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APPROXIMATING THE DISTRIBUTION OF PARETO SUMS

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**Abstract.** Heavy tailed random variables (rvs) have proven to be an essential element in modeling a wide variety of natural and human induced processes, and the sums of heavy tailed rvs represent a particularly important construct in such models. Oriented toward both geophysical and statistical audiences, this paper discusses the appearance of the Pareto law in seismology and addresses the problem of the statistical approximation for the sums of independent rvs with common Pareto distribution  $F(x) = 1 - x^{-\alpha}$  for  $1/2 < \alpha < 2$ . Such variables have infinite second moment which prevents one from using the Central Limit Theorem to solve the problem. This paper presents five approximation techniques for the Pareto sums and discusses their respective accuracy. The main focus is on the median and the upper and lower quantiles of the sum's distribution. Two of the proposed approximations are based on the Generalized Central Limit Theorem, which establishes the general limit for the sums of independent identically distributed rvs in terms of stable distributions; these approximations work well for large numbers of summands. Another approximation, which replaces the sum with its maximal summand, has less than 10% relative error for the upper quantiles when  $\alpha < 1$ . A more elaborate approach considers the two largest observations separately from the rest of the observations, and yields a relative error under 1% for the upper quantiles and less than 5% for the median. The last approximation is specially tailored for the lower quantiles, and involves reducing the non-Gaussian problem to its Gaussian equivalent; it too yields errors less than 1%. Approximation of the observed cumulative seismic moment

in California illustrates developed methods.

**Short running title:** PARETO SUMS

**Key words:** Pareto distribution, Pareto truncated distribution, Seismic moment distribution for earthquakes, Stable distributions, Approximation of Pareto sums

# 1 Introduction

Statistical data analysis, a significant part of modern Earth Sciences research, is led by the intuition of researchers traditionally trained to think in terms of “averages”, “means”, and “standard deviations”. Curiously, an essential part of relevant natural processes does not allow such an interpretation, and appropriate statistical models do not have finite values of these characteristics. Seismology presents a superb example of such a situation with one of its fundamental laws, describing the distribution of the moment (or energy) released in earthquakes as a power-law, having both infinite mean and standard deviation. This paper concerns the problem of the statistical approximation for sums of power-law variables, which are used to describe the total moment or energy released by multiple earthquakes, for example.

## 1.1 Pareto distribution

Many natural and human-induced phenomena exhibit power-law behavior: for instance the power-law is observed to approximate the distribution of sizes of earthquakes and volcanic eruptions, solar-flares, lightning strikes, river networks, forest fires, extinctions of biological species, war casualties, internet traffic, stock returns, insurance pay-offs, and cities (see e.g. MANDELBROT, 1983; RICHARDSON, 1960; TURCOTTE, 1997; BARTON and LA POINTE, 1995; SORNETTE, 2000; NEWMAN *et al.*, 1994); this list can easily be extended. The power-law size distribution implies that the number  $N(x)$  of objects of size larger than  $x$  decreases as a power of  $x$ :

$$N(x) \sim x^{-\alpha}, \quad \alpha > 0. \quad (1)$$

The power-law is scale invariant since the scale change  $y = ax$  affects only the normalization constant in (1). Properly normalized, the law (1) is known as the *Pareto distribution*, its cumulative distribution function (cdf)  $F(x)$  and probability distribution function (pdf)  $f(x)$  given by

$$\begin{aligned} F(x) &= 1 - x^{-\alpha}, \\ f(x) &= \alpha x^{-1-\alpha}, \quad x > 1, \quad \alpha > 0. \end{aligned} \quad (2)$$

In many cases, the distribution (2) appears to fit well to the largest observations,  $x > x_0$ , but not for the whole sample. We describe such processes, where the survivor function  $1 - F(x) \approx x^{-\alpha}$ , as having power-law *tails*. This description is especially useful when describing and modeling processes with large deviations, a situation where one is primarily interested in the largest possible observations, and the specific distribution of the smaller observations may be neglected. A distribution that assigns a non-ignorable probability to extremely large observations is called *heavy tailed*. One refers to a random variable (rv) as heavy tailed if it has an infinite second moment (infinite variation). For Pareto random variables, this corresponds to the case  $0 < \alpha < 2$ ; for  $\alpha \leq 1$  a Pareto rv has also infinite first moment (expectation).

## 1.2 Earthquake size distribution

The distribution of earthquake sizes is described by the well-known Gutenberg-Richter (GR) magnitude-frequency relation (GUTENBERG and RICHTER, 1941, 1944; SCHOLZ, 2002):

$$\log_{10} [N(m)] = a - b m, \quad b \approx 1, \quad (3)$$

where  $N(m)$  is the annual number of earthquakes with magnitude equal or larger than  $m$ . Recalling that earthquake magnitude is related to the scalar seismic moment  $M$  via

$$m = \frac{2}{3} \log_{10} M + c, \quad (4)$$

we see that the GR relation is nothing but a power law for the number  $N(M)$  of earthquakes with seismic moment above  $M$ :

$$N(M) \sim M^{-\alpha}, \quad \alpha = \frac{2}{3} b. \quad (5)$$

Introducing the appropriate normalization, one obtains the Pareto pdf for seismic moments:

$$f(M) = \alpha M_t^\alpha M^{-1-\alpha}, \quad M_t \leq M, \quad (6)$$

where  $M_t$  is a catalog completeness threshold (or observational cutoff) and  $\alpha \approx \frac{2}{3}$  is the index parameter of the distribution. In order for the total seismic moment to be finite, the

distribution density must decay faster than  $M^{-2}$ . Thus, simple considerations of the finiteness of the seismic moment flux or deformational energy available for earthquake generation (KNOPOFF and KAGAN, 1977) require the Pareto relation (6) to be modified for large seismic moments. This can be done, for example, by applying an exponential taper to the survivor function  $1 - F(M)$ , so that it takes the form

$$1 - F(M) = (M_t/M)^\alpha \exp\left(-\frac{M_t - M}{M_c}\right), \quad M_t \leq M < \infty. \quad (7)$$

Here  $M_c$  is the *maximum* or *corner* magnitude, a parameter that primarily controls the distribution in the upper ranges of  $M$  (VERE-JONES *et al.*, 2001; KAGAN and SCHOENBERG, 2001; KAGAN, 2002a). To illustrate the above reasoning, Fig. 1 displays cumulative histograms for the scalar seismic moment of shallow earthquakes in the Harvard catalog (DZIEWONSKI *et al.*, 2001) during 1977-2002. The curves display a scale-invariant (Pareto) segment (linear in the log-log plot) for small and moderate values of the seismic moment  $M$ . But for large  $M$ , the curve clearly bends downward.

Although the index parameter for the seismic moment distribution appears to have a universal value  $\alpha \approx \frac{2}{3}$  (KAGAN, 2002a; 2002b), some geometrical variables which depend on seismic moments, such as slips during earthquakes, or widths and lengths of earthquake ruptures, appear to have power-law distributions with varying index values (KAGAN, 1994; ANDERSON and LUCO, 1983; WELLS and COPPERSMITH, 1994). Thus it is relevant to consider Pareto random variables with a broad range of indices, and in this work we consider the range  $1/2 \leq \alpha < 2$ .

For the reasons explained above, for index  $\alpha \leq 1$  one is often interested in the case of the Pareto distribution with an upper bound or taper. MCCAFFREY (1997), SHEN-TU *et al.* (1998) and HOLT *et al.* (2000) considered the statistical bounds for the sums of upper-truncated Pareto rvs, for the case when several earthquakes approaching the maximum size are observed. In these cases the distribution of the sum can be approximated using the Central Limit Theorem (see more in Section 5). However, in actual earthquake catalogs these largest events are very rarely observed, so that bounds on the sums must often be estimates even when no earthquake of size approximately  $M_c$  is registered. Similar problems are encountered in the estimation of the parameters of the seismic moment-frequency law (7).

KAGAN and SCHOENBERG (2001) and KAGAN (2002a) discuss the difficulty of estimating the maximum or corner moment with insufficient data.

### 1.3 Heavy tailed sums

In many situations one is interested in *sums* of power-law distributed variables. For example, total deformation of rock materials or of the Earth's surface due to earthquakes may be modeled as sum of seismic moment tensors (KOSTROV, 1974; KAGAN, 2002b), the latter obeying the Pareto distribution; cumulative economic losses or casualties due to natural catastrophes are modeled as sums of power-law distributed variables (KAGAN, 1997; PISARENKO, 1998; RODKIN and PISARENKO, 2000); the total pay-off of an insurance company is modeled as the sum of individual pay-offs, each of which is distributed according to a power law, etc. The Pareto distribution is the simplest of heavy-tailed distributions, thus the properties of Pareto sums are easier to study.

For  $0 < \alpha < 2$  the second moment of a Pareto rv is infinite which renders useless many of the conventional statistical techniques commonly used since the early 19-th century. The most prominent of these is the Central Limit Theorem (FELLER, 1971, **2**, VIII, 4), which justifies approximating the sum of independent rvs by a Gaussian (normal) distribution. The sums of heavy tailed rvs cannot be approximated this way. However, such sums, when suitably normalized, typically approach a well-defined limiting distribution which depends on  $\alpha$ . The entire family of such limits is known as *stable probability distributions with parameter  $\alpha$*  (SAMORODNITSKY and TAQQU, 1994; UCHAIKIN and ZOLOTAREV, 1999; see also Section 2.1 below.) The Gaussian distribution is a special case corresponding to  $\alpha = 2$ .

Stable distributions, which (except for the Gaussian case) have power-law tails, recently became an object of intense mathematical and practical development (MANDELROT, 1983; ZOLOTAREV, 1986; UCHAIKIN and ZOLOTAREV, 1999; NOLAN, 2004; RACHEV and MITTNIK, 2000; RACHEV, 2003). Their use is widespread in physics, finance, and other disciplines. The sums of large numbers of heavy-tailed rvs are investigated in many publications, especially in finance, and are well described in terms of stable distributions (MITTNIK *et al.*, 1998; RACHEV and MITTNIK, 2000; RACHEV, 2003; EMBRECHTS *et al.*, 1997.) However, little is known about the distribution of the sum of an arbitrary (intermediate to small)

number of Pareto rvs.

## 1.4 Approximation problem

The aim of this study is to approximate the distribution of Pareto sums with arbitrary numbers of summands. Formally, let  $X_i$ ,  $i = 1, \dots, n$  be iid rvs with a common Pareto distribution (2), and let  $S_n$  denote their sum

$$S_n = \sum_{i=1}^n X_i. \quad (8)$$

We seek to approximate certain quantiles  $z_q(n)$  of  $S_n$ :

$$\text{Prob}\{S_n < z_q(n)\} = q, \quad 0 \leq q \leq 1, \quad n > 1, \quad (9)$$

where  $\text{Prob}\{A\}$  means the probability of event  $A$ . The *median*  $z_{1/2}$  as well as *upper and lower bounds*  $z_q$ ,  $z_{1-q}$ , for  $q \ll 1$ , are of special interest. As mentioned above, the problem becomes non-trivial for  $0 < \alpha < 2$  when  $X_i$  have infinite second moments; for  $0 < \alpha \leq 1$  these rvs also have infinite expectation.

This paper is organized as follows. In Section 2 we construct two approximations for the sum (8) by using stable distributions. Stable distributions are discussed in more detail in Section 2.1; this discussion should help to develop a feeling for these complex distributions which sometimes display quite counter-intuitive behavior. Section 3 describes a different approach, involving the use of the distribution of the largest observation to approximate that of the entire sum (8). It is shown that for  $\alpha < 1$  this seemingly crude replacement may result in fairly good approximations for the upper (0.98) quantile. The results of Sections 2-3 are asymptotic; they give good approximations for the sums of a large number  $n$  of summands. However, unlike the Gaussian case, when “large” typically means  $n > 30$ , the sum of heavy-tailed variables converge very slowly, so that a sufficient number of summands allowing one to apply the asymptotics may in some cases be greater than  $n = 10^4$ . To address this problem, Section 4 describes two techniques specially tailored for approximating the sum  $S_n$  of an arbitrary number  $n$  of summands. Section 4.1 introduces an approach based on the analysis of order statistics to approximate the median and upper quantiles of  $S_n$ . Section



4.2 shows how to transform our non-Gaussian problem to a Gaussian equivalent and use the Central Limit Theorem to approximate the lower quantile. Approximation for the sum of truncated Pareto rvs is considered briefly in Section 5. The study's results are discussed in Section 6. We keep in the main text only the essential formulae; the more complex mathematical derivations are placed in Appendices.

## 2 Approximating by stable distributions

In this section we approximate Pareto sums with *stable distributions*, which is advocated by the Generalized Central Limit Theorem discussed in Section 2.2 below. We start with a brief discussion of stable distributions and their basic properties.

### 2.1 Univariate stable distributions

Stable distributions are a rich class of probability distributions that allow skewness (asymmetry) and heavy tails and have many intriguing mathematical properties (ZOLOTAREV, 1986; SAMORODNITSKY and TAQQU, 1994; UCHAIKIN and ZOLOTAREV, 1999). A random variable  $X$  is said to have a stable distribution if for any  $n \geq 2$ , there is a positive number  $C_n$  and a real number  $D_n$  such that

$$X_1 + X_2 + \dots + X_n \stackrel{d}{=} C_n X + D_n, \quad (10)$$

where  $X_1, X_2, \dots, X_n$  are independent copies of  $X$ , and  $\stackrel{d}{=}$  means that the variables have the same distribution. This property makes stable distributions of special importance when working with sums of rvs. The lack of closed formulas for densities and distribution functions for all but a few stable distributions (Gaussian, Cauchy and Lévy) has been a major drawback to the use of stable distributions by practitioners. There are now reliable computer programs to compute stable densities, distribution functions and quantiles (NOLAN, 1997; MCCULLOCH and PANTON, 1997; 1998). With these programs, it is possible to use stable models in a variety of practical problems.

Different authors have provided several quite distinct representations for stable distributions (see Introduction in ZOLOTAREV, 1986; pp. 7-9 in SAMORODNITSKY and TAQQU,

1994; Section 3.6 in UCHAIKIN and ZOLOTAREV, 1999), which complicates the practical use of these distributions and often leads to confusion (HALL, 1981). Moreover, there are many misprints and other errors in published formulae for stable distributions (see, for instance, remark 2.12 in ZOLOTAREV, 1986). Below we use only the expressions which have been tested by comparing with tabulated values (MCCULLOCH and PANTON, 1997; 1998).

Stable distributions have been proposed as a model for many types of physical and economic systems. There are several reasons for using a stable distribution to describe a system. The first is the presence of solid theoretical reasons for expecting non-Gaussian stable behaviour, e.g., reflection of a rotating mirror yielding the Cauchy distribution, hitting times for a 1D Brownian motion yielding the Lévy distribution, the gravitational field of stars yielding the Holtsmark 3D distribution, the stress distribution in a solid with defects following the Cauchy law (see FELLER, 1971 and UCHAIKIN and ZOLOTAREV, 1999 for other examples). The second reason is the Generalized Central Limit Theorem (see Section 2.2, Eq. 19) which states that the only possible non-trivial limit of normalized sums of independent identically distributed (iid) terms is a stable law. Some observed quantities are posited to be the sum of many individual terms — the price movements of a stock, the noise in a communication system, etc., and hence a stable model may be used to describe such systems. A third argument for the use of stable distributions in modeling physical systems is empirical: many large data sets exhibit heavy tails and skewness. The strong empirical evidence for these features combined with the Generalized Central Limit Theorem is used by many to justify the use of stable models.

The univariate stable distribution is generally characterized by four parameters: its index  $\alpha$ , the parameter  $\beta$  characterizing the degree of skewness, a scale parameter, and a shift parameter (SAMORODNITSKY and TAQQU, 1994). In this work we will consider only normalized stable distributions with the scale parameter equal to 1.0 and the shift parameter equal zero. Moreover, since the Pareto variables we consider here are positive, the sums of these variables converge to a maximally asymmetrical (maximally-skewed) stable distribution, corresponding to  $\beta = 1$ . Thus, the stable variables considered below depend only on one parameter  $\alpha$ , and their pdf and cdf are denoted as  $f_\alpha$  and  $F_\alpha$  respectively.

In Fig. 2 we show an example of the pdf for the stable distribution with index  $\alpha = 2/3$

(15) and a corresponding Pareto distribution. The stable distributions with  $\alpha < 1$  and  $\beta = 1$  are concentrated on the positive  $x$ -axis, whereas maximally-skewed distributions with  $\alpha \geq 1$  and  $\beta = 1$  have support over the whole  $x$ -axis.

Recall that the standard Gaussian (normal) cdf with expectation  $\mu$  and standard deviation  $\sigma$  is given by

$$\Phi(x; \mu, \sigma^2) = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy. \quad (11)$$

The Gaussian *stable* distribution,  $\alpha = 2$ , for which the skewness index  $\beta$  is not defined, is usually written in a slightly different form:  $F_2(x) = \Phi(x; 0, 2)$  so its pdf becomes

$$f_2(x) = \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{x^2}{4}\right). \quad (12)$$

Only one of the stable maximally-skewed distributions, the Lévy, with  $\alpha = 1/2$ , can be expressed through elementary functions. This distribution has density

$$f_{1/2}(x) = \frac{1}{x\sqrt{2\pi x}} \exp\left(-\frac{1}{2x}\right), \quad (13)$$

and cdf

$$F_{1/2}(x) = 2 \left[ 1 - \Phi\left(\frac{1}{\sqrt{x}}; 0, 1\right) \right], \quad (14)$$

where  $\Phi$  is defined in (11).

Two more maximally-skewed distributions, for  $\alpha = 2/3$  and  $\alpha = 3/2$ , can be expressed through special functions (ZOLOTAREV, 1954):

$$f_{2/3}(x) = \frac{\sqrt{3}}{x\sqrt{\pi}} \exp\left(-\frac{16}{27x^2}\right) W_{1/2, 1/6}\left(\frac{32}{27x^2}\right) \quad \text{for } x > 0, \quad (15)$$

where  $W_{1/2, 1/6}$  is a Whittaker function (GRADSHTEYN and RYZHIK, 1980, p. 1059).

The Whittaker function  $W_{k, \mu}(z)$  can be calculated using the confluent hypergeometric function  $U(a, b, z)$  (WOLFRAM, 1999, pp. 770-771)

$$W_{k, \mu}(z) = \frac{z^{0.5+\mu} U(0.5 - k + \mu, 1 + 2\mu, z)}{e^{0.5z}}. \quad (16)$$

Similar expressions for  $f_{3/2}(x)$  are (ZOLOTAREV, 1954)

$$f_{3/2}(x) = -\frac{\sqrt{3}}{x\sqrt{\pi}} \exp\left(\frac{x^3}{27}\right) W_{1/2, 1/6}\left(-\frac{2x^3}{27}\right) \quad \text{for } x < 0, \quad (17)$$

and

$$f_{3/2}(x) = \frac{1}{2x\sqrt{3\pi}} \exp\left(\frac{x^3}{27}\right) W_{-1/2,1/6}\left(\frac{2x^3}{27}\right) \quad \text{for } x > 0. \quad (18)$$

The cumulative functions for distributions (15), and (17, 18) can be obtained by numerical integration (WOLFRAM, 1999).

Though stable distributions usually cannot be summarized via elementary functions, their tails often allow fairly simple approximations given in Appendix A. These approximations may be useful for  $q$  close to 0 or 1.

## 2.2 Generalized Central Limit Theorem approximation

Statistical inference about the sum (8) can be made using the **Generalized Central Limit Theorem** (GCLT) (SAMORODNITSKY and TAQQU, 1994, p. 50; UCHAIKIN and ZOLOTAREV, 1999, p. 62), which states that the properly normalized sum  $S_n$  (8) of a large number  $n$  of iid Pareto rvs with common distribution (2) may be approximated by a stable distribution:

$$\lim_{n \rightarrow \infty} \text{Prob} \left\{ \frac{S_n - b_n}{n^{1/\alpha} C_\alpha} < x \right\} = F_\alpha(x), \quad (19)$$

where  $F_\alpha(x)$  is a stable cdf with index  $\alpha$ . The normalization and shift coefficients are given by

$$C_\alpha = \begin{cases} [\Gamma(1 - \alpha) \cos(\pi\alpha/2)]^{1/\alpha}, & \alpha \neq 1, \\ \pi/2, & \alpha = 1, \end{cases} \quad (20)$$

$$b_n = \begin{cases} 0, & 0 < \alpha < 1, \\ \frac{\pi n^2}{2} \int_1^\infty \sin\left(\frac{\pi x}{2n}\right) dF(x), & \alpha = 1, \\ n\alpha/(\alpha - 1), & 1 < \alpha < 2, \end{cases} \quad (21)$$

where  $\Gamma(x)$  is the gamma function. The integral in (21) for  $\alpha = 1$  can be expanded as (GRADSHTEYN and RYZHIK, 1980, Eqs. 3.761.3, 8.231.2)

$$\begin{aligned} b_n &= n \log n + n \left[ \frac{n\pi}{2} \sin\left(\frac{2}{n\pi}\right) - C - \log \frac{2}{\pi} - \int_0^{2/(n\pi)} \frac{\cos t - 1}{t} dt \right] \\ &\cong n \log n + n \left[ 1 - C - \log \frac{2}{\pi} \right], \end{aligned} \quad (22)$$

where

$$C \approx 0.5772\dots \quad (23)$$

is the Euler constant.

It follows from the GCLT (19) that an arbitrary quantile  $z_q$  (9) of the sum  $S_n$  can be approximated as

$$z_q \approx z_q^{(1)} \equiv n^{1/\alpha} x_q C_\alpha + b_n, \quad (24)$$

where  $x_q$  solves the equation

$$F_\alpha(x_q) = q. \quad (25)$$

In some cases, the applicability of the approximation  $z_q^{(1)}$  is seriously affected by the low convergence rate in the GCLT (19); we investigate the quality of this approximation in detail in Section 2.3 below.

An approximation for upper quantiles  $z_q$ ,  $q > 0.95$ , can be obtained by noticing that the tail  $1 - F_\alpha(x)$  of a stable distribution has a simple asymptotic (SAMORODNITSKY and TAQQU, 1994; UCHAIKIN and ZOLOTAREV, 1999):

$$\lim_{x \rightarrow \infty} (1 - F_\alpha) = (C_\alpha x)^{-\alpha}, \quad (26)$$

where  $C_\alpha$  is defined by (20). Applying the GCLT for the tail

$$\lim_{n \rightarrow \infty} \text{Prob} \left\{ \frac{S_n - b_n}{n^{1/\alpha} C_\alpha} > x \right\} = 1 - F_\alpha(x), \quad (27)$$

and equalizing the probability in (27) to  $(1 - q)$ , one obtains

$$z_q^{(2)} \approx n^{1/\alpha} q^{-1/\alpha} + b_n, \quad (28)$$

where  $b_n$  is defined in (21), (22).

The approximation  $z_q^{(2)}$  is easier to use than  $z_q^{(1)}$  since the former is written in a closed form and does not involve solving the equation (25). This convenience is achieved at a price:  $z_q^{(2)}$  only provides a satisfactory approximation for the upper quantiles, while  $z_q^{(1)}$  is applicable for any value of  $q$ .

### 2.3 Evaluating approximation quality

The quality of the approximation  $z_q^{(1)}$  (Eq. 24) may be evaluated by simulating the sums of Pareto rvs with different  $\alpha$ -values. Note that the result depends essentially on the quality of the approximation (Eq. 19), so in assessing this approximation, we are simultaneously evaluating the convergence rate in the GCLT.

Simulation of Pareto rvs is especially easy since a synthetic realization may be constructed via

$$X_i = R^{-1/\alpha}, \quad (29)$$

where  $R$  is a rv uniformly distributed in the interval  $(0, 1]$ . For each fixed value of  $n$  and  $\alpha$  we simulated  $\geq 10^6$  realizations of the sum  $S_n$  to compute its quantiles  $z_q$  (9). We define the approximation relative error as

$$\Delta_q^{(1)} = z_q^{(1)}/z_q - 1. \quad (30)$$

To abbreviate the notation, we do not explicitly note the dependence on  $n$  and  $\alpha$  of the quantiles and their approximations and errors.

As mentioned previously, the approximation of  $S_n$  using the GCLT requires a very large number of summands. For instance, Figure 3 displays the quality of the approximation  $z_q^{(1)}$  for  $\alpha = 1/2$  and  $n = 1000$ . Even for 1000 summands, disagreements between the distribution of the sum and its approximation are easily discerned. This disagreement is even larger for smaller values of  $x$  (not shown in the figure). Indeed, the relative errors  $\Delta_q^{(1)}$  for the upper quantiles approach the limit of 0 rather quickly as the number  $n$  of summands increases, whereas for lower quantiles the convergence is quite slow. This is demonstrated in Figure 4, which shows the dependence of the relative error  $\Delta_q^{(1)}$  for  $q = 0.02$  and  $0.98$  on the number  $n$  of summands for  $\alpha = 2/3$  and  $\alpha = 1.5$ . We follow MCCULLOCH and PANTON (1997) in evaluating the quantiles 0.02 and 0.98 rather than the more commonly used 0.025 and 0.975.

Such behavior is easy to understand in light of Fig. 2. The upper tail of the Pareto distribution decays as  $x^{-\alpha}$ , similar to the decay of the upper tail of the stable distribution. On the other hand, the lower tail of the Pareto distribution has an abrupt truncation at  $x = 1$  (see Eq. 2), whereas the stable distributions are smooth everywhere. Fig. 2 demonstrates

the similarity of the Pareto and stable distributions at the upper tail and their difference at the small  $x$ -values. Thus, a large number of variables must be summed in order for the lower quantiles of  $S_n$  to be reasonably approximated by those of  $F_\alpha$ .

Table 1 collects the simulation results for selected values of  $\alpha$ . Synthetic values of the sum are compared to stable quantiles as tabulated in MCCULLOCH and PANTON (1997), except for  $\alpha = 2/3$  where we calculate  $F_\alpha$  by integrating (15). The stable distribution approximates the Pareto sum upper 0.98 quantile quite well: only for  $\alpha$  approaching 1.0 and 2.0 from below does the number of summands necessary to achieve 10% relative accuracy exceed two. As mentioned above, the approximation deteriorates substantially for the lower quantile: even in the best cases several dozens of summands are needed to yield  $< 10\%$  relative error.

Table 2 (columns 6, 8, and 10) shows the relative error  $\Delta_q^{(1)}$  of the approximation  $z_q^{(1)}$  for selected values of  $\alpha$ ,  $q$ , and relatively small values of  $n$ . In each case  $S_n$  is calculated by using  $10^7$  simulated Pareto summands. The error  $\Delta_q^{(2)}$  of the approximation  $z_q^{(2)}$  is given in the eleventh column.

### 3 Replacing the sum with the maximum

Approximation of the sums of Pareto rvs can be obtained by noticing that in many important cases the largest of  $n$  such observations,  $M_n$ , has the same order of magnitude as the entire sum  $S_n$ .

To gather some intuition, let us define

$$r_n = S_n/M_n. \quad (31)$$

In the case of iid Pareto summands, the expectation of the ratio  $r_n$  takes the form (see Appendix B)

$$E(r_n) = \begin{cases} [1 - nB(n, \alpha^{-1})]/(1 - \alpha), & \alpha \neq 1 \\ \sum_{k=1}^n 1/k, & \alpha = 1, \end{cases} \quad (32)$$

where  $B(\cdot, \cdot)$  is the beta-function (see Appendix B, Eq. 82).

For  $\alpha = 2/3$  this expression can be simplified (ABRAMOWITZ and STEGUN, 1972, Eq. 6.1.12) as

$$E(r_n) = 3 \left[ 1 - \frac{2^n n!}{(2n+1)!!} \right], \quad (33)$$

where we use the standard notation  $(2n + 1)!! = 1 \cdot 3 \cdot \dots \cdot (2n + 1)$ . It is easy to derive the asymptotics of  $E(r_n)$  as  $n \rightarrow \infty$ :

$$E(r_n) \cong \begin{cases} 1/(1 - \alpha), & \alpha < 1 \\ n^{1-1/\alpha} \Gamma(1/\alpha) / (\alpha - 1), & \alpha > 1 \\ C + \log(n), & \alpha = 1, \end{cases} \quad (34)$$

where  $C$  is given by (23). In Fig. 5 we display dependence of  $E(r_n)$  on the number  $n$  of summands using  $\alpha = 2/3$ . The asymptotic value  $E(r_n) = 3$  is reached relatively slowly: for  $n = 100$ ,  $E(r_n) = 2.73$ ; and even for  $n = 1000$ ,  $E(r_n) = 2.92$ .

For  $\alpha < 1$  the coefficient  $r_n$  is on the order of unity which means that the major contribution to the sum  $S_n$  is made by the maximal observation  $M_n$ . It is important to note, however, that the variation of  $r_n$  may be significant. To illustrate the relation between  $S_n$  and  $M_n$  we show their scatterplot in Fig. 6 for  $\alpha = 2/3$ ,  $n = 1000$ . One can see that departures from the relation  $S_n \sim M_n/(1 - \alpha)$  suggested by (31, 34) are significant, especially at the lower and upper tails of  $S_n$  and  $M_n$ . Moreover, for any fixed value of  $S_n$  or  $M_n$  the ratio  $r_n$  varies substantially.

Fig. 6 suggests the possibility of approximating the upper quantiles of  $S_n$  with those of  $M_n$ , since one clearly sees that the largest possible values of the sum lie close to the line  $S_n = M_n$ . Using the well-known distribution of the maximum (FELLER, 1971):

$$\text{Prob} \{M_n < x\} = F^n(x) = (1 - x^{-\alpha})^n, \quad (35)$$

one can approximate the distribution of the normalized sum as

$$\text{Prob} \left\{ \frac{S_n}{C_\alpha n^{1/\alpha}} < z \right\} \approx \text{Prob} \left\{ \frac{M_n}{C_\alpha n^{1/\alpha}} < z \right\} \approx \exp \left\{ - (z C_\alpha)^{-\alpha} \right\}. \quad (36)$$

Equating the probability (36) to  $q$  and adding the shift  $b_n$  (21), which is important for  $\alpha \geq 1$ , we obtain

$$z_q^{(3)} = n^{1/\alpha} [\log(1/q)]^{-1/\alpha} + b_n. \quad (37)$$

The relative error  $\Delta_q^{(3)}$  of this approximation is shown in column 12 of Table 2. One sees that for  $\alpha < 1$  the approximation is quite close, whereas for  $\alpha > 1$  the relative errors are large.



## 4 More precise techniques

This Section develops techniques that allow one to approximate the sums  $S_n$  with a higher degree of accuracy (relative error  $< 1\%$ ) for an arbitrary number  $n$  of summands.

### 4.1 Approximating the median and upper bound

Here we expand on the idea of using the largest observations to approximate the sum (8) of Pareto rvs. Our approach is based on considering the *variational series*

$$X_{1,n} \leq X_{2,n} \leq \dots \leq X_{n,n} \quad (38)$$

formed by the *order statistics*  $X_{1,n} = \min\{X_1, \dots, X_n\}$ , ...,  $X_{n,n} = \max\{X_1, \dots, X_n\}$ . Introducing the notations

$$S_{k,n} = \sum_{i=1}^k X_{i,n}, \quad T_{k,n} = \sum_{i=k+1}^n X_{i,n}, \quad k \leq n \quad (39)$$

we may rewrite (8) as

$$S_n = \sum_{i=1}^p X_{i,n} + \sum_{i=p+1}^n X_{i,n} = S_{p,n} + T_{p,n}. \quad (40)$$

Essential properties of this representation follow from the following result.

**Theorem 1** *The  $m$ -th moment  $E(X_{n,k})^m$  of the  $k$ -th order statistic from Pareto distribution (2) is finite for  $m < \alpha(n - k + 1)$  and is given by*

$$EX_{k,n}^m = \frac{n! \Gamma(n - k + 1 - m/\alpha)}{(n - k)! \Gamma(n + 1 - m/\alpha)}. \quad (41)$$

*The joint second moment  $E(X_{r,n}X_{s,n})$ ,  $r > s$  is finite if*

$$\min\{n - r + 1, (n - s + 1)/2\} > 1/\alpha,$$

*and is given by*

$$E(X_{r,n}X_{s,n}) = \frac{n! \Gamma(n - r + 1 - 1/\alpha) \Gamma(n - s + 1 - 2/\alpha)}{(n - r)! \Gamma(n - s + 1 - 1/\alpha) \Gamma(n + 1 - 2/\alpha)}. \quad (42)$$

**Proof.** See NEVZOROV, (2001), Assignment 6.2.

An important consequence of Theorem 1 is the following.

**Corollary 1** *If  $p = n - \lfloor m/\alpha \rfloor$ , where  $\lfloor x \rfloor$  is the integer nearest to  $x$  from below, then all the summands in  $S_{p,n}$  have finite  $m$ -th moment, and all the summands in  $T_{p,n}$  have infinite  $m$ -th moments.*

As follows from this corollary, the number  $n_m = \lfloor m/\alpha \rfloor$  of order statistics with infinite  $m$ -th moments does not depend on the sample size  $n$ . Seemingly counter-intuitive, this fact is well-known in statistics (see SEN, 1959); its direct proof is given in Appendix C. Importantly, for  $1/2 \leq \alpha < 1$  there are only two (upper) order statistics with infinite mathematical expectations, and for  $2/3 \leq \alpha < 1$  there are only two (upper) order statistics with infinite second moment.

Below we use the representation (40) with  $p = n - 2$  which gives  $T_{n-2,n} = X_{n-1,n} + X_{n,n}$ . Recalling that order statistics obey the Markov property (NEVZOROV, 2001, Remark 4.3.)

$$\text{Prob}\{X_{k+1,n} < x | X_{1,n}, \dots, X_{k,n}\} = \text{Prob}\{X_{k+1,n} < x | X_{k,n}\}, \quad (43)$$

and for the common distribution  $F(x)$  of  $X_i$

$$\text{Prob}\{X_{k+1,n} > x | X_{k,n} = u\} = \left( \frac{1 - F(x)}{1 - F(u)} \right)^{n-k}, \quad (44)$$

one can prove the following result.

**Theorem 2** *The sum  $T_{n-2,n}$  of the upper two order statistics of  $n$  Pareto rvs has the following distribution:*

$$\begin{aligned} T(x, \alpha, n) &= \text{Prob}\{X_{n-1,n} + X_{n,n} < x\} = \\ &= n(n-1) \alpha^2 \int_2^x \int_1^{z/2} y^{-\alpha-1} (z-y)^{-\alpha-1} (1-y^{-\alpha})^{n-2} dy dz \end{aligned} \quad (45)$$

$$= n(n-1) \alpha^2 \sum_{k=0}^{n-2} (-1)^k C_{n-2}^k \int_2^x z^{-\alpha(k+2)-1} B(1/z, 1/2, -\alpha(k+1), -\alpha) dz, \quad (46)$$

where  $B(x_0, x_1, a, b)$  is the generalized incomplete beta function (WOLFRAM, 1999)

$$B(x_0, x_1, a, b) = \int_{x_0}^{x_1} t^{a-1} (1-t)^{b-1} dt. \quad (47)$$

**Proof** is given in Appendix D.

The Markov property (43) can be used as well to derive the distribution of a larger number of the upper order statistics. We restrict ourselves with the two upper ones and now proceed with the sum  $S_{n-2,n}$  of the lower  $(n-2)$  order statistics. Using (41), (42) we find that for  $\alpha \geq 1/2$  the sum  $S_{n-2,n}$  has finite expectation

$$m_1(\alpha, n) = \sum_{k=1}^{n-2} \frac{n! \Gamma(n-k+1-1/\alpha)}{(n-k)! \Gamma(n+1-1/\alpha)}. \quad (48)$$

For  $\alpha > 2/3$  it has also finite second moment

$$\begin{aligned} m_2(\alpha, n) &= \sum_{k=1}^{n-2} \frac{n! \Gamma(n-k+1-2/\alpha)}{(n-k)! \Gamma(n+1-2/\alpha)} + \\ &+ \sum_{r=1}^{n-2} \sum_{s=1}^{r-1} \frac{n! \Gamma(n-r+1-1/\alpha) \Gamma(n-s+1-2/\alpha)}{(n-r)! \Gamma(n-s+1-1/\alpha) \Gamma(n+1-2/\alpha)}. \end{aligned} \quad (49)$$

For  $\alpha = 1/2$  and 1 the expressions (48), (49) can be simplified:

$$m_1(1, n) = n \sum_{k=2}^{n-1} \frac{1}{k} = n(C-1 + \log(n-1) + o(1)); \quad (50)$$

$$m_1(1/2, n) = n(n-2); \quad (51)$$

$$\begin{aligned} m_2(1, n) &= n(n-2) + 2n(n-1) \times \\ &\times \left[ \log(n-2)(C-1 + \log(n-2)) - \sum_{j=2}^{n-2} \frac{\log(j-1)}{j} + o(\log(n)) \right]. \end{aligned} \quad (52)$$

In particular, it follows that for  $\alpha = 1$ , as  $n \rightarrow \infty$ ,

$$\begin{aligned} m_1(1, n) &= n \log(n) + o(n \log(n)), \\ m_2(1, n) - m_1^2(1, n) &= \text{Var}(S_{n-2,n}) = n^2 \log(n)^2 + o(n^2 \log(n)^2). \end{aligned} \quad (53)$$

To obtain bounds on the distribution of  $S_{n-2,n}$ , we note that its summands can be considered iid rvs with the common distribution

$$\text{Prob}\{Y < x\} = \text{Prob}\{X_i < x | X_i < X_{n-1,n}\}. \quad (54)$$

From Theorem 1 it follows that for  $\alpha > 2/3$  they have finite first and second moments, and one may apply the Central Limit Theorem (FELLER, 1971, **2**, VIII, 4) to obtain

**Corollary 2** *For  $\alpha > 2/3$  the sum  $S_{n-2,n}$  of the  $(n-2)$  lower order statistics of  $n$  Pareto rvs converges in probability, as  $n \rightarrow \infty$ , to a normally distributed rv with first two moments  $m_1(\alpha, n)$ ,  $m_2(\alpha, n)$  given by (48), (49).*

Thus, to approximate the sum  $S_n$  we propose the representation (40) with  $p = n - 2$ . Then the distribution of  $T_{n-2,n}$  is given by Theorem 2, while  $S_{n-2,n}$  allows the normal approximation of Corollary 2. The median  $z_{1/2}$  and the upper quantile  $z_q$ ,  $q > 0.95$  of  $S_n$  are approximated as

$$z_{1/2}^{(4)} = m_1(\alpha, n) + T^{-1}(1/2, \alpha, n); \quad (55)$$

$$z_q^{(4)} = m_1(\alpha, n) + \kappa(\alpha, n) + T^{-1}(q, \alpha, n), \quad (56)$$

where

$$\kappa(\alpha, n) = \begin{cases} \sqrt{m_2(\alpha, n) - m_1^2(\alpha, n)}, & \alpha \geq 2/3, \\ 0, & \alpha < 2/3. \end{cases} \quad (57)$$

Both of these approximations involve inversion of the distribution  $T(x, \alpha, n)$ , which must be done numerically. Numerical integration is more efficient using the representation (45) of Theorem 2. The relative errors  $\Delta_q^{(4)}$  of these approximation are shown in columns 9 and 13 of Table 2. The approximations appear to be very close for all values of  $\alpha < 2$ , even for small numbers of summands. For  $n = 2$ ,  $m_1$  and  $m_2$  become zero and approximations  $z^{(4)}$  coincide with theoretical quantiles of the distribution  $T$ .

In constructing the approximations  $z^{(4)}$  we obtain the exact quantiles for the sum of the upper two statistics  $T_{n-2,n}$  by inverting its distribution  $T$ , and assume that, due to the normal asymptotic of Corollary 2, the quantiles of  $S_{n-2,n}$ , can be expressed via its first two moments. For instance, in (55) the median of  $S_{n-2}$  is naturally approximated by the mathematical expectation  $m_1(\alpha, n)$ . Indeed, such a construction is seriously affected by the fact that  $S_{n-2,n}$  and  $T_{n-2,n}$  are statistically dependent rvs, so the direct summation of their quantiles is a purely heuristic device. We notice though that for  $\alpha < 1$  the major contribution to the sum  $S_n$  is made by the two upper statistics,  $T_{n-2,n}$ , while the contribution from the rest of the summands ( $S_{n-2,n}$ ) becomes negligible, and so does their statistical variation. On the other hand, for  $\alpha > 1$ , the contribution from  $T_{n-2,n}$  becomes less important comparing to that of the large number of summands in  $S_{p,n}$ . In both cases, the statistical variation of one of the terms in (40) is negligible compared to the second one, so the direct summation of their quantiles results in a reasonable approximation.

## 4.2 Approximating the lower bound

An approximation for the  $q$ -quantile  $z_q$  (9) of the sum  $S_n$ , for small  $q$ , may be constructed based on the following idea. For any value  $y_n$ ,

$$\begin{aligned} \text{Prob}\{S_n < z_q\} &= \text{Prob}\{S_n < z_q | M_n \leq y_n\} \text{Prob}\{M_n \leq y_n\} \\ &+ \text{Prob}\{S_n < z_q | M_n > y_n\} \text{Prob}\{M_n > y_n\}. \end{aligned} \quad (58)$$

The term  $\text{Prob}\{S_n < z_q | M_n \leq y_n\}$  in (58) can be approximated using the Central Limit Theorem (FELLER, 1971, **2**, VIII, 4), since the variables being summed are now truncated and hence have finite moments. The resulting approximation will be close for sufficiently small values of  $y_n$  and sufficiently large  $n$ . At the same time, for any reasonably large value of  $y_n$ , and for small  $q$ , the quantity  $\text{Prob}\{S_n < z_q | M_n > y_n\}$  in (58) will be infinitesimal; hence the entire final term in (58) may be considered negligible. Thus, in approximating  $z_q$ , we suggest choosing an appropriate value of  $y_n$ , and considering the approximation

$$\begin{aligned} \text{Prob}\{S_n < z_q\} &\approx \text{Prob}\{S_n < z_q | M_n \leq y_n\} \text{Prob}\{M_n \leq y_n\} \\ &\approx \Phi\{(z_q - n\mu_{y_n}; 0, 1)/(\sigma_{y_n}\sqrt{n})\} \text{Prob}\{M_n \leq y_n\}, \end{aligned} \quad (59)$$

where  $\Phi$  is the standard normal distribution function (11), and

$$\mu_y = E[X_1 | X_1 \leq y] = \begin{cases} \frac{\alpha}{1-\alpha} (y^{1-\alpha} - 1) / (1 - y^{-\alpha}), & \alpha \neq 1, \\ \log(y)/(1 - y^{-1}), & \alpha = 1, \end{cases} \quad (60)$$

$$\sigma_y^2 = V[X_1 | X_1 \leq y] = \begin{cases} \frac{\alpha}{2-\alpha} (y^{2-\alpha} - 1) / (1 - y^{-\alpha}) - \mu_y^2, & \alpha \neq 2, \\ 2 \log(y)/(1 - y^{-2}) - \mu_y^2, & \alpha = 2 \end{cases} \quad (61)$$

are the conditional mean and variance of each summand, given the restriction on the maximum.

Note that there is a tradeoff in choosing  $y_n$  in (59): if one selects too small a value of  $y_n$ , then the term  $\text{Prob}\{S_n < z_q | M_n > y_n\}$  in (58) is not negligible, so the resulting approximation may not be satisfactory. On the other hand, if  $y_n$  is too large, then the approximation of  $\text{Prob}\{S_n < z_q | M_n < y_n\}$  using the Central Limit Theorem may be unsatisfactory; this is particularly true for small  $n$ .

One option is to choose some value  $p^*$  to represent the probability  $\text{Prob}\{S_n < z_q | M_n \leq y_n\}$  in (59). From (9), one then has  $q/p^* = \text{Prob}\{M_n \leq y_n\} = \{1 - (\frac{1}{y_n})^\alpha\}^n$ , and solving this for  $y_n$  one obtains

$$y_n = \left[1 - (q/p^*)^{1/n}\right]^{-1/\alpha}. \quad (62)$$

One may then obtain an approximation of  $z_q$  by plugging this value of  $y_n$  from (62) into the first term in (59), yielding

$$z_q^{(5)} = \sigma_{y_n} \sqrt{n} \Phi^{-1}(p^*) + n \mu_{y_n}, \quad (63)$$

where  $\mu_{y_n}$  and  $\sigma_{y_n}$  are given by (60-62).

A naive choice for  $p^*$  is  $\sqrt{q}$ ; this seems to balance the aforementioned tradeoff, since then  $\text{Prob}\{S_n < z_q | M_n \leq y_n\} = \text{Prob}\{M_n \leq y_n\} = \sqrt{q}$ . A substantial improvement can be obtained by choosing  $p^*$  in the form

$$p^* = 0.136 + 0.235q + q^2 + 0.0066 \min(n, 10) - 0.05 \max(a, 1). \quad (64)$$

The dotted curve in Figure 7 shows the relative errors, as a function of  $n$ , for the resulting approximation  $z_q^{(5)}$  with  $p^*$  given by (64). For comparison, the dashed curve in Figure 7 shows relative errors for the solution to (63) with  $p^* = \sqrt{q}$ . Here  $\alpha = 2/3$  and  $q = 0.02$ ; the results for other values of  $\alpha$  and other small values of  $q$  are similar.

The values reported in column 7 of Table 2 reflect the approximation  $z_q^{(5)}$  with  $p^*$  as in (64). One sees that the approximation matches the true quantile quite closely, with relative errors less consistently less than 1%, even for small  $n$ .

## 5 Cumulative Pareto sums: Linear vs. non-linear regimes

As mentioned in Section 1.2, it is important to consider the case of the Pareto distribution with an upper bound or taper, especially for  $\alpha \leq 1.0$ . An example of such a distribution is shown in Fig. 1, which demonstrates that an exponential taper applied to the cdf of the Pareto distribution results in a close approximation to the empirical distribution of seismic

moments. A commonly employed alternative is to use a Pareto distribution that is simply truncated at some value  $y$ ; the corresponding cdf is

$$F(x) = \frac{1 - x^{-\alpha}}{1 - y^{-\alpha}}, \quad 1 < x \leq y. \quad (65)$$

For the truncated Pareto distribution, simulated values may be constructed via

$$X_i = \left\{ R \left[ 1 - y^{-\alpha} \right] + y^{-\alpha} \right\}^{-1/\alpha}, \quad (66)$$

where  $R$  is a uniform rv as in (29). The ratio  $E(r_n) = E(S_n/M_n)$  for the truncated Pareto distribution and  $\alpha < 1$  can be evaluated as in (32), and is given in Appendix B, Eq. 82.

Fig. 8 displays an example of simulated sums ( $S_n$ ) for the Pareto distribution truncated at  $y = 3.4 \times 10^4$  compared to the stable distribution quantiles. The upper quantiles depart from the theoretical curve for the stable distribution starting with  $n = 2$ , whereas the behavior of the lower quantile is essentially unaffected by the truncation until  $n$  exceeds  $10^3$ .

When the number of summands is large, the truncation point  $y$  dominates the behavior of the quantiles. The sum is then distributed asymptotically according to the Gaussian law:

$$\lim_{n \rightarrow \infty} F_{S_n}(x) = \Phi \left( \frac{x - n\mu_y}{\sigma_y \sqrt{n}}; 0, 1 \right), \quad (67)$$

where  $\Phi$  is the normal cdf (11), and the parameters  $\mu_y$  and  $\sigma_y$  are given by (60), (61). In Fig. 8 the Gaussian approximation for the truncated Pareto sums is shown to provide a satisfactory approximation to the sum of truncated Pareto rvs, when the number  $n$  of terms being summed exceeds 500.

The median of the distribution of  $S_n$  for truncated Pareto rvs increases non-linearly with  $n$ : for  $n < 100$  the median increases at rate proportional to  $n^{1/\alpha}$ , (PISARENKO, 1998; RODKIN and PISARENKO, 2000; HUILLET and RAYNAUD, 2001). This behavior differs sharply from the linear increase of all quantiles of the distributions of sums of rvs with finite first moment (i.e., for distributions with  $\alpha \geq 1$ ). For larger numbers of summands, quantiles of sums of truncated Pareto distribution increase linearly.

RODKIN and PISARENKO (2000, their Eqs. 19 and 21) introduce two particular values of  $n$ :

$$n_1 = \frac{1 - \alpha}{\alpha} \times y^\alpha \log 2, \quad (68)$$

and

$$n_2 = \frac{9(1-\alpha)^2 y^\alpha}{\alpha(2-\alpha)}. \quad (69)$$

They argue that if  $n < n_1$  the sums behave statistically as if the distribution has not been truncated, whereas for  $n > n_2$  the Gaussian approximation for the sums is appropriate. For the example displayed in Fig. 8,  $n_1 = 350$  and  $n_2 = 1152$ . The curves in Fig. 8 appear to confirm the validity of the conclusions of RODKIN and PISARENKO (2000).

## 6 Discussion

We presented five approximations for the quantiles of the distribution of the sums of Pareto variables. The approximations  $z_q^{(1,2)}$  (Eqs. 24, 28) are based on stable asymptotics for Pareto sums;  $z_q^{(3)}$  (Eq. 37) uses replacing the sum  $S_n$  with its maximal summand;  $z_q^{(4)}$  (Eqs. 55-56) elaborates the latter idea considering separately the sum of the largest two observations and the rest of the lower terms to approximate the median and the upper quantiles; and  $z_q^{(5)}$  (Eq. 63) approximates the lower quantiles by introducing an upper truncation point on the maximum observed value. We consider the relative error  $\Delta_q^{(k)}$  (30) for all our approximations: Table 2 collects the errors for selected values of  $\alpha$ ,  $n$ , and  $q$ .

The accuracy of the figures given in Table 2 is three digits. This accuracy is comparable with that of our best approximations and therefore is sufficient for this study. We notice that for  $n = 2$  the approximations  $z^{(4)}$  coincide with the theoretical quantiles for the distribution of the sum of the two upper statistics  $X_{n-1,n} + X_{n,n}$ . Therefore, the errors  $\Delta^{(4)}$  for  $n = 2$  evaluate the precision of our simulated quantiles given in columns 3, 4, and 5 (lines corresponding to  $n = 2$ ) of Table 2. These errors are always lower than or equal to 0.003, the latter upper bound is reached for  $\alpha = 1$ .

**The lower quantile** approximation  $z_q^{(5)}$ ,  $q \ll 1$ , indeed provides a closer approximation than  $z_q^{(1)}$ . The former uniformly has a relative error less than 1% for any  $n$ , while the latter starts producing reliable approximations only for  $n > 50$ . This is due to that fact that the normal approximation used in  $z_q^{(5)}$  is much better suited for the small observations at the lower tail of the  $S_n$  distribution than the stable approximation used in  $z_q^{(1)}$ . For large  $n$  ( $n > 10^3$ ) the quality of  $z_q^{(1)}$  becomes acceptable, as seen in Fig. 4.



The quality of **the median** approximation  $z_{1/2}^{(1)}$  increases rapidly with  $n$ , providing acceptable results for  $n \geq 20$ . For  $n < 100$  an order of magnitude better approximation is provided by  $z_{1/2}^{(4)}$ , whose relative error is less than 1% for  $\alpha \geq 1$ , and less than 10% for  $1/2 \leq \alpha < 1$ . As discussed in Section 4.1, the quality of this approximation is affected by the existence of statistical dependence between the highest and lower summands within  $S_n$ , and as a result its accuracy decreases with  $n$  for  $\alpha < 1$ .

**The upper quantile** approximations are all quite accurate, unlike the case of medians and lower quantiles where the choice of approximation appears to be more important. The most rough approximation  $z_q^{(3)}$  which simply replaces the sum  $S_n$  with its maximal summand provides less than 5% error for  $\alpha \leq 1$ . Indeed, for  $\alpha > 1$  its performance deteriorates, since the largest observation no longer makes an important contribution to the sum  $S_n$ . The stable-law approximations  $z_q^{(1,2)}$  demonstrate similar performance for  $\alpha > 3/2$  and  $\alpha < 2/3$ , while for  $2/3 \leq \alpha \leq 3/2$ ,  $z_q^{(1)}$  appears to be preferable. The most accurate approximation, with less than 1% relative error for all the cases considered, is demonstrated by the approximation  $z_q^{(4)}$ . By comparison  $z_q^{(4)}$  with  $z_q^{(3)}$ , one observes how important it is to consider more than just the single largest observation, as well as the importance of the contribution from the smaller summands, which becomes crucial for  $\alpha > 1$ .

In searching for a best approximation among the five proposed above, we observe a trade-off between the quality and simplicity of the approximations. This is especially prevalent for the upper quantiles, where the directly calculated approximations  $z_q^{(2,3)}$  are clearly inferior to the more elaborate approximation  $z_q^{(4)}$ , which involves solving an integral equation. For large samples with  $n \geq 10^3$  we recommend using  $z_q^{(1)}$ . When  $n$  is intermediate to small,  $n < 10^3$ , the upper quantiles can be well approximated by  $z_q^{(2)}$ ; for the lower quantiles  $z_q^{(5)}$  appears to perform well. The upper quantile is best approximated by  $z_q^{(4)}$ , which can be replaced in favor of the simpler  $z_q^{(1)}$  when  $\alpha < 1$ . Generally, the choice of approximation should be dictated by the range of  $\alpha$ ,  $n$ , and  $q$  considered, as well as consideration of the relative error rate and computational simplicity desired.

To illustrate the application of our results, Fig. 9 shows the cumulative seismic moment for Californian seismicity,  $M \geq 5.5$ , during the last two centuries. We use the TOPPOZADA *et al.* (2000) earthquake catalog. The total number of events shown in the figure is 163. The

largest earthquake in the plot is the San Francisco 1906 event ( $M7.8$ ). Similar displays are often used in geophysics to study accumulation of seismic moment (PETERSON and SENO, 1984; JAUMÉ and SYKES, 1996; TRIEP and SYKES, 1997; RODKIN and PISARENKO, 2000). Fig. 10 shows the same data and approximations in semilogarithmic scale with the number of earthquakes shown at the  $x$ -axis.

If the seismic moment distribution is strictly Pareto, with  $\alpha = 2/3$ , the observed cumulative moment would be well described by the statistical bounds for Pareto sums given in columns 3 – 5 of Table 2 and shown (dashed lines) in Figs. 9 and 10. For comparison, our approximation by the stable distribution,  $z_q^{(1)}$  (dotted lines), is also displayed. As one can see from Table 2, other approximations work much better than  $z_q^{(1)}$  (especially for the lower quantile and the median) so their respective deviations from the simulated Pareto quantiles in Figs. 9 and 10 would be even smaller.

From approximately 1840 onward, the observed cumulative moment appears to be well within the bounds given by both approximations. The observed values of the cumulative seismic moment lie close to their approximated median and their variation is within the prescribed bounds. However, these bounds indicate that seismic moment release for such data can be estimated only with a great deal of uncertainty: the lower 2% bound is an order of magnitude smaller than the median, whereas the upper 98% bound exceeds the median almost by a factor 100. Note that the seismic data prior to 1840 are thought to be of inferior quality, with significant numbers of events missing and their magnitudes possibly substantially underestimated; this may explain why the observed cumulative line starts out of the prescribed statistical bounds.

Several methods can be used to improve the estimate of the total seismic moment release rate. One can increase the size of the available earthquake dataset and thus include more large earthquakes, either by using older data or by increasing the spatial size of the region under consideration. Both of these methods have clear disadvantages: in the former case the additional data may be of inferior quality; in the latter case the resolution of the study deteriorates. As another alternative, one can use tectonic and geodetic data to constrain the moment release rate (KAGAN, 2002b) by integrating the moment-frequency relation (7). If the maximum seismic moment is known or can be estimated (KAGAN and SCHOENBERG,

2001; KAGAN, 2002a, see more in Section 5), we can use the methods developed in that section to evaluate the bounds.

Figs. 9 and 10 suggest that for seismological applications the quality of the approximation  $z_q^{(1)}$  may be sufficient for an adequate description of the bounds on the distribution of the total seismic moment release. Indeed, when the quality of the available data subject to approximation are of such low accuracy and when the variability of the sum  $S_n$  is so extreme, even a relative error in the approximation as large as 10% - 20% may be acceptable.

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## Appendices:

### A Asymptotic formulas for stable distributions

In comparing the Pareto sums to the appropriately scaled stable distributions, the upper and lower tails of these distributions are of special relevance to us. For maximally-skewed distributions, the lower tail is ‘light’, i.e., its density decays faster than any power of  $x$  (ZOLOTAREV, 1986; SAMORODNITSKY and TAQQU, 1994; UCHAIKIN and ZOLOTAREV, 1999). On the other hand, the upper tail is governed by a power-law decay with the exponent  $\alpha < 2$ , such tails are called ‘heavy’.

There exist series expansions for stable distributions that are known to converge relatively quickly in the tails. One may take the leading term  $F_\alpha^{(0)}(x)$  of these series as an approximation of the distribution at the tails. The form of the leading terms depends on the index  $\alpha$  and on the tail (upper or lower) to be approximated. Below we present three approximations for the lower quantiles (Eqs. 70-73) and two approximations for the upper quantiles (Eqs. 74-75).

For  $\alpha < 1$  by integrating the pdf from LINNIK (1954, Eq. 1) we obtain

$$F_\alpha^{(0)}(x) = \frac{1}{\sqrt{2\alpha}} \left\{ 1 - \operatorname{erf} \left[ \sqrt{\frac{1-\alpha}{\alpha}} \left( \frac{\alpha}{\cos(\pi\alpha/2)} \right)^{\frac{1}{2(1-\alpha)}} x^{\frac{\alpha}{2(1-\alpha)}} \right] \right\}, \quad x > 0, \quad (70)$$

where the error function  $\operatorname{erf}(x)$  is given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt. \quad (71)$$

For  $\alpha = 1$  integrating the pdf from IBRAGIMOV and LINNIK (1971, Theorem 2.4.4) yields

$$F_\alpha^{(0)}(x) = \frac{1}{\sqrt{2}} \left\{ 1 - \operatorname{erf} \left[ \sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2} - \frac{\pi x}{4}\right) \right] \right\}, \quad -\infty < x < \infty \quad (72)$$

and for  $\alpha > 1$  integrating the pdf from SKOROHOD (1954, Eq. VI) yields

$$F_\alpha^{(0)}(x) = \frac{1}{\sqrt{2\alpha}} \left\{ 1 - \operatorname{erf} \left[ \sqrt{\frac{\alpha-1}{\alpha}} \left( \frac{-\alpha}{\cos(\pi\alpha/2)} \right)^{\frac{1}{2(1-\alpha)}} (-x)^{\frac{\alpha}{2(\alpha-1)}} \right] \right\}, \quad -\infty < x < \infty. \quad (73)$$

As UCHAIKIN and ZOLOTAREV (1999, p. 127) note, equation (70) is exact for  $\alpha = 0.5$  (see Eq. 14), and (73) corresponds to the Gaussian distribution  $\Phi(x, 0, 2\sigma)$  for  $\alpha = 2.0$ .

For the upper quantile,  $x \rightarrow \infty$ , one may use the approximation (SAMORODNITSKY and TAQQU, 1994, Eqs. 1.2.8, 1.2.9)

$$F_\alpha^{(0)}(x) = 1 - [x^\alpha \Gamma(1-\alpha) \cos(\pi\alpha/2)]^{-1}, \quad (74)$$

if  $\alpha \neq 1$ , and

$$F_\alpha^{(0)}(x) = 1 - \frac{2}{\pi x}, \quad (75)$$

for  $\alpha = 1$ .

In Fig. 11a we display an example of approximating the lower quantiles using the Eq. 70. In general, the approximations (Eqs. 70–73) work reasonably well, with relative errors of just

a few percent. For the upper quantiles (Eqs. 74–75) the fit is good only for certain ranges of  $\alpha$ . Fig. 11b demonstrates, for instance, that for the 0.98 quantile the expression (74) is within 10% of the theoretical values for  $\alpha < 0.85$  and for  $1.1 < \alpha < 1.53$ , whereas for values of  $\alpha$  closer to 2 the fit of the approximation is only satisfactory for very high quantiles such as 0.999.

## B Proof of (34)

Consider the ratio  $X_1/M_n$ . Its cdf  $G(z)$  can be expressed in terms of the conditional distribution  $F(x|y)$  of  $X_1$  conditioned on the fixed maximum  $M_n = y$  (PISARENKO, 2003, private communication):

$$\begin{aligned} G(z) = \text{Prob} \left\{ \frac{X_1}{M_n} < z \right\} &= \int_0^\infty \text{Prob} \left\{ \frac{X_1}{y} < z|y \right\} dF^n(y) = \\ &= \int_0^\infty \text{Prob} \{ X_1 < yz|y \} dF^n(y) = \\ &= \int_0^\infty F(yz|y) dF^n(y), \end{aligned}$$

and the density  $g(z)$  is obtained by taking the derivative with respect to  $z$ :

$$g(z) = \int_0^\infty y f(yz|y) dF^n(y). \quad (76)$$

The conditional density  $f(x|y)$  corresponding to the distribution  $F(x|y)$  is given by

$$f(x|y) = \frac{\delta(y-x)}{n} + \left(1 - \frac{1}{n}\right) \frac{f(x)H(y-x)}{F(y)}. \quad (77)$$

Here  $H(x) = 1$  for  $x > 0$  and 0 for  $x \leq 0$ . The first term on the rhs of (77) corresponds to the case  $M_n = X_1$ , whose probability is  $1/n$  by symmetry; the complementary event  $X_1 < M_n$  occurs with probability  $(1 - 1/n)$ . Substituting the pdf (77) into (76) we obtain

$$g(z) = \frac{\delta(z-1)}{n} + (n-1) \int_0^\infty y f(yz) f(y) F^{n-2}(y) dy, \quad 1 \leq z \leq 1. \quad (78)$$

This expression can be used to calculate the expectation of the ratio  $r_n = S_n/M_n$ :

$$\begin{aligned} E(r_n) &= n \int_0^1 z g(z) dz = \\ &= 1 + n(n-1) \int_0^\infty y f(y) F^{n-2}(y) dy \int_0^1 z f(yz) dz = \end{aligned}$$

$$\begin{aligned}
&= 1 + n(n-1) \int_0^\infty y f(y) F^{n-2}(y) dy \left( \frac{F(y)}{y} - \frac{1}{y^2} \int_0^y F(x) dx \right) = \\
&= 1 + n(n-1) \left( \int_0^\infty f(y) F^{n-1}(y) dy - \int_0^\infty T(y) f(y) F^{n-2}(y) dy \right) = \\
&= 1 + \frac{n(n-1)}{n} \left( \int_0^\infty dF^n(y) - \int_0^\infty T(y) dF^{n-1}(y) \right) = \\
&= n \left( 1 - \int_0^\infty T(y) dF^{n-1}(y) \right), \tag{79}
\end{aligned}$$

where  $T(y) = 1/y \int_0^y F(u) du$ . For the Pareto distribution (2) this becomes

$$T(y) = \frac{1}{y} \int_1^y (1 - x^{-\alpha}) dx = 1 - \frac{1}{1-\alpha} (y^{-\alpha} - \alpha y^{-1}), \tag{80}$$

and plugging (80) into (79) one obtains

$$E(r_n) = E\left(\frac{S_n}{M_n}\right) = \begin{cases} \frac{1-nB(n, \alpha^{-1})}{1-\alpha}, & \alpha \neq 1 \\ \sum_{k=1}^n \frac{1}{k}, & \alpha = 1. \end{cases} \tag{81}$$

For the truncated Pareto distribution (65) we have (*cf.* PISARENKO, 1998, Eq. 12)

$$\begin{aligned}
E(r_n) &= n \left\{ 1 - (n-1) (1 - X_p^{-\alpha})^{-n} \left[ B(z, 1, 1, n-1) \right. \right. \\
&\quad \left. \left. - \frac{1}{1-\alpha} B(z, 1, 2, n-1) + \frac{\alpha}{1-\alpha} B(z, 1, 1 + 1/\alpha, n-1) \right] \right\}, \tag{82}
\end{aligned}$$

where  $z = X_p^{-\alpha}$  and  $B(\cdot, \cdot, \cdot, \cdot)$  is defined by (47).

## C Existence of moments for ordered statistics

The general distribution of the order statistics  $X_{k,n}$  from a sample with a common cdf  $F(x)$  is given by (NEVZOROV, 2001, Eq. (2.1)):

$$F_{n-k,n}(x) = P\{X_{n-k,n} < x\} = \sum_{m=n-k}^n C_m^n (F(x))^m (1 - F(x))^{n-m}. \tag{83}$$

For the Pareto distribution (2) this gives

$$F_{n-k,n}(x) = 1 - C_{n-k-1}^n x^{-(k+1)\alpha} + o\left(x^{-(k+1)\alpha}\right), \tag{84}$$

so the decay rate of the tail

$$[1 - F_{n-k,n}(x)] \sim x^{-(k+1)\alpha}$$

does not depend on the sample volume  $n$ , hence the existence of moments for  $X_{n-k,n}$  is determined by  $k$  only.

In general, the existence of finite moments for order statistics is described by SEN (1959):

**Theorem 3** *If  $E(|X|^\alpha) < \infty$  for some  $\alpha$ , then the moment  $\mu_{k,n}^{(r)} = E(X_{k,n})^r$  exists for all  $k$  such that*

$$\frac{r}{\alpha} \leq k \leq n + 1 - \frac{r}{\alpha}.$$

## D Proof of Theorem 2

Denote by  $F_{n-1,n}$  the distribution of the order statistic  $X_{n-1,n}$  and by  $\bar{F}_{n,n}(x|u)$  the conditional distribution of the upper order statistic  $X_{n,n}$ :

$$1 - \bar{F}_{n,n}(x|u) = \text{Prob}\{X_{n,n} > x | X_{n-1,n} = u\}. \quad (85)$$

Let  $\bar{f}_{n,n}(x|u) = \frac{d}{dx}\bar{F}_{n,n}(x|u)$  and  $f_{n-1,n}(x) = \frac{d}{dx}F_{n-1,n}(x)$ . Then

$$\begin{aligned} f_{n-1,n}(x) &= \frac{d}{dx} [n(1 - F(x))F^{n-1}(x) + F^n(x)] \\ &= n(n-1)f(x)F^{n-2}(x)(1 - F(x)); \end{aligned} \quad (86)$$

$$\bar{f}_{n,n}(x|u) = \frac{d}{dx} \left[ 1 - \frac{1 - F(x)}{1 - F(u)} \right] \frac{f(x)}{1 - F(u)}. \quad (87)$$

The distribution of  $T_{n-2,n} = X_{n-1,n} + X_{n,n}$  is calculated using the Markov property (43):

$$\begin{aligned} \text{Prob}\{X_{n-1,n} + X_{n,n} < x\} &= \\ &= \int_2^x \int_1^{z/2} f_{n-1,n}(y)\bar{f}_{n,n}(z-y)dydz = \\ &= n(n-1) \int_2^x \int_1^{z/2} f(y)f(z-y)F^{n-2}(y)dydz. \end{aligned} \quad (88)$$

The internal upper integration limit is  $z/2$  due to the inequality  $X_{n-1,n} \leq X_{n,n}$ . For the Pareto distribution this yields

$$\begin{aligned} \text{Prob}\{X_{n-1,n} + X_{n,n} < x\} &= \\ &= n(n-1)\alpha^2 \int_2^x \int_1^{z/2} y^{-\alpha-1}(z-y)^{-\alpha-1} (1-y^{-\alpha})^{n-2} dydz = \\ &= n(n-1)\alpha^2 \sum_{k=0}^{n-2} (-1)^k C_{n-2}^k \int_2^x \int_1^{z/2} y^{-\alpha(k+1)-1}(z-y)^{-\alpha-1} dy. \end{aligned} \quad (89)$$

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Figure 1: Number of earthquakes with seismic moment larger than or equal to  $M$  as a function of  $M$  for the shallow earthquakes in the Harvard catalog during 1977/1/1–2002/12/31. Power-law approximation (Gutenberg-Richter law, Eq. 5) is shown by dotted line. Tapered Gutenberg-Richter distribution (Eq. 7) which is the GR law restricted at large magnitudes by an exponential taper is shown by dashed line. The slope of the linear part of the curve corresponds to  $\alpha$ -value  $0.673 \pm 0.011$  and the corner moment  $M_c = 1.6 \times 10^{21}$  Nm.

Figure 2: Probability distributions for the stable (Eq. 15, blue) and Pareto (Eq. 2, red) distributions, both with  $\alpha = 2/3$ .

Figure 3: Quality of approximation  $z_q^{(1)}$ , Eq. 24 (stable distribution approximation). Red line — cdf for simulated Pareto sum,  $n = 1000$ , blue line — calculation by (14), magenta circles — from MCCULLOCH and PANTON (1997) tables.

Figure 4: Relative error  $\Delta_q^{(1)}$  of approximation  $z_q^{(1)}$  (Eq. 24, stable distribution approximation). Blue (dotted) curve is for lower 0.02 quantile, green (solid) curve for the median, and (red) dashed curve for the 0.98 quantile.

(a)  $\alpha = 2/3$ ,  $n = 3 \times 10^6$ ,  $F_\alpha$  is obtained by integration of expression (15).

(b)  $\alpha = 1.5$ ,  $n = 10^6$ ,  $F_\alpha$  is taken from from MCCULLOCH and PANTON (1997) tables.

Figure 5: Dependence of  $E(r_n)$  on the number of summands  $n$  for Pareto summands with  $\alpha = 2/3$ .

Figure 6: Value of the sum  $S_n$  vs. the maximal summand  $M_n$  for the Pareto distribution with  $\alpha = 2/3$ , the number of summands  $n = 1000$ . Red line is  $S_n = M_n$ , the green line is  $S_n = M_n/(1 - \alpha)$ .

Figure 7: Relative errors  $\Delta_q^{(5)}$  for the approximation  $z_q^{(5)}$  with  $p^* = q^{1/2}$  (dashed line) and  $p^*$  given by (64) (dotted line) as functions of the number of summands  $n$ , for  $\alpha = 2/3$  and  $q = 0.02$ .

Figure 8: Quantiles for the sum  $S_n$  of truncated Pareto variables (upper limit  $y = 3.4 \times 10^4$ , Eq. 65) and their approximations as functions of the number of summands,  $n$ . Two approximations are considered: via the stable distribution, Eq. 24 (dashed lines) and Gaussian, Eq. 67 (dotted lines). Solid lines represent quantiles of simulated Pareto sums. Red upper curves are for the 0.98 quantile, green middle curves are for the median, and blue lower curves are for 0.02 quantile.

Figure 9: Cumulative seismic moment (solid black line) for California as a function of time. Latitude limits  $37.0^\circ \text{ N} - 39.0^\circ \text{ N}$ , magnitude threshold 5.5. TOPPOZADA *et al.* (2000) catalog is used. We display quantiles of simulated Pareto ( $\alpha = 2/3$ ) sums with  $q = 0.02, 0.5$ , and 0.98 (dashed blue, green and red curves, respectively). Bounds obtained using the stable distribution approximation ( $z_q^{(1)}$ , Eq. 24) are shown by dotted curves of respective colors.

Figure 10: Logarithmic cumulative seismic moment (solid black line) for California as a function of the number of events,  $n$ . Latitude limits  $37.0^\circ \text{ N} - 39.0^\circ \text{ N}$ , magnitude threshold 5.5. TOPPOZADA *et al.* (2000) catalog is used. We display quantiles of simulated Pareto ( $\alpha = 2/3$ ) sums with  $q = 0.02, 0.5$ , and 0.98 (dashed blue, green and red curves, respectively). Bounds obtained using the stable distribution approximation ( $z_q^{(1)}$ , Eq. 24) are shown by dotted curves of respective colors.

Figure 11: Approximations for the tails of stable distributions.

(a) Stable cdf (MCCULLOCH and PANTON, 1987) with  $\alpha = 0.66$  (blue) and its approximation by Eq. 70 (red).

(b) Ratio between theoretical and approximated values of the upper quantiles 0.98 and 0.999 as functions of the  $\alpha$ -index.

Table 1: Number  $n$  of summands necessary to approximate the sum  $S_n$  by stable distributions  $(z_q^{(1)})$  with error  $\Delta_q^{(1)} < 0.1$ .

$\alpha$	Quantile		
	0.02	0.50	0.98
0.50	20 $\uparrow$	2 $\uparrow$	2 $\downarrow$
0.60	25 $\downarrow$	3 $\downarrow$	2 $\downarrow$
0.66	270 $\downarrow$	18 $\downarrow$	2 $\downarrow$
2/3	320 $\downarrow$	25 $\downarrow$	2 $\downarrow$
0.80	> 10000 $\downarrow$	3000 $\downarrow$	2 $\downarrow$
0.90	>> 10000 $\downarrow$	> 10000 $\downarrow$	8 $\downarrow$
0.94	>> 10000 $\downarrow$	>> 10000 $\downarrow$	2000 $\downarrow$
0.98	>> 10000 $\downarrow$	>> 10000 $\downarrow$	>> 10000 $\downarrow$
1.00	55 $\uparrow$	2 $\downarrow$	2 $\downarrow$
1.02	65 $\uparrow$	2 $\downarrow$	2 $\downarrow$
1.10	75 $\uparrow$	2 $\downarrow$	2 $\downarrow$
1.20	85 $\uparrow$	2 $\downarrow$	2 $\downarrow$
1.30	100 $\uparrow$	2 $\downarrow$	2 $\downarrow$
1.50	140 $\uparrow$	2 $\downarrow$	3 $\downarrow$
1.66	230 $\uparrow$	2 $\downarrow$	8 $\downarrow$
1.80	300 $\uparrow$	3 $\downarrow$	30 $\downarrow$
1.90	1000 $\uparrow$	6 $\downarrow$	35 $\downarrow$
1.94	1800 $\uparrow$	7 $\downarrow$	500 $\downarrow$
1.98	6500 $\uparrow$	9 $\downarrow$	4000 $\downarrow$

**Table 2. Relative errors of approximations considered in the paper for selected values of  $\alpha$ ,  $n$ , and  $q$ .**

$\alpha$	$n$	Simulated quantiles <sup>1</sup>			Approximation relative errors							
		$q_{0.02}$	$q_{0.5}$	$q_{0.98}$	$q=0.02$		$q=0.5$		$q=0.98$			
					$\Delta^{(1)}_{0.02}$ (Eq. 24)	$\Delta^{(5)}_{0.02}$ (Eq. 63)	$\Delta^{(1)}_{0.5}$ (Eq. 24)	$\Delta^{(4)}_{0.5}$ (Eq. 55)	$\Delta^{(1)}_{0.98}$ (Eq. 24)	$\Delta^{(2)}_{0.98}$ (Eq. 28)	$\Delta^{(3)}_{0.98}$ (Eq. 37)	
1/2	2	2.50	14.94	10000.17	-0.535	-0.002	-0.076	-0.001	0.000	0.000	0.000	-0.020
1/2	5	10.32	89.24	62436.65	-0.297	0.003	-0.033	0.037	0.001	0.001	0.001	-0.019
1/2	10	34.92	351.03	249338.20	-0.169	0.004	-0.016	0.057	0.002	0.003	0.003	-0.017
1/2	20	127.78	1392.25	1004949.00	-0.091	-0.002	-0.008	0.068	-0.005	-0.005	-0.005	-0.025
1/2	50	753.85	8654.80	6237558.00	-0.037	-0.008	-0.003	0.075	0.002	0.002	0.002	-0.018
1/2	100	2960.02	34628.25	25040967.00	-0.019	-0.002	-0.003	0.075	-0.002	-0.002	-0.002	-0.022
2/3	2	2.36	8.63	1012.34	0.257	-0.001	0.336	0.000	0.013	0.013	-0.012	-0.027
2/3	5	8.44	37.29	4029.96	0.389	-0.003	0.221	0.017	0.006	0.006	-0.019	-0.034
2/3	10	24.14	111.27	11406.04	0.373	0.005	0.158	0.023	0.006	0.006	-0.020	-0.035
2/3	20	71.32	327.36	32489.57	0.314	0.003	0.113	0.027	-0.002	-0.002	-0.027	-0.041
2/3	50	302.14	1345.80	128040.70	0.226	0.001	0.070	0.026	0.001	0.001	-0.024	-0.038
2/3	100	896.63	3882.27	363796.40	0.168	0.000	0.049	0.026	-0.003	-0.003	-0.028	-0.043
1	2	2.23	5.11	104.72	-1.658	0.001	-0.020	-0.001	0.043	0.043	-0.015	-0.055
1	5	6.99	16.90	271.33	-0.870	0.000	0.012	0.007	0.023	0.023	-0.033	-0.088
1	10	17.28	40.49	555.47	-0.493	0.002	0.016	0.006	0.012	0.012	-0.043	-0.109
1	20	42.92	94.75	1127.16	-0.269	0.001	0.015	0.004	0.009	0.009	-0.044	-0.122
1	50	140.46	283.08	2888.23	-0.115	-0.002	0.011	0.001	0.001	0.001	-0.052	-0.143
1	100	337.76	636.17	5851.76	-0.059	-0.001	0.008	-0.001	0.000	0.000	-0.052	-0.154
3/2	2	2.15	3.67	24.01	-2.453	0.004	0.064	0.000	0.141	0.141	0.147	-0.109
3/2	5	6.21	10.60	50.39	-1.291	0.003	0.050	0.004	0.080	0.080	0.085	-0.218
3/2	10	14.13	23.00	88.03	-0.765	-0.004	0.037	0.002	0.051	0.051	0.056	-0.289
3/2	20	32.03	48.98	153.80	-0.449	-0.002	0.026	-0.001	0.036	0.036	0.040	-0.354
3/2	50	92.64	130.02	326.42	-0.223	0.005	0.016	-0.003	0.020	0.020	0.024	-0.439
3/2	100	203.00	268.74	583.28	-0.132	0.005	0.010	-0.004	0.012	0.012	0.016	-0.502

<sup>1</sup> using  $10^7$  numerical simulations for each fixed  $\alpha$ ,  $n$



Fig. 1

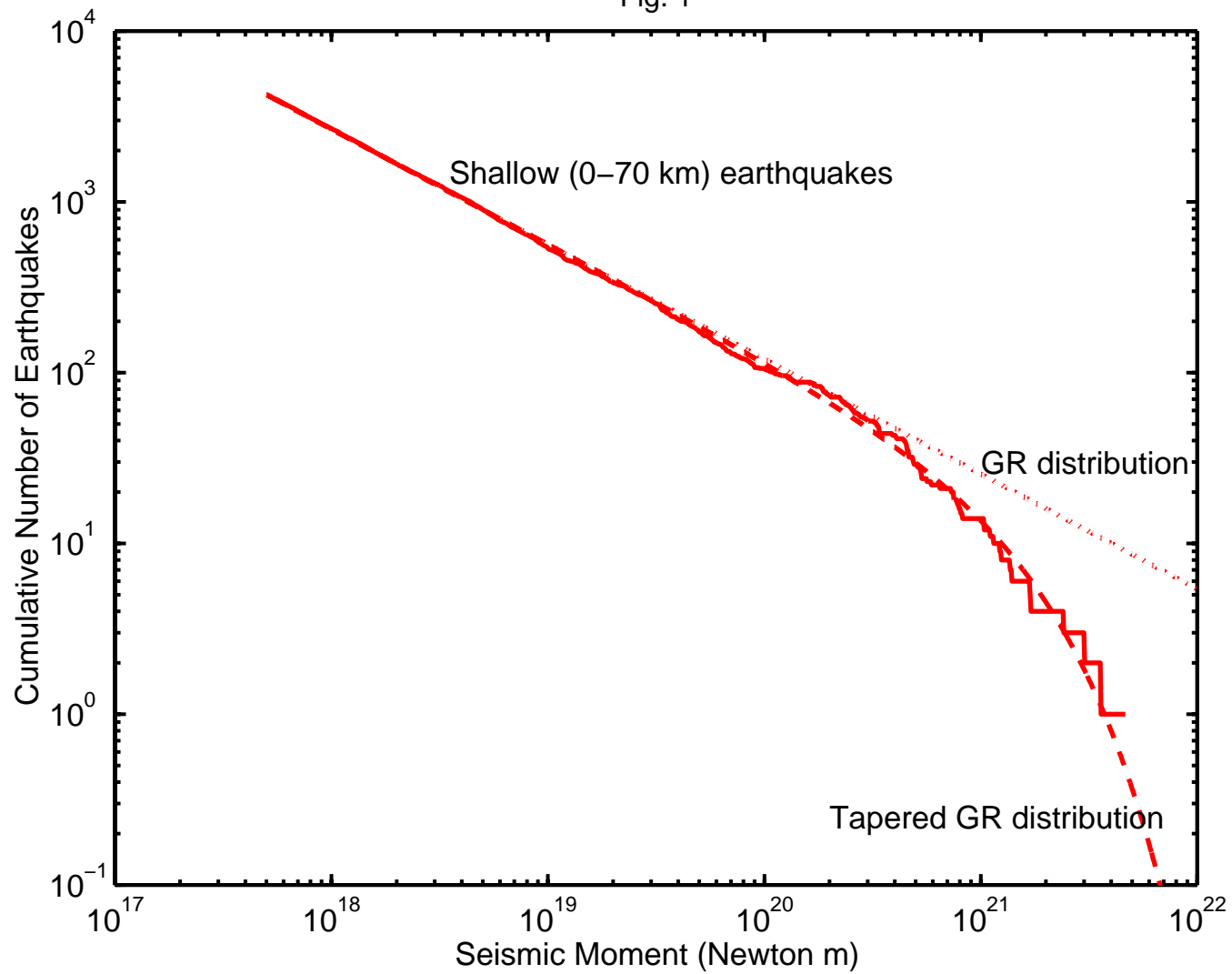


Fig. 2

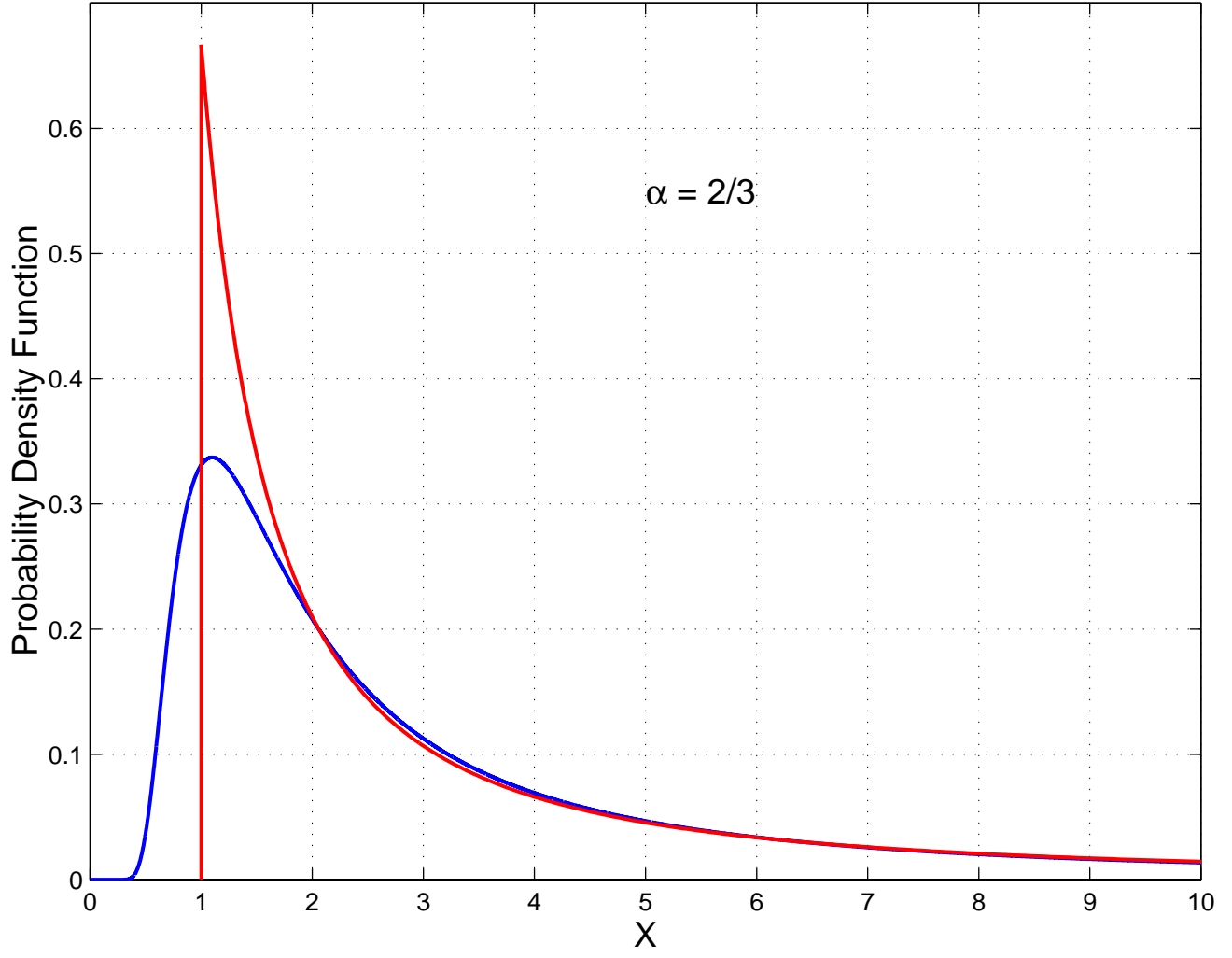


Fig. 3

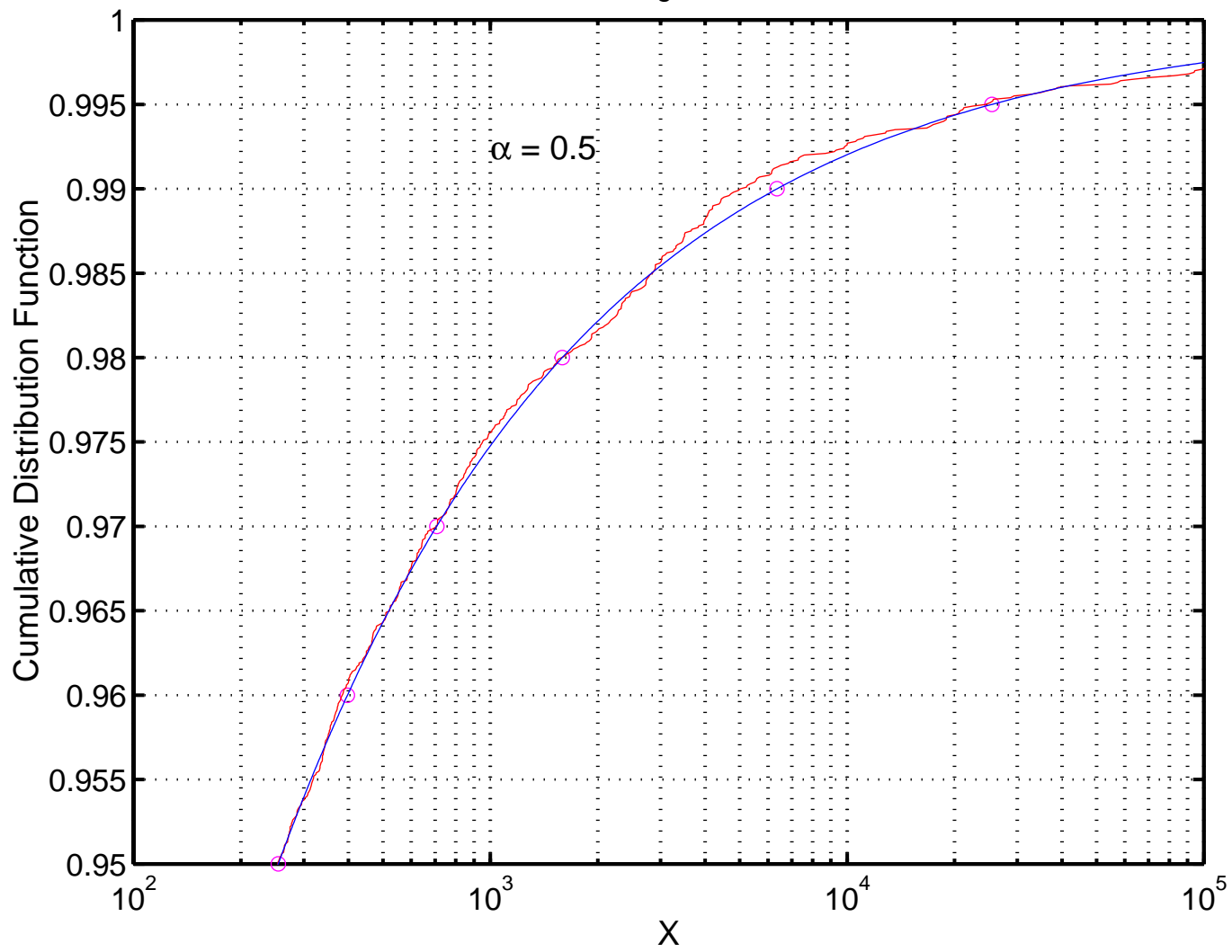


Fig. 4a

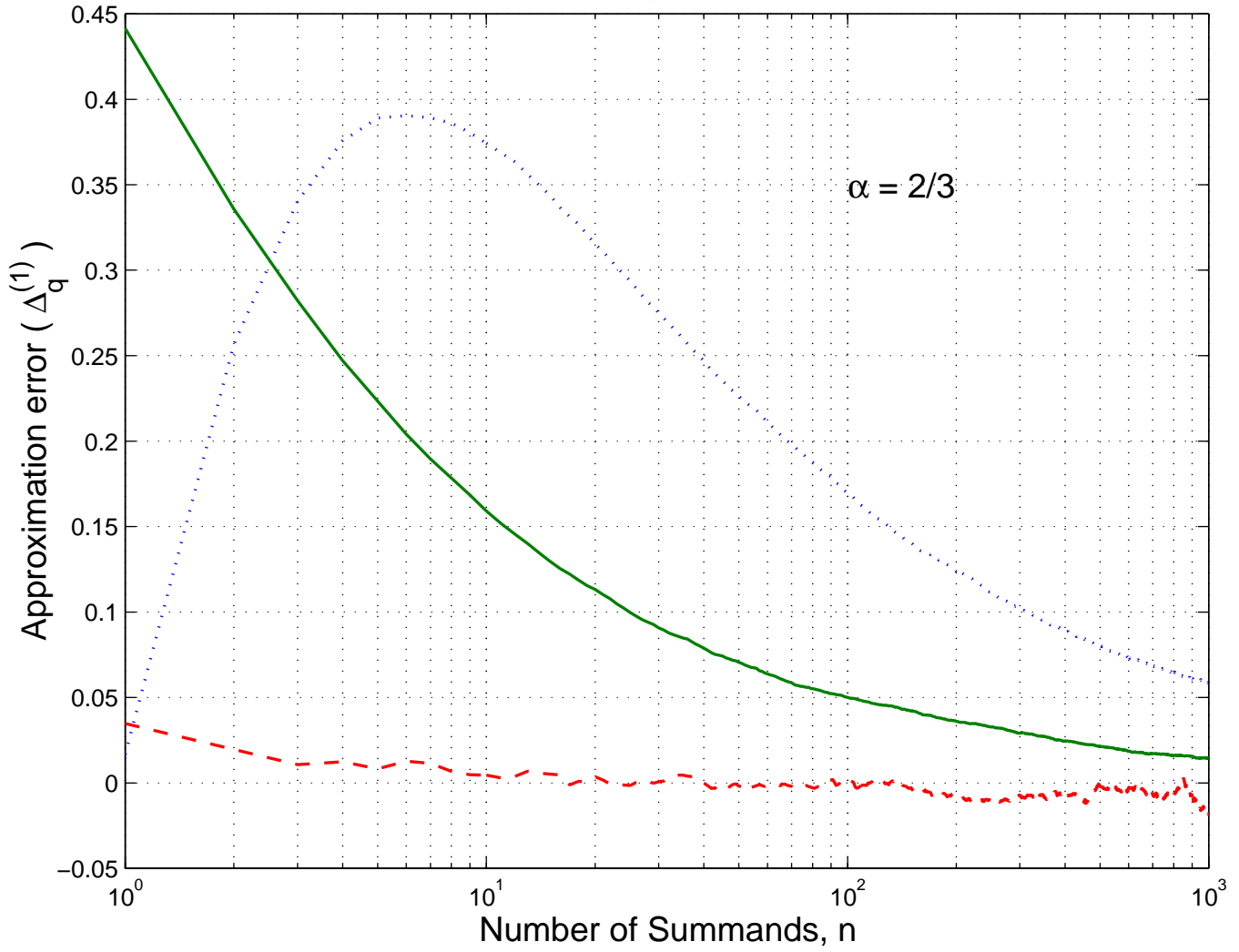


Fig. 4b

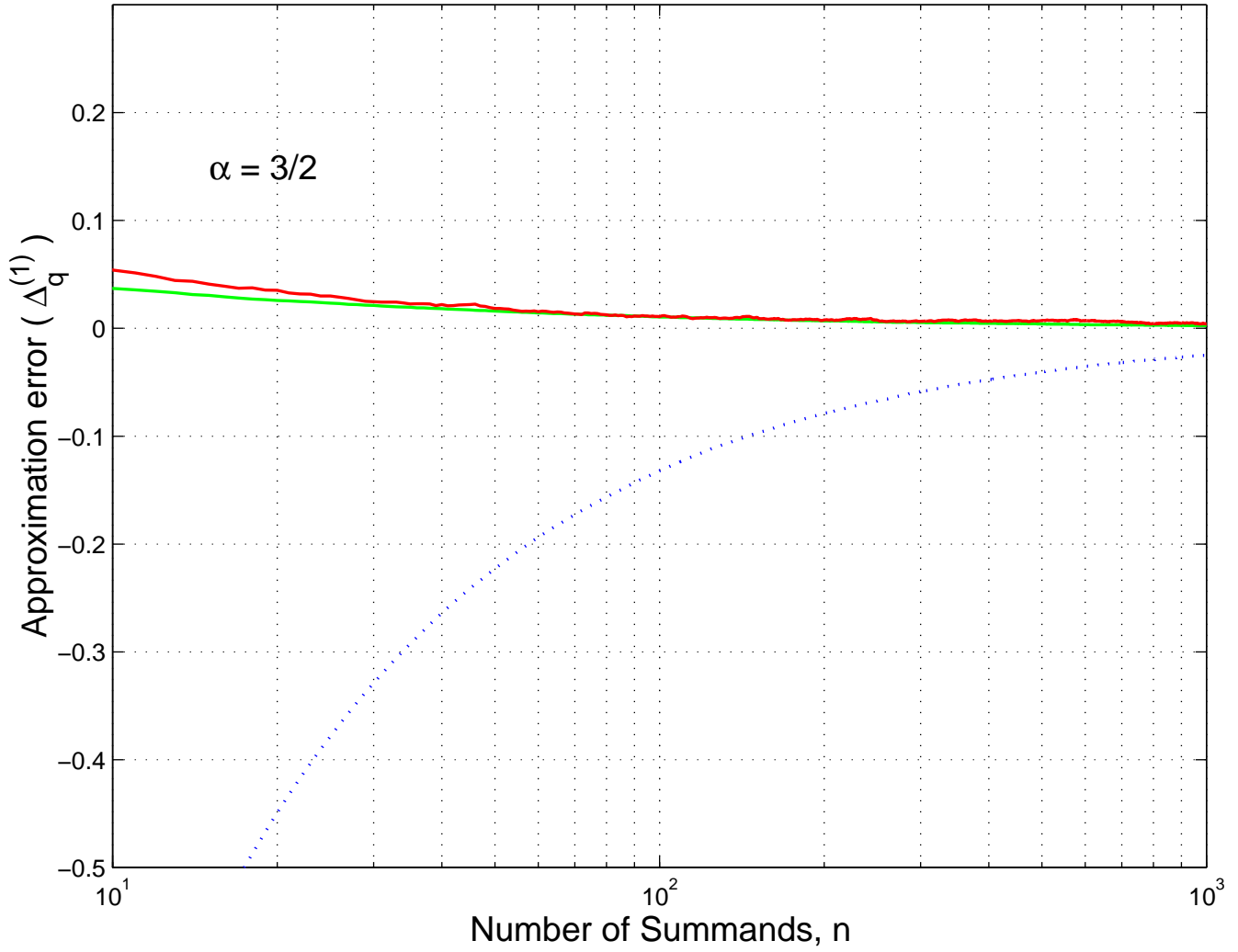


Fig. 5

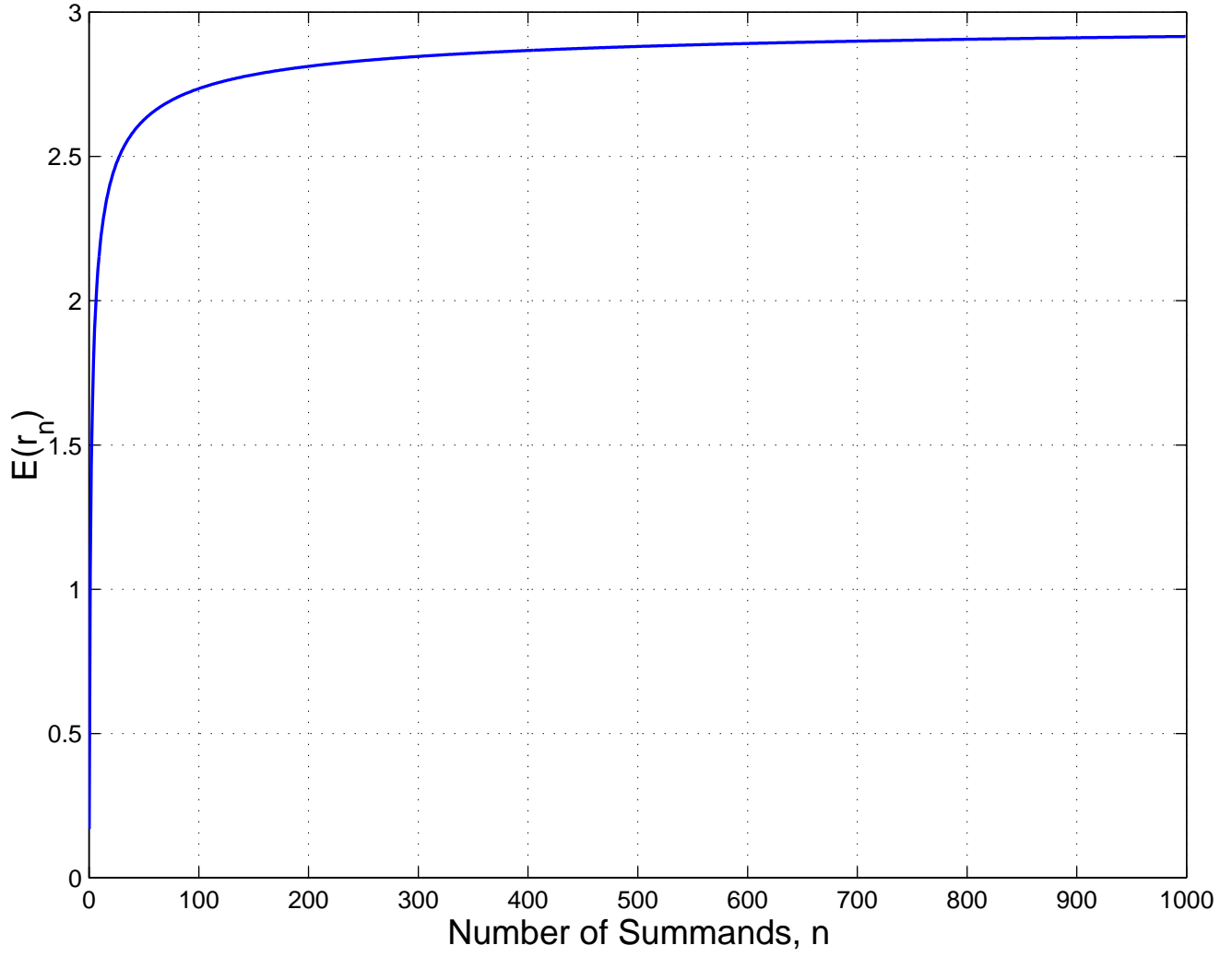


Fig. 6

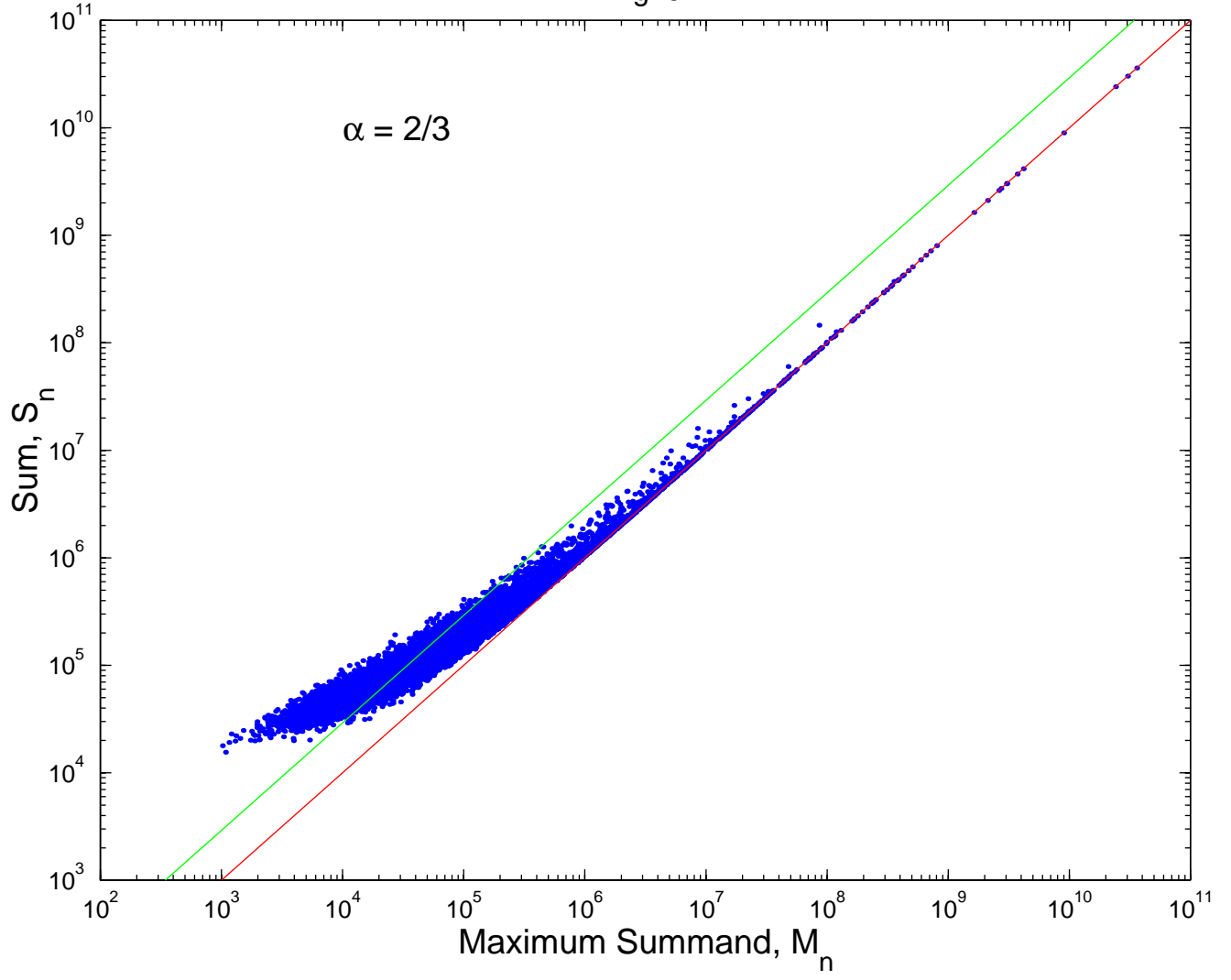


Fig. 7

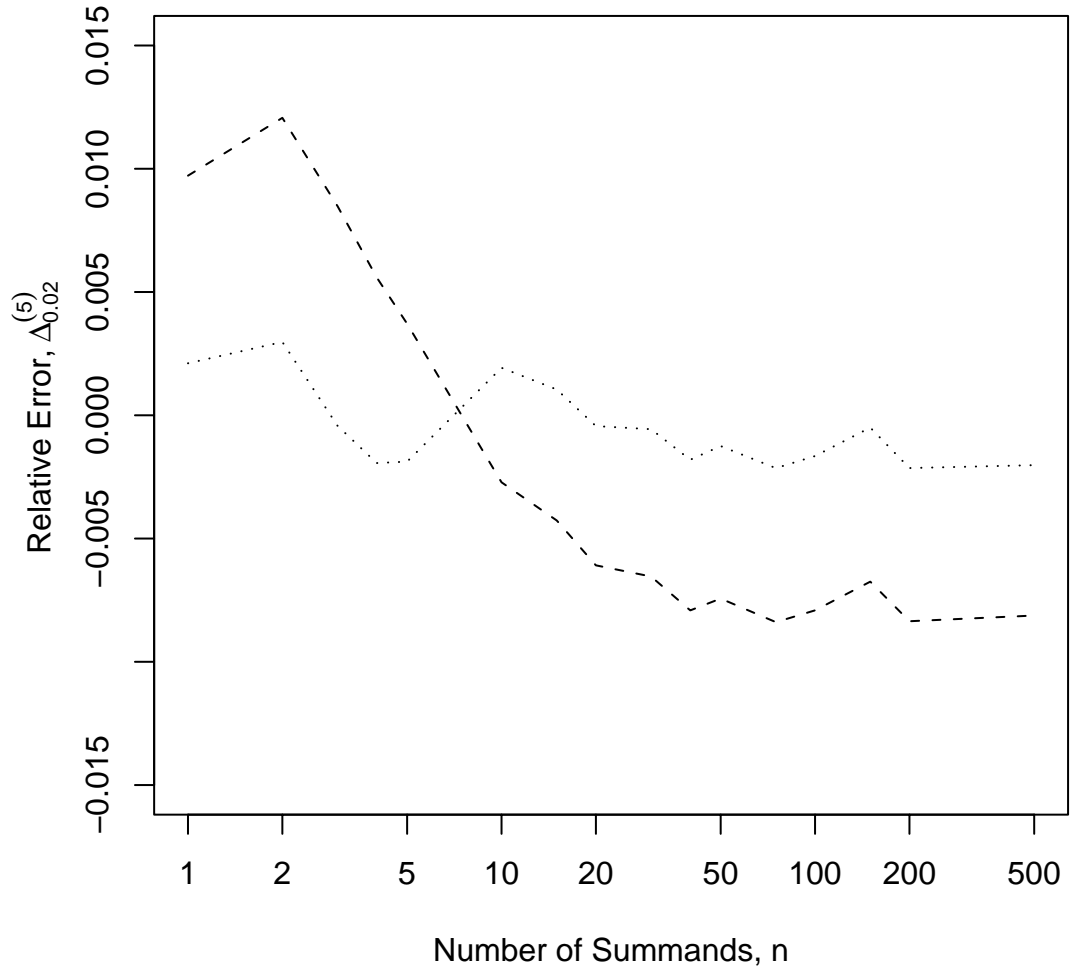




Fig. 8

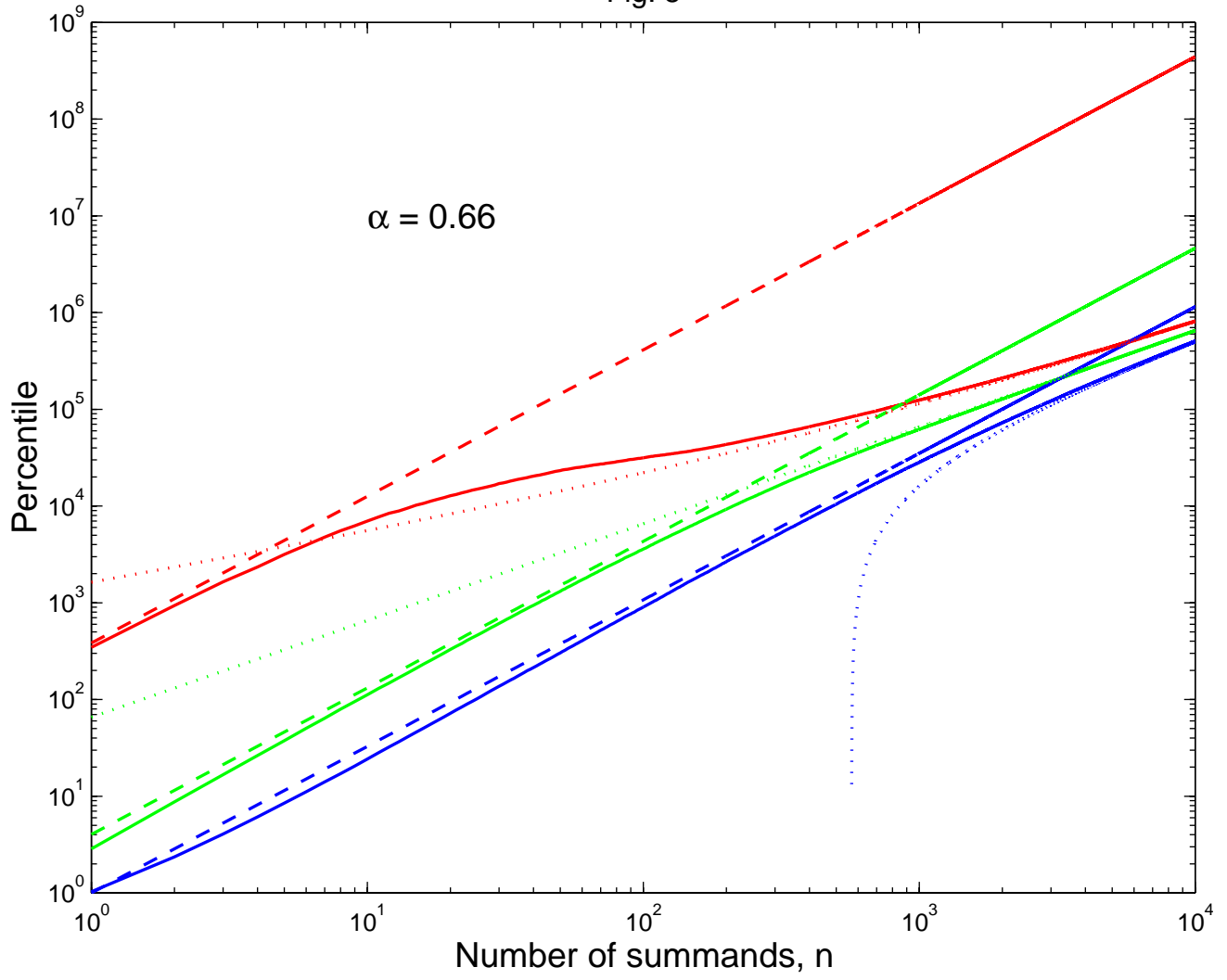


Fig. 9

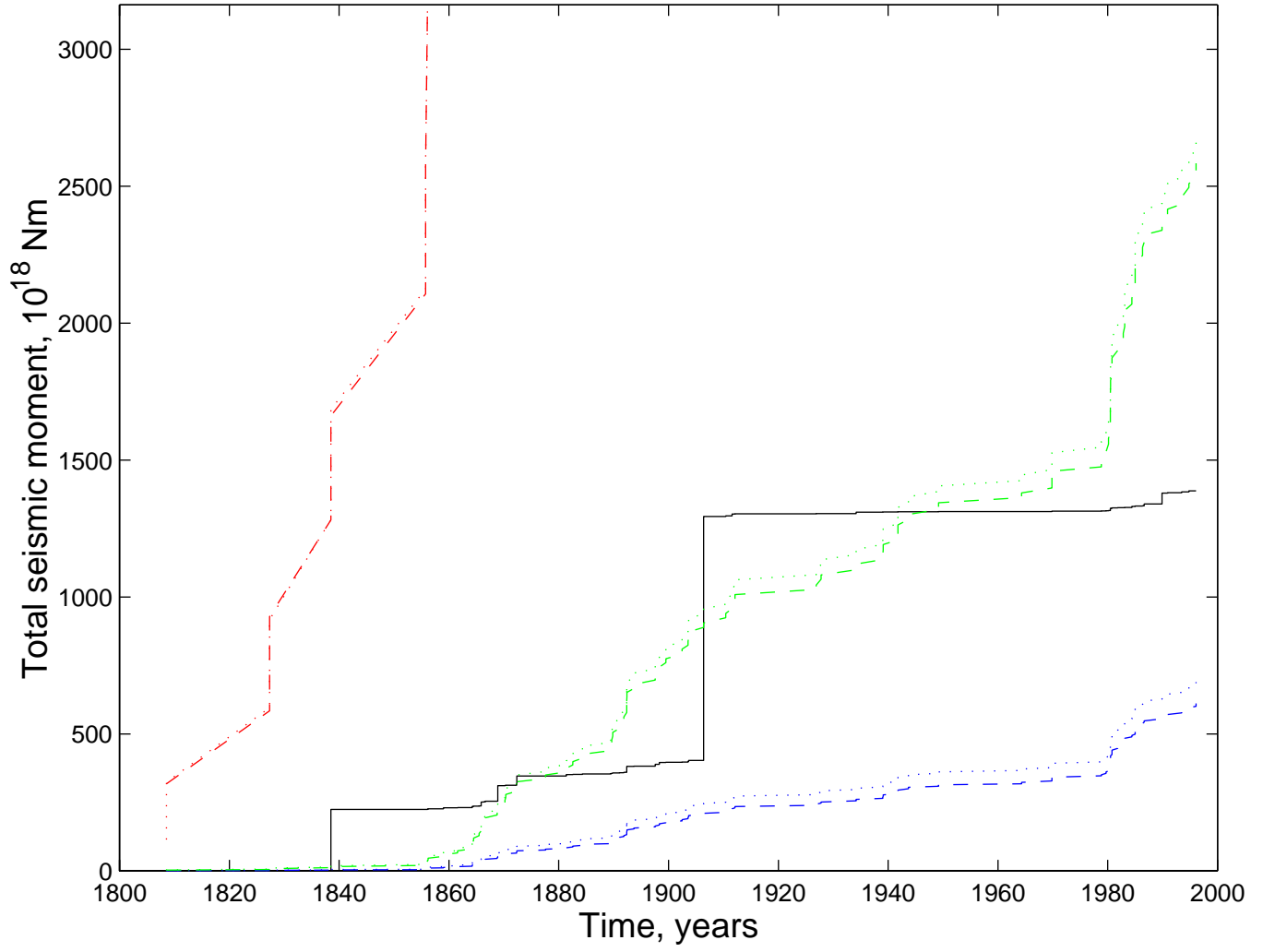


Fig. 10

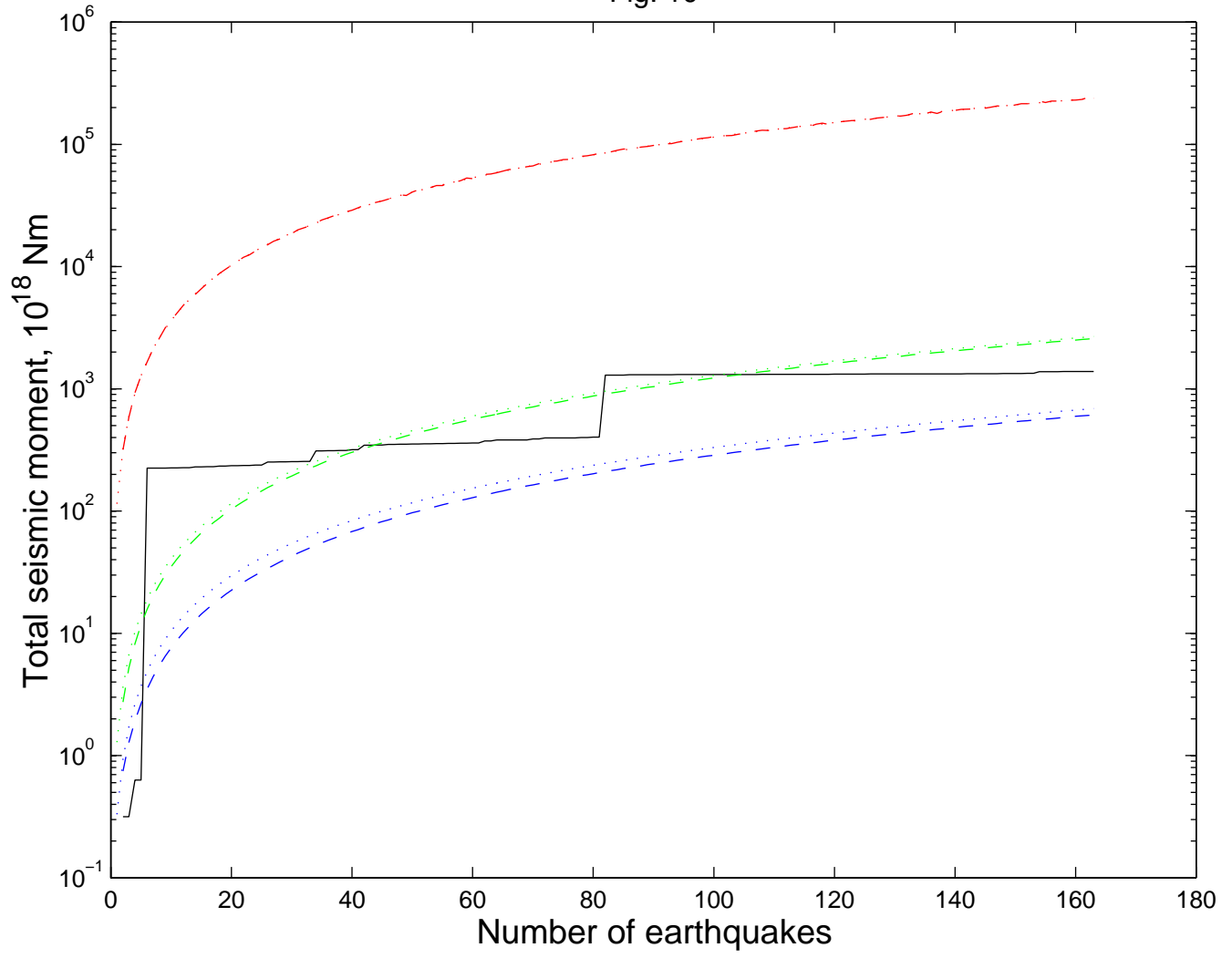


Fig. 11a

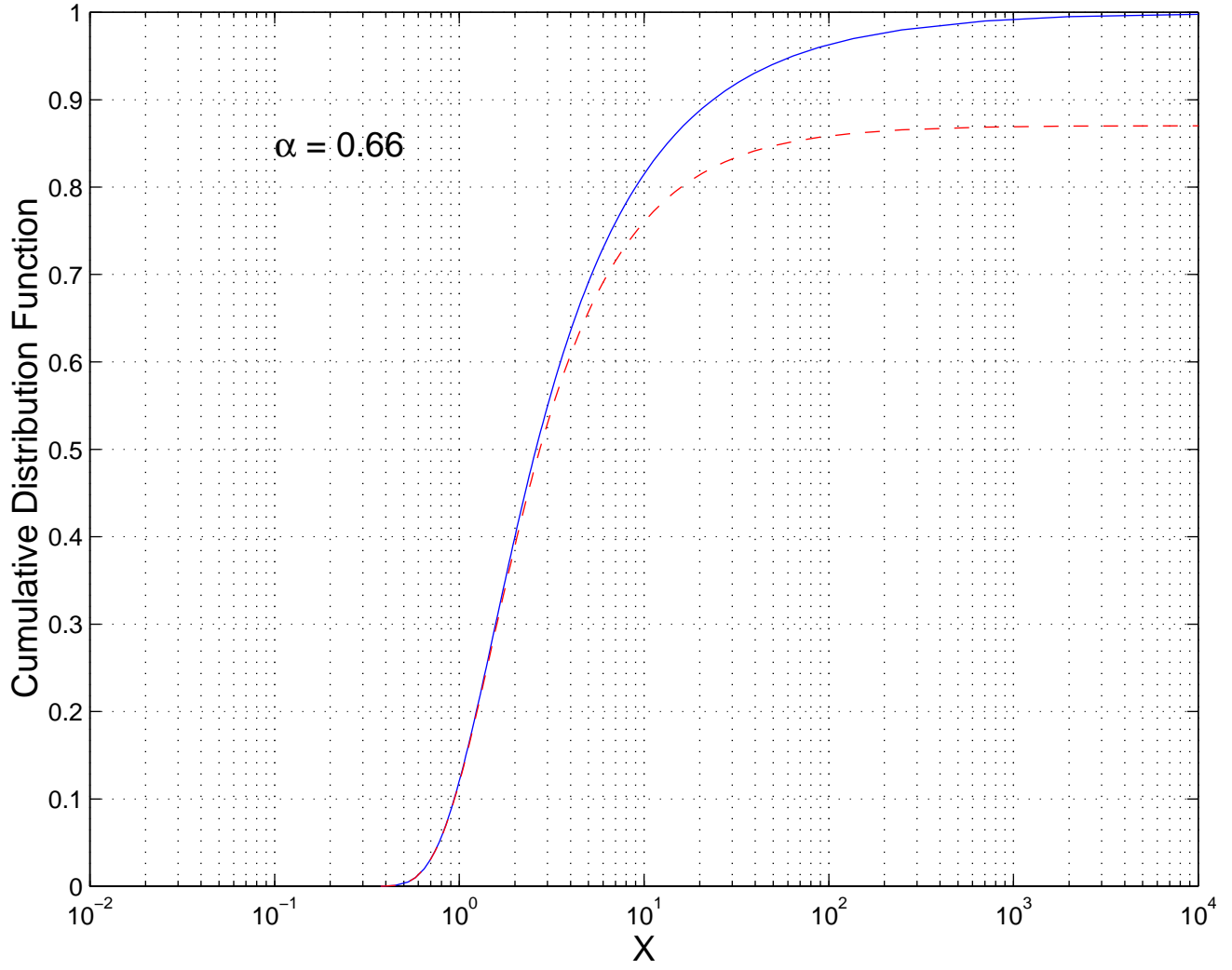


Fig. 11b

