centration changes in the diffusion bulbs even when there is marker motion. The derivation is complicated by the fact that the marker motion causes volume changes. The concentration of species 1 molecules in one of the diffusion bulbs is

$$n_1 = N_1/V, \tag{A.1}$$

where $V$ is the volume of the bulb and $N_1$ is the total number of species 1 molecules in the bulb. Differentiation of the expression gives

$$\frac{\partial n_1}{\partial t} = \frac{1}{V}\frac{\partial N_1}{\partial t} - \frac{N_1}{V^2}\frac{\partial V}{\partial t}. \tag{A.2}$$

The change of $N_1$ with time can only be caused by the influx of molecules through diffusion, so that

$$\frac{\partial N_1}{\partial t} = aJ_1 = -a\, D_1 \frac{\partial n_1}{\partial z} + an_1 v. \tag{A.3}$$

The value of $\partial V/\partial t$ is given by Eq. (5) for the marker motion. Substituting this expression and the expression for $\partial N_1/\partial t$ into Eq. (A.2), we obtain the final expression,

$$\frac{\partial n_1}{\partial t} = -\frac{a}{V}(x_2 D_1 + x_1 D_2)\frac{\partial n_1}{\partial z} = -\frac{a}{V} D_{12}\frac{\partial n_1}{\partial z}, \tag{A.4}$$

which is of the same form as that for an ordinary diffusion experiment without marker motion.

No theory has been used in deriving Eq. (A.4) except equations expressing definitions of diffusion coefficients and conservation of molecules. It is thus somewhat surprising that $v$ drops out explicitly and that there is no apparent influence of marker motion in Eq. (A.4). Both these quantities, as well as any real physical theory, are actually hidden implicitly in the term $\partial n_1/\partial z$, which depends strongly on $v$ and the marker motion. The kinetic theory expression for the variation of $n_1$ with $z$ can be derived from Eq. (41), and is

$$\frac{n_1(z) - n_1(0)}{n_1(L) - n_1(0)} = \frac{\exp(uz/D_{12}) - 1}{\exp(uL/D_{12}) - 1}. \tag{A.5}$$

From this it follows that $(\partial n_1/\partial z)_{z=L} = (\partial n_1/\partial z)_{z=0} \exp(uL/D_{12})$. Since we have been consistently using the subscript 1 for the heavy gas and referring to the arrangement shown in Fig. 1, this means that the largest concentration change takes place in the heavy gas bulb at $z = L$. This is just what we would expect in terms of the simplified molecular picture discussed in the Introduction.

THE PHYSICS OF FLUIDS VOLUME 3, NUMBER 6 NOVEMBER-DECEMBER, 1960

# Kinetic Equation with a Constant Magnetic Field

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(Received June 13, 1960)

The collision operator is derived for the case of a spatially homogeneous plasma subject to a constant external magnetic field. A generalization of Lenard's method is employed.

## I. INTRODUCTION

STARTING from the Liouville equation a chain of equations is obtained by integrating out the coordinates of all but one, two, etc. particles. A solution can be obtained in the form of an expansion in powers of $e$, $m$, and $1/n$. This method was previously developed[1] and applied to test particle problems, i.e., only a particular initial departure from thermal equilibrium was considered in which all particles but one were in thermal equilibrium at $t = 0$. In the case of an arbitrary departure from thermal equilibrium a formal solution was obtained which is the starting point of the present paper.

The expansion of the $s$-body function was obtained in the form

$$f_s(X_1, X_2 \cdots X_s; t) = \prod_{i=1}^{s} f(X_i, t) + \sum_{\text{pairs}} [\prod f^{(0)}(X_i, t)] P(X_j X_k, t), \tag{1}$$

[1] N. Rostoker and M. N. Rosenbluth, Phys. Fluids **3**, 1 (1960).

where $X_i = (\mathbf{x}_i, \mathbf{v}_i)$ means the position and velocity of the $i$th particle. $f^{(0)}(X_i t)$ is the zero-order one-body function that satisfies the equation

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} - \frac{e}{m}\left(\mathbf{E}_M^{(0)} + \frac{1}{c}\,\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial}{\partial \mathbf{v}}\right]f^{(0)}(X, t) = 0, \qquad (2)$$

where

$$\mathbf{E}_M^{(0)} = ne\int \frac{\partial}{\partial \mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\, f^{(0)}(X't)\, dX'.$$

$P(X, X'; t)$ is the pair correlation function that is determined by the equation

$$\begin{aligned}&\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \mathbf{v}'\cdot\frac{\partial}{\partial \mathbf{x}'} - \frac{e}{mc}(\mathbf{v}\times\mathbf{B})\cdot\frac{\partial}{\partial \mathbf{v}} - \frac{e}{mc}\,\mathbf{v}'\times\mathbf{B}\cdot\frac{\partial}{\partial \mathbf{v}'}\right]P(X, X'; t)\\ &- \frac{ne^2}{m}\frac{\partial f^{(0)}}{\partial \mathbf{v}}\cdot\int\frac{\partial}{\partial \mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}''|}\,P(X', X''; t)\, dX''\\ &- \frac{ne^2}{m}\frac{\partial f^{(0)}}{\partial \mathbf{v}'}\cdot\int\frac{\partial}{\partial \mathbf{x}'}\frac{1}{|\mathbf{x}'-\mathbf{x}''|}\,P(X, X''; t)\, dX''\\ &- \frac{e^2}{m}\frac{\partial}{\partial \mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\cdot\left[f^{(0)}(X't)\,\frac{\partial f^{(0)}}{\partial \mathbf{v}} - f^{(0)}(X, t)\,\frac{\partial f^{(0)}}{\partial \mathbf{v}'}\right] = 0. \qquad (3)\end{aligned}$$

$f(Xt) = f^{(0)}(Xt) + f^{(1)}(Xt)$, and $f(Xt)$ satisfies the equation

$$\left\{\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} - \frac{e}{m}\left(\mathbf{E}_M + \frac{1}{c}\,\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial}{\partial \mathbf{v}}\right\}f(Xt) = \frac{ne^2}{m}\int\frac{\partial}{\partial \mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\cdot\frac{\partial}{\partial \mathbf{v}}P(X, X'; t)\, dX',$$

where

$$\mathbf{E}_M = ne\int\frac{\partial}{\partial \mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\, f(X't)\, dX'. \qquad (4)$$

We have assumed a plasma consisting of electrons and infinite mass ions that simply provides a neutralizing background. This avoids a considerable amount of writing. The generalization to finite mass ions is quite trivial and will be indicated in the final results.

Cylindrical coordinates $(v_\perp, \theta, v_z)$ will be employed for $\mathbf{v}$ where the $z$ axis is in the direction of $\mathbf{B}$. The spatially homogeneous solution of Eq. (2) is

$$f^{(0)}(X, t) = f^{(0)}(v_\perp, v_z). \qquad (5)$$

The object of this paper is to solve Eq. (3) and finally obtain an explicit expression for the right-hand side of Eq. (4) which is the collision operator. The method will be a generalization of that developed by Lenard[2] for the case $\mathbf{B} = 0$.

## II. THE PAIR CORRELATION FUNCTION

The spatial dependence of $P(X, X'; t)$ must be on $\mathbf{x}' - \mathbf{x}$ so that it is convenient to Fourier analyze, i.e.,

$$P(X, X'; t) = \int\frac{d\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}P_{\mathbf{k}}(\mathbf{v}, \mathbf{v}'; t). \qquad (6)$$

$P_{\mathbf{k}}(\mathbf{v}, \mathbf{v}'; t)$ has the symmetry properties

$$P_{\mathbf{k}}(\mathbf{v}, \mathbf{v}'; t) = P_{-\mathbf{k}}(\mathbf{v}', \mathbf{v}; t) = P_{\mathbf{k}}^*(\mathbf{v}', \mathbf{v}; t) \qquad (7)$$

which follow from the facts that $P(X, X'; t)$ is real and symmetric with respect to particle interchange.

Equation (3) is an integro-differential equation. It can be converted into an integral equation by integrating along the unperturbed particle trajectories or characteristics

$$\begin{aligned}P(X, X'; t) = \int_{\tau=0}^{t} d\tau\int\frac{d\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot[\mathbf{x}'(\tau)-\mathbf{x}(\tau)]}\,\frac{4\pi i e^2}{mk^2}\\ \cdot\Bigg\{\left[k_\perp\cos(\theta'(\tau)-\alpha)\frac{\partial f^{(0)}}{\partial v_\perp'} + k_z\frac{\partial f^{(0)}}{\partial v_z'}\right]\Bigg[f^{(0)}(v_\perp, v_z)\\ + n\int P_{\mathbf{k}}(v_\perp, v_z, \theta(\tau); \mathbf{v}''; t-\tau)\, d\mathbf{v}''\Bigg]\\ - \left[k_\perp\cos(\theta(\tau)-\alpha)\frac{\partial f^{(0)}}{\partial v_\perp} + k_z\frac{\partial f^{(0)}}{\partial v_z}\right]\Bigg[f^{(0)}(v_\perp', v_z')\\ + n\int P_{-\mathbf{k}}(v_\perp', v_z', \theta'(\tau); \mathbf{v}''; t-\tau)\, d\mathbf{v}''\Bigg]\Bigg\}. \qquad (8)\end{aligned}$$

The unperturbed orbits are described by the equations

$$\begin{aligned}v_\perp(\tau) &= v_\perp,\\ v_z(\tau) &= v_z,\\ \theta(\tau) &= \theta - \omega_c\tau, \qquad (9)\\ \mathbf{x}(\tau) &= \mathbf{x} + \mathbf{e}_x\frac{v_\perp}{\omega_c}[\sin\theta(\tau) - \sin\theta]\\ &\quad - \mathbf{e}_y\frac{v_\perp}{\omega_c}[\cos\theta(\tau) - \cos\theta] - \mathbf{e}_z v_z\tau,\end{aligned}$$

with a similar set for the primed quantities. $\omega_c = eB/mc$ is the cyclotron frequency. Cylindrical coordinates $(k_\perp, \alpha, k_z)$ are employed for $\mathbf{k}$.

For present purposes we require only the quantity

$$h_{\mathbf{k}}(\mathbf{v}, t) = n\int P_{\mathbf{k}}(\mathbf{v}, \mathbf{v}'; t)\, d\mathbf{v}', \qquad (10)$$

[2] A. Lenard, Ann. Phys., **10**, 390 (1959).

since the collision operator can be expressed in terms of $h_{\mathbf{k}}$, i.e., the right-hand side of Eq. (4) is

$$\begin{aligned}\mathrm{St}\,\{f\} &= \frac{ne^2}{m}\int \frac{\partial}{\partial \mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\cdot\frac{\partial}{\partial \mathbf{v}}P(X, X'; t)\,dX' \\ &= \frac{e^2}{m}\frac{\partial}{\partial \mathbf{v}}\cdot\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{4\pi i\mathbf{k}}{k^2}h_{\mathbf{k}}(\mathbf{v}, t) \\ &= -\frac{e^2}{m}\frac{\partial}{\partial \mathbf{v}}\cdot\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{4\pi\mathbf{k}}{k^2}\,\mathrm{Im}\,[h_{\mathbf{k}}(\mathbf{v}, t)]. \end{aligned}\tag{11}$$

The equation for $h_{\mathbf{k}}(\mathbf{v}, t)$ is obtained by Fourier transforming Eq. (8) and integrating over $\mathbf{v}'$.

$$\begin{aligned} h_{\mathbf{k}}(\mathbf{v}, t) &= \frac{i\omega_p^2}{k^2}\int d\mathbf{v}'\int_{\tau=0}^{t} d\tau \\ &\cdot \exp\,[i(k_\perp a'\{\sin\,[\theta'(\tau)-\alpha] - \sin\,(\theta'-\alpha)\} \\ &- k_\perp a\{\sin\,[\theta(\tau)-\alpha] - \sin\,(\theta-\alpha)\})] \\ &\cdot \exp\,[-ik_z(v_z'-v_z)\tau] \\ &\cdot\left(\left\{k_\perp\cos\,[\theta'(\tau)-\alpha]\frac{\partial f^{(0)}}{\partial v_\perp'} + k_z\frac{\partial f^{(0)}}{\partial v_z'}\right\}\right. \\ &\cdot\{f^{(0)}(v_\perp, v_z) + h_{\mathbf{k}}[v_\perp, v_z, \theta(\tau); t-\tau]\} \\ &- \left\{k_\perp\cos\,[\theta(\tau)-\alpha]\frac{\partial f^{(0)}}{\partial v_\perp} + k_z\frac{\partial f^{(0)}}{\partial v_z}\right\} \\ &\left.\cdot\{f^{(0)}(v_\perp', v_z') + h_{\mathbf{k}}^*[v_\perp', v_z', \theta'(\tau); t-\tau]\}\right), \end{aligned}\tag{12}$$

where $\omega_p^2 = 4\pi ne^2/m$, $a = v_\perp/\omega_c$, and $a' = v_\perp'/\omega_c$. Henceforth we shall denote $h_{\mathbf{k}}(\mathbf{v}, t)$ simply by $h(\mathbf{v}, t)$. An expansion for $h(\mathbf{v}, t)$ is assumed of the form

$$h(\mathbf{v}, t) = \sum_{\substack{n,n'\\=-\infty}}^{\infty} h_n(v_\perp, v_z; t)J_n(k_\perp a)J_{n'}(k_\perp a)e^{i(n-n')(\theta-\alpha)}, \tag{13}$$

where $J_n$ are Bessel functions of the first kind.

Equation (12) can thus be reduced to an equation for the coefficients $h_n$;

$$\begin{aligned} h_n(v_\perp, v_z; t) &= \frac{i\omega_p^2}{k^2}\int_0^t d\tau\int d\mathbf{v}'\exp\,[-ik_z(v_z'-v_z)\tau] \\ &\cdot\sum_{n'} J_{n'}^2(k_\perp a')\exp\,[i(n-n')\omega_c\tau] \\ &\cdot\left\{\left(\mathbf{k}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}'}\right)_{n'}[f^{(0)}(v_\perp, v_z) + h_n(v_\perp, v_z; t-\tau)]\right. \\ &\left.- \left(\mathbf{k}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}}\right)_n[f^{(0)}(v_\perp', v_z') + h_{n'}^*(v_\perp', v_z'; t-\tau)]\right\}, \end{aligned}\tag{14}$$

where

$$\left(\mathbf{k}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}}\right)_n = k_z\frac{\partial f^{(0)}}{\partial v_z} + \frac{n}{a}\frac{\partial f^{(0)}}{\partial v_\perp}.$$

This integral equation is of a convolution type. By taking Laplace transforms and making use of the "Faltung" theorem, it can be expressed as

$$\begin{aligned} h_n(v_\perp, v_z; p) &= \frac{i\omega_p^2}{k^2}\sum_{n'}\int\frac{d\mathbf{v}'J_{n'}^2(k_\perp a')}{p + i(\mathbf{k}\cdot\mathbf{v}')_{n'} - i(\mathbf{k}\cdot\mathbf{v})_n} \\ &\cdot\left\{\left(\mathbf{k}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}'}\right)_{n'}\left[\frac{f^{(0)}(v_\perp, v_z)}{p} + h_n(v_\perp, v_z; p)\right]\right. \\ &\left.- \left(\mathbf{k}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}}\right)_n\left[\frac{f^{(0)}(v_\perp', v_z')}{p} + h_{n'}^*(v_\perp', v_z'; p)\right]\right\}, \end{aligned}\tag{15}$$

where

$$h_n(v_\perp, v_z; p) = \int_0^\infty e^{-pt}h_n(v_\perp, v_z; t)\,dt$$

$$(\mathbf{k}\cdot\mathbf{v})_n = k_zv_z + \frac{n}{a}v_\perp.$$

In this paper we shall consider only the asymptotic solution[3]

$$\lim_{t\to\infty} h_n(v_\perp, v_z; t) = h_n(v_\perp, v_z) = \lim_{p\to 0} ph_n(v_\perp, v_z; p).$$

$h_n(v_\perp, v_z)$ satisfies the integral equation

$$\begin{aligned} h_n(v_\perp, v_z) &= \lim_{p\to 0}\frac{\omega_p^2}{k^2}\sum_{n'}\int\frac{d\mathbf{v}'J_{n'}^2(k_\perp a')}{[(\mathbf{k}\cdot\mathbf{v}')_{n'} - (\mathbf{k}\cdot\mathbf{v})_n - ip]} \\ &\cdot\left\{\left(\mathbf{k}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}'}\right)_{n'}[f^{(0)}(v_\perp, v_z) + h_n(v_\perp, v_z)]\right. \\ &\left.- \left[\mathbf{k}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}}\right]_n[f^{(0)}(v_\perp', v_z') + h_{n'}^*(v_\perp', v_z')]\right\}. \end{aligned}\tag{16}$$

The meaning of the integration is such that

$$\lim_{p\to 0}\left[\frac{1}{(\mathbf{k}\cdot\mathbf{v}')_{n'} - (\mathbf{k}\cdot\mathbf{v})_n - ip}\right] = \pi i\,\delta[(\mathbf{k}\cdot\mathbf{v}')_{n'} - (\mathbf{k}\cdot\mathbf{v})_n] + \frac{P}{(\mathbf{k}\cdot\mathbf{v}')_{n'} - (\mathbf{k}\cdot\mathbf{v})_n}, \tag{17}$$

where $P$ stands for the principal part. We can now follow a procedure that is an obvious generalization of Lenard's method.[2] Consider the function

$$\begin{aligned} H(u) &= \sum_{n,n'}\int h_n(v_\perp, v_z)J_n(k_\perp a) \\ &\cdot J_{n'}(k_\perp a)e^{i(n-n')(\theta-\alpha)}\,\delta\left[u - \frac{(\mathbf{k}\cdot\mathbf{v})_n}{k}\right]d\mathbf{v} \\ &= \sum_n\int h_n(v_\perp, v_z)J_n^2(k_\perp a)\,\delta\left[u - \frac{(\mathbf{k}\cdot\mathbf{v})_n}{k}\right]d\mathbf{v}, \end{aligned}\tag{18}$$

[3] It is assumed that $h_n(v_\perp, v_z; p)$ has no poles in which $p$ has a positive real part. The class of admissible functions $f^{(0)}(v_\perp, v_z)$ is therefore restricted to stable distribution functions, i.e., functions for which there are no solutions of $\epsilon(\mathbf{k}, p) = 0$ in which $p$ has a positive real part. $\epsilon(\mathbf{k}, p)$ is defined in Eq. (24).

that satisfies the equation

$$H(u) = \lim_{\lambda\to 0} \frac{\omega_p^2}{k^2} \int \frac{du'}{u' - u - i\lambda} \sum_{n,n'} \int d\mathbf{v}\, d\mathbf{v}' \cdot \delta\left[u - \frac{(\mathbf{k}\cdot\mathbf{v})_n}{k}\right] \delta\left[u' - \frac{(\mathbf{k}\cdot\mathbf{v}')_{n'}}{k}\right] \cdot J_n^2(k_\perp a) J_{n'}^2(k_\perp a') \cdot \left\{\left(\frac{\mathbf{k}}{k}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}'}\right)_n [f^{(0)}(v_\perp, v_z) + h_n(v_\perp, v_z)] - \left(\frac{\mathbf{k}}{k}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}}\right)_n [f^{(0)}(v_\perp', v_z') + h_{n'}^*(v_\perp', v_z')]\right\}. \quad (19)$$

To obtain a more manageable expression we introduce the functions

$$F(u) = \sum_n \int f^{(0)}(v_\perp, v_z) J_n^2(k_\perp a) \cdot \delta\left[u - \frac{(\mathbf{k}\cdot\mathbf{v})_n}{k}\right] d\mathbf{v}, \quad (20)$$

$$G(u) = \sum_n \int \left(\frac{\mathbf{k}}{k}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}}\right)_n J_n^2(k_\perp a) \cdot \delta\left[u - \frac{(\mathbf{k}\cdot\mathbf{v})_n}{k}\right] d\mathbf{v}. \quad (21)$$

Equations (16) and (19) may thus be written as

$$h_n(v_\perp, v_z) = \frac{\omega_p^2}{k^2} \int \frac{du'}{\{u' - [(\mathbf{k}\cdot\mathbf{v})_n/k] - i\lambda\}} \cdot \left\{G(u')[f^{(0)}(v_\perp, v_z) + h_n(v_\perp, v_z)] - \left(\frac{\mathbf{k}}{k}\cdot\frac{\partial f^{(0)}}{\partial v}\right)_n [F(u') + H^*(u')]\right\}, \quad (16.1)$$

$$H(u) = \frac{\omega_p^2}{k^2} \int \frac{du'}{(u' - u - i\lambda)} \{G(u')[F(u) + H(u)] - G(u)[F(u') + H^*(u')]\}. \quad (19.1)$$

Let $H(u) = H_R(u) + iH_I(u)$ and consider the imaginary part of Eq. (19.1)

$$H_I(u) = \frac{\omega_p^2}{k^2} P \int \frac{du'}{u' - u} \cdot [G(u')H_I(u) + G(u)H_I(u')]. \quad (22)$$

We observe that $\int H_I(u)\, du = 0$. Moreover, if $H_I(u)$ is a solution so is $\psi(u) = uH_I(u)$. To see this multiply both sides of Eq. (22) by $u$:

$$\psi(u) = \frac{\omega_p^2}{k^2} P \int \frac{du'}{u' - u} \{G(u')\psi(u) + [(u - u') + u']G(u)H_I(u')\} = \frac{\omega_p^2}{k^2} P \int \frac{du'}{u' - u} [G(u')\psi(u) + G(u)\psi(u')].$$

The only admissible solution of Eq. (22) is a function for which all moments vanish so that for all practical purposes $H_I(u) = 0$.

Now, multiply Eq. (16.1) by $G[(\mathbf{k}\cdot\mathbf{v})_n/k]$, Eq. (19.1) with $u = (\mathbf{k}\cdot\mathbf{v})_n/k$ by $(\mathbf{k}/k\cdot\partial f^{(0)}/\partial\mathbf{v})_n$, and subtract

$$h_n(v_\perp, v_z)G[(\mathbf{k}\cdot\mathbf{v})_n/k] - H[(\mathbf{k}\cdot\mathbf{v})_n/k]\left(\frac{\mathbf{k}}{k}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}}\right)_n = \frac{\omega_p^2}{k^2}\left\{G[(\mathbf{k}\cdot\mathbf{v})_n/k] \int \frac{du'}{u' - [(\mathbf{k}\cdot\mathbf{v})_n/k] - i\lambda} \cdot [f^{(0)}(v_\perp, v_z) + h_n(v_\perp, v_z)]G(u') - \left(\frac{\mathbf{k}}{k}\cdot\frac{\partial f}{\partial \mathbf{v}}\right)_n \int \frac{du'}{u' - [(\mathbf{k}\cdot\mathbf{v})_n/k] - i\lambda} \cdot [F([\mathbf{k}\cdot\mathbf{v}]_n/k) + H([\mathbf{k}\cdot\mathbf{v}]_n/k)]G(u'). \quad (23)$$

It is convenient to introduce the dielectric coefficient

$$\epsilon(\mathbf{k}, p) = 1 - \frac{\omega_p^2}{k^2} \int d\mathbf{v}' \sum_{n'} \frac{J_{n'}^2(k_\perp a')i[\mathbf{k}\cdot(\partial f^{(0)}/\partial \mathbf{v}')]_{n'}}{p + i(\mathbf{k}\cdot\mathbf{v}')_{n'}}, \quad (24)$$

so that

$$\epsilon(\mathbf{k}, -i[\mathbf{k}\cdot\mathbf{v}]_n) = 1 - \frac{\omega_p^2}{k^2} \int \frac{du'G(u')}{\{u' - [(\mathbf{k}\cdot\mathbf{v})_n/k] - i\lambda\}},$$

and

$$\mathrm{Im}\,\{\epsilon[\mathbf{k}, -i(\mathbf{k}\cdot\mathbf{v})_n]\} = -\frac{\pi\omega_p^2}{k^2} G[(\mathbf{k}\cdot\mathbf{v})_n/k].$$

In terms of these quantities Eq. (23) is expressed as

$$h_n(v_\perp, v_z) = -\frac{\pi\omega_p^2}{k^2} \frac{[(\mathbf{k}/k)\cdot(\partial f^{(0)}/\partial\mathbf{v})]_n H((\mathbf{k}\cdot\mathbf{v})_n/k)}{\mathrm{Im}\,\epsilon[\mathbf{k}, -i(\mathbf{k}\cdot\mathbf{v})_n]} + \left\{\frac{1}{\epsilon[\mathbf{k}, -i(\mathbf{k}\cdot\mathbf{v})_n]} - 1\right\}\left\{f^{(0)}(v_\perp v_z) + \frac{\pi\omega_p^2}{k^2} \frac{F[(\mathbf{k}\cdot\mathbf{v})_n/k]}{\mathrm{Im}\,\epsilon[\mathbf{k}, -i(\mathbf{k}\cdot\mathbf{v})_n]} \left[\frac{\mathbf{k}}{k}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}}\right]_n\right\}.$$

Since $\mathrm{Im}\,[H(u)] = 0$,

$$-\mathrm{Im}\,[h_n(v_\perp, v_z)] = \frac{1}{|\epsilon[\mathbf{k}, -i(\mathbf{k}\cdot\mathbf{v})_n]|^2} \cdot \{f^{(0)}(v_\perp, v_z)\, \mathrm{Im}\,\{\epsilon[k, -i(\mathbf{k}\cdot\mathbf{v})_n]\} + \frac{\pi\omega_p^2}{k^2} F[(\mathbf{k}\cdot\mathbf{v})_n/k]\left(\frac{\mathbf{k}}{k}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}}\right)_n. \quad (25)$$

## III. THE COLLISION OPERATOR

By referring to Eqs. (11) and (13)

$$\mathrm{St}\,(f) = \frac{e^2}{m}\frac{\partial}{\partial \mathbf{v}}\cdot\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{4\pi i\mathbf{k}}{k^2} \cdot \sum_{n,n'} h_n(v_\perp, v_z)J_n(k_\perp a)J_{n'}(k_\perp a)e^{i(n-n')(\theta-\alpha)},$$

$$\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}} = \frac{k_\perp}{v_\perp}\sin(\alpha-\theta)\frac{\partial}{\partial\theta} + k_\perp\cos(\alpha-\theta)\frac{\partial}{\partial v_\perp} + k_z\frac{\partial}{\partial v_z},$$

so that the $\alpha$ integration can be carried out immediately. The collision operator reduces to

$$\mathrm{St}\,(f) = -\frac{4\pi e^2}{m}\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{k^2} \cdot\sum_n\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right)_n J_n^2(k_\perp a)\ \mathrm{Im}\,[h_n(v_\perp, v_z)], \tag{26}$$

where

$$\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right)_n = k_z\frac{\partial}{\partial v_z} + \frac{n}{a}\frac{\partial}{\partial v_\perp}\cdot$$

Since $\mathrm{Im}\, h_n(v_\perp, v_z)$ is given by Eq. (25), the collision operator is completely determined.

The generalization to a plasma consisting of electrons and finite mass ions is accomplished as follows:

$$\mathrm{St}\,(f_i) = -\frac{4\pi q_i^2}{m_i}\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{k^2}\sum_n\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right)_n \cdot J_n^2(k_\perp a_i)\ \mathrm{Im}\,[h_n^{(i)}(v_\perp, v_z)], \tag{27}$$

where $i$ is the particle species index,

$$\mathrm{Im}\,[h_n^{(i)}(v_\perp, v_z)] = \frac{1}{|\epsilon[\mathbf{k}, -i(\mathbf{k}\cdot\mathbf{v})_n]|^2} \cdot\left(f_i^{(0)}(v_\perp, v_z)\ \mathrm{Im}\,\epsilon[\mathbf{k}, -i(\mathbf{k}\cdot\mathbf{v})_n] + \pi\frac{\omega_{pi}^2}{k^2}\left\{\sum_j F_j[(\mathbf{k}\cdot\mathbf{v})_n/k]\right\}\left(\frac{\mathbf{k}}{k}\cdot\frac{\partial f_i^{(0)}}{\partial\mathbf{v}}\right)_n\right), \tag{28}$$

$$\omega_{pi}^2 = 4\pi n q_i^2/m_i,\quad (\mathbf{k}\cdot\mathbf{v})_n = k_z v_z + (n/a_i),\quad a_i = v_\perp/\omega_i$$

$$\omega_i = q_i B/m_i c,$$

$$\epsilon(k, p) = 1 - \frac{1}{k^2}\sum_j\omega_{pj}^2\int d\mathbf{v}' \cdot\sum_{n'}\frac{J_{n'}^2(k_\perp a_j')i[\mathbf{k}\cdot(\partial f_j^{(0)}/\partial\mathbf{v}')]_{n'}}{p + i(\mathbf{k}\cdot\mathbf{v}')_{n'}}, \tag{29}$$

$$F_j(u) = \sum_n\int f_j^{(0)}(v_\perp', v_z')J_n^2(k_\perp a_j') \cdot\delta\left[u - \frac{(\mathbf{k}\cdot\mathbf{v}')_{n'}}{k}\right]d\mathbf{v}'. \tag{30}$$

It is not our purpose to obtain numerical results in this paper. However we shall express the collision operator in a form that can be conveniently compared with the results previously obtained for a test particle.[1] The test particle results were expressed in the form

$$\mathrm{St}\,(w_i) = -\frac{1}{m_i}\frac{\partial}{\partial\mathbf{v}}\cdot\left(\mathbf{F}^{(i)}w_i^{(0)} + \mathsf{T}^{(i)}\cdot\frac{\partial}{\partial\mathbf{v}}w_i^{(0)}\right).$$

$w_i$ is the test particle distribution function,

$$\mathbf{F}^{(i)} = \mathbf{e}_\rho F_\rho^{(i)} + \mathbf{e}_\beta F_\beta^{(i)} + \mathbf{e}_z F_z^{(i)},$$

$$\mathsf{T}^{(i)} = T_{\rho\rho}^{(i)}\mathbf{e}_\rho\mathbf{e}_\rho + T_{\beta\beta}^{(i)}\mathbf{e}_\beta\mathbf{e}_\beta + T_{zz}^{(i)}\mathbf{e}_z\mathbf{e}_z + T_{\rho\beta}^{(i)}(\mathbf{e}_\rho\mathbf{e}_\beta - \mathbf{e}_\beta\mathbf{e}_\rho) + T_{\beta z}^{(i)}(\mathbf{e}_\beta\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_\beta) + T_{z\rho}^{(i)}(\mathbf{e}_z\mathbf{e}_\rho - \mathbf{e}_\rho\mathbf{e}_z),$$

and

$$\frac{\partial}{\partial\mathbf{v}} = -\frac{\mathbf{e}_\rho}{v_\perp}\frac{\partial}{\partial\beta} + \mathbf{e}_\beta\frac{\partial}{\partial v_\perp} + \mathbf{e}_z\frac{\partial}{\partial v_z}\cdot$$

The unit vectors $\mathbf{e}_\rho$, $\mathbf{e}_\beta$, $\mathbf{e}_z$ form a local coordinate system such that $\mathbf{e}_z$ is in the direction of $\mathbf{B}$ and

$$\mathbf{e}_x = \mathbf{e}_\rho\cos\beta - \mathbf{e}_\beta\sin\beta$$

$$\mathbf{e}_y = \mathbf{e}_\rho\sin\beta + \mathbf{e}_\beta\sin\beta.$$

$\beta$ is related to the angle $\theta$ employed in this paper by the relation $\theta = (\pi/2) + \beta$. The collision operator can be written in terms of components as follows

$$\begin{aligned}\mathrm{St}\,\{w_i\} = -\frac{1}{m_i}\Bigg[&\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}(v_\perp F_\beta^{(i)}w_i^{(0)}) + \frac{\partial}{\partial v_z}(F_z^{(i)}w_i^{(0)}) \\ &- \frac{1}{v_\perp}F_\rho^{(i)}\frac{\partial w_i^{(0)}}{\partial\beta} + \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left(v_\perp T_{\beta\beta}^{(i)}\frac{\partial w_i^{(0)}}{\partial v_\perp}\right) \\ &+ \frac{\partial}{\partial v_z}\left(T_{zz}^{(i)}\frac{\partial}{\partial v_z}w_i^{(0)}\right) + \frac{T_{\rho\rho}^{(i)}}{v_\perp^2}\frac{\partial 2w_i^{(0)}}{\partial\beta^2} \\ &+ \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left(v_\perp T_{\beta z}^{(i)}\frac{\partial w_i^{(0)}}{\partial v_z}\right) + \frac{\partial}{\partial v_z}T_{\beta z}^{(i)}\frac{\partial w_i^{(0)}}{\partial v_\perp} \\ &+ \frac{1}{v_\perp}\left(\frac{\partial T_{\rho\beta}^{(i)}}{\partial v_\perp} - \frac{\partial T_{z\rho}^{(i)}}{\partial v_z}\right)\frac{\partial w_i^{(0)}}{\partial\beta}\Bigg].\end{aligned} \tag{31}$$

Equation (27) can be placed in this form. $f_i^{(0)}$ replaces $w_i^{(0)}$, and since $f_i^{(0)}$ is independent of $\beta$ or $\theta$, the terms involving $F_\rho^{(i)}$, $T_{\rho\rho}^{(i)}$, $T_{\rho\beta}^{(i)}$, and $T_{z\rho}^{(i)}$ are absent. For the remaining terms, the following results are obtained from Eq. (27):

$$F_z^{(i)} = 4\pi q_i^2\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{k_z}{k^2} \cdot\sum_n\frac{J_n^2(k_\perp a_i)\ \mathrm{Im}\,\epsilon[\mathbf{k}, -i(\mathbf{k}\cdot\mathbf{v})_n]}{|\epsilon[k, -i(\mathbf{k}\cdot\mathbf{v})_n]|^2},$$

$$F_\beta^{(i)} = 4\pi q_i^2 \int \frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{k^2} \cdot \sum_n \frac{n}{a_i}\frac{J_n^2(k_\perp a_i)\ \mathrm{Im}\ \epsilon[\mathbf{k}, -i(\mathbf{k}\cdot\mathbf{v})_n]}{|\epsilon[\mathbf{k}, -i(\mathbf{k}\cdot\mathbf{v})_n]|^2},$$

$$T_{zz}^{(i)} = 4\pi^2 q_i^2\omega_{pi}^2 \int \frac{d\mathbf{k}}{(2\pi)^3}\frac{k_z^2}{k^5} \cdot \sum_n \frac{J_n^2(k_\perp a_i)\ \sum_j F_j[(\mathbf{k}\cdot\mathbf{v})_n/k]}{|\epsilon[\mathbf{k}, -i(\mathbf{k}\cdot\mathbf{v})_n]|^2},$$

$$T_{\beta\beta}^{(i)} = 4\pi^2 q_i^2\omega_{pi}^2 \int \frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{k^5} \cdot \sum_n \left(\frac{n}{a_i}\right)^2 \frac{J_n^2(k_\perp a_i)\ \sum_j F_j[(\mathbf{k}\cdot\mathbf{v})_n/k]}{|\epsilon[\mathbf{k}, -i(\mathbf{k}\cdot\mathbf{v})_n]|^2},$$

$$T_{\beta z}^{(i)} = 4\pi^2 q_i^2\omega_{pi}^2 \int \frac{d\mathbf{k}}{(2\pi)^3}\frac{k_z}{k^5} \cdot \sum_n \frac{n}{a_i} \frac{J_n^2(k_\perp a_i)\ \sum_j F_j[(\mathbf{k}\cdot\mathbf{v})_n/k]}{|\epsilon[\mathbf{k}, -i(\mathbf{k}\cdot\mathbf{v})_n]|^2}. \tag{32}$$

For the case that $f_i^{(0)}$ are Maxwell distributions these results are the same as the previous results obtained for the test particle problem.[1]

**ACKNOWLEDGMENTS**

The author is indebted to Dr. A. Lenard and Dr. M. N. Rosenbluth for constructive discussions. This work was done under a joint General Atomic-Texas Atomic Energy Research Foundation program on controlled thermonuclear reactions.

THE PHYSICS OF FLUIDS VOLUME 3, NUMBER 6 NOVEMBER-DECEMBER, 1960

# Electric Field Distribution in a Dense Plasma

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(Received June 21, 1960)

A calculation is presented of the probability distribution function of the electric field at the center of an ion or atom in a plasma in the high ion density limit. In this limit, it is possible to take into account rigorously the effect of the Coulomb interactions on the distribution function. The distribution function in the high-density limit is Gaussian. The Coulomb interaction decreases the mean square electric field by the multiplicative factor $[1 + \kappa a_0 + \frac{1}{3}(\kappa a_0)^2]^{-1}$, where $\kappa$ is the reciprocal Debye length and $a_0$ the radius of the ion or atom at whose center the field is evaluated.

## I. INTRODUCTION

In this paper a calculation is presented of the probability distribution function of the electric field in a plasma in the limit of extreme high densities. The plasma is assumed to consist of two charged species only, and it is further assumed that the constituents of the plasma are point particles which obey classical statistics. The field distribution function is evaluated at the center of a spherical ion or atom in the plasma. This particular ion or atom will be referred to as the "radiator."

A great deal of theoretical work has been done recently on the problem of the field distribution at a point in a plasma. One of the principal motivations for this recent work is the relationship between the field distribution and the shape of spectral lines emitted or absorbed by constituents of the plasma. Indeed, in the so-called statistical theory of line broadening the line shape is attributed to the Stark effect and the problem of calculating the line shape is reduced to the problem of determining the probability distribution function of the electric field at the ion or atom that is undergoing a transition. A comprehensive discussion of line broadening in plasmas, both with respect to the underlying theory and with respect to recent detailed calculations may be found in a review article by Margenau and Lewis.[1]

In the original calculations by Holtsmark[2] of the probability distribution function of the electric field in a plasma, it was assumed that all the perturbing charges were statistically independent. Thus

[1] H. Margenau and M. Lewis, Revs. Modern Phys. **31**, 569 (1959).
[2] J. Holtsmark, Ann. Physik **58**, 577 (1919); Physik Z. **20**, 162 (1919); **25**, 73 (1924).