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Spectral Zeta Functions of Laplacians on Self-Similar Fractals

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Spectral Zeta Functions of Laplacians on Self-Similar Fractals

A Dissertation submitted in partial satisfaction
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in

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by

Nishu Lal

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I dedicate my thesis to the loving memory of my mother, Neelam Lal. You will always remain very close to my heart. I miss you everyday, mom.

ABSTRACT OF THE DISSERTATION

Spectral Zeta Functions of Laplacians on Self-Similar Fractals

by

Nishu Lal

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2012
Professor Michel L. Lapidus, Chairperson

This thesis investigates the spectral zeta function of fractal differential operators such as the Laplacian on the unbounded (i.e., infinite) Sierpinski gasket and a self-similar Sturm–Liouville operator associated with a fractal self-similar measure on the half-line. In the latter case, C. Sabot discovered the relation between the spectrum of this operator and the iteration of a rational map of several complex variables, called the renormalization map. We obtain a factorization of the spectral zeta function of such an operator, expressed in terms of the Dirac delta hyperfunction, a geometric zeta function, and the zeta function associated with the dynamics of the corresponding renormalization map, viewed either as a polynomial function on $\mathbb{C}$ (in the first case) or (in the second case) as a polynomial on the complex projective plane, $\mathbb{P}^2(\mathbb{C})$. Our first main result extends to the case of the fractal Laplacian on the unbounded Sierpinski gasket a factorization formula obtained by M. Lapidus for the spectral zeta function of a fractal string and later extended by A. Teplyaev to the bounded (i.e., finite) Sierpinski gasket and some other decimable fractals. Furthermore, our second main result generalizes these factorization formulas to the renormalization maps of several complex variables associated with fractal Sturm–Liouville operators. Moreover, as a corollary, in the very special

case when the underlying self-similar measure is Lebesgue measure on $[0, 1]$, we obtain a representation of the Riemann zeta function in terms of the dynamics of a certain polynomial in $\mathbb{P}^2(\mathbb{C})$, thereby extending to several variables an analogous result by A. Teplyaev.

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Chapter 1

Introduction

In functional analysis, the second order differential equations describe many physical phenomena such as heat flow of a region over time with the assumption that the underlying space is smooth. As we come to learn, there are many objects in the world that are not so regular. How should one continue to carry out such analysis if the smoothness condition is removed? Considering the geometry of irregular objects, B. Mandelbrot introduced a mathematical framework for such objects, now known as the theory of analysis on fractals.

The spectral analysis on fractals was first studied by R. Rammal and G. Toulouse in physics literature. We begin with the question of how to define the Laplacian on objects that are not smooth. The Sierpinski gasket is a commonly used fractal to study the construction of Laplacian on fractals. The first mathematical approach by S. Goldstein and S. Kusuoka [12, 20] introduced the Laplacian on the Sierpinski gasket as the generator of the diffusion process. Later on, J. Kigami [18] proposed a direct analytic approach using the theory of Dirichlet forms which was extended to a class of fractals called post-critically finite self-similar sets.

The physicists R. Rammal [32] and R. Rammal and G. Toulouse [33] studied the spectrum of the Laplacian of the Sierpinski gasket, in particular, the eigenvalue equation $-\Delta_\mu u = \lambda u$, and discovered the decimation method which establishes the relations between the spectrum of the Laplacian and the dynamics of the iteration of some polynomial on $\mathbb{C}$. Later on, T. Shima [40] and M. Fukushima and T. Shima [11] gave a precise mathematical statement of their results.

The development of the study of spectral zeta functions on fractals is inspired by M. Lapidus [22, 27] in light of the theory of fractal strings. A spectral zeta function is a zeta function that comes from the eigenvalue spectrum of a suitable differential operator. A fractal string $\mathcal{L}$ is a countable collection of disjoint intervals of lengths. He discovered a factorization of the spectral zeta function of the Dirichlet Laplacian L on a fractal string $\mathcal{L}$ of the following form

$$\zeta_L(s) = \pi^{-s}\zeta(s)\zeta_{\mathcal{L}}(s), \tag{1.1}$$

where $\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s$ is the geometric zeta function on $\mathcal{L}$ and $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. Later, A. Teplyaev [44, 45] studied the spectral zeta function of the Laplacian on a class of symmetric finitely ramified fractals extending the factorization for the fractal strings. He expressed the spectral zeta function of the Laplacian on the bounded Sierpinski gasket in terms of a new zeta function associated with the iterates of a polynomial in $\mathbb{C}$.

In this thesis, we extend the notion of factorization of the spectral zeta function to the Laplacian of the unbounded (i.e., infinite) Sierpinski gasket and to several complex variables in the case of the fractal Sturm–Liouville operators, a class of generalized second order differential operators of the form $\frac{d}{dm}\frac{d}{dx}$. First, we show that the spectral zeta function of the Laplacian of the unbounded Sierpinski gasket can be written as the product of the Dirac delta hyperfunction and the spectral zeta function associated with the bounded Sierpinski gasket, which is further expressed by A. Teplyaev [45] as the product of the zeta function of a quadratic polynomial in one complex variable and a suitable geometric zeta function. In joint work with M. Lapidus [21], we introduce a zeta function ζ_ρ associated with a map $\rho : \mathbb{P}^2\mathbb{C} \to \mathbb{P}^2\mathbb{C}$ of several complex variables which generates the spectrum of the generalized fractal differential operators. It is defined by

$$\zeta_\rho(s) = \sum_{p=0}^{\infty} \sum_{\{\lambda\in\mathbb{C}:\ \rho^p(\phi(\gamma^{-(p+1)}\lambda))\in D\}} (\gamma^p\lambda)^{-\frac{s}{2}}, \tag{1.2}$$

where D is a suitable subset of the Fatou set of ρ, and it contributes to the spectral zeta function ζ_{sp} of the Sturm–Liouville operator, via a product formula of the form

$$\zeta_{sp}(s) = \zeta_\rho(s)\zeta_{\mathcal{L}}(s). \tag{1.3}$$

Here, $\zeta_{\mathcal{L}}$ is the geometric zeta function of some underlying fractal string $\mathcal{L} = \{\ell_j\}_{j=1}^{\infty}$, viewed as a sequence of scales naturally associated with the Sturm–Liouville problem.

More specifically, we consider a sequence of second order differential operators of the form $H_{<n>} = \frac{d}{dm_{<n>}}\frac{d}{dx}$, with a self-similar measure $m_{<n>}$ on a sequence of intervals obtained by blowing up the unit interval with certain scaling ratios. It has been shown by C. Sabot [35, 36] that the limit operator, denoted by $H_{<\infty>}$ on $[0,\infty)$, gives rise to a spectrum which tends to 0 for one part and to ∞ for the other. Due to this peculiar behavior of the spectrum, the factorization of the spectral zeta function involves the Dirac delta hyperfunction $\delta_{\mathbb{T}}$ on the unit circle:

$$\zeta_{H_{<\infty>}} = \zeta_S \cdot \delta_{\mathbb{T}},$$

where ζ_S is the geometric zeta function associated with the infinite set S which generates the spectrum.

Furthermore, we obtain a representation of the Riemann zeta function in terms of ζ_ρ, defined in Equation (1.2), when the underlying self-similar measure m is Lebesgue measure on $[0, 1]$. This very special case extends the result of A. Teplyaev for the one-dimensional case.

Chapter 2

Background

2.1 Self-Similar Sets

Mandelbrot described a set to be a self-similar set if it a union of scaled copies of the initial set. Later, Hutchinson [15] gave a more precise definition of a self-similar set. He described it as a compact set which is the fixed point of the iterated function system. We recall the contraction mapping theorem in $\mathbb{R}^n$ as a motivation to Hutchinson's theory involving Hausdorff metric on the space of compacta. If (X, d) is a metric space, then $f : X \to X$ is a contraction mapping if $d(f(x), f(y)) \leq cd(x, y)$ for all $x, y \in X$, where $0 < c < 1$. We can think of the space X as a Euclidean space $\mathbb{R}^n$.

Theorem 1. *(Contraction Mapping Theorem) If (X, d) is a complete metric space and $f : X \to X$ is a contraction mapping, then there exists a unique $y \in X$ such that $f(y) = y$. Moreover, $\lim_{n\to\infty} f^n(x) = y$ for all $x \in X$.*

2.1.1 Hausdorff Metric

We denote the set of all compact subsets of X by $K(X)$. We define the Hausdroff metric on $K(X)$ as a distance function between two compact sets $A, B \in K(X)$,

$$d_H(A,B) = inf\{\epsilon > 0 : B \subseteq A_\epsilon \text{ and } A \subseteq B_\epsilon\}$$

where $A_\epsilon = \{x \in X : d(x,A) \leq \epsilon\}$. The completeness property is preserved from the space (X,d) to $(K(X),d_H)$, i.e, if (X,d) is complete, then so is $(K(X),d_H)$. The same is true of the compactness property. Hence, if (X,d) is a compact metric space, then so is $(K(X),d_H)$, in particular, $(K(X),d_H)$ is a complete and compact metric space in this case.

2.1.2 Iterated Function System (IFS)

Given $N \geq 2$, consider the contraction mappings $\{\Phi_1, \Phi_2, ...\Phi_N\}$ on (X,d). We define a new function $\Phi : K(X) \to K(X)$ by $\Phi(A) = \bigcup_{j=1}^N \Phi_j(A)$. The map Φ is contractive on $K(X)$ with respect to the Haursdorff metric d_H.

Definition 2. *A compact set $F \in K(X)$ is a self-similar set with respect to the contraction mappings $\{\Phi_1, \Phi_2, ..., \Phi_N\}$ on (X,d) if it is a fixed point of Φ, namely,*

$$F = \Phi(F) = \bigcup_{j=1}^{N} \Phi_j(F).$$

The existence of this unique set is proved by Hutchinson and it follows from the Contraction Mapping Principle applied to the complete metric space, $K(X)$ of non-empty compact subsets of X.

Theorem 3. *(Hutchinson) Let $(K(X),d_H)$ be complete and let $\Phi_1, \Phi_2, ...\Phi_N$ be contraction mappings on X. Then there exists a unique self-similar set F with respect to $\Phi_1, \Phi_2, ..., \Phi_N$. In addition, if $\{F_n\}$ is a sequence of compact sets in $K(X)$ such that $F_{n+1} = \bigcup_{j=1}^N \Phi_j(F_n)$, then F_n converges to the self-similar set F as $n \to \infty$ in the Hausdroff metric d_H.*

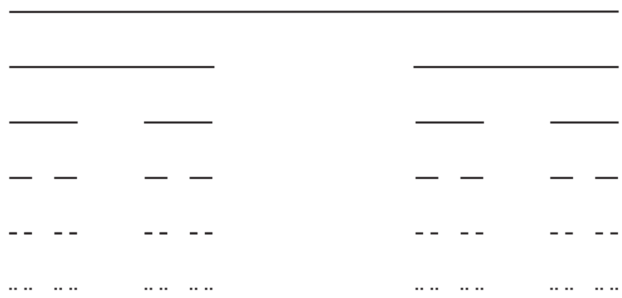


Figure 2.1: Approximations to the Cantor set

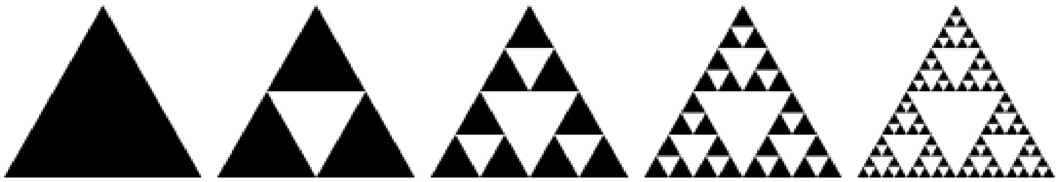


Figure 2.2: Approximations to the Sierpinski gasket

2.1.3 Examples

Example 4. *(The middle-third Cantor set) The classic Cantor set is a self-similar set on* $\mathbb{R}$ *which can be described by two contraction mappings* $\Phi_1,\ \Phi_2 : \mathbb{R} \to \mathbb{R}$ *defined by* $\Phi_1(x) = \frac{1}{3}x$ *and* $\Phi_2(x) = \frac{1}{3}x + \frac{2}{3}$.

Example 5. *(The Sierpinski gasket, SG) The Sierpinski gasket is generated by the iterated function system consisting of three contraction mappings* $\Phi_j : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by*

$$\Phi_j(x) = \frac{1}{2}(x - q_j) + q_j \tag{2.1}$$

for $j = 0, 1, 2$*, where* q_0, q_1, q_2 *are the vertices of an equilateral triangle. Note that each* Φ_j *has a unique fixed point, namely,* q_j*. The Sierpinski gasket is the unique compact set of* $\mathbb{R}^2$ *such that* $SG = \Phi_1(SG) \cup \Phi_2(SG) \cup \Phi_3(SG)$.

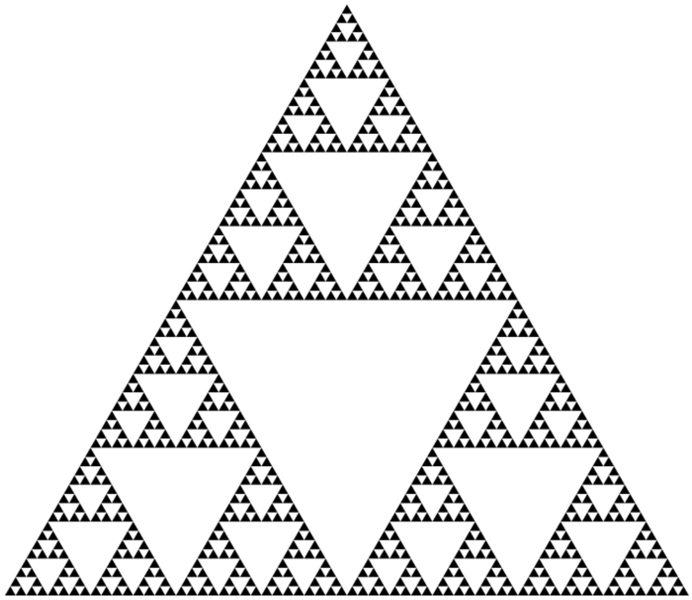


Figure 2.3: The Sierpinski gasket

2.2 Laplacians on P.C.F. Fractals

We describe a class of postcritically finite (p.c.f.) self-smilar fractals which was orginally introduced by Kigami. The self-similarity and finite ramification are the two strong characteristics of this class of fractals. Roughly speaking, a fractal is finitely ramified if a removal of finite number of points makes the fractal disconnected. Kigami [18] developed the theory of Laplacian operators on p.c.f. fractals by constructing Dirichlet forms on graphs approximating the fractal.

We begin by assuming the fractal F is a self-similar set for a finite iterated function system of contraction mappings $\{\Phi_0, \Phi_1, ..., \Phi_N\}$. We construct fractals by the approach of using discrete approximations of graphs $\Gamma_0, \Gamma_1, ...$ Let Γ_0 be the complete graph on V_0.

Definition 6. *The vertices of* Γ_m *are defined recursively*

$$V_m := \bigcup_{j=1}^{N} \Phi_j(V_{m-1}).$$

Define $V_* = \bigcup_{m \geq 0} V_m$ and $\Phi_\omega := \Phi_{\omega_1} \circ \Phi_{\omega_2} \circ ... \circ \Phi_{\omega_n}$ for a word $\omega = (\omega_1 \omega_2 ... \omega_n)$ of length n with each $\omega_j \in \{1, 2, ..., n\}$. A typical cell of level m has the form $\Phi_\omega(F)$ with $|\omega| = m$. We define the edge relation $x \underset{m}{\sim} y$ if there is a word of length m such that $x, y \in \Phi_\omega V_0$. The set of vertices form an increasing chain $V_0 \subseteq V_1 \subseteq V_2 \subseteq ....$ Note that V_* is dense in F under the usual Euclidean metric; so it is enough to use the sequence of finite sets V_m to approximate F.

Now we will give a precise definition of the p.c.f. fractals.

Definition 7. *F is a postcritically finite (p.c.f.) if F is connected and if there exists a finite set $V_0 \subseteq F$, called the boundary set, such that $\Phi_\omega F \bigcap \Phi_{\omega'} F \subseteq \Phi_\omega V_0 \bigcap \Phi_{\omega'} V_0$ for $\omega \neq \omega'$ with $|\omega| = |\omega'|$.*

2.2.1 Self-Similar Measures

To define the Laplacian on p.c.f. fractals, we use a self-similar measure. We start with a regular probability measure μ on the p.c.f. self-similar fractal F.

Definition 8. *μ is a regular probability measure on F if it satisfies the following conditions:*

- *(positivity) $\mu(C) > 0$ for every cell $C = \Phi_\omega(F)$*
- *(additivity) If $C = \bigcup_{j=1}^{n} C_j$ with the cells $\{C_j\}$ intersecting only at the boundary points, then $\mu(C) = \sum_{j=1}^{n} \mu(C_j)$.*
- *(continuity) $\mu(C) \to 0$ as the size of C approaches 0.*
- *(probability) $\mu(F) = 1$.*

For the construction of a self-similar measure, we choose a set of probability weights $\{\mu_j\}_{j=1}^{N}$ on $\{1, 2, ..., N\}$, each corresponding to the contraction mappings Φ_j,

$j = 1, ..., N$ such that $\mu_j > 0$ and $\sum_{j=1}^{n} \mu_j = 1$.

Definition 9. *μ is a self-similar measure if*

$$\mu(\Phi_\omega F) = \mu_\omega := \mu_{\omega_1}\mu_{\omega_2}...\mu_{\omega_m},$$

where $\omega = (\omega_1\omega_2...\omega_m)$ is a word of length m. Moreover, it satisfies the self-similar identity for measures, which is

$$\mu(A) = \sum_{j}^{N} \mu_j \mu(\Phi_j^{-1} A), \tag{2.2}$$

for all $A \subseteq F$.

Integration of continuous functions with respect to a self-similar measure can easily be computed by mimicking the theory of Riemann integration. We can subdivide the fractal F into all cells of level m as $F = \bigcup_{|\omega|=m} \Phi_\omega F$; then the integral of any continuous function f on F can be expressed by

$$\int_F f d\mu = \lim_{m\to\infty} \sum_{|\omega|=m} f(x_\omega)\mu(\Phi_\omega F).$$

Since any continuous function on a compact space F is uniformly continuous, the choice of $x_\omega \in \Phi_\omega F$ does not make a difference. The self-similar identity for the measure μ can be transformed into an identity involving integrals of functions through a characteristic function $f = \chi_A$. The self-similar identity of μ becomes

$$\int_F f d\mu = \sum_{j=1}^{n} \mu_j \int_F f \circ \Phi_j d\mu,$$

where f is an arbitrary continuous (and, more generally, Borel measurable) function on F. (Note that in our earlier notation, we have $f \circ \Phi_j = \chi_{\Phi_j^{-1} A}$.)

2.2.2 Dirichlet Forms and Laplacians

Dirichlet forms play a fundamental role in constructing Laplacians on p.c.f. fractals. We define discrete Laplacians on an increasing sequence of finite sets and take the limit to construct a Laplacian on the self-similar set. This construction was discovered by Kigami [18]. There are two ways to formulate the Laplacian, the pointwise formulation and the weak formulation which involves the Dirichlet form. We will first discuss the pointwise formulation of the Laplacian Δ, then we will show that the pointwise formulation and the weak formuation are equivalent.

2.2.2.1 Weak Formulation

We start with an iterated function system of contraction mappings to define a self-similar set, which is often a fractal. So far, we have fractals approximated by a sequence of graphs $\{\Gamma_m\}$ and the corresponding sequence of vertices $\{V_m\}$ are finite sets. The discrete Laplacians are defined on this sequence of finite sets to produce the fractal Laplacian in the limit. The discrete Dirichlet forms are also known as energy forms.

Definition 10. *The discrete Dirichlet (energy) form on V_m is*

$$\mathcal{E}_m(u,u) = r^{-m} \sum_{x \underset{m}{\sim} y} (u(x) - u(y))^2$$

where u is any function on V_m and $0 < r < 1$ is chosen so that the restriction of $\mathcal{E}_m$ to V_{m-1} is equal to $\mathcal{E}_{m-1}$. The Dirichlet (energy) form on F is

$$\mathcal{E}(u,u) = \lim_{m\to\infty} \mathcal{E}_m(u,u)$$

with the domain $Dom(\mathcal{E}) = \{u : \mathcal{E}(u,u) < \infty\}$ and where the limit is non-decreasing.

For any u defined on V_m, the harmonic extension $\tilde{u}$ to V_{m+1} is the extension of u that minimizes the energy $\mathcal{E}_m(\tilde{u}, \tilde{u})$. A function is called *harmonic* h on F if it minimizes the energy $\mathcal{E}$ for the given set of boundary values. Any function h_0 on V_0 has a unique continuation to a harmonic function h with the property $\mathcal{E}(h, h) = \mathcal{E}_m(h, h)$ for all m. It is also clear that for any $g \in Dom(\mathcal{E})$, $\mathcal{E}_0(g, g) \leq \mathcal{E}(g, g)$. In addition, we have the associated bilinear form on V_m, $\mathcal{E}_m(u, v) = r^{-m} \sum_{x \underset{m}{\sim} y} (u(x) - u(y))(v(x) - v(y))$. In general, the Harmonic Extension Lemma states that for any u, v defined on V_m such that $\tilde{u}$ is the harmonic extension of u and v' is any extension of v to V_{m+1}, we have $\mathcal{E}_{m+1}(\tilde{u}, v') = \mathcal{E}_m(u, v)$.

Let μ be the self-similar probability measure on F satisfying Equation (2.2). We define the Laplacian Δ_μ via a weak formulation as follows: Let $u \in Dom(\mathcal{E})$ and $f \in C(F)$, where $C(F)$ is the space of all continuous functions on F. Then $\Delta_\mu u = f$ with $u \in Dom(\Delta_\mu)$ means that

$$\mathcal{E}(u, v) = -\int_F f v d\mu,$$

for all $v \in Dom_0(\mathcal{E})$ ($v \in Dom_0(\mathcal{E})$ means that $v \in Dom(\mathcal{E})$ and $v|_{V_0} = 0$). A motivation for the weak formulation of the Laplacian is integration by parts involving the Laplacian on the unit interval. We will discuss the example of the unit interval, viewed as a self-similar set, later on.

2.2.2.2 The Pointwise Formulation

The pointwise definition of the Laplacian Δ_m on V_m is defined as

$$\Delta_m u(x) = \sum_{x \underset{m}{\sim} y} c_m \big(u(x) - u(y)\big),$$

for all functions $u : V_m \to \mathbb{R}$.

In order to show that the pointwise and the weak formulations are equivalent, we define a piecewise harmonic function. For $x \in V_m \backslash V_0$, let $\Psi_x^{(m)}(y)$ denote the piecewise harmonic defined by $\Psi_x^{(m)}(y) = \delta_{xy}$ on V_m. The pointwise formula can be derived from the weak formulation. It is easy to observe that $\mathcal{E}(u, \Psi_x^m) = r^{-m}\Delta_m u(x)$. Using the equation for the weak formulation,

$$r^{-m}\Delta_m = -\int f \Psi_x^m d\mu.$$

By taking into account the fact that f is continuous and the support of Ψ_x^m is close to x, we have $r^{-m}\Delta_m \approx f(x) \int \Psi_x^m d\mu$. In conclusion, it leads us to

$$\lim_{m\to\infty} r^{-m} \left(\int_F \Psi_x^m d\mu \right)^{-1} \Delta_m u(x) = f(x) = \Delta_\mu u(x),$$

where the limit holds uniformly in $x \in F$.

We will next return to the example of the Sierpinski gasket. It is one of the first fractals for which the Laplacian operator is rigorously defined and it plays a fundamental role in constructing a Laplacian for the more general class of p.c.f. fractals. (See the exposition in [41].)

2.2.3 Examples

Example 11. *(The Sierpinski gasket, SG) The Sierpinski gasket is generated by the IFS defined in Equation* (2.1)*. Denote by Γ_m the mth level graph and let Γ_0 be the complete graph on $V_0 := \{q_1, q_2, q_3\}$. The vertices of Γ_m are defined recursively as follows: $V_m := \bigcup_{j=1}^n \Phi_j(V_{m-1})$. Let $\ell^2(V_m)$ be the space of real-valued functions on V_m*

with the standard inner product $(u, v) = \sum_{x\in V_m} u(x)v(x)$. *The discrete Laplacian on* $\ell^2(V_m)$ *is defined by*

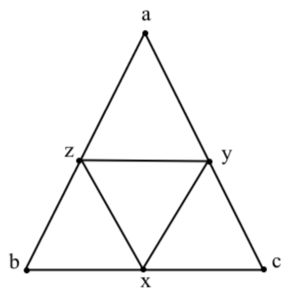
$$\Delta_m f(x) = \frac{1}{4}\sum_{x \underset{m}{\sim} y} f(x) - f(y).$$

We define the Laplacian on the SG as a limit

$$\Delta f(x) = \lim_{m\to\infty} 5^m \Delta_m f(x).$$

The factor 5 is the product of the scaling factor 3 for the Haursdorff measure and the renormalized factor $\frac{5}{3}$ *for the energy. The graph energy on* V_m *is defined by* $\mathcal{E}_m(u, u) = \left(\frac{3}{5}\right)^{-m} \sum_{x \underset{m}{\sim} y}(u(x) - u(y))^2$ *which does not change under the harmonic extension, and the graph energy on SG is defined by* $\mathcal{E}(u, u) = \lim_{m\to\infty} \mathcal{E}_m(u, u)$.

Suppose u is a function defined on $V_0 = \{a, b, c\}$ *and we want to extend u to* V_1 *to minimize the energy.*



Let ũ be the extension of u to V_1. *Since ũ minimizes* $\mathcal{E}_1(\tilde{u})$, *we can take the derivatives with respect to* x, y, z *and set them equal to zero to obtain*

$$4x = b+c+y+z$$

$$4y = a+c+x+z$$

$$4z = a+b+x+y.$$

These equations express the mean value property which states that the function value at *each of the junction point is the average of the function values of the four neighboring points in the graph. We can use the matrix representation of these equations*

$$\begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b+c \\ a+c \\ a+b \end{pmatrix}$$

to get the following solutions:

$$x = \tfrac{1}{5}a + \tfrac{2}{5}b + \tfrac{2}{5}c$$

$$y = \tfrac{2}{5}a + \tfrac{1}{5}b + \tfrac{2}{5}c$$

$$z = \tfrac{2}{5}a + \tfrac{2}{5}b + \tfrac{1}{5}c.$$

The harmonic extension $\tilde{u}$ satisfies the $\frac{1}{5} - \frac{2}{5}$ rule. In general, this rule holds for the harmonic extension on each m-cell.

Definition 12. *A harmonic function on SG is a continuous function whose restriction to any Γ_m or V_m is the harmonic extension.*

2.3 The Decimation Method

We have seen in the above section that often fractals are described as a limit of a sequence of graphs. The Laplacian operators on fractals are defined similarly via a suitable approximation. The decimation method is a process through which we find the spectrum of the Laplacian on the fractals (self-similar sets) via iterations of a rational function.

To study the spectrum of the Laplacian, we consider the equation $-\Delta u = \lambda u$, where u is a continuous function. The spectrum was first studied by the physicists R. Rammal and G. Toulouse [32, 33]. Later, M. Fukushima and T. Shima [11, 40] gave a mathematical description of the eigenvalues and the eigenfunctions. In the case of the Sierpinski gasket, Rammal and Toulouse discovered interesting relations between the spectrum of the discrete Laplace operator and the dynamics of the iteration of a polynomial. More precisely, if λ is an eigenvalue of $-\Delta_{m+1}$ on Γ_{m+1} then $\lambda(5 - 4\lambda)$ is an eigenvalue of $-\Delta_m$ on Γ_m. Thus, the relationship between the eigenvalues of the Laplacians on one graph to another can be described by a quadratic equation $\lambda_m = \lambda_{m+1}(5-4\lambda_{m+1})$ (e.g., R(z)=z(5-4z)). The restriction to V_m of any eigenfunction belonging to λ_{m+1} is an eigenfunction belonging to λ_m. The relationship between the eigenvalues λ_m and λ_{m+1} of $-\Delta_m$ and $-\Delta_{m+1}$, respectively, can be found by comparing the corresponding eigenvalue problem for a point common to both V_m and V_{m+1}.

Theorem 13 (Fukushima–Shima, [11], [40])**.**

(i) If u is an eigenfunction of $-\Delta_{m+1}$ with eigenvalue λ, that is, $-\Delta_{m+1}u = \lambda u$, and if $\lambda \notin B$, then $-\Delta_m(u|_{V_m}) = R(\lambda)u|_{V_m}$, where $B = \{\frac{1}{2}, \frac{5}{4}, \frac{3}{2}\}$ is the set of 'forbidden' eigenvalues.

(ii) If $-\Delta_m u = R(\lambda)u$ and $\lambda \notin B$, then there exists a unique extension $\tilde{u}$ of u such that $-\Delta_{m+1}\tilde{u} = \lambda\tilde{u}$.

At any given level m, there are two kind of eigenvalues of $-\Delta_m$ called the initial and continued. The continued eigenvalues arise from the spectrum of $-\Delta_{m-1}$ via the decimation method and the remaining eigenvalues are called the initial eigenvalues. The forbidden eigenvalues $\{\frac{1}{2}, \frac{5}{4}, \frac{3}{2}\}$ in Proposition 13 have no predecessor, i.e., they are the initial eigenvalues. Furthermore, the exclusion of the eigenvalue $\frac{1}{2}$ can be explained by showing that $\frac{1}{2}$ is an eigenvalue of $-\Delta_m$ only for $m = 1$ (see Figure 2.4).

Given the eigenvalues of $-\Delta_0$ to be $\{0, \frac{3}{2}\}$, we consider the inverse images of $R(z)$: $R_-(z) = \frac{5-\sqrt{25-16z}}{8}$ and $R_+(z) = \frac{5+\sqrt{25-16z}}{8}$ to obtain the eigenvalues of $-\Delta_1$.

The continuation of this process generates the entire set of eigenvalues for each level. The diagram on the next page illustrates the eigenvalues associated with each graph Laplacian in terms of inverse iterates of the polynomial R.

The spectrum of the Sierpinski gasket is the renormalized limit of the spectra of $-\Delta_m$. Each eigenvalue satisfying the equation $-\Delta_\mu u = \lambda u$ can be written as

$$\lambda = \lim_{m\to\infty} 5^m \lambda_m$$

for a sequence $\{\lambda_m\}_{m=m_0}^{\infty}$ such that $\lambda_m = \lambda_{m+1}(5 - 4\lambda_{m+1})$ for $m \geq m_0$. Note that λ_m does not equal any of the forbidden eigenvalues $\{\frac{1}{2}, \frac{5}{4}, \frac{3}{4}\}$ while λ_{m_0} belongs to the set $\{\frac{1}{2}, \frac{5}{4}, \frac{3}{4}\}$. Furthermore, the values λ_m are determined by the solutions of $\lambda_m = \lambda_{m+1}(5 - 4\lambda_{m+1})$: $\lambda_{m+1} = \frac{5+\epsilon_m\sqrt{25-16\lambda_m}}{8}$, where $\epsilon_m = \pm 1$ provided that the limit exists. The limit only exists if $\epsilon_m = -1$ for all but a finite number of $m's$.

We will see the generalization of the decimation method when presenting the case of fractal Sturm–Liouville operators in Chapter 4. The decimation method is not valid for fractals in general, however, but a somewhat larger class of self-similar set with a symmetry condition has been explored by Teplyaev.

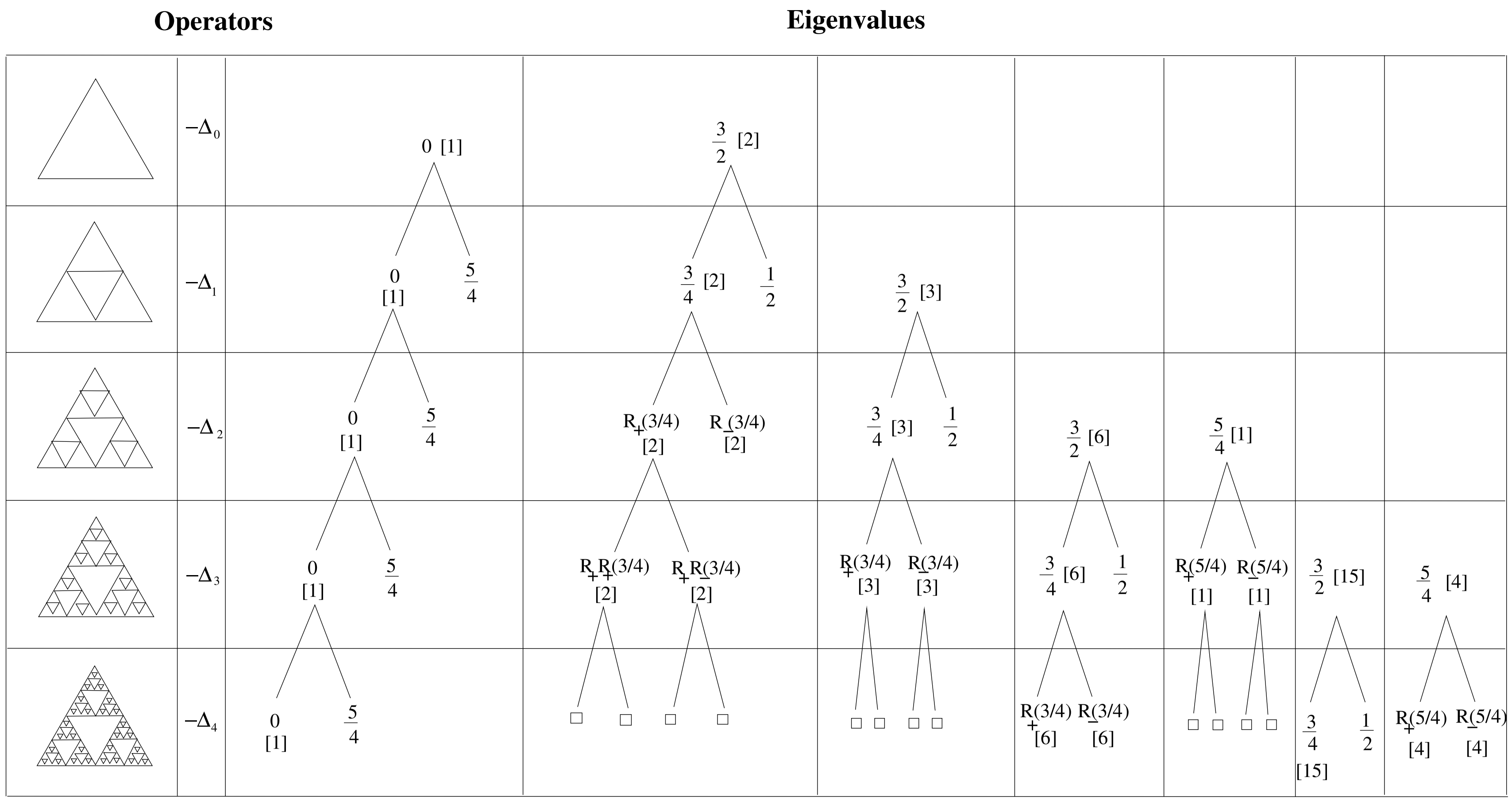


Figure 2.4: The eigenvalue diagram for SG

2.4 Dirichlet Spaces

In this section, we define the Dirichlet space in a more general setting and describe its connection with operator theory. The following diagram explains the direction we are interested in for this topic. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{E}$ be a Dirichlet form.

$$\{(\mathcal{E}, Dom(\mathcal{E}) : (\mathcal{E}, Dom(\mathcal{E}) \text{ is a closed form on } \mathcal{H}\}$$

$$\Updownarrow$$

$$\{H : H \text{ is a non-negative self-adjoint operator on } \mathcal{H}\}$$

$$\Updownarrow$$

$$\{\{T_t\}_{t\geq 0} : \{T_t\}_{t\geq 0} \text{ is a strongly continuous semigroup on } \mathcal{H}\}$$

Note that $-H$ is a generator of $\{T_t\}_{t\geq 0}$. We will review some of the definitions before investigating the connections shown in the diagram.

2.4.1 Self-Adjoint Operators and Quadratic Forms

Let $\mathcal{H}$ be a real separable Hilbert space with inner product $(\cdot, \cdot)$. The operator $H : Dom(H) \to \mathcal{H}$ is a linear operator on $\mathcal{H}$ if the domain of H, $Dom(H)$ is a dense subspace of $\mathcal{H}$.

A linear operator on H is *symmetric* if and only if $(Hf, g) = (f, Hg)$ for any $f, g \in Dom(H)$. H is a *self-adjoint* operator if and if H is symmetric and $Dom(H) = \{g \in \mathcal{H} : \exists h \in \mathcal{H} \text{ such that } (Hf, g) = (f, h)\ \forall f \in Dom(H)\}$. A symmetric operator H is non-negative if $(Hf, f) \geq 0$ for all f in $Dom(H)$.

Definition 14. *$Q(\cdot, \cdot) : Dom(Q) \times Dom(Q) \to \mathbb{R}$ is called a non-negative quadratic form on $\mathcal{H}$ if*

- *$Dom(Q)$ is a dense subspace of $\mathcal{H}$.*

- *Q is bilinear and symmetric: $Q(af+bg,h) = aQ(f,h)+bQ(g,h) = aQ(h,f)+bQ(h,g)$, $\forall f,g,h \in DomQ$.*

- *$Q(f,f) \geq 0$, $f \in Dom(Q)$.*

We call Q a closed form if it satisfies the following condition: $Dom(Q)$, equipped with the inner product $Q_\star(f,g) := Q(f,g)+(f,g)$ for any $f,g \in Dom(Q)$, is a Hilbert space.

Theorem 15. *(Friedrichs) If H is a non-negative symmetric operator $\mathcal{H}$ and $Q : Dom(H) \times Dom(H) \to \mathbb{R}$ is a symmetric bilinear form defined by $Q(f,g) = (Hf,g)$ for f and $g \in Dom(H)$, then Q is closable.*

In order to relate the operator with the Dirichlet forms, we need more definitions. We will consider $\mathcal{H}$ to be the space $\ell(V)$ where V is a finite set and H to be the Laplacian operator. Let $\ell(V) = \{f : f : V \to \mathbb{R}\}$, equipped with the standard inner product $(u,v) = \sum_{p \in V} u(p)v(p)$, $u,v \in \ell(v)$.

Definition 16. *A symmetric bilinear form $\mathcal{E}$ on $\ell(V)$ is called a* Dirichlet form *on V if it satisfies the following*

- *$\mathcal{E}(u,u) \geq 0$, $u \in \ell(V)$.*

- *$\mathcal{E}(u,u) = 0$ if and only if u is constant on V.*

- *Markov property: for any $u \in \ell(V)$, $\mathcal{E}(u,u) \geq \mathcal{E}(\bar{u},\bar{u})$ where $\bar{u}(p) = 1$ if $u(p) \geq 1$, $\bar{u}(p) = u(p)$ if $0 < u(p) < 0$, and $\bar{u}(p) = 0$ if $u(p) \leq 0$.*

Definition 17. *A symmetric linear operator $H : \ell(V) \to \ell(V)$ is a* Laplacian *on V if*

- *H is non-positive definite.*

- $Hu = 0$ *if and only if* $u = c \in V$.

- *if* $H_{pq} \geq 0$ $\forall p \neq q \in V$, *then* $(Hf)(p) = \sum_{q \in V} H_{pq} f(q)$ *where* $H_{pq} = (H\chi_q)(p)$.

There is a natural correspondence between the space of symmetric forms and the space of Laplacians. Given $H : \ell(V) \to \ell(V)$, define a symmetric quadratic form $\mathcal{E}_H(\cdot, \cdot)$ on $\ell(V)$ by $\mathcal{E}_H(u, v) = -(u, Hv)$ for u and $v \in \ell(V)$. Consider X is a locally compact metric space and ν is a σ-finite Borel measure on X that satisfies $\nu(A) < \infty$ for any compact set A with $\nu(A) > 0$. A closed form $\mathcal{E}$ on $L^2(X, \nu)$ is a Dirichlet form on $L^2(X, \nu)$ if and only if it has the Markov property.

Definition 18. *A family of bounded symmetric operator* $\{T_t\}_{t>0}$, $T_t : \mathcal{H} \to \mathcal{H}$ *is a semigroup on* $\mathcal{H}$ *if*

- $T_{t+s} = T_t T_s$, $t, s > 0$.

- $(T_t u, T_s u) \geq (u, u)$, $u \in \mathcal{H}$, $t > 0$.

If $\{T_t\}_{t>0}$ satisfies $lim_{t \to 0}(T_t u - u, T_t u - u) = 0$ then $\{T_t\}_{t>0}$ is called a strongly continuous semigroup.

The semigroup is the diffusion process X_t which (in the discrete case) means that if a particle is located at point p at time t, it will move to one of its neighbors with a certain probability at time $t + 1$. We consider the generator of a strongly continuous semigroup.

Definition 19. *The generator* $A : Dom(A) \to \mathcal{H}$ *of* $\{T_t\}_{t>0}$ *is a non-negative and self-adjoint operator defined by*

$$Au = \lim_{t \to 0}\left(\frac{T_t u - u}{t}\right)$$

for any $u \in Dom(A) = \{u \in \mathcal{H} : \exists f \in \mathcal{H}$ *such that* $\lim_{t \to 0} \|f - \frac{T_t u - u}{t}\| = 0\}$.

The corresponding Laplacian is the infinitesimal generator of the semigroup which is associated with the diffusion process.

2.5 Introduction To Hyperfunctions

The notion of hyperfunctions was introduced by a Japanese mathematican Mikio Sato. Roughly speaking, the hyperfunctions are the distributional generalization of analytic functions.

2.5.1 Motivation

We are familiar with functions known as the Dirac delta function and the Heaviside function. The Dirac delta function $\delta : \mathbb{R} \to \mathbb{R}$ is defined by

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

with the property

$$\int_{-\infty}^{\infty} \delta(x)dx = 1.$$

According to the classical theory of integration, δ would have an integral which is zero. On the other hand, the Heaviside function H on the real line is defined by

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

whose derivative is the Dirac delta function: $H'(x) = \delta(x)$. To justify the derivative statement of the Heaviside function, Laurent Schwartz introduced the theory of distributions which generalizes the classical notions of functions and the derivative. Basically, he defined a distribution as a continuous linear functional (an element of the dual space) on the space of test functions which are smooth functions with bounded support. The inner product is defined by

$$< Y, \phi >:= \int_{-\infty}^{\infty} Y(x)\phi(x)dx$$

where ϕ is a test function. Hence, we have $< H, \phi >:= \int_0^\infty \phi(x)dx$ and $< \delta, \phi >= \int_{-\infty}^{\infty} \delta(x)\phi(x)dx = \phi(0)$.

Moreover, now it can be shown rigorously that the derivative of the Heaviside function is the delta function through the method of integration by parts:

$$\begin{aligned} < H', \phi >: &= \int_{-\infty}^{\infty} H'(x)\phi(x)dx \\ &= -\int_{-\infty}^{\infty} H(x)\phi'(x)dx \\ &= -\int_{0}^{\infty} \phi'(x)dx = \phi(0) =< \delta, \phi > . \end{aligned}$$

Sato [38, 39] used the basic complex function theory to describe the notion of generalized functions. One of the advantages of the theory of hyperfunctions is that any hyperfunction can be differentiated arbitrary many times as well as integrated on any finite interval.

2.5.2 Definitions and Examples

Let Ω be a subset of $\mathbb{R}$. A complex neighborhood of Ω is an open subset $U \subset \mathbb{C}$ such that Ω is a closed subset of U. We denote by $\mathcal{C}(U)$ and $\mathcal{C}(U \setminus \Omega)$ the vector spaces of holomorphic functions on U and $U \setminus \Omega$, respectively. The functions of $\mathcal{C}(U \setminus \Omega)$ can be written as

$$F(z) = \begin{cases} F_+(z), & z \in U_+ \\ \\ F_-(z), & z \in U_- \end{cases}$$

where $U_+ := U \cap \mathbb{C}_+$ and $U_- := U \cap \mathbb{C}_-$ are called the upper and the lower half-neighborhoods of Ω, respectively. The quotient space $\mathcal{C}(U \setminus \Omega)/\mathcal{C}(U)$ is equipped with the equivalence relation according to which any holomorphic function in $\mathcal{C}(U \setminus \Omega)$ that extends holomorphically to all of U is identified with the zero function. Now, we define an equivalence relation on the set $\mathcal{C}(U \setminus \Omega)/\mathcal{C}(U)$. Two functions $F(z)$ and $G(z)$ from $\mathcal{C}(U \setminus \Omega)/\mathcal{C}(U)$ are equivalent if $F(z) = G(z) + h(z)$ with $h(z) \in \mathcal{C}(U)$. By definition, each equivalence class represents a *hyperfunction*.

Alternatively, a hyperfunction on the real line, $f(x) = [F(z)] = [F_+, F_-]$, consists of two functions, $F_+(z)$ and $F_-(z)$, which are analytic in the upper and the lower half-planes, respectively, and such that the following limit exists:

$$\lim_{\epsilon \to 0^+} \Big(F_+(x + i\epsilon) - F_-(x - i\epsilon) \Big). \tag{2.3}$$

Every hyperfunction $[F_+, F_-]$ forms an equivalence class of the form $[h + F_+, h + F_-]$, where h is a holomorphic function on U.

Any holomorphic function g on the real line represents the zero hyperfunction. This is easily seen by Equation (2.3). The addition of hyperfunctions and the muliti-plication by complex numbers are well defined: for given any two hyperfunctions $[F(z)]$ and $[G(z)]$,

- $\lambda[F(z)] = [\lambda F(z)]$ for $\lambda \in \mathbb{C}$
- $[F(z)] + [G(z)] = [F(z) + G(z)]$.

Moreover, the hyperfunctions are closed under multiplication by holomorphic functions. Suppose we have a hyperfunction $[F(z)]$ and a holomorphic function h on the real line. Then the product $h[F(z)] = [hF(z)]$ is a well defined hyperfunction. In particular, $h[F(z)] = h[F_+(z), F_-(z)] = [hF_+(z), hF_-(z)]$.

Now we will present some examples of hyperfunctions.

Example 20. *A hyperfunction $f(x)$ is a function on the real line described as a difference of two holomorphic functions defined on the upper and lower half-planes. Consider the following functions:*

$$I_1(z) = \begin{cases} 1, & Im(z) > 0 \\ 0, & Im(z) < 0, \end{cases}$$

$$I_2(z) = \begin{cases} 0, & Im(z) > 0 \\ -1, & Im(z) < 0, \end{cases}$$

$$I_3(z) = \begin{cases} \frac{1}{2}, & Im(z) > 0 \\ -\frac{1}{2}, & Im(z) < 0. \end{cases}$$

These three functions define the same hyperfunction $f(x) = 1$, which is the ordinary constant function. (*See* [13].)

Example 21. *(The Dirac delta hyperfunction)*

The Dirac delta function on the real line $\mathbb{R}$ is defined by

$$\delta_{\mathbb{R}}(z) := [-\frac{1}{2\pi iz}, -\frac{1}{2\pi iz}]$$

We will show that this is just a different representation of the function

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0. \end{cases}$$

By definition, for $x \neq 0$,

$$\begin{aligned} \delta(x) &= \lim_{\epsilon \to 0^+} \Big(F_+(x + i\epsilon) - F_-(x - i\epsilon) \Big) \\ &= \lim_{\epsilon \to 0^+} \left(\frac{-1}{2\pi i(x + i\epsilon)} - \frac{-1}{2\pi i(x - i\epsilon)} \right) \\ &= \lim_{\epsilon \to 0^+} \frac{\epsilon}{\pi(x^2 + \epsilon^2)} = 0. \end{aligned}$$

For $x = 0$, however, the above limit does not exist, and this is the point at which the delta 'function' has an isolated singularity.

As a motivation for Sato's theory of hyperfunctions, the Cauchy integral formula can be used to produce the delta function. By the Cauchy integral formula, for any holomorphic function ϕ and for any closed region γ about the origin, we have

$$\begin{aligned} \phi(0) &= -\frac{1}{2\pi i} \int_{\gamma} \frac{\phi(z)}{z} dz \\ &= -\frac{1}{2\pi i} \left(\int_{\gamma^+} \frac{\phi(z)}{z} dz - \int_{\gamma^-} \frac{\phi(z)}{z} dz \right) \\ &= -\frac{1}{2\pi i} \left(\int_{a+i0}^{b+i0} \frac{\phi(z)}{z} dz - \int_{a-i0}^{b-i0} \frac{\phi(z)}{z} dz \right) \\ &= -\frac{1}{2\pi i} \int_{a}^{b} \left(\frac{1}{x + i0} - \frac{1}{x - i0} \right) \phi(x) dx, \end{aligned}$$

where γ^+ and γ^- are the connected paths from the endpoints of the interval $[a, b]$. Hence, it becomes clear that $\delta(x) = -\frac{1}{2\pi i}\left(\frac{1}{x+i0} - \frac{1}{x-i0}\right)$, which is the difference $F_+(x +$

$i\epsilon) - F_-(x - i\epsilon)$ of the boundary values of holomorphic functions in the definition of a hyperfunction. As a result, a hyperfunction on the real line is a difference of holomorphic functions on the boundary (see [16]).

Example 22. *(The Heaviside function)*

The Heaviside function can be interpreted as

$$H(x) := [-\frac{1}{2\pi i}\log(-z)].$$

With the branch cut of the complex plane along the negative real x-axis, $\log(-z)$ *takes the value* $\log(|z|) - \pi i$ *on the upper side of the positive x-axis and* $\log(|z|) + \pi i$ *on the lower side of the positive x-axis. Note that it is holomorphic on the negative real x-axis. Then it is easy to see that*

$$H(x) = \lim_{\epsilon \to 0^+}\Big(F_+(x+i\epsilon) - F_-(x-i\epsilon)\Big) = \begin{cases} 0, & x < 0 \\ 1, & x > 0. \end{cases}$$

Since the limit does not exist at $x = 0$, the hyperfunction has a singular point at $x = 0$.

2.6 Complex Dynamics of Functions of Several Complex Variables

In this section, we briefly present the theory of complex dynamics in higher dimensions. In particular, we give definitions of the Fatou set and the Julia set.

Consider the quotient space on $\mathbb{C}^{k+1}$ with the equivalence relation:

$$(z_0, z_1, ...z_k) \sim (z'_0, z'_1, ...z'_k)$$

if there exists $\lambda \in \mathbb{C}$ such that $\lambda(z_0, z_1, ...z_k) = (z'_0, z'_1, ...z'_k)$. The quotient space $\mathbb{C}^{k+1} \setminus \sim$ represents the *complex projective space of dimension* k denoted by $\mathbb{P}^k$. The map π : $\mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ is the canonical projection map given by $\pi((z_0, z_1, ...z_k)) = [z_0, z_1, ...z_k]$ for all $(z_0, z_1, ...z_k) \in \mathbb{C}^{k+1} \setminus \{0\}$. π defines a holomorphic C^*-bundle over $\mathbb{P}^k$. For $p \in \mathbb{P}^k$, $\pi^{-1}(p) = L \setminus \{0\}$ is a complex line in $\mathbb{C}^{k+1}$ through the origin.

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic map. For such a map, there always exists a homogeneous map $R : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ with $R(0) = 0$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{C}^{k+1} \setminus \{0\} & \xrightarrow{R} & \mathbb{C}^{k+1} \setminus \{0\} \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{P}^k & \xrightarrow[f]{} & \mathbb{P}^k \end{array}$$

e.g., $R \circ \pi = \pi \circ f$. A holomorphic map $R : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ is a *homogenous map of degree* d if R is defined by $R = (R_0, R_1, ...R_k)$ of homogenous polynomials of degree d.

The map f is not defined on the set of points of the form $\pi(p)$ where $R(p) = 0$. A point $p \in \mathbb{P}^k$ is called an *indeterminacy point* for f if $R(\pi^{-1}(p)) = 0$. Denote by I the set of indeterminacy points of f. Thus, f is a holomorphic map on $\mathbb{P}^k \setminus I$.

From now on, we will work with meromorphic maps on $\mathbb{P}^2$ and define the corresponding Fatou sets and Julia sets.

Definition 23 ([8]). *Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a meromorphic map. We say that a point $p \in \mathbb{P}^2$ belongs to the* Fatou set *if for every $\epsilon > 0$ there exists some neighborhood $U(p)$ such that $diam(f^n(U \setminus I_n)) < \epsilon$ for all n. (Here, $diam(A)$ denotes the diameter of A.)*

The *Julia set* is the complement of the Fatou set. Note that $\bigcup_{n=1}^{\infty} I_n$ is a part of the Julia set, where each I_n denotes the indeterminacy set of R^n.

In general, the Fatou set of a holomorphic map $f : \mathbb{P}^k \to \mathbb{P}^k$ is defined by

$$\Big\{ p \in \mathbb{P}^k : \{f^j|V\} \text{ is a normal family on a neighborhood V of p} \Big\}.$$

A complex curve V is called a *f-constant curve* if $f(V)$ is a single point. Every f-constant curve contains a point of indeterminacy. A f-constant curve V is a degree lowering curve if $f^n(V)$ is an indeterminacy point.

As an example, consider the map $f([z_0, z_1.z_2]) = [z_0 z_1 : z_0^2 + z_1 z_2 : z_2^2]$. It is easy to see that f has a point of indeterminacy at $[0 : 1 : 0]$. There is no f-constant curve.

Later in Chapter 4, we will investigate these various notions of the complex dynamics of a map on $\mathbb{P}^2$ when studying the spectrum of a fractal differential Sturm–Liouville operator on the unit interval.

Chapter 3

The Spectral Zeta Function of the Laplacian on Fractals

In this chapter, we prove our first main result regarding the factorization of the spectral zeta function of the Laplacian on the infinite (or unbounded) Sierpinski gasket. First, we discuss some of the known theorems on the spectral zeta functions of Laplacians on self-similar fractals, including the finite (or bounded) Sierpinski gasket, which we will extend to more generalized second order differential operators on fractals such as the Sturm–Liouville operators in Chapter 4.

We begin with the concept of zeta function associated with an elliptic differential operator on a fractal. Roughly speaking, the spectrum of a second order differential operator with corresponding boundary conditions may not be computable explicitly. In general, when the spectrum of a differential operator A need not be expressed explicitly and under some trace-class conditions, the corresponding zeta function is given by

$$\zeta_A(s) = \mathrm{Tr}\ e^{-s \ln A}$$

in some suitable domain of the complex plane s. When the spectrum is discrete and the eigenvalues λ_j are repeated according to multiplicity, the zeta function is therefore defined (for Re(s) sufficiently large) by the following infinite sum:

$$\zeta_A(s) = \sum_{j=1}^{\infty} \lambda_j^{-s}.$$

The use of the Mellin transform is required to note that the zeta function $\zeta_A(s)$ is a meromorphic function on the complex plane, $s \in \mathbb{C}$.

As we all know, the Riemann zeta function is a classical and well known zeta function in the study of number theory. It is a complex function $\zeta : \mathbb{C} \to \mathbb{C}$ defined as

$$\zeta(s) = \sum_{j=1}^{\infty} j^{-s}$$

for $Re(s) > 1$. $\zeta(s)$ has a meromorphic extension to the complex plane $\mathbb{C}$ with a simple pole at the point $s = 1$. As a special case, we obtain the Riemann zeta function when the eigenvalues of an elliptic differential operator with boundary conditions are of the form aj with $a = 1$ for $j = 1, 2, ...$

The rest of this chapter will be devoted to analyzing the spectral zeta function in regards with a factorization formula induced by the decimation method.

3.1 Examples of Factorization of the Spectral Zeta Function

We begin by a basic example of the Laplacian on the unit interval to study the corresponding spectral zeta function. First, we will give a precise definition of a spectral zeta function as follows:

Definition 24. *The* spectral zeta function *of a positive self-adjoint operator L with compact resolvent (and hence, with discrete spectrum) is given (for $Re(s)$ sufficiently large) by*

$$\zeta_L(s) = \sum_{j=1}^{\infty} (\kappa_j)^{-s/2}, \tag{3.1}$$

where the positive real numbers κ_j are the eigenvalues of the operator written in nonincreasing order and counted according to their multiplicities.

Consider the differential operator $\Delta = -\frac{d^2}{dx^2}$ on the unit interval $[0, 1]$ with Dirichlet boundary conditions $u(0) = 0$ and $u(1) = 0$. The eigenvalues of Δ are of the form $\lambda = \pi^2 j^2$ for $j = 1, 2, ...$ The spectral zeta function of Δ can be expressed as

$$\zeta_\Delta(s) = \sum_{j=1}^{\infty} (\pi^2 j^2)^{-s/2} = \pi^{-s} \sum_{j=1}^{\infty} j^{-s} = \pi^{-s} \zeta(s) \tag{3.2}$$

where $\zeta(s)$ is the Riemann zeta function.

3.1.1 The Spectral Zeta Function Associated with a Fractal String

Now we will present an another example from the theory of fractal strings. A fractal string is a bounded open subset Ω of the real line of finite length. We denote by $\mathcal{L} = \{\ell_j\}_{j=1}^{\infty}$ the associated nonincreasing sequence of lengths $\{\ell_j\}$ of I_j such that $\Omega = \bigcup_{j=1}^{\infty} I_j$.

Definition 25. *The geometric zeta function of a fractal string Ω with lengths $\mathcal{L}$ is*

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s$$

where $s \in \mathbb{C}$.

We define the abscissa of convergence of $\zeta_{\mathcal{L}}$ by

$$\mathcal{D}_{\mathcal{L}} = \inf\{\sigma \in \mathbb{R} : \sum_{j=1}^{\infty} \ell_j^{\sigma} < \infty\}.$$

One should point out that the Minkowski dimension of a fractal string is equal to $\mathcal{D}_{\mathcal{L}}$. Hence, the geometric zeta function $\zeta_{\mathcal{L}}$ converges for $Re(s) > \mathcal{D}_{\mathcal{L}}$. The complex dimensions of a fractal string Ω are given by

$$\mathcal{C} = \{\omega \in W : \zeta_{\mathcal{L}} \text{ has a pole at } \omega\}$$

where $W \subset \mathbb{C}$ is a certain region in the complex plane.

Now we focus our attention on the spectral zeta function of the fractal strings. We begin by considering the Dirichlet Laplacian $\Delta = -\frac{d^2}{d^2x}$ on $\Omega = (a, b)$. We analyze the following partial differential equation

$$\Delta u - \lambda u = 0 \text{ with } u(a) = 0, u(b) = 0.$$

The associated eigenvalues are given by $\frac{n^2\pi^2}{\ell^2}$ for $n \in \mathbb{N}$ where $\ell = b - a$. In particular, in the case of a fractal string $\mathcal{L} = \{\ell_j\}_{j=1}^{\infty}$, the eigenvalues of the Dirichlet Laplacian Δ on each I_J with $|I_J| = \ell_j$ are $\frac{n^2\pi^2}{\ell_j}$. Hence, the spectrum of $\mathcal{L}$ is $\sigma(\Delta) = \bigcup_{k=1}^{\infty} \sigma(\Delta; I_j) = \{\frac{\pi^2 n^2}{\ell_j^2} : n \geq 1, j \geq 1\}$.

The spectral zeta function of Δ on $\mathcal{L}$ is

$$\zeta_{\Delta}(s) \quad = \textstyle\sum_{k,n\geq 1} (\frac{\pi^2 n^2}{\ell_k^2})^{-s/2} = \pi^{-s} \sum_{k=1}^{\infty} \ell_k^s \sum_{n=1}^{\infty} n^{-s}.$$

The spectral zeta function of the Dirichlet Laplacian Δ on a fractal string with length

$\mathcal{L}$ has a factorization in terms of the Riemann zeta function and the geometric zeta function as shown in the following theorem.

Theorem 26 (Lapidus). *The spectral zeta function of* Δ *on* $\mathcal{L}$ *is* $\zeta_A(s) = \pi^{-s}\zeta_{\mathcal{L}}(s)\zeta(s)$, *where* $\zeta(s)$ *is the Riemann zeta function.*

3.1.2 Cantor Fractal String

We consider yet another interesting example of the Cantor self-similar fractal string, which is the complement of the middle third Cantor set in $[0,1]$. Basically,

$$CS = (\frac{1}{2}, \frac{2}{3}) \cup (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}) \cup (\frac{1}{27}, \frac{9}{27}) \cup (\frac{7}{27}, \frac{8}{27}) \cup ....$$

The corresponding lengths are $\ell_1 = \frac{1}{3}$, $\ell_2 = \ell_3 = \frac{1}{9}$, $\ell_4 = \ell_5 = \ell_6 = \ell_7 = \frac{1}{27}, ...$ Then the geometric zeta function is given by

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty}(\ell_j)^{-s} = \sum_{j=1}^{\infty}(2^{j-1}(3^{-j})^s = \frac{1}{3^s - 2}$$

(see [28]) and so the factorization formula for the associated spectral zeta function is

$$\zeta_L(s) = \pi^{-s}\zeta(s)\frac{1}{3^s - 2}.$$

We will see that this product formula is analogous to the one obtained for the spectral zeta function on SG.

3.2 The Spectral Zeta Function of the Laplacian on the Bounded Sierpinski Gasket

A. Teplyaev studied the spectral zeta function of the Laplacian on SG and explored interesting connections between the spectral zeta function and the iteration of the polynomial induced by the decimation method. Before we further discuss the spectral zeta function of the Laplacian of the Sierpinski gasket, we present the definition of the zeta function of a polynomial.

Definition 27. *Let R be a polynomial of degree N satisfying $R(0) = 0$, $c := R'(0) > 1$, and with Julia set $\mathcal{J} \subset [0, \infty)$. Then the zeta function of R is defined by*

$$\zeta_{R,z_0}(s) = \lim_{n\to\infty} \sum_{z \in R^{-n}\{z_0\}} (c^n z)^{-\frac{s}{2}},$$

for $Re(s) > d_R := \frac{2\log N}{\log c}$. Here, R^{-n} denotes the nth inverse iterate of R.

In addition, in the case of the Laplacian on the compact Sierpinski gasket, he discovered the product structure of the spectral zeta function that involves the zeta function of a polynomial. We recall the Laplacian of the Sierpinski gasket and its spectrum from Chapter 2. The set of eigenvalues of the Laplacian can easily be read from the eigenvalue diagram, Figure 2.4, which is

$$\{\frac{3}{2}\} \bigcup \Big(\bigcup_{j=0}^{\infty} R^{-j}\{\frac{3}{4}\}) \Big) \bigcup \Big(\bigcup_{j=0}^{\infty} R^{-j}\{\frac{5}{4}\} \Big).$$

Then the factorization formula for the associated spectral zeta function is described in the next theorem. For notational simplicity, we write Δ instead of $-\Delta$.

Theorem 28 (Teplyaev, [45]). *The spectral zeta function of the Laplacian on SG is*

$$\zeta_{\Delta_\mu}(s) = \zeta_{R,\frac{3}{4}}(s)\frac{5^{-\frac{s}{2}}}{2}\left(\frac{1}{1-3\cdot 5^{-\frac{s}{2}}} + \frac{3}{1-5^{-\frac{s}{2}}}\right) + \zeta_{R,\frac{5}{4}}(s)\frac{5^{-s}}{2}\left(\frac{3}{1-3\cdot 5^{-\frac{s}{2}}} - \frac{1}{1-5^{-\frac{s}{2}}}\right), \tag{3.3}$$

where $R(z) = z(5-4z)$. *Furthermore, there exists* $\epsilon > 0$ *such that* $\zeta_{\Delta_\mu}(s)$ *has a meromorphic continuation for* $Re(s) > -\epsilon$, *with poles contained in* $\left\{\frac{2in\pi}{\log 5}, \frac{\log 9+2in\pi}{\log 5} : n \in \mathbb{Z}\right\}$.

Derfel, Grabner, and Vogel have also worked independently on the same zeta function associated with a polynomial and proved that it has a meromorphic continuation on the whole complex plane. They expressed the spectral zeta function in terms of this zeta function and a zeta function related to the generating set associated with the multiplicities of the eigenvalues of the operator which is defined as

$$B_w(x) = \sum_{m=0}^{\infty} \beta_m(w)x^m$$

where $\beta_m(w)$ is the recurrent sequence of multiplicities of the eigenvalue w at each level m. Then, in [6], the spectral zeta function is defined by

$$\sum_{w\in W} B_w(c^{-s})\zeta_{R,w}(s)$$

for a suitable set W depending on R.

Now we will use this equation to derive Equation (3.3) in Theorem 28. In the case of two-dimensional Sierpinski gasket, $W = \{\frac{3}{4}, \frac{5}{4}\}$ and $c = 5$. The multiplicities of the eigenvalues $\frac{3}{4}$ and $\frac{5}{4}$ at each level m can be determined by the eigenvalue diagram to be $\beta_m(\frac{3}{4}) = \frac{3^{m-1}+3}{2}$ for $m \geq 1$ and $\beta_m(\frac{5}{4}) = \frac{3^{m-1}-1}{2}$ for $m \geq 2$ (see [3]). Hence, we have the following calculations:

$$
\begin{aligned}
\zeta_{\Delta}(s) &= \zeta_{R,\frac{3}{4}}(s)\sum_{m=1}^{\infty}\beta_m(\frac{3}{4})(5^{-\frac{s}{2}})^m + \zeta_{R,\frac{5}{4}}(s)\sum_{m=2}^{\infty}\beta_m(\frac{5}{4})(5^{-\frac{s}{2}})^m \\
&= \zeta_{R,\frac{3}{4}}(s)\sum_{m=1}^{\infty}\frac{3^{m-1}+3}{2}(5^{-\frac{s}{2}})^m + \zeta_{R,\frac{5}{4}}(s)\sum_{m=2}^{\infty}\frac{3^{m-1}-1}{2}(5^{-\frac{s}{2}})^m \\
&= \zeta_{R,\frac{3}{4}}(s)\Big[\sum_{m=1}^{\infty}\frac{1}{2}3^{m-1}5^{-\frac{sm}{2}} + \sum_{m=1}^{\infty}\frac{3}{2}(5^{-\frac{s}{2}})^m\Big] \\
&\quad + \zeta_{R,\frac{5}{4}}(s)\left[\sum_{m=2}^{\infty}\frac{1}{2}3^{m-1}5^{-\frac{sm}{2}} - \sum_{m=2}^{\infty}\frac{1}{2}5^{-\frac{sm}{2}}\right] \\
&= \zeta_{R,\frac{3}{4}}(s)\Big[\sum_{m=0}^{\infty}\frac{1}{2}3^{m}5^{-\frac{s(m+1)}{2}} + \sum_{m=0}^{\infty}\frac{3}{2}5^{-\frac{s}{2}(m+1)}\Big] \\
&\quad + \zeta_{R,\frac{5}{4}}(s)\left[\sum_{m=0}^{\infty}\frac{1}{2}3^{m+1}5^{\frac{-(m+2)s}{2}} + \sum_{m=0}^{\infty}\frac{1}{2}5^{\frac{-(m+2)s}{2}}\right] \\
&= \zeta_{R,\frac{3}{4}}(s)\Big[\sum_{m=0}^{\infty}\frac{1}{2}(\frac{3}{5^{\frac{s}{2}}})^m 5^{-\frac{s}{2}} + \sum_{m=0}^{\infty}\frac{3}{2}(\frac{1}{5^{\frac{s}{2}}})^m 5^{-\frac{s}{2}}\Big] \\
&\quad + \zeta_{R,\frac{5}{4}}(s)\left[\sum_{m=0}^{\infty}\frac{3}{2}(\frac{3}{5^{\frac{s}{2}}})^m 5^{-s} + \sum_{m=0}^{\infty}\frac{1}{2}(5^{-\frac{s}{2}})^m 5^{-s}\right] \\
&= \zeta_{R,\frac{3}{4}}(s)\frac{5^{-\frac{s}{2}}}{2}\left[\frac{1}{1-3\cdot 5^{-\frac{s}{2}}} + \frac{3}{1-5^{-\frac{s}{2}}}\right] + \zeta_{R,\frac{5}{4}}(s)\frac{5^{-s}}{2}\left[\frac{3}{1-3\cdot 5^{-\frac{s}{2}}} - \frac{1}{1-5^{-\frac{s}{2}}}\right]
\end{aligned}
$$

for $|\frac{3}{5^{\frac{s}{2}}}| < 1$.

3.3 Results for the Infinite (Unbounded) Sierpinski Gasket

We have already discussed the Laplacian and its spectral properties of the finite Sierpinski gasket SG. In this section, we analyze the construction of an infinite Sierpinski gasket and its spectral zeta function of the Laplacian. Since the infinite Sierpinski gasket can be viewed as a blow up of the finite Sierpinski gasket, we extend the result from the previous section to the infinite Sierpinski gasket to study the factorization formula of the spectral zeta function of the Laplacian.

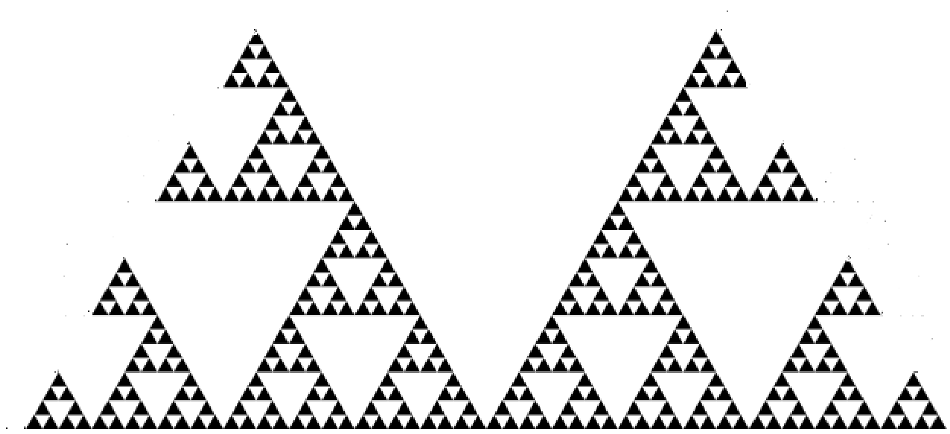


Figure 3.1: An infinite Sierpinski gasket

Let $k = \{k_n\}_{n\geq 1}$ be a fixed sequence with $k_n \in \{1,2,3\}$ for $n \geq 1$. We construct a sequence $SG^{(n)} = \Psi_{k,n}^{-1}(SG)$ where $\Psi_{k,n} = \Psi_{k_n,\ldots k_1} = \Psi_{k_n} \circ \ldots \circ \Psi_{k_1}$. The infinite Sierpinski gasket is defined by

$$SG^{(\infty)} = \bigcup_{n=0}^{\infty} SG^{(n)},$$

viewed as a blow-up of SG. The mth pre-gasket approximating $SG^{(n)}$ and $SG^{(\infty)}$ are $V_m^{(n)} = \Psi_{k,n}^{-1}(V_{n+m})$ and $V_m^{(\infty)} = \cup_{n=0}^{\infty}\Psi_{k,n}^{-1}(V_{n+m})$, respectively. Note that $V_m = V_m^{(0)}$ and $SG = SG^{(0)}$.

We define the Laplacian $\Delta^{(n)}$ on $SG^{(n)}$ as follows: $\Delta^{(n)}u = f \in L^2(SG^{(n)}, \mu)$ if and only if $\mathcal{E}_{SG^{(n)}}(u,v) = \int_{SG^{(n)}} \Delta^{(n)}uvd\mu$, where $\mathcal{E}_{SG^{(n)}}$ is a scaled copy of $\mathcal{E}_{n+m}$ on V_{n+m} for the finite Sierpinski pre-gasket. The pointwise Laplacian $\Delta^{(\infty)}$ can then be defined by the following pointwise limit $5^n\Delta^{(n)}u \to \Delta^{(\infty)}u$ as $n \to \infty$ (see [11], [18]). Here, it should be noted that the Laplacian on the finite Sierpinski gasket coincides with the restriction of $\Delta^{(\infty)}$ to the interior of the finite (or bounded) SG.

3.3.1 Main Lemma Regarding the Dirac Delta Hyperfunction

Now, we present a lemma which will be used to prove some of the main results later in this thesis.

We consider the bi-infinite series $\sum_{p=-\infty}^{\infty}(\gamma^p)^{-\frac{s}{2}}$ with $\gamma > 1$ and $s \in \mathbb{C}$. The most peculiar behavior of this series is that it seems to be equal to zero, even though it does not make sense to sum up a complex series with one part converging and the other part diverging. However, this series can be represented by a suitable version of the Dirac hyperfunction, as we shall soon see. For now, we carry out the naive computation as follows:

$$\begin{aligned}\sum_{p=-\infty}^{\infty}(\gamma^p)^{-\frac{s}{2}} &= \sum_{p=-\infty}^{-1}(\gamma^p)^{-\frac{s}{2}} + \sum_{p=0}^{\infty}(\gamma^p)^{-\frac{s}{2}}\\ &= \frac{\gamma^{\frac{s}{2}}}{1-\gamma^{\frac{s}{2}}} + \frac{1}{1-\gamma^{-\frac{s}{2}}}\\ &= \frac{1}{1-\gamma^{-\frac{s}{2}}} - \frac{1}{1-\gamma^{-\frac{s}{2}}} = 0.\end{aligned}$$

Note that this computation is meaningless, unless it is properly interpreted. Indeed, we have added two infinite series, one of which is convergent only for $Re(s) > 0$, whereas the other series is convergent only for $Re(s) < 0$. In fact, fortunately, the geometric part $\sum_{p=-\infty}^{\infty}(\gamma^p)^{-\frac{s}{2}}$ can be interpreted in terms of the *Dirac delta hyperfunction on the unit circle*, $\delta_{\mathbb{T}}(w) = [\delta_{\mathbb{T}}^{+}(w), \delta_{\mathbb{T}}^{-}(w)]$, by means of a suitable change of variable; namely, $w = \gamma^{-\frac{s}{2}}$. The Dirac delta hyperfunction on the unit circle $\mathbb{T}$ is defined as $\delta_{\mathbb{T}} = [\delta_{\mathbb{T}}^{+}, \delta_{\mathbb{T}}^{-}] = [\frac{1}{1-z}, \frac{1}{z-1}]$. It consists of two analytic functions, $\delta_{\mathbb{T}}^{+} : E \to \mathbb{C}$ and $\delta_{\mathbb{T}}^{-} : \mathbb{C}\backslash\bar{E} \to \mathbb{C}$, where $E = \{z \in \mathbb{C} : |z| < 1 + \frac{1}{N}\}$ for a large natural number N. In other words, a hyperfunction on $\mathbb{T}$ can be viewed as a suitable pair of holomorphic functions, one on the unit disk $|z| < 1$, and one on its exterior, $|z| > 1$. (See, for example, [13], §1.3 and [29], §3.3.2 for a discussion of various changes of variables in a hyperfunction.

Moreover, see [42] for a detailed discussion of $\delta_{\mathbb{T}}$ and, more generally, of hyperfunctions on the unit circle $\mathbb{T}$.)

Lemma 29. *Let* $\gamma > 1$. *Then*

$$\sum_{p=-\infty}^{\infty} (\gamma^p)^{-\frac{s}{2}} = \delta_{\mathbb{T}}(\gamma^{-\frac{s}{2}}), \tag{3.4}$$

where $\delta_{\mathbb{T}}$ *is the Dirac delta hyperfunction on the unit circle* $\mathbb{T}$.

Proof. We introduce the formal expression $\psi(w) = \sum_{p=-\infty}^{\infty} w^p$. Note that

$$\psi(w) = \begin{cases} \sum_{p=0}^{\infty} w^p, & \text{if } |w| < 1 \\ \sum_{p=-\infty}^{-1} w^p, & \text{if } |w| > 1 \end{cases}$$

or equivalently,

$$\psi(w) = \begin{cases} \frac{1}{1-w}, & \text{if } |w| < 1 \\ \frac{1}{w-1}, & \text{if } |w| > 1. \end{cases}$$

Now, essentially by definition, $\psi(w)$ is in fact equal to the Dirac delta hyperfunction $\delta_{\mathbb{T}}(w) = [\delta_{\mathbb{T}}^{+}(w), \delta_{\mathbb{T}}^{-}(w)]$ on the unit circle $\mathbb{T}$. We refer the interested reader to [42] for a precise mathematical discussion of the delta hyperfunction $\delta_{\mathbb{T}}$.

We conclude the proof of Equation (3.4) by making the change of variable $w = \gamma^{-\frac{s}{2}}$ and noting that $|w| < 1$ and $|w| > 1$ correspond to the upper and lower half-planes $Re(s) > 0$ and $Re(s) < 0$, respectively. Indeed, $\log \gamma > 0$ since $\gamma \geq 4$. $\square$

It follows from the construction that the spectrum of the Laplacian $\Delta^{(\infty)}$ on the infinite Sierpinski gasket can be generated by the spectrum of the Laplacian on the finite SG.

Theorem 30 (see, e.g., [43])**.** *Let $R(z) = z(5-4z)$, as in Theorem. 28. Then the spectrum of the self-adjoint operator $\Delta^{(\infty)}$ acting on $L^2(SG^{(\infty)}, \mu)$ is pure point and the set of compactly supported eigenfunctions is complete. Furthermore, the set of eigenvalues is given by*

$$\bigcup_{n=-\infty}^{\infty} 5^n \mathcal{R}\{\Sigma\},$$

where $\Sigma = \{\frac{3}{2}\} \cup (\cup_{j=0}^{\infty} R^{-j}\{\frac{3}{4}\}) \cup (\cup_{j=0}^{\infty} R^{-j}\{\frac{5}{4}\})$ is the set of eigenvalues of the Laplacian Δ_μ on the finite SG, $\mathcal{R}(z) := \lim_{m\to\infty} 5^m R_-^{-m}(z)$ and R_-^{-m} is the branch of the mth inverse iterate of R that passes through the origin.

In particular, the spectrum of $\Delta^{(\infty)}$ has the following form

$$\bigcup_{n=-\infty}^{\infty} \bigcup_{j=0}^{\infty} 5^n \mathcal{R}(R^{-j}(z_0)) \tag{3.5}$$

where $z_0 = \frac{3}{4}, \frac{5}{4}$. Every eigenvalue λ of $\Delta^{(\infty)}$ can be expressed as

$$\lambda = 5^n \lim_{m\to\infty} 5^m R_-^{-m}(z_m)$$

for some $n \in \mathbb{Z}$, with z_m in the spectrum $\sigma(\Delta_m)$ of the finite mth Sierpinski pre-gasket.

We now state our result regarding the unbounded Sierpinski gasket.

Theorem 31. *The spectral zeta function $\zeta_{\Delta^{(\infty)}}$ of the Laplacian $\Delta^{(\infty)}$ on the infinite Sierpinski gasket $SG^{(\infty)}$ is given by*

$$\zeta_{\Delta^{(\infty)}}(s) = \delta_{\mathbb{T}}(5^{-\frac{s}{2}})\zeta_{\Delta_\mu}(s), \tag{3.6}$$

where $\delta_{\mathbb{T}}$ is the Dirac hyperfunction and ζ_{Δ_μ} is the spectral zeta function of the Laplacian on the finite SG as given and factorized explicitly in Equation (3.3) *of Theorem 28.*

Proof. In light of Equation (3.5) and Theorem 28, this result follows from Lemma 29. Indeed, note that λ is an eigenvalue of Δ_μ iff $5^n\lambda$ is an eigenvalue of $\Delta^{(n)}$. Furthermore, if λ is an eigenvalue of $\Delta^{(\infty)}$, then $5^n\lambda$ is also an eigenvalue of $\Delta^{(\infty)}$ for $n \in \mathbb{Z}$ (see [11]). Combining these two facts, we obtain

$$\zeta_{\Delta^{(\infty)}}(s) = \Big(\sum_{n=-\infty}^{\infty} (5^{-\frac{s}{2}})^n \Big) \zeta_{\Delta_\mu}(s).$$

By Lemma 29 and using the substitution $\gamma = 5$, we now deduce the desired result. □

3.4 A Representation of the Riemann Zeta Function

Revisiting the example of fractal strings, A. Teplyaev [45] proved that the Riemann zeta function can be described in terms of the zeta function of a quadratic polynomial of one complex variable. In the case of the unit interval, the normalized Laplacian is defined as a limit of the graph Laplacians:

$$\Delta u(x) = 2 \lim_{n\to\infty} 4^n \Delta_n u(x)$$

where $\Delta_n u(\frac{k}{2^n}) = u(\frac{k}{2^n}) - \frac{1}{2}(u(\frac{k-1}{2^n}) + u(\frac{k+1}{2^n}))$ is defined on $V_n = \{0, ..., \frac{k}{2^n}, \frac{k+1}{2^n}, ..., 1\}$. One could find the set of eigenvalues of each Δ_n by analyzing the corresponding Laplacian matrix. It turns out that the spectrum of the Laplacian on the unit interval can be described by the polynomial $R(z) = 2z(2 - z)$ (see [3]).

Moreover, the spectral zeta function of the Dirichlet Laplacian on the unit interval can be expressed in terms of the zeta function $\zeta_{R,0}$ as

$$\zeta_\Delta(s) = 2^{\frac{s}{2}-1}\zeta_{R,0}(s) \tag{3.7}$$

since the set $R^{-n-1}(0)$ contains the spectrum of Δ_n. Combining Equations (3.2) and (3.7), the Riemann zeta function $\zeta(s)$ can be represented by

$$\zeta(s) = \frac{1}{2} C^s \zeta_{R,0}(s), \tag{3.8}$$

where $C = \sqrt{2}\pi$ and $\zeta_{R,0}(s)$ is the zeta function of the polynomial $R(z) = 2z(2-z)$. One of the theorems to be presented in Chapter 4 will generalize this results to several complex variables.

Chapter 4

Factorization of the Spectral Zeta Function of the Generalized Differential Operators

In this chapter, we present our main results obtained regarding the factorization of the spectral zeta function of Laplacian-like operators on fractals with blow-ups, which also appears in [21].

We start by presenting the generalized second order differential operators of the form $\frac{d}{d\mu}\frac{d}{d\nu}$, where μ and ν are finite Borel measures with compact support on the real line. Later on, our main focus will be on the Sturm–Liouville operator of the form $\frac{d}{d\mu}\frac{d}{dx}$, where μ is a self-similar measure and dx is the Lebesgue measure. The differentiation with respect to these measures requires to define suitable L_2-spaces. Assume that ν and μ are finite Borel measures on the interval $[a, b]$, with compact support K_1 and K_2 respectively. Define $L_2(K_1, \nu)$ to be the separable Hilbert space with inner product

$< f, g >= \int_b^a fgd\nu$. Furthermore, we define

$$\mathcal{D}_1 := \{f : K_1 \to \mathbb{R} : \exists f' \in L_2(K_1, \nu),\ f(x) = f(a) + \int_a^x f'(y)d\nu(y),\ x \in K_1\}. \quad (4.1)$$

Every function $f \in \mathcal{D}_1$ is continuous on K_1 and we denote the derivative of f with respect to the measure ν by $\frac{d}{d\nu} := f'$. The Dirichlet form on $\mathcal{D}_1$ is defined by $a(f, g) =< f', g' >$ as usual in the Lebesgue theory.

By repeating the construction with respect to the other measure, we define a subspace $\mathcal{D}_2$ of $L_2(K_1, \nu)$ as

$$\mathcal{D}_2 := \{f \in \mathcal{D}_1 : \exists f'' \in L_2(K_2, \mu),\ f'(x) = f'(a) + \int_a^x f''(y)d\mu(y),\ x \in K_2\}. \quad (4.2)$$

The product rule and the integration by parts formula are valid here, just as in the Lebesgue measure theory. By combining Equations (4.1) and (4.2), we obtain the following definition.

Definition 32. *The Laplacian* $\Delta : \mathcal{D}_2 \to L_2(K_1, \mu)$ *is defined by* $\Delta f = \frac{d}{d\mu}\frac{d}{d\nu} f = g$ *if*

$$f(x) = f(a) + f'(a)\nu([a, x)) + \int_a^x \int_x^y g(z)d\mu(z)d\nu(y).$$

In this chapter, we will study the special case where ν is the Lebesgue measure and μ is a self-similar measure, which is known as the Sturm–Liouville operator.

4.1 The Fractal Sturm–Liouville Operator

We investigate a class of self-similar sets and measures in terms of the spectrum and the spectral zeta function of the associated fractal differential operators. C. Sabot, in a series of papers ([34]–[36]), extended the decimation method to Laplacians defined on a class of finitely-ramified self-similar sets with blow-ups, involving dynamics in several complex variables. We discuss the prototypical example he studied, fractal Laplacians on the blow-up $I_{<\infty>} = [0, \infty)$ of the unit interval $I = I_{<0>} = [0, 1]$. From now on, we will assume that

$$0 < \alpha < 1,\ b = 1 - \alpha,\ \delta = \frac{\alpha}{1-\alpha},\ \text{and } \gamma = \frac{1}{\alpha(1-\alpha)}. \tag{4.3}$$

Consider the contraction mappings from $I = [0, 1]$ to itself given by

$$\Psi_1(x) = \alpha x,\ \ \Psi_2(x) = 1 - (1 - \alpha)(1 - x),$$

and the unique self-similar measure m on $[0, 1]$ such that for all $f \in C([0, 1])$,

$$\int_0^1 f dm = b \int_0^1 f \circ \Psi_1 dm + (1 - b) \int_0^1 f \circ \Psi_2 dm. \tag{4.4}$$

Define $H_{<0>} = -\frac{d}{dm}\frac{d}{dx}$, the free Hamiltonian with Dirichlet boundary conditions on $[0, 1]$, by $H_{<0>} f = g$ on the domain

$$\left\{ f \in L^2(I, m) : \exists g \in L^2(I, m), f(x) = cx + d + \int_0^x \int_0^y g(z)dm(z)dy, f(0) = f(1) = 0 \right\}.$$

The operator $H_{<0>}$ is the infinitesimal generator associated with the Dirichlet form $(a, \mathcal{D})$ given by

$$a(f, g) = \int_0^1 f' g' dx,\ \text{for } f, g \in \mathcal{D},$$

where

$$\mathcal{D} = \{f \in L^2(I, m) : f' \in L^2(I, dx)\}.$$

As can be easily checked, the Dirichlet form a satisfies the similarity equation

$$a(f) = \alpha^{-1} a(f \circ \Psi_1) + (1-\alpha)^{-1} a(f \circ \Psi_2), \tag{4.5}$$

where we denote the quadratic form $a(f,f)$ by $a(f)$. (See, e.g., [9] for an exposition.)

The idea is that having the unit interval I as a self-similar set, we construct an increasing sequence $I_{<n>}$ of intervals by blowing up the initial unit interval with a certain scaling ratio α^{-n}. Hence, extend I to $I_{<n>} = \Psi_1^{-n}(I) = [0, \alpha^{-n}]$, which can be expressed as a self-similar set as follows: $I_{<n>} = \bigcup_{i_1,...,i_n} \Psi_{i_1 \ldots i_n}(I_{<n>})$, where $(i_1, ..., i_n) \in \{1,2\}^n$. Here, we have set $\Psi_{i_1 \ldots i_n} = \Psi_{i_n} \circ \ldots \circ \Psi_{i_1}$. We define the self-similar measure $m_{<n>}$ by

$$\int_{I_{<n>}} f dm_{<n>} = (1-\alpha)^{-n} \int_I f \circ \Psi_1^{-n} dm,$$

for all $f \in C(I_{<n>})$. Similarly, the corresponding differential operator, $H_{<n>} = -\frac{d}{dm_{<n>}} \frac{d}{dx}$ on $I_{<n>} = [0, \alpha^{-n}]$, can be defined as the infinitesimal generator of the Dirichlet form $(a_{<n>}, \mathcal{D}_{<n>})$ given by

$$a_{<n>}(f,f) = \int_0^{\alpha^{-n}} (f')^2 dx = \alpha^n a(f \circ \Psi_1^{-n}), \text{ for } f \in \mathcal{D}_{<n>},$$

where

$$\mathcal{D}_{<n>} = \{f \in L^2(I_{<n>}, m_{<n>}) : f' \text{ exists and } f' \in L^2(I_{<n>}, dx)\}.$$

We define $H_{<\infty>}$ as the operator $-\frac{d}{dm_{<\infty>}} \frac{d}{dx}$ with Dirichlet boundary conditions on $I_{<\infty>} = [0, \infty)$. It is clear that the (projective system of) measures $m_{<n>}$ give rise to a measure $m_{<\infty>}$ on $I_{<\infty>}$ since for any $f \in \mathcal{D}_{<n>}$ with $supp(f) \subset [0,1]$,

$a_{<n>}(f,f) = a(f,f)$ and $\int_{I_{<n>}} f dm_{<n>} = \int_I f dm$. Furthermore, we define the corresponding Dirichlet form $(a_{<\infty>}, \mathcal{D}_{<\infty>})$ by

$$a_{<\infty>}(f,f) = \lim_{n\to\infty} a_{<n>}(f|_{I_{<n>}}, f|_{I_{<n>}}), \text{ for } f \in \mathcal{D}_{<\infty>},$$

where

$$\mathcal{D}_{<\infty>} = \{f \in L^2(I_{<\infty>}, m_{<\infty>}) : \sup_n a_{<n>}(f|_{I_{<n>}}, f|_{I_{<n>}}) < \infty\}.$$

Clearly, $a_{<\infty>}$ satisfies a self-similar identity analogous to Equation (4.5) and its infinitesimal generator is $H_{<\infty>}$.

4.2 The Eigenvalue Problem

The study of the eigenvalue problem

$$H_{<n>}f = -\frac{d}{dm_{<n>}}\frac{d}{dx}f = \lambda f \tag{4.6}$$

for the Sturm–Liouville operator with Dirichlet boundary condition on $I_{<n>}$ revolves around a map ρ, called the *renormalization map*, which is initially defined on a space of quadratic forms associated with the fractal and then, via analytic continuation, on $\mathbb{C}^3$ as well as (by homogeneity) on $\mathbb{P}^2(\mathbb{C})$. The propagator of the above differential equation (4.6) is very useful in producing this rational map, initially on $\mathbb{C}^3$, and later on, as the polynomial map

$$\rho([x,y,z]) = [x(x+\delta^{-1}y) - \delta^{-1}z^2, \delta y(x+\delta^{-1}y) - \delta z^2, z^2], \tag{4.7}$$

defined on the complex projective plane $\mathbb{P}^2(\mathbb{C})$. Here, $[x, y, z]$ denote the homogeneous coordinates of a point in $\mathbb{P}^2(\mathbb{C})$, where $(x, y, z) \in \mathbb{C}^3$ is identified with $(\beta x, \beta y, \beta z)$ for any $\beta \in \mathbb{C}$, $\beta \neq 0$. Note that in the present case, ρ is a homogeneous polynomial of total degree two. As we shall see later on, the spectrum of the fractal Sturm–Liouville operator is intimately related to the iteration of ρ. In the sequel, we shall assume that $\delta \leq 1$ in order for the spectrum of $H_{<0>}$, $H_{<n>}$ $(n = 1, 2, ...)$ and $H_{<\infty>}$ to be purely discrete.

4.3 The Renormalization Map and the Spectrum of the Operator

The renormalization map f is a function from 2-dimensional projective space to itself which is induced by the homogeneous polynomial map on $\mathbb{C}^3$ to itself. The nature of the spectrum of the operator defined above is heavily dependent on the iteration of the renormalization map. It can be derived from studying the propagator on the eigenvalue problem associated with the operator.

We define the *propagator* $\Gamma_\lambda(s, t)$ for the eigenvalue problem $-\frac{d}{dm_{<\infty>}}\frac{d}{dx}f = \lambda f$ associated with the operator $H_{<\infty>}$ on $I_{<\infty>} = [0, \infty)$ as a time evolution function which for each $0 \leq s \leq t$ is a 2×2 matrix with nonzero determinant such that the solution of the equation satisfies

$$\begin{bmatrix} f(t) \\ f'(t) \end{bmatrix} = \Gamma_\lambda(s, t) \begin{bmatrix} f(s) \\ f'(s) \end{bmatrix}.$$

Using the self-similarity relations (4.4) and (4.5) satisfied by the measure m and the Dirichlet form a, respectively, and recalling that γ is given by Equation (4.3), we obtain

$\Gamma_{<n>,\lambda} = D_{\alpha^n} \circ \Gamma_{\gamma^n\lambda} \circ D_{\alpha^{-n}}$ for the eigenvalue problem $-\frac{d}{dm_{<n>}}\frac{d}{dx}f = \lambda f$, where

$$D_{\alpha^n} = \begin{bmatrix} 1 & 0 \\ 0 & \alpha^n \end{bmatrix}.$$

We will consider the case when $n = 1$ to derive the renormalization map. So we have $\Gamma_{<1>,\lambda} = D_\alpha \circ \Gamma_{\gamma\lambda} \circ D_{\alpha^{-1}}$. Let

$$\Gamma_\lambda = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda). \end{bmatrix}$$

We will go through the following calculations:

$$\Gamma_{<1>,\lambda} = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} a(\gamma\lambda) & b(\gamma\lambda) \\ c(\gamma\lambda) & d(\gamma\lambda) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{bmatrix} = \begin{bmatrix} a(\gamma\lambda) & \alpha^{-1}b(\gamma\lambda) \\ \alpha c(\gamma\lambda) & d(\gamma\lambda) \end{bmatrix}.$$

On the other hand, we also have

$$\begin{aligned}
\Gamma_{<1>,\lambda} &= \Gamma_\lambda(1,\alpha^{-1}) \circ \Gamma_\lambda(0,1) = D_\delta \circ \Gamma_\lambda \circ D_{\delta^1} \circ \Gamma_\lambda \\
&= \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \delta^{-1} \end{bmatrix} \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{bmatrix} \\
&= \begin{bmatrix} a(\lambda)^2 + \delta^{-1}b(\lambda)c(\lambda) & a(\lambda)b(\lambda) + \delta^{-1}d(\lambda)b(\lambda) \\ \delta a(\lambda)c(\lambda) + c(\lambda)d(\lambda) & \delta b(\lambda)c(\lambda) + d(\lambda)d(\lambda) \end{bmatrix}.
\end{aligned}$$

Using the fact that $\Gamma_\lambda \in Sl_2(\mathbb{C})$ and $a(\lambda)d(\lambda) - b(\lambda)c(\lambda) = 1$, two of the diagonal terms can be rewritten as

$$a(\lambda)^2 + \delta^{-1}b(\lambda)c(\lambda) = a(\lambda)\left[a(\lambda) + \delta^{-1}\left(\frac{d(\lambda)a(\lambda) - 1}{a(\lambda)}\right)\right] = a(\lambda)(a(\lambda) + \delta^{-1}d(\lambda)) - \delta^{-1},$$

$$\delta b(\lambda)c(\lambda) + d(\lambda)d(\lambda) = \delta d(\lambda)\left[\frac{a(\lambda)d(\lambda)-1}{d(\lambda)} + \delta^{-1}d(\lambda)\right] = \delta d(\lambda)(a(\lambda)+\delta^{-1}d(\lambda)) - \delta.$$

We define the renormalization map $\rho : \mathbb{C}^2 \to \mathbb{C}^2$ in terms of the above diagonal entries as

$$\rho(x,y) = (x(x+\delta^{-1}y) - \delta^{-1}, \delta y(x+\delta^{-1}y) - \delta).$$

and the map $\phi : \mathbb{C} \to \mathbb{C}^2$ as

$$\phi(\lambda) = \begin{bmatrix} a(\lambda) \\ d(\lambda) \end{bmatrix}$$

Note that $\rho \circ \phi(\lambda)) = \phi(\gamma\lambda)$ for all $\lambda \in \mathbb{C}$.

Going back to the 2-dimensional projective space $\mathbb{P}^2$, any point $[x,y,z] \in \mathbb{P}^2$ is equivalent to $[\frac{x}{z}, \frac{y}{z}, 1]$ for $z \neq 0$. We can represent $\mathbb{P}^2$ by

$$\mathbb{P}^2 = \{(q_1, q_2, 1) : (q_1, q_2) \in \mathbb{C}^2\} \cup \{[x, y, 0]\}.$$

We can naturally extends the map ρ to $\mathbb{P}^2$ as follows:

$$\rho([x,y,z]) = [x(x+\delta^{-1}y) - \delta^{-1}z^2, \delta y(x+\delta^{-1}y) - \delta z^2, z^2].$$

The invariant curve ϕ can be viewed as a map $\phi : \mathbb{C} \to \mathbb{C}^3$ defined by $\phi(\lambda) = (a(\lambda), b(\lambda), 1)$ which satisfies the functional equation

$$\rho \circ \phi(\lambda) = \phi(\gamma\lambda). \tag{4.8}$$

Now we study the spectrum of the eigenvalue equation (4.6). In the sequel, we shall assume that $\delta < 1$ in order for the spectrum of $H_{<0>}$, $H_{<n>}$ $(n = 1, 2, ...)$ and $H_{<\infty>}$ to be purely discrete. An *attractive fixed point* x_0 of ρ is a point such that

$\rho x_0 = x_0$ and for any other point x in some neighborhood of x_0, the sequence $\{\rho^n x\}_{n=0}^{\infty}$ converges to x_0. The *basin of attraction* of a fixed point is contained in the Fatou set of ρ. For $\delta > 1$, $x_0 = [0, 1, 0]$ is an attractive fixed point of ρ. The set

$$D = \{[x, y, z] : x + \delta^{-1} y = 0\} \tag{4.9}$$

is part of the Fatou set of ρ since it is contained in the basin of attraction of x_0. (For various notions of higher-dimensional complex dynamics, see, e.g., [7] and [8].) The set D and the invariant curve ϕ of ρ together determine the spectrum of $H_{<n>}$ and of $H_{<\infty>}$. Moreover, the set of eigenvalues can be described by the set

$$S = \{\lambda \in \mathbb{C} : \phi(\gamma^{-1}\lambda) \in D\}, \tag{4.10}$$

the 'time intersections' of the curve $\phi(\gamma^{-1}\lambda)$ with D. It turns out that S is countably infinite and contained in $\mathbb{R}^+$. We write $S = \{\lambda_k\}_{k=1}^{\infty}$, with $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k \leq ...$ repeated accordingly to multiplicity. Furthermore, we call S the *generating set* for the spectrum of $H_{<n>}$, with $n = 0, 1, ..., \infty$.

Let $S_p = \gamma^p S$, for each $p \in \mathbb{Z}$. As was noted earlier, the spectrum of $H_{<\infty>}$ with Dirichlet boundary conditions is pure point for $\alpha < \frac{1}{2}$ (hence, for $\delta < 1$ and $\gamma \geq 4$), an hypothesis we will make from now on, and can be deduced from the spectrum of $H_{<0>}$:

Theorem 33 (Sabot, [36]). *The spectrum of $H_{<0>}$ on $I = I_{<0>}$ is $\bigcup_{p=0}^{\infty} S_p$ and the spectrum of $H_{<\infty>}$ on $\mathbb{R}^+$ is $\bigcup_{p=-\infty}^{\infty} S_p$. Moreover, for any $n \geq 0$, the spectrum of $H_{<n>}$ is equal to $\bigcup_{p=-n}^{\infty} S_p$. For $n = 0, 1, ..., \infty$, each eigenvalues of $H_{<n>}$ is simple.*

The diagram of the set of eigenvalues of the operator $H_{<\infty>}$ is as follows:

$$
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \\
\gamma^{-2}\lambda_1 & \gamma^{-2}\lambda_2 & \gamma^{-2}\lambda_3 & \gamma^{-2}\lambda_4 & \cdots \\
\gamma^{-1}\lambda_1 & \gamma^{-1}\lambda_2 & \gamma^{-1}\lambda_3 & \gamma^{-1}\lambda_4 & \cdots \\
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \cdots \\
\gamma\lambda_1 & \gamma\lambda_2 & \gamma\lambda_3 & \gamma\lambda_4 & \cdots \\
\gamma^{2}\lambda_1 & \gamma^{2}\lambda_2 & \gamma^{2}\lambda_3 & \gamma^{2}\lambda_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Sabot's work ([34]–[37]) has sparked an interest in generalizing the decimation method to a broader class of fractals and therefore, to the iteration of rational functions of several complex variables. For each $k \geq 1$, we denote by f_k the solution of the equation $H_{<\infty>}f = \lambda_k f$ for $\lambda_k \in S$. In other words, f_k is an eigenfunction of $H_{<\infty>}$ associated with the eigenvalue $\lambda_k \in S$. (Note that f_k is uniquely determined, up to a nonzero multiplicative constant which can be fixed by a suitable normalization.)

Theorem 34 (Sabot, [36])**.**

(i) Given any $k \geq 1$, if f_k is the normalized solution of the equation $H_{<\infty>}f = \lambda_k f$ for $\lambda_k \in S$, then $f_{k,p} := f_k \circ \Psi_1^{-p}$ is the solution of the equation $H_{<\infty>}f = \lambda_{k,p} f$, where $\lambda_{k,p} := \gamma^p \lambda_k$ and $p \in \mathbb{Z}$ is arbitrary.

(ii) Moreover, if $f_{k,p} = f_k \circ \Psi_1^{-p}$ is the solution of the equation $H_{<\infty>}f = \lambda_{k,p} f$, then $f_{k,p,<n>} := f_{k,p}|_{I_{<n>}}$, the restriction of $f_{k,p}$ to $I_{<n>}$, is then the solution of the equation $H_{<n>}f = \lambda_{k,p} f$.

Finally, for each fixed $n \geq 0$, $\{f_{k,p,<n>} : k \geq 1, p \geq -n\}$ is a complete set of eigenfunctions of $H_{<n>}$ in the complex Hilbert space $L^2(\mathbb{R}^+, m_{<\infty>})$.

4.4 The Zeta Function Associated with the Renormalization Map

We now introduce a multivariable analog of the polynomial zeta function of Definition 27.

Definition 35 ([21]). *We define the* zeta function of the renormalization map ρ *to be*

$$\zeta_\rho(s) = \sum_{p=0}^{\infty} \sum_{\{\lambda\in\mathbb{C}:\, \rho^p(\phi(\gamma^{-(p+1)}\lambda))\in D\}} (\gamma^p\lambda)^{-\frac{s}{2}}, \tag{4.11}$$

for Re(s) sufficiently large.

Recall Definition 24 of the spectral zeta function $\zeta_L(s)$ of a positive self-adjoint operator L with discrete spectrum. We can now state our results regarding the fractal Sturm–Liouville operator:

Theorem 36 ([21]). *The zeta function $\zeta_\rho(s)$ of the renormalization map ρ is equal to the spectral zeta function $\zeta_{H_{<0>}}(s) = \sum_{\lambda\in S}\sum_{p=0}^{\infty}(\gamma^p\lambda)^{-\frac{s}{2}}$ of $H_{<0>}(s)$: $\zeta_\rho(s) = \zeta_{H_{<0>}}(s)$. (An expression for $\zeta_{H_{<0>}}(s)$ is given by the $n=0$ case of Proposition 37 below.)*

Proof. We have successively:

$$\begin{aligned}\zeta_\rho(s) &= \sum_{p=0}^{\infty} \sum_{\{\lambda\in\mathbb{C}:\, \rho^p(\phi(\gamma^{-(p+1)}\lambda))\in D\}} (\gamma^p\lambda)^{-\frac{s}{2}} \\ &= \sum_{p=0}^{\infty}\sum_{\lambda\in S}^{\infty}(\gamma^p\lambda)^{-\frac{s}{2}} \\ &= \zeta_{H_{<0>}}(s).\end{aligned}$$

In order to justify the first equality, we show that the set $\{\lambda\in\mathbb{C} : \rho^p(\phi(\gamma^{-(p+1)}\lambda))\in D\}$ is exactly equal to $S = \{\lambda\in\mathbb{C} : \phi(\gamma^{-1}\lambda)\in D\}$. Recall from the Equation (4.8), the relation $\rho(\phi(\lambda)) = \phi(\gamma\lambda)$, for all $\lambda \in \mathbb{C}$. After p iterations, this equation becomes $\rho^p(\phi(\lambda)) = \phi(\gamma^p\lambda)$. Therefore, we get $\rho^p(\phi(\gamma^{-(p+1)}\lambda)) = \phi(\gamma^p\gamma^{-p-1}\lambda) = \phi(\gamma^{-1}\lambda)$, for $p = 0, 1, 2...$ $\square$

We have a sequence of operators $H_{<n>} = -\frac{d}{dm_{<n>}}\frac{d}{dx}$, starting with $H_{<0>}$ on $[0, 1]$, which converges to the Sturm–Liouville operator $H_{<\infty>}$ on $[0, \infty)$. We will now consider the associated spectral zeta functions and their product formulas. Recall that given an integer $n \geq 0$, the spectral zeta $\zeta_{H_{<n>}}(s)$ of $H_{<n>}$ on $[0, \alpha^{-n}]$ is $\zeta_{H_{<n>}}(s) = \sum_{\lambda\in S}\sum_{p=-n}^{\infty}(\gamma^p\lambda)^{-\frac{s}{2}}$. Then, a simple computation yields the following result.

Proposition 37 ([21])**.** *For $n \geq 0$ and $Re(s)$ sufficiently large and positive, we have*

$$\zeta_{H_{<n>}}(s) = \frac{(\gamma^n)^{\frac{s}{2}}}{1-\gamma^{-\frac{s}{2}}}\zeta_S(s), \tag{4.12}$$

where $\zeta_S(s)$ is the geometric zeta function of the generating set S. *Namely, $\zeta_S(s) := \sum_{j=1}^{\infty}(\lambda_j)^{-\frac{s}{2}}$ (for $Re(s)$ large enough) or is given by its meromorphic continuation thereof.*

Proof. Recall that given an integer $n \geq 0$, the spectral zeta $\zeta_{H_{<n>}}(s)$ of $H_{<n>}$ on $[0, \alpha^{-n}]$ is

$$\zeta_{H_{<n>}}(s) = \sum_{\lambda\in S}\sum_{p=-n}^{\infty}(\gamma^p\lambda)^{-\frac{s}{2}}.$$

By separating the positive and negative powers of γ and using the geometric series with the substitution $\tau = \gamma^{-\frac{s}{2}}$ we obtain:

$$\begin{aligned}
\zeta_{H_{< n >}}(s) &= \left[(\gamma^n)^{\frac{s}{2}} + ...\gamma^s + \gamma^{\frac{s}{2}} + \sum_{p=0}^{\infty}(\gamma^p)^{-\frac{s}{2}}\right]\sum_{j=1}^{\infty}(\lambda_j)^{-\frac{s}{2}} \\
&= \left[\sum_{p=-n}^{-1}(\gamma^{-\frac{s}{2}})^p + \sum_{p=0}^{\infty}(\gamma^{-\frac{s}{2}})^p\right]\sum_{j=1}^{\infty}(\lambda_j)^{-\frac{s}{2}} \\
&= \left[\sum_{p=0}^{n-1}(\tau)^{-p-1} + \sum_{p=0}^{\infty}(\tau)^p\right]\sum_{j=1}^{\infty}(\lambda_j)^{-\frac{s}{2}} \\
&= \left[\frac{\tau^{-1}(1-\tau^{-n})}{1-\tau^{-1}} + \frac{1}{1-\tau}\right]\sum_{j=1}^{\infty}(\lambda_j)^{-\frac{s}{2}} \\
&= \frac{1}{\tau^n(1-\tau)}\sum_{j=1}^{\infty}(\lambda_j)^{-\frac{s}{2}} = \frac{(\gamma^n)^{\frac{s}{2}}}{1-\gamma^{-\frac{s}{2}}}\sum_{j=1}^{\infty}(\lambda_j)^{-\frac{s}{2}}
\end{aligned}$$

for $|\gamma^{-\frac{s}{2}}| < 1$. □

4.5 The Sturm–Liouville Operator on the Half-Line and the Dirac Hyperfunction

In the case of the operator $H_{<\infty>}$ we discovered that the geometric part of the product formula of the spectral zeta function $\zeta_{H_{<\infty>}}$ is the Dirac delta hyperfunction.

Theorem 38 ([21])**.** *The spectral zeta function $\zeta_{H_{<\infty>}}$ is factorized as follows*:

$$\zeta_{H_{<\infty>}}(s) = \delta_{\mathbb{T}}(\gamma^{-\frac{s}{2}}) \cdot \zeta_S(s). \tag{4.13}$$

Proof. Observe that

$$\zeta_{H_{<\infty>}}(s) = \sum_{p=-\infty}^{\infty}(\gamma^p)^{-\frac{s}{2}}\sum_{j=1}^{\infty}(\lambda_j)^{-\frac{s}{2}}.$$

Let $w = \gamma^{-\frac{s}{2}}$. Then we have successively:

$$\zeta_{H_{<\infty>}}(s) = \left(\sum_{p=-\infty}^{\infty} w^p \right) \zeta_S(s) = \delta_{\mathbb{T}}(w) \cdot \zeta_S(s),$$

where the last equality follows from Lemma 29. □

Next, we revisit and extend some of the earlier results obtained in [45]. More precisely, we show that the zeta function associated with the renormalization map coincides with the Riemann zeta function for a special value of α.

4.6 The case $\alpha = \frac{1}{2}$: Connection with the Riemann zeta function

When $\alpha = \frac{1}{2}$, the self-similar measure m coincides with Lebesgue measure on $[0, 1]$ and hence, $H = H_{<0>}$ coincides with the usual Dirichlet Laplacian on the unit interval $[0, 1]$.

Theorem 39 ([21])**.** *When $\alpha = \frac{1}{2}$, the Riemann zeta function ζ is equal (up to a trivial factor) to the zeta function ζ_ρ associated with the renormalization map ρ on $\mathbb{P}^2(\mathbb{C})$. More specifically, we have*

$$\zeta(s) = \pi^s \zeta_\rho(s) = \frac{\pi^s}{1 - 2^{-s}} \, \zeta_S(s), \tag{4.14}$$

where ζ_ρ is given by Definition 35 and the polynomial map $\rho : \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ is given by Equation (4.7) *with $\alpha = \frac{1}{2}$ (and hence, in light of* (4.3)*, with $\delta = 1$ and $\gamma = 4$):*

$$\rho([x, y, z]) = [x(x + y) - z^2, y(x + y) - z^2, z^2]. \tag{4.15}$$

Proof. First, we consider the case when $n = 0$ in Theorem 36 and Proposition 37. Then, since $\gamma = 4$ in the present situation, we have

$$\zeta_{H_{<0>}}(s) = \zeta_\rho(s) = \frac{1}{1 - \gamma^{-\frac{s}{2}}}\ \zeta_S(s) = \frac{1}{1 - 2^{-s}}\ \zeta_S(s). \tag{4.16}$$

Next, we recall that the eigenvalues of the Dirichlet Laplacian $L = -\frac{d^2}{dx^2}$ on $[0, 1]$ are $\kappa_j = \pi^2 j^2$, for $j = 1, 2, ...$ Therefore, in light of Definition 24, the associated spectral zeta function is

$$\zeta_L(s) = \sum_{j=1}^{\infty} (\pi^2 j^2)^{-\frac{s}{2}} = \pi^{-s}\zeta(s),$$

where ζ is the Riemann zeta function. We recall that in the present case, the Sturm–Liouville operator $H_{<0>}$ and the Dirichlet Laplacian L on $[0, 1]$ coincide; hence, the corresponding spectral zeta functions are equal: $\zeta_{H_{<0>}}(s) = \zeta_L(s)$. In light of Theorem 36, $\zeta_{H_{<0>}}(s) = \zeta_\rho(s)$ and we therefore obtain the relation

$$\zeta(s) = \pi^s \zeta_\rho(s),$$

with ζ_ρ given by Equation (4.11) and ρ defined by Equation (4.15), as desired. □

This is an extension to several complex variables of A. Teplyaev's result [45] discussed in Equation (3.8) in chapter 3. Still assuming that $\alpha = \frac{1}{2}$ and since Equation (4.14) implies that $\zeta_\rho(s) = \pi^{-s}\zeta(s)$, we deduce that the factorization formula (1.1) for the spectral zeta function $\zeta_L(s) = \zeta_{sp}(s)$ of a fractal string $\mathcal{L}$ can be rewritten as follows:

$$\zeta_L(s) = \zeta_\rho(s) \cdot \zeta_{\mathcal{L}}(s), \tag{4.17}$$

in agreement with Equation (1.3). (Compare with [23], [24] and [28], Theorem 1.19.) Here, ρ is the homogeneous quadratic polynomial on $\mathbb{C}^3$ (or rather, on $\mathbb{P}^2(\mathbb{C})$) given by Equation (4.15).

4.7 Concluding Comments

The research done in this thesis and in joint work with M. Lapidus [21] has initiated the study of factorization of the spectral zeta function of differential operators on fractals with blow-ups, via zeta functions associated with multivariable complex rational maps. It is immediate that the obtained results would apply to the modified Koch curve for which the decimation method is well established with a rational map of one complex variable and we expect to recover a similar hyperfunction in the factorization formula of the spectral zeta function of the Laplacian. Another goal would be to analyze a large class of finitely-ramified self-similar sets with (possibly random) blow-ups; a special case of that is the infinite Sierpinski gasket, which corresponds to the blow-up of the classic Sierpinski gasket and was studied in the deterministic case in §3.2. A study of random infinite gaskets and other self-similar fractals, as in [37] but along the lines of §3.3 and §4.4, remains to be carried out.

Using Sabot's multivariable extension of the decimation method, one should be able to obtain an analogous factorization formula for such fractals. Such a generalization would also enable us to better understand the nature of the spectrum of the Laplacian and to formulate and possibly solve suitable direct and inverse spectral problems in this context. In a more familiar language, and appropriately interpreted, this would enable us in certain situations to "hear the shape of a fractal drum". (See, e.g., [19]–[28].)

Thus far, we have only considered differential operators with discrete spectrum.

It would be interesting, both mathematically and physically, to obtain related results in the situation where the Laplacian under investigation has a continuous spectrum or, more generally, a combination of continuous and discrete spectra. We would then have to work with a suitably defined notion of density of states, both at the geometric and spectral levels. (Compare, e.g., [28], §6.3.1 and [19], [35].)

Moreover, as we have seen, Sabot discovered in [34]–[37] some striking relationships between the spectral properties of certain differential operators on fractals and the iteration of rational maps of several complex variables. The further study of the connections between these rational maps and the spectral zeta functions of fractal Laplacians is one of the main goals of future research on this topic and should lead to a deeper exploration of complex dynamics in higher dimensions, in relation to the spectral theory of fractal drums. (See [19], [22]–[27], [28], §12.5, and the relevant references therein.) It may also have applications to condensed matter physics ([1], [5], [14], [30]–[33]), particularly, the study of random and fractal media.

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