Title
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Publication Date
1968-05-01

University of California

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DISCLAIMER

UCRL-18217
UC-34 Physics
TID-4500 (51st Ed.)

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

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Printed in the United States of America
Available from
Clearinghouse for Federal Scientific and Technical Information
National Bureau of Standards, U. S. Department of Commerce
Springfield, Virginia 22151
Price: Printed Copy $3.00; Microfiche $0.65

WAIST-TO-WAIST TRANSFER IN THIN-LENS OPTICS

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ABSTRACT

Graphs are presented, which enable one to calculate quickly the waist-to-waist transport of a charged particle beam through a line in the thin-lens approximation. The method can be used also in dealing with accelerating electrostatic lenses (immersion optics).

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*On leave from University of Milan, Milan, Italy.

An expression for the waist-to-waist transfer matrix in the case of a thin lens is given, for instance, in Banford's book.[1] However, for many applications it is useful to derive formulas for thin accelerating lenses, and to draw graphs that enable us to calculate quickly an ion-beam transfer line.

We will use the following definitions (see Fig. 1):

$\epsilon = x_0 x_0'$	emittance of a waist
$X = x_0/x_0'$	characteristic length of a waist
$\mu = x_{02}/x_{01}$	linear magnification
$\gamma^2 = V_2/V_1$	acceleration factor
z_1, z_2	distances of a point from the lens, upstream and downstream
z_{01}, z_{02}	distances of the waists from the lens
f	focal length (positive for a converging lens).

The thin lens we are discussing can be considered as composed of a thin nonaccelerating lens (L) followed or preceded by a thin accelerating gap (A), which changes the energy eV_1 of an incoming particle into eV_2. This lens changes the emittance of the beam, and therefore the characteristic length of the second waist, in the following way:

$$\begin{cases} \epsilon_2 = \epsilon_1/\gamma \\ X_2 = \mu^2 \gamma X_1 \end{cases} . \qquad (1)$$

In matrix notations, a field-free (drift = D) space can be represented by

$$D = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} ,$$

while a thin nonaccelerating lens is represented by

$$L = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} .$$

Across the accelerating gap, the x component of the phase-space representation of the beam is not changed, while for the x' component it is

$$\begin{cases} x_2' = \frac{P_x}{P_2} = \frac{P_x}{P_1}\frac{P_1}{P_2} = \frac{x_1'}{\gamma} \\ P^2 = 2mqV \end{cases} \tag{2}$$

Accordingly, the matrix for the accelerating gap can be written

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\gamma} \end{pmatrix} .$$

For a lens <u>followed</u> by A, the overall transfer matrix is

$$M = DALD = \begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\gamma} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{z_2}{\gamma f} & z_1 + \frac{z_2}{\gamma} - \frac{z_1 z_2}{\gamma f} \\ -\frac{1}{\gamma f} & \frac{1}{\gamma} - \frac{z_1}{\gamma f} \end{pmatrix} .$$

The equations

$$\begin{cases} \begin{pmatrix} 0 \\ x_2' \end{pmatrix} = DALD \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = DALD \begin{pmatrix} 0 \\ x_1' \end{pmatrix} \end{cases}$$

can be used to find the relationship between f and the measurable focal lengths before and after the lens. The result* is

$$\begin{cases} f_1 = f \\ f_2 = \gamma f \end{cases} . \qquad (M = DALD) \qquad (3)$$

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*Analogously, for a thin nonaccelerating lens preceded by A, one obtains

$$\begin{cases} f_1 = \frac{f}{\gamma} \\ f_2 = f \end{cases} . \qquad (M = DLAD) \qquad (3')$$

The transport equations, waist-to-waist, can be written explicitly, with z_{01} and z_{02} in the place of z_1 and z_2:

$$\begin{cases} x_2 = M_{11}(z_{02})x_1 + M_{12}(z_{01}, z_{02})x_1' \\ x_2' = M_{21}x_1 + M_{22}(z_{01})x_1' \quad . \end{cases} \tag{4}$$

At a waist, we can choose x_1, x_1', x_2, x_2' as coordinates of points on the contour of upright phase-space ellipses; namely:

$$\begin{cases} \dfrac{x_1^2}{x_{01}^2} + \dfrac{x_1'^2}{x_{01}'^2} = 1 \qquad \text{or} \qquad x_1^2 + X_1^2 x_1'^2 = x_{01}^2 \\ \dfrac{x_2^2}{x_{02}^2} + \dfrac{x_2'^2}{x_{02}'^2} = 1 \qquad \text{or} \qquad x_2^2 + X_2^2 x_2'^2 = x_{02}^2 \end{cases} \tag{5}$$

Squaring and summing the two equations (4) with the conditions (5), and equating the coefficients of x_1^2, $x_1'^2$, and $x_1 x_1'$, one obtains

$$\begin{cases} \left(1 - \dfrac{z_{02}}{\gamma f}\right)^2 + \dfrac{X_2^2}{\gamma^2 f^2} = \mu^2 \\ \left(z_{01} + \dfrac{z_{02}}{\gamma} - \dfrac{z_{01} z_{02}}{\gamma f}\right)^2 + \left(\dfrac{1}{\gamma} - \dfrac{z_{01}}{\gamma f}\right)^2 X_2^2 = \mu^2 X_1^2 \\ \left(1 - \dfrac{z_{02}}{\gamma f}\right)\left(z_{01} + \dfrac{z_{02}}{\gamma} - \dfrac{z_{01} z_{02}}{\gamma f}\right) - \left(\dfrac{1}{\gamma} - \dfrac{z_{01}}{\gamma f}\right)\dfrac{X_2^2}{\gamma f} = 0 \end{cases}$$

which are satisfied by

$$\begin{cases} \mu^2 = \dfrac{z_{02}-\gamma f}{\gamma(z_{01}-f)} \\ f^2 = \mu^2[(z_{01}-f)^2 + X_1^2] \end{cases} \tag{6}$$

Equations (6) reduce to Banford's equations for $\gamma = 1$ (no acceleration).

It is convenient to rearrange Eq. (6) in the following fashion:

$$\begin{cases} z_{02} = \gamma f \dfrac{(z_{01}-f)z_{01} + X_1^2}{(z_{01}-f)^2 + X_1^2} \\ \mu^2 = \dfrac{f^2}{(z_{01}-f)^2 + X_1^2} \end{cases} \tag{6'}$$

Equations (6') together with Eq. (1) can be used to design a transport line, from the starting values of

$$\epsilon_1, X_1 \quad \text{or} \quad x_{01}, x'_{01} \quad ,$$

which determine completely the initial phase-space configuration of the beam.

Before developing a graphical method to find quickly solutions of the system (6'), it is interesting to rewrite the (6') in a dimensionless form:

$$\begin{cases} t = \dfrac{s(s-1) + \alpha^2}{(s-1)^2 + \alpha^2} \\ \mu^2 = \dfrac{1}{(s-1)^2 + \alpha^2} \end{cases} , \qquad (6'')$$

with the following definitions:

$$s = \frac{z_{01}}{f}, \quad t = \frac{z_{02}}{\gamma f}, \quad \alpha = \frac{X_1}{f} .$$

The behaviour of the functions $t(s,\alpha)$ and $\mu^2(s,\alpha)$ is sketched in Fig. 2.

The formulas derived so far must reduce to the classical ones for thin-lens--point-object optics. In the classical limit,

$$\alpha \ll 1, s, s-1,$$

we have, to second order in α/s and $\alpha/(s-1)$,

$$\begin{cases} \dfrac{1}{s} + \dfrac{1}{t} = 1 + \dfrac{\alpha^2}{s^2(s-1)} \\ \mu = \dfrac{1}{s-1}\left[1 - \dfrac{\alpha^2}{2(s-1)^2}\right] \end{cases} . \qquad (7)$$

For $\gamma = 1$, Eqs. (7) become (see Fig. 3)

$$\begin{cases} \frac{1}{z_{01}} + \frac{1}{z_{02}} = \frac{1}{f}\left[1 + \frac{f}{z_{01}-f}\,\frac{X_1^2}{z_{01}^2}\right] = \frac{1}{f}\left[1 + \frac{z_{02}}{z_{01}}\,\frac{X_1^2}{z_{01}^2}\right] \\ \mu = \frac{f}{z_{01}-f}\left[1 - \frac{X_1^2}{2(z_{01}-f)^2}\right] = \frac{z_{02}}{z_{01}}\left[1 - \frac{1}{2}\left(\frac{z_{01}+z_{02}}{z_{01}}\right)^2 \frac{X_1^2}{z_{01}^2}\right] \end{cases} \quad (8)$$

In another, more useful form, obtained by using the definitions

$$u = \frac{z_{01}}{X_1}, \quad v = \frac{z_{02}}{X_1}, \quad \text{and } \phi = \frac{f}{X_1},$$

Eqs. (1) and (6') are

$$\begin{cases} X_2 = \mu^2\gamma X_1 \\ v = \gamma\phi\frac{u(u-\phi) + 1}{(u-\phi)^2 + 1} \\ \mu^2 = \frac{\phi^2}{(u-\phi)^2 + 1} \end{cases} \quad (6''')$$

The function $v(u,\phi)/\gamma$ is plotted in Figs. 4 (for positive values) and 5 (negative values). It is easy to recognize that such negative values exist only for $\phi > 2$. The function $\mu^2(u,\phi)$ is plotted in Fig. 6.

An example of application can be useful to understand the convenience of the plots. Let us consider a line composed by an ion gun with proper (accelerating) optics, plus another (nonaccelerating) lens, e.g., an "Einzel" lens. The purpose of the line is to transfer a crossover (waist) of given characteristics near the ion source to a given point at a given distance (Fig. 7). If we refer to the geometrical data of Fig. 7 and to the following properties of the starting waist:

$$\begin{cases} x_{01} = 0.5 \text{ cm} \\ x'_{01} = 0.100 \text{ rad} \end{cases} \qquad \begin{cases} \epsilon_1 = 0.05 \text{ cm-rad} \\ X_1 = 5 \text{ cm} \end{cases} ,$$

the calculation can run as follows:

First lens For $\gamma = 2$, $z_{01} = 10$ cm and accordingly for

$$u = \frac{z_{01}}{X_1} = 10{:}5 = 2$$

a reasonable value for z_{02}, corresponding to a second waist midway between the first and the second lens, is $z_{02} = 24$ cm. This value corresponds to

$$\frac{v}{\gamma} = \frac{z_{02}}{\gamma X_1} = 2.4$$

which can be obtained from the graph of Fig. 4 with

$$\phi = 1.5; \text{ or } f = 7.5 \text{ cm}, \ f_2 = 15 \text{ cm}.$$

The corresponding values for μ^2 and X_2 obtained from the graph of Fig. 6 are

$$\mu^2 = 1.8,\ X_2 = \mu^2 \gamma X_1 = 18 \text{ cm}.$$

Second lens For $\gamma = 1$, $z_{01} = 25$ cm and accordingly for

$$u = \frac{z_{01}}{X_1} = 25:18 = 1.4,$$

we can obtain (Fig. 4) a value $z_{02} = 21$ cm, or $\frac{v}{\gamma} = \frac{z_{02}}{\gamma X_1} = 21:18 = 1.17$,

with $\phi = 0.95$; or $f = f_2 = 17$ cm,

the resulting magnification (Fig. 6) is

$$\mu^2 = 0.8$$

The overall magnification is

$$\mu_{ov} = (0.8 \times 1.8)^{\frac{1}{2}} = 1.2$$

and the final emittance is

$$\epsilon_{ov} = \epsilon_{in}/\gamma = 0.025 \text{ cm-rad}.$$

ACKNOWLEDGMENT

This work was done under the auspices of the U. S. Atomic Energy Commission.

REFERENCE

1. A. P. Banford, "The Transport of Charged Particle Beams" (E. and F. N. Spon, Ltd., London, 1966).

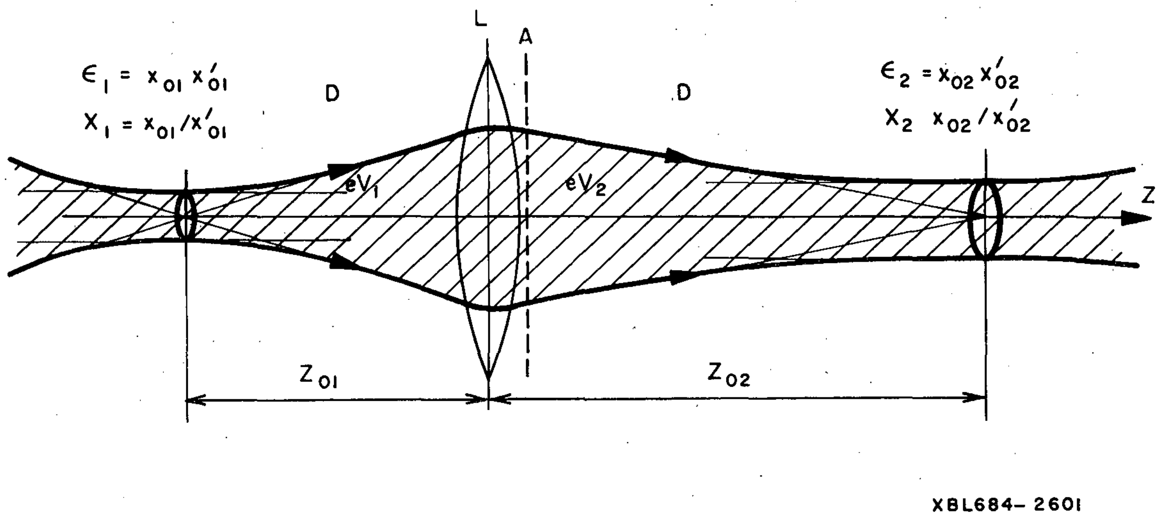

Fig. 1. A thin accelerating lens.

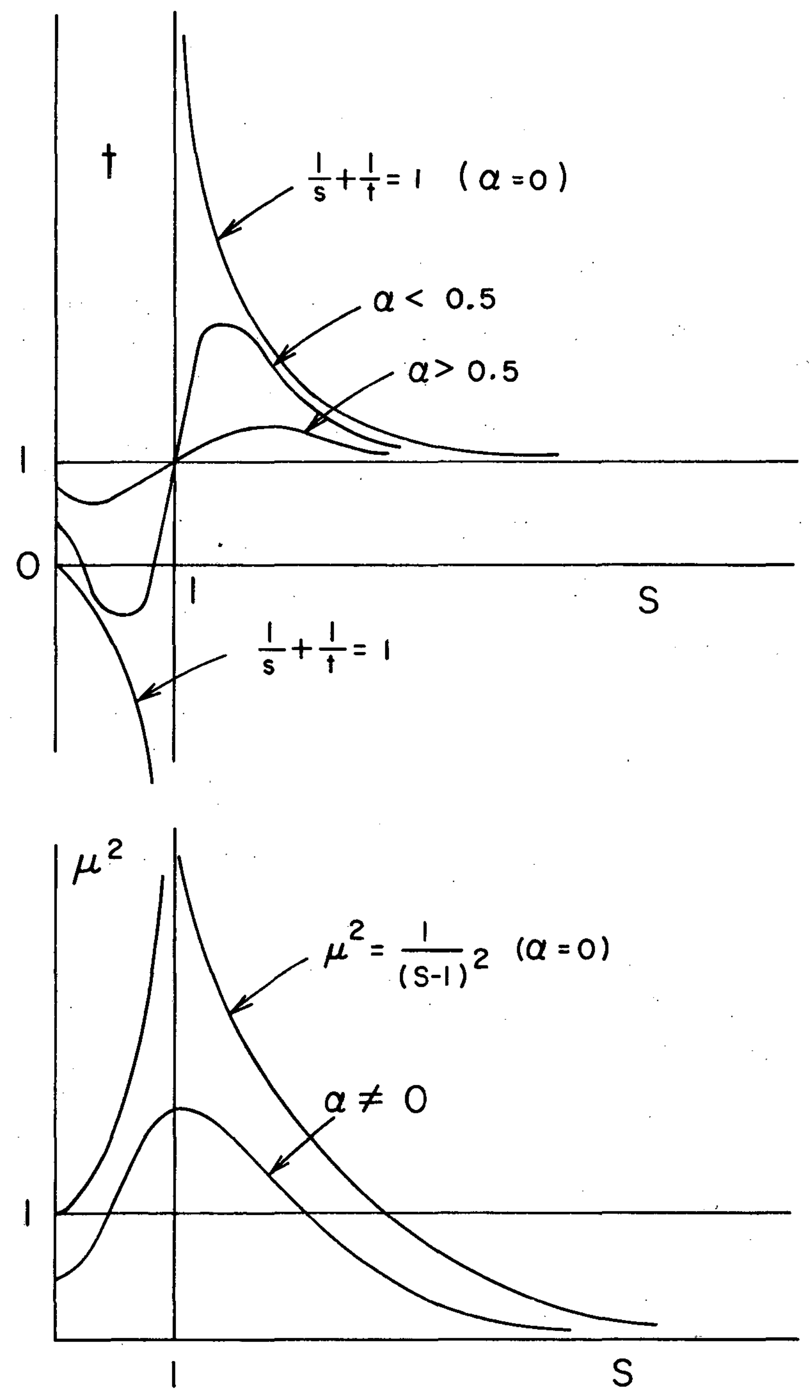

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Fig. 2. The functions $t(s,\alpha)$ and $\mu^2(s,\alpha)$ and their point-optics limit.

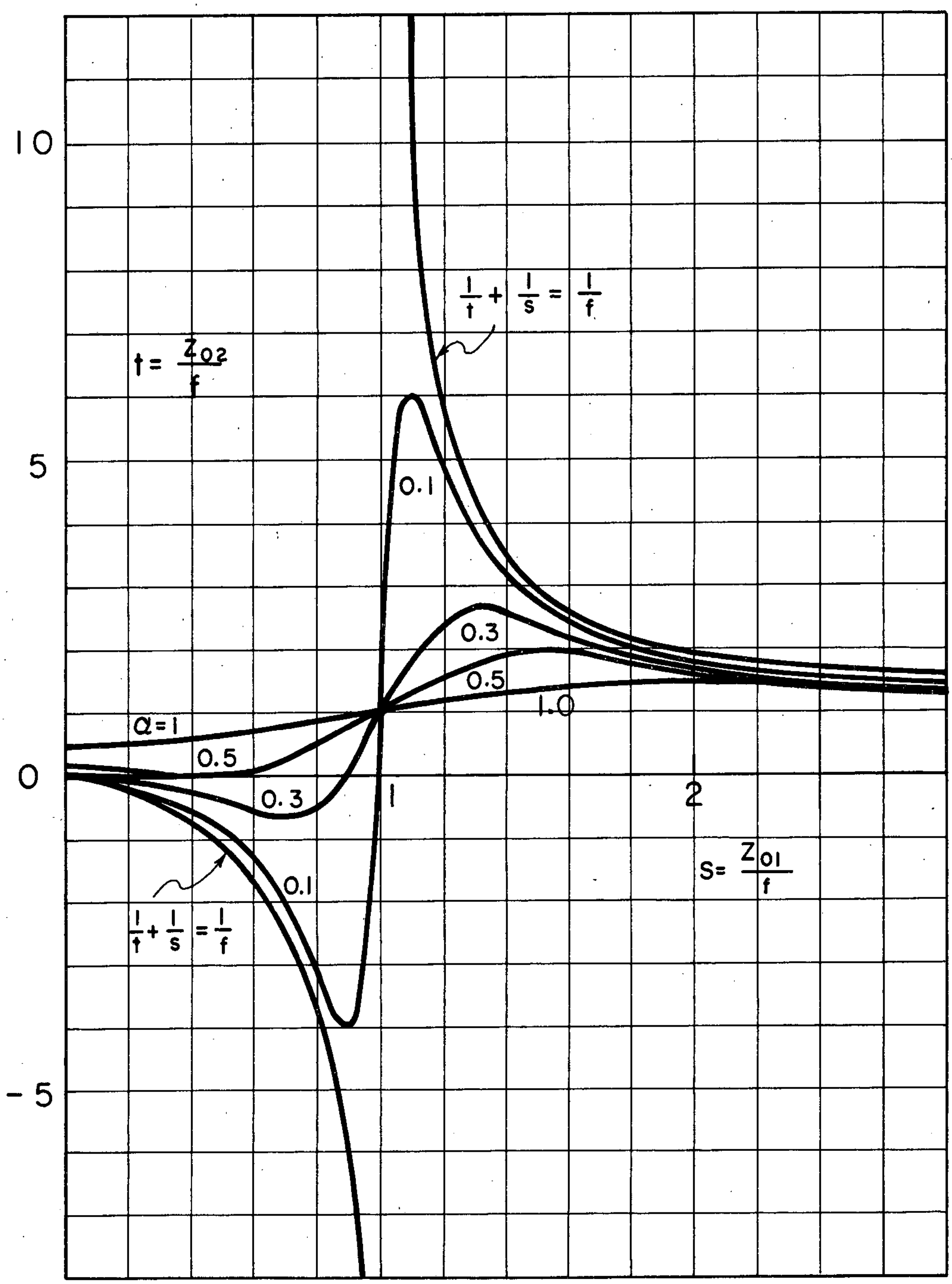

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Fig. 3. The function $t(s,\alpha) = [s(-1) + \alpha^2]/[(s-1)^2 + \alpha^2]$.

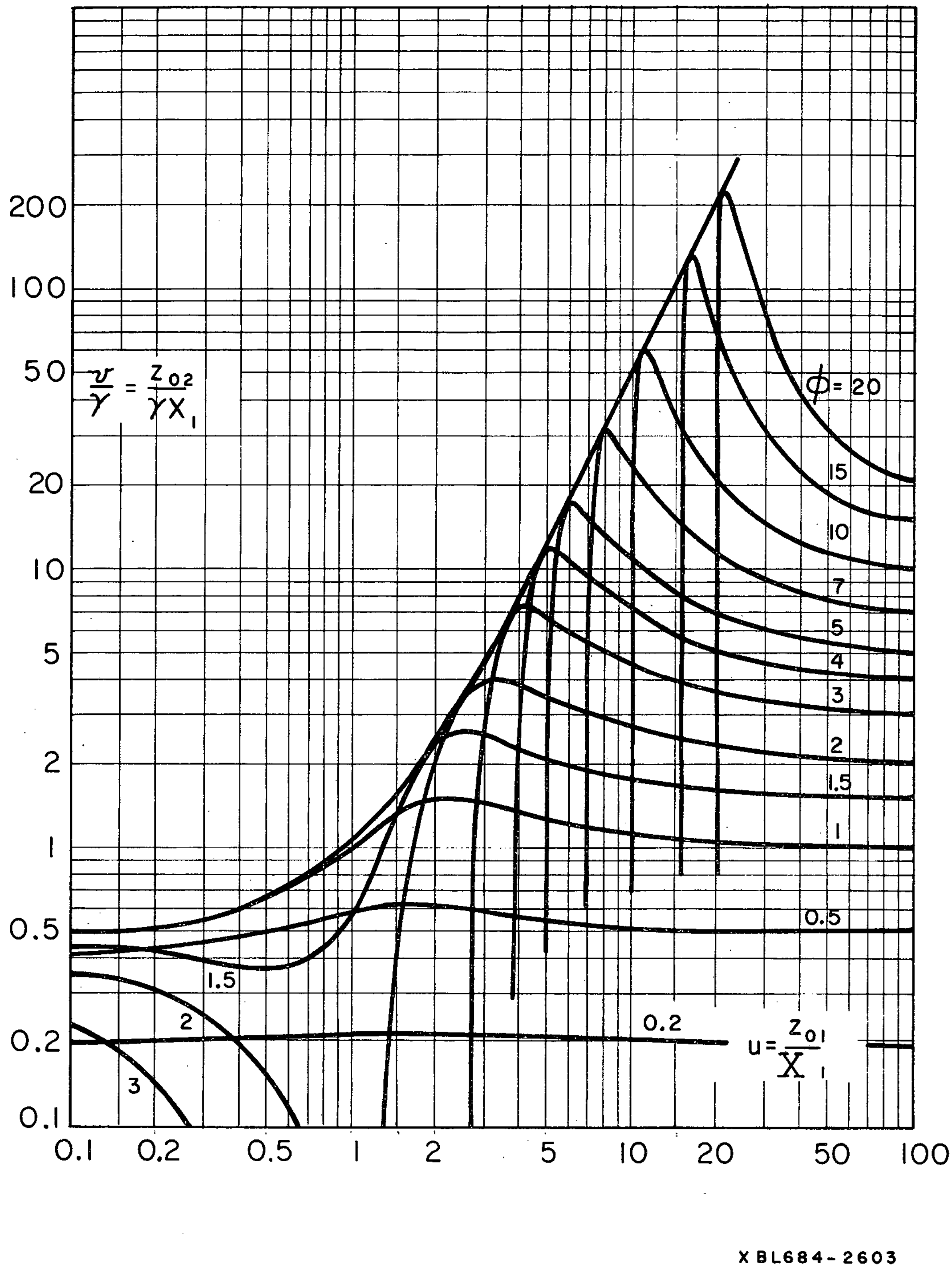

Fig. 4. The function $v(u,\phi)/\gamma$. Positive values.

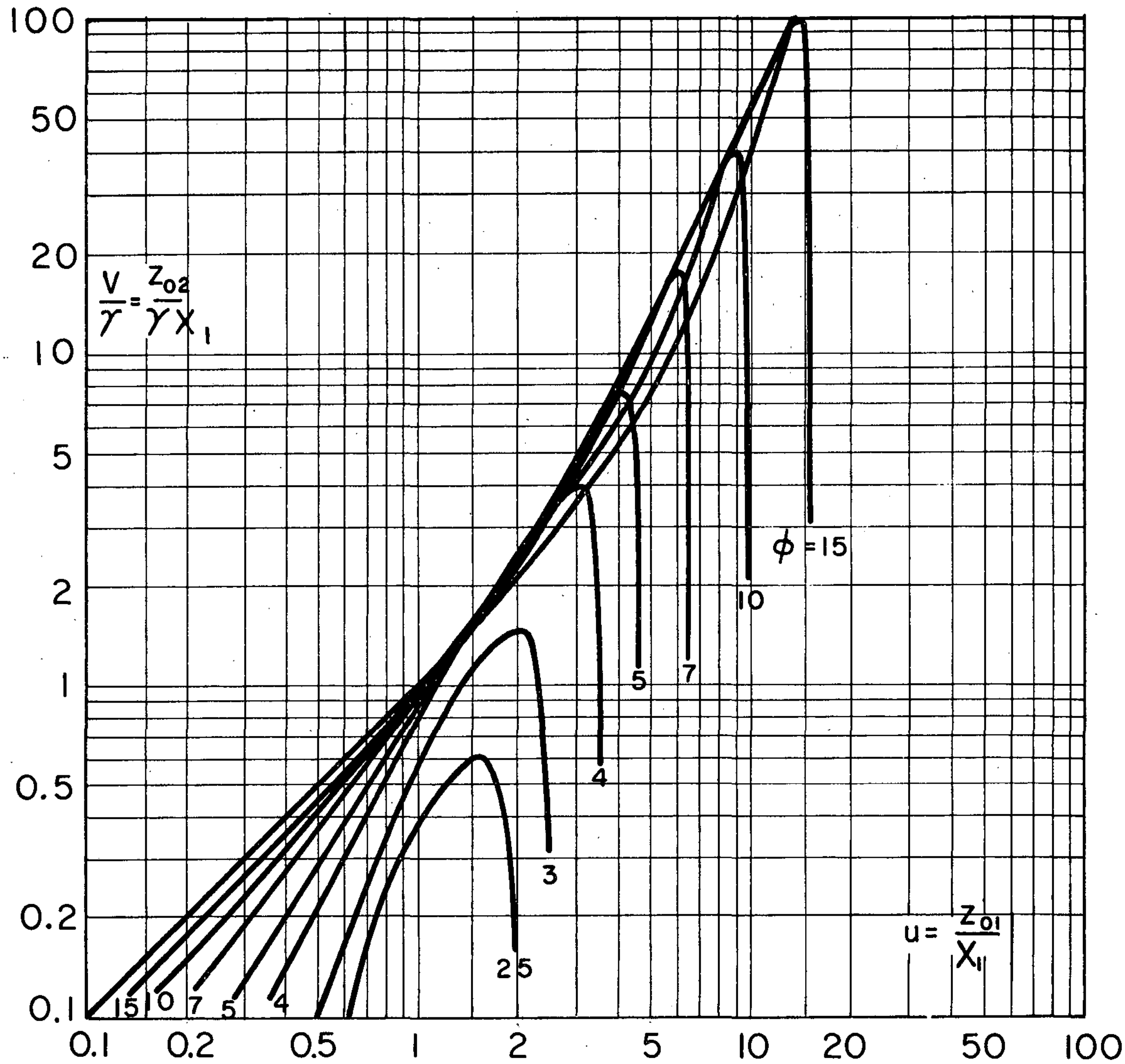

Fig. 5. The function $v(u,\phi)/\gamma$. Negative values, $\phi > 2$.

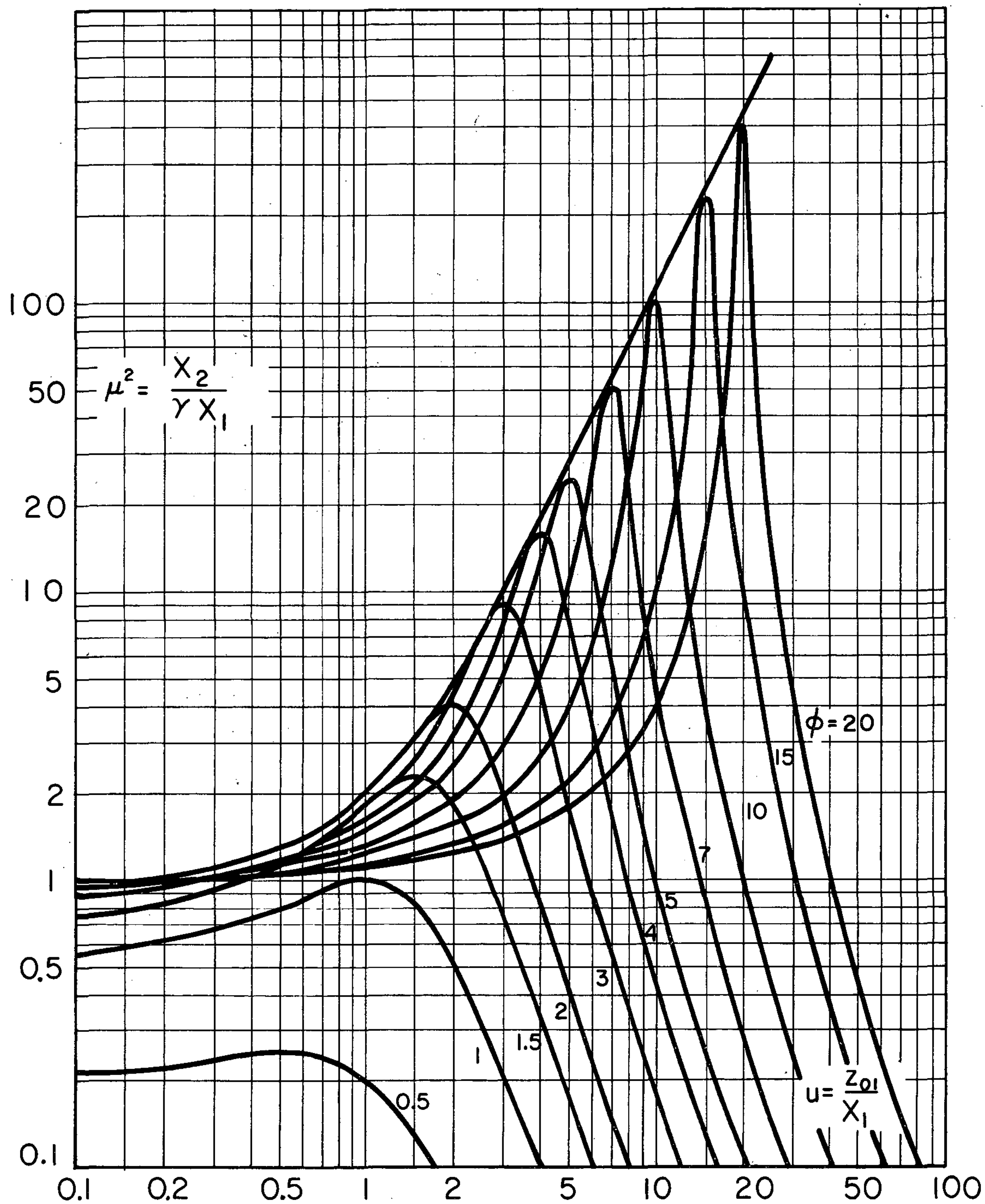

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Fig. 6. The function magnification $\mu^2(u,\phi)$.

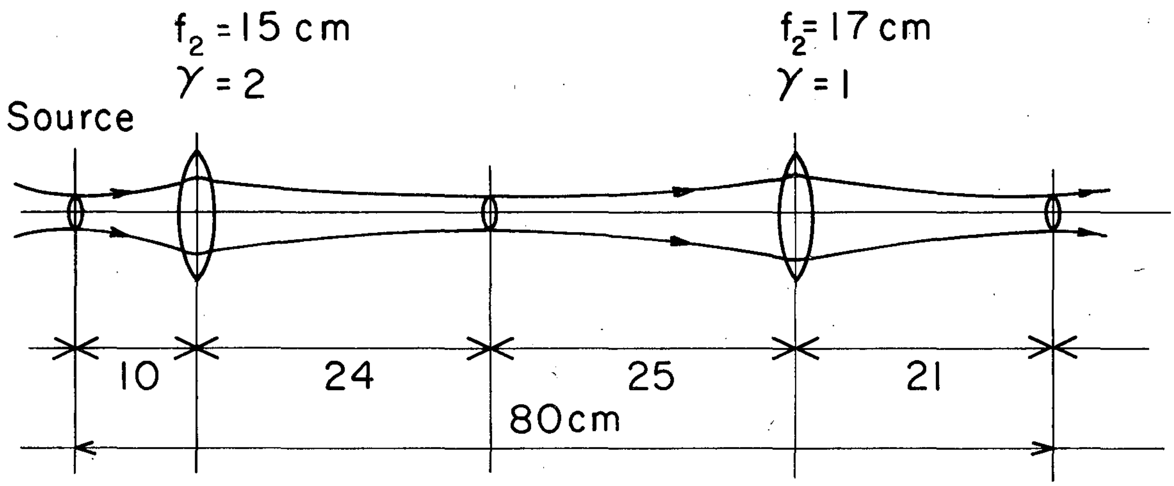

Fig. 7. Example of application.