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Hook Length Polynomials for Plane Forests of a Certain Type

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Abstract. The original motivation for the study of hook length polynomials was to find a combinatorial proof for a hook length formula for binary trees given by Postnikov, as well as a proof for a hook length polynomial formula conjectured by Lascoux. In this paper, we define the hook length polynomial for plane forests of a given degree sequence type and show that it can be factored into a product of linear forms. Some other enumerative results on forests are also given.

Keywords: hook length, plane forest

1. Introduction

In [3], Du and the author defined the hook length polynomials for m-ary trees and showed that they can be written as simple binomial expressions. In this paper, we extend this result to plane forests of a given degree sequence type.

The original motivation for the work on hook length formulas was to seek a combinatorial proof of an identity derived by Postnikov [7, 8]:

$$\sum_T \frac{n!}{2^n} \prod_v \left(1+\frac{1}{h_v}\right) = (n+1)^{n-1}, \tag{1.1}$$

where the sum is over all complete binary trees with n internal vertices, the product is over all internal vertices of T, and h_v is the "hook length" of v in T, namely, the number of internal vertices in the subtree of T rooted at v.

Chen and Yang [2] and Seo [9] both gave direct bijective proofs for (1.1). Moreover, based on (1.1), Lascoux replaced 1 with x and conjectured a hook length polynomial formula for binary trees:

$$\sum_T \prod_v \left(x+\frac{1}{h_v}\right) = \frac{1}{(n+1)!} \prod_{i=0}^{n-1} ((n+1+i)x+n+1-i). \tag{1.2}$$

Du and the author [3] generalized Lascoux's conjecture and proved hook length polynomial formulas for m-ary trees and plane forests. Analogous results were also given by Gessel and Seo [5].

In Section 2, we define hook length polynomial for forests corresponding to a given degree sequence and show that it has a simple binomial form. In Section 3, we study another form of hook length polynomials (3.2) and get an enumerative result on colored labelled forests (3.4) by using the idea of proper vertices introduced by Seo [9]. These techniques allow a fully bijective proof of (3.4), which then yields new fully bijective proofs for the formulas (2.1) and (2.2).

2. Hook Length Polynomials for Plane Forests of Type r

A *tree* is an acyclic connected graph. For any vertices v and u in a tree, we call v a *descendant* of u (or u an *ancestor* of v) if u lies on the unique path from the root to v. In particular, if u and v are adjacent, we call v a *child* of u. For any vertex v, we use $Des(v)$ to denote the set of descendants of v.

For any vertex v in a tree, the *degree* of v is the number of children of v. A vertex is an *internal* vertex if it is not a leaf, i.e., its degree is not zero. A *plane tree* is an unlabelled rooted tree whose vertices are regarded as indistinguishable, but the subtrees at any vertex are linearly ordered. A *plane forest* is a finite set of ordered plane trees. For any plane forest F, we denote by $I(F)$ the set of internal vertices of F.

For any plane forest F, let r_i be the number of vertices of degree i and $\mathbf{r} = (r_0, r_1, r_2, \ldots)$, then we say that F is of *type* $\mathbf{r}$. Given a nonnegative integer sequence $\mathbf{r} = (r_0, r_1, r_2, \ldots)$ with $\sum_{d\geq 0} r_d < \infty$, we use $\mathscr{F}(\mathbf{r})$ to denote all the forests F of type $\mathbf{r}$.

There is a well-known result on the cardinality of $\mathscr{F}(\mathbf{r})$ [1,4,6,10], denoting by $n = \sum_{d\geq 1} r_d = |I(F)|$ the number of internal vertices and $\ell = -\sum_{d\geq 0}(d-1)r_d$ the number of trees in F:

$$|\mathscr{F}(\mathbf{r})| = \frac{\ell}{n+r_0}\binom{n+r_0}{r_0, r_1, r_2, \ldots}. \tag{2.1}$$

Definition 2.1. *For any vertex v of a forest F, let d_v be its degree and h_v its hook length, i.e., the number of descendants it has. We define the hook length polynomial of v as*

$$P_v(x) = \frac{((d_v-1)h_v+1)x+1-h_v}{d_v h_v}.$$

Definition 2.2. *We define the hook length polynomial for plane forests of type* $\mathbf{r}$ *as*

$$\mathcal{H}_{\mathbf{r}}(x) = \sum_{F\in\mathscr{F}(\mathbf{r})} \prod_{v\in I(F)} P_v(x).$$

Then $\mathcal{H}_{\mathbf{r}}(x)$ can be written as a binomial expression:

Theorem 2.3.

$$\mathcal{H}_{\mathbf{r}}(x) = \frac{\ell}{r_0}\binom{r_0 x}{r_0 x - n, r_1, r_2, \ldots}. \tag{2.2}$$

Proof. If we replace x by k, then the right side of (2.2) becomes

$$\frac{\ell}{r_0}\binom{kr_0}{kr_0-n,\, r_1,\, r_2,\ldots} = \frac{k\ell}{kr_0}\binom{kr_0}{kr_0-n+r_1,\, r_2,\, r_3,\ldots}\binom{kr_0-n+r_1}{r_1}.$$

Applying (2.1), one can see that it counts the number of plane forests of type $\mathbf{r}' = (r'_0, r'_1, r'_2, \ldots)$, with r_1 leaves circled, where $r'_0 = kr_0 - n + r_1$, $r'_{ik} = r_{i+1}$, $\forall i \geq 1$ and $r'_j = 0$ for all $j \neq ik$ for any i. (Note that $(kr_0 - n + r_1) + \sum_{d\geq 2} r_d = kr_0$ and $(kr_0 - n + r_1) - \sum_{d\geq 2}((d-1)k-1)r_d = k\ell$.)

Because both sides of (2.2) are polynomials in x, it's enough to prove that

$$\mathcal{H}_{\mathbf{r}}(k) = \frac{\ell}{r_0}\binom{r_0k}{r_0k-n,\, r_1,\, r_2,\ldots}. \tag{2.3}$$

We prove this by induction on n, the number of internal vertices of F.

When $n = 0$, we have $\mathbf{r} = (r_0, 0, 0, \ldots)$ and $\ell = r_0$, so

$$\mathcal{H}_{\mathbf{r}}(k) = 1 = \frac{\ell}{r_0}\binom{r_0k}{r_0k}.$$

Assume (2.3) holds for $n < n_0$. Now we consider $n = n_0$.

If $\ell = 1$, then $\forall F \in \mathscr{F}(\mathbf{r})$, F is just a tree, say, T. Let v_0 be the root of T. Then

$$\mathcal{H}_{\mathbf{r}}(k) = \sum_{d\geq 1,\, r_d\neq 0} \frac{((d-1)n+1)k+1-n}{dn}\mathcal{H}_{\mathbf{r}^{(d)}}(k),$$

where $\mathbf{r}^{(d)} = (r_0, r_1, \ldots, r_{d-1}, r_d - 1, r_{d+1}, \ldots)$.

By the induction hypothesis,

$$\begin{aligned}\mathcal{H}_{\mathbf{r}^{(d)}}(k) &= \frac{d}{r_0}\binom{r_0k}{r_0k-n+1,\, r_1,\ldots,\, r_{d-1},\, r_d-1,\, r_{d+1},\ldots}\\ &= \frac{d}{r_0}\frac{r_d}{r_0k-n+1}\binom{r_0k}{r_0k-n,\, r_1,\ldots,\, r_{d-1},\, r_d,\, r_{d+1},\ldots}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{H}_{\mathbf{r}}(k) &= \sum_{d\geq 1,\, r_d\neq 0} \frac{((d-1)n+1)k+1-n}{r_0n}\frac{r_d}{r_0k-n+1}\binom{r_0k}{r_0k-n,\, r_1,\, r_2,\ldots}\\ &= \frac{1}{r_0}\binom{r_0k}{r_0k-n,\, r_1,\, r_2,\ldots}\sum_{d\geq 1,\, r_d\neq 0}\frac{(((d-1)n+1)k+1-n)r_d}{n(r_0k-n+1)}\\ &= \frac{1}{r_0}\binom{r_0k}{r_0k-n,\, r_1,\, r_2,\ldots}.\end{aligned}$$

If $\ell > 1$,

$$\begin{aligned}
\mathcal{H}_{\mathbf{r}}(k) &= \sum_{\mathbf{r}=\mathbf{r}^{(1)}+\mathbf{r}^{(2)}+\cdots+\mathbf{r}^{(\ell)}} \prod_{i=1}^{\ell} \mathcal{H}_{\mathbf{r}^{(i)}}(k) \\
&= \sum_{\mathbf{r}=\mathbf{r}^{(1)}+\mathbf{r}^{(2)}+\cdots+\mathbf{r}^{(\ell)}} \prod_{i=1}^{\ell} \#\text{ of forests of type } \mathbf{r}^{(i)\prime} \text{ with } \mathbf{r}_1^{(i)} \text{ leaves circled} \\
&= \#\text{ of forests of type } r' \text{ with } r_1 \text{ leaves circled} \\
&= \frac{\ell}{r_0}\binom{r_0 k}{r_0 k - n, r_1, r_2, \ldots}.
\end{aligned}$$

∎

3. Colored Labelled Forests

In this section, we will consider *labelled* plane forests. Given a plane forest F with n internal vertices, a *labelling* is a bijection from $I(F)$ to $[n]$. A *labelled forest* is a plane forest with a labelling. For a vertex v in a labelled forest, following [9], we call v a *proper* vertex if none of its descendants has smaller label than v, and an *improper* vertex otherwise. For a labelled forest F, we use $\mathrm{Prop}(F)$ to denote the set of proper vertices and $\mathrm{Improp}(F)$ the set of improper vertices.

Suppose that we have two sets of colors $\{c_1, c_2, \ldots\}$ and $\{c'_1, c'_2, \ldots\}$. Fix $k \geq 0$ and given a labelled forest F (which means we know the two sets $\mathrm{Prop}(F)$ and $\mathrm{Improp}(F)$ of F), a *proper k-coloring* of F is a way of coloring all the internal vertices of F so that for any $v \in I(F)$, if v is proper then it can be colored by any color in $\{c_1, c_2, \ldots, c_{d_v}\}$; otherwise it can be colored by any color in $\{c_1, c_2, \ldots, c_{d_v}\} \cup \{c'_1, c'_2, \ldots, c'_k\}$. (Note that $c_1, c_2, \ldots c_{d_v}$ can be considered corresponding to the d_v edges of v, and the colors $c'_1, c'_2, \ldots, c'_k$ for improper vertices are considered as "special" colors.) Therefore, given a labelled forest F, there are $\prod_{v\in\mathrm{Prop}(F)} d_v \prod_{v\in\mathrm{Improp}(F)} (d_v + k)$ proper k-colorings.

A *k-colored labelled forest* is a labelled forest with a proper k-coloring. Given a degree sequence $\mathbf{r}$ and $k \geq 0$, let $\mathcal{CF}_{\mathbf{r},k}$ be the set of all k-colored labelled forests F of type $\mathbf{r}$.

Lemma 3.1. *$\mathcal{CF}_{\mathbf{r},k}$ is counted by* $\sum_{F\in\mathscr{F}(\mathbf{r})} n! \prod_{v\in I(F)} \left((d_v + k) - \frac{k}{h_v}\right)$.

Proof. For any $F \in \mathscr{F}(\mathbf{r})$,

$$\begin{aligned}
n! \prod_{v\in I(F)} \left((d_v+k) - \frac{k}{h_v}\right) &= n! \sum_{J\subset I(F)} \left(\prod_{v\in J} -\frac{k}{h_v}\right)\left(\prod_{v\in I(F)\setminus J} (d_v+k)\right) \\
&= \sum_{J\subset I(F)} \frac{n!}{\prod_{v\in J} h_v} \left(\prod_{v\in J} -k\right)\left(\prod_{v\in I(F)\setminus J} (d_v+k)\right)
\end{aligned}$$

However, $n!/\prod_{v\in J} h_v$ is the number of labellings of F so that all the vertices in J are proper. Therefore,

$$
\begin{aligned}
& n! \prod_{v\in I(F)} \left((d_v+k) - \frac{k}{h_v} \right) \\
&= \sum_{F'} \sum_{J\subset \mathrm{Prop}(F')} \left(\prod_{v\in J} -k \right) \left(\prod_{v\in I(F)\setminus J} (d_v+k) \right) \quad (F' \text{ is } F \text{ with a labelling}) \\
&= \sum_{F'} \sum_{J\subset \mathrm{Prop}(F')} \left(\prod_{v\in J} -k \right) \left(\prod_{v\in \mathrm{Prop}(F')\setminus J} (d_v+k) \right) \left(\prod_{v\in \mathrm{Improp}(F')} (d_v+k) \right) \\
&= \sum_{F'} \left(\prod_{v\in \mathrm{Prop}(F')} (-k+d_v+k) \right) \left(\prod_{v\in \mathrm{Improp}(F')} (d_v+k) \right) \\
&= \sum_{F'} (\text{the number of proper } k\text{-colorings } F' \text{ has}).
\end{aligned}
$$

Summing over all of the forests in $\mathscr{F}(\mathbf{r})$ gives us

$$
|\mathcal{CF}_{\mathbf{r},k}| = \sum_{F\in\mathscr{F}(\mathbf{r})} n! \prod_{v\in I(F)} \left((d_v+k) - \frac{k}{h_v} \right). \tag{3.1}
$$

∎

Now we look back at our hook length polynomials. We change (2.2) into another form which is more closely related to Postnikov's identity.

Lemma 3.2. *Identity* (2.2) *has the following equivalent form*:

$$
\sum_{F\in\mathscr{F}(\mathbf{r})} \prod_{v\in I(F)} \frac{(d_v+x)h_v - x}{d_v h_v} = \frac{\ell}{r_1!r_2!\cdots} \prod_{i=1}^{n-1} (r_0 + i(1+x)). \tag{3.2}
$$

Note that when $\mathbf{r} = (n+1, 0, n, 0, 0, \ldots)$ and $x = -1$, (3.2) is the same as (1.1).

Proof.

$$
\begin{aligned}
(2.2) &\Leftrightarrow \sum_{F\in\mathscr{F}(\mathbf{r})} \prod_{v\in I(F)} \frac{((d_v-1)h_v+1)x+1-h_v}{d_v h_v} = \frac{\ell x}{r_1!r_2!\cdots} \prod_{i=1}^{n-1} (r_0 x - i) \\
&\Leftrightarrow \sum_{F\in\mathscr{F}(\mathbf{r})} \prod_{v\in I(F)} \frac{((d_v-1)h_v+1)x+(1-h_v)y}{d_v h_v} = \frac{\ell x}{r_1!r_2!\cdots} \prod_{i=1}^{n-1} (r_0 x - iy) \\
&\Leftrightarrow \sum_{F\in\mathscr{F}(\mathbf{r})} \prod_{v\in I(F)} \frac{d_v h_v x + (1-h_v)(x+y)}{d_v h_v} = \frac{\ell x}{r_1!r_2!\cdots} \prod_{i=1}^{n-1} (r_0 x - iy) \\
&\Leftrightarrow \sum_{F\in\mathscr{F}(\mathbf{r})} \prod_{v\in I(F)} \frac{d_v h_v y - (1-h_v)x}{d_v h_v} = \frac{\ell y}{r_1!r_2!\cdots} \prod_{i=1}^{n-1} (r_0 y + i(x+y)) \\
&\Leftrightarrow (3.2)
\end{aligned}
$$

∎

If we replace x with k in (3.2) and rearrange it a bit, we have

$$\sum_{F\in\mathscr{F}(\mathbf{r})} n!\prod_{v\in I(F)}\left((d_v+k)-\frac{k}{h_v}\right)=\binom{n}{r_1,r_2,\ldots}\ell 1^{r_1}2^{r_2}\cdots\prod_{i=1}^{n-1}(r_0+i(1+k)). \quad (3.3)$$

Comparing (3.1) and (3.3), we get the following proposition.

Proposition 3.3.

$$|\mathcal{CF}_{\mathbf{r},k}|=\binom{n}{r_1,r_2,\ldots}\ell 1^{r_1}2^{r_2}\cdots\prod_{i=1}^{n-1}(r_0+i(1+k)).$$

However, we have a stronger result than Proposition 3.3. For any degree sequence $\mathbf{r}$, we denote by $V_{\mathbf{r}}$ the set of all ordered partitions $\mathbf{S}=(S_1,S_2,\ldots)$ of $[n]$ such that $|S_i|=r_i$. We let $\mathbf{S}=(S_1,S_2,\ldots)$ and $\mathcal{CF}_{\mathbf{r},k,\mathbf{S}}$ be the set of all the forests F in $\mathcal{CF}_{\mathbf{r},k}$ such that $\forall v\in I(F)$, the label of v is in S_{d_v}, that is, the label i can only be used to label an internal vertex v of degree d if $i\in S_d$.

We call two partitions $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ *adjacent* if there exists $i\in[n-1]$, such that we can obtain $\mathbf{S}^{(1)}$ by swapping i and $i+1$ in $\mathbf{S}^{(2)}$.

We construct a graph $G_{\mathbf{r}}$ with vertex set $V_{\mathbf{r}}$ and $\left\{\mathbf{S}^{(1)},\mathbf{S}^{(2)}\right\}$ forming an edge in $G_{\mathbf{r}}$ if and only if they are adjacent. It's not hard to see that $G_{\mathbf{r}}$ is connected.

Lemma 3.4. *For any two partitions $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ in $V_{\mathbf{r}}$,*

$$\left|\mathcal{CF}_{\mathbf{r},k,\mathbf{S}^{(1)}}\right|=\left|\mathcal{CF}_{\mathbf{r},k,\mathbf{S}^{(2)}}\right|.$$

Proof. It's enough to prove the case when $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ are adjacent.

Suppose that we obtain $\mathbf{S}^{(1)}$ by swapping i and $i+1$ in $\mathbf{S}^{(2)}$, for some $i\in[n-1]$, and $i\in S^{(1)}_{d_1}$, $i+1\in S^{(1)}_{d_2}$. $\left(\text{So } i\in S^{(2)}_{d_2}, i+1\in S^{(2)}_{d_1}.\right)$

We define a map ψ from $\mathcal{CF}_{\mathbf{r},k,\mathbf{S}^{(1)}}$ to $\mathcal{CF}_{\mathbf{r},k,\mathbf{S}^{(2)}}$. For any colored labelled forest $F\in\mathcal{CF}_{\mathbf{r},k,\mathbf{S}^{(1)}}$, let v_1 be the vertex with label i and v_2 be the vertex with label $i+1$:

(i) If $v_1\notin Des(v_2)$ and $v_2\notin Des(v_1)$, then $\psi(F)=(i,i+1)F$, where $(i,i+1)F$ means swap the labels i and $i+1$.
(ii) If $v_2\in Des(v_1)$ and v_1 is improper, then let $\psi(F)=(i,i+1)F$.
(iii) If $v_2\in Des(v_1)$ and v_1 is proper (so v_2 is proper too), then let $\psi(F)=(i,i+1)F$.
(iv) If $v_1\in Des(v_2)$ and $\exists j<i$ such that the vertex with label j is in $Des(v_2)$, then let $\psi(F)=(i,i+1)F$.
(v) If $v_1\in Des(v_2)$, any vertex with label $j<i$ is not in $Des(v_2)$, and color of v_2 is not one of the k special colors, then let $\psi(F)=(i,i+1)F$.
(vi) If $v_1\in Des(v_2)$, any vertex with label $j<i$ is not in $Des(v_2)$ (so v_1 is proper and has a color c_α), and the color of v_2 is one of the k special colors, then we obtain $\psi(F)$ in the following way: Suppose that v_1 and v_2 are in tree T with root r. Let u be the βth child of v_2 that is an ancestor of v_1 and w be the αth child of v_1 that corresponds to the color of v_1. We separate T at v_2, u, v_1, and w to get five trees T_1,T_2,T_3,T_4, and T_5 with roots r, v_2, u, v_1, and w, respectively, denote by

v_2', u', v_1', and w' the leaves of T_1, T_2, T_3, and T_4 obtained from v_2, u, v_1, and w, respectively. Attach the root w of T_5 to u', attach the root v_2 of T_2 to v_1', attach the root u of T_3 to w', and attach the root v_1 of T_4 to v_2'. Color v_2 with the color c_β corresponding to u and v_1 with the original color for v_2. Finally, swap labels i and $i+1$.

One can check that ψ gives a bijection between $\mathcal{CF}_{\mathbf{r},k,\mathbf{S}^{(1)}}$ and $\mathcal{CF}_{\mathbf{r},k,\mathbf{S}^{(2)}}$. ∎

We observe that Proposition 3.3 and Lemma 3.4 together are equivalent to the following Theorem:

Theorem 3.5.

$$|\mathcal{CF}_{\mathbf{r},k,\mathbf{S}}| = \ell 1^{r_1} 2^{r_2} \cdots \prod_{i=1}^{n-1} (r_0 + i(1+k)). \tag{3.4}$$

We also provide another proof of Theorem 3.5, which is bijective and combinatorial.

Proof. Let $\mathcal{CF}_{\mathbf{r},k,\mathbf{S},1} \subset \mathcal{CF}_{\mathbf{r},k,\mathbf{S}}$ be the set with all the forests with label 1 appearing in the first tree. Clearly, (3.4) is equivalent to

$$|\mathcal{CF}_{\mathbf{r},k,\mathbf{S},1}| = 1^{r_1} 2^{r_2} \cdots \prod_{i=1}^{n-1} (r_0 + i(1+k)). \tag{3.5}$$

Let $g_i \in [d_i], \forall 1 \le i \le n$, $f_j \in [r_0 + j(1+k)], \forall 1 \le j \le n-1$. Then there are $1^{r_1} 2^{r_2} \cdots \prod_{i=1}^{n-1} (r_0 + i(1+k))$ choices for the g_i's and f_j's.

We will construct a bijection between $\{g_i, f_j\}$ and $\mathcal{CF}_{\mathbf{r},k,\mathbf{S},1}$ inductively on $n = \sum_{d \ge 1} r_d$, the number of internal vertices of the forests.

When $n = 1$, we have only one vertex. Suppose it has degree d. Then $r_0 = d + \ell - 1$. We don't have f_j's and $g_1 \in [d]$. Clearly, there's a natural bijection between the value of g_1 and the color of vertex 1 in any forest in $\mathcal{CF}_{\mathbf{r},k,\mathbf{S},1}$.

Now we assume for $n < n_0$, we have a bijection between $\{g_i, f_j\}$ and $\mathcal{CF}_{\mathbf{r},k,\mathbf{S},1}$, and consider $n = n_0$. For any $F \in \mathcal{CF}_{\mathbf{r},k,\mathbf{S},1}$, let T be it's first tree. We know that 1 is in T. We have two cases:

If the root of T is 1, then let g_1 be the value corresponding to the color of 1. By removing 1 from T, F becomes a forest with $\ell + d_1 - 1$ trees and $n-1$ vertices. However, the smallest vertex 2 is not necessarily in the first tree. Let f_{n-1} be the position number of the tree containing 2; then $f_{n-1} \in [\ell + d_1 - 1]$. We cyclicly rotate the order of the trees so that that tree becomes the first tree in the forest and call the resulting forest F'. Hence, $F' \in \mathcal{CF}_{\mathbf{r}',k,\mathbf{S}',1}$, where $\mathbf{r}'$ is obtained by subtracting 1 from r_1 in $\mathbf{r}$, and $\mathbf{S}'$ is obtained from $\mathbf{S}$ by removing 1 from S_{d_1}. By the induction hypothesis, we can associate $\{g_i, f_j\}_{2 \le i \le n, 1 \le j \le n-2}$ to F'. Including g_1 and f_{n-1}, we obtain a bijection between forests of this type and the set $\{g_i, f_j\}$ with $f_{n-1} \in [\ell + d_1 - 1]$.

If the root of T is i ($i \ne 1$), then i is improper and it can have $d_i + k$ choices of colors. We can associate these $d_i + k$ colors with choosing f_{n-1} in the interval $\left[\ell + d_1 + \sum_{j=2}^{i-1} (d_j + k), \ell + d_1 - 1 + \sum_{j=2}^{i} (d_j + k)\right]$. Because 1 is in T, it is a descendant of i. Let g_i be the number corresponding to the child of i that is ancestor of 1. Similar to the first case, remove i and rotate the first d_i trees so that 1 becomes contained in the first tree in the new forest F'. Then $F' \in \mathcal{CF}_{\mathbf{r}',k,\mathbf{S}',1}$, where $\mathbf{r}'$ is obtained by subtracting 1 from r_i in $\mathbf{r}$, and $\mathbf{S}'$ is obtained from $\mathbf{S}$ by removing i from S_{d_i}. Again, by a similar argument, we can get a bijection between forests of this type and the set $\{g_i, f_j\}$ with $f_{n-1} \in \left[\ell + d_1 + \sum_{j=2}^{i-1} (d_j + k), \ell + d_1 - 1 + \sum_{j=2}^{i} (d_j + k)\right]$.

Note that $\ell + d_1 - 1 + \sum_{j=2}^{n} (d_j + k) = \ell + \sum_{d \geq 1} (d-1) r_d + (n-1)(1+k) = r_0 + (n-1)(1+k)$. Therefore, we have constructed a bijection for $n = n_0$. ∎

Remark 3.6. Because Theorem 3.5 implies Proposition 3.3, Proposition 3.3 together with Lemma 3.1 imply (3.2), and (3.2) is equivalent to (2.2), the above proof of Theorem 3.5 can be considered as another proof of Theorem 2.3. It also gives a new proof of (2.1) by substituting -1 for x in (2.2) or substituting 0 for x in (3.2).

Remark 3.7. One can modify our definition of k-colorings of a labelled forest so that it makes sense for $k = -1$. The proof of Theorem 3.5 works as well for $k = -1$.

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