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1. Introduction. The Duffing Problem. The motion of a mass under the influence of a nonlinear restoring force and a periodic externally-applied force is described by Duffing's differential equation (Duffing, [1]). We restrict our attention to the case where no damping is present and write the differential equation in the following form:

$$\gamma \cdot y''(\theta) + y(\theta) + g(y(\theta)) = f(\theta). \tag{1}$$

We further focus our attention on the case where $y(\theta)$ has a representation as a sum of odd cosines; this restriction introduces the boundary conditions

$$y(\pi/2) = 0, \qquad y'(0) = 0. \tag{2}$$

In the formulation most often investigated (Duffing, [1]; Heinbockel, Struble [2]), eq. (1) is written with $g(y(\theta)) = \beta \cdot y^3(\theta)$ and $f(\theta) = \cos\theta$. We shall consider this case in detail as a specific example, but many of the results developed here apply equally well in the more general case so long as $f(\theta)$ and $g(y(\theta))$ can be represented as sums of odd cosines, and subject to some restrictions on g' which we will encounter as we proceed and which are summarized among the hypotheses of Theorem 1. Note especially that we are not restricting our attention to the "weakly nonlinear" case: that is, we are not assuming that g is "small".

2. Perturbation analysis when γ is small: High-order resonance. We seek to determine the behavior of solutions to eqs. (1, 2) when γ is small: in particular, the relation of such solutions to the solution for $\gamma = 0$. When $\gamma = 0$, we have the "reduced equation" (which is not

even a differential equation), whose solution we call y_o:

(3) $$y_o(\theta) + g(y_o(\theta)) = f(\theta).$$

Linear perturbation of eq. (1) about y_o produces the following "variational equation," which is a special case of Hill's equation:

(4) $$\gamma \cdot z''(\theta) + (1 + g'(y_o(\theta))) \cdot z(\theta) = 0; \quad (0 \leqq \theta \leqq \pi/2)$$
$$z(\pi/2) = 0, \qquad z'(0) = 0.$$

We summarize certain results obtained by applying standard methods (Magnus, Winkler [3], p. 51) to this case. Let

(5) $$M^4(\theta) = 1 + g'(y_o(\theta)). \qquad (0 \leqq \theta \leqq \pi/2)$$

Assume that $M^4 > 0$ on $(0, \pi/2)$. Then, the Liouville transformation, normalized so that $t(0) = 0$ and $t(\pi/2) = \pi/2$, is

(6) $$t(\theta) = (1/C) \int_0^\theta M^2(s)\, ds;$$
$$C = (2/\pi) \int_0^{\pi/2} M^2(\theta)\, d\theta.$$

This transformation brings the differential equation into the canonical Sturm-Liouville form

(7) $$w''(t) + (C^2/\gamma + q(t)) \cdot w(t) = 0; \qquad (0 \leqq t \leqq \pi/2)$$
$$w(\pi/2) = 0, \qquad w'(0) = 0,$$

where

(8) $$w(t) = M(\theta(t)) \cdot z(\theta(t)),$$
$$q(t) = (1/M(\theta(t))) \frac{d^2}{dt^2} M(\theta(t)).$$

If q is bounded on $(0, \pi/2)$ by $k_1 \leq q(t) \leq k_2$, then we may apply the Comparison Theorem (Ince [4], p. 232) with the boundary condition $\cos (C^2/\lambda_n + k^2)^{\frac{1}{2}} \cdot (\pi/2) = 0$ to show that the values $\gamma = \lambda_n$, at which eq. (4) has a non-trivial solution, satisfy the conditions

$$(9) \qquad (2n - 1)^2 - k_2 \leq C^2/\lambda_n \leq (2n - 1)^2 - k_1,$$

and as $n \to \infty$ we have the asymptotic behavior $\lambda_n \sim C^2/4 n^2$.

The standard "singular perturbation" analysis for problems of this type, where the order of the differential equation changes by more than one at the limiting value of a parameter, is given by Wasow [5]. However, Wasow's "assumption C" (page 322) requires that $\sqrt{(-M^4)}$ must never be purely imaginary. This assumption is directly contradictory to our $M^4 > 0$, which is the essential condition for the existence of real eigenvalues. In fact, it should not be considered surprising that "smooth" behavior as $\gamma \to 0$ may be incompatible with the existence of arbitrarily small values of γ at which the variational equation has a non-trivial solution.

The phenomenon called "nonlinear resonance" is associated with the existence of these non-trivial solutions of the variational equation. A qualitative description of the difference between this phenomenon and ordinary (linear) resonance is given by Stoker [6] (see page 88) for some examples including the Duffing problem with γ near 1. Resonance when γ is small (compare Hayashi [7], page 35) is associated with large amplitudes in the higher harmonics of y, even though f may consist only of the single "base" frequency; hence, in this case, the phenomenon is often called "superharmonic" or "higher-order" nonlinear resonance. Actually, in this paper, we are interested in locating these regions of high-order resonance only in order to avoid them.

From eq. (9), we derive the following lower bound for the separation between adjacent eigenvalues:

(10) $$\lambda_n - \lambda_{n+1} \geqq C^2(8n + k_1 - k_2)/((2n - 1)^2 - k_2)((2n + 1)^2 - k_1),$$

which is positive for sufficiently large n and asymptotically $(\lambda_n - \lambda_{n+1}) \sim C^2/2\, n^3$. Thus, if we choose γ_n halfway between λ_n and λ_{n+1}, we have a lower bound (which depends upon n, and is asymptotically $C^2/4\, n^3$) for the distance of γ_n from its nearest eigenvalue. Subject to the assumptions used in establishing these bounds we shall use them to demonstrate (see Theorem 1) that a solution to the Duffing problem exists for each γ_n sufficiently far out in this sequence and, furthermore, that these solutions tend to the solution y_0 of the reduced equation, eq. (3), as $\gamma_n \to 0$.

3. <u>The Duffing problem in operator form.</u> We now find it necessary to introduce some notational conventions.

Let α be the interval $(0, \pi/2)$ and let E_α be the following subset of $\mathcal{L}^2(\alpha)$:

(11) $$E_\alpha = \{y_E \,\varepsilon\, \mathcal{L}^2(\alpha) : \; y_E(\pi/2) = 0,\; y_E'(0) = 0\}.$$

Then E_α is a Banach space, with $\|y_E\| = (\int_\alpha y_E^2)^{\frac{1}{2}}$. Let the operators G_E, D_E^2, I_E, and finally P_E be defined on E_α as follows:

(12a) $$(G_E\, y_E)(\theta) = y_E(\theta) + g(y_E(\theta)) - f(\theta);$$

(12b) $$(D_E^2\, y_E)(\theta) = (-\, d^2/d\theta^2)\, y_E(\theta); \qquad (\theta \,\varepsilon\, \alpha)$$

(13) $$P_E = G_E - \gamma \cdot D_E^2$$

Then the Duffing problem may be written as: $P_E\, y_E = 0$.

Subscripts in this notation will be used to indicate the domain of an operator or function, and we will use a superscript with number sign as an index; that is, to specify a particular element of a space or of a set of operators; for example, $y^{\#1}$, $P^{\#2}$, etc. The superscript (#0) will be reserved to indicate the solution of the reduced equation, eq. (3):

(14)
$$G_E\, y_E^{\#0} = 0;$$
$$P_E\, y_E^{\#0} = -\gamma\, D_E^2\, y_E^{\#0}.$$

We also need to be able to discuss the first derivative of $y_E \in E_\alpha$. Although, the second derivative, which appears in eq. (1), causes no special difficulty, the first derivative of our odd cosine series is a series of odd sines. These satisfy a different set of boundary conditions and, hence, lie in a different space. We call this space F_α and we define the operators D_{FE} and D_{EF} as follows:

(15a)
$$(D_{FE}\, y_E)(\theta) = (-\, d/d\theta)\, y_E(\theta), \qquad (y_E \in E_\alpha)$$
$$(D_{EF}\, x_F)(\theta) = (d/d\theta)\, x_F(\theta). \qquad (x_F \in F_\alpha)$$

Now $(D_{FE}\, y_E) \in F_\alpha$ and $(D_{EF}\, x_F) \in E_\alpha$, so $D_E^2 = D_{EF} \circ D_{FE}$. The inverses of these operators are the integral operators $J_{EF} = (D_{FE})^{-1}$, $J_{FE} = (D_{EF})^{-1}$, and $J_{EF} \circ J_{FE} = (D_E^2)^{-1}$, where

(15b)
$$(J_{FE}\, y_E)(\theta) = \int_{\pi/2}^{\theta} y_E(s)\, ds, \qquad (y_E \in E_\alpha)$$
$$(J_{EF}\, x_F)(\theta) = -\int_0^{\theta} x_F(s)\, ds. \qquad (x_F \in F_\alpha)$$

4. <u>Normal and self-adjoint operators</u>. Let us review some properties of linear operators on $E_\alpha \subset \mathcal{L}^2(\alpha)$. We define the norm of a linear operator H_E as

(16a) $$\|H_E\| = \sup \{\|H_E\, y_E\|/\|y_E\| \ : \ y_E \in E_\alpha\}.$$

The adjoint of H_E is the operator $H_E{}^*$ such that for any $y_E^{\#1}$, $y_E^{\#2}$ in E_α,

(16b) $$\int_{\theta\epsilon\alpha} (H_E\, y_E^{\#1}(\theta)) \cdot y_E^{\#2}(\theta)\, d\theta = \int_{\theta\epsilon\alpha} y_E^{\#1}(\theta) \cdot (H_E{}^*\, y_E^{\#2}(\theta))\, d\theta.$$

We know that $\|H_E\|^2$ is equal to the l.u.b. of the eigenvalues of $H_E \circ H_E{}^*$ and $\|H_E\|$ is not less than the l.u.b. of the eigenvalues of H_E. If H_E is a normal operator; that is, $H_E \circ H_E{}^* = H_E{}^* \circ H_E$, then $\|H_E\|$ is equal to the l.u.b. of its eigenvalues and so, in this case, we can bound $\|H_E\|$ if we can bound the eigenvalues. We shall be especially interested in self-adjoint operators; that is, normal operators in the special case $H_E{}^* = H_E$.

A linear operator of the form

(16c) $$(H_E\, y_E)(\theta) = r_E(\theta) \cdot y_E(\theta)$$

is obviously self-adjoint. Furthermore, if H_E is a self-adjoint operator on E_α, then $J_{FE} \circ H_E \circ J_{EF}$ is self-adjoint on F_α. For, any linear operator H_E has the following effect:

(16d) $$H_E(\cos\,(2j - 1)\,\theta) = \Sigma_k\ h_{jk} \cos\,(2k - 1)\,\theta \qquad j = 1, 2, \ldots$$

and the self-adjoint of H_E is equivalent to the condition $h_{jk} = h_{kj}$ for all j, k. But we may see that this same condition on H_E is exactly what is needed to show that the composite operator is self-adjoint on F_α.

5. <u>Kantorovich's Theorem: Solutions of an operator equation in a Banach space</u>. The following theorem is given by Kantorovich [ref. 8, p. 708].

Hypotheses:

P is defined on a neighborhood $S(y^{\#0}, \rho)$ in a Banach space F.

The linear operator $\Gamma^{\#0} = (\mathcal{D}P(y^{\#0}))^{-1}$ exists, where $\mathcal{D}P(y^{\#0})$ is the Frechet derivative (see Dieudonne [9], p. 141) of P at $y^{\#0}$.

(17a) $$\|\Gamma^{\#0} \circ P\, y^{\#0}\| \leq \eta$$

(17b) $$\|\Gamma^{\#0} \circ \mathcal{D}^2 P\, (y^{\#\rho})\| \leq K \text{ for all } y^{\#\rho} \varepsilon\ S(y^{\#0}, \rho)$$

(17c) $$h = K \cdot \eta \leq 1/2$$

$$\rho \leq \rho_o = (1 - \sqrt{(1 - 2 \cdot h)}) \cdot \eta/h.$$

Conclusion:

A solution to $Py = 0$ <u>exists</u> and is in $S(y^{\#0}, \rho)$.

Furthermore, if

for $h < 1/2$, $\rho < \rho^{\#1} = (1 + \sqrt{(1 - 2 \cdot h)})\, \eta/h$, or

for $h = 1/2$, $\rho \leq \rho^{\#1}$,

then the solution is <u>unique</u> in $S(y^{\#0}, \rho)$.

We note, as a particular consequence of Kantorovich's theorem, that the solution exists and is unique if

(18) $$\rho = 2\eta = 1/K.$$

6. <u>Existence of Solutions as $\gamma \to 0$</u>. We apply this theorem to the Duffing problem in the operator form $P_E\, y_E = 0$, where P_E is given by eq. (13) and

(19) $$\mathcal{D}P_E(y_E{}^{\#0}) = \mathcal{D}G_E(y_E{}^{\#0}) - \gamma D_E{}^2.$$

Here $\mathcal{D}G_E(y_E^{\#0})$, the Frechet derivative of G_E at $y_E^{\#0}$, is a linear operator on the Banach space E_α and since g is an ordinary function of a real variable, it follows from eq. (12a) that

$$(20)\qquad (\mathcal{D}G_E(y_E^{\#0})\ y_E)\ (\theta) = (1 + g'(y_E^{\#0}\ (\theta)))\cdot y_E(\theta). \qquad (\theta\ \varepsilon\ \alpha)$$

Comparing eq. (16c), we see that $\mathcal{D}G_E(y_E^{\#0})$ is self-adjoint.

We introduce the operator $T_F^{\#0}$:

$$(21)\qquad T_F^{\#0} = J_{FE} \circ \mathcal{D}G_E(y_E^{\#0}) \circ J_{EF} - \gamma I_F = J_{FE} \circ P_E(y_E^{\#0}) \circ J_{EF}.$$

We see that $T_F^{\#0}$ is also self-adjoint and, furthermore, that $(\gamma - \lambda)$ is an eigenvalue of $T_F^{\#0}$ whenever λ is an eigenvalue of $J_{FE} \circ \mathcal{D}G_E(y_E^{\#0}) \circ J_{EF}$; that is, whenever there is a nontrivial solution of $(J_{FE} \circ \mathcal{D}G_E(y_E^{\#0}) \circ J_{EF} - \lambda I_F)\ x_F = 0$, or of $(\mathcal{D}G_E(y_E^{\#0}) - \lambda\ D_E^2)\ y_E = 0$. But, we see from eq. (20) that the latter equation is the same as the variational equation, eq. (4), whose eigenvalues we estimated in Section 2. Let $\lambda^{\#n}$ be the eigenvalue of the variational equation nearest to γ; then

$$(22)\qquad \|(T_F^{\#0})^{-1}\| = 1/|\gamma - \lambda^{\#n}|.$$

We use the decomposition $A^{-1} = (I + \gamma\ A^{-1})(A + \gamma\ I)^{-1}$ to write:

$$(23)\qquad (T_F^{\#0})^{-1} = (I_F + \gamma\cdot(T_F^{\#0})^{-1}) \circ (J_{FE} \circ \mathcal{D}G_E(y_E^{\#0}) \circ J_{EF})^{-1}$$

$$= (I_F + \gamma\cdot(T_F^{\#0})^{-1}) \circ D_{FE} \circ (\mathcal{D}G_E(y_E^{\#0}))^{-1} \circ D_{EF};$$

$$(24)\qquad \Gamma_E^{\#0} = (\mathcal{D}P(y_E^{\#0}))^{-1} = J_{EF} \circ (T_F^{\#0})^{-1} \circ J_{FE}$$

$$= J_{EF} \circ (I_F + \gamma\cdot(T_F^{\#0})^{-1}) \circ D_{FE} \circ (\mathcal{D}G_E(y_E^{\#0}))^{-1}.$$

Because of eq. (22), we can expect trouble (resonance) when γ is too close to an eigenvalue of the variational equation and it is not surprising that Kantorovich's theorem fails to prove the existence of solutions near $y_E^{\#0}$ in these cases. On the other hand, if we can choose a sequence of values $\gamma^{\#n}$, each of which is sufficiently far from the nearest eigenvalue $\lambda^{\#n}$, then we may be able to meet the conditions of Kantorovich's theorem as $\gamma^{\#n} \to 0$. These observations are made precise in the following theorem:

Theorem 1. Let M^4 be given by eq. (5), C by eq. (6), and q by eq. (8), and suppose that $M^4 > 0$ and $k_1 \leq q \leq k_2$; also, let $y_E^{\#0}$ be the solution of the reduced equation, eq. (3). If the operator $\mathcal{D}G_E(y_E^{\#0})$ is self-adjoint, and if bounds A, B exist such that

(25a) $\|J_{EF}\| \cdot \|D_{FE}\, (\mathcal{D}G_E(y_E^{\#0}))^{-1}\, y_E^{\#0}\| < A,$

(25b) $\|J_{EF}\| \cdot \|D_{FE}\, (\mathcal{D}G_E(y_E^{\#0}))^{-1}\, \mathcal{D}^2G_E(y_E^{\#0})\| < B,$

(25c) $ABC^2 < 2,$

then there exists a sequence of values $\{\gamma^{\#n}\}$ tending to zero, such that a solution to eqs. (1, 2) in the operator form $P_E\, y_E = 0$ exists at $\gamma = \gamma^{\#n}$ for each sufficiently large n; and, furthermore, the solution, for each $\gamma^{\#n}$ in this sequence, exists and is unique in a neighborhood $S(y_E^{\#0}, \rho)$ where $\rho^{\#n} \to 0$ as $n \to \infty$.

Proof: Note that $P_E\, y_E^{\#0} = \gamma\, y_E^{\#0}$, and $\mathcal{D}^2P_E(y_E^{\#0}) = \mathcal{D}^2G_E(y_E^{\#0})$. According to the estimates of eqs. (9, 10) we may, by choosing $\gamma^{\#n}$ halfway between $\lambda^{\#n}$ and $\lambda^{\#(n+1)}$, use the following asymptotic formulas for sufficiently large n:

(26) $$|\gamma^{\#n}| \sim C^2/4\,n^2;$$

$$|\gamma^{\#n} - \lambda^{\#n}| = \tfrac{1}{2}|\lambda^{\#n} - \lambda^{\#(n+1)}| \sim C^2/4\,n^3.$$

Hence, we have

(27) $$\|I_F + \gamma^{\#n}(T_F^{\#0})^{-1}\| \leq 1 + |\gamma^{\#n}|/|\gamma - \lambda^{\#n}|$$
$$\sim 1 + (C^2/4\,n^2)\cdot(4\,n^3/C^2) \sim n,$$

$$\|\Gamma_E^{\#0} \circ P_E\, y_E^{\#0}\| \sim (C^2/4\,n^2)\cdot n \cdot A,$$

$$\|\Gamma_E^{\#0} \circ \mathcal{D}^2 P_E(y_E^{\#0})\| \sim n \cdot B,$$

and $((C^2/4\,n^2)\cdot n \cdot A)\cdot(n\,B) = ABC^2/4$, independent of n. Hence, in Kantorovich's theorem we may take

(28) $$\eta > AC^2/4\,n,$$
$$K > B\,n.$$

Then $K \cdot \eta = ABC^2/4$, so that $K \cdot \eta < \frac{1}{2}$ if $ABC^2 < 2$. Also $\rho = (1/K) \sim (1/n\,B) \sim 0$ so it suffices in estimating $\mathcal{D}^2 P_E$ to consider the single element $(y_E^{\#0})$ rather than all of $S(y_E^{\#0}, \rho)$ in establishing the bound for K, provided that we make ABC^2 strictly less than 2.

Remark: The requirement that $\mathcal{D}G_E(y_E^{\#0})$ be self-adjoint, which makes it possible to equate the norm of $(T_F^{\#0})^{-1}$ with its largest eigenvalue is, of course, satisfied automatically since the theorem refers explicitly to eq. (1) where g is assumed to be an ordinary function of a real variable. This may not be the case, however, in generalizations of this investigation to other kinds of operators.

7. Numerical Example. Some calculations have been made for the case $g(y(\theta)) = \beta \cdot y^3(\theta)$, $f(\theta) = \cos\theta$, to see how large β can be made without violating the hypotheses of Theorem 1.

We represent $y_E^{\#0}$ by the vector of coefficients of its odd-cosine expansion. Of course, we must truncate this vector to some finite number of components; we determine this number by repeating all of the calculations with various vector lengths until no further change appears in the results. All of the functions in E_α and F_α are represented by coefficient vectors and the linear operators are replaced by matrices.

In order to numerically execute a nonlinear operation upon a coefficent vector, we first perform discrete harmonic synthesis to obtain a function in "sampled" form; that is, defined at a number of equally-spaced sample points. We then apply the nonlinear operation to the sampled function and perform discrete harmonic analysis upon the result.

The first calculation is the solution of the reduced equation, eq. (3), by Newton's method in the Banach space of coefficient vectors. The procedure, which is described in greater detail in the author's Ph.D. thesis [10], involves the nonlinear operator $\mathcal{D}G_E(\bar{y}_E^{\#0})$ (compare eq. 20):

$$(\mathcal{D}G_E(\bar{y}_E^{\#0})\, y_E)(\theta) = (1 + \beta \cdot (\bar{y}_E^{\#0}(\theta))^2) \cdot y_E(\theta). \qquad (\theta \in \alpha) \tag{29}$$

We note that the function $(1 + \beta \cdot (\bar{y}_E^{\#0})^2)$ consists of even cosines and we find that the matrix which corresponds to this operator can be generated from the even-cosine harmonic coefficients: the element in row i and column j is one-half the sum of the coefficient of $\cos(2 \cdot |i - j|) \cdot \theta$ and of $\cos(2 \cdot (i + j - 1)) \cdot \theta$.

From the sampled function $y^{\#0}$, the related functions $(1 + g'(y^{\#0})) = (1 + 3\ \beta \cdot (y^{\#0})^2)$, $\sqrt{(1 + g'(y^{\#0}))}$, and $y^{\#0}/(1 + g'(y^{\#0}))$ are also obtained.

The quantity C involved in the eigenvalue estimates is the average value (see eq. (6)) of $\sqrt{(1 + g'(y^{\#0}))}$; it is convenient to obtain this value from the first coefficient (the constant term) in the "even-cosine" discrete harmonic analysis of the sampled function.

Differential and integral operators in the coefficient vector spaces have particularly simple forms. The operator D_{FE} (see eq. 15a) operates on cos $(2k - 1)\theta$ to produce $(2k - 1)$ sin $(2k - 1)\theta$. When this calculation is performed upon the vector corresponding to $y^{\#0}/(1 + g'(y^{\#0}))$, we obtain the odd-sine coefficient vector for $D_{FE}\ (\mathcal{D}G(y_E^{\#0}))^{-1}\ y_E^{\#0}$. The norm which appears in eq. (25a) is obtained directly from this coefficient vector.

The norm of $D_{FE}\ (\mathcal{D}G_E(y_E^{\#0}))^{-1}\ \mathcal{D}^2 G_E(y_E^{\#0})$, which is a bilinear operator, is the maximum on $(0, \pi/2)$ of the magnitude of the function

$$q(\theta) = \frac{d}{d\theta}\,(y^{\#0}(\theta)/(1 + g'(y^{\#0}(\theta)))), \tag{29}$$

since

$$\|q \cdot x^{\#1} \cdot x^{\#2}\| = \left(\int_0^{\pi/2} (q(\theta) \cdot x^{\#1}(\theta) \cdot x^{\#2}(\theta)\ d\theta)^2\right)^{1/2}, \tag{30}$$

and this norm is largest with $\|x^{\#1}\| = \|x^{\#2}\| = 1$ when $x^{\#1} = x^{\#2}$ and

$$x^{\#1}(\theta) \cdot x^{\#2}(\theta) = \delta(\theta - \theta^*), \tag{31}$$

where θ^* is the value of θ at which $q(\theta)$ attains its maximum magnitude. It is easy to show that $q(\pi/2) = -1$; when the leading term of the odd-sine expansion is sufficiently dominant (as in our case), this is the point of maximum magnitude. Since $\|J_{EF}\| = 1$, we thus obtain the bounds A and B for Theorem 1.

Numerical values for $\beta = 1.0$ and $\beta = 0.4$ are shown in Tables 1 and 2. Column 1 contains the coefficients of $\cos (2k - 1)\theta$, for $k = 1, 2, \ldots, 20$, in the expansion of $y_E^{\#0}$, and Column II shows the corresponding sampled function at $\theta = k \cdot \pi/40$ for $k = 0, 1, 2, \ldots, 20$. In Columns III and IV are the sampled functions $(1 + g'(y^{\#0}))$ and $\sqrt{(1 + g'(y^{\#0}))}$. Column V lists the sampled function q defined in eq. (29). The numerical values of A, B, and C for Theorem 1 are also shown. The quantity ABC^2 is labelled "TEST". We find that this quantity is almost directly proportional to β, and is less than 2 for $\beta = 0.4$.

8. Note. The studies reported here were inspired by the author's ealier research on the Duffing problem, which was suggested by Professor R. J. DeVogelaere of the University of California, Berkeley. These earlier results (see the author's Ph.D. thesis [10]) include a theorem inferior to Theorem 1, along with a detailed examination of some low-order and intermediate-order resonances (up to 15th harmonic, $\gamma = 0.089$).

Some effort has been made to obtain a result similar to Theorem 1, but with J_{EF} and D_{FE} in adjacent positions so that their effects would cancel. (It is conjectured that the theorem, if it could be so modified, would hold for $\beta = 1$ or larger.) No success has been achieved in this direction since the obvious procedures applied to eq. (24) destroy the normality of $(T_F^{\#0})^{-1}$ upon which is based the estimation of the norm of this operator from the eigenvalues of the variational equation.

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	I	II	III	IV	V
0		.6823	2.3967	1.5481	.0000
1	7.2899E-01	.6810	2.3914	1.5464	.0022
2	-5.6189E-02	.6772	2.3757	1.5413	.0044
3	1.2145E-02	.6707	2.3495	1.5328	.0063
4	-3.4453E-03	.6615	2.3129	1.5208	.0078
5	1.1130E-03	.6497	2.2662	1.5054	.0088
6	-3.8820E-04	.6350	2.2096	1.4865	.0088
7	1.4244E-04	.6174	2.1434	1.4640	.0076
8	-5.4181E-05	.5966	2.0679	1.4380	.0045
9	2.1172E-05	.5726	1.9837	1.4084	-.0014
10	-8.4485E-06	.5451	1.8915	1.3753	-.0113
11	3.4281E-06	.5138	1.7920	1.3387	-.0275
12	-1.4101E-06	.4783	1.6864	1.2986	-.0529
13	5.8667E-07	.4383	1.5763	1.2555	-.0922
14	-2.4644E-07	.3932	1.4638	1.2099	-.1523
15	1.0437E-07	.3425	1.3519	1.1627	-.2423
16	-4.4504E-08	.2857	1.2449	1.1157	-.3723
17	1.9057E-08	.2224	1.1484	1.0717	-.5467
18	-8.1190E-09	.1529	1.0701	1.0345	-.7495
19	3.2795E-09	.0780	1.0182	1.0091	-.9271
20	-8.7723E-10	.0000	1.0000	1.0000	-1.0000

A= 6.0000
B= .4389
C= 1.3295
TEST= 4.6549

TABLE 1. NUMERICAL RESULTS FOR BETA=1.0

	I	II	III	IV	V
0		.7973	1.7628	1.3277	.0000
1	8.3395E-01	.7955	1.7594	1.3264	-.0035
2	-4.1713E-02	.7903	1.7494	1.3227	-.0073
3	5.9821E-03	.7815	1.7328	1.3164	-.0120
4	-1.1316E-03	.7691	1.7098	1.3076	-.0179
5	2.4424E-04	.7531	1.6805	1.2963	-.0258
6	-5.6971E-05	.7333	1.6453	1.2827	-.0362
7	1.3988E-05	.7097	1.6044	1.2666	-.0501
8	-3.5614E-06	.6821	1.5583	1.2483	-.0686
9	9.3178E-07	.6504	1.5076	1.2278	-.0933
10	-2.4898E-07	.6144	1.4529	1.2054	-.1261
11	6.7657E-08	.5739	1.3952	1.1812	-.1694
12	-1.8640E-08	.5287	1.3354	1.1556	-.2258
13	5.1945E-09	.4786	1.2749	1.1291	-.2983
14	-1.4617E-09	.4236	1.2153	1.1024	-.3895
15	4.1472E-10	.3635	1.1585	1.0764	-.4999
16	-1.1851E-10	.2984	1.1068	1.0521	-.6264
17	3.4076E-11	.2287	1.0627	1.0309	-.7593
18	-9.8342E-12	.1549	1.0288	1.0143	-.8809
19	2.7915E-12	.0783	1.0074	1.0037	-.9681
20	-5.9167E-13	.0000	1.0000	1.0000	-1.0000

A= 2.4000
B= .5454
C=-1.1855
TEST= 1.8394

TABLE 2. NUMERICAL RESULTS FOR BETA= .4