INF-SUP ESTIMATES FOR THE STOKES PROBLEM IN A PERIODIC CHANNEL

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Abstract. We derive estimates of the Babuška-Brezzi inf-sup constant β for two-dimensional incompressible flow in a periodic channel with one flat boundary and the other given by a periodic, Lipschitz continuous function h. If h is a constant function (so the domain is rectangular), we show that periodicity in one direction but not the other leads to an interesting connection between β and the unitary operator mapping the Fourier sine coefficients of a function to its Fourier cosine coefficients. We exploit this connection to determine the dependence of β on the aspect ratio of the rectangle. We then show how to transfer this result to the case that h is $C^{1,1}$ or even $C^{0,1}$ by a change of variables. We avoid non-constructive theorems of functional analysis in order to explicitly exhibit the dependence of β on features of the geometry such as the aspect ratio, the maximum slope, and the minimum gap thickness (if h passes near the substrate). We give an example to show that our estimates are optimal in their dependence on the minimum gap thickness in the $C^{1,1}$ case, and nearly optimal in the Lipschitz case.

Key words. Incompressible flow, Stokes equations, Babuška-Brezzi inf-sup condition, gradient, divergence, Sobolev space, dual space

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1. Introduction.

Many problems of industrial and biological importance involve fluid flow in narrow channels with moving boundaries [7, 11]. Examples include the flow of oil in journal bearings or between moving machine parts, the flow of air between disk drive platters and read-write heads, or the flow of mucus under a crawling gastropod [13]. A primary objective in all these problems is to solve for the pressure required to maintain incompressibility. Indeed, it is the pressure that determines the load sustainable by a journal bearing, and that provides propulsion against viscous drag forces in peristaltic locomotion. However, only the *gradient* of pressure enters directly into the Stokes or Navier-Stokes equations; thus, regardless of the method used to solve the equations, the pressure must be determined via its gradient.

The fundamental fact that makes it possible to extract p from ∇p is that the gradient is an isomorphism from $L^2_\#(\Omega)$, the space of mean-zero square integrable functions, onto the subspace of linear functionals in $H^{-1}(\Omega)^2$ that annihilate the divergence free vector fields $\mathbf{u} \in H^1_0(\Omega)^2$; see Section 2 below. The inf-sup constant β (or rather, its inverse) gives a bound on the norm of the inverse of this operator. Thus the magnitude of p (and our ability to estimate errors in p) depends to a large extent on the size of β^{-1}. However, to the author's knowledge, every existing proof (e.g. [4, 8]) that β^{-1} is finite relies on Rellich's compactness theorem to extract a subsequence whose lower order derivatives converge, making it impossible to determine how large β^{-1} might be or how it depends on Ω. The proof in [4] also uses the closed graph theorem, which, like Rellich's theorem, leads to constants that depend on Ω in an uncontrollable way. These proofs are appropriate for pathological domains with bulbous regions connected by thin, circuitous pathways; however, for "nice domains", it should be possible to obtain better estimates of the constants — existing theorems are of limited practical use.

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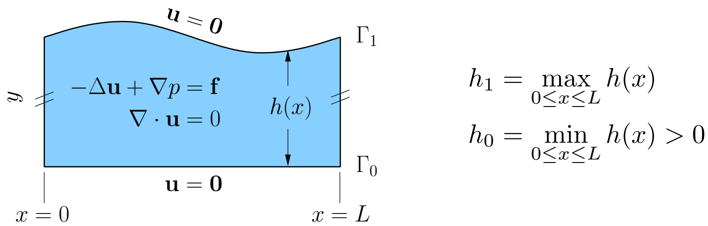


Fig. 2.1: Two dimensional Stokes flow in a periodic channel. The left and right boundaries have been identified and are considered to be part of the interior of the domain.

In this paper, we derive explicit estimates of the inf-sup constant β for two-dimensional incompressible flow in a periodic channel with one flat boundary and the other given by a periodic, Lipschitz continuous function $h(x)$. Our goal is to determine how β^{-1} depends on features of the geometry such as the aspect ratio, the maximum slope, and the minimum gap thickness (if h passes near the substrate). Although these requirements on Ω are fairly restrictive, such geometries do cover a wide range of interesting applications.

Our interest in this problem arose in the course of deriving a-priori error estimates for Reynolds' lubrication approximation (and its higher order corrections) with constants that depend on Ω in an explicit, intuitive way; see [12] and also [7, 10, 11] for background on lubrication theory. These a-priori estimates were used by the author and A. E. Hosoi to monitor errors in the lubrication approximation while studying shape optimization of swimming sheets over thin liquid films; see [13].

2. Preliminaries. In this section we briefly review the weak formulation of the Stokes equations, emphasizing the role played by the Babuška-Brezzi inf-sup condition; see e.g. [2, 3, 6] for a more detailed account.

Consider the two-dimensional, x-periodic Lipschitz domain Ω shown in Figure 2.1:

$$\Omega = \{(x,y) \;:\; x \in T,\;\; 0 < y < h(x)\}, \qquad h \in C^{0,1}(T), \qquad T = [0,L]_p. \tag{2.1}$$

The case of non-zero Dirichlet boundary conditions may be reduced to the homogeneous case by subtracting off an appropriate function to transfer the inhomogeneity from the boundary conditions to the body force $\mathbf{f}$; see e.g. [2]. We treat Ω and T as C^∞ manifolds by identifying the points

$$\begin{aligned} \Omega: &\quad (0,y) \sim (L,y) \qquad 0 < y < h(0), \\ T: &\qquad\quad 0 \sim L \end{aligned} \tag{2.2}$$

and adding a coordinate chart to each that "wraps around". In particular: a function in $C^k(\Omega)$ or $C^k(T)$ is understood to have k continuous periodic derivatives; $\partial\Omega = \Gamma_0 \cup \Gamma_1$; $\partial T = \varnothing$; the support of a function $\phi \in C_c^k(\Omega)$ vanishes near Γ_0 and Γ_1 but not necessarily at $x = 0$ and $x = L$; and the Sobolev spaces $H^k(\Omega)$ and $H_0^k(\Omega)$ are the completions of $C^k(\overline{\Omega})$ and $C_c^k(\Omega)$ in the $\|\cdot\|_k$ norm, and thus contain only x-periodic functions with appropriate smoothness at $x = 0, L$.

In the weak formulation of the Stokes equations, we seek the velocity $\mathbf{u}$ and pressure p in the spaces

$$X = H_0^1(\Omega)^2, \qquad M = L^2_{\#}(\Omega) = \Big\{p \in L^2(\Omega) \;:\; \int_\Omega p\,dA = 0\Big\}, \tag{2.3}$$

respectively, such that

$$a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = \langle \mathbf{f},\mathbf{v}\rangle \tag{2.4a}$$

$$b(\mathbf{u},q) \qquad\qquad = 0 \tag{2.4b}$$

for all $\mathbf{v} \in X$ and $q \in M$, where the body force $\mathbf{f}$ may be any linear functional in the dual space $X' = H^{-1}(\Omega)^2$ and

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\,dA, \qquad b(\mathbf{u},p) = -\int_\Omega p\,\nabla\cdot\mathbf{u}\,dA. \tag{2.5}$$

We endow M with the L^2 norm $\|\cdot\|_0$ and X with the energy norm (i.e. the H^1 semi-norm) $\|\mathbf{u}\|_a = \sqrt{a(\mathbf{u},\mathbf{u})}$, which is equivalent to the H^1 norm $\|\mathbf{u}\|_1 = \sqrt{\|\mathbf{u}\|_0^2 + \|\mathbf{u}\|_a^2}$ due to the Poincaré-Friedrichs inequality (see Lemma A.2):

$$\|\mathbf{u}\|_0 \le \frac{h_1}{\sqrt{8}}\|\mathbf{u}\|_a, \qquad (\mathbf{u}\in X), \qquad\qquad h_1 = \max_{0\le x\le L} h(x). \tag{2.6}$$

Next we define the operators $B : X \to M'$ and $B' : M \to X'$ via

$$\langle B\mathbf{u}, p\rangle = b(\mathbf{u},p) = \langle B'p,\mathbf{u}\rangle, \qquad (B = \text{div},\;\; B' = \text{grad}). \tag{2.7}$$

B and B' are clearly bounded and satisfy

$$\|B\| = \|B'\| = \sup_{p\in\dot M}\sup_{\mathbf{u}\in\dot X} \frac{|b(\mathbf{u},p)|}{\|p\|_0\,\|\mathbf{u}\|_a} \le \sqrt{2}, \tag{2.8}$$

where $\dot M = M\setminus\{0\}$ and $\dot X = X\setminus\{\mathbf{0}\}$. We note that if $\mathbf{u}\in X$ then $\nabla\cdot\mathbf{u}\in M$, i.e. the divergence of $\mathbf{u}$ has zero mean; hence,

$$V := \ker B = \{\mathbf{u}\in X \;:\; \nabla\cdot\mathbf{u} = 0\}. \tag{2.9}$$

The Babuška-Brezzi inf-sup condition

$$\exists\,\beta > 0 \quad \text{such that} \quad \inf_{p\in\dot M}\sup_{\mathbf{u}\in\dot X} \frac{|b(\mathbf{u},p)|}{\|p\|_0\,\|\mathbf{u}\|_a} \ge \beta \tag{2.10}$$

is precisely the condition required for B' to be an isomorphism onto its range with inverse bounded by $\|(B')^{-1}\| \le \beta^{-1}$. Once we know the range of B' is closed, we may take the polar of the equation $\operatorname{ran}(B')^0 = \ker(B) = V$ to conclude

$$\operatorname{ran}(B') = V^0 = \{\mathbf{f}\in X' \;:\; \langle\mathbf{f},\mathbf{u}\rangle = 0 \text{ whenever } \mathbf{u}\in V\}. \tag{2.11}$$

As V^0 is naturally isomorphic to $(X/V)'$, we see that $\widetilde{B} : X/V \to M' : (\mathbf{u}+V) \mapsto B\mathbf{u}$ is the adjoint of the composite map $M \xrightarrow{B'} V^0 \xrightarrow{\cong} (X/V)'$, and is therefore itself an isomorphism with the same bound on the inverse. Identifying X/V with

$$V^\perp = \{\mathbf{u}\in X \;:\; a(\mathbf{u},\mathbf{v}) = 0 \text{ whenever } \mathbf{v}\in V\}, \tag{2.12}$$

we learn that the restriction of B to $V^\perp$ is an isomorphism onto M', which would be essential to the analysis of the Stokes equations if the right hand side of (2.4b) were inhomogeneous. Other interesting solutions of $B\mathbf{u} = \varphi$ with $\varphi \in M'$ (requiring e.g. $\mathbf{u} \in L^\infty(\Omega)^2 \cap X$ or $\nabla \times \mathbf{u} = 0$ rather than $\mathbf{u} \in V^\perp$) are studied in [1]. Finally, we define $A : X \to X'$ and $\tilde{A} : V \to V'$ via

$$\langle A\mathbf{u}, \mathbf{v}\rangle = a(\mathbf{u}, \mathbf{v}), \quad (\mathbf{u}, \mathbf{v} \in X), \qquad \langle \tilde{A}\mathbf{u}, \mathbf{v}\rangle = a(\mathbf{u}, \mathbf{v}), \quad (\mathbf{u}, \mathbf{v} \in V). \tag{2.13}$$

Both are isometric isomorphisms in the $\|\cdot\|_a$ norm.

The weak solution $(\mathbf{u}, p)$ of (2.4) must satisfy $\mathbf{u} \in V$ so that $B\mathbf{u} = 0$. But then $A\mathbf{u} + B'p = \mathbf{f}$ requires

$$(*) \ \ \tilde{A}\mathbf{u} = \tilde{\mathbf{f}}, \qquad (\dagger) \ \ B'p = \mathbf{f} - A\mathbf{u}, \tag{2.14}$$

where $\tilde{\mathbf{f}} = \mathbf{f}|_V \in V'$ and we note that $(\mathbf{f} - A\mathbf{u}) \in \text{range}(B') = V^0$ iff $\mathbf{u}$ satisfies $(*)$. Since $\tilde{A}$ and B' are isomorphisms onto their ranges, a unique solution of (2.4) exists and we have the estimates

$$\|\mathbf{u}\|_a = \|\tilde{\mathbf{f}}\|_{V'} \le \|\mathbf{f}\|_{X'} = \sup_{\mathbf{u} \in \dot{X}} \frac{|\langle \mathbf{f}, \mathbf{u}\rangle|}{\|\mathbf{u}\|_a}, \qquad \|p\|_0 \le 2\beta^{-1}\|\mathbf{f}\|_{X'}. \tag{2.15}$$

In summary, the inf-sup condition (2.10) is the key to analyzing the weak formulation of the Stokes equations — it is equivalent to the assertion that the gradient B' is an isomorphism from $M = L^2_\#(\Omega)$ onto the polar set V^0 of linear functionals in X' that annihilate the divergence free vector fields $\mathbf{u} \in V$.

It is instructive to compare the inf-sup condition written in the form

$$\beta\|p\|_0 \le \|B'p\|_{X'} = \|\nabla p\|_{-1} \le \sqrt{2}\|p\|_0 \qquad (p \in L^2_\#(\Omega)), \tag{2.16}$$

to the Poincaré-Friedrichs inequality for mean-zero functions:

$$\|p\|_0 \le C\|\nabla p\|_0 \quad \Rightarrow \quad (1 + C^2)^{-1/2}\|p\|_1 \le \|\nabla p\|_0 \le \|p\|_1 \qquad (p \in H^1_\#(\Omega)). \tag{2.17}$$

Whereas (2.17) is easy to prove for $p \in H^1_0(\Omega)$ (with $C = \frac{1}{\sqrt{8}}h_1$ in our case), it is more challenging to prove for mean zero functions $p \in H^1_\#(\Omega)$. The usual proof [2, 5] relies on Rellich's theorem that $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$. As a result, the proof does not tell us how large the constant C might be or how it depends on Ω. Similarly, the usual proof [4] of (2.16) makes use of Rellich's theorem that $L^2(\Omega)$ is compactly embedded in $H^{-1}(\Omega) = H^1_0(\Omega)'$; however, there is an added complication not present in proving (2.17): it must first be established that

$$\|p\|_0 \le C(\|p\|_{-1} + \|\nabla p\|_{-1}), \qquad (p \in L^2(\Omega)). \tag{2.18}$$

This can be done in our case (if $h \in C^{1,1}(T)$) by flattening out the boundary and constructing appropriate extension operators from $H^{-1}(\Omega)$ to $H^{-1}(T \times \mathbb{R})$ to reduce the problem to a case that can be solved using the Fourier transform; see Duvaut and Lions [4] and also Nitsche [9], who used a similar technique to prove Korn's inequality. In this paper, we show how to bypass (2.18) and prove (2.16) directly *without invoking Rellich's theorem*, which allows us to determine how the constant β depends on Ω. We present two versions of the proof: one assuming $h \in C^{1,1}(T)$, and the other assuming only that $h \in C^{0,1}(T)$, i.e. that h is a periodic, Lipschitz continuous function. Our proof does rely on the boundary of Ω being the graph of a function $h(x)$; however, we feel this is a sufficiently important case to warrant a separate analysis. We sketch a proof of (2.17) that avoids Rellich's theorem in Appendix B for comparison.

3. A rectangular channel. In the following theorem, we prove that B' in (2.7) is an isomorphism onto its range $\big(\text{with } \beta = \frac{1}{3}\min(1, 4\frac{H}{L})\big)$ when Ω is the x-periodic rectangle $R = T \times (0, H)$ of height H. In Sections 4 and 5, we will transfer this result to a general x-periodic domain Ω by a change of variables. It is useful in this change of variables to know that the constant C_2 in Theorem 3.1 (and especially in Corollary 3.4) does not diverge as H approaches zero.

The periodicity of the domain in one direction but not the other leads to an interesting relationship between the inf-sup condition and the unitary operator mapping the Fourier sine coefficients of a function of one variable to its Fourier cosine coefficients. By studying this operator, we can obtain explicit estimates of β and its dependence on L/H.

Recall that every $u \in H_0^1(R)$ must be zero (in the trace sense) on the top and bottom walls but not necessarily on the side walls, where it is only required to be periodic. Such a function can be expanded in a sine or cosine series in the y-direction and differentiated term by term. (If $u \in H^1(R)$ is not zero on the top and bottom walls, only the cosine series can be differentiated term by term).

THEOREM 3.1. *For all* $q \in L^2_\#(R)$,

$$\|q\|_0^2 \leq C_1 \|\partial_x q\|_{-1}^2 + C_2 \|\partial_y q\|_{-1}^2, \tag{3.1}$$

where $C_1 = \max\left(9, \frac{9}{16}\frac{L^2}{H^2}\right)$, $C_2 = 9$, *and* $\|f\|_{-1} = \sup_{u \in H_0^1(R)} \frac{|\langle f, u\rangle|}{\|u\|_a}$.

Proof. We may expand any $q \in L_0^2(R)$ and $u \in H_0^1(R)$ in a Fourier series

$$q(x,y) = \sum_{n\in\mathbb{Z}} \left(a_{n0} + \sum_{j=1}^\infty a_{nj}\sqrt{2}\cos\frac{\pi j y}{H}\right) e^{\frac{2\pi i n x}{L}} = \sum_{n\in\mathbb{Z}} \left(\sum_{j=1}^\infty b_{nj}\sqrt{2}\sin\frac{\pi j y}{H}\right) e^{\frac{2\pi i n x}{L}},$$

$$u(x,y) = \sum_{n\in\mathbb{Z}} \left(c_{n0} + \sum_{j=1}^\infty c_{nj}\sqrt{2}\cos\frac{\pi j y}{H}\right) e^{\frac{2\pi i n x}{L}} = \sum_{n\in\mathbb{Z}} \left(\sum_{j=1}^\infty d_{nj}\sqrt{2}\sin\frac{\pi j y}{H}\right) e^{\frac{2\pi i n x}{L}}$$

so that

$$\|q\|_0^2 = \sum_{\mathbb{Z}\times\mathbb{N}_0} LH|a_{nj}|^2 = \sum_{\mathbb{Z}\times\mathbb{N}} LH|b_{nj}|^2, \tag{3.2}$$

$$\|u\|_a^2 = \sum_{\mathbb{Z}\times\mathbb{N}_0} LH\Big[\Big(\frac{2\pi n}{L}\Big)^2 + \Big(\frac{\pi j}{H}\Big)^2\Big]|c_{nj}|^2 = \sum_{\mathbb{Z}\times\mathbb{N}} LH\Big[\Big(\frac{2\pi n}{L}\Big)^2 + \Big(\frac{\pi j}{H}\Big)^2\Big]|d_{nj}|^2. \tag{3.3}$$

Here $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and the sums are over ordered pairs (n, j). Let us denote $(\mathbb{Z}\times\mathbb{N}_0)' = \mathbb{Z}\times\mathbb{N}_0 \setminus \{(0,0)\}$. We claim that

$$\begin{array}{ccccc}
 & & A_1 & A_2 & \|\partial_y q\|_{-1}^2 \\
 & & \downarrow & \searrow & \| \\
\|q\|_0^2 & = & \displaystyle\sum_{(\mathbb{Z}\times\mathbb{N}_0)'} \frac{LH(2\pi n/L)^2|a_{nj}|^2}{(2\pi n/L)^2 + (\pi j/H)^2} & + & \displaystyle\sum_{\mathbb{Z}\times\mathbb{N}} \frac{LH(\pi j/H)^2|a_{nj}|^2}{(2\pi n/L)^2 + (\pi j/H)^2} \\
 & & \vee| & & \wedge| \\
\|q\|_0^2 & = & \displaystyle\sum_{\mathbb{Z}\times\mathbb{N}} \frac{LH(2\pi n/L)^2|b_{nj}|^2}{(2\pi n/L)^2 + (\pi j/H)^2} & + & \displaystyle\sum_{\mathbb{Z}\times\mathbb{N}} \frac{LH(\pi j/H)^2|b_{nj}|^2}{(2\pi n/L)^2 + (\pi j/H)^2} \\
 & & \| & \nwarrow & \uparrow \\
 & & \|\partial_x q\|_{-1}^2 & B_1 & B_2
\end{array} \tag{3.4}$$

Here A_1, A_2, B_1 and B_2 are labels to represent the indicated sums. The horizontal assertions clearly hold (since $q \in L^2_0(R) \Rightarrow a_{00} = 0$) while the vertical assertions follow from the Cauchy-Schwarz inequality and a particular choice of u to show that two of the upper bounds are least upper bounds:

$$\langle \partial_x q, u\rangle = \int_R (q)(-\partial_x u)\, dA = \sum_{\mathbb{Z}\times\mathbb{N}} LH\, b_{-n,j}\Big(-\frac{2\pi i n}{L} d_{nj}\Big) \le B_1^{1/2}\|u\|_a,$$

$$\langle \partial_x q, u\rangle = \sum_{(\mathbb{Z}\times\mathbb{N}_0)'} LH\, a_{-n,j}\Big(-\frac{2\pi i n}{L} c_{nj}\Big) \le A_1^{1/2}\|u\|_a,$$

$$\langle \partial_y q, u\rangle = \int_R (q)(-\partial_y u)\, dA = \sum_{\mathbb{Z}\times\mathbb{N}} LH\, a_{-n,j}\Big(-\frac{\pi j}{H} d_{nj}\Big) \le A_2^{1/2}\|u\|_a,$$

$$\langle \partial_y q, u\rangle = \sum_{\mathbb{Z}\times\mathbb{N}} LH\, b_{-n,j}\Big(\frac{\pi j}{H} c_{nj}\Big) \le B_2^{1/2}\|u\|_a,$$

$$d_{nj} = \frac{(2\pi i n/L)\,\bar{b}_{-n,j}}{(2\pi n/L)^2 + (\pi j/H)^2} \quad\Rightarrow\quad \|u\|_a = B_1^{1/2}, \quad \langle \partial_x q, u\rangle = B_1, \tag{3.5}$$

$$d_{nj} = \frac{-(\pi j/H)\,\bar{a}_{-n,j}}{(2\pi n/L)^2 + (\pi j/H)^2} \quad\Rightarrow\quad \|u\|_a = A_2^{1/2}, \quad \langle \partial_y q, u\rangle = A_2. \tag{3.6}$$

The choices of c_{nj} analogous to (3.5) and (3.6) do not generally lead to functions u that satisfy the boundary conditions on the top and bottom walls; hence, we cannot replace the inequalities in (3.4) by equalities.

The theorem will be proved if we can show that

$$\theta(A_1 + A_2) + (1-\theta)(B_1 + B_2) \le C_1 B_1 + C_2 A_2 \tag{3.7}$$

for some $\theta \in [0,1]$. The result (3.15) below turns out to be independent of θ, so we set $\theta = 1$ here for simplicity. We will prove (3.7) by slicing the lattices $(\mathbb{Z}\times\mathbb{N}_0)'$ and $\mathbb{Z}\times\mathbb{N}$ into vertical strips and showing that

$$A_{1,n} \le C_1 B_{1,n} + (C_2 - 1)A_{2,n}, \qquad (n \in \mathbb{Z}), \tag{3.8}$$

where the subscript n indicates that only the terms in strip n should be included in the sum, e.g. $A_{1,3} = \sum_{j=0}^\infty LH(6\pi/L)^2|a_{3j}|^2/[(6\pi/L)^2 + (\pi j/H)^2]$. Since $A_{1,0} = 0$, the $n = 0$ case holds trivially. If we freeze $n \in \mathbb{Z}\setminus\{0\}$, we find that

$$\frac{1}{L}\int_0^L q(x,y)e^{-\frac{2\pi i n x}{L}}\, dx \;=\; a_{n0} + \sum_{k=1}^\infty a_{nk}\sqrt{2}\cos\frac{\pi k y}{H} \;=\; \sum_{j=1}^\infty b_{nj}\sqrt{2}\sin\frac{\pi j y}{H}. \tag{3.9}$$

Thus, the coefficients a_{nk} and b_{nj} are related to each other by a unitary transformation

$$a_{nk} = \sum_{j=1}^\infty E_{kj} b_{nj}, \qquad (n \in \mathbb{Z},\ k \ge 0). \tag{3.10}$$

The entries of E can be computed explicitly: for $j \ge 1$ we have

$$E_{kj} = \left\{\begin{array}{ll} \int_0^1 \sqrt{2}\sin(\pi j\eta)\, d\eta, & k = 0 \\ \int_0^1 2\sin(\pi j\eta)\cos(\pi k\eta)\, d\eta, & k \ge 1 \end{array}\right\} = \begin{cases} 2\sqrt{2}/(j\pi), & k = 0,\ j \text{ odd} \\ \dfrac{4j}{(j^2 - k^2)\pi}, & k > 0,\ j - k \text{ odd} \\ 0, & \text{otherwise} \end{cases} \tag{3.11}$$

Keeping $n \in \mathbb{Z} \setminus \{0\}$ frozen and dividing (3.8) by LH, we must show that

$$\sum_{k=0}^{\infty} \frac{(2\pi n/L)^2 |a_{nk}|^2}{(2\pi n/L)^2+(\pi k/H)^2} \le C_1 \sum_{j=1}^{\infty} \frac{(2\pi n/L)^2 |b_{nj}|^2}{(2\pi n/L)^2+(\pi j/H)^2} + (C_2-1) \sum_{k=1}^{\infty} \frac{(\pi k/H)^2 |a_{nk}|^2}{(2\pi n/L)^2+(\pi k/H)^2}.$$

This is accomplished via the following lemma using $\nu = 2|n|H/L$ and $\nu_0 = 2H/L$. ∎

LEMMA 3.2. *Suppose $b \in \ell^2(\mathbb{N})$ and let $a = Eb \in \ell^2(\mathbb{N}_0)$, where E maps the Fourier sine coefficients of a function to its Fourier cosine coefficients; see (3.11) above. Then for $\nu > 0$ there holds*

$$\sum_{k=0}^{\infty} \frac{\nu^2}{\nu^2+k^2} |a_k|^2 \le C_1 \sum_{j=1}^{\infty} \frac{\nu^2}{\nu^2+j^2} |b_j|^2 + (C_2-1) \sum_{k=1}^{\infty} \frac{k^2}{\nu^2+k^2} |a_k|^2, \tag{3.12}$$

with $C_1 = \max\left(9, \frac{9}{4}\nu^{-2}\right)$ and $C_2 = 9$. If $\nu \ge \nu_0 > 0$, $C_1 = \max\left(9, \frac{9}{4}\nu_0^{-2}\right)$ also works .

Proof. It suffices to show that (3.12) holds whenever b is a unit vector in $\ell^2(\mathbb{N})$. The general case follows by re-scaling this result. We will split each sum into terms of low and high index and use different arguments to handle the two cases. Let $k_0 \ge 0$, $j_0 \ge 1$, $k_1 = k_0 + 1$ and $j_1 = j_0 + 1$. If we discard terms on the right hand side with $j \ge j_1$ and $k \le k_0$, we obtain a sufficient condition for (3.12) to hold. Also, on the left hand side, $\sum_{k=k_1}^{\infty} \frac{\nu^2}{\nu^2+k^2} |a_k|^2 \le \frac{\nu^2}{k_1^2} \sum_{k=k_1}^{\infty} \frac{k^2}{\nu^2+k^2} |a_k|^2$, so it suffices to show that

$$\sum_{k=0}^{k_0} \frac{\nu^2}{\nu^2+k^2} |a_k|^2 \le C_1 \sum_{j=1}^{j_0} \frac{\nu^2}{\nu^2+j^2} |b_j|^2 + \left(C_2 - 1 - \frac{\nu^2}{k_1^2}\right) \sum_{k=k_1}^{\infty} \frac{k^2}{\nu^2+k^2} |a_k|^2. \tag{3.13}$$

Next, we see that (3.13) will hold if we can show that

$$\alpha^2 \le \frac{C_1 \nu^2}{\nu^2 + j_0^2} \beta^2 + \frac{(C_2 - 1 - \nu^2/k_1^2) k_1^2}{\nu^2 + k_1^2} (1 - \alpha^2), \tag{3.14}$$

where $\alpha^2 = \sum_{k=0}^{k_0} |a_k|^2$, $\beta^2 = \sum_{j=1}^{j_0} |b_j|^2$, and $1 - \alpha^2 = \sum_{k=k_1}^{\infty} |a_k|^2$. Note that β here is not the inf-sup constant β, but rather a measure of the relative weight of low frequency modes in comparison to high frequency modes in a sine series expansion. Solving (3.14) for α^2, we require

$$\alpha^2 \le \frac{C_1}{C_2} \frac{1 + \nu^2/k_1^2}{1 + j_0^2/\nu^2} \beta^2 + 1 - \frac{1 + \nu^2/k_1^2}{C_2}. \tag{3.15}$$

Our goal is to show that for each $\nu > 0$ there is a choice of $j_0 \ge 1$, $k_1 \ge 1$, $C_1 \le \max(9, (9/4)\nu^{-2})$ and $C_2 \le 9$ such that (3.15) and consequently (3.12) holds for all unit vectors $b \in \ell^2(\mathbb{N})$; (b determines a, α and β). C_1 and C_2 can then be increased if necessary to the values stated in the lemma without violating (3.12).

We now use the fact that a and b are unit vectors related by a known unitary transformation to obtain a bound on α in terms of β. Let S, T, x, y, z be the sub-matrices and sub-vectors

$$\begin{aligned} &S = E(0{:}k_0,\, 1{:}j_0), \qquad T = E(0{:}k_0,\, j_1{:}\infty), \qquad E(0{:}k_0,\, :) = [S, T]. \\ &z = a(0{:}k_0), \qquad x = b(1{:}j_0), \qquad y = b(j_1{:}\infty), \qquad z = Sx + Ty. \end{aligned} \tag{3.16}$$

We have $\alpha = \|z\|$ and $\beta = \|x\| = \sqrt{1 - \|y\|^2}$. Since $\|S\| \le 1$ and $\|y\| \le 1$, the estimate $\|z\| \le \|Sx\| + \|Ty\|$ gives

$$\alpha \le \beta + t, \qquad t = \|T\| \le 1. \tag{3.17}$$

If $t < 1$, this can be used to derive a bound on α^2 of the form (3.15). However, we can obtain a sharper estimate as follows. First, we compute the singular value decomposition $S = U\Sigma V^*$ and rotate the rows of T by a unitary operator Q such that

$$U^*[S,T]\begin{bmatrix} V & 0 \\ 0 & Q \end{bmatrix} = \left(\begin{array}{cccccc|cccc} \sigma_0 & & 0 & 0 & \cdots & 0 & t_0 & & 0 & 0 & \cdots \\ & \ddots & & \vdots & & \vdots & & \ddots & & \vdots & \\ 0 & & \sigma_{k_0} & 0 & \cdots & 0 & 0 & & t_{k_0} & 0 & \cdots \end{array}\right), \quad (3.18)$$

where $\sigma_k^2 + t_k^2 = 1$ for $0 \le k \le k_0$. We assume here that $j_0 \ge k_0 + 1$; otherwise we will not be able to derive a sufficient condition for (3.15) to hold, for if S has more rows than columns, we can produce a unit vector $a = [z; 0]$ with $S^* z = 0$ so that $b = E^* a$ yields $\alpha = 1$ and $\beta = 0$. Next we define $\tilde{z} = U^* z$, $\tilde{x} = V^* x$, $\tilde{y} = Q^* y$ so that

$$\alpha^2 = \sum_{k=0}^{k_0} |\tilde{z}_k|^2, \quad \beta^2 = \sum_{j=1}^{j_0} |\tilde{x}_j|^2, \quad 1 - \beta^2 = \sum_{j=1}^{\infty} |\tilde{y}_j|^2, \qquad \tilde{z}_k = \sigma_k \tilde{x}_{k+1} + t_k \tilde{y}_{k+1} \quad (3.19)$$

and, by Lemma A.1 below,

$$|\tilde{z}_k|^2 \le \frac{1}{1-t_k}|\sigma_k \tilde{x}_{k+1}|^2 + \frac{1}{t_k}|t_k \tilde{y}_{k+1}|^2 = (1+t_k)|\tilde{x}_{k+1}|^2 + t_k|\tilde{y}_{k+1}|^2. \quad (3.20)$$

Hence, majorizing t_k by $\|T\| = t_{\max} = \sqrt{1 - \sigma_{\min}^2}$ and summing over k, we obtain

$$\alpha^2 \le (1+t)\beta^2 + t(1-\beta^2) = \beta^2 + t, \qquad t = \|T\| \le 1. \quad (3.21)$$

Thus, (3.15) holds if we define C_1 and C_2 via $\left(1 - \frac{1+\nu^2/k_1^2}{C_2}\right) = t$ and $\left(\frac{C_1}{C_2}\frac{1+\nu^2/k_1^2}{1+j_0^2/\nu^2}\right) = 1$:

$$C_1(\nu) = \frac{1 + j_0^2/\nu^2}{1-t}, \qquad C_2(\nu) = \frac{1+\nu^2/k_1^2}{1-t}. \quad (3.22)$$

Next we look for choices of k_1 and j_0 that lead to a window of values of ν over which C_1 and C_2 remain small. We need enough such windows to cover the positive real line $\nu > 0$. The trade-off is that choosing $j_0 \gg k_1$ makes t small but also makes one of the numerators in (3.22) large. We consider 3 cases:

- *Case 1:* $(0 < \nu \le 1/2)$. We set $k_1 = j_0 = 1$ so that $t = \sqrt{1 - E_{01}^2} = .4352$ and

$$\begin{aligned} C_2(\nu) &= (1+\nu^2)/(1-t) \le (5/4)/(1-t) = 2.2133 \le 9/4, \\ C_1(\nu) &= (1+\nu^{-2})/(1-t) = C_2\nu^{-2} \le (9/4)\nu^{-2}, \end{aligned} \qquad (0 < \nu \le 1/2). \quad (3.23)$$

- *Case 2:* $(1/2 \le \nu \le 100)$. We wrote a program to compute the singular value decomposition of S for all pairs of small integers k_1 and j_0 satisfying $k_1 \le j_0 \le 4k_1 \le 400$ to determine $t = \sqrt{1 - \sigma_{\min}^2}$ for each pair. We then choose a threshold C_{thresh} and find the values $\nu_{\min}$ and $\nu_{\max}$ such that $C_1(\nu_{\min}) = C_{\text{thresh}}$ and $C_2(\nu_{\max}) = C_{\text{thresh}}$. We then discard all cases with $\nu_{\max} < \nu_{\min}$ and sort the remaining intervals $[\nu_{\min}, \nu_{\max}]$ by their first entry. Finally, we discard all intervals for which $\nu_{\min}$ of the next interval is smaller than $\nu_{\max}$ of the previous interval (to avoid redundancy). The results with $C_{\text{thresh}} = 4.9$ and $C_{\text{thresh}} = 8.9$ are shown in Figure 3.1. The method breaks down (i.e. there are gaps between some of the windows) for $C_{\text{thresh}} < 5.83$.

Table 3.1: Parameters used to construct $C_1(\nu)$ and $C_2(\nu)$ with $C_{\text{thresh}} = 8.9$. The corresponding table with $C_{\text{thresh}} = 5.9$ has 32 lines corresponding to the smaller windows shown in Figure 3.1.

k_1	j_0	t	$\nu_{\min}$	$\nu_{\max}$
1	1	.43524	0.498	2.007
3	3	.57904	1.810	4.972
6	8	.54892	4.608	10.42
13	17	.58222	10.31	21.43
25	37	.54766	21.27	43.49
50	76	.54321	43.41	87.54
99	155	.53535	87.54	175.3

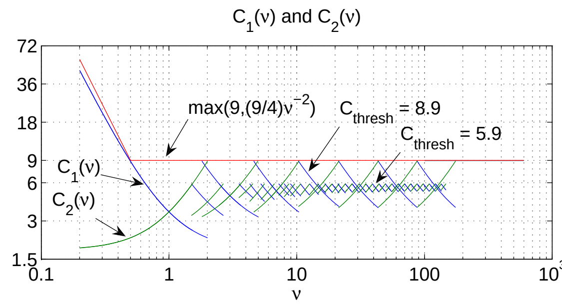


Fig. 3.1: Plot of $C_1(\nu)$ and $C_2(\nu)$ over the range $0.2 \le \nu \le 600$. Each criss-cross corresponds to a different window $\nu_{\min} \le \nu \le \nu_{\max}$ in Table 3.1.

• *Case 3:* ($\nu \ge 100$). We set $k_1 = \lfloor \nu/\sqrt{3} \rfloor$, $j_0 = 3k_1$ and bound t by the Frobenius norm:

$$\begin{aligned} t^2 &\le \|T\|_F^2 = \sum_{k=0}^{k_0} \sum_{j=j_1}^{\infty} |E_{kj}|^2 = \frac{8}{\pi^2} \sum_{j=j_1}^{\infty} \frac{\delta_{j,odd}}{j^2} + \frac{16}{\pi^2} \sum_{k=1}^{k_0} \sum_{j=j_1}^{\infty} \frac{j^2 \delta_{j-k,odd}}{(j^2-k^2)^2} \qquad (3.24) \\ &\le \frac{4}{\pi^2} \int_{j_0-1}^{\infty} \frac{1}{x^2}\,dx + \frac{8}{\pi^2} k_0 \int_{j_0-1}^{\infty} \frac{x^2}{(x^2-k_0^2)^2}\,dx, \qquad (j_0 - 1 = j_1 - 2) \\ &= \frac{4}{\pi^2(j_0-1)} + \frac{4}{\pi^2}\left[\frac{\kappa}{\kappa^2-1} + \frac{1}{2}\log\frac{\kappa+1}{\kappa-1}\right], \qquad \left(\kappa = \frac{j_0-1}{k_0} > \frac{j_0}{k_1} = 3\right) \\ &\le \frac{4}{170\pi^2} + \frac{4}{\pi^2}\left[\frac{3}{8} + \frac{1}{2}\log 2\right] = (.54298)^2, \qquad \left(k_1 \ge \left\lfloor \frac{100}{\sqrt{3}} \right\rfloor = 57,\ j_0 \ge 171\right). \end{aligned}$$

Here we represent sums of decreasing functions sampled at even or odd integers by staircases of width two and half the height of the function at the right endpoint. Each choice of k_1 and j_0 will cover the range $\sqrt{3}k_1 \le \nu < \sqrt{3}(k_1+1)$; over this range, we have

$$\frac{\nu}{k_1} \le \left(\frac{k_1+1}{k_1}\right)\left(\frac{\nu}{k_1+1}\right) \le \frac{58}{57}\sqrt{3}, \qquad \frac{j_0}{\nu} \le \left(\frac{j_0}{k_1}\right)\left(\frac{k_1}{\nu}\right) \le (3)\frac{1}{\sqrt{3}} = \sqrt{3} \qquad (3.25)$$

and we learn that C_1 and C_2 are bounded by $\frac{1+3(58/57)^2}{1-.54298} = 8.985$.

Thus, for all $\nu > 0$ we have $C_1 \le \max(9, (9/4)\nu^{-2})$ and $C_2 \le 9$, as claimed. □

COROLLARY 3.3. *For all* $q \in L^2_\#(R)$, $\|q\|_{-1}^2 \le \frac{L^2}{4\pi^2}\|\partial_x q\|_{-1}^2 + \frac{H^2}{\pi^2}\|\partial_y q\|_{-1}^2$.

Proof. Arguing as in (3.4)–(3.6), it is readily shown that

$$\|q\|_{-1}^2 = \sum_{\mathbb{Z}\times\mathbb{N}} \frac{LH|b_{nj}|^2}{(2\pi n/L)^2 + (\pi j/H)^2} \le \frac{L^2}{4\pi^2} \sum_{\mathbb{Z}\times\mathbb{N}} \frac{LH(2\pi n/L)^2|b_{nj}|^2}{(2\pi n/L)^2 + (\pi j/H)^2} + \sum_{j=1}^{\infty} \frac{LH|b_{0j}|^2}{(\pi j/H)^2}.$$

The first term on the right hand side is simply $\frac{L^2}{4\pi^2}\|\partial_x q\|_{-1}^2$ while the second satisfies

$$\sum_{j=1}^{\infty} \frac{LH|b_{0j}|^2}{(\pi j/H)^2} \le \frac{H^2}{\pi^2} \sum_{j=1}^{\infty} LH|b_{0j}|^2 = \frac{H^2}{\pi^2} \sum_{j=1}^{\infty} LH|a_{0j}|^2 \le \frac{H^2}{\pi^2}\|\partial_y q\|_{-1}^2, \qquad (3.26)$$

where the middle equality follows from the fact that $a_{00} = 0$. ∎

Corollary 3.4. *Suppose $q \in L^2_\#(R)$ and $\zeta \in L^\infty(T)$. Then for any aspect ratio H/L, we have*

$$\|\partial_y(\zeta q)\|_{-1}^2 \le C_2 M^2 \left(\|\partial_x q\|_{-1}^2 + \|\partial_y q\|_{-1}^2\right), \tag{3.27}$$

where $C_2 = 9$ and $M = \|\zeta\|_\infty$.

Proof. Since ζ does not depend on y, the Fourier coefficients of $\tilde{q} = \zeta q$ are related to the those of q via column-by-column convolution with the Fourier coefficients of ζ:

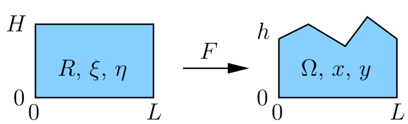
$$\tilde{a}_{nk} = \sum_{m\in\mathbb{Z}} \hat{\zeta}_{n-m} a_{mk}, \qquad \tilde{b}_{nj} = \sum_{m\in\mathbb{Z}} \hat{\zeta}_{n-m} b_{mj}, \qquad (n \in \mathbb{Z},\ k \ge 0,\ j > 0). \tag{3.28}$$

Since multiplication by ζ is bounded in $L^2(T)$ by M, convolution with $\hat{\zeta}$ is bounded in $\ell^2(\mathbb{Z})$ by M. Thus, by (3.4), we have

$$\|\partial_y(\zeta q)\|_{-1}^2 \le LH \sum_{k>0}\sum_{n\in\mathbb{Z}} |\tilde{a}_{nk}|^2 \le M^2 \left(LH \sum_{k>0}\sum_{n\in\mathbb{Z}} |a_{nk}|^2 \right). \tag{3.29}$$

The key point is that entries a_{nk} with $k = 0$ are absent from the right hand side. The quantity in parentheses may be written as $A_1 + A_2$ just as in (3.4), but omitting the $k = 0$ terms from A_1. Thus, it suffices to show that (3.12) holds with C_1 replaced by C_2 if the $k = 0$ term is omitted from the sum on the left. For $\nu \ge 1$, the result has already been proved without omitting this term. But for $\nu < 1$, we see that $C_1 = 0$ and $C_2 = 2$ suffice (since $\nu^2 \le k^2$ for $k \ge 1$). Thus $C_1 = C_2 = 9$ works for all $\nu > 0$, as claimed. ∎

4. Curved boundaries. We now perform a change of variables to transfer the result of Theorem 3.1 from the rectangle R to a domain Ω bounded on one side by a periodic, Lipschitz continuous function $h \in C^{0,1}(T)$:



$$x = \xi, \quad y = \frac{h(\xi)}{H}\eta, \quad dx\,dy = \frac{h(\xi)}{H}\,d\xi\,d\eta,$$

$$\xi = x, \quad \eta = \frac{H}{h(x)}y, \quad d\xi\,d\eta = \frac{H}{h(x)}\,dx\,dy,$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \frac{\eta}{h}h_x\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial y} = \frac{H}{h}\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} + \frac{y}{h}h_x\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial \eta} = \frac{h}{H}\frac{\partial}{\partial y}. \tag{4.1}$$

The main challenges involve avoiding lower order terms that have to be dealt with using Rellich's compactness theorem, balancing the sources of error to avoid excessive overestimation of the constants in the error bounds, and dealing with various subtleties of the dual space $H^{-1}(\Omega)$ such as the fact that if $p \in L^2(\Omega)$ and $\zeta \in L^\infty(\Omega)$ then $\|\zeta p\|_{-1}$ need not be smaller than $\|\zeta\|_\infty \|p\|_{-1}$. For clarity, we postpone the case that h is only Lipschitz continuous to Section 5 and begin with the simplifying assumption $h \in C^{1,1}(T)$. The aspect ratio of the rectangle R plays an essential role in the Lipschitz case but only a minor role (improving our estimate of β) here.

Theorem 4.1. *Suppose $h \in C^{1,1}(T)$ and $0 < h_0 \le h(x) \le h_1$ for $0 \le x \le L$. Then for every $p \in L^2_\#(\Omega)$ we have*

$$\big\|p\big\|_{0,\Omega} \le \beta^{-1}\big\|\nabla p\big\|_{-1,\Omega}, \qquad \beta^{-1} = \frac{9}{4}\big(1 + M^2\big)\left(\frac{h_1}{h_0}\right)^{1/2} \max\left(4, \frac{L}{h_0}, \frac{h_1}{h_0}\right), \tag{4.2}$$

where $M^2 = \max\Big(\|h_x\|_\infty^2, \|\tfrac{1}{2}hh_{xx}\|_\infty\Big)$.

REMARK 4.2. The quantity $\frac{1}{2}hh_{xx}$ arises naturally in the study of Reynolds' lubrication approximation and its higher order corrections on a periodic domain [12].

REMARK 4.3. In many practical applications, the aspect ratio L/h_0 is large while $M \ll 1$ and $h_1/h_0 \approx 1$; in this regime, (4.2) shows that β^{-1} scales linearly with L/h_0. If the geometry has a narrow gap so that $h_1/h_0 \gg 1$, we learn that β^{-1} depends on the gap size as $h_0^{-3/2}$. This dependence is shown to be optimal in Example 4.5 below. We do not know if the quadratic dependence on M is optimal; it seems to be an unavoidable artifact of changing variables to a rectangular geometry.

Proof of Theorem 4.1. The coordinate transformation $(x, y) = F(\xi, \eta)$ defined in (4.1) provides a one-to-one correspondence between functions $p \in L^2(\Omega)$, $u \in H_0^1(\Omega)$ and their counterparts $\tilde{p} = p \circ F \in L^2(R)$, $\tilde{u} = u \circ F \in H_0^1(R)$:

$$\tilde{p}(\xi,\eta) = p\left(\xi, \frac{h(\xi)}{H}\eta\right), \qquad \tilde{u}(\xi,\eta) = u\left(\xi, \frac{h(\xi)}{H}\eta\right). \tag{4.3}$$

F does not map $L^2_\#(\Omega)$ to $L^2_\#(R)$; however, the norm of $p \in L^2_\#(\Omega)$ does not decrease if we add a constant to p to enforce $\int_\Omega h^{-1}p\,dA = 0$ instead of $\int_\Omega p\,dA = 0$. By Theorem 3.1, this new p satisfies

$$\|p\|_{0,\Omega}^2 \le \left\|\left(\frac{h_1}{h}\right)^{1/2} p\right\|_{0,\Omega}^2 = \frac{h_1}{H}\|\tilde{p}\|_{0,R}^2 \le C_1\frac{h_1}{H}\|\partial_\xi\tilde{p}\|_{-1,R}^2 + C_2\frac{h_1}{H}\|\partial_\eta\tilde{p}\|_{-1,R}^2, \tag{4.4}$$

where $C_1 = \frac{9}{16}\max(16, \frac{L^2}{H^2})$ and $C_2 = 9$. But since the right hand side does not change when a constant is added to $\tilde{p}$, the original p also satisfies this equation (dropping the intermediate inequalities). We can relate the action of $\nabla_\xi\tilde{p}$ on $\tilde{\mathbf{u}}$ to that of $\nabla_x p$ on $\mathbf{u}$:

$$\langle\partial_\xi\tilde{p}, \tilde{u}\rangle_R = \left\langle\frac{H}{h}p, \Big(-\partial_x - \frac{y}{h}h_x\partial_y\Big)u\right\rangle_\Omega = H\left\langle\partial_x p, h^{-1}u\right\rangle_\Omega + H\left\langle\partial_y p, h_x\frac{y}{h^2}u\right\rangle_\Omega, \tag{4.5}$$

$$\langle\partial_\eta\tilde{p}, \tilde{v}\rangle_R = \left\langle\frac{H}{h}p, -\frac{h}{H}v_y\right\rangle_\Omega = \langle\partial_y p, v\rangle_\Omega, \tag{4.6}$$

where we used $\partial_x(h^{-1}) + \partial_y(yh^{-2}h_x) = 0$ in (4.5). If we had not introduced the factor of $h^{-1/2}$ in (4.4), this cancellation would not have occurred and the proof would become much more complicated; see Remark 4.4 below. It will be shown in Lemmas A.3, A.4 and A.5 that

$$H\|h^{-1}u\|_{a,\Omega} \le C_3\|\tilde{u}\|_{a,R}, \qquad C_3^2 = \max\left(3\frac{H}{h_0}, \big(1+3M^2\big)\frac{H^3}{h_0^3}\right), \tag{4.7}$$

$$H\left\|h_x\frac{y}{h^2}u\right\|_{a,\Omega} \le C_4\|\tilde{u}\|_{a,R}, \qquad C_4^2 = \max\left(8M^2\frac{H}{h_0}, \big(2M^2+6M^4\big)\frac{H^3}{h_0^3}\right), \tag{4.8}$$

$$\|v\|_{a,\Omega} \le C_5\|\tilde{v}\|_{a,R}, \qquad C_5^2 = \max\left(2\frac{h_1}{H}, \big(1+2M^2\big)\frac{H}{h_0}\right). \tag{4.9}$$

If h only belongs to $C^{0,1}(T)$, then (4.8) does not hold and we have to replace the last term in (4.5) by $\langle\partial_y(h_x p), Hyh^{-2}u\rangle_\Omega$, which requires a more difficult analysis; see Section 5 below. Combining (4.5)–(4.9), we obtain

$$\begin{gathered}\big|\langle\partial_\xi\tilde{p}, \tilde{u}\rangle_R\big| \le \Big(C_3\|\partial_x p\|_{-1,\Omega} + C_4\|\partial_y p\|_{-1,\Omega}\Big)\|\tilde{u}\|_{a,R},\\ \big|\langle\partial_\eta\tilde{p}, \tilde{v}\rangle_R\big| \le C_5\|\partial_y p\|_{-1,\Omega}\|\tilde{v}\|_{a,R}.\end{gathered} \tag{4.10}$$

It follows that

$$\|\partial_\xi \tilde{p}\|^2_{-1,R} \le 3C_3^2\|\partial_x p\|^2_{-1,\Omega} + \frac{3}{2}C_4^2\|\partial_y p\|^2_{-1,\Omega}, \quad \|\partial_\eta \tilde{p}\|^2_{-1,R} \le C_5^2\|\partial_y p\|^2_{-1,\Omega}, \quad (4.11)$$

which, together with (4.4), gives

$$\|p\|^2_{0,\Omega} \le \beta^{-2}\Big(\|\partial_x p\|^2_{-1,\Omega} + \|\partial_y p\|^2_{-1,\Omega}\Big), \quad \beta^{-2} = \frac{h_1}{H}\max\left(3C_1C_3^2,\ \frac{3}{2}C_1C_4^2 + C_2C_5^2\right).$$

Next, we choose $H = h_0$ so that

$$3C_1C_3^2 \le 9(1+M^2)C_1, \quad \frac{3}{2}C_1C_4^2 \le \left(12M^2 + 9M^4\right)C_1, \quad C_2C_5^2 \le 18\frac{h_1}{h_0} + 18M^2.$$

Finally, we observe that $\frac{h_1}{h_0} \le \frac{1}{4}\max\left(16, \frac{h_1^2}{h_0^2}\right)$ regardless of whether $\frac{h_1}{h_0} \ge 4$. As a result, $C_2C_5^2 \le (8+2M^2)\frac{9}{16}\max\left(16, \frac{h_1^2}{h_0^2}\right)$ and

$$\beta^{-2} \le \frac{h_1}{h_0}\max\left\{9(1+M^2),\ 8 + 14M^2 + 9M^4\right\}\frac{9}{16}\max\left(16, \frac{L^2}{h_0^2}, \frac{h_1^2}{h_0^2}\right), \quad (4.12)$$

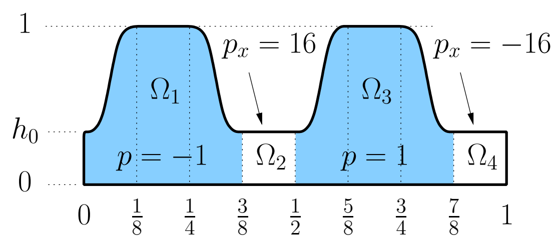
which yields (4.2) when we majorize the terms in braces by $9(1+M^2)^2$. ∎

Remark 4.4. One might hope to improve (4.2) by working directly with $\|p\|_0$ in (4.4) instead of via $\|h^{-1/2}p\|_0$. The main difference is that (4.5) acquires a lower order term

$$\left\langle \partial_\xi(h^{1/2}\tilde{p}), \tilde{u}\right\rangle_R = H\left\langle \partial_x p, h^{-1/2}u\right\rangle_\Omega + H\left\langle \partial_y p, y h_x h^{-3/2}u\right\rangle_\Omega + \frac{H}{2}\left\langle p, h_x h^{-3/2}u\right\rangle_\Omega$$

that would normally be dealt with by invoking a compactness argument to bound $\|p\|_{-1,\Omega}$ by a constant times $\|\nabla p\|_{-1,\Omega}$. This is not acceptable in the current calculation as this constant depends on Ω, and hence h. It is possible to bound $\|p\|_{-1,\Omega}$ in terms of $\|\tilde{p}\|_{-1,R}$ and then use Corollary 3.3. But the final step of bounding $\|\nabla_\xi \tilde{p}\|_{-1,R}$ by $\|\nabla_x p\|_{-1,\Omega}$ brings us back to the proof given above. The following example shows that the power of $h_0^{-3/2}$ in the formula (4.2) for β^{-1} is the best possible.

Example 4.5. Suppose $0 < h_0 < 1$ and consider a periodic function $h(x)$ that transitions smoothly and symmetrically between h_0 for $x \in [3/8, 1/2] \cup [7/8, 1]$ and 1 for $x \in [1/8, 1/4] \cup [5/8, 3/4]$. Let Ω_1, Ω_2, Ω_3, and Ω_4 be the regions under the curve h with $x \in [0, 3/8]$, $x \in [3/8, 1/2]$, $x \in [1/2, 7/8]$ and $x \in [7/8, 1]$, respectively. Let $p(x,y)$ be the continuous, piecewise linear function that equals -1 on Ω_1, 1 on Ω_3, and satisfies $p_x = \pm 16$, $p_y = 0$ on Ω_2 and Ω_4.



Then for any $u \in H_0^1(\Omega)$, we have $\left|\langle \partial_y p, u\rangle\right| = 0$ and

$$\begin{aligned}\left|\langle \partial_x p, u\rangle\right| &\le \int_{\Omega_2\cup\Omega_4} 16|u(x,y)|\,dA \le 16\sqrt{\text{area}(\Omega_2\cup\Omega_4)}\,\|u\|_{0,\Omega_2\cup\Omega_4} \\ &\le 8h_0^{1/2}\left(h_0/\sqrt{8}\right)\|u_y\|_{0,\Omega_2\cup\Omega_4} \le \sqrt{8}h_0^{3/2}\|u\|_{a,\Omega}, \quad (4.13)\end{aligned}$$

where we used the Cauchy-Schwarz and Poincaré-Friedrichs inequalities; see Lemma A.2. Thus $\|\nabla p\|_{-1,\Omega} \le \sqrt{8}h_0^{3/2}$ while $\|p\|_{0,\Omega} \ge 1/2$, showing that β^{-1} in (4.2) must be at least $(2\sqrt{8})^{-1}h_0^{-3/2}$, i.e. the power $h_0^{-3/2}$ is optimal.

5. Lipschitz boundaries. In this section we show how to modify the proof of Theorem 4.1 to handle the case that h only belongs to $C^{0,1}(T)$. The main difference is that $yh^{-2}h_x u$ no longer belongs to $H_0^1(\Omega)$ in (4.5), so a different strategy is required to deal with the term $\langle \partial_y p, Hyh^{-2}h_x u\rangle_\Omega$. The idea is to show that when $h^{-2}h_x$ is grouped with p, this term can be made small in comparison to the other two terms in (4.5) by choosing the aspect ratio of the rectangle R small enough. The loss of a power of $h_0^{1/2}$ in the estimate of β^{-1} when M is not small is discussed in Remark 5.3 below.

THEOREM 5.1. *Suppose $h \in C^{0,1}(T)$ and $0 < h_0 \le h(x) \le h_1$ for $0 \le x \le L$. Then for every $p \in L^2_\#(\Omega)$ we have*

$$\|p\|_{0,\Omega} \le \beta^{-1}\|\nabla p\|_{-1,\Omega}, \qquad \beta^{-1} = 2\max\left(4, \frac{L}{\sqrt{h_0h_1}}, 8\frac{L}{h_0}M\right)\max(1, 8M)\frac{h_1}{h_0}, \tag{5.1}$$

where $M = \|h_x\|_\infty$.

Proof. As before, (4.4) holds for all $p \in L^2_\#(\Omega)$:

$$\|p\|^2_{0,\Omega} \le C_1\frac{h_1}{H}\|\partial_\xi\tilde{p}\|^2_{-1,R} + C_2\frac{h_1}{H}\|\partial_\eta\tilde{p}\|^2_{-1,R}, \qquad C_1 = \frac{9}{16}\max\left(16, \frac{L^2}{H^2}\right), \quad C_2 = 9. \tag{5.2}$$

We now transform the problematic term in (4.5) back to the ξ, η coordinate system:

$$\langle f, \tilde{u}\rangle_R - \langle g_1, \tilde{u}\rangle_R := \langle \partial_\xi\tilde{p}, \tilde{u}\rangle_R - \langle \partial_\eta(h^{-1}h_x\tilde{p}), \eta\tilde{u}\rangle_R = \langle \partial_x p, Hh^{-1}u\rangle_\Omega, \tag{5.3}$$

$$\langle g, \tilde{v}\rangle_R := \langle \partial_\eta\tilde{p}, \tilde{v}\rangle_R = \langle \partial_y p, v\rangle_\Omega. \tag{5.4}$$

So we can bound $\|p\|_{0,\Omega}$ in terms of f and g and we can bound $(f - g_1)$ and g in terms of $\|\nabla_x p\|_{-1,\Omega}$; thus, we need a bridge from f to $(f - g_1)$ and g. By Corollary 3.4 and Lemma A.6,

$$|\langle g_1, \tilde{u}\rangle_R| \le \|\partial_\eta(h^{-1}h_x\tilde{p})\|_{-1,R}\|\eta\tilde{u}\|_{a,R} \le \left(3Mh_0^{-1}\|\nabla_\xi\tilde{p}\|_{-1,R}\right)\left(\frac{4}{3}H\|\tilde{u}\|_{a,R}\right),$$

$$\Rightarrow \qquad \|g_1\|^2_{-1,R} \le \theta^2\left(\|f\|^2_{-1,R} + \|g\|^2_{-1,R}\right), \qquad \theta = 4\frac{H}{h_0}M. \tag{5.5}$$

As a result, $\|f\|^2 \le 2\|f - g_1\|^2 + 2\theta^2(\|f\|^2 + \|g\|^2)$, which implies

$$\|f\|^2 \le 4\|f - g_1\|^2 + 4\theta^2\|g\|^2, \qquad (\theta^2 \le 1/4). \tag{5.6}$$

Equation (5.2) now becomes

$$\|p\|^2_{0,\Omega} \le 4C_1\frac{h_1}{H}\|f - g_1\|^2_{-1,R} + (4\theta^2C_1 + C_2)\frac{h_1}{H}\|g\|^2_{-1,R}, \qquad (HM \le h_0/8). \tag{5.7}$$

From (5.3) and (5.4) we see that

$$|\langle f - g_1, \tilde{u}\rangle| \le \|\partial_x p\|_{-1,\Omega}\|Hh^{-1}u\|_{a,\Omega}, \qquad |\langle g, \tilde{v}\rangle| \le \|\partial_y p\|_{-1,\Omega}\|v\|_{a,\Omega}. \tag{5.8}$$

By Lemmas A.3 and A.5 below, we then have

$$\|f-g_1\|_{-1,R}\le C_3\|\partial_x p\|_{-1,\Omega},\qquad C_3^2=\max\left(\frac{9}{8}\frac{H}{h_0},(1+16M^2)\frac{H^3}{h_0^3}\right), \tag{5.9}$$

$$\|g\|_{-1,R}\le C_5\|\partial_y p\|_{-1,\Omega},\qquad C_5^2=\max\left(\frac{9}{8}\frac{h_1}{H},(1+9M^2)\frac{H}{h_0}\right). \tag{5.10}$$

It follows from (5.7) that

$$\big\|p\big\|_{0,\Omega}^2\le\beta^{-2}\Big(\big\|\partial_x p\big\|_{-1,\Omega}^2+\big\|\partial_y p\big\|_{-1,\Omega}^2\Big),\quad \beta^{-2}=\frac{h_1}{H}\max\left(4C_1C_3^2,\,(4\theta^2C_1+C_2)C_5^2\right).$$

Finally, we choose $H=\min\left(h_0,\frac{1}{8M}h_0\right)$ so that if $M\ge 1/8$ we have

$$4\frac{h_1}{H}C_3^2\le\max\left(\frac{9}{2},\frac{4}{64M^2}+1\right)\frac{h_1}{h_0}\le 5\frac{h_1}{h_0},\quad \frac{h_1}{H}C_5^2\le\max\left(72\frac{h_1^2}{h_0^2}M^2,(1+9M^2)\frac{h_1}{h_0}\right)\le 73M^2\frac{h_1^2}{h_0^2}$$

and if $M\le 1/8$ we have

$$4\frac{h_1}{H}C_3^2\le\max\left(\frac{9}{2},4+64M^2\right)\frac{h_1}{h_0}\le 5\frac{h_1}{h_0},\quad \frac{h_1}{H}C_5^2\le\max\left(\frac{9}{8}\frac{h_1^2}{h_0^2},(1+9M^2)\frac{h_1}{h_0}\right)\le\frac{73}{64}\frac{h_1^2}{h_0^2}.$$

Moreover, $C_1=\max\left(9,\frac{9}{16}\frac{L^2}{h_0^2},36\frac{L^2}{h_0^2}M^2\right)$ and $4\theta^2C_1+C_2\le 2\max\left(9,36\frac{L^2}{h_0^2}M^2\right)$ regardless of whether $M\le 1/8$. Combining these results, we obtain

$$\begin{aligned}4C_1\frac{h_1}{H}C_3^2&\le 5\max\left(9,\frac{9}{16}\frac{L^2}{h_0^2},36\frac{L^2}{h_0^2}M^2\right)\frac{h_1}{h_0},\\ (4\theta^2C_1+C_2)\frac{h_1}{H}C_5^2&\le\frac{73}{32}\max\left(9,36\frac{L^2}{h_0^2}M^2\right)\max(1,64M^2)\frac{h_1^2}{h_0^2}.\end{aligned} \tag{5.11}$$

Formula (5.1) for β^{-1} follows by taking the square root of the maximum of these expressions after increasing the constants and consolidating terms. ∎

Remark 5.2. Inequality (5.5) is the key to this proof. For fixed u, both $\langle f,\tilde u\rangle$ and $\langle g_1,\tilde u\rangle$ in (5.3) scale like H while $\langle g,\tilde u\rangle$ in (5.4) is independent of H. Because of the way $\|\tilde u\|_{a,R}$ depends on H, it follows that if $R_1=T\times H_1$, $R_2=T\times H_2$, and $H_1<H_2$, then

$$\|g_1\|_{-1,R_1}^2\le\frac{H_1}{H_2}\|g_1\|_{-1,R_2}^2,\quad \|f\|_{-1,R_1}^2\le\frac{H_1}{H_2}\|f\|_{-1,R_2}^2,\quad \|g\|_{-1,R_1}^2\le\frac{H_2}{H_1}\|g\|_{-1,R_2}^2.$$

Thus, $\|g_1\|^2$ and $\theta^2\|g\|^2$ are both $O(H)$ quantities and the surprising aspect of (5.5) is that the $O(H^3)$ term $\theta^2\|f\|^2$ is sufficient to help $\theta^2\|g\|^2$ bound $\|g_1\|^2$.

Remark 5.3. We believe the optimal bound in the Lipschitz case should scale like $\beta^{-1}\sim h_0^{-3/2}$, just as in the $C^{1,1}$ case; however, proving this would require eliminating (or at least finding a better bound for) the cross term $4\theta^2C_1\frac{h_1}{H}\|g\|_{-1,R}^2$ in (5.7). As it stands, θ^2C_1 and $H^{-1}\|g\|^2$ each contribute a factor of h_0^{-2} to this cross term due to the requirement $HM\le h_0/8$, which yields $\beta^{-2}\sim h_0^{-4}$. We suspect that the functions p that require C_1 to diverge as $H\to 0$ are distinct from the functions p for which $\|f-g_1\|\ll\|f\|$ in (5.6), but we have not found a way to make this idea rigorous.

Appendix A. Useful Lemmas. In this section we gather several results that are either elementary but used frequently in our proofs or are tedious and distract from the main argument.

Lemma A.1. *Suppose $\gamma_1,\ \ldots,\ \gamma_n$ are positive real numbers such that $\sum_1^n\gamma_j^{-1}\le 1$. Then $|w_1+\cdots+w_n|^2\le\sum_{j=1}^n\gamma_j|w_j|^2$, for all $w\in\mathbb{C}^n$.*

Proof. This is a consequence of the Cauchy-Schwarz inequality:

$$\Big|\sum_j w_j\Big|^2 = \Big|\sum_j \left(\gamma_j^{-1/2}\right)\left(\gamma_j^{1/2}w_j\right)\Big|^2 \le \left(\sum_j \gamma_j^{-1}\right)\left(\sum_j \gamma_j|w_j|^2\right). \tag{A.1}$$

□

LEMMA A.2. *(Poincaré-Friedrichs inequality). If Ω has the geometry of Figure 2.1 with $h \in C^{0,1}(T)$ and if R is the x-periodic rectangle of width L and height H, then*

$$\|u\|_{0,\Omega}^2 \le \frac{h_1^2}{8}\|u_y\|_{0,\Omega}^2, \qquad \|\tilde u\|_{0,\Omega}^2 \le \frac{H^2}{\pi^2}\|\tilde u_y\|_{0,R}^2, \qquad \Big(u \in H_0^1(\Omega),\ \tilde u \in H_0^1(R)\Big). \tag{A.2}$$

The former inequality also works over the subregion $\Omega_2 \cup \Omega_4$ in Example 4.5 with h_1 replaced by the maximum height of that subregion, namely h_0.

Proof. The latter inequality follows by expanding $\tilde u = \sum d_{nj}\sqrt{2}\exp(\frac{2\pi i n x}{L})\sin\frac{\pi j y}{H}$ and comparing the formulas for $\|u\|_{0,R}^2$ and $\|u_y\|_{0,R}^2$. If $u \in C_c^1(\Omega)$, the former inequality follows by integrating

$$\begin{aligned} |u(x,y)|^2 &\le \Big|\int_0^y u_y(x,y')\,dy'\Big|^2 \le y\int_0^{h/2}|u_y(x,y')|^2\,dy', && \big(0\le y\le \tfrac12 h(x)\big)\\ |u(x,y)|^2 &\le \Big|\int_y^h u_y(x,y')\,dy'\Big|^2 \le (h-y)\int_{h/2}^h |u_y(x,y')|^2\,dy', && \big(\tfrac12 h(x)\le y\le h(x)\big)\end{aligned}$$

over the lower and upper halves of Ω, respectively, and combining the results. The result for $u \in H_0^1(\Omega)$ then follows by a standard density argument. □

LEMMA A.3. *Suppose $h \in C^{0,1}(T)$, $R = T \times H$ and $u \in H_0^1(\Omega)$. Then*

$$\big\|Hh^{-1}u\big\|_{a,\Omega}^2 \le C_3^2\|\tilde u\|_{a,R}^2, \tag{A.3}$$

where $\tilde u$ expresses u in the ξ, η coordinate system of R and C_3^2 is given by (4.7) or (5.9).

Proof. Using the change of variables formulas (4.1) and (4.3), we obtain

$$\begin{aligned}\big\|\partial_x(Hh^{-1}u)\big\|_{0,\Omega}^2 &= \int_R H^2\left(-h^{-2}h_x\tilde u + h^{-1}\tilde u_\xi - h^{-1}\frac{\eta}{h}h_x\tilde u_\eta\right)^2\frac{h}{H}\,d\xi\,d\eta\\ &\le \gamma_1 H h_0^{-3}M^2\|\tilde u\|_{0,R}^2 + \gamma_2 H h_0^{-1}\|\tilde u_\xi\|_{0,R}^2 + \gamma_3 H^3 h_0^{-3}M^2\|\tilde u_\eta\|_{0,R}^2,\end{aligned} \tag{A.4}$$

$$\big\|\partial_y(Hh^{-1}u)\big\|_{0,\Omega}^2 = \int_R \frac{H^2}{h^2}\Big(\frac{H}{h}\tilde u_\eta\Big)^2\frac{h}{H}\,d\xi\,d\eta \le H^3h_0^{-3}\|\tilde u_y\|_{0,R}, \tag{A.5}$$

where $M = \|h_x\|_\infty$, $h_0 = \min_{0\le x\le L} h(x)$, and $\gamma_1^{-1}+\gamma_2^{-1}+\gamma_3^{-1} \le 1$. Combining these and using the Poincaré-Friedrichs inequality (with 9 instead of π^2), we find that

$$\big\|Hh^{-1}u\big\|_{a,\Omega}^2 \le \max\left(\gamma_2\frac{H}{h_0}, \left(1+\left(\gamma_3+\frac{\gamma_1}{9}\right)M^2\right)\frac{H^3}{h_0^3}\right)\|\tilde u\|_{a,R}^2, \tag{A.6}$$

which yields (4.7) with $\vec\gamma = (9,3,2)$ and (5.9) with $\vec\gamma = (36, 9/8, 12)$. □

LEMMA A.4. *Suppose $h \in C^{1,1}(T)$, $R = T \times H$ and $u \in H_0^1(\Omega)$. Then*

$$\big\|Hh^{-2}yh_xu\big\|_{a,\Omega}^2 \le C_4^2\|\tilde u\|_{a,R}^2, \qquad C_4^2 = \max\left(8M^2\frac{H}{h_0},\ \big(2M^2+6M^4\big)\frac{H^3}{h_0^3}\right), \tag{A.7}$$

where $M^2 = \max\Big(\big\|h_x\big\|_\infty^2, \big\|\frac{1}{2}hh_{xx}\big\|_\infty\Big)$.

Proof. Using the change of variables formulas (4.1) and (4.3) as well as the Poincaré-Friedrichs inequality, we obtain

$$\begin{aligned}
\Big\|\partial_x\Big(\frac{Hyh_xu}{h^2}\Big)\Big\|_{0,\Omega}^2 &= \int_R \Big[-2\frac{\eta h_x^2}{h^2}\tilde{u} + 2\frac{\eta}{h^2}\Big(\frac{1}{2}hh_{xx}\Big)\tilde{u} + \frac{\eta h_x}{h}\Big(\tilde{u}_\xi - \frac{\eta h_x}{h}\tilde{u}_\eta\Big)\Big]^2 \frac{h}{H}\,d\xi\,d\eta \\
&\le 4(\gamma_1+\gamma_2)\frac{HM^4}{h_0^3}\|\tilde{u}\|_{0,R}^2 + \gamma_3\frac{HM^2}{h_0}\|\tilde{u}_\xi\|_{0,R}^2 + \gamma_4\frac{H^3M^4}{h_0^3}\|\tilde{u}_\eta\|_{0,R}^2, \\
&\le \gamma_3\frac{HM^2}{h_0}\|\tilde{u}_\xi\|_{0,R}^2 + \Big(\gamma_4 + \frac{4}{\pi^2}(\gamma_1+\gamma_2)\Big)\frac{H^3M^4}{h_0^3}\|\tilde{u}_\eta\|_{0,R}^2 \qquad \text{(A.8)}
\end{aligned}$$

$$\begin{aligned}
\Big\|\partial_y\Big(\frac{Hyh_xu}{h^2}\Big)\Big\|_{0,\Omega}^2 &= \int_R \Big(\frac{Hh_x}{h^2}\tilde{u} + \frac{H\eta h_x}{h^2}\tilde{u}_\eta\Big)^2\frac{h}{H}\,d\xi\,d\eta \qquad \text{(A.9)} \\
&\le \delta_1\frac{HM^2}{h_0^3}\|\tilde{u}\|_{0,R}^2 + \delta_2\frac{H^3M^2}{h_0^3}\|\tilde{u}_\eta\|_{0,R}^2 \le \Big(\frac{\delta_1}{9}+\delta_2\Big)\frac{H^3M^2}{h_0^3}\|\tilde{u}_\eta\|_{0,R}^2,
\end{aligned}$$

where $\sum_1^4 \gamma_j^{-1} \le 1$ and $\delta_1^{-1} + \delta_2^{-1} \le 1$. Now we set $\vec{\gamma} = \big(\frac{3}{8}\pi^2, \frac{3}{8}\pi^2, 8, 3\big)$ and $\vec{\delta} = \big(\frac{9}{2}, \frac{3}{2}\big)$ to obtain (A.7). ∎

LEMMA A.5. *Suppose* $h \in C^{0,1}(T)$, $R = T \times H$ *and* $v \in H_0^1(\Omega)$. *Then* $\|v\|_{a,\Omega} \le C_5\|\tilde{v}\|_{a,R}$ *with* C_5 *given by (4.9) or (5.10).*

Proof. Let $M = \|h_x\|_\infty$. For any γ_1, γ_2 satisfying $\gamma_1^{-1} + \gamma_2^{-1} \le 1$, we have

$$\|\partial_x v\|_{0,\Omega}^2 = \int_R \Big(\tilde{v}_\xi - \frac{\eta h_x}{h}\tilde{v}_\eta\Big)^2\frac{h}{H}\,d\xi\,d\eta \le \gamma_1\frac{h_1}{H}\|\tilde{v}_\xi\|_{0,R}^2 + \gamma_2\frac{HM^2}{h_0}\|\tilde{v}_\eta\|_{0,R}^2, \qquad \text{(A.10)}$$

$$\|\partial_y v\|_{0,\Omega}^2 = \int_R \Big(\frac{H}{h}\tilde{v}_\eta\Big)^2\frac{h}{H}\,d\xi\,d\eta \le \frac{H}{h_0}\|\tilde{v}_\eta\|_{0,R}^2. \qquad \text{(A.11)}$$

It follows that $\|v\|_{a,\Omega} \le C_5\|\tilde{v}\|_{a,R}$ with $C_5 = \max\big(\gamma_1\frac{h_1}{H}, (1+\gamma_2M^2)\frac{H}{h_0}\big)$. We obtain (4.9) using $\vec{\gamma} = (2,2)$ and (5.10) using $\vec{\gamma} = (9/8, 9)$. ∎

LEMMA A.6. *On the* ξ*-periodic rectangle* R,

$$\|\eta\tilde{u}\|_a^2 \le \frac{16}{9}H^2\|\tilde{u}\|_a^2, \qquad \big(\tilde{u} \in H_0^1(R)\big). \qquad \text{(A.12)}$$

Proof. Using the Poincaré-Friedrichs inequality, we find that

$$\|\partial_\xi(\eta\tilde{u})\|_{0,R}^2 \le H^2\|\tilde{u}_\xi\|_{0,R}^2, \qquad \|\partial_\eta(\eta\tilde{u})\|_{0,R}^2 \le \Big(\frac{\gamma_1}{9}+\gamma_2\Big)H^2\|\tilde{u}_\eta\|_{0,R}^2 \qquad \text{(A.13)}$$

provided $\gamma_1^{-1} + \gamma_2^{-1} \le 1$. Choosing $\vec{\gamma} = (4, 4/3)$, the result follows. ∎

Appendix B. The Poincaré-Friedrichs inequality on $H_\#^1(\Omega)$. In this section we present a simple proof of the Poincaré-Friedrichs inequality for H^1 functions with zero mean. Our proof does not rely on Rellich's compactness theorem, but does require the boundary of Ω to be the graph of a Lipschitz continuous function h; see Figure 2.1 above. The main difference between the estimates

$$\|p\|_{1,\Omega} \le K\|\nabla p\|_{0,\Omega}, \quad \big(p \in H_\#^1(\Omega)\big), \qquad \|p\|_{0,\Omega} \le \beta^{-1}\|\nabla p\|_{-1,\Omega}, \quad \big(p \in L_\#^2(\Omega)\big) \qquad \text{(B.1)}$$

proved below and in Theorems 4.1 and 5.1 above is that $K \sim h_0^{-1/2}$ while $\beta^{-1} \sim h_0^{-3/2}$; (we were only able to prove $\beta^{-1} \sim h_0^{-2}$ in the Lipschitz case). A narrow gap causes K

to grow because a large gradient of p in the gap region can lead to a large change in p across the gap with relatively little cost (in terms of $\|\nabla p\|_{0,\Omega}$) due to the small area of the gap region. The effect on β^{-1} is more severe than on K because, in addition to the small area of the gap region, the test functions (u,v) that ∇p acts on belong to $H_0^1(\Omega)^2$, i.e. they are zero on Γ_0 and Γ_1. These boundary conditions cause u and v to be small in the gap region, which reduces their ability to penalize large gradients of p there. This was illustrated in Example 4.5 above.

To keep the equations dimensionally correct, we define the norm on $H^1_\#(\Omega)$ to be

$$\|p\|_{1,\Omega}^2 = L^{-2}\|p\|_{0,\Omega}^2 + \|p\|_{a,\Omega}^2 = \int_\Omega \frac{|p|^2}{L^2} + |p_x|^2 + |p_y|^2\,dx\,dy, \tag{B.2}$$

i.e. we use L as a length scale to compare $\|p\|_0$ to $\|p\|_a = \|\nabla p\|_0$.

THEOREM B.1. *Suppose $h \in C^{0,1}(T)$ and $0 < h_0 \le h(x) \le h_1$ for $0 \le x \le L$. Then for every $p \in H^1_\#(\Omega)$, we have*

$$L^{-1}\|p\|_{0,\Omega} \le C\|\nabla p\|_{0,\Omega}, \qquad C = \frac{1+M}{2\pi}\max\left(1, 2\frac{\sqrt{h_0h_1}}{L}\right)\sqrt{\frac{h_1}{h_0}}, \tag{B.3}$$

where $M = \|h_x\|_\infty$. The constant K in (B.1) is given by $K = (1+C^2)^{1/2}$.

Proof. On the ξ-periodic rectangle $R = T \times (0,H)$, the expansion

$$\tilde p(\xi,\eta) = \sum_{n\in\mathbb{Z}}\left(a_{n0} + \sum_{k=1}^\infty a_{nk}\sqrt{2}\cos\frac{\pi k\eta}{H}\right)e^{\frac{2\pi i n\xi}{L}} \qquad \big(\tilde p \in H^1(R)\big) \tag{B.4}$$

can be differentiated term by term and we have

$$\|\tilde p\|_{0,R}^2 = \sum_{n,k} LH|a_{nk}|^2, \qquad \|\nabla\tilde p\|_{0,R}^2 = \sum_{n,k} LH\left[\Big(\frac{2\pi n}{L}\Big)^2 + \Big(\frac{\pi k}{H}\Big)^2\right]|a_{nk}|^2. \tag{B.5}$$

Assuming $\tilde p \in H^1_\#(R)$, i.e. $a_{00} = 0$, we learn that

$$\|\tilde p\|_{0,R}^2 \le L^2\widetilde{C}^2\|\nabla\tilde p\|_{0,R}^2, \qquad L^2\widetilde{C}^2 = \max\left\{\Big(\frac{L}{2\pi}\Big)^2, \Big(\frac{H}{\pi}\Big)^2\right\}. \tag{B.6}$$

Now we transfer this result to Ω by the change of variables (4.1) and (4.3). To avoid Rellich's theorem, we estimate

$$\|p\|_{0,\Omega}^2 \le \left\|\Big(\frac{h_1}{h}\Big)^{1/2}p\right\|_{0,\Omega}^2 = \frac{h_1}{H}\|\tilde p\|_{0,R}^2 \le L^2\widetilde{C}^2\frac{h_1}{H}\|\nabla\tilde p\|_{0,R}^2. \tag{B.7}$$

This inequality holds for all p such that $\tilde p \in H^1_\#(R)$. Arguing as in (4.4), we find that if we drop the intermediate inequalities, (B.7) also holds for $p \in H^1_\#(\Omega)$. Next, we bound $\|\nabla\tilde p\|_{0,R}$ in terms of $\|\nabla p\|_{0,R}$:

$$\|\tilde p_\xi\|_{0,R}^2 = \int_\Omega \Big(p_x + \frac{y}{h}h_xp_y\Big)^2\frac{H}{h}\,dx\,dy \le \gamma_1\frac{H}{h_0}\|p_x\|_{0,\Omega}^2 + \gamma_2M^2\frac{H}{h_0}\|p_y\|_{0,\Omega}^2 \tag{B.8}$$

$$\|\tilde p_\eta\|_{0,R}^2 = \int_\Omega \Big(\frac{h}{H}p_y\Big)^2\frac{H}{h}\,dx\,dy \le \frac{h_1}{H}\|p_y\|_{0,\Omega}^2, \tag{B.9}$$

where $\gamma_1^{-1} + \gamma_2^{-1} = 1$ and $M = \|h_x\|_\infty$. It follows that

$$L^{-2}\|p\|_{0,\Omega}^2 \le C^2\|\nabla p\|_{0,\Omega}^2, \qquad C^2 = \widetilde{C}^2\frac{h_1}{H}\max\Big(\gamma_1\frac{H}{h_0}, \frac{h_1}{H} + \gamma_2M^2\frac{H}{h_0}\Big). \tag{B.10}$$

Next, we choose $H = \sqrt{h_0 h_1}$ and minimize $\max\big(\gamma_1, 1 + \gamma_2 M^2\big)$ over all choices of γ_j such that $\gamma_1^{-1} + \gamma_2^{-1} = 1$. The result is

$$\gamma_1 = 1 + \gamma_2 M^2 = \frac{1}{4}\big(\sqrt{M^2+4} + M\big)^2 \le (1+M)^2, \tag{B.11}$$

which yields $C^2 = \frac{1}{4\pi^2}\max\left(1, 4\frac{h_0 h_1}{L^2}\right)\frac{h_1}{h_0}(1+M)^2$ as claimed. □

Remark B.2. Example 4.5 shows that the scaling $C \sim h_0^{-1/2}$ is optimal: for that function p, we have

$$L^{-2}\|p\|_{0,\Omega}^2 \ge L^{-2} h_1 \frac{L}{4} = \frac{h_1}{256 h_0}\left(h_0 \frac{L}{4}\Big(\frac{16}{L}\Big)^2\right) \ge \frac{h_1}{256 h_0}\|p_x\|_{0,\Omega}^2, \tag{B.12}$$

which shows that C in (B.3) is at least $\frac{1}{16}\sqrt{\frac{h_1}{h_0}}$. We do not know if the linear dependence of C on M is optimal — it seems to be an unavoidable artifact of changing variables to a rectangular geometry.

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