above corollary shows that

$$\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right\}_s = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \right\} \tag{235}$$

and thus

$$\{\gamma\}_s = \{\gamma\} \cup \{\gamma'\} \tag{236}$$

explaining the above example.

Let us begin by clarifying how $\{w\gamma w^{-1}\}$ makes sense for $w \in N_M(T)(\mathbb{C})$ as an element of $\{G\}$. This is settled by the following:

Lemma 2.5.24 ([She79, Theorem 2.1])**.** *Let notation be as in the beginning previous proposition. Then, the group*

$$\{g \in G(\mathbb{C}) : \mathrm{Int}(g) : T_{\mathbb{C}} \to G_{\mathbb{C}} \text{ is defined over } \mathbb{R}\} \tag{237}$$

is equal to the group $G(\mathbb{R})N_M(T)(\mathbb{C})$.

In particular, for any $\gamma \in T(\mathbb{R})$ and $g \in N_M(T)(\mathbb{C})$ we have that the map $\mathrm{Int}(g) : T_{\mathbb{C}} \to T_{\mathbb{C}}$ is defined over $\mathbb{R}$, and thus $g\gamma g^{-1}$ is an element of $T(\mathbb{R})$. Thus, $\{g\gamma g^{-1}\}$ is a well-defined element of $\{G\}$.

Remark 2.5.25. Note that, a priori, the conjugacy class $\{g\gamma g^{-1}\}$ may depend on the choice of γ in $\{\gamma\}$. Thus, the notation $w \cdot \{\gamma\}$ doesn't a priori make sense for $w \in W_C(M,T)$. In fact, the well-definedness of $w \cdot \{\gamma\}$ (the independence of choice representative in $\{\gamma\}$ in $T(\mathbb{R})$) is equivalent to the normality of $W_{\mathbb{R}}(M,T)$ in $W_{\mathbb{C}}(M,T)$ which needn't necessarily hold. That said, the right-hand side of (230) doesn't depend on a choice of γ.

To begin to prove Proposition 2.5.21 we begin with the following observation:

Lemma 2.5.26. *Suppose that $\gamma \in T(\mathbb{R})$ is strongly regular. Suppose that $\gamma' \in G(\mathbb{R})$ is stably conjugate to γ. Then, γ' is strongly regular and the tori $T' := Z_G(\gamma')$ and T are stably conjugate (i.e there is a $g \in G(\mathbb{C})$ such that $\mathrm{Int}(g) : T_{\mathbb{C}} \to T'_{\mathbb{C}}$ and the map is defined over $\mathbb{R}$).*

Proof. The fact that γ' is strongly regular is clear since $Z_G(\gamma')$ and $Z_G(\gamma)$ are forms of each other, and thus $Z_G(\gamma')$ is a torus. Now, by assumption, there is $g \in G(\mathbb{C})$ such that $g\gamma g^{-1} = \gamma'$. In particular, for $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$,

$$\begin{aligned} \sigma(g)\gamma\sigma(g)^{-1} &= \sigma(g\gamma g^{-1}) \\ &= \sigma(\gamma') \\ &= \gamma' \\ &= g\gamma g^{-1} \end{aligned} \tag{238}$$

Hence, $\sigma(g)^{-1}g = t_1 \in T(\mathbb{C})$.

Now, we need to show $\mathrm{Int}(g) : T \to T'$ is defined over $\mathbb{R}$. In particular, we need to show that $\sigma \circ \mathrm{Int}(g) \circ \sigma^{-1} = \mathrm{Int}(g)$. But we have for all $t \in T(\mathbb{C})$,

$$\begin{aligned} (\sigma \circ \mathrm{Int}(g) \circ \sigma^{-1})(t) &= \sigma(g\sigma^{-1}(t)g^{-1}) \\ &= \sigma(g)t\sigma(g)^{-1} \\ &= gt_1^{-1}tt_1g^{-1} \\ &= gtg^{-1} \\ &= \mathrm{Int}(g)(t) \end{aligned} \tag{239}$$

from where the result follows since T and T' are separated. ∎

The last preliminary result we need is the following:

Theorem 2.5.27 ([She79, Cor 2.3])**.** *Let G be a reductive group over $\mathbb{R}$ and let T and T' be maximal tori in G. Then, if T and T' are stably conjugate, then they are conjugate.*

We now prove the main proposition as follows:

Proof. (Proposition 2.5.21) Evidently

$$\{\gamma\}_s \supseteq \bigcup_{w\in W_C(M,T)} \{w\gamma w^{-1}\} \tag{240}$$

Conversely, suppose that $\gamma' \in G(\mathbb{R})$ is stably conjugate to γ. Since γ is strongly regular we know from Lemma 2.5.26 that T and $T' := Z_G(\gamma')$ are stably conjugate. Thus, by Lemma 2.5.27 we know that T and T' are conjugate. Thus, we may assume without loss of generality (without changing the conjugacy class) that $\gamma' \in T(\mathbb{R})$. Let $g \in G(\mathbb{C})$ be such that $g\gamma g^{-1} = \gamma'$. Since γ is strongly regular this implies, by Lemma 2.5.26, that $\mathrm{Int}(g)$ maps $T_{\mathbb{C}} \to T_{\mathbb{C}}$ and, in fact, is defined over $\mathbb{R}$. By Lemma 2.5.24 this implies that $g \in G(\mathbb{R})N_M(T)(\mathbb{C})$. But, since conjugation by $G(\mathbb{R})$ evidently doesn't effect conjugacy classes, we may assume that $g \in N_M(T)(\mathbb{C})$. The first part of (230) follows.

Suppose now that $w_1\gamma w_1^{-1}$ is conjugate to $w_2\gamma w_2^{-1}$. Then, there exists some $g \in G(\mathbb{R})$ such that

$$w_2\gamma w_2^{-1} = g w_1\gamma w_1^{-1} g^{-1} \tag{241}$$

so that $g \in N_G(T)(\mathbb{R})$ and $w_2^{-1} g w_1$ fixes γ. Since γ is strongly regular this implies that $w_2^{-1} g w_1 \in T(\mathbb{R})$ which means that $w_2^{-1} g w_1$ is the trivial element of $W_{\mathbb{C}}$. This says that $w_2 = g w_1$ as elements of $W_{\mathbb{C}}$. Since $g \in N_G(T)(\mathbb{R})$ we see that $g \in W_{\mathbb{R}}$ and the second equality of (230) follows. ☐

2.5.1.4. *Reflex fields and a construction of Kottwitz.* In this appendix we record, for the ease of the reader, the following extension of a classicl construction of Kottwitz (see [Kot84a, Lemma 2.1.2]) to the setting of not necessarily quasi-split groups.

Let us fix a field F and G a reductive group over F. Let $\boldsymbol{\mu}$ be a conjugacy class of cocharacters over $\overline{F}$. Recall that Γ_F acts on the set of conjugacy class of cocharacters of $G_{\overline{F}}$ and we define the *reflex field* of $\boldsymbol{\mu}$, denoted by $E(\boldsymbol{\mu})$ (or just E when $\boldsymbol{\mu}$ is clear from context), to be the fixed field.

Let G^* denote the quasi-split inner form of G over F. Choose an inner twisting $f : \mathbf{G}_{\overline{F}} \to \mathbf{G}^*_{\overline{F}}$ and let us specify that $\sigma \mapsto g_\sigma$ is the $G^{\mathrm{ad}}(\overline{F})$-valued cocycle such that for all $\sigma \in \Gamma_F$ we have that

$$f \circ \sigma_{G_{\overline{F}}} \circ f^{-1} \circ \sigma^{-1}_{G^*_{\overline{F}}} = \mathrm{Inn}(g_\sigma)$$

We then have the following observation:

Lemma 2.5.28. *The reflex field of the $G^*(\overline{F})$ - conjugacy class of cocharacters*

$$f(\boldsymbol{\mu}) := \{f \circ \mu : \mu \in \boldsymbol{\mu}\}$$

is $E(\boldsymbol{\mu})$.

Proof. To see this it suffices to show that for any σ in Γ_F we have that $\sigma \cdot (f \circ \mu)$ is conjugate to $f \circ \mu$ since, by symmetry, the reverse direction will also follow. To see this we merely note that for any $\sigma \in \Gamma$ we have that

$$\begin{aligned}\sigma \cdot (f \circ \mu) &= \sigma_{G^*_{\overline{F}}} \circ f \circ \mu \circ \sigma^{-1}_{\mathbb{G}_{m,\overline{F}}}\\ &= \mathrm{Inn}(g_\sigma^{-1}) \circ f \circ \sigma_{G_{\overline{F}}} \circ \mu \circ \sigma^{-1}_{\mathbb{G}_{m,\overline{\mathbb{Q}}}}\\ &= \mathrm{Inn}(g_\sigma^{-1}) \circ f \circ \mathrm{Inn}(h_\sigma) \circ \mu)\\ &= \mathrm{Inn}(g_\sigma^{-1} f(h_\sigma)) \circ f \circ \mu\end{aligned}$$

where we have used the fact that $\boldsymbol{\mu}$ is Γ_F-stable to obtain the element h_σ. □

It's also clear that if we choose another inner twisting (G^*, f') of G that $f'(\boldsymbol{\mu}) = f(\boldsymbol{\mu})$ since for all μ in $\boldsymbol{\mu}$ we have that $f \circ \mu$ is conjugate to $f' \circ \mu$ by definition. Thus, we see that this conjugacy class of cocharacters of $G^*_{\overline{F}}$ depends only on G^* and not on the inner twist (G, f). Thus, we denote this conjugacy class $\boldsymbol{\mu}^*$. By the above we have that $E(\boldsymbol{\mu}) = E(\boldsymbol{\mu}^*)$. Also note that for any $\boldsymbol{\mu}$ we have that $(-\boldsymbol{\mu})^* = -\boldsymbol{\mu}^*$.

Let us now choose a rational Borel-torus pair (B, T) of G^* over F. To $\boldsymbol{\mu}^*$ we associate a $\overline{\mathbb{Q}_\ell}$-representation $r_{\boldsymbol{\mu}}$ of $\widehat{\mathbf{G}^*} \rtimes W_{E(\boldsymbol{\mu}^*)}$ where $W_{E(\boldsymbol{\mu})^*}$ acts on $\widehat{\mathbf{G}^*}$ via the pair (B, T). To do this note that since $\mathbf{G}^*$ is quasi-split we have that $\boldsymbol{\mu}$ is actually defined over $E(\boldsymbol{\mu})$ (see [Kot84a, Lemma 1.1.3]). Let μ be the unique B-dominant representative of $\boldsymbol{\mu}^*$ defined over $E(\boldsymbol{\mu}^*)$. Let $V(\mu)$ be the irreducible $\overline{\mathbb{Q}_\ell}$-representation with highest weight μ and then define

$$r_{\boldsymbol{\mu}^*} : \widehat{\mathbf{G}^*} \rtimes W_{E(\boldsymbol{\mu}^*)} \to \mathrm{GL}(V(\mu))$$

to be such that its restriction to $\widehat{\mathbf{G}^*}$ is the usual action and such that the action of $W_{E(\boldsymbol{\mu}^*)}$ on the weight space $V_\mu \subseteq V(\mu)$ is trivial. The existence of such a representation is precisely [Kot84a, Lemma 2.1.2].

Note though that there is an isomorphism

$$\widehat{G^*} \rtimes W_{E(\boldsymbol{\mu}^*)} \cong \widehat{G} \rtimes W_{E(\boldsymbol{\mu})} \tag{242}$$

unique up to inner automorphism. Thus, associated to $r_{\boldsymbol{\mu}^*}$ is a representation

$$\widehat{G} \rtimes W_{E(\boldsymbol{\mu})} \xrightarrow{\approx} \widehat{G^*} \rtimes W_{E(\boldsymbol{\mu}^*)} \to \mathrm{GL}(V(\mu)) \tag{243}$$

unique up to isomorphism which we denote $r_{\boldsymbol{\mu}}$. Of course, up to isomorphism, this representation doesn't depend on the choice of (B,T) and, in particular, depends only on $\boldsymbol{\mu}$ not the choice of an element $\mu \in \boldsymbol{\mu}$. Thus, we will often times write $r_{\boldsymbol{\mu}}$ as a representation $\widehat{G} \rtimes W_{E(\boldsymbol{\mu})} \to \mathrm{GL}(V(\boldsymbol{\mu}))$.

We now record some results in the case of F being a global field. To begin we note thatfor any place v of F and any choice of embedding $\overline{F} \hookrightarrow \overline{F_v}$ one gets an induced conjugacy class $\boldsymbol{\mu}_v$ of cocharacters of $G_{\overline{F_v}}$. The following claim is then simple:

Lemma 2.5.29. *There is an equality of fields $E(\boldsymbol{\mu})_w = E(\boldsymbol{\mu}_v)$.*

In particular, we see the following:

Corollary 2.5.30. *Let v be an element of $S^{\mathrm{ur}}(\mathbf{G})$. Then, $E(\boldsymbol{\mu})_w/F_v$ is unramified.*

Proof. Note that by Lemma 2.5.29 it suffices to show that $E(\boldsymbol{\mu}_v)/F_v$ is unramified. But, since $\mathbf{G}_v$ splits over F_v^{ur} we evidently have an inclusion $E(\boldsymbol{\mu}_v) \subseteq F_v^{\mathrm{ur}}$ from where the claim follows. $\square$

The following lemma is equally as simple as Lemma 2.5.29:

Lemma 2.5.31. *There is an equality (up to isomorphism) of representations*

$$r_{\boldsymbol{\mu}} \mid_{\widehat{G} \rtimes W_{E(\boldsymbol{\mu})_w}} = r_{\boldsymbol{\mu}_v} \tag{244}$$

2.5.1.5. *The Kottwitz group.* We record in this section, for the convenience of the reader, the basic definitions and properties we would like to use concerning the *Kottwitz group* associated to a local or global field F.

To make sense of the definition of this group, it is useful to first recall the following basic lemma:

Lemma 2.5.32. *Let F be a field of characteristic 0 and let G be a connected reductive group over F. Let H be any connected reductive subgroup of G of the same rank. The choice of a maximal torus T in H induces a natural Γ_F-equivariant inclusion $Z(\widehat{G}) \subseteq Z(\widehat{H})$, and this embedding is, in fact, independent of T.*

Remark 2.5.33. See [Bor79, §2] for a recollection of dual groups and their associated Galois actions.

Proof. (Lemma 2.5.32) Let us first consider the case when H is a maximal torus defined over F, in which case we will take T to be equal to H. Then, essentially by definition of the dual group, there exists an embedding $\widehat{H} \hookrightarrow \widehat{G}$ of complex algebraic groups identifying the image of $\widehat{H}$ with a maximal torus of $\widehat{G}$. In particular, we see that the image of $\widehat{H}$ contains $Z(\widehat{G})$. Let us denote by Z' the preimage of $Z(\widehat{G})$ in $\widehat{H}$. We then claim that the isomorphism of complex algebraic groups $Z' \to Z(\widehat{G})$ is actually Γ-equivariant.

To see this, note that induced map of root datum from the morphism $\widehat{H} \hookrightarrow \widehat{G}$ can be identified with the natural inclusion

$$(X_*(H), 0, X^*(H), 0) \hookrightarrow (X_*(H), \Phi^\vee(G), X^*(H), \Phi(G)) \tag{245}$$

which is patently Γ-equivariant. Thus, we see that for all $\gamma \in \Gamma$ the action of γ on $\widehat{H}$ and the action of γ on the image of $\widehat{H}$ in $\widehat{G}$ differ by inner automorphisms of G. In particular, it follows that the map $Z' \to \widehat{G}$ is Γ-equivariant, and thus is the map $Z' \to Z(\widehat{G})$, as claimed.

The desired Γ-equivariant embedding $Z(\widehat{G}) \hookrightarrow Z(\widehat{H}) = \widehat{H}$ can thus be taken to be the inverse of the induced Γ-equivariant isomorphism $Z' \xrightarrow{\approx} Z(\widehat{G})$ discussed above.

Suppose now that H is an arbitrary reductive subgroup of G of the same rank. Let us fix a maximal torus T of H. From the initial case when H was assumed to be a torus, we see that we obtain separate Γ-equivariant embeddings $Z(\widehat{G}) \hookrightarrow \widehat{T}$ and $Z(\widehat{H}) \hookrightarrow \widehat{T}$. But, since $Z(\widehat{G})$ is clearly contained in $Z(\widehat{H})$ as complex algebraic subgroups of $\widehat{T}$ we thus obtain a Γ-equivariant embedding $Z(\widehat{G}) \hookrightarrow Z(\widehat{H})$ as desired.

Finally, observe that changing the maximal torus T to T' doesn't affect the embedding $Z(\widehat{G}) \hookrightarrow Z(\widehat{H})$ since $\widehat{T}$ and $\widehat{T'}$ are conjugate in $\widehat{H}$ and this conjugation doesn't alter the embedding $Z(\widehat{G}) \hookrightarrow Z(\widehat{H})$. $\square$

Suppose now that F is a number field and $\mathbf{G}$ is a reductive group over F. Assume further that $\mathbf{H}$ is a reductive subgroup of G of the same rank. Clearly then for all places v of F we have that $\mathbf{H}_v$ is a reductive subgroup of $\mathbf{G}_v$ of the same rank. Thus, from Lemma 2.5.32 we obtain a Γ_F-equivariant inclusion $Z(\widehat{\mathbf{G}}) \hookrightarrow Z(\widehat{\mathbf{H}})$ and Γ_{F_v}-equivariant inclusions $Z(\widehat{\mathbf{G}_v}) \hookrightarrow Z(\widehat{\mathbf{H}_v})$ for all places v of F. Given our particular embeddings of $\overline{F} \hookrightarrow \overline{F_v}$ we obtain a diagram

$$\begin{tikzcd} Z(\widehat{\mathbf{G}_v}) \arrow[r] \arrow[d, "\wr"'] & Z(\widehat{\mathbf{H}_v}) \arrow[d, "\wr"] \\ Z(\widehat{\mathbf{G}}) \arrow[r] & Z(\widehat{\mathbf{H}}) \end{tikzcd} \tag{246}$$

where the vertical maps are isomorphisms of complex Lie groups equivariant for the Γ_v action where $Z(\widehat{\mathbf{G}})$ is endowed with the Γ_v action inherited from the inclusion $\Gamma_v \subseteq \Gamma$ induced by our choice of embedding $\overline{F} \hookrightarrow \overline{F_v}$.

From the maps $Z(\widehat{\mathbf{G}}) \to Z(\widehat{\mathbf{H}})$ of Γ-modules obtain a short exact sequence of Γ-modules

$$0 \to Z(\widehat{\mathbf{G}}) \to Z(\widehat{\mathbf{H}}) \to Z(\widehat{\mathbf{H}})/Z(\widehat{\mathbf{G}}) \to 0 \tag{247}$$

Moreover, for each place v of F we obtain from the map $Z(\widehat{\mathbf{G}_v}) \to Z(\widehat{\mathbf{H}_v})$ of Γ_{F_v}-modules we obtain a short exact sequences of Γ_{F_v}-modules

$$0 \to Z(\widehat{\mathbf{G}_v}) \to Z(\widehat{\mathbf{H}_v}) \to Z(\widehat{\mathbf{H}_v})/Z(\widehat{\mathbf{G}_v}) \to 0 \tag{248}$$

with similar compatibilities as in (246).

We further denote by

$$\mathrm{inv} : Z(\widehat{\mathbf{H}})/Z(\widehat{\mathbf{G}}) \to H^1(\Gamma, Z(\widehat{\mathbf{G}})) \tag{249}$$

and

$$\mathrm{inv}_v : Z(\widehat{\mathbf{H}_v})/Z(\widehat{\mathbf{G}_v}) \to H^1(\Gamma_v, Z(\widehat{\mathbf{G}_v})) \tag{250}$$

the connecting homomorphisms associated to (247) and (248) respectively. Under the aforementioned Γ_v-equivariant local-global identifications it's easy to see that inv_v can be identified with with the composition of inv and the localization map $H^1(\Gamma, Z(\widehat{\mathbf{G}})) \to H^1(\Gamma_v, Z(\widehat{\mathbf{G}})$.

With this setup, we can define the Kottwitz group as follows:

Definition 2.5.34. *Let F be a number field and let $\mathbf{G}$ be a reductive group over F. Let $\mathbf{H}$ be a reductive subgroup of G of the same rank. Define the* Kottwitz group $\mathfrak{K}(\mathbf{G}, \mathbf{H}, F)$ *as follows:*

$$\mathfrak{K}(\mathbf{G}, \mathbf{H}, F) := \left\{\alpha \in (Z(\widehat{\mathbf{H}})/Z(\widehat{\mathbf{G}}))^{\Gamma} : \mathrm{inv}(\alpha) \in \ker^1(\Gamma, Z(\widehat{\mathbf{G}}))\right\} \tag{251}$$

If $\gamma \in \mathbf{G}(F)$ is semisimple, we denote by $\mathfrak{K}(I_\gamma/F)$ the group $\mathfrak{K}(\mathbf{G}, I_\gamma, F)$.

It will be helpful later to note that our definition of $\mathfrak{K}(\mathbf{G}, \mathbf{H}, F)$ differs from the definition given in [Kot84b] and [Kot90] where, instead, Kottwitz uses the group $\pi_0(\mathfrak{K}(\mathbf{G}, \mathbf{H}, F))$ where $\mathfrak{K}(\mathbf{G}, \mathbf{H}, F)$ is given the Hausdorff topology inheirted from the complex Lie group $Z(\widehat{\mathbf{H}})$.

The definition we have chosen to use is more in line with the later work of Kottwitz and other authors (e.g. see [Shi10]). That said, since we would like to make use of the material in [Kot84b] and [Kot86b] we would like to verify that our two definitions agree when $\mathbf{G}^{\mathrm{ad}}$ is F-anisotropic.

Namely, we have the following:

Lemma 2.5.35. *Let F be a number field and $\mathbf{G}$ a reductive group over F such that $\mathbf{G}^{\mathrm{ad}}$ is F-anisotropic. If $\mathbf{H}$ is a connected reductive subgroup of $\mathbf{G}$ of the same rank, then $\mathfrak{K}(\mathbf{G},\mathbf{H},F)$ is finite and, in particular, is equal to $\pi_0(\mathfrak{K}(\mathbf{G},\mathbf{H},F))$.*

To prove this, it will be helpful to make the following basic observation:

Lemma 2.5.36. *Let F be a number field and $\mathbf{G}$ a reductive group over F. Let $\mathbf{H}$ be a reductive subgroup of G of the same rank. Let $\mathbf{T}$ be a maximal torus of $\mathbf{H}$. Then, there is a natural inclusion*

$$\mathfrak{K}(\mathbf{G},\mathbf{H},F) \hookrightarrow \mathfrak{K}(\mathbf{G},\mathbf{T},F) \tag{252}$$

Proof. Let us merely observe that, by the proof of Lemma 2.5.32, we have a Γ-equivariant inclusions

$$Z(\widehat{\mathbf{G}}) \hookrightarrow Z(\widehat{\mathbf{H}}) \hookrightarrow \widehat{\mathbf{T}} \tag{253}$$

which gives rise to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Z(\widehat{\mathbf{G}})^{\Gamma} \arrow[r] \arrow[d, "\mathrm{id}"'] & Z(\widehat{\mathbf{H}})^{\Gamma} \arrow[r] \arrow[d, hook] & (Z(\widehat{\mathbf{H}})/Z(\widehat{\mathbf{G}}))^{\Gamma} \arrow[r] \arrow[d, hook] & H^1(\Gamma, Z(\widehat{\mathbf{G}})) \arrow[d] \\
0 \arrow[r] & Z(\widehat{\mathbf{G}})^{\Gamma} \arrow[r] & \widehat{\mathbf{T}}^{\Gamma} \arrow[r] & (\widehat{\mathbf{T}}/Z(\widehat{\mathbf{G}}))^{\Gamma} \arrow[r] & H^1(\Gamma, Z(\widehat{\mathbf{G}}))
\end{tikzcd}$$

from where it's clear that we get the desired inclusion $\mathfrak{K}(\mathbf{G},\mathbf{H},F) \hookrightarrow \mathfrak{K}(\mathbf{G},\mathbf{T},F)$. ∎

From Lemma 2.5.36 the proof of Lemma 2.5.35 follows immediately from the following:

Lemma 2.5.37. *Let F be a number field and $\mathbf{G}$ a reductive group over F. Let $\mathbf{T}$ be a torus in $\mathbf{G}$ containing $Z(\mathbf{G})$ which is elliptic. Then $(\widehat{\mathbf{T}}/Z(\widehat{\mathbf{G}}))^{\Gamma}$ is finite.*

Proof. Let us begin by showing that for any torus $\mathbf{S}$ over F there is a natural identification of $\widehat{\mathbf{S}}^{\Gamma}$ and $D(\mathbb{C})$ where D is the diagonalizable $\mathbb{C}$-group with character lattice $X_*(\mathbf{S})_{\Gamma_F}$ (the Γ_F-coinvariants of $X_*(\mathbf{S})$).

Now, we write $\mathbf{G}^{\mathrm{Sc}}$ to denote the simply connected cover of $\mathbf{G}^{\mathrm{ad}}$. Then denote by $\mathbf{T}^{\mathrm{ad}}$ the projection of $\mathbf{T}$ to $\mathbf{G}^{\mathrm{ad}}$ and $\mathbf{T}^{\mathrm{sc}}$ the pre-image of $\mathbf{T}^{\mathrm{ad}}$ under the surjection $\mathbf{G}^{\mathrm{sc}} \to \mathbf{G}^{\mathrm{ad}}$. Then $\mathbf{T}^{\mathrm{ad}} = \mathbf{T}/Z(\mathbf{G})$ and the projection $\mathbf{T}^{\mathrm{sc}} \to \mathbf{T}^{\mathrm{ad}}$ is an isogeny so that we have a Γ_F-equivariant isomorphism

$$X_*(\mathbf{T}^{\mathrm{ad}})_{\mathbb{Q}} \cong X_*(\mathbf{T}^{\mathrm{sc}})_{\mathbb{Q}}. \tag{254}$$

Taking coinvariants and applying the previous paragraph as well as basic theory of actions of finite groups on $\mathbb{Q}$-spaces, we get

$$(255)\quad X^*(\widehat{\mathbf{T}^{\mathrm{sc}}}^{\Gamma_F})_{\mathbb{Q}} = X_*(\mathbf{T}^{\mathrm{sc}})_{\Gamma} \otimes \mathbb{Q} \cong X_*(\mathbf{T}^{\mathrm{ad}})_{\Gamma_F} \otimes \mathbb{Q} = X_*(\mathbf{T}^{\mathrm{ad}})^{\Gamma_F}_{\mathbb{Q}}.$$

Now, $X_*(\mathbf{T}^{\mathrm{ad}})^{\Gamma_F}_{\mathbb{Q}} = 0$ since $\mathbf{T}^{\mathrm{ad}}$ is anisotropic. Then, a diagonalizable group D is finite if and only if $X^*(D)_{\mathbb{Q}}$ is trivial which implies that $\widehat{\mathbf{T}^{\mathrm{sc}}}^{\Gamma_F}$ is finite. But $\widehat{\mathbf{T}^{\mathrm{sc}}}^{\Gamma_F} = (\widehat{\mathbf{T}}^{\mathrm{ad}})^{\Gamma_F} = (\widehat{\mathbf{T}}/Z(\widehat{\mathbf{G}}))^{\Gamma_F}$ so this is the desired result. □

2.5.1.6. *Preservation of properties under Weil restriction.* In this appendix we merely collect the verification that several properties of algebraic groups used in this note are preserved under Weil restriction:

Lemma 2.5.38. *Let F/F' be a finite extension. Let H be a reductive gorup over a field F' such that H^{ad} is F'-anisotropic. Then, $(\mathsf{Res}_{F/\mathbb{Q}}H)^{\mathrm{ad}}$ is F-anisotropic.*

Proof. The claim is trivial given Lemma 2.5.83 since we have the equality $(\mathsf{Res}_{F/\mathbb{Q}}H)(F) = H(F')$. □

Lemma 2.5.39. *Let F'/F be an extension of number fields. Let $\mathbf{H}$ be a reductive group over F' which satisfies the Hasse principle. Then, $\mathsf{Res}_{F'/F}\mathbf{H}$ satisfies the Hasse principle.*

Proof. Begin by noting that we have the following commutative diagram

$$(256)\quad \begin{tikzcd} H^1(F',\mathbf{H}) \arrow[r] \arrow[dd, "(1)"', "\wr"] & \prod\limits_w H^1(F'_w,\mathbf{H}) \arrow[d, "||"] \\ & \prod\limits_v \prod\limits_w H^1(\mathbf{F}'_w, H) \arrow[d, "(2)"', "\wr"] \\ H^1(F,\mathsf{Res}_{F'/F}\mathbf{H}) \arrow[r] & \prod_v H^1(F_v,\mathsf{Res}_{F'/F}\mathbf{H}) \end{tikzcd}$$

The isomorphism in arrow (1) is just Shapiro's lemma. To see the isomorphism in arrow (2) we proceed as follows:

$$\begin{aligned} H^1(F_v, \mathsf{Res}_{F'/F}\mathbf{H}) &= H^1_{\text{ét}}(F_v, (\mathsf{Res}_{F'/F}\mathbf{H})_{F_v}) \\ &\cong H^1_{\text{ét}}(F_v, \mathsf{Res}_{F'_v/F_v}\mathbf{H}_{F'_v}) \\ &\cong H^1_{\text{ét}}\left(F_v, \prod_{w|v} \mathsf{Res}_{F'_w/F_v}\mathbf{H}_{F'_w}\right) \\ &\cong \prod_{w|v} H^1_{\text{ét}}(F_v, \mathsf{Res}_{F'_w/F_v}\mathbf{H}_{F'_w}) \\ &\underset{(3)}{\cong} \prod_{w|v} H^1_{\text{ét}}(F'_w, H_{F_w}) \\ &= \prod_{w|v} H^1(F'_w, \mathbf{H}) \end{aligned} \tag{257}$$

where, obviously, the isomorphism labeled (3) is just Shapiro's lemma.

The commutativity of this diagram, and the fact that the vertical maps are isomorphisms, gives an isomorphism

$$\ker^1(F', \mathbf{H}) \cong \ker^1(F, \mathsf{Res}_{F'/F}\mathbf{H}) \tag{258}$$

from where the conclusion follows. □

Lemma 2.5.40. *Let F'/F be an extension of number fields. Let $\mathbf{H}$ be a reductive F'-group such that $\mathbf{H}^{\mathrm{ad}}$ is F'-anisotropic, $\mathbf{H}$ satisfies the Hasse principle, and $\mathbf{H}$ has no relevant global endoscopy. Then, $\mathsf{Res}_{F'/F}\mathbf{H}$ has no relevant global endoscopy.*

Proof. By Proposition 2.2.29 it suffices to show that for all maximal F'-tori $\mathbf{T}' \subseteq \mathsf{Res}_{F'/F}\mathbf{H}'$ that the equality

$$Z(\widehat{\mathsf{Res}_{F'/F}\mathbf{H}})^{\Gamma_F} = \widehat{\mathbf{T}'}^{\Gamma_F} \tag{259}$$

holds. Note though that $\mathbf{T}' = \mathsf{Res}_{F'/F}\mathbf{T}$ for some maximal torus $\mathbf{T}$ in H (e.g. see [CGP15, Proposition A.5.15 (2)]). Note now though that since

$$\widehat{\mathbf{T}'} \cong \widehat{\mathbf{T}}^{[F':F]} \tag{260}$$

with Γ_F acting through its quotient $\mathrm{Gal}(F'/F)$ which acts by permutation of the factors, that

$$\widehat{\mathbf{T}'}^{\Gamma_F} = \widehat{\mathbf{T}}^{\Gamma_{F'}} \tag{261}$$

and similarly

$$Z(\widehat{\mathsf{Res}_{F'/F}\mathbf{H}})^{\Gamma_F} = Z(\widehat{\mathbf{H}})^{\Gamma_{F'}} \tag{262}$$

from where the equality follows from Lemma 2.2.29 and the fact that $\mathbf{H}$ has no relevant global endoscopy. ☐

Lemma 2.5.41. *Let F'/F be an extension of fields. Let H be a areductive group over a field F' with H^{der} simply connected. Then, $\mathsf{Res}_{F'/F}H$ has simply connected derived subgroup.*

Proof. Begin by noting that $(\mathsf{Res}_{F'/F}H)^{\text{der}} \cong \mathsf{Res}_{F'/F}H^{\text{der}}$. Note though that we can check derived subgroup over algebraic closure. But

$$(\mathsf{Res}_{F'/F}H^{\text{der}})_{\overline{F}} \cong (H^{\text{der}}_{\overline{F}})^{[F':F]} \tag{263}$$

Since we're in characteristic zero, the fundamental group splits across direct products and so

$$\pi_1^{\text{ét}}\left((H^{\text{der}}_{\overline{F}})^{[F':F]}, \overline{x}\right) \cong \pi_1^{\text{ét}}((H^{\text{der}}_{\overline{F}}), \overline{x})^{[F':F]} = 0 \tag{264}$$

as desired. ☐

2.5.1.7. *Some lemmas about transfer.* In this subsection we establish several results concerning transferability of conjugacy classes. We begin with the following observation:

Lemma 2.5.42. *Let F be a field of characteristic 0 and let G be a quasi-split group over F. Let $\psi : G_{\overline{F}} \to G'_{\overline{F}}$ be an inner twist. Let T be a torus of G which transfers to G' (in the sense of [Kal16, §3.2]) then for any $\gamma \in T(F)$ the conjugacy class of γ transfers to a conjugacy class in $G'(F)$ (in the sense of [Shi10, §2.3]).*

Proof. By definition there exists some $g \in G(\overline{F})$ such that the map $\psi \circ \mathrm{Int}(g)_{|T_{\overline{F}}} : T_{\overline{F}} \to G'_{\overline{F}}$ is defined over F. Let T' be the image of T under the descent of $\psi \circ \mathrm{Int}(g)_{|T_{\overline{F}}}$ to F. Note then that taking $\mathbb{T}_H := T_{\overline{F}}$ and $\mathbb{T} := T'_{\overline{F}}$ as in [Shi10, §2.3] we have that θ can be taken to be $\mathrm{Int}(\psi(g)) \circ \psi$. Then, by definition, γ transfers to a conjugacy class in $G'(F)$ if and only if $\theta(g) \in T'(\overline{F})$ has an element of its associated $G(\overline{F})$-conjugacy class defined over F. But, evidently we can take the image of γ under the descent of $\psi \circ \mathrm{Int}(g)_{|T_{\overline{F}}}$ to F. The conclusion follows. ☐

One thing that follows immediately from this is the following:

Corollary 2.5.43. *Let F be a p-adic local field let G be a quasi-split group over F. Let $\psi : G_{\overline{F}} \to G'_{\overline{F}}$ be an inner twist. Let T be an elliptic maximal torus of G. Then, any element $\gamma \in T(F)$ transfers to a conjugacy class in $G'(F)$.*

Proof. This follows immediately by combining Lemma 2.5.42 and [Kot86b, §10] (see also [Kal16, Lemma 3.2.1] ☐

2.5.2. **Appendix 2: The trace formula in the anisotropic case and its pseudo-stabilization.** In this appendix we record, mostly for the convenience of the reader and to set notation, the Arthur-Selberg trace formula in the compact case or, said differently, for a reductive group $\mathbf{G}$ over $\mathbb{Q}$ such that $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic (which is a blanket assumption throughout this assumption assuming throughout this section unless stated otherwise). We will often times assume that $\mathbf{G}^{\mathrm{der}}$ is simply connected to simplify the discussion, but this is rarely strictly necessary.

We then write out the pseudo-stabilization of this trace formula under the assumption that $\mathbf{G}$ has no relevant global elliptic endoscopy (in the sense of §2.2.5).

2.5.2.1. *The trace formula in the compact case.* In this subsection we recall the Arthur-Selberg trace formula in the case when $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic. For the beginning part of this section, one can put no restrictions on $\mathbf{G}$ other than it being reductive.

We begin with the following lemma that will be continually useful in the following:

Lemma 2.5.44. *Let $\mathbf{G}$ be a reductive group over $\mathbb{Q}$. Then, the group $\mathbf{G}(\mathbb{A})$ is an internal direct product of $A_{\mathbf{G}}(\mathbb{R})^0$ and $\mathbf{G}(\mathbb{A})^1$. In particular the natural map*

$$[\mathbf{G}] \to \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/A_{\mathbf{G}}(\mathbb{R})^0 \tag{265}$$

is an isomorphism of topological measure spaces.

Before we begin the proof, Let us note that we will often times shorten the notation for an element $\mathbf{G}(\mathbb{Q})x$ in $[\mathbf{G}]$ to the notation $[x]$.

Proof. (Lemma 2.5.44) Since $A_{\mathbf{G}}(\mathbb{R})^0$ and $\mathbf{G}(\mathbb{A})^1$ are normal we need to show that the equality $A_{\mathbf{G}}(\mathbb{R})^0\mathbf{G}(\mathbb{A})^1 = \mathbf{G}(\mathbb{A})$ holds and $A_{\mathbf{G}}(\mathbb{R})^0 \cap \mathbf{G}(\mathbb{A})^1$ is trivial. This latter fact is clear. The former follows easily from the decomposition $\mathbf{G} = \mathbf{G}^{\mathrm{der}}Z(\mathbf{G})$ which shows that the natural map $X^*(\mathbf{G}) \to X^*(A_{\mathbf{G}})$ is injective with finite cokernel. The second claim readily follows. □

Because of this lemma we will conflate $[\mathbf{G}]$ with $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/A_{\mathbf{G}}(\mathbb{R})^0$ and, in particular, call this latter topological measure space (with the measure induced from the Haar measure on $\mathbf{G}(\mathbb{A})$) the adelic quotient.

Let us now set up some of the necessary notation. Namely, let us fix a smooth character $\chi : A_{\mathbf{G}}(\mathbb{R})^+ \to \mathbb{C}$ and let us make the following definition:

Definition 2.5.45. *We denote by* $L^2_\chi(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$ *the space of functions* $\phi : \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}) \to \mathbb{C}$ *such that* $\phi(ax) = \chi(a)\phi(x)$ *for all* $a \in A_{\mathbf{G}}(\mathbb{R})^0$ *and such that* $\phi\chi^{-1}$ *is square-integrable on* $[\mathbf{G}]$.

Note that combining the fact that $\mathbf{G}(\mathbb{Q}) \cap A_{\mathbf{G}}(\mathbb{R})^0$ is trivial with Lemma 2.5.44 we see that every element $\alpha \in \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ can be written in the form $\alpha = \mathbf{G}(\mathbb{Q})ax$ with $a \in A_{\mathbf{G}}(\mathbb{R})^0$ and $x \in \mathbf{G}(\mathbb{A})^1$ and, moreover, a and $\mathbf{G}(\mathbb{Q})x$ are unique. In particular, the function $(\phi\chi^{-1})(\alpha) := \chi^{-1}(a)\phi(\mathbf{G}(\mathbb{Q})x)$ makes sense as a function $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}) \to \mathbb{C}$. Moreover, it's clear that since $\phi\chi^{-1}$ is $A_{\mathbf{G}}(\mathbb{R})^0$ invariant it descends to a function $[\mathbf{G}] \to \mathbb{C}$ which we also denote $\phi\chi^{-1}$.

Let us now set the following notation:

Definition 2.5.46. *We denote by* $\mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$ *the set of* $\mathbb{C}$*-linear combinations of functions* $f = f_\infty f^\infty : \mathbf{G}(\mathbb{A}) \to \mathbb{C}$ *where:*

(1) $f^\infty : \mathbf{G}(\mathbb{A}_f) \to \mathbb{C}$ *is locally constant and compactly supported.*
(2) $f_\infty : \mathbf{G}(\mathbb{R}) \to \mathbb{C}$ *is smooth, satisfies* $f(ax) = \chi(a)^{-1}f(x)$ *for all* $a \in A_{\mathbf{G}}(\mathbb{R})^0$, *and for which* $f\chi$ *is compactly supported as a function on* $\mathbf{G}(\mathbb{R})/A_{\mathbf{G}}(\mathbb{R})^0$.

If $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$ note that we get a compactly supported function $f\chi : \mathbf{G}(\mathbb{A})^1 \to \mathbb{C}$ defined by $(f\chi)(ax) := f(x)$ where $a \in A_{\mathbf{G}}(\mathbb{R})^0$ and $x \in \mathbf{G}(\mathbb{A})^1$ (again using Lemma 2.5.44).

We now make a definition of the operators $R_\chi(f)$ and $R(f\chi)$ for an element $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$. Namely:

Definition 2.5.47. *The* right convolution operator $R_\chi(f)$ *on* $L^2_\chi(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$ *is defined by taking* $\phi \in L^2_\chi(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$ *to*

$$R_\chi(f)(\phi)(\mathbf{G}(\mathbb{Q})x) := \int_{\mathbf{G}(\mathbb{A})/A_{\mathbf{G}}(\mathbb{R})^0} f(g)\phi(\mathbf{G}(\mathbb{Q})xg)\,dg \tag{266}$$

which is well-defined since f *and* ϕ *transform by inverse characters and* f *is compactly supported on* $\mathbf{G}(\mathbb{A})/A_{\mathbf{G}}(\mathbb{R})^+$.

We also define the operator $R(f\chi^{-1})$ *on* $L^2([\mathbf{G}])$ *as*

$$R(f\chi)(\psi)([x]) := \int_{\mathbf{G}(\mathbb{A})^1} (f\chi)(g)\psi([xg])\,dg \tag{267}$$

We then have the following elementary observation:

Lemma 2.5.48. *We have a natural isomorphism of* $\mathbb{C}$*-spaces*

$$L^2_\chi(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})) \xrightarrow{\approx} L^2([\mathbf{G}]) : \phi \mapsto \phi\chi^{-1} \tag{268}$$

which is equivariant for the natural $\mathbf{G}(\mathbb{A})^1$*-action on both sides and such that*

$$R_\chi(f)(\phi) = R(f\chi)(\phi\chi^{-1}) \tag{269}$$

Proof. We can define an inverse of the above map by pulling back a function $\phi \in L^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/A_{\mathbf{G}}(\mathbb{R})^+)$ along the quotient map

$$\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}) \to \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/A_{\mathbf{G}}(\mathbb{R})^0, \tag{270}$$

and twisting by χ.

Now, we have

$$R_\chi(f)(\phi)(\mathbf{G}(\mathbb{Q})x) = \int_{\mathbf{G}(\mathbb{A})/A_G(\mathbb{R})^0} f(g)\phi(\mathbf{G}(\mathbb{Q})xg)dg \tag{271}$$

$$= \int_{\mathbf{G}(\mathbb{A})^1} (f\chi)(g)(\phi\chi^{-1})(xg)dg \tag{272}$$

$$= R(f\chi)(\phi\chi^{-1})(x). \tag{273}$$

from where the lemma follows. □

From this point on we assume that $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic and $\mathbf{G}^{\mathrm{der}}$ is simply connected. This has the benefit of implying that $I_\gamma = Z_{\mathbf{G}}(\gamma)$ for all $\gamma \in \mathbf{G}(\mathbb{Q})$ and thus $a(\gamma) = 1$ for all semi-simple $\gamma \in \mathbf{G}(\mathbb{Q})$.

Let us now appeal to the following result which justifies our terminology of calling the situation when $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic the 'compact case':

Theorem 2.5.49 (Borel, Harish-Chandra)**.** *Let $\mathbf{H}$ be a reductive group over $\mathbb{Q}$. Then, the space $[\mathbf{H}]$ is compact if and only if $\mathbf{H}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic.*

Proof. The desired result is contained in [Con12a, §A.5]. Note, in particular, that since $\mathbf{H}$ was assumed reductive that [Con12a, Lemma A.5.2] shows that conditions a) and b) are equivalent to $\mathbf{H}^{\mathrm{ad}}$ being $\mathbb{Q}$-anisotropic. □

Note then that we have the following well-known result:

Theorem 2.5.50. *For any function $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$ the operator $R(f\chi)$ on $L^2([\mathbf{G}])$ is trace class. Moreover, there is a decomposition*

$$L^2([\mathbf{G}]) = \bigoplus_{\pi' \in \Pi(\mathbf{G}(\mathbb{A})^1)} m(\pi')\pi' \tag{274}$$

where $\Pi(\mathbf{G}(\mathbb{A})^1)$ denotes the set of irreducible unitary $\mathbf{G}(\mathbb{A})^1$ - subrepresentations and $m(\pi')$ is some integer (possibly zero).

Proof. This is a classical, and well-known result that follows from easy function analysis since $[\mathbf{G}]$ is compact. For example, see [Whi, §3]. □

From this we deduce the following:

Corollary 2.5.51. *The operator $R_\chi(f)$ on the space $L^2_\chi(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$ is trace class and there is a decomposition*

$$L^2_\chi(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})) = \bigoplus_{\pi\in\Pi_\chi(\mathbf{G}(\mathbb{A}))} m(\pi)\pi \tag{275}$$

where $\Pi_\chi(\mathbf{G}(\mathbb{A}))$ denotes the set of irreducible unitary $\mathbf{G}(\mathbb{A})$ - representations acting by the character χ on $A_{\mathbf{G}}(\mathbb{R})^+$ and $m(\pi)$ is some integer (possibly zero).

Proof. The fact that $R_\chi(f)$ is trace class follows from the map constructed in 2.5.48. The decomposition follows from this map as well as the fact that $A_{\mathbf{G}}(\mathbb{R})^0$ is central in $\mathbf{G}(\mathbb{A})$, hence extending $\mathbf{G}(\mathbb{A})^1$ representations to $\mathbf{G}(\mathbb{A})$ via a character of $A_{\mathbf{G}}(\mathbb{R})^0$ does not affect the decomposition into irreducible representations. $\square$

We would now like to state the Arthur-Selberg trace formula in this context. Before we do this, it's useful to note the following trivial finiteness result.

Lemma 2.5.52. *Let $\mathbf{H}$ be a reductive group over a global field F and let $C \subset \mathbf{H}(\mathbb{A}_F)$ a compact subset. Then $\mathbf{H}(F) \cap C$ is finite.*

Proof. This is essentially trivial. It suffices to show that $\mathbf{H}(F) \cap C$ is discrete and compact. The group $\mathbf{H}(F) \subset \mathbf{H}(\mathbb{A}_F)$ is discrete, therefore so is $\mathbf{H}(F) \cap C$. But, $\mathbf{H}(F)$ is also closed in $\mathbf{H}(\mathbb{A}_F)$ (as any discrete subgroup of a Hausdorff group is closed) and thus $\mathbf{H}(F) \cap C$, being a closed subset of C, is also compact. The conclusion follows. $\square$

From this we deduce the following:

Corollary 2.5.53. *Let $\mathbf{H}$ be a reductive group over a global field F. Suppose that $C \subseteq \mathbf{H}(\mathbb{A})$ is such that its projection to $\mathbf{H}(\mathbb{A})/A_{\mathbf{H}}(\mathbb{R})^0$ is compact. Then, C meets finitely many $\mathbf{H}(F)$-conjugacy classes.*

Proof. Note that since $\mathbf{H}(F)$-conjugacy classes are separated by the natural map $\mathbf{H}(F) \to \mathbf{H}^{\mathrm{ad}}(F)$ it suffices to show that the projection of C along the projection $\mathbf{H}(\mathbb{A}) \to \mathbf{H}^{\mathrm{ad}}(\mathbb{A})$ intersects only finitely many $\mathbf{H}^{\mathrm{ad}}(F)$ conjugacy classes. But, note that C has compact image in $\mathbf{H}^{\mathrm{ad}}(\mathbb{A})$, since the map $\mathbf{H}(\mathbb{A}) \to \mathbf{H}^{\mathrm{ad}}(\mathbb{A})$ factors through the map $\mathbf{H}(\mathbb{A}) \to \mathbf{H}(\mathbb{A})/A_{\mathbf{H}}(\mathbb{R})^0$, and thus the claim follows easily from the previous lemma. $\square$

Let us now assume that $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$. We then define, as in the notation at the beginning of this article, the notion of an orbital integral:

Definition 2.5.54. *Let $\gamma \in \mathbf{G}(\mathbb{Q})$ be given. Then, the* orbital integral *of f relative to γ is the following:*

$$O_\gamma(f) := \int_{I_\gamma(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})} f(g^{-1}\gamma g)\, dg \tag{276}$$

This integral converges because of our assumption that f lies in the set $\mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$ (and, in particular, has compact support modulo $A_{\mathbf{G}}(\mathbb{R})^0$).

Let us also note that $[I_\gamma]$ is compact since I_γ, being a closed subgroup of $\mathbf{G}$, also satisfies $I_\gamma/Z(I_\gamma)$ is $\mathbb{Q}$-anisotropic. Thus, $v_\gamma := \mathrm{vol}([I_\gamma])$, which is equal (by definition) to $\tau(I_\gamma)$, is finite. Note that both $O_\gamma(f)$ and $\mathrm{vol}([I_\gamma])$ only depend on the conjugacy class $\{\gamma\}$ in $\mathbf{G}(\mathbb{Q})$.

Definition 2.5.55. *For $(\pi, V) \in \Pi_\chi(\mathbf{G}(\mathbb{A}))$ and $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$, we define the* trace $\mathrm{tr}(f|\pi)$ *to be the trace of the operator $\pi(f)$ on V given by*

$$\pi(f) := v \mapsto \int_{\mathbf{G}(\mathbb{A})/A_{\mathbf{G}}(\mathbb{R})^0} f(g)\pi(g)v dg. \tag{277}$$

Let us note that any element of $\Pi_\chi(\mathbf{G}(\mathbb{A}))$ is admissible (as follows from Harish-Chandra's finiteness results as in [BJ79, THeorem 1.7]), and thus this trace is a well-defined complex number.

Before we finally state the trace formula, we record the following fact implicitly used in the sequel:

Lemma 2.5.56. *Let $\mathbf{H}$ be a reductive group over $\mathbb{Q}$. Suppose that γ is an elliptic element of $\mathbf{H}(\mathbb{Q})$. Then $I_\gamma(\mathbb{A})^1 = I_\gamma(\mathbb{A}) \cap \mathbf{H}(\mathbb{A})^1$.*

Proof. First note that we really do need the assumption that γ is elliptic as the example in [AEK05, §4, pg20] indicates.

To prove the lemma, we first show that $X^*_{\mathbb{Q}}(\mathbf{H})_{\mathbb{Q}} = X^*_{\mathbb{Q}}(I_\gamma)_{\mathbb{Q}}$. Indeed, we have isogenies

$$Z(\mathbf{H}) \to \mathbf{H}^{\mathrm{ab}}, \qquad Z(I_\gamma) \to I_\gamma^{\mathrm{ab}} \tag{278}$$

and hence isomorphisms

$$X^*_{\mathbb{Q}}(Z(\mathbf{H}))_{\mathbb{Q}} \cong X^*_{\mathbb{Q}}(\mathbf{H})_{\mathbb{Q}}, \qquad X^*_{\mathbb{Q}}(Z(I_\gamma))_{\mathbb{Q}} \cong X^*_{\mathbb{Q}}(I_\gamma)_{\mathbb{Q}} \tag{279}$$

Additionally, since γ is elliptic, we have

$$X^*_{\mathbb{Q}}(Z(I_\gamma)) = X^*_{\mathbb{Q}}(Z(\mathbf{H})) \tag{280}$$

Putting these isomorphisms together, gives the desired equality.

Now, we then have

$$I_\gamma(\mathbb{A})^1 := \{h \in I_\gamma(\mathbb{A}) : |\chi(h)| = 1\, \forall \chi \in X^*_{\mathbb{Q}}(I_\gamma)_{\mathbb{Q}}\} \tag{281}$$
$$= \{h \in I_\gamma(\mathbb{A}) : |\chi(h)| = 1\, \forall \chi \in X^*_{\mathbb{Q}}(\mathbf{H})_{\mathbb{Q}}\} \tag{282}$$
$$= I_\gamma(\mathbb{A}) \cap \mathbf{H}(\mathbb{A})^1 \tag{283}$$

as desired. □

We then have the following:

Theorem 2.5.57. *Assume that $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic. Then, for any function $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$ we have an equality*

$$\sum_{\{\gamma\} \in \{\mathbf{G}\}^{\mathrm{s.s.}}} v_\gamma O_\gamma(f) = \mathrm{tr}(R_\chi(f)) \tag{284}$$

Let us note that by Corollary 2.5.53 the sum on the left-hand side of (284) is a finite sum, and thus is convergent. The right-hand side of (284) is convergent since $R_\chi(f)$ is trace class by Corollary 2.5.51.

Proof. (Theorem 2.5.57) This follows from the discussion in [AEK05, §1.1]. Namely, from the discussion therein, since $[\mathbf{G}]$ is compact we get an equality of $\mathrm{tr}(R(f\chi))$ with

$$\sum_{\{\gamma\} \in \{\mathbf{G}\}^{\mathrm{s.s.}}} \mathrm{vol}(I_\gamma(\mathbb{Q}) \backslash I(\mathbb{A})^1_\gamma) \int_{I(\mathbb{A})^1_\gamma \backslash \mathbf{G}(\mathbb{A})^1} (f\chi)(g^{-1}\gamma g)\, dg \tag{285}$$

But, from Lemma 2.5.48 we know that $\mathrm{tr}(R_\chi(f)) = \mathrm{tr}(R(f\chi))$. Moreover, it's easy to see that (285) agrees with the left hand side of (284) for $f\chi$ in place of f with the only subtle point being the contents of Lemma 2.5.56. The conclusion follows. □

Finally, we use Corollary 2.5.51 to deduce:

Corollary 2.5.58. *Assume that $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic. Then, for any $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$ we have an equality*

$$\sum_{\{\gamma\} \in \{\mathbf{G}\}^{\mathrm{s.s.}}} v_\gamma O_\gamma(f) = \sum_{\pi \in \Pi_\chi(\mathbf{G})} m(\pi)\, \mathrm{tr}(f \mid \pi) \tag{286}$$

where $\Pi_\chi(\mathbf{G})$ and $m(\pi)$ are as in Corollary 2.5.51.

2.5.2.2. *Pseudo-stabilization.* Our goal is now to rewrite Corollary 2.5.58 in terms of stable orbital integrals. Namely, we aim to prove the following:

Proposition 2.5.59. *Suppose that $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$ anisotropic and $\mathbf{G}$ has no relevant global elliptic endoscopy (in the sense of §2.2.5). Let $f \in \mathscr{H}(\mathbf{G}(\mathbb{A}), \chi^{-1})$. Then,*

$$\tau(\mathbf{G}) \sum_{\{\gamma\} \in \{\mathbf{G}\}^{\text{s.s.}}} SO_\gamma(f) = \sum_{\pi \in \Pi_\chi(\mathbf{G})} m(\pi) \operatorname{tr}(f \mid \pi) \tag{287}$$

where $m(\pi)$ is as in Corollary 2.5.58.

To prove this, we will manipulate the left hand side of (286) into the left hand side of (287). We will mainly be following the material in [Kot86b, §6].

To start, let us first write

$$\sum_{\{\gamma\} \in \{\mathbf{G}\}^{\text{s.s.}}} v_\gamma O_\gamma(f) = \sum_{\{\gamma_0\} \in \{\mathbf{G}\}_s^{\text{s.s.}}} \sum_{\{\gamma\} \in S(\gamma_0)} v_\gamma O_\gamma(f) \tag{288}$$

We now have the following

Lemma 2.5.60 ([Kot84b])**.** *Let $\mathbf{H}$ and $\mathbf{H}'$ reductive groups over $\mathbb{Q}$ which are inner forms. Then, $\tau(\mathbf{H}) = \tau(\mathbf{H}')$.*

Proof. By [Kot84b, (5.1.1)], (since $\tau(\mathbf{H}_{\text{sc}}) = 1$ by the resolution of the Tamagawa conjecture by Kottwitz in [Kot88]) we have

$$\tau(\mathbf{H}) = |\pi_0(Z(\widehat{\mathbf{H}})^\Gamma| \cdot |\ker^1(F, Z(\widehat{\mathbf{H}}))|^{-1}. \tag{289}$$

Since we have a Γ-equivariant isomorphism $\widehat{\mathbf{H}} \cong \widehat{\mathbf{H}'}$, this formula immediately implies the desired result. □

Hence, we see that $v_\gamma = v_{\gamma_0}$ for all $\{\gamma\} \in S(\gamma_0)$. Thus, the above becomes

$$\sum_{\{\gamma\} \in \{\mathbf{G}\}^{\text{s.s.}}} v_\gamma O_\gamma(f) = \sum_{\{\gamma_0\} \in \{\mathbf{G}\}_s^{\text{s.s.}}} v_{\gamma_0} \sum_{\{\gamma\} \in S(\gamma_0)} O_\gamma(f) \tag{290}$$

To continue, we recall the following lemma of Kottwitz (see §2.5.1.5 for notation concerning the Kottwitz group):

Lemma 2.5.61 (Kottwitz)**.** *Let $\mathbf{H}$ be a reductive group over a number field F. Let $\gamma_0 \in \mathbf{H}(F)$ be a given semi-simple element. Then, for a given semi-simple element $(\gamma_v) = \gamma \in \mathbf{H}(\mathbb{A})$ such that for all places v, we have $\gamma_v \sim_s \gamma_{0v}$ one has that $\gamma \sim \gamma'$ for some $\gamma' \in \mathbf{H}(F)$ if and only if the equality holds*

$$\sum_v \operatorname{obs}(\gamma_0, \gamma_v) \mid_{\mathfrak{K}(I_\gamma/F))} = 0 \tag{291}$$

where both sides are considered as elements of $\mathfrak{K}(I_\gamma/F)$. Moreover, if there exist such a γ' then the number of such γ' (up to $\mathbf{H}(F)$-conjugacy) is the quantity $|\mathfrak{K}(I_\gamma/F)|\tau(\mathbf{H})v_{\gamma_0}^{-1}$.

Proof. For the first claim see [Kot86b, Theorem 6.6]. For the second claim see the discussion succeeding Equation (9.6.3) on page 394 and the discussion preceding (9.6.5) on page 395 noting, again, that the resolution of the Tamagawa conjecture by Kottwitz in [Kot88] shows that $\tau_1(\mathbf{M}) = \tau(\mathbf{M})$ for any reductive group $\mathbf{M}$ over $\mathbb{Q}$. $\square$

In particular, we see that since $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-anisotropic and $\mathbf{G}$ has no relevant global endoscopy we see that the following holds:

Corollary 2.5.62. *Let $\gamma_0 \in \mathbf{G}(F)$ be a given semi-simple element. Then, for a given semi-simple $(\gamma_v) = \gamma \in \mathbf{G}(\mathbb{A})$ such that for all places v, we have $\gamma_v \sim_s \gamma_{0_v}$ one has that $\gamma \sim \gamma'$ for some $\gamma' \in G(F)$. Moreover, the number of such γ' (up to $G(F)$-conjugacy) is $\tau(G)v_{\gamma_0}^{-1}$.*

From this we see that we can rewrite (290) as follows:

$$\sum_{\{\gamma\}\in\{\mathbf{G}\}^{\text{s.s.}}} v_\gamma O_\gamma(f) = \tau(\mathbf{G}) \sum_{\{\gamma_0\}\in\{\mathbf{G}\}_s^{\text{s.s.}}} \sum_{\gamma\in S_{\mathbb{A}}(\gamma_0)} O_\gamma(f) \tag{292}$$

where $S_{\mathbb{A}}(\gamma_0)$ are the $\mathbf{G}(\mathbb{A})$-conjugacy classes which are stably $\mathbf{G}(\mathbb{A})$-conjugate to $\{\gamma_0\}$. Proposition 2.5.59 then follows considering the term on the right hand side is almost the definition of the term on the left hand side of (287). In particular, we see that in this case, $e(\gamma) = 1$ because at each place v, we have $\gamma_v \sim \gamma'$ for some $\gamma' \in \mathbf{G}(F)$, so that $e(\gamma) = e(I_{\gamma'}) = 1$ from which the claimed equality holds.

2.5.3. **Appendix 3: Base change for unitary groups.** We record here the version of base change necessary for our purposes. We are essentially following the results in [Lab09].

For this appendix we fix a CM number field $\mathbf{E}$ and let $\mathbf{F}$ be its maximal real subfield. We assume that $\mathbf{F} \supsetneq \mathbb{Q}$. Let us also fix an integer $n \geqslant 1$ and let $\mathbf{U}$ be an inner form of $U_{E/F}(n)^*$. We then set $\mathbf{G} := \mathrm{Res}_{F/\mathbb{Q}}\mathbf{U}$ and $\mathbf{H} := \mathsf{Res}_{E/\mathbb{Q}}\mathrm{GL}_{n,E}$. We fix a cofinite set S_{unram} of primes p of $\mathbb{Q}$ over which $\mathbf{G}$ is unramified, and for each $p \in S_{\text{unram}}$ we fix a hyperspecial subgroup $K_{0,p} \subseteq \mathbf{G}(\mathbb{Q}_p)$.

Next, let us fix an automorphic representation π of $\mathbf{U}(\mathbb{A}_F) = \mathbf{G}(\mathbb{A})$. We then denote denote by $S_{\text{ram}}(\pi)$ the union of the complement of S_{unram} and the finitely many $p \in S_{\text{unram}}$ for which π_p is ramified relative to $K_{0,p}$.

For every prime $p \notin S_{\text{ram}}(\pi)$ let us note that we have an unramified base change map

$$\mathrm{BC}_p : \left\{ \begin{matrix} \text{Irreducible and smooth} \\ K_{0,p}\text{-unramified} \\ \text{representations of } \mathbf{G}(\mathbb{Q}_p) \end{matrix} \right\} \to \left\{ \begin{matrix} \text{Irreducible and smooth} \\ K'_{0,p} - \text{unramified} \\ \text{representations of } \mathbf{H}(\mathbb{Q}_p) \end{matrix} \right\} \tag{293}$$

(where $K'_{0,p}$ is the unique hyperspecial subgroup of $\mathbf{H}(\mathbb{Q}_p)$) as in [Mın11, §2.7] (see also [Mın11, §4.1]).

With this setup, we then have the following result:

Theorem 2.5.63 ([Lab09, Corollaire 5.3])**.** *Fix ξ to be a regular algebraic representation of $\mathbf{G}_{\mathbb{C}}$. Then, there exists a map*

$$\mathrm{BC}: \left\{\begin{matrix}\textit{Irreducible discrete}\\ \textit{automorphic representations}\\ \textit{of } U_{E/F}(V)(\mathbb{A}_F) \textit{ such that}\\ \pi_\infty \textit{ is } \xi\textit{-cohomological}\end{matrix}\right\} \to \left\{\begin{matrix}\textit{Irreducible discrete}\\ \textit{automorphic}\\ \textit{representations}\\ \textit{of } \mathrm{GL}_n(\mathbb{A}_E)\end{matrix}\right\}$$

such that for all primes $p \notin S_{\mathrm{ram}}(\pi)$ we have that

- $\mathrm{BC}(\pi)_p = \mathrm{BC}_p(\pi_p)$.
- $\mathrm{BC}(\pi)^\vee \cong \mathrm{BC}(\pi) \circ c$ *(where c is the conjugation operator corresponding to the non-trivial element of* $\mathrm{Gal}(E/F)$*).*
- *The infinitesimal character of* $\mathrm{BC}(\pi)_\infty$ *is* $(\xi \otimes \xi)^\vee$.

2.5.4. **Appendix 4: Unitary groups.** In this appendix we recall the basic theory of unitary groups, their local-to-global construction, and when such groups have no relevant endoscopy as in §2.2.5.

2.5.4.1. *Decomposition of the forms of a split group.* Before we begin discussing unitary groups in earnest, it will be helpful to first recall the decomposition of the forms of a split group G into classes corresponding to inner and outer forms.

To begin, let F be any field, assumed perfect for convenience, and let G be a reductive group over F. Recall then the following well-known definition:

Definition 2.5.64. *A* form *or* twist *of G is an algebraic group H over F such that $H_{\overline{F}}$ is isomorphic to $G_{\overline{F}}$. An isomorphism of forms is merely an isomorphism of algebraic groups over F.*

Let us denote by $\mathsf{Form}(G)$ the set of (isomorphism classes of) forms of G. The set $\mathsf{Form}(G)$ is a pointed set with identity element the isomorphism class of G itself.

We recall the cohomological characterization of the pointed set $\mathsf{Form}(G)$. The group functor sending an F-algebra R to the group $\mathrm{Aut}(G_R)$ of R-automorphisms of G_R is representable by a separated and smooth group scheme denoted $\mathrm{Aut}(G)$ (e.g. see [Con14, Theorem 7.1.9]). Note then that associated to this group scheme $\mathrm{Aut}(G)$ there are two pointed sets. The étale cohomology set $H^1_{\text{ét}}(\mathrm{Spec}(F), \mathrm{Aut}(G))$ (cf [Mil80, Page 122]) and the Galois cohomology set $H^1(F, \mathrm{Aut}(G))$.

We have a natural map of pointed sets

$$\mathsf{Form}(G) \to H^1_{\text{ét}}(\operatorname{Spec}(F), \operatorname{Aut}(G)) \tag{294}$$

and a natural map

$$\mathsf{Form}(G) \to H^1(F, \operatorname{Aut}(G)) \tag{295}$$

defined as follows. The first map takes a twist H of G to the $\operatorname{Aut}(G)$-torsor $\underline{\operatorname{Isom}}(H, G)$ (where, here, we have used the identification given by [Mil80, Proposition 4.6]). The second map is defined as follows. Let H be an element of $\mathsf{Form}(G)$ and let $f : G_{\overline{F}} \to H_{\overline{F}}$ be an isomorphism. Then, the association

$$\iota_f : \sigma \mapsto \iota_f(\sigma) := f^{-1} \circ \sigma_H \circ f \circ \sigma_G^{-1} \tag{296}$$

defines a map $\iota_f : \Gamma_F \to Z^1(F, \operatorname{Aut}(G))$. Differing choices of f or H (within the same F-isomorphism class) define cohomologous elements of $Z^1(F, \operatorname{Aut}(G))$ and thus we get a well-defined map as in (295).

We then have the following well-known proposition:

Proposition 2.5.65. *There is a commuting triangle of isomorphisms of pointed sets*

$$\begin{tikzcd} \mathsf{Form}(G) \arrow[r] \arrow[d] & H^1_{\text{ét}}(\operatorname{Spec}(F), \operatorname{Aut}(G)) \arrow[dl] \\ H^1(F, \operatorname{Aut}(G)) & \end{tikzcd} \tag{297}$$

where the two arrows emanating from $\mathsf{Form}(G)$ *are* (294) *and* (295), *and the remaining arrow is the one from [Stacks, Tag03QQ].*

Proof. The proof of the bijectivity of the maps (294) and (295) follows easily from the fact that affine morphisms satisfy effective descent (e.g. see [Ser13, §1.3, Chapter III]). The commutivity of the diagram is easy and left to the reader. □

We would like to refine the set of forms of G by decomposing it into its constituents corresponding to whether a form is so-called *inner*. Namely, we make the following well-known definition:

Definition 2.5.66. *An* inner twist *of a group G is a pair (H, ξ) where H is an algebraic group over F and $\xi : G_{\overline{F}} \to H_{\overline{F}}$ is an isomorphism such that $\iota_\xi(\sigma)$ is an inner automorphism of $G_{\overline{F}}$ (i.e. conjugation by some element of $G(\overline{F})$) for every $\sigma \in \Gamma_F$. Two inner twists (H, ξ) and (H', ξ') are* equivalent *if there exists an isomorphism $\phi : H \to H'$ such that $\phi_{\overline{F}} \circ \xi = \operatorname{Int}(h') \circ \xi'$ for some $h' \in H(\overline{F})$.*

The equivalence classes of inner twists of G form a pointed set denoted $\mathsf{InnTwist}(G)$.

We can also classify inner twists of G cohomologically. To do this, begin by noting that we have a natural map of algebraic groups $G^{\mathrm{ad}} \to \mathrm{Aut}(G)$. Indeed, it suffices to give a map $G \to \mathrm{Aut}(G)$ which annihilates $Z(G)$. This map, on R-points, takes an R-point $g \in G(R)$ to the the obvious associated inner automorphism of G_R which is an element of $\mathrm{Aut}(G_R) = \mathrm{Aut}(G)(R)$. From this we obtain a maps of pointed sets

$$H^1_{\text{ét}}(\mathrm{Spec}(F), G^{\mathrm{ad}}) \to H^1_{\text{ét}}(\mathrm{Spec}(F), \mathrm{Aut}(G)) \tag{298}$$

and

$$H^1(F, G^{\mathrm{ad}}) \to H^1(F, \mathrm{Aut}(G)) \tag{299}$$

Notice that we also have a natural map

$$\mathsf{InnTwist}(G) \to \mathsf{Form}(G) \tag{300}$$

given by sending (H, ξ) to H.

Note that we also have a map of pointed sets

$$\mathsf{InnTwist}(G) \to H^1(F, G^{\mathrm{ad}}) \tag{301}$$

given by associating to (H, ξ) the element $\iota_\xi \in Z^1(F, G^{\mathrm{ad}})$. Again, one can check that changing (H, ξ) within its equivalence class corresponds to a cohomologous cocycle and thus we get a well-defined map as in (301).

We then have the following (also well-known) proposition:

Proposition 2.5.67. *The following diagram of maps of pointed sets is commutative with the horizontal arrows being isomorphisms*

$$\begin{tikzcd} \mathsf{InnTwist}(G) \arrow[r] \arrow[d] & H^1(F, G^{\mathrm{ad}}) \arrow[r] \arrow[d] & H^1_{\text{ét}}(\mathrm{Spec}(F), G^{\mathrm{ad}}) \arrow[d] \\ \mathsf{Form}(G) \arrow[r] & H^1(F, \mathrm{Aut}(G)) \arrow[r] & H^1_{\text{ét}}(\mathrm{Spec}(F), \mathrm{Aut}(G)) \end{tikzcd} \tag{302}$$

where all maps are defined as before this proposition.

Now, the map $\mathsf{InnTwist}(G) \to \mathsf{Form}(G)$ needn't be injective, and we denote by $\mathsf{InnForm}(G)$ its image and call such forms (in the image) *inner forms* of G. Evidently $\mathsf{InnForm}(G)$ can be a proper subset of $\mathsf{Form}(G)$. But, while not every form of G is an inner form, there is a partition of the forms of G in to groupings of the inner forms of certain special forms of G. We now elaborate on this point. While it is not strictly necessary, we assume from this point out that $\overline{G \text{ is split}}$. To this end, we also fix a pair (B, T) consisting of a Borel subgroup B and a split maximal subtorus T of B. We denote the triple (G, B, T) by $\mathcal{P}$.

Begin by recalling that a reductive group H over F is *quasi-split* if it possesses an F-rational Borel subgroup (i.e. a subgroup B of H such that $B_{\overline{F}}$ is a maximal smooth connected solvable subgroup of $H_{\overline{F}}$). We denote the set of (isomorphism classes of) quasi-split forms of G by $\mathsf{QS}(G)$ and thus, by definition, we have an inclusion $\mathsf{QS}(G) \subseteq \mathsf{Form}(G)$. These quasi-split forms of G are the previously alluded to 'special forms' for which every form of G will be an inner form of.

Before we state the decomposition of $\mathsf{Form}(G)$ in terms of these quasi-split forms, we explain how to cohomologically classify the subset $\mathsf{QS}(G)$ of $\mathsf{Form}(G)$. To begin, note that the inclusion of G^{ad} into $\mathrm{Aut}(G)$ has normal image and thus we can form the quotient group scheme which we denote $\mathrm{Out}(G)$. This group scheme is constant, and is finite whenever $Z(G)$ has rank at most 1 (e.g. see [Con14, Proposition 7.1.9]). Note that by definition we have the defining short exact sequence

$$1 \to G^{\mathrm{ad}} \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1 \tag{303}$$

which gives rise to the diagram

$$\begin{tikzcd} \mathrm{Out}(G)(F) \arrow[r] & H^1(F, G^{\mathrm{ad}}) \arrow[r] \arrow[d] & H^1(F, \mathrm{Aut}(G)) \arrow[r, "\mathrm{cl}"] \arrow[d] & H^1(F, \mathrm{Out}(G)) \\ & \mathsf{InnTwist}(G) & \mathsf{Form}(G) & \end{tikzcd} \tag{304}$$

where the verital maps are bijections and the horizontal maps form an exact sequence of pointed sets. Moreover, we have an idenficiation

$$H^1(F, \mathrm{Out}(G)) = \mathrm{Hom}_{\mathrm{cont.}}(\Gamma_F, \mathrm{Out}(G)(\overline{F}))/\sim \tag{305}$$

where $\sim$ denotes conjugation by $\mathrm{Out}(G)(\overline{F})$. One also has a natural identification of $\mathrm{Out}(G)(\overline{F})$ with the group of automorphisms of the based root datum associated to (G, B, T) (e.g. see [Con14, §1.5] as well as [Con14, Theorem 7.1.9]).

Let us denote by $\mathrm{Aut}(\mathcal{P})$ the subpresheaf of $\mathrm{Aut}(G)$ consisting of those automorphisms preserving $\mathcal{P}$ (i.e. preserving B and T). Note then that we get a natural map

$$H^1(F, \mathrm{Aut}(\mathcal{P})) \to H^1(F, \mathrm{Aut}(G)) \tag{306}$$

coming from this inclusion.

We then have the following cohomological classification of $\mathsf{QS}(G)$:

Proposition 2.5.68. *The natural map*

$$H^1(F, \mathrm{Aut}(\mathcal{P})) \to H^1(F, \mathrm{Aut}(G)) \tag{307}$$

is injective with image $\mathsf{QS}(G)$. *Moreover, the natural map*

$$\mathsf{QS}(G) \to H^1(F, \mathrm{Out}(G)) \tag{308}$$

is a bijection. Thus, we have natural bijections

$$H^1(F, \mathrm{Aut}(\mathcal{P})) \xrightarrow{\approx} \mathsf{QS}(G) \xrightarrow{\approx} H^1(F, \mathrm{Out}(G)) \tag{309}$$

Proof. Let us begin by showing that the image of the map in (307) is precisely $\mathsf{QS}(G)$. To do this, let ι is a cocycle of $\mathrm{Aut}(G)(\overline{F})$ with corresponding form H. Suppose now that ι lies in the image of $H^1(F, \mathrm{Aut}(\mathcal{P})$. Then, ι also gives rise (by restriction) to a cocycle in $H^1(F, \mathrm{Aut}(B))$ and thus, by definition, B descends to a form B' of B over F. Since we obtained the cocycle of $H^1(F, \mathrm{Aut}(B))$ by restriction of a cocycle in $H^1(F, \mathrm{Aut}(G))$ we see that we have an embedding $B' \hookrightarrow H$. It's not hard then to see that the image of this B' is a Borel subgroup of H, and thus H is quasi-split.

Suppose now that $H \in \mathsf{QS}(G)$ and fix a pair (B', T') of an F-rational Borel subgroup of H and a maximal torus T' contained in B'. Select an isomorphism $f : G_{\overline{F}} \to H_{\overline{F}}$. Note that by standard algebraic group theory the pair $(f^{-1}(B'_{\overline{F}}), f^{-1}(T'_{\overline{F}}))$ must be conjugate to the pair $(B_{\overline{F}}, T_{\overline{F}})$ by some element $g \in G(\overline{F})$. Note that H corresponds to the cocycle ι_f in $H^1(F, \mathrm{Aut}(G))$. Note then that ι_f is cohomologous to the cocycle $\iota' : \sigma \mapsto g\iota_f(\sigma)\sigma(g)^{-1}$. But, note that ι' (by construction) lands in the image of $H^1(F, \mathrm{Aut}(\mathcal{P}))$ as desired.

If we can show that the map $H^1(F, \mathrm{Aut}(\mathcal{P})) \to H^1(F, \mathrm{Out}(G))$ is an isomorphism then, since the diagram

$$\begin{tikzcd} H^1(F, \mathrm{Aut}(\mathcal{P})) \arrow[r] \arrow[d] & \mathsf{QS}(G) \arrow[dl] \\ H^1(F, \mathrm{Out}(G)) & \end{tikzcd} \tag{310}$$

commutes the injectivity of $H^1(F, \mathrm{Aut}(\mathcal{P}))$ and the bijectivity of the map $\mathsf{QS}(G) \to H^1(F, \mathrm{Out}(G))$ will follow. Thus, we focus on this.

Let us note that the map $\mathrm{Aut}(\mathcal{P}) \to \mathrm{Out}(G)$ is split (by any pinning of the triple (G, B, T)) and thus so is the map $H^1(F, \mathrm{Aut}(\mathcal{P})) \to H^1(F, \mathrm{Out}(G))$. This shows that the map $H^1(F, \mathrm{Aut}(\mathcal{P})) \to H^1(F, \mathrm{Out}(G))$ is surjective. To show the map is injective note that we have a short exact sequence of group schemes

$$1 \to T/Z(G) \to \mathrm{Aut}(\mathcal{P}) \to \mathrm{Out}(G) \to 1 \tag{311}$$

and thus (by the twisting trick of [Ser13, I, §5.7]) it suffices to show that for all $\mathrm{Out}(G)(\overline{F})$-valued cocycles a one has that $H^1(F, (T/Z(G))_a) = 0$. But, since T is split and the action of a on $X^*(T/Z(G))$ is by permutation of roots, we see that $(T/Z(G))_a$ is an induced torus, and thus the vanishing follows from Shapiro's lemma and Hilbert's theorem 90. $\square$

As a final observation, we give a decomposition of $\mathsf{Form}(G)$ into inner forms of the quasi-split forms of G. Namely, we have the following:

Proposition 2.5.69. *There is a decomposition*

$$\mathsf{Form}(G) = \bigsqcup_{H_0 \in \mathsf{QS}(G)} \mathsf{InnForm}(H_0) \tag{312}$$

Proof. Let us note that we have the exact sequence

$$1 \to G^{\mathrm{ad}} \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1 \tag{313}$$

which gives rise to the exact sequence

$$H^1(F, G^{\mathrm{ad}}) \to H^1(F, \mathrm{Aut}(G)) \xrightarrow{p} H^1(F, \mathrm{Out}(G)) \tag{314}$$

Then, clearly, we have a decomposition

$$H^1(F, \mathrm{Aut}(G)) = \bigsqcup_{a \in H^1(F, \mathrm{Aut}(G))} p^{-1}(a) \tag{315}$$

But, by the contents of [Ser13, I, §5.5] we know that $p^{-1}(a)$ is identified of a quotient of $H^1(F, G_a^{\mathrm{ad}})$. But, it's not hard to see that if a corresponds to $H \in \mathsf{QS}(G)$ by Proposition 2.5.68 then $G_a^{\mathrm{ad}} = H^{\mathrm{ad}}$, and the conclusion follows. $\square$

The above decomposition gives us a map $\mathsf{Form}(G) \to \mathsf{QS}(G)$. For an element H of $\mathsf{Form}(G)$ we denote by H^*, an element of $\mathsf{QS}(G)$, the image of H under this map. For a split group G over F we call an element H of $\mathsf{Form}(G)$ an *outer form* if $H^* \neq G$. Equivalently, H is an outer form if its image in $H^1(F, \mathrm{Out}(G))$ is non-trivial.

The last useful lemma we record is the following, which is easy (it follows from the proof of Proposition 2.5.69) and is left to the reader:

Lemma 2.5.70. *Let H be an element of $\mathsf{Form}(G)$ and H_0 an element of $\mathsf{QS}(G)$. Then, $H^* = H_0$ if and only if $\mathrm{cl}(H) = \mathrm{cl}(H_0)$.*

2.5.4.2. *Unitary groups: basic definitions and properties.* We now specialize and elaborate the discussion from the previous subsection in the case when $G = \mathrm{GL}_{n,F}$. In particular, we recall the theory of *unitary groups* over F by which we mean forms of $\mathrm{GL}_{n,F}$. For simplicity we assume that F has characteristic 0.

To begin, let us fix the pair (B, T) in the case of $\mathrm{GL}_{n,F}$ to be the standard Borel B_n of upper triangular matrices, and the standard torus

T_n of diagonal matrices. It is then not hard to check that the automorphisms of the associated based root datum are isomorphic to $\mathbb{Z}/2\mathbb{Z}$. From this we deduce that we have natural bijections

$$\begin{aligned} H^1(F, \mathrm{Out}(\mathrm{GL}_{n,F})) &\cong \mathrm{Hom}_{\mathrm{cont.}}(\Gamma_F, \mathbb{Z}/2\mathbb{Z}) \\ &\cong \{\text{étale algebras of degree 2 over } F\} \end{aligned} \tag{316}$$

which are identifications we freely make. Here an étale algebra of degree 2 over F means either $F \times F$, the *split étale algebra*, or a degree 2 extension E over F.

Before we continue, it will be helpful to clarify some notation concerning central simple algebras (or their generalizations *Azumaya algebras*) and their involutions. We begin by recalling the following definition.

Definition 2.5.71. *Let R be a (commutative unital) ring. Then an* Azumaya algebra *over R is a (possibly non-commutative) unital R-algebra A such that there exists some faithfully flat (commutative unital) R-algebra R' such that $A_{R'}$ is isomorphic to* $\mathrm{Mat}_n(R')$ *as an R'-algebra.*

We will only be interested in dealing with Azumaya algebras over degree 2 étale algebras over F, in which case such objects take a particularly simple form.

Namely, we have the following easy lemma:

Lemma 2.5.72. *Let R be a (commutative unital) ring.*

(1) If $R \to S$ is a ring map, and A is an Azumaya algebra over R, then A_S is an Azumaya algebra over S.

(2) If R is a field, then an R-algebra A is an Azumaya algebra if and only if it's a central simple R-algebra.

(3) If $R = F \times F$, where F is a field, then an R-algebra A is an Azumaya algebra if and only if $A \cong \Delta_1 \times \Delta_2$ where Δ_1 and Δ_2 are central simple F-algebras.

Azumaya algebras can support involutions of particular interest to us, ones of the so-called *second kind*. We record here the rigourous definition:

Definition 2.5.73. *Let F be a field of characteristic* 0 *and E a degree 2 étale algebra over F and let us write σ for the non-trivial element of* $\mathrm{Gal}(E/F)$*. If A is an Azumaya algebra over E, then an* involution of the second kind *is a morphism $A \to A$, denoted $x \mapsto x^*$, satisfying the following properties:*

(1) $(x+y)^* = x^* + y^*$ for all $x, y \in A$.

(2) $(xy)^* = y^* x^*$ *for all* $x, y \in A$.
(3) $x^* = \sigma(x)$ *for all* $x \in E$.

We shall often write $(A, *)$ for a pair of an Azumaya algebra and an involution of the second kind. To such a pair $(A, *)$ we can associate a *unitary group*:

Definition 2.5.74. *Let F be a field of characteristic* 0 *and E a* 2*-dimensional étale algebra over F. Then, for a pair $(A, *)$ of an an Azumaya algebra A over E and $*$ is an involution of the second kind we define the* unitary group of $(A, *)$, *denoted $U(A, *)$, to be the algebraic F-group whose R-points are given by*

$$U(A, *)(R) := \{x \in A_R : xx^* = 1\} \tag{317}$$

Let us now make the following elementary observation

Lemma 2.5.75. *Let F be a field of characteristic* 0 *and $E = F \times F$. Then, up to isomorphism, the only Azumaya algebras over E with an involution of the second kind are those of the form $(\Delta \times \Delta^{\mathrm{op}}, *_{switch})$ where Δ is a central simple F-algebra and*

$$*_{\text{switch}}(x, y) = (y, x) \tag{318}$$

Moreover,

$$U(\Delta \times \Delta, *_{\text{switch}}) \cong \Delta^\times \tag{319}$$

as algebraic groups over F.

Proof. The first claim is [Knu+98, Proposition 2.14]. The second claim is then clear. ∎

From this, we immediately deduce the following:

Lemma 2.5.76. *Let F be a field of characteristic* 0 *and let E be a degree* 2 *extension of F. Let $(\Delta, *)$ be a central simple E-algebra and let $U(\Delta, *)$ be its associated unitary group. Then, $U(\Delta, *)_E \cong \Delta^\times$.*

Proof. It's not hard to see that

$$U(\Delta, *)_E \cong U(\Delta_E, *_E) \tag{320}$$

where Δ_E is now an Azumaya algebra over $E \otimes_F E = E \times E$. By the previous lemma we know that

$$(\Delta_E, *_E) \cong (\Delta' \times \Delta', *_{\text{switch}}) \tag{321}$$

for some central simple E-algebra Δ'. Since Δ naturally embeds into Δ_E it's not hard to see that $\Delta' \cong \Delta$ and thus $U(\Delta, *)_E \cong \Delta^\times$ from the previous lemma. ∎

The last definition we require before returning to our analysis of the forms of $\mathrm{GL}_{n,F}$ is the follwing:

Definition 2.5.77. *Let F be a field of characteristic 0 and E a 2-dimensional étale algebra over F. A* Hermitian space relative to E/F *is a pair $(V, \langle -,- \rangle)$ consisting of a free E-module V together a non-degenerate F-linear pairing*

$$\langle -,- \rangle : V \times V \to E \tag{322}$$

such that $\langle -,- \rangle$ is E-linear in the first entry and satisfies

$$\langle v, w \rangle = \sigma(\langle w, v \rangle) \tag{323}$$

where σ is the non-trivial element of $\mathrm{Gal}(E/F)$.

For a Hermitian space $(V, \langle -,- \rangle)$ we define $U(V, \langle -,- \rangle)$ to be the algebraic F-group so that on F-algebras R we have the following:

$$U(V, \langle -,- \rangle)(R) := \tag{324}$$
$$\{g \in \mathrm{GL}_R(V_R) : \langle gv, gw \rangle = \langle v, w \rangle \text{ for all } v, w \in V_E\}$$

Now, combining (316) with Proposition 2.5.68 we see that we have a bijection

$$\mathsf{QS}(\mathrm{GL}_n) \cong \{\text{étale algebras of degree 2 over } F\} \tag{325}$$

For an étale algebra E over F of degree 2 let us denote by $U_{E/F}(n)^*$ the element of $\mathsf{QS}(\mathrm{GL}_n)$ corresponding to E. We then have the following description of $U^*_{E/F}(n)$ which is well-known, and whose proof is elementary and left to the reader:

Lemma 2.5.78. *Let E be an etale algebra of degree 2 over F. If E is split then $U_{E/F}(n)^* \cong \mathrm{GL}_n$. If E is a degree 2 extension of F then there is an isomorphism*

$$U_{E/F}(n)^* \cong U(E^n, \langle -,- \rangle_0) \tag{326}$$

where

$$\langle x, y \rangle_0 := \overline{x}^\top J_N y \tag{327}$$

where

$$J_N = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & -1 & 0 \\ 0 & \cdots & 1 & 0 & 0 \\ \vdots & \iddots & \iddots & 0 & 0 \\ 0 & \cdots & \vdots & 0 & 0 \\ (-1)^{N-1} & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{328}$$

Thus, combining this lemma with Proposition 2.5.69 we deduce that

$$\mathsf{Form}(\mathrm{GL}_{n,F}) = \bigsqcup_E \mathsf{InnForm}(U_{E/F}(n)^*) \tag{329}$$

and, in particular, the outer forms of GL_n are precisely the inner forms of some $U_{E/F}(n)^*$ where E is a degree 2 extension of F.

The last thing we would like to do is explicate the structure of the pointed set $\mathsf{InnForm}(U_{E/F}(n)^*)$. Namely, we would like to claim the following:

Lemma 2.5.79. *The elements of $\mathsf{InnForm}(U_{E/F}(n)^*)$ are precisely $U(A,*)$ where A is an Azumaya algebra over E of F-dimension $2n^2$ over F.*

Proof. Let us first note that the fact that every form of $\mathrm{GL}_{n,F}$ is of the form $U(A,*)$ for some Azumaya algebra over a degree 2 etale algebra over F is classical (e.g. see [PS92, §2.3.4]). The fact that $\mathsf{InnForm}(\mathrm{GL}_{n,F})$ is just $\Delta^\times$ for a central simple algebra over F is also well-known (see loc. cit.).

Let us now deal with the non-split case. Let us note that by Lemma 2.5.70 that an element $H = U(A,*)$ of $\mathsf{Form}(\mathrm{GL}_{n,F})$ is in $\mathsf{InnForm}(U_{E/F}(n)^*)$ if and only if $\mathrm{cl}(H) = \mathrm{cl}(U_{E/F}(n)^*) = E$. Moreover, by functoriality we know that $\mathrm{cl}(H_E) = \mathrm{cl}(H)_E$ and since E is the unique non-trivial element of $H^1(F,\mathbb{Z}/2\mathbb{Z})$ with trivial image in $H^1(E,\mathbb{Z}/2\mathbb{Z})$. Thus, we see that H is in $\mathsf{InnForm}(U_{E/F}(n)^*)$ if and only if $\mathrm{cl}(H_E)$ is trivial. But, this is equivalent to saying that H_E is in $\mathsf{InnForm}(\mathrm{GL}_{n,F})$ which, by the previous paragraph, shows that $H_E \cong \Delta^\times$ for some central simple algebra Δ over E. Note then that this implies that $Z(H)_E$ is split. But, if A is an Azumaya algebra over E' then one can easily show compute that $Z(H)$ is the unique 1-dimensional torus over F split over E'. Thus, $E = E'$ as desired. $\square$

We end this section with the well-known classification of unitary groups over local fields. We begin with the classification over $\mathbb{R}$:

Lemma 2.5.80. *There is a natural decomposition*

$$\mathsf{Form}(\mathrm{GL}_{n,\mathbb{R}}) = \mathsf{InnForm}(\mathrm{GL}_{n,\mathbb{R}}) \sqcup \mathsf{InnForm}(U_{\mathbb{C}/\mathbb{R}}(n)^*) \tag{330}$$

Moreover, we have that

$$\mathsf{InnForm}(\mathrm{GL}_{n,\mathbb{R}}) = \begin{cases} \{\mathrm{GL}_{n,\mathbb{R}}\} & \text{if} \quad \text{n odd} \\ \{\mathrm{GL}_{n,\mathbb{R}}, \mathrm{GL}_{\frac{n}{2}}(\mathbb{H})\} & \text{if} \quad \text{n even} \end{cases} \tag{331}$$

where $\mathbb{H}$ is the Hamiltonian quaternions and

$$\mathsf{InnForm}(U_{\mathbb{C}/\mathbb{R}}(n)^*) = \{U(p,q) : 0 \leqslant p \leqslant q \leqslant n \text{ and } p+q=n\} \tag{332}$$

where $U(p,q) = U(\mathbb{R}^n, \langle -,-\rangle_{(p,q)}$ *where*
(333)
$\langle (x_1,\dots,x_n),(y_1,\dots,y_n)\rangle_{(p,q)} := x_1y_1+\dots+x_py_p-x_{p+1}y_{p+1}-\dots-x_ny_n$

Proof. The claim concerning the inner forms of $\mathrm{GL}_{n,\mathbb{R}}$ follows immediately from the observation that $H^1(\mathbb{R}, \mathrm{PGL}_n)$ injects in to $\mathrm{Br}(\mathbb{R})[2n]$ and since $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ the claim follows quite easily.

The second claim follows from a computation of $H^1(\mathbb{R}, (U_{\mathbb{C}/\mathbb{R}}(n)^*)^{\mathrm{ad}})$. Let us note that $U(n) := U(0,n)$ is an inner form of $U_{\mathbb{C}/\mathbb{R}}(n)$, since it's not an inner form of $\mathrm{GL}_{n,\mathbb{R}}$, and thus it suffices to compute $H^1(\mathbb{R}, U(n)^{\mathrm{ad}})$. Note though that by [Bor14, Theorem 9] this is equal to $H^1(\mathbb{R}, T)/W_T(\mathbb{R})$ where T is a fundamental torus (i.e. a maximal torus of minimal split rank) in $U(n)^{\mathrm{ad}}$. But, $U(n)^{\mathrm{ad}}$ is $\mathbb{R}$-anisotropic so we can take T to be any maximal torus, namely $T = U(1)^n/Z(U(n))$ (where $U(1)$ is the unique non-split torus over $\mathbb{R}$). But, as can be easily calculated $H^1(\mathbb{R}, U(1)) = \mathbb{Z}/2\mathbb{Z}$ and thus $H^1(\mathbb{R}, T) = ((\mathbb{Z}/2\mathbb{Z})^n/(\mathbb{Z}/2\mathbb{Z}))$ where $\mathbb{Z}/2\mathbb{Z}$ is embedded diagonally in $(\mathbb{Z}/2\mathbb{Z})^n$. But, as can be easily checked (and as holds for any elliptic maximal torus in an $\mathbb{R}$-group), the group scheme W_T is constant. Thus, $W_T(\mathbb{R}) = W_T(\mathbb{C}) = S_n$. It's easy to check that the S_n action on $H^1(\mathbb{R}, T)$ is the obvious one and thus

$$
\begin{aligned}
H^1(\mathbb{R}, U(n)^{\mathrm{ad}}) &\cong ((\mathbb{Z}/2\mathbb{Z})^n/(\mathbb{Z}/2\mathbb{Z}))/S_n \\
&\cong \{(p,q) \in \mathbb{N}^2 : 0 \leqslant p \leqslant q \text{ and } p+q = n\}
\end{aligned}
\tag{334}
$$

It's then easy to check that $U(p,q)$, which is an inner form of $(U_{\mathbb{C}/\mathbb{R}}(n)^*$ is sent to (p,q) under the natural map $\mathsf{InnForm}(U(n)^{\mathrm{ad}}) \to H^1(\mathbb{R}, U(n)^{\mathrm{ad}})$ from where the conclusion follows. □

We now state the analogous classification of unitary groups over p-adic local fields:

Lemma 2.5.81. *Let F be a p-adic local field. There is a natural decomposition*

$$\mathsf{Form}(\mathrm{GL}_{n,F}) = \mathsf{InnForm}(\mathrm{GL}_{n,F}) \sqcup \bigsqcup_E \mathsf{InnForm}(U_{E/F}(n)^*) \tag{335}$$

where E travels over the degree 2 extensions of F (of which there are only finitely many). Moreover,

$$\mathsf{InnForm}(\mathrm{GL}_{n,F})\{\mathrm{GL}_k(D_{\frac{i}{j}}) : (i,j) = 1 \textit{ and } jk = n\} \tag{336}$$

where $D_{\frac{i}{j}}$ is the division algebra over F of invariant $\dfrac{i}{j}$ and

$$\mathsf{InnForm}(U_{E/F}(n)^*) \cong \begin{cases} \{e\} & \text{if} \quad n \,\text{odd} \\ \mathbb{Z}/2\mathbb{Z} & \text{if} \quad n \,\text{even} \end{cases} \tag{337}$$

Proof. The first claim follows quite easily from the fact (see [Mil, Chapter IV,§4]) that the inner forms of $\mathrm{GL}_{n,F}$ are of the form $\Delta^\times$ where Δ is a central simple F-algebra of dimension n^2 and that such division algebras are all of the form $\mathrm{Mat}_m(D_{\frac{i}{j}})$ where $D_{\frac{i}{j}}$ is the division algebra of invariant $\frac{i}{j}$ (in the sense loc. cit.).

The second claim follows, again, by explicitly computing the pointed set $H^1(F,(U_{E/F}(n)^*)^{\mathrm{ad}})$. Let us set $H := (U_{E/F}(n)^*)^{\mathrm{ad}}$. We use [Kot86b, Theorem 1.2] to equate this to the computation of $\pi_0(Z(\widehat{H})^{\Gamma_F})$. But, $Z(\widehat{H}) \cong \mathbb{Z}/n\mathbb{Z}$ and it's not hard to check that Γ_F acts through its quotient $\mathrm{Gal}(E/F)$ and the non-trivial element of $\mathrm{Gal}(E/F)$ acts by multiplication by -1. The conclusion easily follows. ☐

2.5.4.3. *Anisotropicity and unitary groups.* In this subsection we list some natural conditions that guarantee anisotropicity (modulo center) of unitary groups as well as the existence of elliptic maximal tori.

We start with the following:

Lemma 2.5.82. *Let E be a degree 2 étale algebra over F and set G^* to be $U^*_{E/F}(n)$. Let us set then set $G := U(A,*)$ to be an inner form of G^* Then:*

1. *If $E \cong F \times F$ then G satisfies that G^{ad} is F-anisotropic if and only if $G \cong D^\times$ where $D^\times$ is an F-central division algebra over F.*
2. *If E is a degree 2 extension of F, then G satisfies that G^{ad} is F-ansiotropic if $G \cong U(D,*)$ where D is an E-central division algebra.*

Before we prove this, it's useful to first recall the following:

Lemma 2.5.83. *Let F be a field of characteristic 0 and let G be a connected reductive group over F. Then, G^{ad} is F-anisotropic if and only if $G(F)$ contais no non-trivial unipotent elements.*

Proof. This follows from the contents of [BT72, §8]. ☐

Lemma 2.5.82. Suppose first that $E \cong F \times F$ and that G^{ad} is F-anisotropic. Then, we know from (or rather via the proof of) Lemma 2.5.79 that $G \cong \Delta^\times$ for some F-central simple algebra Δ. Note then that by the Artin-Wedderburn theorem that $\Delta^\times \cong \mathrm{GL}_m(D)$ for some (necessarily unique) F-central division algebra D. If $m > 1$ then $G(F) = \mathrm{GL}_m(D)$ contains $\mathrm{GL}_m(F)$ which implies that $G(F)$ contains a unipotent element which contradicts Lemma 2.5.83. Thus $m = 1$ and thus $G \cong D^\times$ as desired.

Conversely, if $G \cong D^\times$ then to show that G^{ad} is anisotropic it suffices, by Lemma 2.5.83, to show that $G(F) = D^\times$ contains no non-trivial unipotent elements. But, note that the natural left action of $D^\times$ on itself gives an embedding $\iota : G \hookrightarrow \mathrm{GL}_F(D)$ and so it suffices to show that the map $D^\times \hookrightarrow \mathrm{GL}_F(D)$ on F-points has no unipotent elements in the image. But, if $u \in D^\times$ were unipotent then that would mean that $(\iota(u) - I)^n = 0$ for some $n \geqslant 1$. Note though that ι arises from an algebra embedding $\iota : D \hookrightarrow \mathrm{End}_F(D)$ which allows us to rewrite this equation as $\iota\left((u-1)^n\right) = 0$. Since ι is injective this implies that $(u-1)^n = 0$ and since D is a division algebra this implies that $u = 1$ as desired.

Suppose now that E is a degree 2 extension of F and let $G \cong U(D, *)$ where D is an E-central division algebra. By Lemma 2.5.83 it suffices to show that $U(D, *)(F)$ contais no non-trivial unipotent elements. Note though that, by definition, $U(D, *)$ is contained in $\mathsf{Res}_{E/F} D^\times$. So,

$$U(D, *)(F) \subseteq \mathsf{Res}_{E/F} D^\times = D^\times \tag{338}$$

The same argument as in the last paragraph then shows that no non-trivial unipotent elements exist. $\square$

Remark 2.5.84. One cannot change (2) in Lemma 2.5.82 to an if and only if. Indeed, note that over $\mathbb{R}$, for example, $U(n) := U(0, n)$ is anisotropic but is of the form $U(\mathrm{Mat}_n(\mathbb{R}), *)$.

We now would like to explain when unitary groups over a local field F contain elliptic maximal tori. If F is a p-adic local field this is a non-question by Lemma 2.5.6. Suppose now that $F = \mathbb{R}$ we then have the following:

Lemma 2.5.85. *Let $n > 1$ be an integer. Then, a form G of $\mathrm{GL}_{n,\mathbb{R}}$ has an elliptic maximal torus if:*

(1) If $n = 2$ and G arbitrary.
(2) If $n > 2$ and G is an outer form of $\mathrm{GL}_{n,\mathbb{R}}$.

Proof. By the classification in 2.5.80 and [Kal16, Lemma 3.2.1] it suffices to analyze for which n do $\mathrm{GL}_{n,\mathbb{R}}$ and $U(n) = U(0, n)$ have elliptic maximal tori. In the former case since the elliptic maximal tori in $\mathrm{GL}_{n,F}$, for any field F, are of the form $\mathsf{Res}_{F'/F}\mathbb{G}_{m,E}$ where F' is a degree n extension of F' it's clear that elliptic maximal tori exist if and only if $n = 2$. For the latter case since $U(n)$ is always $\mathbb{R}$-anisotropic the answer is clearly that elliptic maximal tori exist for all n. The deisred conclusion follows. $\square$

2.5.4.4. *Local-to-global definitions of unitary groups.* We now explain the methodology for the construction of global unitary groups from local ones. In other words, we discuss the question of whether or not there is a (unique) unitary group over a number field F whose base change to F_v (for all places v of F) is some pre-perscribed unitary group.

So, let us fix F to be a global field (assumed to be a number field for convenience). From the last section we know that to give a form of $\mathrm{GL}_{n,F}$ is the same as to give a class in $H^1(F, \mathrm{Aut}(\mathrm{GL}_{n,F}))$. Note then that for every place v of F we have the usual localization map

$$H^1(F, \mathrm{Aut}(\mathrm{GL}_{n,F})) \to H^1(F_v, \mathrm{Aut}(\mathrm{GL}_{n,F})) \tag{339}$$

We can then assemble these maps to give a map

$$\mathrm{loc} : H^1(F, \mathrm{Aut}(\mathrm{GL}_{n,F})) \to \prod_v H^1(F_v, \mathrm{Aut}(\mathrm{GL}_{n,F})) \tag{340}$$

To begin, we have the following well-known lemma:

Lemma 2.5.86. *The localizaton map* (339) *is injective.*

Proof. Note that the sequence (303) for $\mathrm{GL}_{n,F}$

$$1 \to \mathrm{PGL}_{n,F} \to \mathrm{Aut}(\mathrm{GL}_{n,F}) \to \mathbb{Z}/2\mathbb{Z} \to 1 \tag{341}$$

splits. Thus, it suffices to prove that the maps

$$H^1(F, \mathrm{PGL}_{n,F}) \to \prod_v H^1(F_v, \mathrm{PGL}_{n,F}) \tag{342}$$

and

$$H^1(F, \mathbb{Z}/2\mathbb{Z}) \to \prod_v H^1(F_v, \mathbb{Z}/2\mathbb{Z}) \tag{343}$$

are injective.

To see that the map in (342) is injective, note that via the sequence

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_{n,F} \to 1 \tag{344}$$

we get a commutative diagram

$$\begin{tikzcd} H^1(F, \mathrm{PGL}_{n,F}) \arrow[r] \arrow[d] & \prod_v H^1(F_v, \mathrm{PGL}_{n,F}) \arrow[d] \\ H^2(F, \mathbb{G}_m) \arrow[r] & \prod_v H^2(F_v, \mathbb{G}_m) \end{tikzcd} \tag{345}$$

where all vertical maps are injective (using Hilbert's theorem 90 together with the theory of twists as in [Ser13, Part I, §5.7]). Thus it suffices to show that the map

$$H^2(F, \mathbb{G}_m) \to \prod_v H^2(F_v, \mathbb{G}_m) \tag{346}$$

is injective. But, there are is an obvious commutative diagram

$$\begin{tikzcd} \mathrm{Br}(F) \arrow[r] \arrow[d] & \prod_v \mathrm{Br}(F_v) \arrow[d] \\ H^2(F, \mathbb{G}_m) \arrow[r] & \prod_v H^2(F_v, \mathbb{G}_m) \end{tikzcd} \tag{347}$$

where the vertical maps are isomorphisms. Thus, it suffices to show that

$$\mathrm{Br}(F) \to \prod_v \mathrm{Br}(F_v) \tag{348}$$

is injective. This follows form the fundamental exact sequence of class field theory (e.g. take the limit of the map in [Mil, Chapter VII, Corollary 4.3]).

The fact that the map

$$H^1(F, \mathbb{Z}/2\mathbb{Z}) \to \prod_v H^1(F_v, \mathbb{Z}/2\mathbb{Z}) \tag{349}$$

is injective follows from basic algebraic number theory. Namely, Kummer theory implies that this is equivalent to the injectivity of the map

$$K^\times/(K^\times)^2 \to \prod_v K_v^\times/(K_v^\times)^2 \tag{350}$$

which is simple to see (e.g. see [Mil, Chapter VII, Theorem 1.1]). □

As a corollary of the above we obtain the following:

Corollary 2.5.87. *For any degree 2 étale algebra E over F the natural map*

$$\mathrm{loc}_E : \mathsf{InnForm}(U_{E/F}(n)^*) \to \prod_v \mathsf{InnForm}(U_{E_v/F_v}(n)^*) \tag{351}$$

is injective.

Here we are abusing notation by denoting $E \otimes_F F_v$ by E_v. Of course, since E is a degree 2 étale algebra over F, E_v is a degree 2 étale algebra over F_v.

We would now like to describe the explicit image of loc_E. In other words, we'd like to discuss when a collection of inner forms of $U_{E_v/F_v}(n)^*$ for all places v of F is the simultaneous base change of some inner form of $U_{E/F}(n)^*$.

To do this it will be helpful to construct a map

$$\epsilon_v : \mathsf{InnForm}(U^*_{E_v/F_v}(n)) \to \mathbb{Z}/2\mathbb{Z} \tag{352}$$

This map is given as follows (where we are using Lemma 2.5.80 and Lemma 2.5.81 without mention):

(1) Assume that E_v is a degree 2 extension of F_v. Then:
  (a) if F_v is a p-adic local field then the map

$$\epsilon_v : \mathsf{InnForm}(U^*_{E_v/F_v}(n)) \to \mathbb{Z}/2\mathbb{Z} \tag{353}$$

  is the unique injective homomorphism.
  (b) if $F_v \cong \mathbb{R}$ then the map

$$\epsilon_v : \mathsf{InnForm}(U^*_{E_v/F_v}(n)) \to \mathbb{Z}/2\mathbb{Z} \tag{354}$$

  is defined as follows:

$$\epsilon_v(U(p,q)) = \begin{cases} 1 & \text{if} \quad n \text{ odd} \\ \left\lfloor \dfrac{p-q}{2} \right\rfloor \mod 2 & \text{if} \quad n \text{ even} \end{cases} \tag{355}$$

  (c) Assume that $E_v \cong F_v \times F_v$. Then:
    (i) if F_v is a p-adic local field then

$$\epsilon_v : \mathsf{InnForm}(U^*_{E_v/F_v}(n)) \to \mathbb{Z}/2\mathbb{Z} \tag{356}$$

    is the quotient map by $2(\mathbb{Z}/n\mathbb{Z})$ after making the identification $\mathsf{InnForm}(U^*_{E_v/F_v}(n)) \cong \mathbb{Z}/n\mathbb{Z}$ as above.
    (ii) if $F_v \cong \mathbb{R}$ then

$$\mathsf{InnForm}(U^*_{E_v/F_v}(n)) \to \mathbb{Z}/2\mathbb{Z} \tag{357}$$

    is the unique injective homomorphism

Of course, we have neglected to say what happens when $F_v \cong \mathbb{C}$ in all cases, but here there are no non-trivial inner forms and so ϵ_v is just the trivial map.

We can now explicitly state which collections of local unitary groups come from a global unitary group:

Proposition 2.5.88. *Let F be a number field and let E be a degree 2 étale algebra over F. Then, the image of the injective map*

$$\mathsf{InnForm}(U_{E/F}(n)^*) \to \prod_v \mathsf{InnForm}(U_{E_v/F_v}(n)^*) \tag{358}$$

is the set of all tuples $(U_v)_v$ in $\prod_v \mathsf{InnForm}(U^*_{E_v/F_v}(n))$ such that the following two conditions hold:

(1) *$U_v \cong U_{E_v/F_v}(n)^*$ for almost all v.*
(2) *The equality*

$$\sum_v \epsilon_v(U_v) = 0 \tag{359}$$

holds as an element of $\mathbb{Z}/2\mathbb{Z}$.

Proof. This is contained in the contents of [Clo91, §2]. □

Remark 2.5.89. Note that ϵ_v is trivial for all v when n is odd, and so we see that in this case the only obstruction to a tuple $(U_v)_v$ of inner forms of $U^*_{E_v/F_v}(n)$ being the simultaneous base change of some inner form of $U^*_{E/F}(n)$ is that $U_v \cong U^*_{E_v/F_v}(n)$ for almost all v.

2.5.4.5. *Unitary groups with no relevant global endoscopy.* We now discuss sufficient conditions for a unitary group $\mathbf{U}$ over a number field F, such that $\mathbf{U}^{\mathrm{ad}}$ is F-anisotropic, to have no relevant global endoscopy as in §2.2.5.

We begin by observing the following:

Lemma 2.5.90. *Let F be a global field and let E be a quadratic extension of E. Let $\mathbf{U}$ be an element of $\mathsf{InnForm}(U^*_{E/F}(n))$. Then, if $\mathbf{U} \cong U(D, *)$ for D an E-central division algebra then U has no relevant elliptic endoscopy.*

Proof. We would like to apply Proposition 2.2.29. To do this we need to show that $\mathbf{U}^{\mathrm{ad}}$ is F-anisotropic and that $\mathbf{U}$ satisfies the Hasse principle. The former condition is Lemma 2.5.82. The latter is contained in [PS92, §6.7].

Now, let T be a maximal torus in $\mathbf{U}$. Then, we need to show that the containment $Z(\widehat{\mathbf{U}}) \subseteq \widehat{T}^{\Gamma_F}$ is an equality or, equivalently, that $\widehat{T}^{\Gamma_F} \subseteq$

$Z(\widehat{\mathbf{U}})$. Note though that evidently

$$\widehat{T}^{\Gamma_F} \subseteq \widehat{T}^{\Gamma_E} = \widehat{T_E}^{\Gamma_E} \tag{360}$$

Note though that, by assumption, T_E is a maximal torus of $\mathbf{U}_E \cong D^\times$. But, all maximal tori of $D^\times$ are induced, say they are equal to $\mathsf{Res}_{M/E}\mathbb{G}_{m,E}$ where M is a degree n extension of E. It is then clear to see that $\widehat{T_E}^{\Gamma_E} \subseteq Z(\widehat{\mathbf{U}})$ as desired. $\square$

Part 3. An Approach to the Characterization of the Local Langlands Correspondence (with Alex Youcis)

3.1. Introduction

The local Langlands conjecture for a reductive group G over a p-adic field F has held a central role in the study of number theory since its initial development by R. Langlands. While the precise formulation of these conjectures for the group $G = \mathrm{GL}_{n,F}$ is classical (e.g. see [HT01] and [Hen00]), such statements for general G, and especially for G which are not quasi-split, have only gradually been made precise over recent years (e.g. see [Kal16b] and the references therein). Such statements often list desiderata that the local Langlands conjecture for G is expected to satisfy but generally make no claim that these properties uniquely characterize the correspondence.

In the case of $G = \mathrm{GL}_{n,F}$ such characterizations classically employ the theory of L-functions and ϵ-factors. For other classical groups G, Arthur realized that one can often use the theory of standard and twisted endoscopy to relate the local Langlands conjecture for G and the local Langlands conjecture for $\mathrm{GL}_{n,F}$, thus reducing the characterization problem for G to the case of $\mathrm{GL}_{n,F}$. This was carried out for quasi-split symplectic and orthogonal groups in [Art13] and for quasi-split unitary groups in [Mok15]. The non quasi-split unitary case was tackled in [Kal+14].

In [Sch13b], Scholze gave an alternate characterization of the local Langlands conjecture for $\mathrm{GL}_{n,F}$. This characterization involves an explicit equation (called the *Scholze–Shin equation(s)* in the article below) which relates the local Langlands conjecture to certain geometrically defined functions $f_{\tau,h}$ which are of geometric provenance. This characterization, unlike that appealing to the theory of L-functions and ϵ-factors, has the property that it is amenable to study for a general group G. Namely, the functions $f_{\tau,h}$ as in [Sch13b] can be defined for a much wider class of groups than just $\mathrm{GL}_{n,F}$ (e.g. see [Sch13a] and [You19]) and thus one can ask whether analogues of Scholze's characterization of the local Langlands conjecture for $\mathrm{GL}_{n,F}$ exist for other groups G.

The goal of this Part can then be stated as giving an affirmative answer to this question for supercuspidal L-parameters (conjecturally the class of parameters whose packets consist entirely of supercuspidal representations). We show that a conjectural Langlands correspondence satisfying a certain list of desiderata including the Scholze-Shin equations is uniquely characterized by these conditions. Our method, as currently stated, cannot hope to handle all groups G but only groups satisfying a certain 'niceness' condition. For example, both unitary and odd special orthogonal groups are 'nice' whereas symplectic and even special orthogonal groups are not in general 'nice'.

3.2. Notation

Let F be a p-adic local field. Fix an algebraic closure $\overline{F}$ and let F^{un} be the maximal unramified extension of F in $\overline{F}$. Let L be the completion of F^{un} and fix an algebraic closure $\overline{L}$.

Let G be a connected reductive group over F. We denote by $G(F)^{\mathrm{reg}}$ the regular semisimple elements in $G(F)$ and by $G(F)^{\mathrm{ell}}$ the subset of elliptic regular semisimple elements. We denote by D, or D_G, the discriminant map on $G(F)$. If $\gamma, \gamma' \in G(F)$ are stably conjugate we denote this by $\gamma \sim_{\mathrm{st}} \gamma'$.

Let $\widehat{G}$ be the connected Langlands dual group of G and let ${}^LG$ be the Weil group version of the L-group of G as defined in [Kot84b, §1]. We denote the set of irreducible smooth representations of $G(F)$ by $\mathrm{Irr}(G(F))$ and by $\mathrm{Irr}^{\mathrm{sc}}(G(F))$ the subset of supercuspidal representations. For a finite group C the notation $\mathrm{Irr}(C)$ means all irreducible $\mathbb{C}$-valued representations of C.

A *supercuspidal Langlands parameter* is an L-parameter (see [Bor79, §8.2]) $\psi : W_F \to {}^LG$ such that the image of ψ is not contained in a proper Levi subgroup of ${}^LG$. We say that supercuspidal parameters ψ and ψ' are equivalent if they are conjugate in $\widehat{G}$ and denote this by $\psi \sim \psi'$. Let C_ψ be the centralizer of $\psi(W_F)$ in $\widehat{G}$. Then by [Kot84b, §10.3.1], ψ is supercuspidal if and only if the identity component C_ψ° of C_ψ is contained in $Z(\widehat{G})^{\Gamma_F}$. We define the group $\overline{C_\psi} := C_\psi / Z(\widehat{G})^{\Gamma_F}$ which is finite by our assumptions on ψ. For the sake of comparison, in [Kal16b, Conj. F], Kaletha defines $S_\psi^\natural := C_\psi/(C_\psi \cap [\widehat{G}]_{\mathrm{der}})^\circ$. For ψ a supercuspidal parameter, we have

$$S_\psi^\natural = C_\psi. \tag{361}$$

Indeed,

$$(C_\psi \cap [\widehat{G}]_{\mathrm{der}})^\circ = (C_\psi^\circ \cap [\widehat{G}]_{\mathrm{der}})^\circ \subset (Z(\widehat{G})^{\Gamma_F} \cap [\widehat{G}]_{\mathrm{der}})^\circ = \{1\}, \tag{362}$$

from where the equality follows.

Define $Z^1(W_F, G(\overline{L}))$ to be the set of continuous cocycles of W_F valued in $G(\overline{L})$ and let $\mathbf{B}(G) := H^1(W_F, G(\overline{L}))$ be the corresponding cohomology group. Let $\kappa : \mathbf{B}(G) \to X^*(Z(\widehat{G})^{\Gamma_F})$ be the Kottwitz map as in [Kot97].

An *elliptic endoscopic datum* of G (cf. [Kot84b, pp. 7.3-7.4]) is a triple (H, s, η) of a quasisplit reductive group H, an element $s \in$

$Z(\widehat{H})^{\Gamma_F}$, and a homomorphism $\eta : \widehat{H} \to \widehat{G}$. We require that η gives an isomorphism

$$\eta : \widehat{H} \to Z_{\widehat{G}}(\eta(s))^\circ, \tag{363}$$

that the $\widehat{G}$-conjugacy class of η is stable under the action of Γ_F, and that $(Z(\widehat{H})^{\Gamma_F})^\circ \subset Z(\widehat{G})$.

An *extended elliptic endoscopic datum* of G is a triple $(H, s, {}^L\eta)$ such that ${}^L\eta : {}^LH \to {}^LG$ and $(H, s, {}^L\eta|_{\widehat{H}})$ gives an elliptic endoscopic datum of G.

An *extended elliptic hyperendoscopic datum* is a sequence of tuples of data $(H_1, s_1, {}^L\eta_1), \ldots, (H_k, s_k, {}^L\eta_k)$ such that $(H_1, s_1, {}^L\eta_1)$ is an extended elliptic endoscopic datum of G, and for $i > 1$, the tuple $(H_i, s_i, {}^L\eta_i)$ is an extended elliptic endoscopic datum of H_{i-1}. An *elliptic hyperendoscopic group* of G is a quasisplit connected reductive group H_k appearing in an extended elliptic hyperendoscopic datum for G as above.

3.3. Statement of the Main Result

Throughout the rest of the paper we assume that our groups G satisfy the following assumption:

(Ext) For each elliptic hyperendoscopic group H of G and each elliptic endoscopic datum (H', s, η') of H, one can extend (H', s, η') to an extended elliptic endoscopic datum $(H', s, {}^L\eta')$ such that ${}^L\eta' : {}^LH' \to {}^LH$.

For a discussion on the severity of this assumption see §3.6.2.

We now state the main result. Let us fix G^* to be a quasi-split reductive group over F. We define a *supercuspidal local Langlands correspondence* for a group G^* to be an assignment

$$\Pi_H : \left\{ \begin{matrix} \text{Equivalence classes of} \\ \text{Supercuspidal } L\text{-parameters} \\ \text{for } H \end{matrix} \right\} \to \left\{ \begin{matrix} \text{Subsets of} \\ \mathrm{Irr}^{\mathrm{sc}}(H(F)) \end{matrix} \right\}, \tag{364}$$

for every elliptic hyperendoscopic group H of G^* satisfying the following properties:

(Dis) If $\Pi_H(\psi) \cap \Pi_H(\psi') \neq \varnothing$ then $\psi \sim \psi'$.

(Bij) For each Whittaker datum $\mathfrak{w}_H$ of H, a bijection

$$\iota_{\mathfrak{w}_H} : \Pi_H(\psi) \to \mathrm{Irr}(\overline{C_\psi}). \tag{365}$$

This bijection $\iota_{\mathfrak{w}_H}$ gives rise to a pairing

$$\langle -, - \rangle_{\mathfrak{w}_H} : \Pi_H(\psi) \times \overline{C_\psi} \to \mathbb{C}, \tag{366}$$

defined as follows:

$$\langle \pi, s\rangle_{\mathfrak{w}_H} := \mathrm{tr}(s \mid \iota_{\mathfrak{w}_H}(\pi)). \tag{367}$$

(St) For all supercuspidal L-parameters ψ of H, the distribution

$$S\Theta_\psi := \sum_{\pi\in\Pi_H(\psi)} \langle \pi, 1\rangle \Theta_\pi, \tag{368}$$

is stable and does not depend on the choice of $\mathfrak{w}_H$.

(ECI) For all extended elliptic endoscopic data $(H', s, {}^L\eta)$ for H and all $h \in \mathscr{H}(H(F))$, suppose ψ^H is a supercuspidal L-parameter of H that factors through ${}^L\eta$ by some parameter $\psi^{H'}$. Then such a $\psi^{H'}$ must be supercuspidal and we assume it satisfies the *endoscopic character identity*:

$$S\Theta_{\psi^{H'}}(h^{H'}) = \Theta^s_{\psi^H}(h), \tag{369}$$

where we define $h^{H'}$ to be a transfer of h to H' (e.g. see [Kal16b, §1.3]) and we define

$$\Theta^s_{\psi^H} := \sum_{\pi\in\Pi_H(\psi^H)} \langle \pi, s\rangle \Theta_\pi. \tag{370}$$

the s-twisted character of ψ^H.

Suppose now that $z_{\mathrm{iso}} \in Z^1(W_F, G(\overline{L}))$ projecting to an element of $\mathbf{B}(G)_{\mathrm{bas}}$. Let G be the inner form of G^* corresponding to the projection of z_{iso} to $Z^1(W_F, \mathrm{Aut}(G)(\overline{F}))$. We then define a *supercuspidal local Langlands correspondence* for the *extended pure inner twist* (G, z_{iso}) (cf. [Kal16b, §2.5]) to be a supercuspidal local Langlands correspondence for G^* as well as a correspondence

$$\Pi_{(G,z_{\mathrm{iso}})} : \left\{\begin{matrix}\text{Supercuspidal } L\text{-parameters}\\ \text{for } G\end{matrix}\right\} \to \left\{\begin{matrix}\text{Subsets of}\\ \mathrm{Irr}^{\mathrm{sc}}(G(F))\end{matrix}\right\}, \tag{371}$$

satisfying

(Bij') For each Whittaker datum $\mathfrak{w}_G$ of G, a bijection

$$\iota_{\mathfrak{w}_G} : \Pi_G(\psi) \to \mathrm{Irr}(C_\psi, \chi_{z_{\mathrm{iso}}}), \tag{372}$$

where $\mathrm{Irr}(C_\psi, \chi_{z_{\mathrm{iso}}})$ denotes the set of equivalence classes of irreducible algebraic representations of C_ψ with central character on $Z(\widehat{G})^{\Gamma_F}$ equal to $\chi_{z_{\mathrm{iso}}} := \kappa(\overline{z_{\mathrm{iso}}})$. This gives rise to a pairing

$$\langle -, -\rangle_{\mathfrak{w}_G} : C_\psi \times \mathrm{Irr}(C_\psi, \chi_{z_{\mathrm{iso}}}) \to \mathbb{C}, \tag{373}$$

defined as

$$\langle \pi, s\rangle_{\mathfrak{w}_G} := \mathrm{tr}(s \mid \iota_{\mathfrak{w}_G}(\pi)). \tag{374}$$

(ECI') For all supercuspidal parameters ψ of G and all extended elliptic endoscopic data $(H, s, {}^L\eta)$ of G such that ψ factors as $\psi = {}^L\eta \circ \psi^H$, there is an equality

$$\Theta^s_{\psi^H}(h^H) = S\Theta_\psi(h), \tag{375}$$

where $h \in \mathscr{H}(G(F))$ and $S\Theta_\psi$ is independent of choice of Whittaker datum in **(Bij')**.

For a supercuspidal local Langlands correspondence Π for (G, z_{iso}) we say that a subset of $\mathrm{Irr}(H(F))$ of the form $\Pi_\psi(H)$ is a *supercuspidal L-packet* for Π_H. We furthermore say that an element π of $\mathrm{Irr}(H(F))$ is Π_H*-accessible* if π is in a supercuspidal L-packet for Π_H.

A priori, the above axioms **(Dis)**, **(Bij)**-**(Bij')**,**(St)**, and **(ECI)**-**(ECI')** are not enough to uniquely specify a supercuspidal local Langlands correspondence Π for G^* even under the specification of the set of Π-accessible representations. The goal of our main theorem is to explain a sufficient extra condition which does uniquely specify a supercuspidal local Langlands correspondence.

In the statement of this condition we need to assume an extra property of G. Namely, we say that G^* is *good* if for every elliptic hyperendoscopic group H of G^* we have:

(Mu) There exists a set S^H of dominant cocharacters of $H_{\overline{F}}$ with the following propery. Let ψ_1^H, ψ_2^H be any pair of supercuspidal parameters of H such that for all dominant cocharacters $\mu \in S^H$, we have an equivalence $r_{-\mu} \circ \psi_1^H \sim r_{-\mu} \circ \psi_2^H$. Then $\psi_1^H \sim \psi_2^H$.

Here $r_{-\mu}$ is the representation of ${}^LH$ as defined in [Kot84a, (2.1.1)].We say that G is *good* if G^* is. We call a set S^H as in assumption **(Mu)** *sufficient*. See §3.6.1 for a discussion of the severity of this assumption.

To this end, let us define a *Scholze–Shin datum* $\{f^\mu_{\tau,h}\}$ for G to consist of the following data for each elliptic hyperendoscopic group H of G:

- A compact open subgroup $K^H \subset H(F)$,
- A sufficient set S^H of dominant cocharacters of $H_{\overline{F}}$,
- For each $\mu \in S^H$ of with reflex field E_μ, each $\tau \in W_{E_\mu}$, and each $h \in \mathscr{H}(K^H)$, a function $f^\mu_{\tau,h} \in \mathscr{H}(H(F))$.

Let us say that a supercuspidal local Langlands correspondence for G satisfies the *Scholze–Shin equations* relative to the Scholze–Shin datum $\{f^\mu_{\tau,h}\}$ if the following holds:

(SS) For all elliptic hyperendoscopic groups H, all $h \in \mathscr{H}(K^H)$, all $\mu \in S^H$, and all parameters ψ^H of H one has that

$$S\Theta_{\psi^H}(f^\mu_{\tau,h}) = \mathrm{tr}(\tau \mid (r_{-\mu} \circ \psi^H)(\chi_\mu))S\Theta_{\psi^H}(h), \tag{376}$$

where $\chi_\mu := |\cdot|^{-\langle \rho,\mu\rangle}$ and ρ is the half-sum of the positive roots of H (for a representation V and character χ we denote by $V(\chi)$ the character twist of V by χ).

We then have the following result:

Theorem 3.3.1. *Let G be a good group and suppose Π^i for $i = 1, 2$ are supercuspidal local Langlands correspondences for (G, z_{iso}) such that*

(1) For every elliptic hyperendoscopic group H of G the set of Π^1_H-accessible representations is contained in the set of Π^2_H - accessible representations.

(2) There exists a Scholze–Shin datum $\{f^\mu_{\tau,h}\}$ such that Π^i satisfies **(SS)** *relative to $\{f^\mu_{\tau,h}\}$ for $i = 1, 2$.*

Then $\Pi^1 = \Pi^2$ and for every (H, z), either equal to $(H, 1)$ where H is an elliptic hyperendoscopic group of G or equal to (G, z_{iso}), and choice of Whittaker datum $\mathfrak{w}_H$, the bijections $\iota^i_{\mathfrak{w}_H}$ for $i = 1, 2$ agree.

3.4. Atomic Stability of L-packets

Before we begin the proof of Theorem 3.3.1 in earnest, we first discuss the following extra assumption one might make on a supercuspidal local Langlands correspondence Π for the group G which, for this section, we assume is quasi-split. Namely, let us say that Π possesses *atomic stability* if the following condition holds:

(AS) If $S = \{\pi_1, ..., \pi_k\}$ is a finite subset of Π-accessible elements of $\mathrm{Irr}^{\mathrm{sc}}(G(F))$ and $\{a_1, ..., a_k\}$ is a set of complex numbers such that $\Theta := \sum_{i=1}^{k} a_i \pi_i$ is a stable distribution, then there is a partition

$$S = \Pi_{\psi_1}(G) \sqcup \cdots \sqcup \Pi_{\psi_n}(G) \tag{377}$$

such that

$$\Theta = \sum_{j=1}^{n} b_j S\Theta_{\psi_j} \tag{378}$$

(i.e. that a_i is constant on $\Pi_{\psi_i}(G)$).

We then have the following result:

Proposition 3.4.1. *Let Π be supercuspidal local Langlands correspondence for a group G. Then, Π automatically possesses atomic stability.*

Proposition 3.4.1 will follow from the following a priori weaker proposition. To state it we make the following definitions. For supercuspidal L-parameters $\psi_1, \ldots, \psi_n$ we denote by $D(\psi_1, \ldots, \psi_n)$ the $\mathbb{C}$-span of the distributions Θ_π for $\pi \in \Pi_G(\psi_1) \cup \cdots \cup \Pi_G(\psi_n)$ and let $S(\psi_1, \ldots, \psi_n)$ be the subspace of stable distributions in $D(\psi_1, \ldots, \psi_n)$.

Proposition 3.4.2. *For any finite set of supercuspidal L-parameters $\{\psi_1, \ldots, \psi_n\}$ one has that $\{S\Theta_{\psi_1}, \ldots, S\Theta_{\psi_n}\}$ is a basis for $S(\psi_1, \ldots, \psi_n)$.*

Let us note that this proposition actually implies Proposition 3.4.1. Indeed, since each $\pi_i \in S$ is accessible we know that we can enlarge S to be a union $\Pi_{\psi_1}(G) \sqcup \cdots \sqcup \Pi_{\psi_n}(G)$ of L-packets. Proposition 3.4.1 is then clear since every stable distribution in the span of S is contained in $S(\psi_1, \ldots, \psi_n)$.

Before we proceed with the proof of Proposition 3.4.2 we establish some further notation and basic observations. For an π element of $\mathrm{Irr}^{\mathrm{sc}}(G(F))$ we denote by f_π the locally constant $\mathbb{C}$-valued function on $G(F)^{\mathrm{reg}}$ given by the Harish-Chandra regularity theorem. We then obtain a linear map

$$R : D(\mathrm{Irr}^{\mathrm{sc}}(G(F))) \to C^\infty(G(F)^{\mathrm{ell}}, \mathbb{C}) \tag{379}$$

given by linearly extending the association $\Theta_\pi \mapsto f_\pi \mid_{G(F)^{\mathrm{ell}}}$. Here $D(\mathrm{Irr}^{\mathrm{sc}}(G(F)))$ is the $\mathbb{C}$-span of the distributions on $\mathcal{H}(G(F))$ of the form Θ_π for $\pi \in \mathrm{Irr}^{\mathrm{sc}}(G(F))$. We also have averaging maps

$$\mathrm{Avg} : C^\infty(G(F)^{\mathrm{ell}}, \mathbb{C}) \to C^\infty(G(F)^{\mathrm{ell}}, \mathbb{C}) \tag{380}$$

given by

$$\mathrm{Avg}(f)(\gamma) := \frac{1}{n_\gamma} \sum_{\gamma'} f(\gamma') \tag{381}$$

where γ' runs over representatives of the conjugacy classes of $G(F)$ stably equal to the conjugacy class of γ and n_γ is the number of such classes (which is finite since F is a p-adic field).

We then have the following well-known lemma concerning R:

Lemma 3.4.3 ([Kaz86, Theorem C])**.** *The linear map R is injective.*

In addition, we have the following observation concerning the interaction between R and Avg, which follows from the well-known fact that Θ is stable implies that $R(\Theta)$ is stable:

Lemma 3.4.4. *Let $\Theta \in D(\mathrm{Irr}^{\mathrm{sc}}(G(F)))$ be stable as a distribution. Then,* $\mathrm{Avg}(R(\Theta)) = R(\Theta)$.

We may now proceed to the proof of Proposition 3.4.2:

Proof. (Proposition 3.4.2) By assumption **(Bij)**, the set of virtual characters $S\Theta^s_{\psi_i}$, as s runs through representatives for the conjugacy classes in $\overline{C_\psi}$ and i runs through $\{1,\ldots,n\}$, is a basis of $D(\psi_1,\ldots,\psi_n)$. It suffices to show this in the case when $n=1$ in which case it is clear. Indeed, writing just ψ instead of ψ_1, we see that it suffices to note that the matrix $(\langle\pi,s\rangle)$, where π runs through the elements of $\Pi_\psi(G)$, is unitary, and thus invertible, by the orthogonality of characters.

We next show that for any supercuspidal L-parameter ψ and any non-trivial s in $\overline{C_\psi}$ we have that $\mathrm{Avg}(R(S\Theta^s_\psi))=0$. Indeed, we begin by observing that by [HS12, Lemma 6.20] we have that

(382)
$$\mathrm{Avg}(R(S\Theta^s_\psi))(\gamma)=\frac{1}{n_\gamma}\sum_{\gamma'}\sum_{\gamma_H\in X(\gamma')/\sim_{\mathrm{st}}}\Delta(\gamma_H,\gamma')\left|\frac{D_H(\gamma_H)}{D_G(\gamma')}\right|S\Theta_{\phi^H}(\gamma_H)$$

where here γ' travels over the set of conjugacy classes of $G(F)$ stably equal to the conjugacy class of γ and, as in loc. cit., $X(\gamma')$ is the set of conjugacy classes in $H(F)$ that transfer to γ, and $\Delta(\gamma_H,\gamma')$ is the usual transfer factor, and D denotes the discriminant function.

Let us note that we can rewrite this sum as

(383)
$$\frac{1}{n_\gamma}\sum_{\gamma_H\in X(\gamma)/\sim_{\mathrm{st}}}\left(\sum_{\gamma'}\Delta(\gamma_H,\gamma')\left|\frac{D_H(\gamma_H)}{D_G(\gamma')}\right|\right)S\Theta_{\phi^H}(\gamma_H).$$

because $X(\gamma')/\sim_{\mathrm{st}}$ is independent of the choice of γ'.

Note that $D_G(\gamma')=D_G(\gamma)$ for all γ' stably conjugate to γ (since $D_G(\gamma')$ is defined in terms of the characteristic polynomial of $\mathrm{Ad}(\gamma')$) and thus we can further rewrite this as

(384)
$$\frac{1}{n_\gamma}\sum_{\gamma_H\in X(\gamma)/\sim_{\mathrm{st}}}\left|\frac{D_H(\gamma_H)}{D_G(\gamma)}\right|\left(\sum_{\gamma'}\Delta(\gamma_H,\gamma')\right)S\Theta_{\phi^H}(\gamma_H)$$

and so it suffices to show that this inner sum $\sum\limits_{\gamma'}\Delta(\gamma_H,\gamma')$ is zero.

For $\gamma'\sim_{st}\gamma$, we have

(385)
$$\Delta(\gamma_H,\gamma')=\langle\mathrm{inv}(\gamma,\gamma'),s\rangle\Delta(\gamma_H,\gamma),$$

where $\mathrm{inv}(\gamma,\gamma')\in\mathfrak{K}(I_\gamma/F)^D$ (as in [Shi10, §2.2]). Since γ is elliptic, $\gamma'\mapsto\mathrm{inv}(\gamma,\gamma')$ gives a bijection between F-conjugacy classes in the stable conjugacy class of γ and $\mathfrak{K}(I_\gamma/F)^D$. Hence

(386)
$$\sum_{\gamma'}\Delta(\gamma_H,\gamma')=\Delta(\gamma_H,\gamma)\sum_{\chi\in\mathfrak{K}(I_\gamma)^D}\chi(s).$$

In particular, it suffices to show that s gives a nontrivial element of $\mathfrak{K}(I_\gamma/F)$. Since (H,s,η) is a nontrivial elliptic endoscopic datum and γ is elliptic, this follows from [Shi10, Lemma 2.8].

Now, since the set $\{S\Theta_{\psi_1},\ldots,S\Theta_{\psi_n}\}$ is independent (by assumption **(Dis)**) it suffices to show that this set generates $S(\psi_1,\ldots,\psi_n)$. But, this is now clear since if $\Theta \in S(\psi_1,\ldots,\psi_n)$ then we know by Lemma 3.4.4 that $\mathrm{Avg}(R(\Theta)) = R(\Theta)$. On the other hand, writing

$$\Theta = \sum_{i=1}^{n}\sum_{s} a_{is} S\Theta_{\psi_i}^{s} \tag{387}$$

we see from the above discussion, as well as combining assumption **(St)** with Lemma 3.4.4, that

$$\mathrm{Avg}(R(\Theta)) = \sum_{i=1}^{n} R(Sa_{ie}\Theta_{\psi_i}) = R\left(\sum_{i=1}^{n} a_{ie} S\Theta_{\psi_i}\right) \tag{388}$$

(identifying $S\Theta_{\psi_i}$ with $S\Theta_{\psi_i}^{e}$ where e is the identity conjugacy class in $\overline{C_\psi}$). The claim then follows from Lemma 3.4.3. □

3.5. Proof of main result

Let us begin by explaining that it suffices to assume G is quasi-split. Indeed, note that the assumptions of Theorem 3.3.1 are also satisfied for (G, z_{iso}) equal to $(G^*,1)$ and so, in particular, if we have proven the theorem in the case of $(G^*,1)$ then we know that $\Pi^1_{G^*} = \Pi^2_{G^*}$. Now, let ψ be any supercuspidal L-parameter for G. By assumption **(ECI')** we have that

$$S\Theta^1_\psi(h) = S\Theta^1_{\psi^{G^*}}(h^{G^*}) = S\Theta^2_{\psi^{G^*}}(h^{G^*}) = S\Theta^2_\psi(h) \tag{389}$$

for all $h \in \mathscr{H}(G(F))$ and where the superscripts correspond to those of Π^i. By independence of characters, this implies that $\Pi^1_{(G,z_{\mathrm{iso}})}(\psi) = \Pi^2_{(G,z_{\mathrm{iso}})}(\psi)$. It remains to show that $\iota^1_{\mathfrak{w}_H} = \iota^2_{\mathfrak{w}_H}$. Since each $\iota^i_{\mathfrak{w}_G}(\pi)$ is algebraic, it suffices to show that for all $\pi \in \Pi^1_{(G,z_{\mathrm{iso}})}(\psi) = \Pi^2_{(G,z_{\mathrm{iso}})}(\psi)$ one has that $\langle \pi, s\rangle^1_{\mathfrak{w}_H} = \langle \pi, s\rangle^2_{\mathfrak{w}_H}$ for all $s \in C_\psi$. By independence of characters, it suffices to show that $\Theta^{1,s}_\psi = \Theta^{2,s}_\psi$ for all $s \in C_\psi$. By the standard bijection $(H, s, {}^L\eta, \psi^H) \iff (\psi, s)$ (cf. [BM20, Prop. 2.10]) and the **(Ext)** assumption, each such s comes from an extended elliptic endoscopic datum $(H, s, {}^L\eta)$. Hence by **(ECI')** we have reduced to the quasi-split setting. We now work in the situation when $(G, z_{\mathrm{iso}}) = (G^*, 1)$.

Let us begin with the following lemma:

Lemma 3.5.1. *Suppose that H is an elliptic hyperendoscopic group of G and suppose that $\Pi^1_H(\psi)$ is a singleton set $\{\pi\}$. Then, in fact, $\{\pi\} = \Pi^2_H(\psi)$.*

Proof. Since $\{\pi\}$ is a superscuspidal packet for Π^1_H, we have by assumption **(St)** that Θ_π is stable. By the assumption of the theorem, π is Π^2_H-accessible and since Π^2_H satisfies **(AS)** (by the contents of §4), we have $\{\pi\} = \Pi^2_H(\psi')$ for some supercuspidal L-parameter ψ' of H. Then, by the assumption of the theorem we have that

$$\begin{aligned} \text{(390)} \qquad \operatorname{tr}(\tau \mid (r_{-\mu} \circ \psi)(\chi_\mu)) \operatorname{tr}(h \mid \pi) &= \operatorname{tr}(f^\mu_{\tau,h} \mid \pi) \\ &= \operatorname{tr}(\tau \mid (r_{-\mu} \circ \psi')(\chi_\mu)) \operatorname{tr}(h \mid \pi) \end{aligned}$$

In particular, choosing $h \in \mathscr{H}(K^H)$ such that $\operatorname{tr}(h \mid \pi) \neq 0$ and letting τ vary we deduce that

$$\text{(391)} \qquad \operatorname{tr}(\tau \mid (r_{-\mu} \circ \psi)(\chi_\mu)) = \operatorname{tr}(\tau \mid (r_{-\mu} \circ \psi')(\chi_\mu))$$

for all $\tau \in W_E$. This implies, since ψ is supercuspidal so that $r_{-\mu} \circ \psi$ and $r_{-\mu} \circ \psi'$ are semi-simple, that $r_{-\mu} \circ \psi \sim r_{-\mu} \circ \psi'$ for all $\mu \in S^H$. By our assumption that S^H is sufficient, we deduce that $\psi \sim \psi'$. In particular, $\{\pi\} = \Pi^2_\psi(H)$ as desired. $\square$

Lemma 3.5.2. *Let H be an elliptic hyperendoscopic group for G. Let ψ be a supercuspidal parameter for H and suppose $\overline{C_\psi} \neq \{1\}$. If ρ is an irreducible representation of $\overline{C_{\psi^H}}$ then there exists a nontrivial $\overline{s} \in \overline{C_\psi}$ such that the trace character χ_ρ of ρ satisfies $\operatorname{tr}(\overline{s} \mid \rho) \neq 0$.*

Proof. Suppose ρ vanishes on all nontrivial $\overline{s}$. Then we have

$$\text{(392)} \quad 1 = \langle \chi_\rho, \chi_\rho \rangle = \frac{1}{|\overline{C_\psi}|} \sum_{\overline{s} \in \overline{C_\psi}} \chi_\rho(\overline{s})^2 = \frac{1}{|\overline{C_\psi}|} \chi_\rho(1)^2 = \frac{1}{|\overline{C_\psi}|} \dim(\rho)^2,$$

so that $|\overline{C_\psi}| = \dim(\rho)^2$. But every irreducible representation ρ' of $\overline{C_\psi}$ is isomorphic to an irreducible factor appearing with multiplicity $\dim(\rho')$ in the regular representation of $\overline{C_\psi}$, which has dimension $|\overline{C_\psi}|$. Hence ρ must be the unique irreducible representation of $\overline{C_\psi}$, which implies that ρ is isomorphic to the trivial representation, and hence that $|\overline{C_\psi}| = 1$ contrary to assumption. $\square$

We now explain the proof of Theorem 3.3.1 in general:

Proof. (of Theorem 3.3.1) We prove this by inducting on the number of roots k for elliptic hyperendoscopic groups H of G. If $k = 0$ then H is a torus. Since every distribution on H is stable, one deduces from assumption **(Dis)** and assumption **(St)** that $\Pi^1_H(\psi)$ is a singleton and thus we are done by Lemma 3.5.1. Suppose now that the result is true for elliptic hyperendoscopic groups of G with at most k roots. Let H be an elliptic hyperendoscopic group of G with $k+1$ roots and let ψ be a supercuspidal parameter of H. We wish to show that $\Pi^1_H(\psi) = \Pi^2_H(\psi)$. If $\Pi^1_H(\psi)$ is a singleton, then we are done again by Lemma 3.5.1. Otherwise, we show that $\Pi^1_H(\psi) \subset \Pi^2_H(\psi)$, which by **(Bij)** will imply that $\Pi^1_H(\psi) = \Pi^2_H(\psi)$. By Lemma 3.5.2, we can find a non-trivial $\overline{s} \in \overline{C_\psi}$ and a lift $s \in C_\psi$ such that $\langle \pi, s \rangle \neq 0$. By definition of $\overline{C_\psi}$, we have that $s \notin Z(\widehat{G})$. Now, it suffices to show that $\Theta^{1,s}_\psi = \Theta^{2,s}_\psi$ since then by indpendence of characters, we deduce that $\pi \in \Pi^2_H(\psi)$ as desired.

To show that $\Theta^{1,s}_\psi = \Theta^{2,s}_\psi$ for all non-trivial $s \in \overline{C_\psi}$ we proceed as follows. We obtain, by combining our assumption **(Ext)** and Proposition 2.2.15 from (ψ, s), an extended elliptic endoscopic quadruple $(H', s, {}^L\eta, \psi^{H'})$ with $\psi^{H'}$ supercuspidal so that $\psi = {}^L\eta \circ \psi^{H'}$. One then has from Assumption **(ECI)** that

$$\Theta^{1,s}_\psi = \Theta^{2,s}_\psi \iff S\Theta^1_{\psi^{H'}} = S\Theta^2_{\psi^{H'}} \tag{393}$$

Moreover, since s is non-central, we know that H' has a smaller number of roots than H and thus $S\Theta^1_{\psi^{H'}} = S\Theta^2_{\psi^{H'}}$ by induction. The conclusion that $\Pi^1 = \Pi^2$ follows.

Let us now show that for any supercuspidal L-parameter ψ one has that $\iota^1_{\mathfrak{w}_H} = \iota^2_{\mathfrak{w}_H}$ for all elliptic hyperendoscopic groups H of G and Whittaker data $\mathfrak{w}_H$ of H. It suffices to show that $\langle \pi, s \rangle^1_{\mathfrak{w}_H} = \langle \pi, s \rangle^2_{\mathfrak{w}_H}$ for all $\pi \in \Pi^1_\psi(H) = \Pi^2_\psi(H)$. By independence of characters, it suffices to show that $\Theta^{1,s}_\psi = \Theta^{2,s}_\psi$ for all $s \in \overline{C_\psi}$. Since $s \in \overline{C_\psi}$, there exists, associated to the pair (ψ, s), a quadruple $(H', s, {}^L\eta, \psi^{H'})$ as in Proposition 2.2.15 (again using also assumption **(Ext)**) where H' is an elliptic endoscopic group of H and $\psi^{H'}$ is a parameter such that $\psi = {}^L\eta \circ \psi^{H'}$. By assumption **(ECI)** it suffices to show that $S\Theta^1_{\psi^{H'}} = S\Theta^2_{\psi^{H'}}$, but this follows from the previous part of the argument since we know that $\Pi^1_{H'}(\psi^{H'}) = \Pi^2_{H'}(\psi^{H'})$. The theorem follows. □

3.6. Examples and Discussion of Assumptions

In this section we discuss examples of groups and correspondences satisfying the assumptions **(Mu)**, **(Ext)**, and **(SS)**. We also comment on some possible rephrasings and generalizations of this work.

3.6.1. **Examples satisfying (Mu).** In this subsection we explain that several classes of classical groups satisfy assumption **(Mu)**. In particular, we have the following:

Proposition 3.6.1 ([GGP12, Theorem 8.1]). *Let G be such that G^* is one of the following: a general linear group, a unitary group, or an odd special orthogonal group. Then G satisfies assumption* **(Mu)**.

We record the following trivial observation:

Lemma 3.6.2. *Suppose that $G_1, \dots, G_m$ are groups satisfying assumption* **(Mu)**, *then $G_1 \times \cdots \times G_m$ satisfies assumption* **(Mu)**.

We can then quickly explain the proof of Proposition 3.6.1:

Proof. By [GGP12, Theorem 8.1], we can recover ψ from $r_{-\mu} \circ \psi$ in the case of GL_n, $U(n)$, SO_{2n+1} where $r_{-\mu}$ corresponds to the standard representation. However, to prove **(Mu)**, one must also prove a result about recovering ψ from $r_{-\mu} \circ \psi$ not only for G^* but for all elliptic hyperendoscopic groups H of G^*. We analyze this for each case.

- There are no nontrivial elliptic hyperendoscopic groups for GL_n and so there is no difficulty in this case.
- The elliptic hyperendoscopic groups of unitary groups are products of unitary groups so we are done by Lemma 3.6.2.
- The elliptic hyperendoscopic groups of odd special orthogonal groups are products of odd special orthogonal groups so we are again done by 3.6.2.

□

3.6.2. **Examples satisfying (Ext).** The authors are not aware of any example for G a group over F where this property does not hold. If G and all its hyperendoscopic groups have simply connected derived subgroup, then **(Ext)** follows from [Lan79, Prop. 1]. In particular, unitary groups satisfy **(Ext)**.

All elliptic endoscopic data (H, s, η) for G a symplectic or special orthogonal group can also be extended to a datum $(H, s, {}^L\eta)$ ([Kal16b, pg.5]). Since the elliptic endoscopic groups of symplectic and special orthogonal groups are products of groups of this type ([Wal10, §1.8]), it follows that symplectic and special orthogonal groups also satisfy **(Ext)**.

One could likely remove the assumption **(Ext)** altogether at the cost of having to consider z-extensions of endoscopic groups (see [KS99]) and perhaps slightly modify the statement of Theorem 3.3.1 to account for these extra groups.

3.6.3. **Examples of Scholze–Shin datum.** In this subsection we explain the origin of the Scholze–Shin datum and equations, explain the extent to which such datum and equations are thought to exist, and discuss known examples and expected examples.

In [Sch13a], Scholze constructs functions $f^{\mu}_{\tau,h}$ for certain unramified groups G and for certain cocharacters μ which together form a datum (G, μ) that one might call of 'PEL type'. In the second named author's thesis [You19] these functions and their basic properties were extended to a larger class of pairs (G, μ) which the author calls 'abelian type'. It seems plausible that functions $f^{\mu}_{\tau,h}$ (and thus Scholze–Shin datum) can be constructed in essentially full generality using the ideas of [Sch13a] and [You19] but using the moduli spaces of shtukas constructed by Scholze et al.

In [SS13], Scholze and Shin, in the course of studying the cohomology of compact unitary similitude Shimura varieties, posit that for the class of groups G showing up in the pair [Sch13a] that the local Langlands conjecture should satisfy the Scholze–Shin equations for the Scholze–Shin datum constructed in ibid.—we refer to such conjectures as the *Scholze–Shin conjectures*. In fact, Scholze and Shin describe endoscopic versions of the Scholze–Shin equations and thus arrive at endoscopic versions of the Scholze–Shin conjectures. They show that their conjectures hold in the case of groups of the form $\mathrm{GL}_n(F)$ (cf. [Sch13b]) and the Harris–Taylor version of the local Langlands conjectures.

Using the functions $f^{\mu}_{\tau,h}$ constructed by the second named author in his thesis [You19] one can construct Scholze–Shin data for a wider class of groups including the groups $G = U(n)^{*}_{E/F}$ where $E/\mathbb{Q}_p$ is an unramified extension of $\mathbb{Q}_p$. In Part 2, we showed that the Scholze–Shin conjectures hold true for such unitary groups (at least in the trivial endoscopic case which is all that is needed in this Part) using the version of the local Langlands conjectures constructed by Mok in [Mok15] and for non-quasisplit unitary groups in [Kal+14].

3.6.4. **Discussion of Extended Pure Inner Twists.** In this paper we have considered only G that arise as extended pure inner twists of G^* (e.g. see [Kal16b]). In general, the map

$$\mathbf{B}(G^*)_{\mathrm{bas}} \to \mathrm{Inn}(G^*), \tag{394}$$

where $\mathrm{Inn}(G^*) := \mathrm{im}[H^1(F, G^*_{\mathrm{ad}}(\overline{F})) \to H^1(F, \mathrm{Aut}(G^*)(\overline{F})]$ denotes the set of inner twists of G^*, need not be surjective. However, when G^* has connected center, this map will be surjective (see [Kal16b, pg.20]). In general, one can likely consider all inner twists by adapting the arguments of this paper to the language of rigid inner twists as in [Kal16a] (cf. [Kal16b]).

3.6.5. **The characterization in the unitary case.** Combining the discussion of §5.1-§5.3 and the results from Part 2 we see in particular the following:

Theorem 3.6.3. *Let $E/\mathbb{Q}_p$ be an unramified extension and F the quadratic subextension of E. Let G be an extended pure inner form twist of the quasi-split unitary group $U_{E/F}(n)^*$ associated to E/F. Then, the local Langlands correspondence for G as in [Mok15] and [Kal+14] satisfies the Scholze–Shin conjecture and thus is characterized by Theorem 3.3.1.*

References

[ABV12] Jeffrey Adams, Dan Barbasch, and David A Vogan. *The Langlands classification and irreducible characters for real reductive groups*. Vol. 104. Springer Science & Business Media, 2012.

[AEK05] James Arthur, David Ellwood, and Robert Kottwitz, eds. *Harmonic analysis, the trace formula, and Shimura varieties*. Vol. 4. Clay Mathematics Proceedings. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2005, pp. x+689. ISBN: 0-8218-3844-X.

[Art02] James Arthur. "A note on the automorphic Langlands group." In: *Canadian Mathematical Bulletin* 45.4 (2002), pp. 466–482.

[Art13] James Arthur. *The endoscopic classification of representations*. Vol. 61. American Mathematical Society Colloquium Publications. Orthogonal and symplectic groups. American Mathematical Society, Providence, RI, 2013, pp. xviii+590. ISBN: 978-0-8218-4990-3. DOI: 10.1090/coll/061. URL: https://doi.org/10.1090/coll/061.

[Art93] James Arthur. "On elliptic tempered characters." In: *Acta Mathematica* 171.1 (1993), pp. 73–138.

[Bad07] Alexandru Ioan Badulescu. "Jacquet-Langlands et unitarisabilité." In: *J. Inst. Math. Jussieu* 6.3 (2007), pp. 349–379. ISSN: 1474-7480. URL: https://doi.org/10.1017/S1474748007000035.

[BC+83] Armand Borel, William Casselman, et al. "L^2-cohomology of locally symmetric manifolds of finite volume." In: *Duke Mathematical Journal* 50.3 (1983), pp. 625–647.

[BJ79] Armand Borel and Hervé Jacquet. "Automorphic forms and automorphic representations." In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part*. Vol. 1. 1979, pp. 189–207.

[BM20] Alexander Bertoloni Meli. "An Averaging formula for Rapoport-Zink Spaces." 2020.

[Bor14] Mikhail Borovoi. "Galois cohomology of reductive algebraic groups over the field of real numbers." In: *arXiv preprint arXiv:1401.5913* (2014).

[Bor79] A. Borel. "Automorphic L-functions." In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math.,* *Oregon State Univ., Corvallis, Ore., 1977), Part 2*. Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I., 1979, pp. 27–61.

[Bor91] Armand Borel. *Linear algebraic groups*. Second. Vol. 126. Graduate Texts in Mathematics. Springer-Verlag, New York, 1991, pp. xii+288. ISBN: 0-387-97370-2. DOI: 10.1007/978-1-4612-0941-6. URL: https://doi.org/10.1007/978-1-4612-0941-6.

[Bou68] N. Bourbaki. *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*. Actualités Scientifiques et Industrielles, No. 1337. Hermann, Paris, 1968, 288 pp. (loose errata).

[BR94] Don Blasius and Jonathan D Rogawski. "Zeta functions of Shimura varieties." In: *Motives (Seattle, WA, 1991)* 55.Part 2 (1994), pp. 525–571.

[BT72] François Bruhat and Jacques Tits. "Groupes réductifs sur un corps local: I. Données radicielles valuées." In: *Publications Mathématiques de l'IHÉS* 41 (1972), pp. 5–251.

[BZ77] I. N. Bernstein and A. V. Zelevinsky. "Induced representations of reductive p-adic groups. I." In: *Ann. Sci. École Norm. Sup. (4)* 10.4 (1977), pp. 441–472. ISSN: 0012-9593. URL: http://www.numdam.org/item?id=ASENS_1977_4_10_4_441_0.

[Car+79] Pierre Cartier et al. "Representations of p-adic groups: a survey." In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part*. Vol. 1. 1979, pp. 111–155.

[CD85] Laurent Clozel and Patrick Delorme. "Pseudo-coefficients et cohomologie des groupes de Lie réductifs réels." In: *C. R. Acad. Sci. Paris Sér. I Math.* 300.12 (1985), pp. 385–387. ISSN: 0249-6291.

[CD90] Laurent Clozel and Patrick Delorme. "Le théorème de Paley - Wiener invariant pour les groupes de Lie réductifs. II." In: *Annales scientifiques de l'École Normale Supérieure*. Vol. 23. 2. 1990, pp. 193–228.

[CGP15] Brian Conrad, Ofer Gabber, and Gopal Prasad. *Pseudo-reductive Groups*. 2nd ed. New Mathematical Monographs. Cambridge University Press, 2015.

[Clo90] Laurent Clozel. “The fundamental lemma for stable base change.” In: *Duke Math. J.* 61.1 (1990), pp. 255–302. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-90-06112-5. URL: https://doi.org/10.1215/S0012-7094-90-06112-5.

[Clo91] Laurent Clozel. “Représentations galoisiennes associées aux représentations automorphes autoduales de $GL(n)$.” In: *Publications Mathématiques de l'IHÉS* 73 (1991), pp. 97–145.

[Con12a] Brian Conrad. “Finiteness theorems for algebraic groups over function fields.” In: *Compositio Mathematica* 148.2 (2012), pp. 555–639.

[Con12b] Brian Conrad. “Weil and Grothendieck approaches to adelic points.” In: *Enseign. Math.(2)* 58.1-2 (2012), pp. 61–97.

[Con14] Brian Conrad. “Reductive group schemes.” In: *Autour des schémas en groupes* 1 (2014), pp. 93–444.

[Far04] Laurent Fargues. “Cohomologie des espaces de modules de groupes p-divisibles et correspondances de Langlands locales.” In: *Astérisque* 291 (2004). Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales, pp. 1–199. ISSN: 0303-1179.

[Fla79] D Flath. “Decomposition of representations into tensor products.” In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part*. Vol. 1. 1979, pp. 179–183.

[GGP12] Wee Teck Gan, Benedict H Gross, and Dipendra Prasad. “Symplectic local root numbers, central critical L-values, and restriction problems in the representation theory of classical groups.” In: *Astérisque* 346.1 (2012), pp. 1–109.

[Har01] Michael Harris. “Local Langlands correspondences and vanishing cycles on Shimura varieties.” In: *European Congress of Mathematics, Vol. I (Barcelona, 2000)*. Vol. 201. Progr. Math. Birkhäuser, Basel, 2001, pp. 407–427.

[Hen00] Guy Henniart. “Une preuve simple des conjectures de Langlands pour GL (n) sur un corps p-adique.” In: *Inventiones mathematicae* 139.2 (2000), pp. 439–455.

[HG] Heekyoung Hahn and Jayce Getz. *An introduction to automorphic representations*. https://sites.duke.edu/heekyounghahn/?page_id=117.

[HS12] Kaoru Hiraga and Hiroshi Saito. *On L-Packets for Inner Forms of SL_n*. American Mathematical Soc., 2012.

[HT01] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties*. Vol. 151. Annals of Mathematics Studies. With an appendix by Vladimir G. Berkovich. Princeton University Press, Princeton, NJ, 2001, pp. viii+276. ISBN: 0-691-09090-4.

[Hum75] James E. Humphreys. *Linear algebraic groups*. Graduate Texts in Mathematics, No. 21. Springer-Verlag, New York-Heidelberg, 1975, pp. xiv+247.

[Hum95] James E. Humphreys. *Conjugacy classes in semisimple algebraic groups*. Vol. 43. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1995, pp. xviii+196. ISBN: 0-8218-0333-6.

[JNS17] Christian Johansson, James Newton, and Claus Sorensen. "Local Langlands correspondence in rigid families." In: *arXiv preprint arXiv:1708.01213* (2017).

[Kal16a] Tasho Kaletha. "Rigid inner forms of real and p-adic groups." In: *Ann. of Math. (2)* 184.2 (2016), pp. 559–632. ISSN: 0003-486X. DOI: 10.4007/annals.2016.184.2.6. URL: https://doi.org/10.4007/annals.2016.184.2.6.

[Kal16b] Tasho Kaletha. "The local Langlands conjectures for non-quasi-split groups." In: *Families of automorphic forms and the trace formula*. Simons Symp. Springer, [Cham], 2016, pp. 217–257.

[Kaz86] David Kazhdan. "Cuspidal geometry of p-adic groups." In: *J. Analyse Math.* 47 (1986), pp. 1–36. ISSN: 0021-7670. DOI: 10.1007/BF02792530. URL: https://doi.org/10.1007/BF02792530.

[Kis10] Mark Kisin. "Integral models for Shimura varieties of abelian type." In: *Journal of The American Mathematical Society* 23.4 (2010), pp. 967–1012.

[Kna01] Anthony W Knapp. *Representation theory of semisimple groups: an overview based on examples*. Vol. 36. Princeton university press, 2001.

[Knu+98] Max-Albert Knus et al. "The book of involutions, AMS Coll." In: *Pub* 44 (1998), p. 17.

[Kot82] Robert E. Kottwitz. "Rational conjugacy classes in reductive groups." In: *Duke Math. J.* 49.4 (1982), pp. 785–806. ISSN: 0012-7094. URL: http://projecteuclid.org/euclid.dmj/1077315531.

[Kot83] Robert E. Kottwitz. "Sign changes in harmonic analysis on reductive groups." In: *Trans. Amer. Math. Soc.* 278.1 (1983), pp. 289–297. ISSN: 0002-9947. URL: https://doi.org/10.2307/1999316.

[Kot84a] Robert E. Kottwitz. “Shimura varieties and twisted orbital integrals.” In: *Math. Ann.* 269.3 (1984), pp. 287–300. ISSN: 0025-5831. URL: https://doi.org/10.1007/BF01450697.

[Kot84b] Robert E. Kottwitz. “Stable trace formula: cuspidal tempered terms.” In: *Duke Math. J.* 51.3 (1984), pp. 611–650. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-84-05129-9. URL: https://doi.org/10.1215/S0012-7094-84-05129-9.

[Kot86a] Robert E Kottwitz. “Base change for unit elements of Hecke algebras.” In: *Compositio Mathematica* 60.2 (1986), pp. 237–250.

[Kot86b] Robert E. Kottwitz. “Stable trace formula: elliptic singular terms.” In: *Math. Ann.* 275.3 (1986), pp. 365–399. ISSN: 0025-5831. DOI: 10.1007/BF01458611. URL: https://doi.org/10.1007/BF01458611.

[Kot88] Robert E Kottwitz. “Tamagawa numbers.” In: *Annals of Mathematics* 127.3 (1988), pp. 629–646.

[Kot90] Robert E. Kottwitz. “Shimura varieties and λ-adic representations.” In: *Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988).* Vol. 10. Perspect. Math. Academic Press, Boston, MA, 1990, pp. 161–209.

[Kot92a] Robert E. Kottwitz. “On the λ-adic representations associated to some simple Shimura varieties.” In: *Invent. Math.* 108.3 (1992), pp. 653–665. ISSN: 0020-9910. URL: https://doi.org/10.1007/BF02100620.

[Kot92b] Robert E Kottwitz. “Points on some Shimura varieties over finite fields.” In: *Journal of the American Mathematical Society* 5.2 (1992), pp. 373–444.

[Kot97] Robert E. Kottwitz. “Isocrystals with additional structure. II.” In: *Compositio Math.* 109.3 (1997), pp. 255–339. ISSN: 0010-437X.

[KS99] Robert E. Kottwitz and Diana Shelstad. “Foundations of twisted endoscopy.” In: *Astérisque* 255 (1999), pp. vi+190. ISSN: 0303-1179.

[KSZ] Mark Kisin, SugWoo Shin, and Yihang Zhu. “The stable trace formula for certain Shimura varieties of abelian type.” In: ().

[Lab09] Jean-Pierre Labesse. “Changement de base CM et séries discretes.” In: *preprint* 486 (2009).

[Lab90] J.-P. Labesse. “Fonctions élémentaires et lemme fondamental pour le changement de base stable.” In: *Duke Math. J.*

61.2 (1990), pp. 519–530. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-90-06120-4. URL: https://doi.org/10.1215/S0012-7094-90-06120-4.

[Lan+97] Robert P Langlands et al. "Representations of abelian algebraic groups." In: *Pacific J. Math* 181 (1997), pp. 231–250.

[Lan79] R. P. Langlands. "Stable conjugacy: definitions and lemmas." In: *Canad. J. Math.* 31.4 (1979), pp. 700–725. ISSN: 0008-414X. DOI: 10.4153/CJM-1979-069-2. URL: https://doi.org/10.4153/CJM-1979-069-2.

[Lor07] Falko Lorenz. *Algebra: Volume II: Fields with Structure, Algebras and Advanced Topics*. Springer Science & Business Media, 2007.

[Man05] Elena Mantovan. "On the cohomology of certain PEL-type Shimura varieties." In: *Duke Math. J.* 129.3 (2005), pp. 573–610. ISSN: 0012-7094. URL: https://doi.org/10.1215/S0012-7094-05-12935-0.

[Man08] Elena Mantovan. "On non-basic Rapoport-Zink spaces." In: *Ann. Sci. Éc. Norm. Supér. (4)* 41.5 (2008), pp. 671–716. ISSN: 0012-9593. URL: https://doi.org/10.24033/asens.2079.

[Mil] James S Milne. "Class field theory (v4. 02), 2013." In: *Available on line at www. jmilne. org/math* ().

[Mil04] James S Milne. "Introduction to Shimura varieties." In: *Harmonic analysis, the trace formula, and Shimura varieties* 4 (2004), pp. 265–378.

[Mil17] James S Milne. *Algebraic groups: The theory of group schemes of finite type over a field*. Vol. 170. Cambridge University Press, 2017.

[Mil80] James S Milne. *Etale cohomology (PMS-33)*. 33. Princeton university press, 1980.

[Mok15] Chung Pang Mok. *Endoscopic classification of representations of quasi-split unitary groups*. Vol. 235. 1108. American Mathematical Society, 2015.

[Mın11] Alberto Mınguez. "Unramified representations of unitary groups." In: *On the stabilization of the trace formula* 1 (2011), pp. 389–410.

[Pau04] Frédéric Paugam. "Galois representations, Mumford-Tate groups and good reduction of abelian varieties." In: *Mathematische Annalen* 329.1 (2004), pp. 119–160.

[Per12] Keerthi Madapusi Pera. “Toroidal compactifications of integral models of Shimura varieties of Hodge type.” In: *arXiv preprint arXiv:1211.1731* (2012).

[Poo17] Bjorn Poonen. *Rational points on varieties*. Vol. 186. American Mathematical Soc., 2017.

[PS92] Vladimir Platonov and Rapinchuk Stepanovich. “Algebraic groups and number theory.” In: *Russian Mathematical Surveys* 47.2 (1992), p. 133.

[RSZ17] Michael Rapoport, Brian Smithling, and Wei Zhang. “Arithmetic diagonal cycles on unitary Shimura varieties.” In: *arXiv e-prints*, arXiv:1710.06962 (2017), arXiv:1710.06962. arXiv: 1710.06962 [math.NT].

[RV14] Michael Rapoport and Eva Viehmann. “Towards a theory of local Shimura varieties.” In: *Münster J. Math.* 7.1 (2014), pp. 273–326. ISSN: 1867-5778.

[RZ96] M. Rapoport and Th. Zink. *Period spaces for p-divisible groups*. Vol. 141. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1996, pp. xxii+324. ISBN: 0-691-02782-X; 0-691-02781-1. URL: https://doi.org/10.1515/9781400882601.

[Sch13a] Peter Scholze. “The Langlands-Kottwitz method and deformation spaces of p-divisible groups.” In: *Journal of the American Mathematical Society* 26.1 (2013), pp. 227–259.

[Sch13b] Peter Scholze. “The local Langlands correspondence for GL_n over p-adic fields.” In: *Invent. Math.* 192.3 (2013), pp. 663–715. ISSN: 0020-9910. DOI: 10.1007/s00222-012-0420-5. URL: https://doi.org/10.1007/s00222-012-0420-5.

[Ser13] Jean-Pierre Serre. *Galois cohomology*. Springer Science & Business Media, 2013.

[Ser93] Jean-Pierre Serre. “Gebres.” In: *Enseign. Math.(2)* 39.1-2 (1993), pp. 33–85.

[She79] Diana Shelstad. “Characters and inner forms of a quasi-split group over R.” In: *Compositio Mathematica* 39.1 (1979), pp. 11–45.

[She83] Diana Shelstad. “Orbital integrals, endoscopic groups and L-indistinguishability for real groups.” In: *Conference on automorphic theory (Dijon, 1981)*. Vol. 15. Publ. Math. Univ. Paris VII. Univ. Paris VII, Paris, 1983, pp. 135–219.

[Shi10] Sug Woo Shin. “A stable trace formula for Igusa varieties.” In: *J. Inst. Math. Jussieu* 9.4 (2010), pp. 847–895. ISSN:

1474-7480. DOI: 10.1017/S1474748010000046. URL: https://doi.org/10.1017/S1474748010000046.

[Shi11] Sug Woo Shin. "Galois representations arising from some compact Shimura varieties." In: *Ann. of Math. (2)* 173.3 (2011), pp. 1645–1741. ISSN: 0003-486X. URL: https://doi.org/10.4007/annals.2011.173.3.9.

[Shi12a] Sug Woo Shin. "Automorphic Plancherel density theorem." In: *Israel Journal of Mathematics* 192.1 (2012), pp. 83–120.

[Shi12b] Sug Woo Shin. "On the cohomology of Rapoport-Zink spaces of EL-type." In: *Amer. J. Math.* 134.2 (2012), pp. 407–452. ISSN: 0002-9327. URL: https://doi.org/10.1353/ajm.2012.0009.

[SS13] Peter Scholze and Sug Woo Shin. "On the cohomology of compact unitary group Shimura varieties at ramified split places." In: *J. Amer. Math. Soc.* 26.1 (2013), pp. 261–294. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-2012-00752-8. URL: https://doi.org/10.1090/S0894-0347-2012-00752-8.

[Stacks] The Stacks Project Authors. *Stacks Project*. http://stacks.math.columbia.edu. 2018.

[Ste06] Robert Steinberg. *Conjugacy classes in algebraic groups*. Vol. 366. Springer, 2006.

[Ste65] Robert Steinberg. "Regular elements of semi-simple algebraic groups." In: *Publications Mathématiques de l'IHÉS* 25 (1965), pp. 49–80.

[Tat79] John Tate. "Number theoretic background." In: *Proc. Symp. Pure Math*. Vol. 33. 2. 1979, pp. 3–26.

[Var07] Yakov Varshavsky. "Lefschetz–Verdier trace formula and a generalization of a theorem of Fujiwara." In: *GAFA Geometric And Functional Analysis* 17.1 (2007), pp. 271–319.

[VZ84] David A Vogan and Gregg J Zuckerman. "Unitary representations with non-zero cohomology." In: *Compositio Mathematica* 53.1 (1984), pp. 51–90.

[Wal08] Jean-Loup Waldspurger. *L'endoscopie tordue n'est pas si tordue*. American Mathematical Soc., 2008.

[Wal10] J.-L. Waldspurger. "Les facteurs de transfert pour les groupes classiques: un formulaire." In: *Manuscripta Math.* 133.1-2 (2010), pp. 41–82. ISSN: 0025-2611. DOI: 10.1007/s00229-010-0363-3. URL: https://doi.org/10.1007/s00229-010-0363-3.

[Wei12] André Weil. *Adeles and algebraic groups*. Vol. 23. Springer Science & Business Media, 2012.

[Whi] David Whitehouse. "An introduction to the trace formula." In: ().

[You19] Alex Youcis. "The Langlands-Kottwitz-Scholze method for deformation spaces of abelian type (In preparation)." In: (2019).

[Zel80] A. V. Zelevinsky. "Induced representations of reductive p-adic groups. II. On irreducible representations of $\mathrm{GL}(n)$." In: *Ann. Sci. École Norm. Sup. (4)* 13.2 (1980), pp. 165–210. ISSN: 0012-9593. URL: http://www.numdam.org/item?id=ASENS_1980_4_13_2_165_0.

[Kal+14] Tasho Kaletha et al. "Endoscopic Classification of Representations: Inner Forms of Unitary Groups." In: *arXiv e-prints*, arXiv:1409.3731 (2014), arXiv:1409.3731. arXiv: 1409.3731 [math.NT].

[Kal16] Tasho Kaletha. "Regular supercuspidal representations." In: *arXiv e-prints*, arXiv:1602.03144 (2016), arXiv:1602.03144. arXiv: 1602.03144 [math.RT].