

# Lawrence Berkeley National Laboratory

## Recent Work

### Title

WAIST-TO-WAIST TRANSFER IN THIN-LENS OPTICS

### Permalink

<https://escholarship.org/uc/item/89p583wd>

### Author

Luccio, Alfredo U.

### Publication Date

1968-05-01

UCRL-18217

1387  
LRL

University of California

Ernest O. Lawrence  
Radiation Laboratory

TWO-WEEK LOAN COPY

*This is a Library Circulating Copy  
which may be borrowed for two weeks*

WAIST-TO-WAIST TRANSFER IN THIN-LENS OPTICS

Alfredo U. Luccio  
May, 1968

Berkeley, California

1387  
LRL

## **DISCLAIMER**

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

UCRL-18217  
UC-34 Physics  
TID-4500 (51st Ed.)

UNIVERSITY OF CALIFORNIA  
Lawrence Radiation Laboratory  
Berkeley, California

AEC Contract No. W-7405-eng-48

WAIST-TO-WAIST TRANSFER IN THIN-LENS OPTICS

Alfredo U. Luccio

May 1968

Printed in the United States of America  
Available from  
Clearinghouse for Federal Scientific and Technical Information  
National Bureau of Standards, U. S. Department of Commerce  
Springfield, Virginia 22151  
Price: Printed Copy \$3.00; Microfiche \$0.65

WAIST-TO-WAIST TRANSFER IN THIN-LENS OPTICS

Alfredo U. Luccio\*

Lawrence Radiation Laboratory  
University of California  
Berkeley, California

May 1968

ABSTRACT

Graphs are presented, which enable one to calculate quickly the waist-to-waist transport of a charged particle beam through a line in the thin-lens approximation. The method can be used also in dealing with accelerating electrostatic lenses (immersion optics).

---

\*On leave from University of Milan, Milan, Italy.

An expression for the waist-to-waist transfer matrix in the case of a thin lens is given, for instance, in Banford's book.<sup>1</sup> However, for many applications it is useful to derive formulas for thin accelerating lenses, and to draw graphs that enable us to calculate quickly an ion-beam transfer line.

We will use the following definitions (see Fig. 1):

- $\epsilon = x_0 x_0'$  emittance of a waist
- $X = x_0 / x_0'$  characteristic length of a waist
- $\mu = x_{02} / x_{01}$  linear magnification
- $\gamma^2 = V_2 / V_1$  acceleration factor
- $z_1, z_2$  distances of a point from the lens, upstream  
and downstream
- $z_{01}, z_{02}$  distances of the waists from the lens
- $f$  focal length (positive for a converging lens).

The thin lens we are discussing can be considered as composed of a thin nonaccelerating lens (L) followed or preceded by a thin accelerating gap (A), which changes the energy  $eV_1$  of an incoming particle into  $eV_2$ . This lens changes the emittance of the beam, and therefore the characteristic length of the second waist, in the following way:

$$\begin{cases} \epsilon_2 = \epsilon_1 / \gamma \\ X_2 = \mu^2 \gamma X_1 \end{cases} \quad (1)$$

In matrix notations, a field-free (drift = D) space can be represented by

$$D = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} ,$$

while a thin nonaccelerating lens is represented by

$$L = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} .$$

Across the accelerating gap, the  $x$  component of the phase-space representation of the beam is not changed, while for the  $x'$  component it is

$$\begin{cases} x_2' = \frac{P_x}{P_2} = \frac{P_x}{P_1} \frac{P_1}{P_2} = \frac{x_1'}{\gamma} \\ P^2 = 2mqV \end{cases} \quad (2)$$

Accordingly, the matrix for the accelerating gap can be written

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\gamma} \end{pmatrix} .$$

For a lens followed by  $A$ , the overall transfer matrix is



$$\begin{aligned}
 M = \text{DALD} &= \begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\gamma} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 - \frac{z_2}{\gamma f} & z_1 + \frac{z_2}{\gamma} - \frac{z_1 z_2}{\gamma f} \\ -\frac{1}{\gamma f} & \frac{1}{\gamma} - \frac{z_1}{\gamma f} \end{pmatrix}
 \end{aligned}$$

The equations

$$\begin{cases} \begin{pmatrix} 0 \\ x_2' \end{pmatrix} = \text{DALD} \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \text{DALD} \begin{pmatrix} 0 \\ x_1' \end{pmatrix} \end{cases}$$

can be used to find the relationship between  $f$  and the measurable focal lengths before and after the lens. The result\* is

$$\begin{cases} f_1 = f \\ f_2 = \gamma f \end{cases} \quad (M = \text{DALD}) \quad (3)$$

---

\* Analogously, for a thin nonaccelerating lens preceded by A, one obtains

$$\begin{cases} f_1 = \frac{f}{\gamma} \\ f_2 = f \end{cases} \quad (M = \text{DLAD}) \quad (3')$$

The transport equations, waist-to-waist, can be written explicitly,

with  $z_{01}$  and  $z_{02}$  in the place of  $z_1$  and  $z_2$ :

$$\begin{cases} x_2 = M_{11}(z_{02})x_1 + M_{12}(z_{01}, z_{02})x'_1 \\ x'_2 = M_{21}x_1 + M_{22}(z_{01})x'_1 \end{cases} \quad (4)$$

At a waist, we can choose  $x_1, x'_1, x_2, x'_2$  as coordinates of points on the contour of upright phase-space ellipses; namely:

$$\begin{cases} \frac{x_1^2}{x_{01}^2} + \frac{x'_1{}^2}{x_{01}^2} = 1 & \text{or} & x_1^2 + X_1^2 x'_1{}^2 = x_{01}^2 \\ \frac{x_2^2}{x_{02}^2} + \frac{x'_2{}^2}{x_{02}^2} = 1 & \text{or} & x_2^2 + X_2^2 x'_2{}^2 = x_{02}^2 \end{cases} \quad (5)$$

Squaring and summing the two equations (4) with the conditions (5), and equating the coefficients of  $x_1^2, x'_1{}^2,$  and  $x_1 x'_1,$  one obtains

$$\begin{cases} \left(1 - \frac{z_{02}}{\gamma f}\right)^2 + \frac{X_2^2}{\gamma^2 f^2} = \mu^2 \\ \left(z_{01} + \frac{z_{02}}{\gamma} - \frac{z_{01} z_{02}}{\gamma f}\right)^2 + \left(\frac{1}{\gamma} - \frac{z_{01}}{\gamma f}\right)^2 X_2^2 = \mu^2 X_1^2 \\ \left(1 - \frac{z_{02}}{\gamma f}\right) \left(z_{01} + \frac{z_{02}}{\gamma} - \frac{z_{01} z_{02}}{\gamma f}\right) - \left(\frac{1}{\gamma} - \frac{z_{01}}{\gamma f}\right) \frac{X_2^2}{\gamma f} = 0 \end{cases}$$

which are satisfied by

$$\begin{cases} \mu^2 = \frac{z_{02} - \gamma f}{\gamma(z_{01} - f)} \\ f^2 = \mu^2 [(z_{01} - f)^2 + X_1^2] \end{cases} \quad (6)$$

Equations (6) reduce to Banford's equations for  $\gamma = 1$  (no acceleration).

It is convenient to rearrange Eq. (6) in the following fashion:

$$\begin{cases} z_{02} = \gamma f \frac{(z_{01} - f)z_{01} + X_1^2}{(z_{01} - f)^2 + X_1^2} \\ \mu^2 = \frac{f^2}{(z_{01} - f)^2 + X_1^2} \end{cases} \quad (6')$$

Equations (6') together with Eq. (1) can be used to design a transport line, from the starting values of

$$\epsilon_1, X_1 \quad \text{or} \quad x_{01}, x'_{01},$$

which determine completely the initial phase-space configuration of the beam.

Before developing a graphical method to find quickly solutions of the system (6'), it is interesting to rewrite the (6') in a dimensionless form:

$$\left\{ \begin{array}{l} t = \frac{s(s-1) + \alpha^2}{(s-1)^2 + \alpha^2} \\ \mu^2 = \frac{1}{(s-1)^2 + \alpha^2} \end{array} \right. , \quad (6'')$$

with the following definitions:

$$s = \frac{z_{01}}{f}, \quad t = \frac{z_{02}}{\gamma f}, \quad \alpha = \frac{x_1}{f} .$$

The behaviour of the functions  $t(s, \alpha)$  and  $\mu^2(s, \alpha)$  is sketched in Fig. 2.

The formulas derived so far must reduce to the classical ones for thin-lens--point-object optics. In the classical limit,

$$\alpha \ll 1, \quad s, \quad s-1,$$

we have, to second order in  $\alpha/s$  and  $\alpha/(s-1)$ ,

$$\left\{ \begin{array}{l} \frac{1}{s} + \frac{1}{t} = 1 + \frac{\alpha^2}{s^2(s-1)} \\ \mu = \frac{1}{s-1} \left[ 1 - \frac{\alpha^2}{2(s-1)^2} \right] \end{array} \right. . \quad (7)$$

For  $\gamma = 1$ , Eqs. (7) become (see Fig. 3)

$$\begin{cases} \frac{1}{z_{01}} + \frac{1}{z_{02}} = \frac{1}{f} \left[ 1 + \frac{f}{z_{01}-f} \frac{X_1^2}{z_{01}^2} \right] = \frac{1}{f} \left[ 1 + \frac{z_{02}}{z_{01}} \frac{X_1^2}{z_{01}^2} \right] \\ \mu = \frac{f}{z_{01}-f} \left[ 1 - \frac{X_1^2}{2(z_{01}-f)^2} \right] = \frac{z_{02}}{z_{01}} \left[ 1 - \frac{1}{2} \left( \frac{z_{01}+z_{02}}{z_{01}} \right)^2 \frac{X_1^2}{z_{01}^2} \right] \end{cases} \quad (8)$$

In another, more useful form, obtained by using the definitions

$$u = \frac{z_{01}}{X_1}, \quad v = \frac{z_{02}}{X_1}, \quad \text{and } \phi = \frac{f}{X_1},$$

Eqs. (1) and (6') are

$$\begin{cases} X_2 = \mu^2 \gamma X_1 \\ v = \gamma \phi \frac{u(u-\phi) + 1}{(u-\phi)^2 + 1} \\ \mu^2 = \frac{\phi^2}{(u-\phi)^2 + 1} \end{cases} \quad (6''')$$

The function  $v(u, \phi)/\gamma$  is plotted in Figs. 4 (for positive values) and 5 (negative values). It is easy to recognize that such negative values exist only for  $\phi > 2$ . The function  $\mu^2(u, \phi)$  is plotted in Fig. 6.

An example of application can be useful to understand the convenience of the plots. Let us consider a line composed by an ion gun with proper

(accelerating) optics, plus another (nonaccelerating) lens, e.g., an "Einzel" lens. The purpose of the line is to transfer a crossover (waist) of given characteristics near the ion source to a given point at a given distance (Fig. 7). If we refer to the geometrical data of Fig. 7 and to the following properties of the starting waist:

$$\left\{ \begin{array}{l} x_{01} = 0.5 \text{ cm} \\ x'_{01} = 0.100 \text{ rad} \end{array} \right. \quad \left\{ \begin{array}{l} \epsilon_1 = 0.05 \text{ cm-rad} \\ X_1 = 5 \text{ cm} \end{array} \right. ,$$

the calculation can run as follows:

First lens For  $\gamma = 2$ ,  $z_{01} = 10$  cm and accordingly for

$$u = \frac{z_{01}}{X_1} = 10:5 = 2$$

a reasonable value for  $z_{02}$ , corresponding to a second waist midway between the first and the second lens, is  $z_{02} = 24$  cm. This value corresponds to

$$\frac{v}{\gamma} = \frac{z_{02}}{\gamma X_1} = 2.4$$

which can be obtained from the graph of Fig. 4 with

$$\phi = 1.5; \text{ or } f = 7.5 \text{ cm, } f_2 = 15 \text{ cm.}$$

The corresponding values for  $\mu^2$  and  $X_2$  obtained from the graph of Fig. 6 are

$$\mu^2 = 1.8, X_2 = \mu^2 \gamma X_1 = 18 \text{ cm.}$$

Second lens For  $\gamma = 1$ ,  $z_{01} = 25$  cm and accordingly for

$$u = \frac{z_{01}}{X_1} = 25:18 = 1.4,$$

we can obtain (Fig. 4) a value  $z_{02} = 21$  cm, or  $\frac{v}{\gamma} = \frac{z_{02}}{\gamma X_1} = 21:18 = 1.17$ ,

with  $\phi = 0.95$ ; or  $f = f_2 = 17$  cm,

the resulting magnification (Fig. 6) is

$$\mu^2 = 0.8$$

The overall magnification is

$$\mu_{ov} = (0.8 \times 1.8)^{\frac{1}{2}} = 1.2$$

and the final emittance is

$$\epsilon_{ov} = \epsilon_{in}/\gamma = 0.025 \text{ cm-rad.}$$

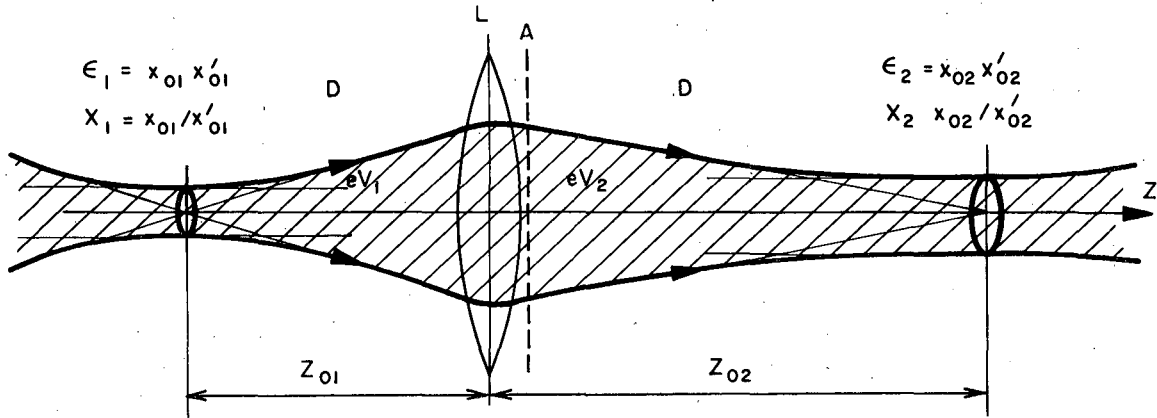
#### ACKNOWLEDGMENT

This work was done under the auspices of the U. S. Atomic Energy Commission.

REFERENCE

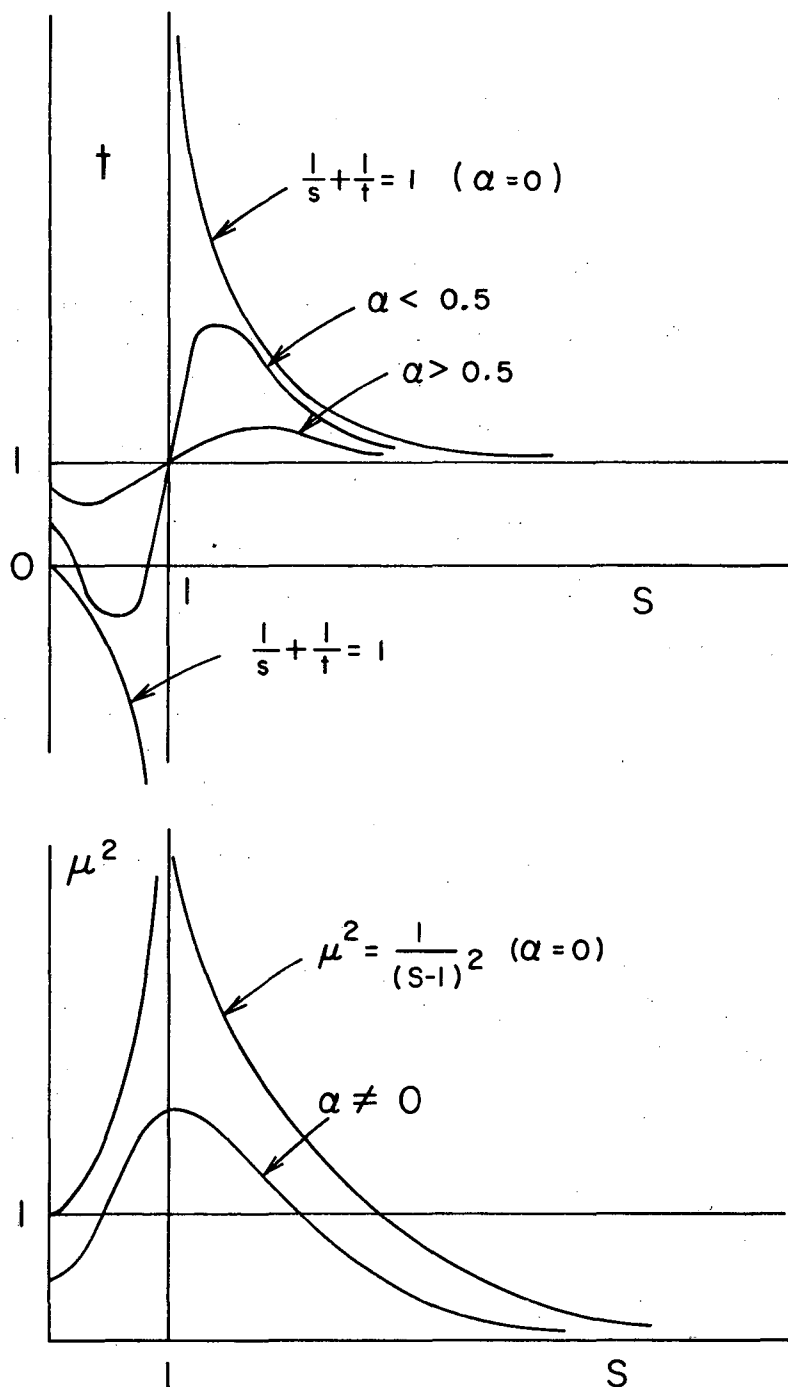
1. A. P. Banford, "The Transport of Charged Particle Beams" (E. and F. N. Spon, Ltd., London, 1966).





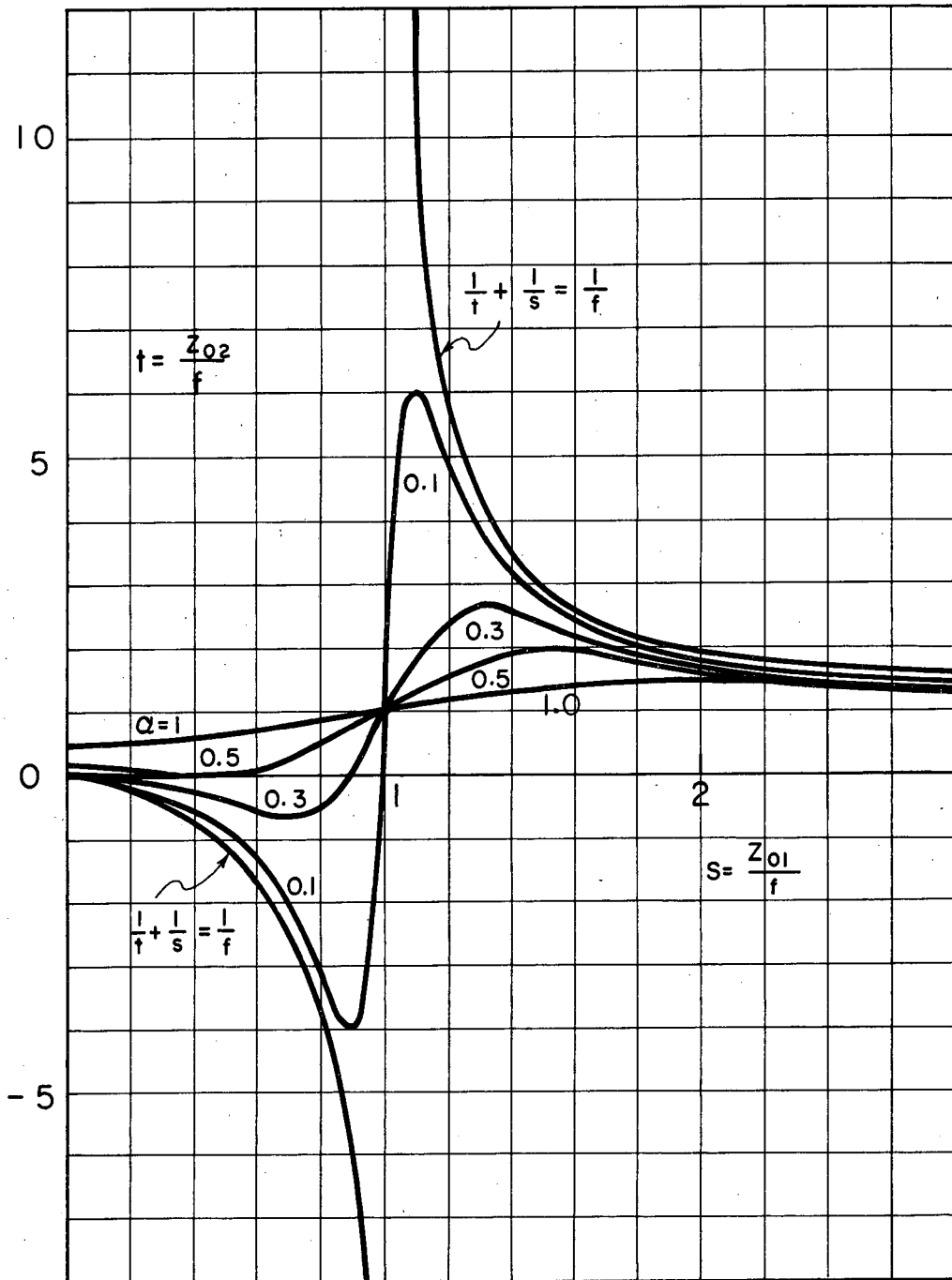
XBL684-2601

Fig. 1. A thin accelerating lens.



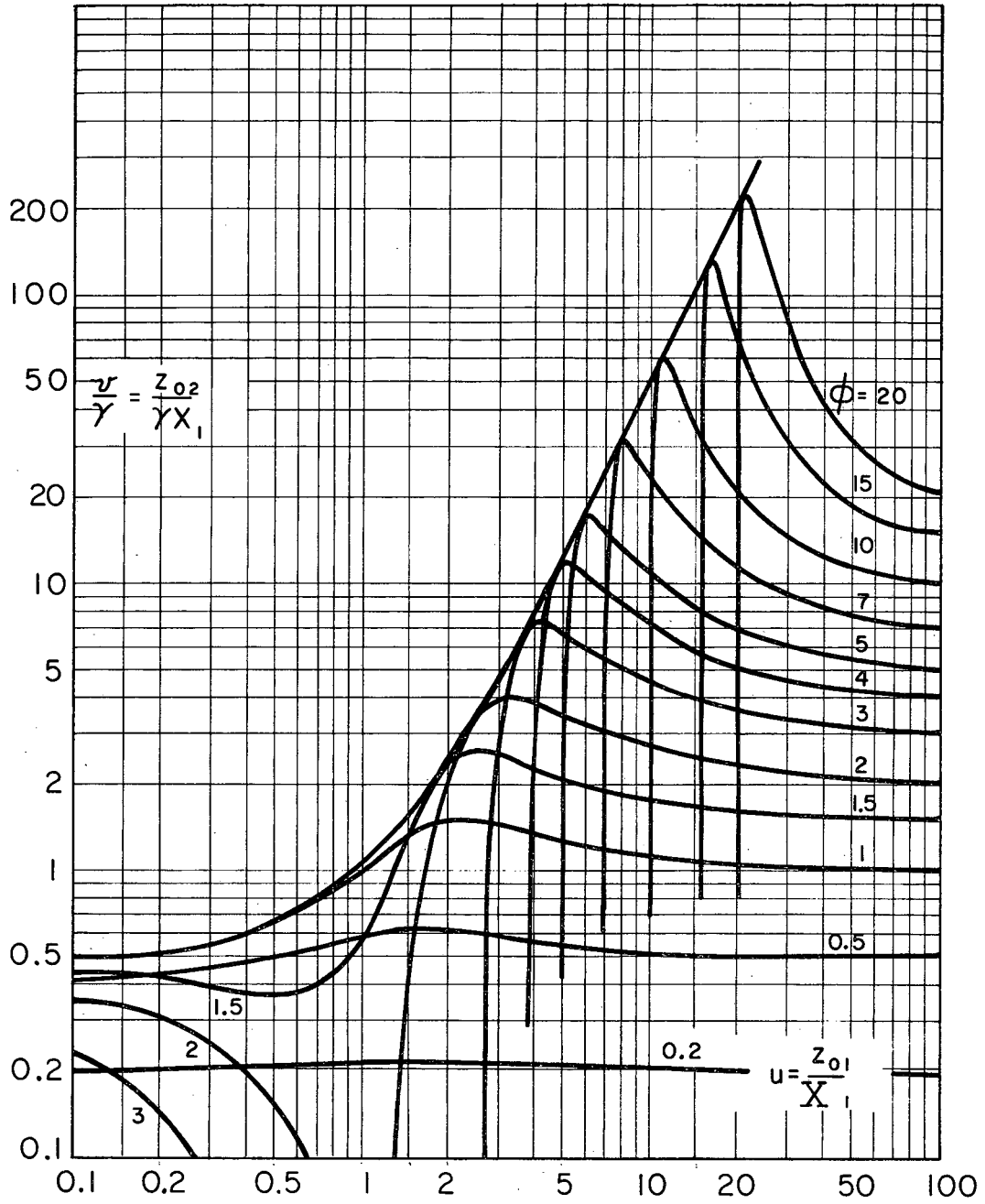
XBL684-2605

Fig. 2. The functions  $t(s, \alpha)$  and  $\mu^2(s, \alpha)$  and their point-optics limit.



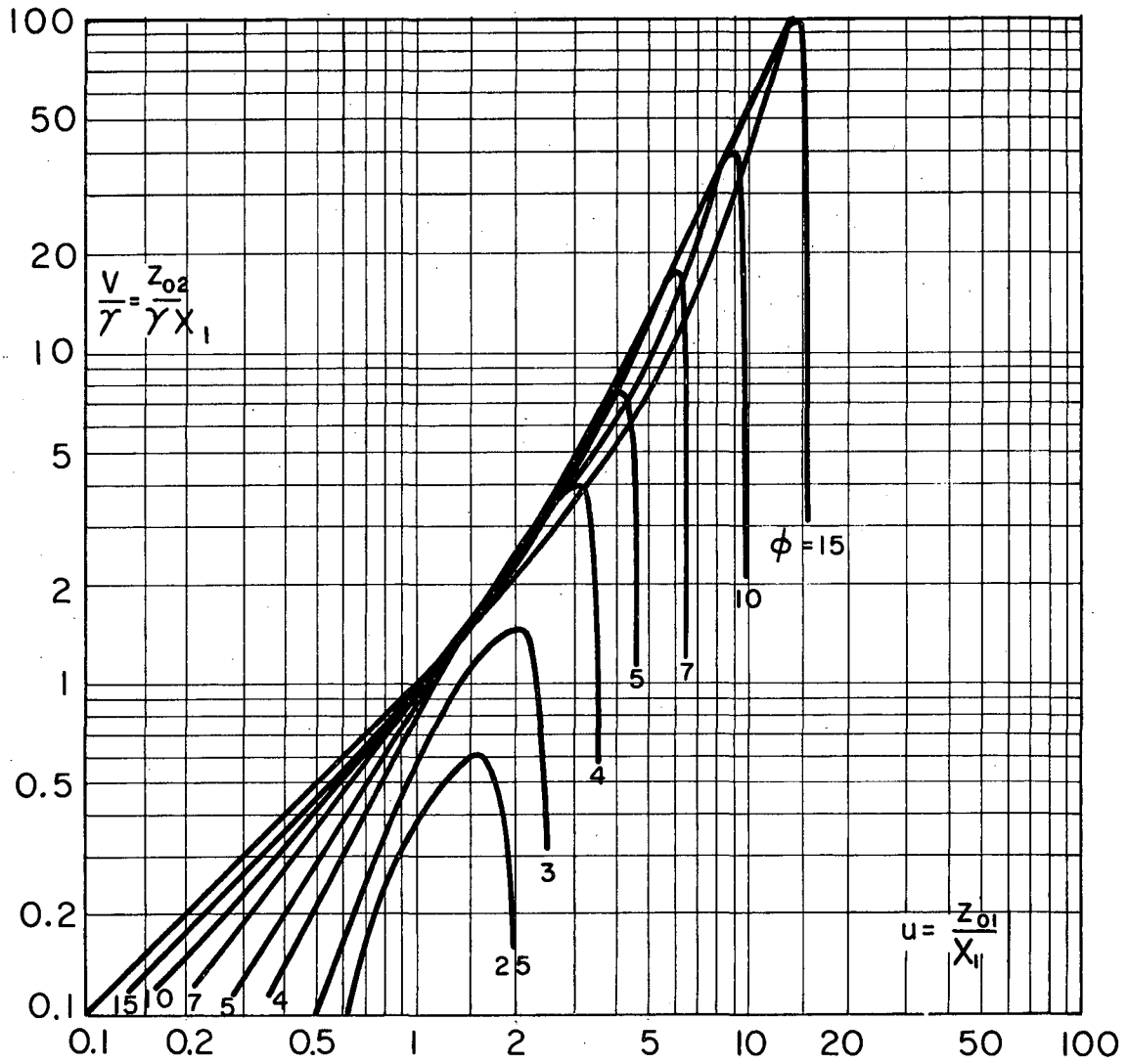
XBL684 - 2599

Fig. 3. The function  $t(s, \alpha) = \frac{[s(-1) + \alpha^2]}{[(s-1)^2 + \alpha^2]}$ .



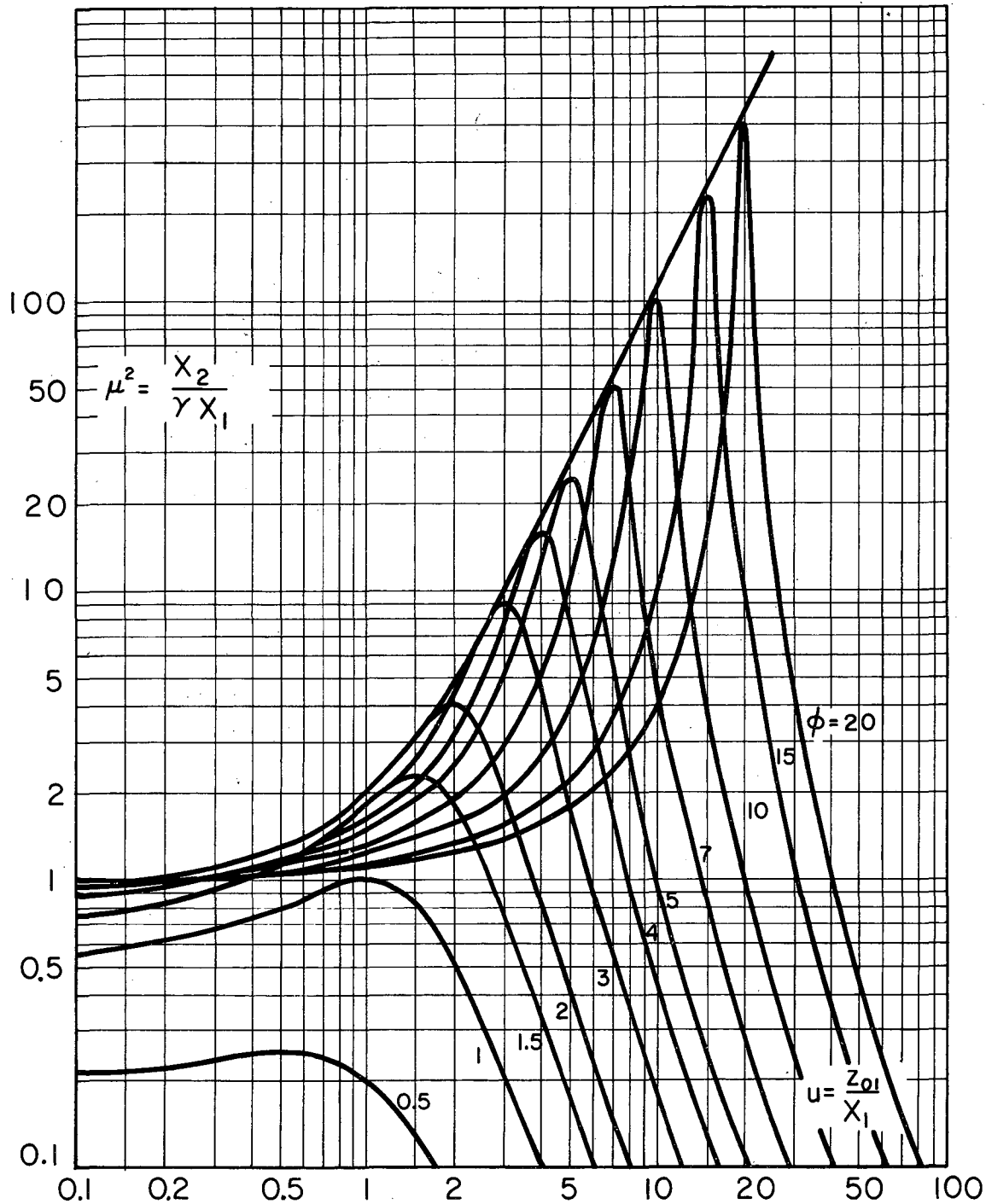
X BL684-2603

Fig. 4. The function  $v(u, \phi) / \gamma$ . Positive values.



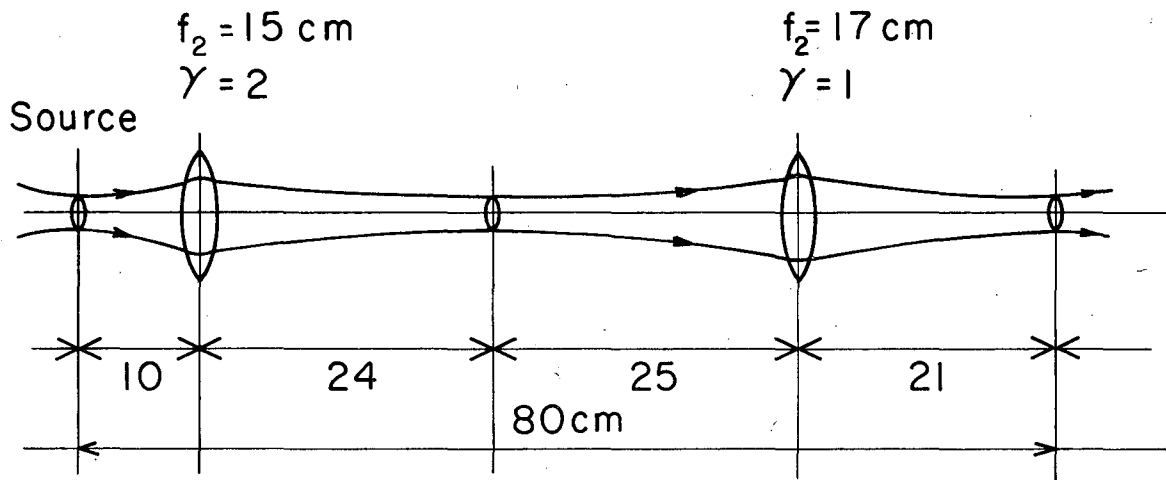
XBL684-2602

Fig. 5. The function  $v(u, \phi) / \gamma$ . Negative values,  $\phi > 2$ .



XBL684 - 2604

Fig. 6. The function magnification  $\mu^2(u, \phi)$ .



XBL684-2600

Fig. 7. Example of application.

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

- A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or
- B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.



