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Efficient idempotent methods for optimal control

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Efficient Idempotent Methods for Optimal Control

A dissertation submitted in partial satisfaction of the
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in

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by

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2009

The dissertation of Ameet Shridhar Deshpande is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2009

DEDICATION

To Aai and Baba, who taught me the value of education, hardwork, honesty and loved me unconditionally always; and to Priyanka, who is my love and my inspiration. Her patient support, impatient urging and constant optimism have made this thesis possible.

EPIGRAPH

Life can only be understood going backward,
but it must be lived going forward.
—Kierkegaard

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As I near the completion of my doctoral studies at UCSD, I am grateful for the shoulders of giants, upon which I stand, and feel compelled to acknowledge them.

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Doctoral studies test a student to the limits, as one is married to his research and has to believe in its promise. One needs to rebuild the resolve to work in it despite the doubts about the self, the guide and the path. Such the journey seems worthwhile when the serendipity smiles, a proof comes forth or a complex equation simplifies. Still, there are times when the creativity, inspiration and enthusiasm ebbs. In such ebbs, the merry company of my roommate Rathinakumar and my labmate Ali, the modern Rumi, have helped me lighten up.

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Chapter 2, in part, has been submitted for publication as it may appear in Automatica. Deshpande, Ameet, Elsevier Press [Des]. The dissertation author was the primary investigator and author of this paper.

Chapter 4, in part, is a reprint of the material as it appears in the proceedings of American Control Conference 2008, McEneaney, William; Deshpande, Ameet; Gaubert, Stéphane, IEEE Press, 2008 [MDG08]. The dissertation author was the coauthor of this paper.

Chapter 5, in part, is a reprint of the material as it appears in the international symposium on Mathematical Theory of Networks and Systems, (MTNS) 2008, McEneaney, William; Deshpande, Ameet, Birkhäuser Press [MD08b]. The same chapter, in full, has been submitted for publication as it may appear in SIAM Journal of Control and Optimization, McEneaney, William; Deshpande, Ameet, SIAM Press [MDb]. The dissertation author was the coauthor of these papers.

Chapter 6, in part, is currently being prepared for submission for publication of the material. McEneaney, William; Deshpande, Ameet [MDa]. The dissertation author was the coauthor of this paper.

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W.M. McEneaney and A.S. Deshpande, "Payoff suboptimality and errors in value induced by approximation of the Hamiltonian", In *IEEE Conference on Decision and Control (CDC)*, pages 3175–3180, Dec. 2008.

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ABSTRACT OF THE DISSERTATION

Efficient Idempotent Methods for Optimal Control

by

Ameet Shridhar Deshpande
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Professor William McEneaney, Chair

Dynamic programming (DP) is a very powerful and robust tool for nonlinear optimization. Nevertheless, the applications have been limited to discrete / low dimensional systems due to the ubiquitous *curse-of-dimensionality* (CoD), which increases computation cost exponentially with the dimensionality of the problem. Application of DP to continuous-time and continuous-space systems gives rise to Hamilton-Jacobi-Bellman (HJB) PDEs, which are nonlinear and can have non-smooth solutions.

Recently, a CoD-free method was developed to solve certain nonlinear semi-convex HJB PDEs. It is based on the linearity of the underlying semigroup on a suitable idempotent algebra. The CoD is avoided as it is grid-less and the solution is expressed as the maximum of quadratic functions. Moreover, the Hamiltonian is approximated by the maximum of M linear-quadratic Hamiltonians, where M is called the complexity. In process, the original problem is approximated by the optimal switching problem between M linear systems. Unfortunately, although the above method avoids the CoD, it suffers from a *curse-of-complexity* (CoC) as the number of quadratic bases used to approximate the value function, grow exponentially with the complexity.

In this thesis, through the use of semidefinite programming based pruning techniques, this CoC has been partially abated. High dimensional, low complexity problems have been solved, pushing the envelope of the applicability of DP. This thesis also carries out the analysis of the error in the solution due to the PDE approximation and suboptimality of feedback control computed using it. This thesis also extends the original method for semiconcave Hamiltonians arising in cost minimization problems without nominal stability.

As a generalization of a sub-problem within the CoD-free method, this thesis also develops the fundamental solution for the time-varying *differential Riccati Equation* (DRE). This is the counterpart of the state transition matrix in time-varying ODEs, and allows analytic computation of a general solution from a particular solution. It is also shown that the semiconvex duality transforms one DRE into another, and compatibility conditions are derived. In time-invariant special case, efficient doubling algorithms and analytic solutions are proposed. These show dramatic improvement over the time marching methods for long time horizon evolution of stiff DREs.

Chapter 1

Introduction

This thesis deals with a subclass of the optimal control problems in which decisions are made in stages. The outcome of each decision is anticipated before the next decision is made. The objective is to minimize a certain cost, or maximize a certain payoff, mathematical expressions of what is considered an undesirable or desirable outcome, respectively.

A key aspect of such situations is that decisions can not be made in isolation since one must balance the desire for low present cost with the undesirability of high future costs. The *dynamic programming* (DP) technique captures this tradeoff. At each stage, it ranks decisions based on the sum of the present cost and the future costs, assuming optimal decision making for subsequent stages. DP can find the globally optimal solution for such a problem, despite the nonlinear dynamics and various state and control constraints. Thus it is an extremely robust tool for solving nonlinear optimal control problems.

There is a variety of practical problems that can be treated by dynamic programming. Significant progress has been made especially in discrete systems. Viterbi algorithm used in communications, algorithms for computer chess and many shortest path and traveling salesman algorithms are the few stellar successes of dynamic programming.

Primary motivation of this thesis is to solve the optimal control and estimation problems arising in continuous-time, continuous-space nonlinear systems using dynamic programming. Section 1.1 gives an overview of DP. A key concept is the *value function*, which is the optimal cost to go as a function of system state. Barring a few technicalities, the optimal control in a feedback form can be derived from the value function. Thus the goal of DP is to compute the value function.

The value function satisfies a functional equation known as the *Dynamic Programming Principle* (DPP). Such DPP can be viewed as an operator mapping the value function at one time to the value function at another time. For deterministic, continuous-time and continuous-space problems, as this time interval goes to zero, the DPP reduces to a *Hamilton-Jacobi-Bellman* partial differential equation (HJB PDE), usually expressed in terms of a *Hamiltonian*. In general, this is a fully nonlinear, first-order PDE and does not have a classical (smooth) solution. Therefore several notions of generalized solutions have been developed. One of them is the theory of *viscosity solutions*. (cf. [BCD97], [MCL87], [FS06]). The value function has been shown to be the viscosity solution of the HJB PDE for many problem classes. Moreover, it usually is the unique solution within a certain class of functions.

Typical approach is to obtain the HJB PDE corresponding to the control problem of interest, and then to apply a numerical method to solve it, thereby obtaining the value function. The difficulty is that one must solve the HJB PDE. Various approaches have been used to solve the HJB PDE. More common methods are the grid-based methods (c.f., [SD02], [FF98]) which still suffer from the *curse-of-dimensionality* , as the number of grid points and computations grow exponentially with the space dimension.

However, in recent years, an entirely new class of numerical methods for HJB PDEs have emerged (cf., [FM00], [McE03], [MAL04], [AGL08], [McE06]). These methods work with the underlying DPP directly instead of the HJB PDE. For the objective maximization problems, these methods exploit the linearity of the associated semigroup under max-plus algebra. Similarly, the problems involving objective minimization exploit the linearity under the min-plus algebra. Recall that the max-plus algebra is a commutative semifield in which addition is the ordinary maximum operation and multiplication is the ordinary addition operation. Min-plus algebra is similarly defined. Both the maximum and the minimum are idempotent operations, which implies that operation on two equal elements gives the same as a result. Hence these methods are called idempotent methods.

In particular, McEneaney (cf. [McE05], [McE06], [McE07]) proposed a new method based on the above semigroup linearity for certain nonlinear HJB PDEs. This method is free from the usually ubiquitous curse-of-dimensionality. Hence it is known as the *curse-of-dimensionality-free* numerical method. In fact, the computational growth in state-space dimension is cubic.

The PDEs considered in this method are the ones which can be approximated by the pointwise maximum of linear-quadratic Hamiltonians. This spans a large class of nonlinear PDES as any semiconvex PDE can be approximated by this approach. Such an approximation is exact for the objective maximization problems involving switched linear systems.

However, there is an exponential computational growth in a certain measure of the complexity of the Hamiltonian. This phenomena is called the *curse-of-complexity.* Under the above measure, the minimal complexity Hamiltonian is the linear/quadratic Hamiltonian – corresponding to a solution by a Riccati equation. If the Hamiltonian is given as the pointwise maximum of M linear/quadratic Hamiltonians, then one could say the complexity of the Hamiltonian is M.

This thesis makes several theoretical and numerical advances to the curse-of-dimensionality-free method described next. Pruning methods are developed to attenuate the curse-of-complexity and make the technique more suitable for practical problems. The curse-of-dimensionality-free method has been extended to objective minimization problems without nominal stability, thus spanning the class of semiconcave Hamiltonians. A new fundamental solution has also been developed for time-varying Riccati equations and used in curse-of-dimensionality-free methods for efficient time evolution of quadratic functions. Finally, theoretical analysis of the error in the value function and the payoff induced by the approximation of the Hamiltonian in the curse-of-dimensionality-free method has been carried out.

The presentation in this chapter will be rather informal to point out the main ideas, and emphasize the common theme in this thesis. First, we give an overview of the dynamic programming in section 1.1 and idempotent analysis in section 1.2. In section 1.3, we describe the optimal control problems we plan to solve. We shall survey existing methods and their limitations in 1.4. In the same section, we shall introduce the idempotent methods designed to solve above problems and refer the reader to subsequent chapters for detailed treatment.

Each of the other chapters is devoted to solve a specific subproblem, and is loosely connected with one another. A brief overview of chapters follows. A detailed overview of chapters in the context of idempotent methods is covered in section 1.4.

Chapter 2, based on [Des], develops a new fundamental solution for the time varying *differential Riccati equation* (DRE) by computing an analytic solution of the max-plus kernel proposed earlier in [FM00]. This solution is the counterpart of the state transition matrix in linear time varying systems, and enables us to find a general solution from a particular solution of a certain bivariate DRE. This max-plus kernel is quadratic and is useful for an efficient evolution of the DRE. Variants of this kernel are used in later chapters in the curse-of-dimensionality-free methods.

Chapter 3 gives an overview of the previous development on the curse-of-dimensionality-free idempotent method by McEneaney based on max-plus algebra (c.f. [McE07], [McE06], [McE09]), and suggests a modified algorithm based on the fundamental solution derived in chapter 2. This method is designed to solve the HJB PDEs which arise in the maximization problems with nominal stability and which can be approximated as the max-plus sum of linear quadratic Hamiltonians. The number of quadratics in the Hamiltonian approximation is called the *complexity*. The value function is approximated by the maximum of quadratic functions as well. Unfortunately, the number of quadratic functions used in the value function approximation, increases exponentially with the complexity. This is the curse-of-complexity reviewed earlier. But not all of the quadratic functions contribute to the maximum, which is the value function.

Chapter 4, based on [MDG08], develops pruning methods which attenuate the aforementioned curse-of-complexity, by efficiently pruning quadratics which do not contribute to the maximum. It also demonstrates the application of the curse-of-dimensionality-free method on complex high dimensional problems, making them solvable on desktop PC in a reasonable amount of time.

Chapter 5, based on [MDb], is devoted to the analysis of the value function error and the payoff suboptimality incurred in the curse-of-dimensionality-free method due to approximation of the Hamiltonian.

Finally, chapter 6, based on [MDa], extends the curse-of-dimensionality-free method for minimization problems without nominal stability, for which the Hamiltonian is approximated by the min-plus sum (pointwise minimum) of quadratic functions.

1.1 Dynamic Programming: An Overview

Dynamic programming is applicable to a surprisingly general class of problems. Here we use a model problem involving a time varying continuous-time, continuous-state system and a finite horizon payoff optimization to present the method. Consider the *control system* with state ξ_t governed by the *state equation*

$$\dot{\xi}_t = f(\xi_t, u_t, t), \quad \xi_s = x \in \mathbb{R}^n. \tag{1.1}$$

Here, the *control* u_t is any measurable function of time $t \in [s, T]$ taking values in U, the *control space*. Assume that the *dynamics* $f : \mathbb{R}^n \times U \times [s, T] \to \mathbb{R}^n$ is such that, the state equation (1.1) has a unique solution in the interval $[s, T]$ for any initial positions $x \in \mathbb{R}^n$ and control u.

The *payoff functional* to be maximized with respect to control u, is additive over time, hence expressed as the sum of the *terminal payoff* at time T, ϕ and an integral of the *running payoff*, l. Such maximum is termed as the *value* function defined as below.

$$V(x, s) = \sup_{u \in \mathcal{U}} J(x, u, s) \doteq \sup_{u \in \mathcal{U}} \int_s^T l(\xi_t, u_t, t)\, dt + \phi(\xi_T) \tag{1.2}$$

Here $t \in [s, T]$ and $\xi_t = x$. $\mathcal{U}$ denotes the set of all square integrable measurable functions $u : [s, T] \to U$. We assume that V exists and is finite for now. For specific problems in the subsequent chapters this will be rigorously proved.

Following DPP holds true. Let $s \le t_1 \le t_2 \le T$, then V satisfies

$$V(x, t_1) = \sup_{u \in \mathcal{U}} \left\{ \int_{t_1}^{t_2} l(\xi_\tau, u_\tau, \tau))\, d\tau + V(\xi_{t_2}, t_2) \right\} \tag{1.3}$$

Above DPP links the value functions at different times together. When the time interval gets smaller, assuming smoothness, V satisfies an infinitesimal version of DPP, called the *Hamilton-Jacobi-Bellman* partial differential equation (HJB PDE). V then satisfies the following HJB PDE with the boundary condition, $V(T, x) = \phi(x)$ for all $x \in \mathbb{R}^n$.

$$\begin{aligned} 0 &= -\nabla_t V(x,t) - \sup_{u \in U} \{ f(x,u,t) \cdot \nabla_x V(x,t) + l(x,u,t) \} \\ &\doteq -\nabla_t V(x,t) - H(x, \nabla_x V(x,t)) \end{aligned} \tag{1.4}$$

Here $H(x, t, p)$ is referred to as the *Hamiltonian.* Given the value function V, both (1.3) and (1.4) contain all the relevant information to generate the *optimal feedback control* when one exists, which achieves the maximum in (1.3) and (1.4). Computing the optimal feedback from the value function is termed the *synthesis procedure.* Verification that a particular feedback control is optimal is known as the *verification procedure.*

HJB is a fully nonlinear, first-order PDE. Consequently, the classical (smooth) solutions may not exist. Hence we use a notion of generalized solution called *viscosity solution* ([BCD97], [MCL87], [FS06]). The value function can be shown to satisfy the HJB PDE (1.4) in the *viscosity* sense. More importantly, for many classes of problems, it is the unique viscosity solution within a certain class of functions satisfying boundary conditions. (c.f. [BCD97], [BL97], [Lio00], [MCL87], [McE98], [Sor96]). Thus the problem of solving the DPP and finding the value function can be reduced to finding the viscosity solution of the HJB PDE. In this thesis, we will employ the underlying DPP directly, to compute the value function of the optimal control problem. Still, the viscosity solution theory will help confirm the value function as the solution of the HJB PDE.

Let us briefly discuss the idea of the viscosity solution. Following [BCD97], a continuous function u is a *viscosity solution* of the nonlinear partial differential equation

$$F(z, u(z), \nabla u(z)) = 0 \tag{1.5}$$

if the following conditions are satisfied:

$$F(z,u(z),p) \le 0 \qquad \forall z, \forall p \in D^+u(z)$$
$$F(z,u(z),q) \ge 0 \qquad \forall z, \forall q \in D^-u(z)$$

where

$$D^+u(z) = \left\{ p \in \mathbb{R}^n : \limsup_{y\to z} \frac{u(y)-u(z)-p\cdot(y-z)}{|y-z|} \le 0 \right\}$$
$$D^-u(z) = \left\{ q \in \mathbb{R}^n : \liminf_{y\to z} \frac{u(y)-u(z)-q\cdot(y-z)}{|y-z|} \ge 0 \right\}$$

$D^+u(z)$ and $D^-u(z)$ are called the *superdifferential* and the *subdifferential* at z respectively. Informally, they can be described as sets of tangent planes touching $u(z)$ from above and below near z, respectively. Note that if both $D^+u(z)$ and $D^-u(z)$ are nonempty at some z, then $D^+u(z) = D^-u(z) = \nabla u(z)$ and u is differentiable at z.

Observe that the general form (1.5) of the HJB PDE covers the specific case of (1.4), in which $z = \{x,t\} \in \mathbb{R}^n \times (t_1,t_2]$, $u(z) = V(x,t)$ and $\nabla u(z) = \nabla_z V(z) = \{\nabla_x V(x,t), \nabla_t V(x,t)\}$. Thus

$$\begin{aligned} F(z,u(z),\nabla u(z)) &= F(\{x,t\}, V(x,t), \{\nabla_x V(x,t), \nabla_t V(x,t)\}) \\ &= -\nabla_t V(x,t) - \sup_{u\in U} \{f(x,u,t)\cdot \nabla_x V(x,t) + l(x,u,t)\} \end{aligned}$$

In this thesis we shall use idempotent methods to solve the DP problem. The efficacy of the idempotent methods lies in the fact that the semigroup underlying DP is linear under certain idempotent algebra. In next section, basics of idempotent analysis will be discussed.

1.2 Idempotent Analysis

Definition 1.2.1. *A binary operation is* idempotent *if, whenever applied to two equal values, it gives that value as the result.*

Specifically we study two idempotent algebras based on maximum and minimum operations. These are called max-plus and min-plus algebras respectively.

The symbols we use for addition and multiplication operations under these algebras are the same. Hence their interpretation will be context specific. The analysis for the min-plus algebra is exactly analogous to the max-plus algebra, with the maximum operator substituted by the minimum operator. Additional information regarding idempotent analysis can be found in [FBQ92], [CG79], [KM97].

1.2.1 Max-Plus algebra

The max-plus algebra is a commutative semifield over $\mathbb{R}^- \doteq \mathbb{R} \cup \{-\infty\}$. The addition and multiplication operations, $\oplus$ and $\otimes$ are defined for $a, b \in \mathbb{R}^-$ as the ordinary maximum operation and the ordinary sum respectively.

$$a \oplus b = \max(a, b), \quad a \otimes b = a + b$$

In particular, we take

$$a \oplus -\infty = a, \quad a \otimes -\infty = -\infty \quad \forall a \in \mathbb{R}^-$$

Thus the additive identity is $-\infty$, and the multiplicative identity is 0. The multiplicative inverse of $a > -\infty$ is $-a$. Note that with the exception of $-\infty$, no other element has an additive inverse, hence this algebra is a *semifield* rather than a field. Also note that the commutative, associative, distributive properties hold.

$$a \oplus b = b \oplus a, \quad a \otimes b = b \otimes a, \quad a \otimes (b \otimes c) = (a \otimes b) \otimes c$$
$$a \oplus (b \oplus c) = (a \oplus b) \oplus c, \quad a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$$

Now we define a vector space over the max-plus algebra, referred to as a max-plus vector space. We say that $\mathcal{X}$ is a max-plus vector space (with zero element denoted by $\phi^0 \in \mathcal{X}$) if given any $a, b \in \mathbb{R}^-$ and any $\phi, \psi \in \mathcal{X}$, following properties hold.

$$a \otimes \phi \in \mathcal{X}, \quad \phi \oplus \psi = \psi \oplus \phi \in \mathcal{X}, \quad (a \otimes b) \otimes \phi = a \otimes (b \otimes \phi),$$
$$(a \oplus b) \otimes \phi = (a \otimes \phi) \oplus (b \otimes \phi), \quad a \otimes (\phi \oplus \psi) = (a \otimes \phi) \oplus (b \otimes \psi),$$
$$\phi \oplus \phi^0 = \phi, \quad a \otimes \phi^0 = \phi^0, \quad -\infty \otimes \phi = \phi^0, \quad 0 \otimes \phi = \phi.$$

If $\mathcal{X}$ is defined as a set of vectors of elements of $\mathbb{R}^-$ indexed by $x \in \Lambda$ for some index set Λ, then we may denote elements of $\mathcal{X}$ as $\phi = \{\phi(x)\}_{x\in\Lambda}$. As with standard vector spaces, the max-plus addition of two functions (or vectors) is done pointwise. Specifically, for functions $\phi_1(x)$ and $\phi_2(x)$, the max-plus sum is

$$[\phi_1 \oplus \phi_2](x) = \phi_1(x) \oplus \phi_2(x) = \max(\phi_1(x), \phi_2(x))$$

for all x. Multiplication by a scalar is analogous to the standard case as well. That is, for $a \in \mathbb{R}^-$ and function $\phi_1(x)$, for all x,

$$[a \otimes \phi_1](x) = a \otimes \phi_1(x) = a + \phi_1(x).$$

Max-Plus Linearity of the Semigroup

Now we revisit the DPP (1.3) which links the value functions at different times together. This can also be meaningfully expressed using the following operator notation. If we define the operator

$$\mathcal{S}_{t_1}^{t_2}[\psi](x) \doteq \sup_{u\in L_2[t_1,t_2]} \left\{ \int_{t_1}^{t_2} l(\xi_\tau, u_\tau, \tau)\, d\tau + \psi(\xi_{t_2}) \right\}, \tag{1.6}$$

with $\xi_{t_1} = x$. The DPP, (1.3), can then be expressed as

$$\mathcal{S}_{t_1}^{t_2}[V(\cdot, t_2)](x) = V(\cdot, t_1)(x) = V(x, t_1).$$

It is well known (c.f. [McE06]) that operators $\mathcal{S}_{t_1}^{t_2}$ form a semigroup. That is

$$\mathcal{S}_{t_1}^{t_2} \cdot \mathcal{S}_{t_2}^{t_3} = \mathcal{S}_{t_1}^{t_3} \text{ and } \mathcal{S}_{t_1}^{t_1} = I. \tag{1.7}$$

Such semigroup is also known to be linear in the max-plus algebra. Restricting ourselves to continuous functions, if we define the domain of $\mathcal{S}_{t_1}^{t_2}$ as

$$\text{Dom}(\mathcal{S}_{t_1}^{t_2}) \doteq \left\{\psi \in C(\mathbb{R}^n) : \mathcal{S}_{t_1}^{t_2}[\psi](x) < \infty \quad \forall x \in \mathbb{R}^n \right\},$$

then $\forall a \in \mathbb{R}^-$ and $\forall \phi, \psi \in \text{Dom}(\mathcal{S}_{t_1}^{t_2})$,

$$\mathcal{S}_{t_1}^{t_2}[\phi \oplus \psi](x) = \mathcal{S}_{t_1}^{t_2}[\phi](x) \oplus \mathcal{S}_{t_1}^{t_2}[\psi](x)$$
$$\mathcal{S}_{t_1}^{t_2}[a \otimes \phi](x) = a \otimes \mathcal{S}_{t_1}^{t_2}[\phi](x).$$

This property makes the max-plus algebra a natural tool to solve the dynamic programming problems involving maximization of the objective function (payoff). Hence it is used in chapters 2, 3, 4 and 5 in the analysis of such problems.

A Max-Plus Fundamental Solution

Now we shall define a *max-plus kernel* $I : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ derived earlier in [FM00] and [Fle03]. Let $s \leq t_1 \leq t_2 \leq T$ and $x, y \in \mathbb{R}^n$, and ξ_t evolves with the dynamics (1.1). Define

$$I_{t_1}^{t_2}(x,y) \doteq \begin{cases} \sup_{u \in \mathcal{U}_{t_1}^{t_2}(x,y)} \int_{t_1}^{t_2} l_t(x_t, u_t, t)\, dt & \text{if } \mathcal{U}_{t_1}^{t_2}(x,y) \neq \emptyset \\ -\infty & \text{otherwise} \end{cases} \tag{1.8}$$

where

$$\mathcal{U}_{t_1}^{t_2}(x,y) \doteq \{u \in L_2[t_1, t_2] : x_{t_1} = x, x_{t_2} = y\}$$

Note that $I_{t_1}^{t_2} = -\infty$ indicates that it is impossible to reach y from x in time interval (t_1, t_2) using any possible control u.

Fleming and McEneaney [FM00] proposed above kernel, and showed that

$$\mathcal{S}_{t_1}^{t_2}[\psi](x) = \sup_{y \in \mathbb{R}^n} \left(I_{t_1}^{t_2}(x,y) + \psi(y) \right) \doteq \int_{\mathbb{R}^n}^{\oplus} I_{t_1}^{t_2}(x,y) \otimes \psi(y)\, dy \tag{1.9}$$

and since $I_{t_1}^{t_2}$ depends only on the dynamics $\dot{\xi}_t = f_t(\xi_t, u_t)$ and running payoff $l_t(\xi_t, u_t)$, it is independent of the terminal payoff $\psi(\xi_{t_2})$. Hence it can serve as a *Fundamental solution*, and obtain $\mathcal{S}_{t_1}^{t_2}[\psi](x)$ for any $\psi(x)$ by a kernel operation.

This kernel and the fundamental solution will prove very useful in the chapters ahead. In chapter 2, we shall prove that such a kernel is bi-quadratic for the time varying linear-quadratic problems, and shall derive analytical formulas for the same. In chapters 3 and 6, we will use a similar fundamental solution to propagate quadratic basis functions under a linear-quadratic Hamiltonian.

1.2.2 Min-Plus Algebra

The min-plus algebra is a commutative semifield over $\mathbb{R}^+ \doteq \mathbb{R} \cup \{+\infty\}$. It is completely analogous to the max-plus algebra discussed above, except that the addition and multiplication operations, $\oplus$ and $\otimes$ respectively, are defined for $a, b \in \mathbb{R}^+$ as the ordinary minimum operation and the ordinary addition respectively.

$$a \oplus b = \min(a, b), \quad a \otimes b = a + b$$

In particular, $a \oplus \infty = a$, $a \otimes \infty = \infty$ and $a \otimes 0 = a$. Thus, additive identity is $+\infty$ and the multiplicative identity is 0. The rest of the formalism runs parallel to the max-plus algebra.

When the objective function (cost) is to be minimized, the semigroup underlying the dynamic programming, is linear in the min-plus algebra. As an example, for the infinite horizon regulator problem (1.21) in section 1.3.3, with $\mathcal{U}$ as the control space, the semigroup is

$$\mathcal{S}_t[\psi](x) \doteq \inf_{u \in L_2[0,t]} \left\{ \int_0^t l(\xi_\tau) + \frac{1}{2}|u_\tau|^2 \, d\tau + \psi(\xi_t) \right\}. \tag{1.10}$$

The min-plus fundamental solution is

$$I_t(x,y) \doteq \begin{cases} \inf_{u \in \mathcal{U}_0^t(x,y)} \int_0^t l_\tau(x_\tau) + \frac{1}{2}|u_\tau|^2 \, d\tau & \text{if } \mathcal{U}_0^t(x,y) \neq \emptyset \\ \infty & \text{otherwise,} \end{cases} \tag{1.11}$$

where

$$\mathcal{U}_0^t(x,y) \doteq \{u \in L_2[0,t] : x_0 = x, x_t = y\}.$$

The fundamental solution and the semigroup operation are related as

$$\mathcal{S}_t[\psi](x) = \inf_{y \in \mathbb{R}^n} \left(I_0^t(x,y) + \psi(y) \right) \doteq \int_{\mathbb{R}^n}^{\oplus} I_0^t(x,y) \otimes \psi(y) \, dy. \tag{1.12}$$

This property is used in chapter 6 for the curse-of-dimensionality-free method designed to solve problems involving infinite horizon cost minimization.

1.3 Some optimal control problems

We briefly indicate the optimal control problems this thesis attempts to solve. More rigorous formulation will appear in later chapters.

1.3.1 Finite Horizon Time varying LQ Problem

Now we shall discuss the well known problem involving a time varying system with linear dynamics and quadratic payoff. Using the notation from section 1.1, consider a deterministic, continuous-time, continuous-state dynamical system with state ξ_t. The state equation with linear dynamics is as below.

$$\dot{\xi}_t = f(\xi_t, u_t, t) \doteq A(t)\xi_t + \sigma(t)u_t, \quad \xi_s = x \in \mathbb{R}^n. \tag{1.13}$$

Here, the *control* $u \in L_2[s,T]$ is any square integrable function taking values in $\mathbb{R}^n$, the *control space*. The *running* payoff and the *terminal* payoff are quadratic, and the *value* function is given by

$$V(x,s) = \sup_{u \in L_2[s,T]} J(x,u,s) = \sup_{u \in L_2[s,T]} \int_s^T l(\xi_t, u_t, t)\, dt + \phi(\xi_T) \tag{1.14}$$

where $l(\xi_t, u_t, t) \doteq \frac{1}{2}\xi_t' C(t)\xi_t - \frac{1}{2}|u_t|^2$ and $\phi(\xi_T) \doteq \frac{1}{2}\xi_T{}' P_T \xi_T$.

Using (1.4), the dynamic programming gives rise to the following HJB PDE.

$$0 = -\nabla_t V(x,t) - H(x,t,\nabla_x V(x,t)), \quad V(x,T) = \phi(x). \tag{1.15}$$

Here $(x,t) \in \mathbb{R}^n \times (-\infty, T]$ and $H(x,t,p)$ is the linear/quadratic *Hamiltonian*, defined as

$$\begin{aligned} H(t,x,p) &\doteq \sup_u \{f(x,u,t) \cdot \nabla_x V(x,t) + l(x,u,t)\} \\ &= \frac{1}{2}x' C(t) x + x' A(t)' p + \frac{1}{2} p' \sigma(t)\sigma(t)' p. \end{aligned} \tag{1.16}$$

This PDE has a classical (smooth), quadratic solution which is the value function,

$$V(x,s) = \frac{1}{2}x' P_s x,$$

and P_s is the solution of the following differential Riccati equation (DRE) backwards in time and starting with P_T at $t = T$, where P_T is the parameter in the terminal cost.

$$-\dot{P}_t = A(t)' P_t + P_t A(t) + C(t) + P_t \sigma(t)\sigma(t)' P_t \tag{1.17}$$

Few analytical and many time marching methods have been developed for solving this important DRE. But they all face numerical problems when applied to a stiff DRE evolution for long time horizons. Chapter 2 is devoted to this problem, and a fundamental solution is derived. Such solution allows analytic computation of a general solution of the DRE (1.17), from any particular solution of a certain bivariate DRE. Extensions of this problem also prove useful in solving a nonlinear infinite horizon problem discussed next and covered in chapters 3 and 6.

1.3.2 An Infinite Horizon Robust Control Problem

We now consider a robust/H_∞ infinite time-horizon control problem. We consider the case with a fixed feedback control and an unknown disturbance process u. The time-invariant dynamics then takes the form

$$\dot{\xi}_t = g(\xi_t, v(\xi_t)) + \sigma(\xi_t)u_t \doteq f(\xi_t) + \sigma(\xi_t)u_t, \quad \xi_0 = x \in \mathbb{R}^n \tag{1.18}$$

Note that $v(\xi_t)$ is the fixed feedback controller, and $f(\xi_t)$ represents the nominal dynamics in absence of a disturbance process. We assume that $f(0) = 0$ and that the origin is an exponentially stable equilibrium point. The state $\xi_t \in \mathbb{R}^n$ and the control u lies in $L_2^{\text{loc}} \doteq \{u : [0,\infty) \to \mathbb{R}^m : u \in L_2[0,T] \quad \forall T < \infty\}$, where $L_2[0,T]$ is the set of square-integrable functions over $[0,T]$.

The associated value function (also known as available storage) is obtained by maximizing the integral payoff

$$V(x) = \sup_{u \in L_2^{\text{loc}}} \sup_{T<\infty} \int_0^T l(\xi_t) - \frac{\gamma^2}{2}|u_t|^2 \, dt$$

As this problem involves maximization, the value function is linear in the max-plus algebra, and this problem is referred to as a *max-plus problem.*

Assumptions that l is strictly nonnegative, $l(0) = 0$, additional stability conditions on f and lower threshold on γ ensure that the value function exists and is reached only at the infinite horizon. Exact assumptions are discussed in detail in chapter 3.

The value function is the solution of the following HJB PDE in the *viscosity* sense satisfying the boundary condition $V(0) = 0$.

$$\begin{aligned} 0 &= -\sup_{u \in \mathbb{R}^m} \left\{ (f(x) + \sigma(x)u) \cdot \nabla V + l(x) - \frac{\gamma^2}{2}|u_t|^2 \right\} \\ &= -\left[f(x) \cdot \nabla V + l(x) + \nabla V' \left(\frac{\sigma(x)\sigma'(x)}{2\gamma^2} \right) \nabla V \right] \\ &\doteq -H(x, \nabla V) \end{aligned} \tag{1.19}$$

Chapters 3 is devoted to building the curse-of-dimensionality-free method for exactly such problems. Such a payoff maximization problem also has a *cost minimization* analogue, which results in an infinite horizon regulator problem, which is discussed next.

1.3.3 An Infinite Horizon Regulator Problem

Consider the problem with dynamics

$$\dot{\xi}_t = f(\xi_t) + \sigma(\xi_t)u_t, \quad \xi_0 = x \in \mathbb{R}^n \tag{1.20}$$

The state $\xi_t \in \mathbb{R}^n$ and the control u lies in

$$L_2^{\text{loc}} \doteq \{u : [0, \infty) \to \mathbb{R}^m : u \in L_2[0, T] \quad \forall T < \infty\},$$

where $L_2[0, T]$ is the set of square-integrable functions over $[0, T]$.

We assume that $f(0) = 0$. Thus the origin is the fixed point of the nominal dynamics. But we do not assume the stability of the origin.

The associated value function is obtained by minimizing the integral payoff

$$V(x) = \inf_{u \in L_2^{\text{loc}}} \sup_{T < \infty} \int_0^T l(\xi_t) + \frac{1}{2}|u_t|^2 \, dt \tag{1.21}$$

As this problem involves minimization, the value function is linear in the min-plus algebra, and this problem is referred to as a *min-plus problem*.

Assumptions that l is strictly nonnegative, $l(0) = 0$, ensure that the value function exists and is reached as the limit of the finite horizon problems as $T \to \infty$. Exact assumptions are discussed in detail in chapter 6.

The value function is the solution of the following HJB PDE in the *viscosity* sense satisfying the boundary condition $V(0) = 0$.

$$\begin{aligned} 0 &= -\inf_{u_t \in \mathbb{R}^m} \left\{ (f(x) + \sigma(x)u) \cdot \nabla V + l(x) + \frac{1}{2}|u_t|^2 \right\} \\ &= -\left[f(x) \cdot \nabla V + l(x) - \nabla V' \left(\frac{\sigma(x)\sigma'(x)}{2} \right) \nabla V \right] \\ &\doteq -H(x, \nabla V) \end{aligned} \tag{1.22}$$

Chapters 6 is devoted to building the curse-of-dimensionality-free method for exactly such problems.

1.4 Overview of Methods

In all the problems discussed before, one is interested in finding the value function at x which solves the corresponding HJB PDE. This in turn, will yield the optimal control $u^{x,\star}(t)$ or the optimal feedback law $u^{\star}(\xi, t)$, if one exists. Now we shall review the conventional methods for solving such problems. These start with HJB PDE and find the value function. The more recent, *idempotent* methods, discussed later, work with the underlying DPP directly to compute V.

1.4.1 Conventional methods

There are two common methods for computing the value function and generating the optimal feedback in dynamic programming. Both these methods face certain difficulties.

Finite element methods: The most common approach is to use grid based *finite element methods* to find the value function for all grid-points in the region of interest. (c.f. [SD02], [FF98]). Unfortunately, they suffer from the curse-of-dimensionality, as the number of grid-points needed to cover the state space, grow exponentially with the dimensionality n. As an example, if we choose 100 grid-points per dimension, computations will grow by a factor of 10^6 as we go from a dimension 3 problem to a dimension 6 problem. Consequently there is little hope that such methods could be used for problems in dimensions greater than 4 or 5 in the foreseeable future.

Method of Characteristics: The second approach uses the classical *method of characteristics* for the HJB PDE, also known as the *Pontryagin's Maximum Principle* (PMP). In this method, one tries to solve a two-point boundary value problem for a system of ordinary differential equations involving state and adjoint variables. Thus one propagates the solution along the one dimensional paths, also called the *characteristic curves*, to obtain the value and the optimal control only at the points on such a path, instead of the whole space. This decreases the computational cost drastically. However, this method assumes smoothness. Nonsmoothness can pose a daunting challenge, as the shooting methods to find the correct initial conditions may fail to converge, the projections of characteristics in the state space can cross and may not cover the entire state space. Also, the solution needs to be recomputed if the system deviates from the optimal path, which can happen in presence of noise. Ways to get around these difficulties using generalized characteristics, have associated bookkeeping costs which have been a deterrent to application of this method for large or complex problems.

1.4.2 Idempotent methods

In recent years, an entirely new class of numerical methods for HJB PDEs have emerged (c.f. [FM00], [McE03], [MAL04], [AGL08], [McE06]). These methods exploit the linearity of the semigroup associated with the optimal control under suitable *idempotent* algebra. Specifically, we deal with the *max-plus* and the *min-plus* algebras. These arise in problems in which the objectives need to be maximized or minimized, respectively. The description of these methods, several of which are the contributions of this thesis, follows. The chapters in which they are covered are also mentioned along with.

Eigenvector methods: The first class of numerical methods based on above linearity, were the eigenvector methods (c.f. [McE03], [McE04], [CM04]). These employ the max-plus basis expansion of the value function, and numerical methods obtain the coefficients in such a basis expansion. But again, the number of basis functions required grows exponentially with space dimension. Thus one still can not avoid the curse-of-dimensionality.

The curse-of-dimensionality-free method for max-plus problems: A second numerical method based on max-plus linearity of above semigroup was proposed for certain class of semiconvex HJB PDEs (c.f. [McE07], [MK], [McE09], [MDG08], [McE06]). These PDEs arise in the infinite horizon payoff maximization problems with stable dynamics and unbounded control and state space. In this method, the Hamiltonian is approximated by the max-plus sum (pointwise maximum) of many linear/quadratic Hamiltonians, and the infinite horizon value function is obtained as the limit of the finite horizon problem as time grows. This method surprisingly is not subject to the curse-of-dimensionality, hence called the *curse-of-dimensionality-free* method. In fact, the computational growth in state-space dimension is of the order of n^3. There is however no *free lunch*, since it exhibits an exponential computational growth in a certain measure of the complexity of the Hamiltonian. Under this measure, the minimal complexity Hamiltonian is linear/quadratic, which is easily solved by a Riccati equation. If the Hamiltonian is the max-plus sum (pointwise maximum) of M linear/quadratic Hamiltonians, then it is said to have the complexity M. Such an approximation is very useful, since the max-plus addition of linear/quadratic Hamiltonians can span the space of all semiconvex Hamiltonians, which includes many problems of practical interest. Chapter 3 describes this method in detail.

This thesis contributes towards the error analysis of the curse-of-dimensionality-free method. Specifically it addresses the error due to an approximation of a nonlinear Hamiltonian with pointwise maximum of many linear-quadratic Hamiltonians. It computes the bounds on the error between the original and approximate value function. It also computes the payoff suboptimality by following the approximate optimal control. This is covered in chapter 5.

Pruning methods for the curse-of-dimensionality-free method: The exponential computational growth with complexity, or the *curse-of-complexity* , which replaces the curse-of-dimensionality discussed above is not always inevitable. This is because, the value function is represented as the max-plus summation (pointwise maximum) of quadratic functions, and those which lie below the maximal envelope can be pruned without introducing any error. This thesis proposes various pruning methods in chapter 4. (c.f. [MDG08]). Such methods make it possible to solve the nonlinear control problems in higher dimensions on a desktop PC.

The curse-of-dimensionality-free method for min-plus problems: Chapter 6, in this thesis extends the above method to solve a semiconcave HJB PDE, which can be approximated as the min-plus sum (pointwise minimum) of linear/quadratic Hamiltonians. These arise in infinite horizon problems with unbounded control and state space, and in which we aim to minimize an integral cost. In such problems, the nominal dynamics does have a fixed point at the origin. But the stability need not be assumed, making it more general.

Fundamental solutions of the differential Riccati equation: A crucial sub-step in the curse-of-dimensionality-free method is the evolution of a quadratic terminal cost function under a linear/quadratic Hamiltonian. This involves solving a Differential Riccati Equation (DRE). As a surprising outcome of the idempotent methods and semiconvex duality theory, a fundamental solution for the time-invariant DRE was discovered in [McE08]. In this thesis, this algorithm is refined, so as not to use the semiconvex dual transformation and thereby avoid some complications and error sources. In this process, a closed form quadratic analytical solution for the fundamental solution originally theorized in [FM00] is also found. Suitable variations of such fundamental solution are also used in chapters 3 and 6 to evolve the quadratic cost function in discrete steps for fast computations.

Chapter 2 in this thesis, extends such a fundamental solution to a time varying DRE. This enables us to compute analytically the general solution from any particular solution of such a DRE. So far, this has been a hard problem, with only time marching solutions available for long time horizon propagation. Chapter 2 covers the development of these algorithms.

Chapter 2

New Fundamental solution for the time varying differential Riccati equation

2.1 Introduction

In this chapter, we consider the matrix *differential Riccati equation* (DRE) of the form

$$\begin{aligned} -\dot{p}(t) &= A(t)'p(t) + p(t)A(t) + C(t) + p(t)'\Sigma(t)p(t) \\ p(T) &= p_T \end{aligned} \tag{2.1}$$

where $t \in (-\infty, T]$ and $A(t)$ is square and $p(t), C(t), \Sigma(t)$ are square and symmetric $n \times n$ matrices. Note that (2.1) can be easily converted into an initial value problem with forward time propagation. Nevertheless, we use this approach because it simplifies the notation when framed as an optimal control problem.

The development in this chapter is mostly based on the paper [Des]. Variations of the fundamental solution derived here will be used in numerical methods for curse-of-dimensionality-free methods developed in chapters 3 and 6. Hence it is considered first.

DREs are widely used in the systems and control theory, especially so in the areas of optimal control, filtering and estimation. Many numerical algorithms have been proposed in the past for solving time-varying DREs. These include carefully redesigned, conventional Runge-Kutta and other explicit linear multi-step methods, as well as nonlinear implicit methods by Choi and Laub [CL90], Dieci [Die92] and many others. Although these methods benefit greatly from the past development in general purpose computer programs for solving *ordinary differential equations* (ODEs), they can become rather complex in code structure and interface. Implicit methods, which are more preferred to the explicit ones for solving stiff problems, also suffer from implementation and computational complexity. Moreover, these methods have to be rerun to solve for each initial condition, which makes it difficult to carry out the sensitivity analysis.

There have also been many unconventional methods designed to solve time-invariant DREs (cf. [Lei85], [Rus88], [Lai76], [And77] and [Kim89]). These comprise of various analytical solutions and doubling algorithms. Nevertheless, it is known that these are not suited for time-varying DREs.

There are two methods which *can* be used. The first is the analytical solution developed by Davison and Maki [DM73]. It solves the following system.

$$\begin{bmatrix} \dot{U} \\ \dot{V} \end{bmatrix} = \begin{bmatrix} -A(t) & -\Sigma(t) \\ C(t) & A(t)' \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} \qquad P_{t_1} = V(t_1)U^{-1}(t_1) \tag{2.2}$$

with the solution to (2.1) obtained as $P_{t_2} = V(t_2)U^{-1}(t_2)$. Thus this method does work for the time-varying systems. But as $t_2 - t_1$ grows, columns of $U(t_2)$ become more and more linearly dependent, which makes the problem ill-conditioned. Therefore, this method can be used only for a small time propagation. Thus until now, there has been no fundamental solution available for the time-varying DRE, which is useful for a long time horizon propagation and for the infinite horizon as a special case. This work attempts to fill this gap.

The second method is by Sorine and Winternitz [SW85]. Their work provides a way to construct a general solution from five particular solutions. The forthcoming method uses just one particular solution of the bivariate DRE, instead of five particular solutions in [SW85], to construct the general solution of the time varying DRE.

Now we shall discuss an overview of the forthcoming development. Recently

McEneaney [McE08] proposed a new fundamental solution for solving the time-invariant DRE and a doubling algorithm. They are based on the tools of max-plus algebra and semiconvex duality. In another recent development, in [FM00], [Fle03], Fleming and McEneaney introduced the concept of the *max-plus fundamental solution* for time invariant systems with nonlinear dynamics. This chapter extends both of these ideas to apply to a time-varying system with linear dynamics and quadratic payoff. It finds surprisingly simple formulas for the fundamental solution, which turns out to be bivariate quadratic and can be represented by three n dimensional square matrices. It can also be viewed as the max-plus kernel which can operate on any initial condition p_0 to obtain p_t analytically.

The fundamental solution is obtained from the time evolution of a bivariate quadratic terminal cost function under optimal control, where one variable is the state and the second variable is used to parametrize the terminal cost. Note that this requires us to solve a bivariate DRE (2.13), hence evolve three parameters instead of just one in case of the DRE (2.1). The fundamental solution itself is invariant with respect to the terminal bivariate cost function, but depends only on the evolution time interval. Thus any particular solution to a bivariate DRE can be converted into the fundamental solution and then into a general solution, analytically. This makes sensitivity/ perturbation analysis for such an initial value problem much easier.

As a special case, the fundamental solution to the time-invariant problem and a new doubling algorithm is derived. Though inspired from the doubling algorithm in [McE08], the new algorithm is more direct and simpler, since it does away with the kernel propagation in the semiconvex dual space. Instead, it propagates the kernel in the primal space of quadratics. One numerical issue with such a fundamental solution as well as the one described in [McE08], is that the kernel parameters blow up as the time step gets smaller. A modification to the formula to avoid such a blowup is suggested. It maintains better solution accuracy for the propagation at a small time step.

Building further on the idea of semiconvex duality, it shall be proved that the semiconvex dual of the solution of DRE (2.1) satisfies another DRE, whose coefficients can be found analytically. One can also choose these coefficients and find an appropriate duality kernel for transformation, as long as certain compatibility/matching conditions are satisfied. Such conditions take the form of coupled Riccati equations or similarity transformations on the primal and dual Hamiltonian matrices. As a special case, one can make the dual DRE linear, by choosing the quadratic term coefficient zero, and solve it analytically. Using this method, the analytical solutions obtained earlier by Leipnik [KL85] and Rusnak [Rus88] can be easily derived, demonstrating the versatility and power of this approach.

We shall also use the fundamental solution developed here to solve stiff DREs with known analytical solutions, and benchmark them for accuracy, numerical stability and speed. These algorithms, being analytical and stable, are very useful for solving the stiff DRE for a long time horizon propagation, unlike the Davison-Maki method.

Finally even though the optimal control problem considered here, does make a number of assumptions to ensure existence of the value function, and avoid singularity, the resulting fundamental solution is valid for a much wider class of problems, since it is purely algebraic. As a special case, it is observed that even in case of unstable DREs, which exhibit the finite time blowup, the above fundamental solution can propagate beyond the singularity. For example, when applied to the special case $\dot{p} = 1 + p^2$, whose solution is $\tan(p)$, it correctly propagates beyond the singularities at $p = (2n+1)\frac{\pi}{2}, n \in \mathbb{W}$, where $\mathbb{W}$ is the set of whole numbers. But this aspect shall not be covered here in order to contain the scope of the thesis.

2.2 Optimal control problem

We shall obtain the fundamental solution for DRE (2.1) through an associated optimal control problem. To ensure the existence and the regularity of the value function, we make following assumptions throughout this section.

Since DREs exhibit a finite time blowup, we assume that for $t \in (\bar{T}, T]$ with $t \leq T$, there exists a solution of DRE (2.1) with the terminal condition P_T. We may have $\bar{T} = -\infty$. We assume that $\Sigma(t) \succeq 0$, $\forall t \in (\bar{T}, T]$. Hence let $\Sigma(t) = \sigma(t)\sigma(t)'$. We also assume controllability, that is given $x, y \in \mathbb{R}^n$, and $\bar{T} < t_1 < t_2 \leq T$, $\exists \tilde{u} \in L_2(t_1, t_2)$ such that the solution $\tilde{x}_t$ of $\dot{\tilde{x}}_t = A(t)\tilde{x}_t + \sigma(t)\tilde{u}_t$, satisfies $\tilde{x}_{t_1} = x$ and $\tilde{x}_{t_2} = y$. We also assume that $A(t), C(t), \Sigma(t)$ are piecewise continuous, locally bounded functions of time t, and $\Sigma(t) \succeq 0$ for all t. (2.3)

Now consider the following optimal control problem. Let $\bar{T} < t_1 \leq T$. We wish to maximize

$$J_{t_1}^{Tz}(x_1, u) \doteq \int_{t_1}^{T} l_t(x_t, u_t)\, dt + \phi^z(x_T) \tag{2.4}$$

where $x_t \in \mathbb{R}^n$ satisfies

$$\begin{aligned} \dot{x}_t &= f_t(x_t, u_t) \doteq A(t)x_t + \sigma(t)u_t && (2.5)\\ x_{t_1} &= x_1 && (2.6) \end{aligned}$$

and $l_t(x_t, u_t) \doteq \frac{1}{2}x_t{}'C(t)x_t - \frac{1}{2}|u_t|^2$ and with $u \in L_2(t_1, T)$ and the terminal payoff is bivariate quadratic in the state variable x and a parameter $z \in \mathbb{R}^n$ as below.

$$\phi^z(x) = \phi(x, z) \doteq \frac{1}{2}x'Px + x'Sz + \frac{1}{2}z'Qz \tag{2.7}$$

where P, Q are symmetric and S is invertible.

The optimal control value function is defined to be

$$V^z(t_1, x_1) = V_{t_1}^z(x_1) \doteq \sup_{u \in L_2(t_1, T)} J_{t_1}^{Tz}(x_1, u) \tag{2.8}$$

for all $x, z \in \mathbb{R}^n$ and $t_1 \in (\bar{T}, T]$. We shall show that the above value function is a solution of a particular Hamilton-Jacobi-Bellman (HJB) PDE.

Let $W^z(t, x)$ be the solution of following Hamilton-Jacobi-Bellman PDE on $(\bar{T}, T] \times \mathbb{R}^n$

$$0 = -\nabla_t W^z(t, x) - H\left(t, x, \nabla_x W^z(t, x)\right), \tag{2.9}$$

and with the boundary condition defined in (2.7)

$$W^z(T,x) = \phi(x,z) \qquad \forall x \in \mathbb{R}^n \tag{2.10}$$

where

$$\begin{aligned} H(t,x,p) &\doteq \sup_{u\in\mathbb{R}^n} \{p' f_t(x,u) + l_t(x,u)\} \\ &= \sup_{u\in\mathbb{R}^n} \left\{ p'\left(A(t)x + \sigma(t)u\right) + \frac{1}{2}x'C(t)x - \frac{1}{2}|u|^2\, dt \right\} \\ &= \frac{1}{2}x'C(t)x + x'A'(t)p + \frac{1}{2}p'\Sigma(t)p \end{aligned} \tag{2.11}$$

and $\Sigma(t) \doteq \sigma(t)\sigma'(t)$.

Lemma 2.2.1. *Assuming (2.3), for any $z \in \mathbb{R}^n$, there exists a solution to (2.9), (2.10), (2.11) in $C^\infty((\bar{T},T] \times \mathbb{R}^n)$, and this is given by*

$$W^z(t,x) = \frac{1}{2}x'P_t x + x'S_t z + \frac{1}{2}z'Q_t z \tag{2.12}$$

where P_t, S_t, Q_t satisfy $P_T = P$, $S_T = S$,$Q_T = Q$ and

$$\begin{aligned} -\dot{P}_t &= A(t)'P_t + P_t A(t) + C(t) + P_t\Sigma(t)P_t \\ -\dot{S}_t &= (A(t) + \Sigma(t)P_t)'S_t \\ -\dot{Q}_t &= S_t{}'\Sigma(t)S_t \end{aligned} \tag{2.13}$$

and S_t is invertible for all $t \in (\bar{T},T]$.

Proof. Existence of the solution $P_t : -\bar{T} < t \le T$ is assumed in (2.3). This combined with local boundedness, and piecewise continuity of coefficients guarantees the existence of S_t, and hence that of Q_t for $-\bar{T} < t \le T$. The proof that it solves HJB PDE, is immediate by substitution in (2.9) and (2.10). Let us define, $B(t) = -(A(t) + \Sigma(t)P_t)$. Then $S_{t_1} = \Phi_B(t_1,T)S_T$, where Φ_B is the state transition matrix of the system $\dot{x}_t = B(t)x_t$. By Abel-Jacobi-Liouville formula

$$\det \Phi_B(t_1,T) = e^{\int_{t_1}^T \mathbf{Tr} B(\tau)\, d\tau} > 0$$

Since both $\Phi_B(t_1,T)$ and $S_T = S$ are invertible, $S_{t_1} = \Phi_B(t_1,T)S_T$ is invertible as well. □

Next we need a verification theorem to connect HJB PDE solution to the control value function.

Theorem 2.2.2. *Assume (2.3). Let* W *and* J *be defined as per (2.12) and (2.4), respectively. Let* $x, z \in \mathbb{R}^n$ *and* $t_1 \in \left(\bar{T}, T\right]$*. One has*

$$W^z(t_1, x) \geq J_{t_1}^{Tz}(x, u) \qquad \forall u \in L_2(t_1, T)$$

and

$$W^z(t_1, x) = J_{t_1}^{Tz}(x, \tilde{u})$$

where $\tilde{u}_t = \tilde{u}(t, x_t) = \sigma(t)'\nabla W^z(t, x_t) = \sigma(t)'(P_t x_t + S_t z)$*, which implies* $W^z = V^z$ *and*

$$V_t(x, z) = V_t^z(x) = W_t^z(x) = \frac{1}{2}x'P_t x + x'S_t z + \frac{1}{2}z'Q_t z \tag{2.14}$$

Proof. Let $u \in L_2(t_1, T)$.

$$\begin{aligned} J_{t_1}^{Tz}(x, u) = & \int_{t_1}^{T} \left(L_t(x_t, u_t) + (A(t)x_t + \sigma(t)u_t)'\nabla W^z(t, x_t)\right)\, dt + \phi(x_T, z) \\ & - \int_{t_1}^{T} (A(t)x_t + \sigma(t)u_t)'\nabla W^z(t, x_t)\, dt \end{aligned}$$

which by definition of H

$$\leq \int_{t_1}^{T} H\left(x_t, \nabla W^z(t, x_t)\right)\, dt + \phi(x_T, z) - \int_{t_1}^{T} (A(t)x_t + \sigma(t)u_t)'\nabla W^z(t, x_t)\, dt$$

which by (2.9) and (2.5)

$$\begin{aligned} = & \int_{t_1}^{T} \left\{ -\frac{\partial}{\partial t} W^z(t, x_t) - \dot{x}_t \nabla W^z(t, x_t) \right\}\, dt + \phi(x_T, z) \\ = & - \int_{t_1}^{T} \frac{d}{dt} W^z(t, x_t)\, dt + \phi(x_T, z) \\ = & W^z(t_1, x) - W^z(T, x_T) + \phi(x_T, z) = W^z(t_1, x) \end{aligned}$$

using (2.10).

Also note that in the above proof, if we substitute $\tilde{u}_t = \sigma(t)'\nabla W^z(t, x_t)$, then using $l_t(x,u) = \frac{1}{2}x'C(t)x - |u_t|^2/2$, $\sigma(t)\sigma'(t) = \Sigma(t)$ and (2.11),

$$
\begin{aligned}
& l_t(x_t, u_t) + (A(t)x_t + \sigma(t)u_t)'\nabla W^z(t, x_t) \\
&= \frac{1}{2}{x_t}'C(t)x_t + \frac{1}{2}\nabla W^{z\prime}(t, x_t)\Sigma(t)\nabla W^z(t, x_t) + A'(t)\nabla W^z(t, x_t) \\
&= H\left(t, x_t, \nabla W^z(t, x_t)\right)
\end{aligned}
$$

This converts the inequality into equality and we get $J_{t_1}^{Tz}(x, \tilde{u}) = W^z(t_1, x)$. □

Now combining (2.8), lemma (2.2.1) and theorem 2.2.2 and substituting $z = 0$ (zero vector), redefining $V_t^0(x)$ as $\mathcal{P}(x)$, and parametrizing the univariate value function by p instead of P for the bivariate value function, we get the following corollary

Corollary 2.2.3. *Given $t \in \left(\bar{T}, T\right]$ and $x \in \mathbb{R}^n$, the value function*

$$
\mathcal{P}_t(x) = \sup_{u \in L_2(t,T)} \left\{ \int_{t_1}^{T} \left(\frac{1}{2}{x_t}'C(t)x_t - \frac{|u_t|^2}{2} \right) dt + \frac{1}{2}{x_T}'px_T) \right\} = \frac{1}{2}x'p_t x \quad (2.15)
$$

subject to the dynamics

$$
\dot{x}_t = A(t)x_t + \sigma(t)u_t
$$

satisfies HJB PDE

$$
-\nabla_t\mathcal{P}_t(x) = \frac{1}{2}x'C(t)x + x'A'(t)\,\nabla_x\mathcal{P}_t(x) + \frac{1}{2}\nabla_x\mathcal{P}_t(x)'\,\Sigma(t)\,\nabla_x\mathcal{P}_t(x)
$$

and p_t satisfies the following DRE in which $\Sigma(t) = \sigma(t)\sigma(t)'$,

$$
-\dot{p}_t = A(t)'p_t + p_t A(t) + C(t) + p_t\Sigma(t)p_t
$$

with boundary condition $p_T = p$.

This gives us the motivation to solve the DRE using the underlying optimal control problem. The optimal control problem defined in (2.8) with bivariate quadratic terminal payoff parametrized by z will be useful in deriving the fundamental solution as will be covered in sections ahead.

2.3 Fundamental Solution

For given t_2 and a general terminal payoff function $\phi(x) : \mathbb{R}^n \to \mathbb{R}$, let us define the operator,

$$\mathcal{S}_{t_1}^{t_2}[\phi](x) = \sup_{u \in L_2(t_1,t_2)} \int_{t_1}^{t_2} l_t(x_t, u_t)\, dt + \phi(x_{t_2}) \tag{2.16}$$

Suppose that $t_1 < t_2$ is such that the solution $\mathcal{S}_{t_1}^{t_2}[\phi](x)$ exists. (Is finite for any $x \in \mathbb{R}^n$.)

We can restate (2.8) and (2.4) using above operator. Noting that $V_T^z(x) = \phi^z(x)$, as defined in (2.7), we have for all $t \in \left(\bar{T}, T\right]$

$$V_t^z(x) = \mathcal{S}_t^T[\phi^z](x) = \mathcal{S}_t^T[V_T^z](x)$$

It is well known that operators $\mathcal{S}_{t_1}^{t_2}$ form a semigroup. That is if $t_1 \le t \le t_2 \le T$, then $\mathcal{S}_{t_1}^{t_2}[\phi] = \mathcal{S}_{t_1}^{t}[\mathcal{S}_t^{t_2}[\phi]]$, which is precisely the celebrated Dynamic programming principle for this problem. That is with $t_2 = T$,

$$\begin{aligned} V_{t_1}^z(x) &= \mathcal{S}_{t_1}^T[\phi^z](x) = \mathcal{S}_{t_1}^t\left[\mathcal{S}_t^T[\phi^z]\right](x) = \mathcal{S}_{t_1}^t\left[V_t^z\right](x) \\ &= \sup_{u \in L_2(t_1,t)} \int_{t_1}^{t} l_t(x_t, u_t)\, dt + V_t^z(x_t) \end{aligned} \tag{2.17}$$

Maslov [Mas87] proved that this semigroup is also linear in max-plus algebra. That is

$$\begin{aligned} \mathcal{S}_{t_1}^{t_2}[\max(\phi_1, \phi_2)](x) &= \max\left\{\mathcal{S}_{t_1}^{t_2}[\phi_1](x), \mathcal{S}_{t_1}^{t_2}[\phi_2](x)\right\} \\ \mathcal{S}_{t_1}^{t_2}[k + \phi_1](x) &= k + \mathcal{S}_{t_1}^{t_2}[\phi_1](x) \end{aligned}$$

where $\phi_1(x)$, $\phi_2(x)$ are functions and $k \in \mathbb{R}$ is a scalar. If we define $a \oplus b \doteq \max(a, b)$ and $a \otimes b \doteq a + b$, then it is well known that $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ forms a commutative semifield which is referred to as the max-plus algebra (see [BCOQ92],[HJ99], [LM98] for a fuller discussion).

We can extend it to functions so as to define the max-plus vector space. Let $[a \oplus b](x) = \max(a(x), b(x))$ and $a(x) \otimes k = a(x) + k$, where $a, b : \mathbb{R}^n \to \mathbb{R}$ and $k \in \mathbb{R}$. Using this notation, we have

$$\begin{aligned} \mathcal{S}_{t_1}^T[\phi_1 \oplus \phi_2](x) &= \mathcal{S}_{t_1}^T[\phi_1](x) \oplus \mathcal{S}_{t_1}^T[\phi_2](x) \\ \mathcal{S}_{t_1}^T[k \otimes \phi_1](x) &= k \otimes \mathcal{S}_{t_1}^T[\phi_1](x) \end{aligned} \tag{2.18}$$

Now we shall define a *max-plus kernel* $I : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ derived earlier in [FM00] and [Fle03]. Let $\bar{T} < t_1 \leq t_2 \leq T$ and $x, y \in \mathbb{R}^n$, and x_t evolve with dynamics (2.5). Define

$$I_{t_1}^{t_2}(x,y) \doteq \begin{cases} \sup_{u \in \mathcal{U}_{t_1}^{t_2}(x,y)} \int_{t_1}^{t_2} l_t(x_t,u_t)\,dt & \text{if } \mathcal{U}_{t_1}^{t_2}(x,y) \neq \emptyset \\ -\infty & \text{otherwise} \end{cases} \tag{2.19}$$

where

$$\mathcal{U}_{t_1}^{t_2}(x,y) \doteq \{u \in L_2(t_1,t_2) : x_{t_1} = x, x_{t_2} = y\}$$

Note that $I_{t_1}^{t_2} = -\infty$ indicates that it is impossible to reach y from x in time interval (t_1, t_2) using any possible control u.

Fleming and McEneaney [FM00] proposed above kernel, and showed that

$$\mathcal{S}_{t_1}^{t_2}[\phi](x) = \sup_{y \in \mathbb{R}^n} \left(I_{t_1}^{t_2}(x,y) + \phi(y) \right) \doteq \int_{\mathbb{R}^n}^{\oplus} I_{t_1}^{t_2}(x,y) \otimes \phi(y)\,dy \tag{2.20}$$

and since $I_{t_1}^{t_2}$ depends only on the dynamics $\dot{x}_t = f_t(x_t,u_t)$ and running payoff $l_t(x_t,u_t)$, it is independent of the terminal payoff $\phi(x_{t_2})$. Hence it can serve as a *Fundamental solution*, and obtain $\mathcal{S}_{t_1}^{t_2}[\phi](x)$ for any $\phi(x)$ by a kernel operation.

Remark 2.3.1. Note that $\mathcal{S}_{t_1}^{t_2}[\phi](x) > -\infty$ for all $x \in \mathbb{R}^n$. Since for any $\tilde{u} \in L_2(t_1,t_2)$, solution to $\tilde{x}_0 = x$ and $\dot{\tilde{x}}_t = f_t(\tilde{x}_t,\tilde{u}_t)$ exists for $t \in [t_1,t_2]$ and $\int_{t_1}^{t_2} l_t(\tilde{x}_t,\tilde{u}_t)\,dt > -\infty$. Hence

$$\mathcal{S}_{t_1}^{t_2}[\phi](x) \geq I_{t_1}^{t_2}(x,\tilde{x}_{t_2}) + \phi(\tilde{x}_{t_2}) \geq \int_{t_1}^{t_2} l_t(\tilde{x}_t,\tilde{u}_t)\,dt + \phi(\tilde{x}_{t_2}) > -\infty$$

Remark 2.3.2. Also note that due to the controllability assumption (2.3), for $t_1 < t_2$, we can always find control $\tilde{u}_t$ which generates the trajectory $\tilde{x}(t)$ satisfying $\tilde{x}_{t_1} = x$ and $\tilde{x}_{t_2} = y$. Thus $\mathcal{U}_{t_1}^{t_2}(x,y) \neq \emptyset$ and $I_{t_1}^{t_2}(x,y) \geq \int_{t_1}^{t_2} l_t(x_t,u_t)\,dt > -\infty$ for all $x, y \in \mathbb{R}^n$. For $t_1 = t_2$, $I_{t_1}^{t_2}(x,y) = -\infty$ for all $y \neq x$ and $I_{t_1}^{t_2}(x,x) = 0$.

2.3.1 Computing the max-plus kernel

First we derive a lemma about the end point of optimal trajectories.

Lemma 2.3.3. *Assume (2.3). Consider the system trajectory* $\tilde{x}_t$ *starting from* $\tilde{x}_{t_1} = x$ *and evolving according to (2.5) under optimal control* $\tilde{u}_t = \sigma(t)'(P_t\tilde{x}_t + S_t z)$ *from theorem 2.2.2. Then for* $\bar{T} < t_1 \le t_2 \le T$,

$$S_{t_2}{}'\tilde{x}_{t_2} + Q_{t_2}z = S_{t_1}{}'\tilde{x}_{t_1} + Q_{t_1}z \tag{2.21}$$

Proof. By time-varying linear system theory, for a system evolving as per

$$\begin{aligned}\dot{\tilde{x}}_t &= A(t)\tilde{x}_t + \sigma(t)\tilde{u}_t \\ &= A(t)\tilde{x}_t + \sigma(t)\sigma(t)'(P_t x_t + S_t z) \\ &= (A(t) + \Sigma(t)P_t)\, x_t + \Sigma(t)S_t z\end{aligned}$$

solution is given as

$$\tilde{x}_{t_2} = \Phi_B(t_2, t_1)\tilde{x}_{t_1} + \int_{t_1}^{t_2} \Phi_B(t_2, \tau)\Sigma(\tau)S_\tau z\, d\tau \tag{2.22}$$

where $\Phi_B(t_2, t_1) = U_{t_2}U_{t_1}^{-1}$, where U_t is the solution of differential equation $\dot{U}_t = B(t)U_t$, with $B(t) = A(t) + \Sigma(t)P_t$.

It is well known that the state transition matrix

$$\Phi_{B(t)}(t_2, t_1) = \Phi'_{-B(t)'}(t_1, t_2)$$

now, noting from (2.13) that $\dot{S}_t = -(A(t) + \Sigma(t)P_t)'S_t = -B(t)'S_t$, and since S_{t_2} is invertible, we have

$$\Phi_{B(t)}(t_2, t_1) = \Phi'_{-B(t)'}(t_1, t_2) = \left(S_{t_1}S_{t_2}^{-1}\right)' = S_{t_2}^{-1\prime}S_{t_1}{}' \tag{2.23}$$

Substituting in (2.22), and noting from (2.13) that $-\dot{Q}_t = S_t'\Sigma(t)S_t$,

$$\begin{aligned}\tilde{x}_{t_2} &= S_{t_2}^{-1\prime}S_{t_1}{}'\tilde{x}_{t_1} + S_{t_2}^{-1\prime}\int_{t_1}^{t_2} S_\tau{}'\Sigma(\tau)S_\tau z\, d\tau \\ &= S_{t_2}^{-1\prime}S_{t_1}{}'\tilde{x}_{t_1} + S_{t_2}^{-1\prime}\left(\int_{t_1}^{t_2} S_\tau{}'\Sigma(\tau)S_\tau\, d\tau\right) z \\ &= S_{t_2}^{-1\prime}S_{t_1}{}'\tilde{x}_{t_1} + S_{t_2}^{-1\prime}\left(Q_{t_1} - Q_{t_2}\right) z\end{aligned}$$

thus we have,

$$S_{t_2}{}'\tilde{x}_{t_2} + Q_{t_2}z = S_{t_1}{}'\tilde{x}_{t_1} + Q_{t_1}z$$

□

Remark 2.3.4. Note that using lemma 2.2.1, since S_{t_1} and S_{t_2} are invertible, (2.21) suggests a one-one and onto relation between start and end of optimal trajectories, x_{t_1} and x_{t_2} for all z. Thus $\forall y \in \mathbb{R}^n$ there exists a $x = S_{t_2}^{-1\prime}(S_{t_1}{}'y + (Q_{t_1} - Q_{t_2})z)$ such that optimal trajectory $\tilde{x}_t$ starting at $\tilde{x}_{t_1} = x$, ends with y. Thus every $y \in \mathbb{R}^n$ is an optimal point for some initial condition.

Remark 2.3.5. Note that due to max-plus linearity, if $k \in \mathbb{R}$, using (2.17),

$$\hat{V}_{t_1}^z = \mathcal{S}_{t_1}^T[\phi^z + k](x) = \mathcal{S}_{t_1}^T[\phi^z](x) + k = V_{t_1}^z(x) + k$$

Thus while keeping the dynamics and the incremental payoff same, adding a constant to the terminal payoff only shifts the value function accordingly. Hence the gradient and therefore the optimal feedback remain the same.

$$\hat{u}_t(x) = \sigma(t)\nabla\hat{V}_{t_1}^z(x) = \sigma(t)\nabla V_{t_1}^z(x) = \tilde{u}_t(x)$$

Hence the optimal trajectory, which is the solution to $\dot{\hat{x}}_t = A(t)\hat{x}_t + \sigma(t)\hat{u}_t(x)$, also stays the same.

Now we shall prove another useful lemma before turning to the main result.

Lemma 2.3.6. *Given $\bar{T} < t_1 < t_2 \leq T$, and Q_t evolving according to (2.13) with terminal value $Q_T = Q$, then*

$$Q_{t_1} - Q_{t_2} \succ 0$$

Proof. Note that we assumed in (2.3) that the system $\dot{x}_t = A(t)x_t + \sigma(t)u_t$ parametrized by $(A(t), \sigma(t))$ is controllable. This is true if and only if the following *controllability grammian* is invertible for any $\bar{T} < t_1 < t_2 \leq T$.

$$\int_{t_1}^{t_2} \Phi_A(t_1,\tau)\sigma(t)\sigma(t)'\Phi_A(t_1,\tau)'\,dt \succ 0 \tag{2.24}$$

Thus for all $x, y \in \mathbb{R}^n$, $\exists$ control $\hat{u}_t$ such that is the trajectory $\dot{\hat{x}} = A(t)\hat{x}_t + \sigma(t)\hat{u}_t$ with $\hat{x}_{t_1} = x$ satisfies $\hat{x}_{t_2} = y$.

Now we claim that system $(A(t) + \Sigma(t)P_t, \sigma(t))$ is also controllable. This is clear because by using control $\bar{u}_t = \hat{u}_t - \sigma(t)'P_t x_t$, we can keep the system trajectory same and reach from x to y.

$$\begin{aligned}\dot{\hat{x}} &= A(t)\hat{x}_t + \sigma(t)\hat{u}_t \\ &= (A(t) + \sigma(t)\sigma(t)'P_t)x_t + \sigma(t)\left(\hat{u}_t - \sigma(t)'P_t x_t\right) \\ &= (A(t) + \Sigma(t)P_t)x_t + \sigma(t)\bar{u}_t\end{aligned}$$

Hence similar to (2.24), using $B(t) = A(t) + \Sigma(t)P_t$ and $\sigma(t)\sigma(t)' = \Sigma(t)$, the following controllability grammian is invertible.

$$\int_{t_1}^{t_2} \Phi_B(t_1,\tau)\Sigma(t)\Phi_B(t_1,\tau)'\,dt \succ 0 \tag{2.25}$$

Substituting $\Phi_B(t_1,\tau) = S_{t_1}^{-1}{}'S_\tau{}'$ from (2.23),

$$\begin{aligned}\int_{t_1}^{t_2} \Phi_B(t_2,\tau)\Sigma(t)\Phi_B(t_2,\tau)'\,d\tau &= \int_{t_1}^{t_2} S_{t_1}^{-1}{}'S_\tau{}'\Sigma(t)S_\tau S_{t_1}^{-1}\,d\tau \\ &= S_{t_1}^{-1}{}'\left\{\int_{t_1}^{t_2} S_\tau{}'\Sigma(t)S_\tau\,d\tau\right\}S_{t_1}^{-1} \\ &= S_{t_1}^{-1}{}'\left(Q_{t_1} - Q_{t_2}\right)S_{t_1}^{-1} \end{aligned} \tag{2.26}$$

where in the last equation, we used Q_t evolution from (2.13). Using (2.25) and since S_{t_1} is invertible by Lemma (2.2.1), we have $Q_{t_1} - Q_{t_2} \succ 0$. □

Theorem 2.3.7. *Assuming (2.3), let V and I be as defined in (2.14) and (2.19), respectively. Given $x, y \in \mathbb{R}^n$ and $\bar{T} < t_1 \le t_2 \le T$, then*

$$\inf_{z\in\mathbb{R}^n}\left[V_{t_1}^z(x) - V_{t_2}^z(y)\right] = I_{t_1}^{t_2}(x,y) \tag{2.27}$$

Since by (2.14), $V_{t_1}^z(x) = \frac{1}{2}x'P_{t_1}x + x'S_{t_1}z + \frac{1}{2}z'Q_{t_1}z$ and $V_{t_2}^z(x) = \frac{1}{2}x'P_{t_2}x + x'S_{t_2}z + \frac{1}{2}z'Q_{t_2}z$, the max-plus kernel $I_{t_1}^{t_2}(x,y)$ is also bivariate quadratic.

$$\begin{aligned} I_{t_1}^{t_2}(x,y) &= \frac{1}{2}x'I_{11}{}_{t_1}^{t_2}x + x'I_{12}{}_{t_1}^{t_2}y + \frac{1}{2}y'I_{22}{}_{t_1}^{t_2}y \qquad \textit{where} \\ I_{11}{}_{t_1}^{t_2} &= P_{t_1} - S_{t_1}(Q_{t_1} - Q_{t_2})^{-1}S_{t_1}{}' \\ I_{12}{}_{t_1}^{t_2} &= S_{t_1}(Q_{t_1} - Q_{t_2})^{-1}S_{t_2}{}' \\ I_{22}{}_{t_1}^{t_2} &= -P_{t_2} - S_{t_2}(Q_{t_1} - Q_{t_2})^{-1}S_{t_2}{}' \end{aligned} \tag{2.28}$$

Proof. Let $x_{t_1} = x$. Since $\Sigma(t) \succeq 0$ and S_t is invertible, by (2.13), $\dot{Q}_t = -S_t'\Sigma(t)S_t \preceq 0$, hence $Q_{t_1} - Q_{t_2} \succeq 0$. For any $z \in \mathbb{R}^n$

$$\begin{aligned} &V_{t_1}^z(x) - V_{t_2}^z(y) \\ &\qquad = \mathcal{S}_{t_1}^{t_2}[V_{t_2}^z](x) - V_{t_2}^z(y) \\ &\qquad = \sup_{u \in L_2(t_1,t_2)} \left\{ \int_{t_1}^{t_2} l_t(x_t, u_t)\, dt + V_{t_2}^z(x_{t_2}) - V_{t_2}^z(y) \right\} \end{aligned}$$

substituting for $V_{t_2}^z$,

$$= \sup_{u \in L_2(t_1,t_2)} \left\{ \int_{t_1}^{t_2} l_t(x_t, u_t)\, dt + \frac{1}{2} x_{t_2}' P_{t_2} x_{t_2} - \frac{1}{2} y' P_{t_2} y + (x_{t_2} - y)' S_{t_2} z \right\}$$

Since $\mathcal{U}_{t_1}^{t_2}(x, y) = \{u \in L_2(t_1, t_2) : x_{t_1} = x, x_{t_2} = y\} \subset L_2(t_1, t_2)$, and $\forall u \in \mathcal{U}_{t_1}^{t_2}(x, y)$, $x_{t_2} = y$.

$$\begin{aligned} &\geq \sup_{u \in \mathcal{U}_{t_1}^{t_2}(x,y)} \left\{ \int_{t_1}^{t_2} l_t(x_t, u_t)\, dt + \frac{1}{2} y' P_{t_2} y - \frac{1}{2} y' P_{t_2} y + (y - y)' S_{t_2} z \right\} \\ &= \sup_{u \in \mathcal{U}_{t_1}^{t_2}(x,y)} \int_{t_1}^{t_2} l_t(x_t, u_t)\, dt = I_{t_1}^{t_2}(x, y) \end{aligned} \tag{2.29}$$

Taking infimum over all $z \in \mathbb{R}^n$,

$$\inf_{z \in \mathbb{R}^n} \left[V_{t_1}^z(x) - V_{t_2}^z(y) \right] \geq I_{t_1}^{t_2}(x, y) \tag{2.30}$$

Since $Q_{t_1} - Q_{t_2} \succ 0$ by 2.3.6, define $\hat{z} = (Q_{t_1} - Q_{t_2})^{-1}(S_{t_2}{}'y - S_{t_1}{}'x)$. Hence

$$S_{t_2}{}'y + Q_{t_2}\hat{z} = S_{t_1}{}'x + Q_{t_1}\hat{z}$$

hence using (2.21) the optimal trajectory $\tilde{x}_t$ starting from $\tilde{x}_{t_1} = x$ and with terminal payoff $V_{t_2}^{\hat{z}}(\cdot)$, ends at $\tilde{x}_{t_2} = y$. Let the corresponding optimal control be $\tilde{u}_t$. Let us define $k = -V_{t_2}^{\hat{z}}(y) = -\left(\frac{1}{2}\hat{z}' Q_{t_2} \hat{z} + \frac{1}{2} y' P_{t_2} y + y' S_{t_2} \hat{z}\right)$ to create a shifted terminal payoff function

$$\begin{aligned} U_{t_2}^{\hat{z}}(x) &= V_{t_2}^{\hat{z}}(x) + k = V_{t_2}^{\hat{z}}(x) - V_{t_2}^{\hat{z}}(y) \\ &= \frac{1}{2} x' P_{t_2} x - \frac{1}{2} y' P_{t_2} y + (x - y)' S_{t_2} \hat{z} \end{aligned} \tag{2.31}$$

From remark 2.3.5, $\tilde{u}_t$, $\tilde{x}_t$ are also the optimal control and trajectory for the following problem with the terminal payoff $U^z_{t_2}$. Hence

$$\begin{aligned}
V^{\hat{z}}_{t_1}(x) - V^{\hat{z}}_{t_2}(y) &= \left\{ \sup_{u \in L_2(t_1,t_2)} \int_{t_1}^{t_2} l_t(x_t, u_t)\, dt + V^{\hat{z}}_{t_2}(x) \right\} - V^{\hat{z}}_{t_2}(y) \\
&= \sup_{u \in L_2(t_1,t_2)} \left\{ \int_{t_1}^{t_2} l_t(x_t, u_t)\, dt + V^{\hat{z}}_{t_2}(x) - V^{\hat{z}}_{t_2}(y) \right\} \\
&= \sup_{u \in L_2(t_1,t_2)} \int_{t_1}^{t_2} l_t(x_t, u_t)\, dt + U^{\hat{z}}_{t_2}(x_{t_2}) \\
&= \int_{t_1}^{t_2} l_t(\tilde{x}_t, \tilde{u}_t)\, dt + U^{\hat{z}}_{t_2}(\tilde{x}_{t_2})
\end{aligned}$$

since $U^{\hat{z}}_{t_2}(\tilde{x}_{t_2}) = U^{\hat{z}}_{t_2}(y) = 0$ from (2.31) and $\tilde{u} \in \mathcal{U}^{t_2}_{t_1}(x, y)$

$$\leq \sup_{u \in \mathcal{U}^{t_2}_{t_1}(x,y)} \int_{t_1}^{t_2} l_t(x_t, u_t)\, dt = I^{t_2}_{t_1}(x, y) \tag{2.32}$$

Thus we have

$$\inf_z \left[V^z_{t_1}(x) - V^z_{t_2}(y) \right] \leq V^{\hat{z}}_{t_1}(x) - V^{\hat{z}}_{t_2}(y) \leq I^{t_2}_{t_1}(x, y) \tag{2.33}$$

Hence (2.30) and (2.33) together give us (2.27) and also the following

$$\inf_{z \in \mathbb{R}^n} \left[V^z_{t_1}(x) - V^z_{t_2}(y) \right] = V^{\hat{z}}_{t_1}(x) - V^{\hat{z}}_{t_2}(y) = I^{t_2}_{t_1}(x, y) \tag{2.34}$$

with $\hat{z} = (Q_{t_1} - Q_{t_2})^{-1}({S_{t_2}}'y - {S_{t_1}}'x)$.

Substituting $\hat{z}$ in (2.34) and expanding, we get (2.28). □

Remark 2.3.8. It is interesting to note that the formulas extend graciously even when assumptions on controllability are violated. In that case $Q_{t_1} - Q_{t_2} \succeq 0$ and may not be invertible. We can do singular value decomposition

$$Q_{t_1} - Q_{t_2} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1' \\ U_2' \end{bmatrix}$$

where $[U_1 \;\; U_2]$ is unitary matrix, and Λ is diagonal matrix of nonzero eigenvalues. We can obtain *Moore-Penrose pseudoinverse* as

$$(Q_{t_1} - Q_{t_2})^+ = U_1 \Lambda^{-1} U_1'$$

If we take the limit of the formula (2.28), by replacing all zero eigenvalues by $k > 0$, and letting $k \to 0$, then we obtain following formulas, which also give us a representation of the reachable set.

$$I_{t_1}^{t_2}(x,y) = \begin{cases} \begin{array}{l} \text{(2.28) with } (Q_{t_1} - Q_{t_2})^{-1} \\ \text{replaced by } (Q_{t_1} - Q_{t_2})^{+} \end{array} & \text{if } {S_{t_1}}'x - {S_{t_2}}'y \in \mathbf{range}(U_1) \\ \\ -\infty & \text{otherwise} \end{cases}$$

In the special case when $\Sigma(t)$ is zero matrix, and no control is possible. U_1 is empty, since there are no nonzero eigenvalues. $(Q_1 - Q_2)$ and $(Q_1 - Q_2)^+$ are zero matrices. Hence $\mathbf{range}(U_1) = 0$, which is the null range. Hence ${S_{t_1}}'x - {S_{t_2}}'y = 0$, and

$$I_{t_1}^{t_2}(x,y) = \begin{cases} x'P_{t_1}x - y'P_{t_2}y & \text{if } y = S_{t_2}^{-1\prime}{S_{t_1}}'x \\ \\ -\infty & \text{otherwise} \end{cases}$$

Thus using (2.23), only state accessible from starting point x is $S_{t_2}^{-1\prime}{S_{t_1}}'x = \Phi_{B(t)}(t_2,t_1)x = \Phi_{A(t)}(t_2,t_1)x$, since $B(t) = A(t) + \Sigma(t)P_t = A(t)$. This is the well-known solution to time-varying linear differential equation, $\dot{x}_t = A(t)x_t$.

Now we shall prove a theorem which can allow us to combine max-plus kernels in time.

Theorem 2.3.9. *Assuming (2.3), let $\bar{T} < t_1 < t_2 < t_3 \leq T$, then max-plus kernel $I_{t_1}^{t_3}$ can be computed from $I_{t_1}^{t_2}$ and $I_{t_2}^{t_3}$ as follows*

$$I_{t_1}^{t_3}(x,y) = \mathcal{S}_{t_1}^{t_2}[I_{t_2}^{t_3}(.,y)](x) = \sup_{z \in \mathbb{R}^n} \left\{ I_{t_1}^{t_2}(x,z) + I_{t_2}^{t_3}(z,y) \right\} \tag{2.35}$$

Thus $I_{t_1}^{t_3}(x,y) = \frac{1}{2}x'{I_{11}}_{t_1}^{t_3}x + x'{I_{12}}_{t_1}^{t_3}y + \frac{1}{2}y'{I_{22}}_{t_1}^{t_3}y$ *where*

$$\begin{aligned} {I_{11}}_{t_1}^{t_3} &= {I_{11}}_{t_1}^{t_2} - {I_{12}}_{t_1}^{t_2} \left({I_{22}}_{t_1}^{t_2} + {I_{11}}_{t_2}^{t_3}\right)^{-1} {{I_{12}}_{t_1}^{t_2}}^T \\ {I_{12}}_{t_1}^{t_3} &= -{I_{12}}_{t_1}^{t_2} \left({I_{22}}_{t_1}^{t_2} + {I_{11}}_{t_2}^{t_3}\right)^{-1} {I_{12}}_{t_2}^{t_3} \\ {I_{22}}_{t_1}^{t_3} &= {I_{22}}_{t_2}^{t_3} - {{I_{12}}_{t_2}^{t_3}}^T \left({I_{22}}_{t_1}^{t_2} + {I_{11}}_{t_2}^{t_3}\right)^{-1} {I_{12}}_{t_2}^{t_3} \end{aligned} \tag{2.36}$$

Proof. Note that by remark 2.3.2, $\mathcal{U}_{t_a}^{t_b}(x,y) \neq \emptyset$ for all $t_a < t_b$ and $x, y \in \mathbb{R}^n$.

$$I_{t_1}^{t_3}(x,y) = \sup_{u \in \mathcal{U}_{t_1}^{t_2}(x,y)} \int_{t_1}^{t_3} l_t(x_t, u_t)\, dt$$

since $\mathcal{U}_{t_1}^{t_3}(x,y) = \bigcup_{z \in \mathbb{R}^n} \left(\mathcal{U}_{t_1}^{t_2}(x,z) \cap \mathcal{U}_{t_2}^{t_3}(z,y)\right)$

$$= \sup_{z \in \mathbb{R}^n} \sup_{u \in U_{t_1}^{t_2}(x,z) \cap U_{t_2}^{t_3}(z,y)} \int_{t_1}^{t_3} l_t(x_t, u_t)\, dt \tag{2.37}$$

Now, we consider the following

$$\begin{aligned}
&\sup_{u \in U_{t_1}^{t_2}(x,z) \cap U_{t_2}^{t_3}(z,y)} \int_{t_1}^{t_3} l_t(x_t, u_t)\, dt \\
&\leq \sup_{u \in U_{t_1}^{t_2}(x,z) \cap U_{t_2}^{t_3}(z,y)} \left\{ \int_{t_1}^{t_2} l_t(x_t, u_t)\, dt + \int_{t_2}^{t_3} l_t(x_t, u_t)\, dt \right\} \\
&\leq \sup_{u \in U_{t_1}^{t_2}(x,z)} \left\{ \int_{t_1}^{t_2} l_t(x_t, u_t)\, dt \right\} + \sup_{u \in U_{t_2}^{t_3}(z,y)} \left\{ \int_{t_2}^{t_3} l_t(x_t, u_t)\, dt \right\} \\
&= I_{t_1}^{t_2}(x,z) + I_{t_2}^{t_3}(z,y)
\end{aligned} \tag{2.38}$$

Now, since $I_{t_1}^{t_2}(x,z) > -\infty$, $\forall \epsilon > 0$, $\exists\ \bar{u} \in \mathcal{U}_{t_1}^{t_2}(x,z)$ and trajectory $\bar{x}_t$ with $\bar{x}_{t_1} = x$ such that

$$\int_{t_1}^{t_2} l_t(\bar{x}_t, \bar{u}_t)\, dt + \epsilon \geq I_{t_1}^{t_2}(x,z) \tag{2.39}$$

Similarly $\exists\ \tilde{u} \in \mathcal{U}_{t_2}^{t_3}(z,y)$ and trajectory $\tilde{x}_t$ with $\tilde{x}_{t_2} = z$ such that

$$\int_{t_2}^{t_3} l_t(\tilde{x}_t, \tilde{u}_t)\, dt + \epsilon \geq I_{t_2}^{t_3}(z,y) \tag{2.40}$$

Now we can create augmented control $\hat{u}$ such that $\hat{u}_t = \bar{u}_t$ for $t \in [t_1, t_2)$ and $\hat{u}_t = \tilde{u}_t$ for $t \in [t_2, t_3]$, and extend it arbitrarily beyond. Note that if $\hat{x}_t$ is corresponding trajectory, then starting with $\hat{x}_{t_1} = x$, $\hat{x}_t = \bar{x}_t$ for $t \in [t_1, t_2]$. Hence $\hat{x}_{t_2} = z$ and $\hat{x}_t = \tilde{x}_t$ for $t \in [t_2, t_3]$, hence $\hat{x}_{t_3} = y$. Hence $\hat{u} \in \mathcal{U}_{t_1}^{t_2}(x,z) \cap \mathcal{U}_{t_2}^{t_3}(z,y)$. Moreover using (2.39) and (2.40),

$$\begin{aligned}
\sup_{u \in U_{t_1}^{t_2}(x,z) \cap U_{t_2}^{t_3}(z,y)} \int_{t_1}^{t_3} l_t(x_t, u_t)\, dt &\geq \int_{t_1}^{t_2} l_t(\hat{x}_t, \hat{u}_t)\, dt + \int_{t_2}^{t_3} l_t(\hat{x}_t, \hat{u}_t)\, dt \\
&= \int_{t_1}^{t_2} l_t(\bar{x}_t, \bar{u}_t)\, dt + \int_{t_2}^{t_3} l_t(\tilde{x}_t, \tilde{u}_t)\, dt \\
&= I_{t_1}^{t_2}(x,z) + I_{t_2}^{t_3}(z,y) - 2\epsilon
\end{aligned} \tag{2.41}$$

Since ϵ is arbitrary, from (2.38) and (2.41), we have

$$\sup_{u \in U_{t_1}^{t_2}(x,z) \cap U_{t_2}^{t_3}(z,y)} \int_{t_1}^{t_3} l_t(x_t, u_t)\, dt = I_{t_1}^{t_2}(x,z) + I_{t_2}^{t_3}(z,y)$$

which with (2.37) proves (2.35). Now, using (2.28) and since $(Q_{t_1} - Q_{t_2}) \succ 0$ and $(Q_{t_2} - Q_{t_3}) \succ 0$,

$$\begin{aligned} & {I_{22}}_{t_1}^{t_2} + {I_{11}}_{t_2}^{t_3} \\ &\quad = \left(-P_{t_2} - S_{t_2}(Q_{t_1} - Q_{t_2})^{-1} {S_{t_2}}'\right) + \left(P_{t_2} - S_{t_2}(Q_{t_2} - Q_{t_3})^{-1} {S_{t_2}}'\right) \\ &\quad = -S_{t_2}\left((Q_{t_1} - Q_{t_2})^{-1} + (Q_{t_2} - Q_{t_3})^{-1}\right) {S_{t_2}}' \\ &\quad \prec 0 \end{aligned}$$

Thus $\left\{I_{t_1}^{t_2}(x,z) + I_{t_2}^{t_3}(z,y)\right\}$ is concave in z. Thus supremum in (2.35) exists, and we get (2.36) by algebraic computation of the local maxima. □

Remark 2.3.10. Note that $I_t^{t_3}(x,z)$ has the same bivariate form as V_t^z given by (2.14), and both $I_t^{t_3}$ and V_t evolve in the time interval (t_1, t_2) according to the semigroup $\mathcal{S}_{t_1}^{t_2}$ as per (2.35). Hence the parameters satisfy DREs similar to the (2.13).

$$\begin{aligned} -\frac{d}{dt}{I_{11}}_t^{t_3} &= A(t)'{I_{11}}_t^{t_3} + {I_{11}}_t^{t_3}A(t) + C(t) + {I_{11}}_t^{t_3}\Sigma(t){I_{11}}_t^{t_3} \\ -\frac{d}{dt}{I_{12}}_t^{t_3} &= (A(t) + \Sigma(t){I_{11}}_t^{t_3})'{I_{12}}_t^{t_3} \\ -\frac{d}{dt}{I_{22}}_t^{t_3} &= {I_{12}}_t^{t_3\prime}\Sigma(t){I_{12}}_t^{t_3} \end{aligned} \tag{2.42}$$

2.3.2 Algorithm

Thus following is the final algorithm to obtain the fundamental solution, and to convert a particular solution of (2.13) into a general solution. It gives us a closed form solution to the DRE (2.1) using max-plus kernel $I_{t_1}^{t_2}$ (2.20). We shall reiterate the formulae derived earlier to make the section self-contained.

- Choose terminal t_2 and the parameters $(P_{t_2}, S_{t_2}, Q_{t_2})$ of the terminal bivariate payoff $V_{t_2}^z(x) = \frac{1}{2}x'P_{t_2}x + x'S_{t_2}z + \frac{1}{2}z'Q_{t_2}z$, such that P_{t_2}, Q_{t_2} are $n \times n$ symmetric matrices, and S_{t_2} is $n \times n$ invertible matrix.

- Propagate (P,S,Q) backwards in time according to (2.13) till time $t_1 < t_2$. That is

$$
\begin{aligned}
-\dot{P}_t &= A(t)'P_t + P_tA(t) + C(t) + P_t\Sigma(t)P_t\\
-\dot{S}_t &= (A(t) + \Sigma(t)P_t)'S_t\\
-\dot{Q}_t &= {S_t}'\Sigma(t)S_t
\end{aligned}
$$

- Compute the *max-plus kernel* or the fundamental solution as per (2.28),

$$I_{t_1}^{t_2}(x,y) = I_{t_1}^{t_2}(x,y) = \frac{1}{2}x'{I_{11}}_{t_1}^{t_2}x + x'{I_{12}}_{t_1}^{t_2}y + \frac{1}{2}y'{I_{22}}_{t_1}^{t_2}y$$

parametrized by triplet $(I_{11}, I_{12}, I_{22})_{t_1}^{t_2}$ where

$$
\begin{aligned}
I_{11} &= P_{t_1} - S_{t_1}(Q_{t_1} - Q_{t_2})^{-1}{S_{t_1}}'\\
I_{12} &= S_{t_1}(Q_{t_1} - Q_{t_2})^{-1}{S_{t_2}}'\\
I_{22} &= -P_{t_2} - S_{t_2}(Q_{t_1} - Q_{t_2})^{-1}{S_{t_2}}'
\end{aligned}
$$

- As per corollary 2.2.3, if terminal payoff is given by $\mathcal{P}_{t_2}(x) = \frac{1}{2}x'p_{t_2}x$ and if p_t evolves as per DRE (2.1),

$$-\dot{p} = A(t)'p_t + p_tA(t) + C(t) + p_t\Sigma(t)p_t$$

and if p_{t_1} exists, i.e. the solution does not blow up during $t_2 \to t_1$ evolution. Then by (2.20)

$$\mathcal{P}_{t_1}(x) = \frac{1}{2}x'p_{t_1}x = \mathcal{S}_{t_1}^{t_2}[\mathcal{P}_{t_2}](x) = \sup_{y\in\mathbb{R}^n} I_{t_1}^{t_2}(x,y) + \mathcal{P}_{t_2}(y) \tag{2.43}$$

Thus algebraically we get

$$\frac{1}{2}x'p_{t_1}x = \sup_{y}\left\{\frac{1}{2}x'I_{11}x + x'I_{12}y + \frac{1}{2}y'(I_{22} + p_{t_2})y\right\} \tag{2.44}$$

$$p_{t_1} = I_{11} - I_{12}(p_{t_2} + I_{22})^{-1}{I_{12}}' \tag{2.45}$$

which is the analytical solution to the DRE (2.1). Thus we have converted a general solution to a bivariate DRE (P,S,Q) as per (2.13) into a fundamental solution $I_{t_1}^{t_2}$, and then into a particular solution p_t of (2.1).

- As seen in (2.28), as $(t_2 - t_1) \to 0$, $(Q_{t_1} - Q_{t_2})^{-1}$ may blow up, as $Q_{t_1} \to Q_{t_2}$. Thus parameters of the max-plus kernel $I_{t_1}^{t_2}$ also blow up, causing numerical inaccuracy in propagation. To remedy this, an alternate form of propagation (2.45) is proposed as follows.

  After substituting kernel parameters from (2.28) in (2.45), with some manipulation we get

$$\begin{aligned}
&\left(S_{t_1}{}'(p_{t_1} - P_{t_1})^{-1} S_{t_1}\right)^{-1} \\
&= -(Q_{t_1} - Q_{t_2})^{-1} \\
&\quad - (Q_{t_1} - Q_{t_2})^{-1} S_{t_2}{}' \left(p_{t_2} - P_{t_2} - S_{t_2}(Q_{t_1} - Q_{t_2})^{-1} S_{t_2}{}'\right)^{-1} S_{t_2}(Q_{t_1} - Q_{t_2})^{-1}
\end{aligned}$$

  which by using Woodbury's matrix identity [Woo50],

$$= \left(-(Q_{t_1} - Q_{t_2}) + S_{t_2}{}'(p_{t_2} - P_{t_2})^{-1} S_{t_2}\right)^{-1}$$

  Thus

$$S_{t_1}{}'(p_{t_1} - P_{t_1})^{-1} S_{t_1} + Q_{t_1} = S_{t_2}{}'(p_{t_2} - P_{t_2})^{-1} S_{t_2} + Q_{t_2}$$

  or rearranging, propagation from p_{t_2} to p_{t_1} is given by

$$p_{t_1} = P_{t_1} - S_{t_1}\left(Q_{t_1} - Q_{t_2} - S_{t_2}{}'(p_{t_2} - P_{t_2})^{-1} S_{t_2}\right)^{-1} S_{t_1}{}' \qquad (2.46)$$

  This formula does not blow up for a small time step propagation, and yields an accurate propagation.

Remark 2.3.11. Note that we assumed that the propagation $\frac{1}{2}x' p_{t_1} x = \mathcal{S}_{t_1}^{t_2}[V_{t_2}](x)$ exists, and derived (2.45). This is also equivalent to $I_{22} + p_{t_2} \prec 0$, so that the supremum in (2.44) exists. Thus ${I_{22}}_{t_1}^{t_2} + p_{t_2} \prec 0$ characterizes all initial conditions p_{t_1} for which DRE propagation $t_2 \to t_1$ does not blow up. Also note that the minimum time $\bar{T}$ for which solution to DRE exists, depends on initial condition. Max-plus kernel obtained from one particular solution, may cause blow up for a different initial condition. Surprisingly it is possible to pass through singularity/ solution blow up using (2.45) without numerical instability (since instead of marching through singularity, we step over it), and generate solution trajectories akin to $\mathbf{tan}(x)$ which is the solution to $\dot{x} = 1 + x^2$.

2.4 Semiconvex dual DRE

Now we shall introduce the concept of semiconvex duality which can help us transform time invariant DREs into semiconvex dual DREs.

2.4.1 Semiconvex duality

A function $\mathcal{P}(x) : \mathbb{R}^n \to \mathbb{R}^- \doteq \mathbb{R} \cup \{-\infty\}$ is defined to be *uniformly semiconvex* with (symmetric) matrix constant K if $\mathcal{P}(x) + \frac{1}{2}x'Kx$ is convex over $\mathbb{R}^n$. We denote this space by $\mathcal{S}^K$.

Semiconvex duality is parametrized by a bivariate quadratic kernel

$$\phi(x, z) = \frac{1}{2}x'Px + x'Sz + \frac{1}{2}z'Qz \tag{2.47}$$

where P and Q are symmetric matrices. We use this kernel to define semiconvex duality.

Theorem 2.4.1. *Let $\mathcal{P} \in \mathcal{S}^{-P}$, S is invertible and $\phi(x, z)$ defined as above. Then $\forall z \in \mathbb{R}^n$ we can define the dual $\mathcal{Q}(z)$ of primal $\mathcal{P}(x)$ as follows.*

$$\mathcal{Q}(z) = \inf_x [\mathcal{P}(x) - \phi(x, z)] \doteq \mathcal{D}_\phi[\mathcal{P}](z) \tag{2.48}$$

from the dual $\mathcal{Q}(z)$, primal can be recovered again using

$$\mathcal{P}(x) = \sup_z [\phi(x, z) + \mathcal{Q}(z)] \doteq \mathcal{D}_\phi^{-1}[\mathcal{Q}](x) \tag{2.49}$$

$\phi(x, z)$ is called the kernel of duality. Thus $\mathcal{D}_\phi^{-1}\mathcal{D}_\phi[\mathcal{P}](x) = \mathcal{P}(x)$.

Proof. Since $\mathcal{P} \in \mathcal{S}^{-P}$, $\mathcal{P}(x) - \phi(x, z)$ is convex in x. Now,

$$\begin{aligned}
\sup_z [\phi(x, z) + \mathcal{Q}(z)] &= \sup_z [\phi(x, z) + \inf_y [\mathcal{P}(y) - \phi(y, z)]] \\
&= \sup_z \inf_y [\mathcal{P}(y) + \phi(x, z) - \phi(y, z)] \\
&= \sup_z \inf_y [\mathcal{P}(y) + \frac{1}{2}x'Px - \frac{1}{2}y'Py + (x - y)'Sz]
\end{aligned}$$

Let $\bar{z} = Sz$. Since S is invertible, $\bar{z}$ also spans $\mathbb{R}^n$

$$= \frac{1}{2}x'Px + \sup_{\bar{z}} \inf_{y} [\mathcal{P}(y) - \frac{1}{2}y'Py + (x-y)'\bar{z}]$$
$$= \frac{1}{2}x'Px + \sup_{z'} [x'\bar{z} + \inf_{y} [\mathcal{P}(y) - \frac{1}{2}y'Py - y'\bar{z}]$$

Since $\mathcal{P}(y) - \frac{1}{2}y'Py$ is convex, by Legendre-Fenchel transform (see e.g. Theorem 11.1 in [RW98])

$$= \frac{1}{2}x'Px + \mathcal{P}(x) - \frac{1}{2}x'Px$$
$$= \mathcal{P}(x)$$

□

If we choose $\mathcal{P}(x) = \frac{1}{2}x'px$, $\phi(x,z) = \frac{1}{2}x'Px + x'Sz + \frac{1}{2}z'Qz$ and assume $p > P$ and S is nonsingular, then $\mathcal{P}(x) \in \mathcal{S}^{-p}$. Hence by substitution in (2.48), we get $\mathcal{Q}(z) = \frac{1}{2}z'qz$, where

$$q = -S'\left(p - P\right)^{-1} S - Q \tag{2.50}$$

We can also derive the following inverse relation

$$p = -S\left(q + Q\right)^{-1} S' + P \tag{2.51}$$

Corollary 2.4.2. *Using very similar methodology, if $\mathcal{Q}(z) + z'Qz$ is concave over $z \in \mathbb{R}^n$, then*

$$\mathcal{D}_\phi \mathcal{D}_\phi^{-1} \mathcal{Q}(z) = \mathcal{Q}(z) \tag{2.52}$$

Remark 2.4.3. Let us observe that a result using (2.20) and (2.17) we saw earlier, can be reposed in the following manner using semiconvex dual notation. With $t_1 < t_2$, since $\mathcal{S}_{t_1}^{t_2}[\phi](x)$ exists, and $I_{t_1}^{t_2}(x,y)$ is bivariate quadratic, we have

$$V_{t_1}^z(x) = \mathcal{S}_{t_1}^{t_2}[V_{t_2}^z](x) = \sup_y \left(I_{t_1}^{t_2}(x,y) + V_{t_2}^z(y)\right) = \mathcal{D}_{I_{t_1}^{t_2}}^{-1}[V_{t_2}^z](x) \tag{2.53}$$

Since above supremum exists for all x, ${I_{22}}_{t_1}^{t_2} + P_{t_2} \preceq 0$, hence by (2.45), and matrix congruence $P_{t_1} - {I_{11}}_{t_1}^{t_2} = -{I_{12}}_{t_1}^{t_2}\left(P_{t_2} + {I_{22}}_{t_1}^{t_2}\right)^{-1} {I_{12}}_{t_1}^{t_2}{}' \succeq 0$. Hence $V_{t_1}^z(x) \in \mathcal{S}^{-I_{11}}$. Hence we can take semiconvex dual. Now using (2.52),

$$\inf_x \left(V_{t_1}^z(x) - I_{t_1}^{t_2}(x,y)\right) = \mathcal{D}_{I_{t_1}^{t_2}}[V_{t_1}^z](y) = \mathcal{D}_{I_{t_1}^{t_2}} \mathcal{D}_{I_{t_1}^{t_2}}^{-1}[V_{t_2}^z](y) = V_{t_2}^z(y) \tag{2.54}$$

As a special case, from (2.43), we have

$$\mathcal{P}_{t_1}(x) = \mathcal{S}_{t_1}^{t_2}[\mathcal{P}_{t_2}](x) = \sup_{y\in\mathbb{R}^n} I_{t_1}^{t_2}(x,y) + \mathcal{P}_{t_2}(y) = \mathcal{D}^{-1}_{I_{t_1}^{t_2}}[\mathcal{P}_{t_2}](x) \tag{2.55}$$

2.4.2 Dual differential Riccati equation

Now let us start with primal space quadratic $\mathcal{P}(x) = \frac{1}{2}x'p_t x$, with p_t varying with time as per (2.1). Let us find its dual $\mathcal{Q}(z) = \frac{1}{2}z'q_t z$, with kernel $\phi(x,z) = \frac{1}{2}x'Px + x'Sz + \frac{1}{2}z'Qz$. We assume $p_t - P > 0$ and S is nonsingular, so that we can use theorem 2.4.1. Using (2.50) and (2.51), we get

$$q_t = -S'\left(p_t - P\right)^{-1} S - Q \tag{2.56}$$

$$p_t = -S\left(q_t + Q\right)^{-1} S' + P \tag{2.57}$$

Differentiating both sides of (2.56),

$$\dot{q}_t = S'\left(p_t - P\right)^{-1} \dot{p}_t \left(p_t - P\right)^{-1} S \tag{2.58}$$

If the primal quadratic evolves according to (2.1), we can track the evolution of the dual. Substituting for $\dot{p}_t$ from (2.1), and for p_t from (2.57) in (2.58), we get

$$\begin{aligned}\dot{q}_t =& (q_t + Q)S^{-1}\left(A(t)'P + PA(t) + P\Sigma(t)P\right) S^{T^{-1}}(q_t + Q)\\ &- (q_t + Q)S^{-1}(A(t) + \Sigma(t)P)^T S\\ &- S^T(A(t) + \Sigma(t)P)S^{T^{-1}}(q_t + Q) + S'\Sigma(t)S\end{aligned}$$

Using (2.13), and after simplification, we get.

$$\begin{aligned}-\dot{q}_t &= q_t S^{-1}\dot{P}S^{-1^T} q_t + q_t S^{-1}(\dot{P}S^{-1^T}Q - \dot{S})\\ &+(\dot{P}S^{-1^T}Q - \dot{S})'S^{-1\prime}q_t + QS^{-1}\dot{P}S^{-1\prime}Q\\ &-QS^{-1}\dot{S} - (QS^{-1}\dot{S})^T + \dot{Q}\end{aligned}$$

This shows that the dual quadratic also satisfies a Riccati equation

$$-\dot{q}_t = \bar{A}(t)'q_t + q_t\bar{A}(t) + \bar{C}(t) + q_t{}'\bar{\Sigma}(t)q_t \tag{2.59}$$

with coefficients

$$\begin{aligned}\bar{A}(t) &= S^{-1}(\dot{P}S^{-1^T}Q - \dot{S}) \\ \bar{\Sigma}(t) &= S^{-1}\dot{P}S^{-1^T} \\ \bar{C}(t) &= QS^{-1}\dot{P}S^{-1\prime}Q \\ &\quad -QS^{-1}\dot{S} - (QS^{-1}\dot{S})^T + \dot{Q}\end{aligned} \tag{2.60}$$

where $\dot{P}$, $\dot{S}$ and $\dot{Q}$ are constants defined by (2.13). Thus $\left(\bar{A}(t), \bar{\Sigma}(t), \bar{C}(t)\right) = f\left(P, S, Q, A(t), \Sigma(t), C(t)\right)$.

2.4.3 Kernel Matching conditions

By using (2.108) and some algebraic manipulation, it can be easily shown that simultaneous equations (2.60) are equivalent to following simultaneous equations,

$$\begin{aligned}-\dot{P} &= A(t)'P + PA(t) + C(t) + P\Sigma(t)P = S\bar{\Sigma}(t)S' \\ -\dot{S} &= (A(t) + \Sigma(t)P)'S = S(-\bar{A}(t) + \bar{\Sigma}(t)Q) \\ -\dot{Q} &= S'\Sigma(t)S = -\bar{A}(t)'Q - Q\bar{A} + \bar{C}(t) + Q\bar{\Sigma}(t)Q\end{aligned} \tag{2.61}$$

These give a neater feasibility condition for finding a kernel parameters, (P, S, Q) to transform Riccati equation (2.105) into any other Riccati equation (2.59).

Remark 2.4.4. Observing the symmetry between the primal and dual DREs motivates us to propose a dual problem with dynamics $\dot{z} = \bar{A}(t)z + \bar{\sigma}(t)u$, with $\bar{\sigma}\bar{\sigma}(t)' = \bar{\Sigma}(t)$ (if $\bar{\Sigma} \succeq 0$), and payoff $\bar{l}_t(z_t, u_t) = \frac{1}{2}z'\bar{C}(t)z$, and a corresponding dual semigroup $\bar{\mathcal{S}}_{t_1}^{t_2}$ similar to (2.16). Using the symmetry of above equations, it can be easily proved that

$$\phi_{t_1}(x, z) = \mathcal{S}_{t_1}^{t_2}[\phi_{t_2}(\cdot, z)](x) = -\bar{\mathcal{S}}_{t_2}^{t_1}[-\phi_{t_2}(x, \cdot)](z) \tag{2.62}$$

Thus given the coefficients of primal and dual Riccati equations, both (P_0, S_0, R_0) and (P_t, S_t, R_t) satisfy (2.61), suggesting that these equations are not independent. Indeed, it can be verified that (2.61) are also equivalent to the following matrix equation, which uses classic Hamiltonian and symplectic matrices.

$$\mathcal{K}\mathcal{H} = \bar{\mathcal{H}}\mathcal{K} \tag{2.63}$$

where

$$\mathcal{K} = \begin{bmatrix} S^{-1}P & -S^{-1} \\ S' - QS^{-1}P & QS^{-1} \end{bmatrix}, \mathcal{H} = \begin{bmatrix} A(t) & \Sigma(t) \\ -C(t) & -A(t)' \end{bmatrix}, \bar{\mathcal{H}} = \begin{bmatrix} \bar{A}(t) & \bar{\Sigma}(t) \\ -\bar{C}(t) & -\bar{A}(t)' \end{bmatrix}$$

$\mathcal{H}$ is the classic Hamiltonian matrix and $\mathcal{K}$ is the classic symplectic matrix. $\mathcal{K}$ is invertible, with the following inverse.

$$\mathcal{K}^{-1} = \begin{bmatrix} S^{-1\prime}Q & S^{-1\prime} \\ -S + PS^{-1\prime}Q & PS^{-1\prime} \end{bmatrix}$$

Hence $\bar{\mathcal{H}} = \mathcal{K}\mathcal{H}\mathcal{K}^{-1}$. Hence a necessary condition to find kernel (P, S, Q) and to convert DRE $(A(t), C(t), \Sigma(t))$ into $(\bar{A}(t), \bar{C}(t), \bar{\Sigma}(t))$ is that matrices $\mathcal{H}$ and $\bar{\mathcal{H}}$ be similar. Sufficiency conditions are being investigated.

With $\mathcal{K}(t) = \mathcal{K}(P(t), S(t), Q(t))$ and using (2.61), it is also easy to verify that

$$-\dot{\mathcal{K}} = \mathcal{K}\mathcal{H} = \bar{\mathcal{H}}\mathcal{K} \tag{2.64}$$

$$\dot{\mathcal{K}^{-1}} = -\mathcal{K}^{-1}\dot{\mathcal{K}}\mathcal{K}^{-1} = \mathcal{H}\mathcal{K}^{-1} = \mathcal{K}^{-1}\bar{\mathcal{H}} \tag{2.65}$$

Thus if $\Phi(t_1, t_2)$ is the state transition matrix associated with the linear time varying system $\dot{x}(t) = -\mathcal{H}(t)x(t)$, then

$$\mathcal{K}_{t_2} = \mathcal{K}_{t_1}\Phi(t_2, t_1) \text{ and } \mathcal{K}_{t_1}^{-1} = \Phi(t_2, t_1)K_{t_2}^{-1}$$

Similarly, if $\bar{\Phi}(t_1, t_2)$ is the state transition matrix associated with the linear time varying system $\dot{x}(t) = -\bar{\mathcal{H}}(t)x(t)$, then

$$\mathcal{K}_{t_2} = \bar{\Phi}(t_2, t_1)\mathcal{K}_{t_1} \text{ and } \mathcal{K}_{t_1}^{-1} = K_{t_2}^{-1}\bar{\Phi}(t_2, t_1)$$

If $\Phi(t_2, t_1)$ and $\bar{\Phi}(t_2, t_1)$ are partitioned into four $n \times n$ blocks, then we have

$$\begin{bmatrix} S_{t_1}^{-1\prime}Q_{t_1} & S_{t_1}^{-1\prime} \\ -S_{t_1} + P_{t_1}S_{t_1}^{-1\prime}Q_{t_1} & P_{t_1}S_{t_1}^{-1\prime} \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} S_{t_2}^{-1\prime}Q_{t_2} & S_{t_2}^{-1\prime} \\ -S_{t_2} + P_{t_2}S_{t_2}^{-1\prime}Q_{t_2} & P_{t_2}S_{t_2}^{-1\prime} \end{bmatrix} \tag{2.66}$$

and

$$\begin{bmatrix} S_{t_1}^{-1\prime}Q_{t_1} & S_{t_1}^{-1\prime} \\ -S_{t_1} + P_{t_1}S_{t_1}^{-1\prime}Q_{t_1} & P_{t_1}S_{t_1}^{-1\prime} \end{bmatrix} = \begin{bmatrix} S_{t_2}^{-1\prime}Q_{t_2} & S_{t_2}^{-1\prime} \\ -S_{t_2} + P_{t_2}S_{t_2}^{-1\prime}Q_{t_2} & P_{t_2}S_{t_2}^{-1\prime} \end{bmatrix} \begin{bmatrix} \bar{\Phi}_{11} & \bar{\Phi}_{12} \\ \bar{\Phi}_{21} & \bar{\Phi}_{22} \end{bmatrix} \tag{2.67}$$

Matching terms, we get following set of equations

$$S_{t_1} = (\Phi_{11} + \Phi_{12}P_{t_2})^{-1\prime}S_{t_2} = S_{t_2}(Q_{t_2}\bar{\Phi}_{12} + \bar{\Phi}_{22})^{-1\prime} \tag{2.68}$$

$$\begin{aligned} Q_{t_1} &= Q_{t_2} - S_{t_2}{}'(\Phi_{11} + \Phi_{12}P_{t_2})^{-1}\Phi_{12}S_{t_2} \\ &= (Q_{t_2}\bar{\Phi}_{12} + \bar{\Phi}_{22})^{-1}(Q_{t_2}\bar{\Phi}_{11} + \bar{\Phi}_{21}) \end{aligned} \tag{2.69}$$

$$\begin{aligned} P_{t_1} &= (\Phi_{21} + \Phi_{22}P_{t_2})\,(\Phi_{11} + \Phi_{12}P_{t_2})^{-1} \\ &= P_{t_2} - S_{t_2}\bar{\Phi}_{12}(Q_{t_2}\bar{\Phi}_{12} + \bar{\Phi}_{22})^{-1}S_{t_2}{}' \end{aligned} \tag{2.70}$$

Remark 2.4.5. Note that (2.70) is equivalent to the traditional Davison-Maki approach, substituting $U = S_t^{-1\prime}$ and $V = P_tS_t^{-1\prime}$, we have $P_t = VU^{-1}$. U and V form the second column on the $\mathcal{K}$ matrix and using (2.66), they evolve as per

$$\begin{bmatrix} U_{t_1} \\ V_{t_1} \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} U_{t_2} \\ V_{t_2} \end{bmatrix}$$

which is the solution to (2.2).

We can substitute (2.68),(2.70) and (2.69) into (2.28), to get

$$\begin{aligned} I_{11}{}_{t_1}^{t_2} &= (\Phi_{21} + \Phi_{22}P_{t_2})(\Phi_{11} + \Phi_{12}P_{t_2})^{-1} + (\Phi_{11} + \Phi_{12}P_{t_2})^{-1\prime}\Phi_{12}^{-1} \\ I_{12}{}_{t_1}^{t_2} &= -(\Phi_{11} + \Phi_{12}P_{t_2})^{-1\prime}\Phi_{12}^{-1}(\Phi_{11} + \Phi_{12}P_{t_2}) \\ I_{22}{}_{t_1}^{t_2} &= \Phi_{12}^{-1}\Phi_{11} \end{aligned}$$

But since (I_{11}, I_{12}, I_{22}) depend only on $(A(t), C(t), \Sigma(t), t_1, t_2)$ and are independent of starting (P, S, Q), we can take $P_{t_2} = 0$ to simplify above formulas.

$$\begin{aligned} I_{11}{}_{t_1}^{t_2} &= \Phi_{21}\Phi_{11}{}^{-1} + \Phi_{11}{}^{-1\prime}\Phi_{12}^{-1} \\ I_{12}{}_{t_1}^{t_2} &= -\Phi_{11}{}^{-1\prime}\Phi_{12}^{-1}\Phi_{11} \\ I_{22}{}_{t_1}^{t_2} &= \Phi_{12}^{-1}\Phi_{11} \end{aligned} \tag{2.71}$$

Remark 2.4.6. Above formulas are useful in deriving analytical solutions for (P, S, Q). Especially for the time invariant case, $\Phi(t_2, t_1) = e^{-\mathcal{H}(t_2 - t_1)}$ and $\bar{\Phi}(t_2, t_1) = e^{-\bar{\mathcal{H}}(t_2 - t_1)}$. But note that the eigenvalues of Hamiltonian $\mathcal{H}$ are symmetric along imaginary axis, thus containing both stable and unstable eigenvalues. For time-invariant case, this leads to more and more ill conditioned $\Phi(t_1, t_2)$, and is thus useful as an analytic solution only for small $t_2 - t_1$.

2.4.4 More Fundamental solutions

Now we shall see how semiconvex duality can help us relate solutions of the primal and the dual DRE in various ways through max-plus kernel operations, in process generating other, possibly easier ways to compute fundamental solutions.

With $t_1 \le t_2$ and $u \in L_2(t_1, t_2)$, first we define a *backward trajectory* of the system, with final point x, which is a unique solution to

$$\dot{x}_t = f_t(x_t, u_t) \qquad t \in [t_1, t_2], \text{with } x_{t_2} = y \tag{2.72}$$

which is guaranteed by assumptions (2.3).

Theorem 2.4.7. *Assume (2.3). With $\bar{T} < t_1 < t_2 \le T$, backward dynamic programming counterpart of (2.17) also holds true. That is if x_t is the backwards trajectory ending at $x_{t_2} = y$, under controls u, as defined in (2.72), then*

$$V^z_{t_2}(y) = \inf_{u \in L_2(t_1,t_2)} \left\{ -\int_{t_1}^{t_2} l_t(x_t, u_t)\, dt + V^z_{t_1}(x_{t_1}) \right\} \tag{2.73}$$

$$= \inf_{x \in \mathbb{R}^n} \left(V^z_{t_1}(x) - I^{t_2}_{t_1}(x, y) \right) \tag{2.74}$$

$$\doteq \mathcal{S}^{t_1}_{t_2}[V^z_{t_1}](y)$$

Note that this also defines the semigroup operation $\mathcal{S}^{t_1}_{t_2}$ for $t_1 \le t_2$. Also we have $V^z_{t_2} = \mathcal{S}^{t_1}_{t_2}[V^z_{t_1}] = \mathcal{S}^{t_1}_{t_2}[\mathcal{S}^{t_2}_{t_1}[V^z_{t_2}]]$. Thus

$$\mathcal{S}^{t_2}_{t_1} = \mathcal{S}^{t_1}_{t_2}{}^{-1} \tag{2.75}$$

Proof. Given $u \in L_2(t_1, t_2)$, let x_t be the backwards trajectory which satisfies (2.72) with $x_{t_2} = y$. Let $x = x_{t_1}$. Then $x = y - \int_{t_1}^{t_2} f_t(x_\tau, u_\tau)\, d\tau$, thus

$$x_t = y - \int_t^{t_2} f_t(x_\tau, u_\tau)\, d\tau = x + \int_{t_1}^{t} f_t(x_\tau, u_\tau)\, d\tau$$

Thus x_t is also a unique solution of $\dot{x}_t = f_t(x_t, u_t)$ for $t \in [t_1, t_2]$ with $x_{t_1} = x$.

Thus we have $y = x_{t_2}$, where $\dot{x}_t = f_t(x_t, u_t)$, with $x_{t_1} = x$. Hence using (2.17),

$$V^z_{t_1}(x) \ge \int_{t_1}^{t_2} l_t(x_t, u_t)\, dt + V^z_{t_2}(x_{t_2}) = \int_{t_1}^{t_2} l_t(x_t, u_t)\, dt + V^z_{t_2}(y)$$

Since $x = x_{t_1}$,

$$V_{t_2}^z(y) \leq V_{t_1}^z(x_{t_1}) - \int_{t_1}^{t_2} l_t(x_t, u_t)\, dt$$

taking infimum $\forall u \in L_2(t_1, t_2)$ and corresponding backward trajectories,

$$\leq \inf_{u \in L2(t_1,t_2)} \left\{ -\int_{t_1}^{t_2} l_t(x_t, u_t)\, dt + V_{t_1}^z(x_{t_1}) \right\} \tag{2.76}$$

Now specifically, we can take $x = S_{t_2}^{-1\prime}(S_{t_1}{}'y + (Q_{t_1} - Q_{t_2}))$ and the forward trajectory $\tilde{x}_t$ starting from $\tilde{x}_{t_1} = x$ as per optimal feedback control discussed in lemma 2.3.3, $\tilde{u}_t = \sigma(t)'(P_t x_t + S_t z)$. By remark 2.3.4, it is clear that $\tilde{x}_{t_2} = y$. Hence

$$\begin{aligned} V_{t_1}^z(x) &= \sup_{u \in L_2(t_1,t_2)} \int_{t_1}^{t_2} l_t(x_t, u_t)\, dt + V_{t_2}^z(x_{t_2}) \\ &= \int_{t_1}^{t_2} l_t(\tilde{x}_t, \tilde{u}_t)\, dt + V_{t_2}^z(\tilde{x}_{t_2}) = \int_{t_1}^{t_2} l_t(\tilde{x}_t, \tilde{u}_t)\, dt + V_{t_2}^z(y) \end{aligned}$$

Since $\tilde{x}_{t_1} = x$,

$$\begin{aligned} V_{t_2}^z(y) &= -\int_{t_1}^{t_2} l_t(\tilde{x}_t, \tilde{u}_t)\, dt + V_{t_1}^z(\tilde{x}_{t_1}) \\ &\geq \inf_{u \in L_2(t_1,t_2)} \left\{ -\int_{t_1}^{t_2} l_t(x_t, u_t)\, dt + V_{t_1}^z(x_{t_1}) \right\} \end{aligned} \tag{2.77}$$

and we prove (2.73) using (2.76) and (2.77). We get (2.74) from (2.73) and (2.54).

□

Now we prove two lemmas which will be useful later.

Lemma 2.4.8. *Let $\phi(x, y)$ be a bivariable function of $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Let us define sets*

$$\mathcal{Y} = \{y' : y' = \operatorname*{argmax}_y \phi(x, y) \text{ for some } x \in \mathbb{R}^n\} \tag{2.78}$$

$$\mathcal{X} = \{x' : x' = \operatorname*{argmax}_x \phi(x, y) \text{ for some } y \in \mathbb{R}^m\} \tag{2.79}$$

Then following inequalities hold true

$$\inf_{y \in \mathcal{Y}} \sup_{x \in \mathbb{R}^n} \phi(x, y) \leq \inf_{x \in \mathbb{R}^n} \sup_{y \in \mathbb{R}^m} \phi(x, y) \tag{2.80}$$

$$\inf_{x \in \mathcal{X}} \sup_{y \in \mathbb{R}^m} \phi(x, y) \leq \inf_{y \in \mathbb{R}^m} \sup_{x \in \mathbb{R}^n} \phi(x, y) \tag{2.81}$$

Proof. Note that (2.81) follows from (2.80) using symmetry in x and y. To prove (2.80), take any $y_1 \in \mathcal{Y}$. From definition of $\mathcal{Y}$, $\exists x_1 \in \mathbb{R}^n$ such that $y_1 = \text{argmax}_{y\in\mathbb{R}^m} \phi(x,y)$. Hence

$$\begin{aligned} \sup_{x\in\mathbb{R}^n} \phi(x,y_1) &\geq \phi(x_1,y_1) = \max_{y\in\mathbb{R}^m} \phi(x_1,y) \\ &\geq \inf_{x\in\mathbb{R}^n} \sup_{y\in\mathbb{R}^m} \phi(x,y) \end{aligned}$$

Taking infimum over all $y_1 \in \mathcal{Y}$, we get (2.78). Hence proved. □

Corollary 2.4.9. *If $\phi(x,y)$ is a bivariable function of $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. With $\mathcal{X}$ and $\mathcal{Y}$ as defined in (2.78) and (2.78) respectively, if $\mathcal{X} = \mathcal{Y} = \mathbb{R}^n$, then using (2.80) and (2.81), we have*

$$\inf_{x\in\mathbb{R}^n} \sup_{y\in\mathbb{R}^n} \phi(x,y) = \inf_{y\in\mathbb{R}^n} \sup_{x\in\mathbb{R}^n} \phi(x,y) \tag{2.82}$$

□

Now for the forthcoming analysis, we assume the following.

Assume that $t_1 < t_2$. $\mathcal{P}_t(x) = \frac{1}{2}x'p_t(x)$, with p_t evolving as per (2.1). We also define a bivariate quadratic function $\phi_t^z(x) = \phi_t(x,z) = \frac{1}{2}x'P_t x + x'S_t z + \frac{1}{2}z'Q_t z$, with parameters (P_t, S_t, Q_t) evolving as per (2.13). A duality kernel $\phi^z(x) = \phi_{t_2}^z(x) = \phi_{t_2}(x,z) = \frac{1}{2}x'P_{t_2}x + x'S_{t_2}z + \frac{1}{2}z'Q_{t_2}z$ transforms $\mathcal{P}_t$ into $\mathcal{D}_{\phi^z}\mathcal{P}_t = \mathcal{Q}_t = \frac{1}{2}x'q_t(x)$. Also assume that $p_{t_2} \succeq P_{t_2}$, and S_{t_2} nonsingular, and that $\mathcal{P}_{t_1}(x) = \mathcal{S}_{t_1}^{t_2}[\mathcal{P}_{t_2}](x)$ and $\phi_{t_1}^z(x) = \mathcal{S}_{t_1}^{t_2}[\phi_{t_2}^z](x)$ exists. (2.83)

Theorem 2.4.10. *Assuming (2.3)-(2.83), the semiconvex dual of $\mathcal{P}_{t_1}(x)$ under kernel $\phi_{t_1}(x,z)$ exists, and is same as the semiconvex dual of $\mathcal{P}_{t_2}(x)$ under kernel $\phi_{t_2}(x,z)$. Using (2.48), $\forall z \in \mathbb{R}^n$*

$$\inf_x [\mathcal{P}_{t_2}(x) - \phi_{t_2}(x,z)] = \inf_y [\mathcal{P}_{t_1}(y) - \phi_{t_1}(y,z)] = \mathcal{Q}_{t_2}(z) \tag{2.84}$$

That is $\mathcal{D}_{\phi_{t_2}}[\mathcal{P}_{t_2}] = \mathcal{D}_{\phi_{t_1}}[\mathcal{P}_{t_1}]$. In terms of parameters, following equation holds true.

$$q_{t_2} = -{S_{t_2}}'(p_{t_2} - P_{t_2})^{-1}S_{t_2} - Q_{t_2} = -{S_{t_1}}'(p_{t_1} - P_{t_1})^{-1}S_{t_1} - Q_{t_1} \tag{2.85}$$

Proof. Existence of the dual $\mathcal{D}_{\phi^z_{t_2}}[\mathcal{P}_{t_2}]$ is evident from theorem 2.4.1. Note that from corollary 2.2.3 and (2.20), with $x = x_0$,

$$\frac{1}{2}x' p_{t_1} x = \mathcal{P}_{t_1}(x_{t_1}) = \mathcal{S}_{t_1}^{t_2}[\mathcal{P}_{t_2}](x_{t_2}) = \sup_{y \in \mathbb{R}^n} \left(I_{t_1}^{t_2}(x, y) + \mathcal{P}_{t_2}(y) \right)$$

using (2.45), $p_{t_1} = I_{11} - I_{12}(p_{t_2} + I_{22})^{-1} I_{12}{}'$. Since $\mathcal{P}_{t_2}$ exists, $p_{t_2} + I_{22}$ is invertible. Similarly since $\phi^z_{t_1}$ exists, $\phi^z_{t_2} = \mathcal{D}_{I_{t_1}^{t_2}} \phi^z_{t_1}$. Thus

$$\frac{1}{2}x' P_{t_2} x + x' S_{t_2} z + \frac{1}{2} z' Q_{t_2} z = \phi^z_{t_2}(x) = \mathcal{S}_{t_2}^{t_1}[\phi^z_{t_1}](x) = \inf_{y \in \mathbb{R}^n} \left(\phi^z_{t_1}(y) - I_{t_1}^{t_2}(y, x) \right)$$

exists, thus by a similar logic, $I_{11} - P_{t_1}$ is also invertible.

Now we shall consider a bivariable function

$$\psi(x, y) = -\phi^z_{t_1}(x) + I_{t_1}^{t_2}(x, y) + \mathcal{P}_{t_2}(y) \tag{2.86}$$

note that given $x \in \mathbb{R}^n$,

$$\begin{aligned} \operatorname*{argmax}_y \psi(x, y) &= \operatorname*{argmax}_y I_{t_1}^{t_2}(x, y) + \mathcal{P}_{t_2}(y) \\ &= (I_{22} + P_{t_2})^{-1} I_{12}{}' x \end{aligned} \tag{2.87}$$

similarly given $y \in \mathbb{R}^n$,

$$\begin{aligned} \operatorname*{argmax}_x \psi(x, y) &= \operatorname*{argmax}_x -\phi^z_{t_1}(x) + I_{t_1}^{t_2}(x, y) \\ &= \operatorname*{argmin}_x I_{t_1}^{t_2}(x, y) - \phi^z_{t_1}(x) \\ &= -(I_{11} - P_{t_1})^{-1} I_{12} y \end{aligned} \tag{2.88}$$

Using 2.3.6 and (2.28), $Q_{t_1} - Q_{t_2} \succ 0$ and hence $I_{12} \succ 0$. Combined with invertibility of $I_{22} + P_{t_2}$ and $I_{11} - P_{t_1}$, from (2.87), (2.88), we have

$$\mathcal{Y} = \{\bar{y} : \bar{y} = \operatorname*{argmax}_y \psi(x, y) \text{ for some } x \in \mathbb{R}^n\} = \mathbb{R}^n$$

$$\mathcal{X} = \{\bar{x} : \bar{x} = \operatorname*{argmax}_x \psi(x, y) \text{ for some } y \in \mathbb{R}^n\} = \mathbb{R}^n$$

Hence by corollary 2.4.9 from appendix,

$$\inf_x \sup_y \psi(x, y) = \inf_y \sup_x \psi(x, y) \tag{2.89}$$

With this preparation, we are ready to prove the main result.

$$\begin{aligned}
\inf_x \left[\mathcal{P}_{t_1}(x) - \phi^z_{t_1}(x)\right] &= \inf_x \left[\mathcal{S}^{t_2}_{t_1}[\mathcal{P}_{t_2}](x) - \phi^z_{t_1}(x_{t_1})\right] \\
&= \inf_x \left[\sup_y \left(I^{t_2}_{t_1}(x,y) + \mathcal{P}_{t_2}(y)\right) - \phi^z_{t_1}(x)\right] \\
&= \inf_x \sup_y \left(-\phi^z_{t_1}(x) + I^{t_2}_{t_1}(x,y) + \mathcal{P}_{t_2}(y)\right) \\
&= \inf_x \sup_y \psi(x,y)
\end{aligned}$$

using (2.89),

$$\begin{aligned}
&= \inf_y \sup_x \psi(x,y) \\
&= \inf_y \left[\mathcal{P}_{t_2}(y) + \sup_x \left(-\phi^z_{t_1}(x) + I^{t_2}_{t_1}(x,y)\right)\right] \\
&= \inf_y \left[\mathcal{P}_{t_2}(y) - \inf_x \left(\phi^z_{t_1}(x) - I^{t_2}_{t_1}(x,y)\right)\right] \\
&= \inf_y \left[\mathcal{P}_{t_2}(y) - \phi^z_{t_2}(y)\right]
\end{aligned}$$

Algebraically, it is easy to see that, minimum occurs at $x = (p_{t_1} - P_{t_1})^{-1} S_{t_1} z$ and $y = (p_{t_2} - P_{t_2})^{-1} S_{t_2} z$, respectively. Plugging this into (2.84) gives us (2.85). Note that these equations have the same form as (2.46) obtained earlier. □

Corollary 2.4.11. *Assume (2.3),(2.83). Using $\phi(x,z) = \phi^z_{t_2}(x)$ as a duality kernel, let $\frac{1}{2} z' q_t z = \mathcal{Q}_t(z) = \mathcal{D}_\phi \mathcal{P}_t(x)$ for all $t \in [t_1, t_2]$. The semiconvex primal of $\mathcal{Q}_{t_2}(z)$ under kernel $\phi_{t_1}(x,z)$ is same as the semiconvex primal of $\mathcal{Q}_{t_1}(x)$ under kernel $\phi_{t_2}(x,z)$. Using (2.48), $\forall x \in \mathbb{R}^n$*

$$\sup_z \left[\mathcal{Q}_{t_2}(z) + \phi_{t_1}(x,z)\right] = \sup_z \left[\mathcal{Q}_{t_1}(z) + \phi_{t_2}(x,z)\right] = \mathcal{P}_{t_1} \tag{2.90}$$

That is $\mathcal{D}^{-1}_{\phi^z_{t_1}}[\mathcal{Q}_{t_2}] = \mathcal{D}^{-1}_{\phi^z_{t_2}}[\mathcal{Q}_{t_1}]$, and

$$p_{t_1} = -S_{t_1}(q_{t_2} + Q_{t_1})^{-1} {S_{t_1}}' + P_{t_1} = -S_{t_2}(q_{t_1} + Q_{t_2})^{-1} {S_{t_2}}' + P_{t_2} \tag{2.91}$$

Proof. We have,

$$\mathcal{Q}_{t_1}(z) = \mathcal{D}_\phi[\mathcal{P}_{t_1}](z) = \mathcal{D}_{\phi_{t_2}}[\mathcal{P}_{t_1}](z) \tag{2.92}$$

$$\mathcal{Q}_{t_2}(z) = \mathcal{D}_\phi[\mathcal{P}_{t_2}](z) = \mathcal{D}_{\phi_{t_2}}[\mathcal{P}_{t_2}](z) \tag{2.93}$$

Also using theorem 2.4.10, $\mathcal{D}_{\phi_{t_2}}[\mathcal{P}_{t_2}](z) = \mathcal{D}_{\phi_{t_1}}[\mathcal{P}_{t_1}](z)$, hence

$$\mathcal{D}^{-1}_{\phi_{t_1}}[\mathcal{D}_{\phi_{t_2}}[\mathcal{P}_{t_2}]] = \mathcal{P}_{t_1} \tag{2.94}$$

Thus using, (2.92), (2.93) and (2.94)

$$\mathcal{D}^{-1}_{\phi_{t_1}}[\mathcal{Q}_{t_2}] = \mathcal{D}^{-1}_{\phi_{t_1}}[\mathcal{D}_{\phi_{t_2}}[\mathcal{P}_{t_2}]] = \mathcal{P}_{t_1} = \mathcal{D}^{-1}_{\phi_{t_2}}[\mathcal{D}_{\phi_{t_2}}[\mathcal{P}_{t_2}]] = \mathcal{D}^{-1}_{\phi_{t_1}}[\mathcal{Q}_{t_2}]$$

Finally we obtain (2.91), using (2.90) and (2.49). □

Now we shall obtain a time-varying version of the result previously obtained in [McE08], in order to complete our picture of kernel relationships between primal and dual DREs. For this result, we make an additional assumption.

We assume that $F(P_t) = A(t)'P_t + P_t A(t) + C(t) + P_t \Sigma(t) P_t = -\dot{P}_t \succ 0$ for $t \in [t_1, t_2]$. Thus we have, $p_{t_1} - p_{t_2} \succ 0$. (2.95)

Theorem 2.4.12. *Assuming (2.3), (2.83) and (2.95), $\mathcal{Q}_{t_1}(z)$ is the semiconvex primal of $\mathcal{Q}_{t_2}(\bar{z})$ under kernel $B^{t_2}_{t_1}(z, \bar{z})$*

$$\begin{aligned} \mathcal{Q}_{t_1}(z) &= \sup_{y \in \mathbb{R}^n} \left[B^{t_2}_{t_1}(z, y) + \mathcal{Q}_{t_2}(y) \right] \quad \textit{or} \\ \mathcal{Q}_{t_1}(z) &= \mathcal{D}^{-1}_{B^{t_2}_{t_1}}[\mathcal{Q}_{t_2}](z) \end{aligned} \tag{2.96}$$

where,

$$\mathcal{B}^{t_2}_{t_1}(z, y) = \inf_{x \in \mathbb{R}^n} \{\phi_{t_1}(x, y) - \phi_{t_2}(x, z)\} \tag{2.97}$$

Hence $\mathcal{B}^{t_2}_{t_1}(z, y) = \frac{1}{2} z' B_{11}{}^{t_2}_{t_1} z + z' B_{12}{}^{t_2}_{t_1} y + \frac{1}{2} y' B_{22}{}^{t_2}_{t_1} y$, *with*

$$\begin{aligned} B_{11}{}^{t_2}_{t_1} &= -S_{t_2}{}'(P_{t_1} - P_{t_2})^{-1} S_{t_2} - Q_{t_2} \\ B_{12}{}^{t_2}_{t_1} &= S_{t_2}{}'(P_{t_1} - P_{t_2})^{-1} S_{t_1} \\ B_{22}{}^{t_2}_{t_1} &= -S_{t_1}{}'(P_{t_1} - P_{t_2})^{-1} S_{t_1} - Q_{t_1} \end{aligned} \tag{2.98}$$

and

$$q_{t_1} = B_{11}{}^{t_2}_{t_1} - B_{12}{}^{t_2}_{t_1}(B_{22}{}^{t_2}_{t_1} + q_{t_2})^{-1} B_{12}{}^{t_2}_{t_1}{}' \tag{2.99}$$

Proof. In corollary 2.4.11, we saw that $\mathcal{D}^{-1}_{\phi_{t_2}}[\mathcal{Q}_{t_1}] = \mathcal{D}^{-1}_{\phi_{t_1}}[\mathcal{Q}_{t_2}]$. hence we have

$$\begin{aligned}
\mathcal{Q}_{t_1}(z) &= \mathcal{D}_{\phi_{t_2}}\mathcal{D}^{-1}_{\phi_{t_1}}[\mathcal{Q}_{t_2}](z) \\
&= \inf_{x\in\mathbb{R}^n}\left(\mathcal{D}^{-1}_{\phi_{t_1}}[\mathcal{Q}_{t_2}](x) - \phi_{t_2}(x,z)\right) \\
&= \inf_{x\in\mathbb{R}^n}\left(\sup_{y\in\mathbb{R}^n}\left(\phi_{t_1}(x,y) + \mathcal{Q}_{t_2}(y)\right) - \phi_{t_2}(x,z)\right) \\
&= \inf_{x\in\mathbb{R}^n}\sup_{y\in\mathbb{R}^n}\left(\phi_{t_1}(x,y) + \mathcal{Q}_{t_2}(y) - \phi_{t_2}(x,z)\right) \\
&= \inf_{x\in\mathbb{R}^n}\sup_{y\in\mathbb{R}^n}\psi(x,y) \qquad (2.100)
\end{aligned}$$

where

$$\psi(x,y) = \frac{1}{2}x'(P_{t_1} - P_{t_2})x + x'S_{t_1}y + \frac{1}{2}y'(Q_{t_2} + q_{t_2})y - x'S_{t_2}z - \frac{1}{2}z'Q_{t_2}z$$

Note that by (2.95), $P_{t_1} - P_{t_2} \succ 0$. Hence $\psi(x,y)$ is strictly convex in x. Also observe that by corollary 2.4.11, $\mathcal{P}_{t_1}(x) = \sup_y(\phi_{t_1}(x,y) + \mathcal{Q}_{t_2}(y))$ exists for any $x \in \mathbb{R}^n$. Hence $Q_{t_2} + q_{t_2} \prec 0$. Thus $\psi(x,y)$ is strictly concave in y. For such a convex-concave function following saddle point exists. By setting $\nabla_x\psi$ and $\nabla_y\psi$ equal to zero, and solving, we get

$$\begin{aligned}
x^0 &= \left((P_{t_1} - P_{t_2}) - S_{t_1}(Q_{t_2} + q_{t_2})^{-1}{S_{t_1}}'\right)^{-1} S_{t_2}z \\
y^0 &= -(Q_{t_2} + q_{t_2})^{-1}{S_{t_1}}'x^0
\end{aligned}$$

For such x^0 and y^0, $\psi(x^0,y) \leq \psi(x^0,y^0) \leq \psi(x,y^0)$. Hence by a well known result,

$$\inf_{x\in\mathbb{R}^n}\sup_{y\in\mathbb{R}^n}\psi(x,y) = \psi(x^0,y^0) = \sup_{y\in\mathbb{R}^n}\inf_{x\in\mathbb{R}^n}\psi(x,y) \qquad (2.101)$$

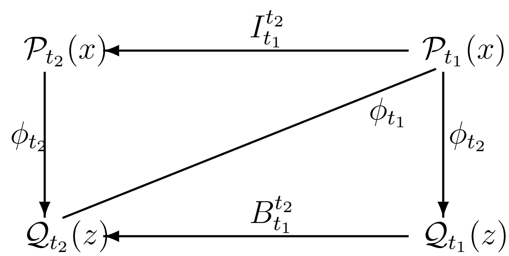
Using (2.100) and (2.101)

$$\begin{aligned}
\mathcal{Q}_{t_1}(z) &= \sup_{y\in\mathbb{R}^n}\inf_{x\in\mathbb{R}^n}\psi(x,y) \qquad (2.102) \\
&= \sup_{y\in\mathbb{R}^n}\left(\mathcal{Q}_{t_2}(y) + \inf_{x\in\mathbb{R}^n}\left(\phi_{t_1}(x,y) - \phi_{t_2}(x,z)\right)\right) \qquad (2.103) \\
&= \sup_{y\in\mathbb{R}^n}\left(\mathcal{Q}_{t_2}(y) + B^{t_2}_{t_1}(z,y)\right) \qquad (2.104)
\end{aligned}$$

(2.98) can be easily obtained from (2.97) by finding local minimum in x (which is global minimum, since infimum exists), substituting and term-wise equating coefficients. Similarly (2.99) results from substituting $\mathcal{Q}_t = \frac{1}{2}z'q_tz$, (2.97), (2.98) into (2.96). □

Thus equations (2.45), (2.55), (2.85), (2.84), (2.91), (2.90), (2.96), (2.99) can be summarized in the diagram below. Note that primal and dual quadratics are on top and bottom respectively. Vertical and diagonal lines show duality transformation with indicated kernel. Arrows are directed from the primal to it semiconvex dual.

Primal DRE: $-\dot{p}_t = A(t)'p_t + p_tA(t) + C(t) + p_t\Sigma(t)p_t$.



Dual DRE: $-\dot{q}_t = \bar{A}(t)'q_t + q_t\bar{A}(t) + \bar{C}(t) + q_t\bar{\Sigma}(t)q_t$.

Figure 2.1: Time varying problem: Duality relationships.

Thus in conclusion, so far we saw three distinct ways of solving (2.1), that is obtaining p_{t_1} from p_{t_2}.

1. Direct method which assumes only (2.3). Formulae are given by (2.45) and (2.28). Propagation is achieved by following transform.

$$\mathcal{P}_{t_1} = \mathcal{D}^{-1}_{I_{t_1}^{t_2}}[\mathcal{P}_{t_2}]$$

   Problem with this method is that as $t_1 \to t_2$, parameters of the kernel $I_{t_1}^{t_2}$ blow up, limiting solution accuracy.

2. Alternate method, which assumes (2.3) and (2.83). Formulas are given by (2.85), which is same as (2.46). Propagation is achieved by following transform.

$$\mathcal{P}_{t_1} = \mathcal{D}^{-1}_{\phi_{t_1}}\mathcal{D}_{\phi_{t_2}}[\mathcal{P}_{t_2}]$$

   This method works better for a small time step propagation, since parameters of kernels ϕ_{t_1} and ϕ_{t_2} do not blow up.

3. Third method assumes (2.3), (2.83) and (2.95). Time invariant version of this method was first proposed in [McE08]. Formulae are (2.98) and (2.99). Propagation is achieved by following transform

$$\mathcal{P}_{t_1} = \mathcal{D}^{-1}_{\phi_{t_2}} \mathcal{D}^{-1}_{B^{t_2}_{t_1}} \mathcal{D}_{\phi_{t_2}}[\mathcal{P}_{t_2}]$$

Problem with this method is similar to the direct method. Namely, $t_1 \to t_2$, parameters of the kernel $I^{t_2}_{t_1}$ blow up, limiting solution accuracy.

2.5 Time Invariant problem

The theory so far developed for DREs with time-varying coefficients, extends readily for the time invariant DRE, in which $A(t) \equiv A$, $C(t) \equiv C$, $\sigma(t) \equiv \sigma$ hence $\Sigma(t) \equiv \sigma\sigma' = \Sigma$. Again we assume (2.3). We state such a DRE again for reference.

$$-\dot{p} = A'p + p + C + p'\Sigma p \tag{2.105}$$

If we define $\mathcal{P}_{t_2}(x) = \frac{1}{2}x'p_{t_2}x$, then with $\bar{T} < t_1 \le t_2 \le T$, $l(x_t, u_t) = \frac{1}{2}x_t{}'Cx_t - \frac{1}{2}|u_t|^2$, and dynamics $\dot{x}_t = f(x_t, u_t) = Ax_t + \sigma u_t$ and starting point $x_{t_1} = x$, value function is

$$\mathcal{P}_{t_1}(x) = \mathcal{S}^{t_2}_{t_1}[\mathcal{P}_{t_2}](x) \doteq \sup_{u \in L_2(t_1,t_2)} \left\{ \int_{t_1}^{t_2} l(x_t, u_t)\, dt + \mathcal{P}_{t_2}(x_{t_2}) \right\} \tag{2.106}$$

Then $\mathcal{P}_{t_1}(x) = \frac{1}{2}x'p_{t_1}x$, where p_t satisfies DRE (2.105). Let us define $\Delta = t_2 - t_1$. Let $\delta \in \mathbb{R}$. Note that using the time invariance of dynamics and incremental payoff, and change of variables $t \to (t - \delta)$,

$$\begin{aligned}
\mathcal{S}^{t_2}_{t_1}[\phi](x) &= \left\{ \sup_{u \in L_2(t_1,t_2)} \left(\int_{t_1}^{t_2} l(x_t, u_t)\, dt + \phi(x_{t_2}) \right) : x_{t_1} = x \right\} \\
&= \left\{ \sup_{u \in L_2(t_1-\delta,t_2-\delta)} \left(\int_{t_1-\delta}^{t_2-\delta} l(x_t, u_t)\, dt + \phi(x_{t_2-\delta}) \right) : x_{t_1-\delta} = x \right\} \\
&= \mathcal{S}^{t_2-\delta}_{t_1-\delta}[\phi](x)
\end{aligned}$$

As a special case, using $\delta = t_2 - t_1$ and $\delta = t_1 - t_2$ respectively,

$$\mathcal{S}^{t_2}_{t_1} = \mathcal{S}^{0}_{t_1-t_2} = \mathcal{S}^{0}_{-\Delta} = \mathcal{S}^{t_2-t_1}_{0} = \mathcal{S}^{\Delta}_{0} \doteq \mathcal{S}^{\Delta}$$

By similar argument, the fundamental solution, or max-plus kernel

$$I_{t_1}^{t_2} = I_{t_1-t_2}^{0} = I_{-\Delta}^{0} = I_0^{t_2-t_1} = I_0^{\Delta} \doteq I^{\Delta} \tag{2.107}$$

Similarly, the bivariate quadratic DRE in (2.13) turns into

$$\begin{aligned} -\dot{P}_t &= A'P_t + P_tA + C + P_t\Sigma P_t \\ -\dot{S}_t &= (A + \Sigma P_t)'S_t \\ -\dot{Q}_t &= {S_t}'\Sigma S_t \end{aligned} \tag{2.108}$$

2.5.1 A doubling algorithm

Using (2.36), we can derive a useful doubling algorithm for solving (2.105). First we propagate the triad (P_0, S_0, Q_0) backwards in time by Δ to obtain $(P_{-\Delta}, S_{-\Delta}, Q_{-\Delta})$, using (2.108). Then we can compute I^{Δ} as defined in (2.107). Thus using (2.28) with $t_1 = -\Delta$ and $t_2 = 0$.

$$\begin{aligned} {I_{11}}^{\Delta} &= P_{-\Delta} - S_{-\Delta}(Q_{-\Delta} - Q_0)^{-1}{S_{-\Delta}}' \\ {I_{12}}^{\Delta} &= S_{-\Delta}(Q_{-\Delta} - Q_0)^{-1}{S_0}' \\ {I_{22}}^{\Delta} &= -P_0 - S_0(Q_{-\Delta} - Q_0)^{-1}{S_0}' \end{aligned} \tag{2.109}$$

We can build up $I^{2\Delta}$ using (2.35) as follows

$$I^{2\Delta}(x,y) = I_0^{2\Delta}(x,y) = \sup_z \left\{ I_0^{\Delta}(x,z) + I_{\Delta}^{2\Delta}(z,y) \right\} = \sup_z \left\{ I_0^{\Delta}(x,z) + I_0^{\Delta}(z,y) \right\}$$

Thus using (2.36)

$$\begin{aligned} {I_{11}}^{2\Delta} &= {I_{11}}^{\Delta} - {I_{12}}^{\Delta}\left({I_{22}}^{\Delta} + {I_{11}}^{\Delta}\right)^{-1}{I_{12}}^{\Delta\prime} \\ {I_{12}}^{2\Delta} &= -{I_{12}}^{\Delta}\left({I_{22}}^{\Delta} + {I_{11}}^{\Delta}\right)^{-1}{I_{12}}^{\Delta} \\ {I_{22}}^{2\Delta} &= {I_{22}}^{\Delta} - {I_{12}}^{\Delta\prime}\left({I_{22}}^{\Delta} + {I_{11}}^{\Delta}\right)^{-1}{I_{12}}^{\Delta} \end{aligned} \tag{2.110}$$

Thus by recursively combining kernel operations, we can get $I^{2^N\Delta}$ in N steps. Hence the name doubling algorithm. Finally the DRE (2.105) can be solved by (2.45). This can give an exponentially fast propagation of the DRE.

Remark 2.5.1. In [McE08], an analogous doubling algorithm was derived in semiconvex dual space. We shall sketch it here without proof. It starts with construction of the dual kernel $B_{t_1}^{t_2} = B^{\Delta}$ using time invariant special case of (2.98).

$$\begin{aligned} {B_{11}}^{\Delta} &= -{S_0}'(P_{-\Delta} - P_0)^{-1} S_0 - Q_0 \\ {B_{12}}^{\Delta} &= {S_0}'(P_{-\Delta} - P_0)^{-1} S_{-\Delta} \\ {B_{22}}^{\Delta} &= -{S_{-\Delta}}'(P_{-\Delta} - P_0)^{-1} S_{-\Delta} + Q_{-\Delta} \end{aligned} \tag{2.111}$$

and kernel is doubled using following formulas.

$$\begin{aligned} {B_{11}}^{2\Delta} &= {B_{11}}^{\Delta} - {B_{12}}^{\Delta} \left({B_{22}}^{\Delta} + {B_{11}}^{\Delta}\right)^{-1} {B_{12}}^{\Delta\prime} \\ {B_{12}}^{2\Delta} &= -{B_{12}}^{\Delta} \left({B_{22}}^{\Delta} + {B_{11}}^{\Delta}\right)^{-1} {B_{12}}^{\Delta} \\ {B_{22}}^{2\Delta} &= {B_{22}}^{\Delta} - {B_{12}}^{\Delta\prime} \left({B_{22}}^{\Delta} + {B_{11}}^{\Delta}\right)^{-1} {B_{12}}^{\Delta} \end{aligned} \tag{2.112}$$

Thus for $T = 2^N \Delta$, B^{2^N} is found using N doubling operations, and propagation is achieved using $\mathcal{P}_{-T} = \mathcal{D}_{\phi_0}^{-1} \mathcal{D}_{B^T}^{-1} \mathcal{D}_{\phi_0}[\mathcal{P}_0]$. Which using (2.48) and (2.49) implies

$$p_{-T} = -S_0 \left(B_{11}^T - B_{12}^T \left(B_{22}^T - {S_0}'(p_0 - P_0)^{-1} S_0 - Q_0 \right)^{-1} {B_{12}^T}' + Q_0 \right)^{-1} {S_0}' + P_0 \tag{2.113}$$

Remark 2.5.2. For both (2.109) and (2.112), as $\Delta \to 0$, $(Q_{-\Delta} - Q_0)^{-1}$ and $(P_{-\Delta} - P_0)^{-1}$ can grow very large, leading to numerical inaccuracy at a very small time step propagation. A work-around is designed in the next subsection.

2.5.2 Alternate doubling algorithm

First we deduce following generalization of theorem 2.4.10.

Lemma 2.5.3. *Consider $t < 0$. Define $\phi_t^z(x) = \phi_t(x, z) = \frac{1}{2} x' P_t x + x' S_t z + \frac{1}{2} z' Q_t z$, with parameters (P_t, S_t, Q_t) evolving as per (2.108). Assume that a primal function $\mathcal{P}_0(x)$ is such that $\mathcal{P}_0(x) - x' P_0 x$ is a convex function. Also assume that $\mathcal{P}_t(x) = \mathcal{S}_t^0[\mathcal{P}_0](x)$. Hence a duality kernel $\phi^z(x) = \phi_0^z(x) = \phi_0(x, z) = \frac{1}{2} x' P_0 x + x' S_0 z + \frac{1}{2} z' Q_0 z$ transforms $\mathcal{P}_t$ into $\mathcal{D}_{\phi^z} \mathcal{P}_t = \mathcal{Q}_t$. Then, the semiconvex dual of $\mathcal{P}_t$ under kernel $\phi_0(x, z)$ exists, and is same as the semiconvex dual $\mathcal{P}_0(x)$ under kernel $\phi_0(x, z)$.*

$$\inf_x \left[\mathcal{P}_0(x) - \phi_0(x, z)\right] = \inf_y \left[\mathcal{P}_t(y) - \phi_t(y, z)\right] = \mathcal{Q}_0(z) \tag{2.114}$$

Proof. The proof proceeds exactly as in theorem 2.4.10 with $t_2 = 0$ and $t_1 = t$. The only difference in the assumption is, instead of assuming $\mathcal{P}_t(x) = \frac{1}{2}x' p_t x$ with $p_0 \succ P_0$, now we assume $\mathcal{P}_0(x) - x' P_0 x$ is a strictly convex function, and $\mathcal{P}_t = \mathcal{S}_t^0 \mathcal{P}_0$. This enables broader choice of $\mathcal{P}_0$, which we shall use in next corollary. □

Corollary 2.5.4. *Let $t_1 < 0$ and $t_2 < 0$. Staring with terminal $\phi_0(x,z)$ we assume that $\phi_{t_1}(x,z)$, $\phi_{t_2}(x,z)$ and $\phi_{t_1+t_2}(x,z)$ exist, and $\phi_{t_2}(x,z) - \phi_0(x,z)$ is strictly convex. With choice of $\mathcal{P}_0(x) = \phi_{t_2}^z(x) = \phi_{t_2}(x.z)$, and $t = t_1$, we have*

$$\mathcal{P}_t = \mathcal{S}_{t_2}^0[\phi_{t_1}^z](x) = \mathcal{S}_{t_1+t_2}^{t_2}[\phi_{t_2}^z](x) = \phi_{t_1+t_2}^z(x) = \phi_{t_1+t_2}(x,z)$$

Hence using lemma 2.5.3 for all z and $\bar{z}$

$$\inf_x [\phi_{t_2}(x,\bar{z}) - \phi_0(x,z)] = \inf_y [\phi_{t_1+t_2}(y,\bar{z}) - \phi_{t_1}(y,z)] \tag{2.115}$$

Hence following is true.

$$\begin{aligned} P_{t_1+t_2} &= P_{t_1} + S_{t_1}\left\{(Q_0 - Q_{t_1}) + S_0{}'(P_{t_2} - P_0)^{-1}S_0\right\}^{-1} S_{t_1}{}' \\ S_{t_1+t_2} &= S_{t_1}\left\{(Q_0 - Q_{t_1}) + S_0{}'(P_{t_2} - P_0)^{-1}S_0\right\}^{-1} S_0{}'(Q_{t_2} - Q_0)^{-1}S_{t_2} \\ Q_{t_1+t_2} &= Q_{t_2} + S_{t_2}{}'\left\{(P_0 - P_{t_2}) + S_0(Q_{t_1} - Q_0)^{-1}S_0{}'\right\}^{-1} S_{t_2} \end{aligned} \tag{2.116}$$

Proof. Substituting the parameters of ϕ_t for $t = 0, t_1, t_2, t_1+t_2$ in (2.115), using (2.50) and (2.51) and matching terms gives us following

$$\begin{aligned} S_0{}'(P_{t_2} - P_0)^{-1}S_0 + Q_0 &= S_{t_1}{}'(P_{t_1+t_2} - P_{t_1})^{-1}S_{t_1} + Q_{t_1} \\ S_{t_2}{}'(P_{t_2} - P_0)^{-1}S_0 &= S_{t_1+t_2}{}'(P_{t_1+t_2} - P_{t_1})^{-1}S_{t_1} \\ -S_{t_2}{}'(P_{t_2} - P_0)^{-1}S_{t_2} + Q_0 &= -S_{t_1+t_2}{}'(P_{t_1+t_2} - P_{t_1})^{-1}S_{t_1+t_2} + Q_{t_1+t_2} \end{aligned}$$

from which (2.116) can be derived by some manipulation, and using Woodbury's matrix inversion formula. □

Corollary 2.5.5. *Using $t_1 = t_2 = -\Delta$, $\Delta \geq 0$. We have following kernel doubling formula, in terms of parameters of the biquadratic duality kernel.*

$$\begin{aligned} P_{-2\Delta} &= P_{-\Delta} + S_{-\Delta}\left\{(Q_0 - Q_{-\Delta}) + S_0{}'(P_{-\Delta} - P_0)^{-1}S_0\right\}^{-1} S_{-\Delta}{}' \\ S_{-2\Delta} &= S_{-\Delta}\left\{(Q_0 - Q_{-\Delta}) + S_0{}'(P_{-\Delta} - P_0)^{-1}S_0\right\}^{-1} S_0{}'(Q_{-\Delta} - Q_0)^{-1}S_{-\Delta} \\ Q_{-2\Delta} &= Q_{-\Delta} + S_{-\Delta}{}'\left\{(P_0 - P_{-\Delta}) + S_0(Q_{-\Delta} - Q_0)^{-1}S_0{}'\right\}^{-1} S_{-\Delta} \end{aligned} \tag{2.117}$$

Using these we can create kernel $\phi_t(x,z)$ parametrized by (P_t, S_t, Q_t) at time $t = -2^N\Delta$ in N steps and achieve Riccati propagation using time-invariant version of (2.46), that is

$$p_t = P_t - S_t\left(Q_t - Q_0 - S_0{}'(p_0 - P_0)^{-1}S_0\right)^{-1} S_t{}' \tag{2.118}$$

2.5.3 Dual DRE and Analytic solutions

Extending (2.59) to time-invariant case, we have if $\mathcal{Q}_t(z) = \mathcal{D}_\phi[\mathcal{P}_t] = \frac{1}{2}z'q_tz$, where $\phi(x,z) = \frac{1}{2}x'P_tx + x'S_tz + \frac{1}{2}z'Q_tz$, then q_t satisfies following dual DRE.

$$-\dot{q}_t = \bar{A}'q_t + q_t\bar{A} + \bar{C} + q_t{}'\bar{\Sigma}q_t \tag{2.119}$$

Similarly extending (2.61) to the time invariant case, coefficients of the dual DRE, $(\bar{A}, \bar{C}, \bar{\Sigma})$ satisfy following matching or compatibility conditions.

$$\begin{aligned} -\dot{P} &= A'P + PA + C + P\Sigma P = S\bar{\Sigma}S' \\ -\dot{S} &= (A + \Sigma P)'S = S(-\bar{A} + \bar{\Sigma}Q) \\ -\dot{Q} &= S'\Sigma S = -\bar{A}'Q - Q\bar{A} + \bar{C} + Q\bar{\Sigma}Q \end{aligned} \tag{2.120}$$

Similarly extending (2.63),

$$\mathcal{K}\mathcal{H} = \bar{\mathcal{H}}\mathcal{K} \tag{2.121}$$

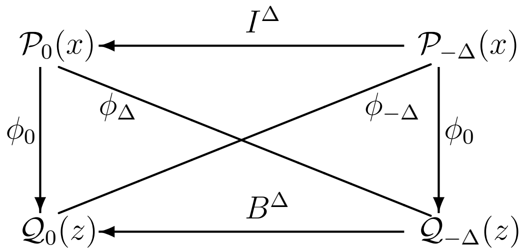
where

$$\mathcal{K} = \begin{bmatrix} S^{-1}P & -S^{-1} \\ S' - QS^{-1}P & QS^{-1} \end{bmatrix}, \mathcal{H} = \begin{bmatrix} A & \Sigma \\ -C & -A' \end{bmatrix}, \bar{\mathcal{H}} = \begin{bmatrix} \bar{A} & \bar{\Sigma} \\ -\bar{C} & -\bar{A}' \end{bmatrix}$$

Note that with a constant duality kernel, hence constant $\mathcal{K}$ (assuming invertibility), and using time invariance of $\mathcal{H}$, $\dot{\bar{\mathcal{H}}} = \mathcal{K}\dot{\mathcal{H}}\mathcal{K}^{-1} = 0$, dual DRE is also time invariant.

Extending the figure 2.5.3 to the time varying case gives us figure 2.4.4. Note that $t_1 = -\Delta$ and $t_2 = 0$, and $\phi_t(x,z) = \frac{1}{2}x'P_tx + x'S_tz + \frac{1}{2}z'Q_tz$, with the triad (P_t, S_t, Q_t) evolving as per (2.120). Note that we have one new relationship in figure (2.4.4), stated as below.

Primal DRE: $-\dot{p}_t = A'p_t + p_tA + C + p_t\Sigma p_t$.



Dual DRE: $-\dot{q}_t = \bar{A}'q_t + q_t\bar{A} + \bar{C} + q_t\bar{\Sigma}q_t$.

Figure 2.2: Time invariant problem: Duality relationships.

Lemma 2.5.6.

$$\mathcal{D}_{\phi_\Delta}[\mathcal{P}_0] = \mathcal{Q}_{-\Delta}$$

Proof. From figure (2.5.3), we have $[\mathcal{P}_{t_2}] = \mathcal{D}^{-1}_{\phi_{t_2}}\mathcal{D}_{\phi_{t_1}}[\mathcal{P}_{t_1}] = \mathcal{S}^{t_2}_{t_1}[\mathcal{P}_{t_1}]$. Because of time invariance, for any $\delta \in \mathbb{R}$,

$$\mathcal{S}^{t_2}_{t_1} = \mathcal{D}^{-1}_{\phi_{t_2}}\mathcal{D}_{\phi_{t_1}} = \mathcal{S}^{t_2+\delta}_{t_1+\delta} = \mathcal{D}^{-1}_{\phi_{t_2+\delta}}\mathcal{D}_{\phi_{t_1+\delta}}$$

Hence using $\delta = \Delta$, $t_1 = -\Delta$ and $t_2 = 0$, we have,

$$\mathcal{P}_{-\Delta} = \mathcal{S}^0_{-\Delta}[\mathcal{P}_0] = \mathcal{S}^\Delta_0[\mathcal{P}_0] = \mathcal{D}^{-1}_{\phi_0}\mathcal{D}_{\phi_\Delta}[\mathcal{P}_0]$$

Thus finally, using $\mathcal{D}_{\phi_0}[\mathcal{P}_{-\Delta}] = \mathcal{Q}_{-\Delta}$,

$$\mathcal{D}_{\phi_\Delta}\mathcal{P}_0 = \mathcal{D}_{\phi_0}[\mathcal{P}_{-\Delta}] = \mathcal{Q}_{-\Delta}$$

□

Remark 2.5.7. Note that letting $\Delta \to \infty$ gives us the relation between the solutions of primal and dual Riccati *equations*. That is, if $\hat{p}$ is stabilizing solution of $= A'\hat{p}_t + \hat{p}_tA + C + \hat{p}_t\Sigma\hat{p}_t$ and $\hat{q}$ is the stabilizing solution of $\bar{A}'\hat{q}_t + \hat{q}_t\bar{A} + \bar{C} + \hat{q}_t\bar{\Sigma}\hat{q}_t$. Then $\hat{q}$ is the semiconvex dual of $\hat{p}$, under kernel $(P, S, Q)_0$. Hence

$$\hat{q} = -{S_0}'(\hat{p} - P_0)^{-1}S_0 - Q_0 \tag{2.122}$$

Remark 2.5.8. Note that $\frac{1}{2}z'q_tz = \mathcal{Q}_t = \mathcal{D}_{\phi_0}[\mathcal{P}_t](z)$ and q_t satisfies dual DRE. Similarly, using lemma 2.5.6, $\mathcal{D}_{\phi_\Delta}[\mathcal{P}_t](z) = \mathcal{Q}_{t-\Delta} = \frac{1}{2}z'q_{t-\Delta}z$. Since $q_{t-\Delta}$ also satisfies same dual DRE (2.119), kenel $\phi_\Delta(x,z)$ also satisfies same matching conditions (2.120). Let us define

$$\mathcal{K}_t \doteq \begin{bmatrix} S_t^{-1}P_t & -S_t^{-1} \\ S_t{}' - Q_t S_t^{-1} P_t & Q_t S_t^{-1} \end{bmatrix}$$

Thus from figure (2.4.4), if $\mathcal{K}_0$ satisfies matching conditions (2.121), then so does $\mathcal{K}_\Delta$, for any $\Delta \geq 0$.

The kernel matching conditions (2.61) allow us to transform one Riccati equation into any other. In particular, we can transform a Riccati equation into a Linear equation amenable to analytical solution. Thus we can derive well known analytical solutions to Riccati equations derived earlier in [KL85] and [Rus88].

Analytical solution 1

We wish to solve $\dot{p_t} = A^T p_t + p_t A + C + p_t \Sigma p_t$. To convert this into a linear equation, let us choose $S_0 = I$, $\bar{C} = \bar{\Sigma} = 0$. Thus kernel matching conditions (2.61) imply that, $\forall t$

$$-\dot{P}_t = A' P_t + P_t A + C + P_t \Sigma P_t = 0 \quad (2.123)$$

$$-\dot{S}_t = (A + \Sigma P_t)' S_t = -S_t \bar{A} \quad (2.124)$$

$$-\dot{Q}_t = S^T \Sigma S_t = -\bar{A}' Q_t - Q_t \bar{A} \quad (2.125)$$

From (2.123), for all t, P_t is the solution to the Riccati equation $A^T P + PA + C + P\Sigma P = 0$. Thus if the Riccati equation has stable and unstable solutions, say P_- and P_+, and if $P_0 = P_+$, then $P_t = P_+\ \forall t$.

From (2.124), since $S_0 = I$, $\bar{A} = -(A + \Sigma P)^T$. Solving $S_0 = I$ and $\dot{S} = -S\bar{A}$, we have $S_{-t} = e^{-\bar{A}t}$.

From (2.125), Q_0 is solution to the Lyapunov equation, $\bar{A}^T Q_0 + Q_0 \bar{A} + \Sigma = 0$. It can be proved that $Q_0 = (P_+ - P_-)^{-1}$. Solving $\dot{Q} = -\bar{A}^T Q - Q\bar{A}$, we get $Q_{-t} = e^{-\bar{A}^T t} Q_0 e^{-\bar{A}t}$.

Combining all above and (2.46), we get

$$\begin{aligned}
&(p_{-t} - P_{-t}) \\
&= \quad -S_{-t}\left[-S_0{}'(p_0 - P_0)^{-1}S_0 + Q_{-t} - Q_0\right]^{-1} S_{-t}{}' \\
&= \quad -e^{-\bar{A}t}[-(p_0 - P_+)^{-1} + e^{-\bar{A}^T t}Q_0 e^{-\bar{A}t} - Q_0]^{-1}e^{-\bar{A}^T t}
\end{aligned}$$

Rearranging and substituting $Q_0 = (P_+ - P_-)^{-1}$ and $\bar{A} = -(A + \Sigma P_+)^T$, we get

$$\begin{aligned}
&e^{t(A+\Sigma P_+)}\left[(p_t - P_+)^{-1} - (P_- - P_+)^{-1}\right] e^{t(A+\Sigma P_+)^T} \\
&\quad = \left[(p_0 - P_+)^{-1} - (P_- - P_+)^{-1}\right]
\end{aligned} \tag{2.126}$$

Equation (2.126) is same as the Method 3 developed by Leipnik in [KL85]. Note that (2.126) assumes that extremal solutions P_+ and P_- are well separated.

This method being analytic, is extremely fast and accurate, and does away with the need to combine operators.

Analytical solution 2

Above method suffers from numerical errors when extremal solutions of the Riccati equations, are not well separated. Following *almost analytic* method can solve this problem. The reason it is called *almost analytics* is because it requires computation of integrals of matrix exponentials instead of exponentials themselves.

If we use $S_0 = I$, $\bar{\Sigma} = 0$, $Q_0 = 0$, then matching conditions (2.61) give us

$$-\dot{P}_t = A'P_t + P_tA + C + P_t\Sigma P_t = 0 \tag{2.127}$$

$$-\dot{S}_t = (A + \Sigma P_t)'S_t = -S_t\bar{A} \tag{2.128}$$

$$-\dot{Q}_t = S'\Sigma S_t = -\bar{A}'Q_t - Q_t\bar{A} + \bar{C} \tag{2.129}$$

Hence again, $P_0 = P_t = P$ for all t, where P is any solution of Riccati equation (2.127). $S_{-t} = e^{-t\bar{A}}$, with $-\bar{A} = (A + \Sigma P)^T$ as earlier. Since $Q_0 = 0$, $\bar{C} = \Sigma$ and Q_t evolves according to (2.129). If T is a solution of Lyapunov equation, $-\bar{A}'T + T\bar{A} = \bar{C}$, then $Q_{-t} = e^{-t\bar{A}'}Te^{-t\bar{A}} - T$. Hence

$$\begin{aligned}
-\dot{Q}_t &= -\bar{A}'e^{-t\bar{A}'}Te^{-t\bar{A}} - e^{-t\bar{A}'}Te^{-t\bar{A}}\bar{A} \\
&= -e^{-t\bar{A}'}(\bar{A}'T + T\bar{A})e^{-t\bar{A}} \\
&= e^{-t\bar{A}'}\bar{C}e^{-t\bar{A}}
\end{aligned}$$

Since $Q_0 = 0$, and $\bar{C} = \Sigma$, $Q_{-t} = \int_0^t e^{-t\bar{A}'}\Sigma e^{-t\bar{A}}$. Substituting in (2.46), and using $-\bar{A} = (A + \Sigma P)^T = B$,

$$e^{tB'}(-P + p_t)^{-1}e^{tB} = (-P + p_0)^{-1} - \int_0^t e^{\tau B'}\Sigma e^{\tau B} d\tau \tag{2.130}$$

This formula works well for problems with extremal solutions not well separated. In the limiting case, when $P = P_+ = P_-$, it can be proved that $-\bar{A} = (A + \Sigma P_+)^T = 0$, which implies

$$(-P + p_{-t})^{-1} = (-P + p_0)^{-1} - \Sigma t$$

Using shorthand for $\bar{p}_t = p_t - P$ and $\bar{p}_0 = p_0 - P$ It can be shown that above is equivalent to

$$\bar{p}_{-t} = e^{tB}\bar{p}_0 \left\{ I - \int_0^t e^{tB'}\Sigma e^{Bt}\bar{p}_0 dt \right\}^{-1} e^{tB'} \tag{2.131}$$

This formula is same as the one developed by Rusnak [Rus88]. Integrals of matrix exponentials can be calculated easily using techniques from [VL78]. For our use, if

$$\exp\left(\begin{bmatrix} -B^T & \bar{\Sigma} \\ 0 & B \end{bmatrix} t \right) = \begin{bmatrix} F_1(t) & F_2(t) \\ 0 & F_3(t) \end{bmatrix}$$

then $\int_0^t e^{tB^T}\bar{\Sigma}e^{Bt}dt = F_3(t)^T F_2(t)$.

2.5.4 Numerical Experiments and Results

Thus we have seen three different approaches to solve the time invariant DRE, (2.105) in this text. Following is the summary of algorithms.

- Starting with (P_0, S_0, Q_0), find the solution of bivariate DRE (2.108) on a smaller time interval, $[-t, 0]$. This can either be done analytically using (2.68),(2.69),(2.70) (for a small time interval, since Davison-Maki theorem is illconditioned for large t), or by time marching using an appropriate solver. Here we use fixed step Runge-Kutta fourth order method for the same. We shall denote the number of steps by Nrk. Note that an often useful initial condition is $P_0 = 0$, $S_0 = I$ and $Q_0 = 0$.

- Using (P_{-t}, S_{-t}, Q_{-t}), we can construct kernels B^t as per (2.111) and I^t as per (2.109). We may also choose to find I analytically using (2.71).

- Having found a kernel at time $-t$ and starting with given p_0, we can choose to perform M kernel doubling operations to solve for kernel at time $2^M t$ and N time-stepping operations to compute p_{-T} for $T = 2^M N t$. We can do this three different ways as follows.

  Method A Construct $I^{2^M t}$ using doubling formula (2.110) recursively M times, and evolve p_0 back in time to get p_{-T} using N stepping operations as per (2.45).

  Method B Construct $B^{2^M t}$ using doubling formula (2.112) recursively M times, and evolve p_0 back in time to get p_{-T} using N stepping operation as per (2.113). This method was first proposed in [McE08].

  Method C Use doubling formula (2.117) M times recursively to compute $(P, Q, S)_{-2^M t}$. Evolve p_0 back in time to get p_{-T} using N stepping operations as per (2.118).

Assuming that we evolve the solution from 0 to t, using Runge-Kutta fourth order method, using Nrk steps, and perform M doubling and N stepping operations, computational complexity (flops) needed for three methods are as follows (found using [Min])

Method A $Nrk(32n^3+3n^2)+(M+1)\left(16n^2 + \frac{37}{3}n^3 + \frac{11}{3}n\right)+N\left(18n^2 + \frac{19}{3}n^3 + \frac{11}{3}n\right)$

Method B $Nrk(32n^3+3n^2)+(M+1)\left(16n^2 + \frac{37}{3}n^3 + \frac{11}{3}n\right)+N\left(54n^2 + 19n^3 + 11n\right)$

Method C $Nrk(32n^3 + 3n^2) + M\left(67n^2 + \frac{94}{3}n^3 + \frac{44}{3}n\right) + N\left(36n^2 + \frac{38}{3}n^3 + \frac{22}{3}n\right)$

Now we shall apply the theory developed so far to the example problems. We shall also benchmark Method B proposed in [McE08] against Method C, which show greater accuracy at a very small time step propagation.

2.5.5 A Stiff Time Invariant Example

We shall show in this section that algorithms discussed so far are applicable to stiff DREs. Usual time marching methods are constrained in step size by the stability requirement. To get a rough idea, let us look at a linear system $\dot{x}(t) = Ax(t)$. Explicit methods like Euler method impose condition $|1 + h\lambda| < 1$ on stepsize h, where λ is an eigenvalue of A. If λ is real and negative, this condition implies $h < \frac{2}{|\lambda|}$. Thus excessively small step sizes may be needed to ensure accuracy if λ is large. Following numerical experiments show that new algorithms give accurate answers with step size $h = 2^M t$ significantly larger than $2/\lambda$ when the initial transient phase is almost over.

We shall choose example 4.1 from [CL90]. Symmetric DRE

$$-\dot{p}(t) = -p^2(t) + k^2 I_n; \qquad p(0) = U\text{diag}[i]U^{-1}, \quad t \geq 0$$

where diag$[i]$ denotes the diagonal matrix with successive diagonal entries $1, 2, \ldots, n$, and U is an orthogonal matrix. Above DRE is solved for different n (size of the equation) and k (larger the value of k, the stiffer the equation). The error of the computed solution is defined as $err = ||p - \bar{p}||_F / ||p||_F$, where X denotes the true and $\bar{X}$ denotes the computed solution. The analytical solution of above DRE is

$$p(t) = U\text{diag}\left[\frac{k\text{sinh } kt + i\text{cosh } kt}{\text{cosh } kt + \frac{i}{k}\text{sinh } kt}\right] U^{-1}$$

where diag$[\cdot]$ here denotes the diagonal matrix whose diagonal entries are generated by letting i take successively the values $1, 2, \ldots, n$.

Let $0 \leq t_1 \leq t_2$. Starting with $\bar{p}_{-t_1} = p(-t_1)$ which is the analytic solution, we computed the solution $\bar{p}_{-t_2}$ using *method A*, for different t_1, t_2, n, k, and using various choices for Nrk, M, N. Note that the solution is marched for $t = T/(N2^M)$, and results are tabulated in table 2.1.

Usually error in the final solution arises out of

1. Error in the initial time marched solution $(P, S, Q)_t$ and hence in I^t, which gets amplified due to doubling algorithms. Higher steps in time marching, Nrk, lead to a more accurate final solution.

Table 2.1: Numerical accuracy test for a stiff DRE.

t_1	t_2	n	k	M	N	Nrk	h	$\frac{2}{\lvert\lambda\rvert}$	err
0	10	10	5	1	10	10	1	0.2	2.94×10^{-5}
0	1000	10	5	10	1	10	1000	0.2	1.50×10^{-5}
0	1000	10	10	10	1	10	1000	0.2	8.07×10^{-6}
0	1000	100	10	10	1	10	1000	0.02	8.07×10^{-6}
0	1000	100	10	10	1	100	1000	0.02	1.55×10^{-12}
0	1000	100	10	10	1	500	1000	0.02	3.24×10^{-15}
0	1000	100	10	16	1	10	1000	0.02	6.65×10^{-10}
0	100000	100	10	22	1	-*	100000	0.02	2.98×10^{-15}
0	100000	100	100	22	1	-*	100000	0.02	2.24×10^{-14}
3	10	100	100	8	3	10	2	0.02	1.52×10^{-6}
3	10	100	100	8	3	1	2	0.02	1.77×10^{-2}
0	10	100	100	11	1	100	10	0.02	9.84×10^{-12}
0	10	100	100	1	2^{11}	100	10	0.02	4.83×10^{-9}

* Note that a blank entry for Nrk implies that the solution for $(P, Q, S)_t$ was found analytically using (2.68),(2.69),(2.70).

2. Error in computing I^t for very small t. Since matrix inverses involved in computation of I^t in (2.110), grow more singular as $t \to 0$. For extremely small timesteps, parameters of I^t can blow up, leading to inaccuracies. Hence it is preferable to keep t above a certain minimum threshold.

Exact error analysis of these methods is beyond the scope of this thesis. But above results show that method 1, is fast, useful and numerically stable fundamental solution for a long time horizon propagation of a stiff DRE. Kernel doubling methods give a quadratic convergence rate, rather than the linear rate of stepping iterations. This enables us to take longer time steps, much higher than the stability threshold of the stiff DRE explicit time marching algorithms.

Time invariant doubling algorithms

Now we shall benchmark the numerical performance of method 1, 2 and 3 using the example from [McE08]. We consider a 2×2 size DRE, with

$$A = \begin{bmatrix} -2 & 1.6 \\ -1.6 & -0.4 \end{bmatrix}, C = \begin{bmatrix} 1.5 & 0.2 \\ 0.2 & -0.4 \end{bmatrix}, \Sigma = \begin{bmatrix} 0.216 & -0.008 \\ -0.008 & 0.216 \end{bmatrix}$$

starting with initial condition

$$p_0 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}$$

and we use the duality kernel $(P, S, Q)_0 = (D, -D, D)$, where

$$D = \begin{bmatrix} -1 & -0.2 \\ -0.2 & -0.1 \end{bmatrix}$$

We also compute a near exact solution using variable step Runge-Kutta (ode45 in MATLAB) method for $T = 4$, with absolute tolerance 10^{-15} and relative tolerance of 10^{-13}. Treating this as the *true* solution, we compare this against solutions by all three methods, using just one step of Runge Kutta fourth order fixed step method for computing $(P, S, Q)_{-t}$ and using only kernel doubling. Thus we choose $N = 1$, $Nrk = 1$ and vary M between 3 and 17. Error in the computed solution is defined as $err = ||p - \bar{p}||_F / ||p||_F$. Here p indicates the *true* solution, and $\bar{p}$ is the computed solution. Figure 2.3 show plots of M against err for all three methods. We can see that the method 3, can achieve maximum accuracy of 9.48×10^{-13} as against to 5.77×10^{-10} for method 1 and 3.85×10^{-10} for method 2 first proposed in [McE08], which is significantly better. Although increasing M, decreases err initially, eventually all methods exhibit an increase in err with M. This is because $t = 4/2^M \to 0$, and $P_{-t} - P_0 \to 0$, $S_{-t} - S_0 \to 0$ and $Q_{-t} - Q_0 \to 0$, and their inverses grow very large and can not be accurately computed. Thus it is advisable to keep t above a minimum threshold. In this example, with $M = 13$, $t = 4.88 \times 10^{-4}$.

We also benchmark above methods against a fourth order Runge-Kutta method, for various stepsizes, and plot err against computational cost measured in flops (using [Min]) in figure 2.4. Specifically, the flopcount for Runge Kutta fixed step 4th order method was taken to be $Nrk(32n^3 + 3n^2)$. Note that the above methods exhibit a faster convergence rate, and yield more accuracy for a given computational cost.

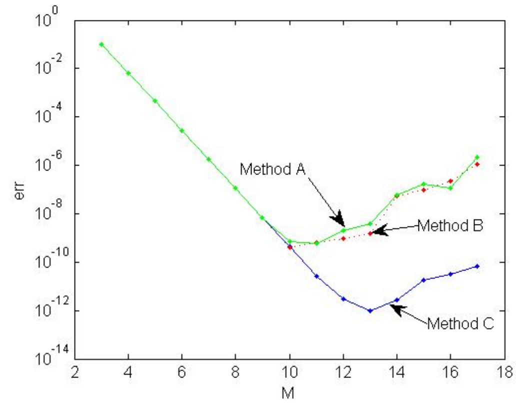


Figure 2.3: M vs. Solution Error

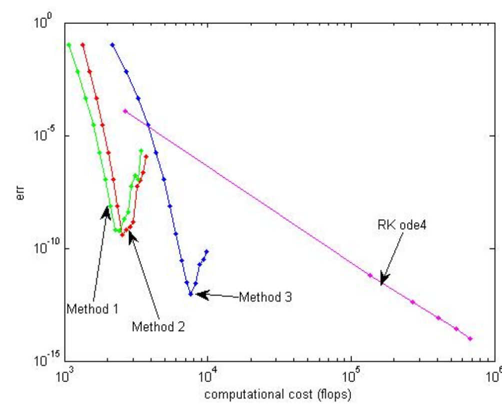


Figure 2.4: Computational cost vs. Solution Error

2.6 Conclusion

Thus in this chapter, we derived a new fundamental solution for the DRE (2.1) with the time varying coefficients. We showed its equivalence with the max-plus based fundamental solution (2.19) and (2.20), proposed in [FM00], which arises out of a two-point boundary value problem and dynamic programming. We proved that such a fundamental solution is a bivariate quadratic for the linear/quadratic problem in (2.28).

We showed that under a semiconvex duality kernel, a primal DRE is transformed into a dual DRE. We derived the compatibility conditions which the coefficients of such DREs should satisfy in (2.61). These conditions can be readily expressed as per (2.63), in terms of classic Hamiltonian matrices of system dynamics and symplectic matrices of kernel parameters. This led us to derive analytical forms of the kernel parameters involving state transition matrix of the Hamiltonian system as per (2.68), (2.69), (2.70).

Using backward dynamic programming principle, we showed the kernel relationships between time shifted primal and the dual DREs, shown in Figure 2.4.4. This enabled us to develop three different methods to propagate a DRE. Time invariant analogue of one of these methods was first proposed in [McE08].

As a special case of the time-varying problem, we derived doubling and stepping algorithms in primal space, for the time-invariant DRE ((2.109),(2.110), (2.45)).

Using the kernel matching conditions, we could design kernels which convert primal DREs into linear differential equations under duality, which have well-known analytical solution. Using this, in sections 2.5.3 and 2.5.3, we rediscovered the analytical solutions for the DRE, previously developed by [KL85] and [Rus88].

Finally in section 2.5.4, we used these methods on the stiff problems and demonstrated their speed, accuracy and numerical stability. We also benchmarked them against previous dual space doubling algorithm developed in [McE08], and demonstrated an order of magnitude improvement in the accuracy for propagation at a very small time step.

Thus this paper provides an elegant fundamental solution for the time-varying DRE, useful for stiff problems and a long time horizon propagation. It also provides a powerful unifying framework based on optimal control formulation, semiconvex duality and max-plus algebra, which enables us to solve Riccati differential equations, and see existing methods in new light.

Chapter 2, in part, has been submitted for publication as it may appear in Automatica. Deshpande, Ameet, Elsevier Press [Des]. The dissertation author was the primary investigator and author of this paper.

Chapter 3

A Curse-of-Dimensionality-Free Numerical Method based on Max-Plus algebra

3.1 Introduction

In this chapter we shall revisit the earlier development (cf. [McE07], [McE06], [McE09]) on the curse-of-dimensionality-free method for solving the maximization problems in which the Hamiltonian can be approximated by the maxima of the linear-quadratic Hamiltonians.

Then we shall derive the max-plus fundamental solution for the constituent linear-quadratic problem, and use it in the context of curse-of-dimensionality-free method.

3.2 The Problem

We consider the problem already discussed in chapter 1, with the dynamics

$$\begin{aligned} \dot{\xi}_t &= f(\xi_t) + \sigma(\xi_t) u_t \\ \xi_0 &= x \in \mathbb{R}^n \end{aligned} \tag{3.1}$$

where the state $\xi_t \in \mathbb{R}^n$, the control u lies in $L_2^{\text{loc}} \doteq \{u : [0,\infty) \to \mathbb{R}^m : u \in L_2[0,T] \quad \forall T < \infty\}$, f represents the nominal dynamics, and σ is the $n \times m$ matrix valued multiplier on the control.

This represents an infinite horizon H_∞ problem with the fixed feedback $f(\xi)$ and u is the disturbance, and we wish to compute the worst case cost/ maximal payoff for the disturbance.

Thus the optimal control objective is to minimize the integral payoff

$$V(x) \doteq \inf_{u \in L_2^{\text{loc}}} \sup_{T<\infty} \int_0^T l(\xi_t) - \frac{\gamma^2}{2}|u_t|^2\, dt. \tag{3.2}$$

Such system is said to satisfy an H_∞ attenuation bound (of γ) if there exists $\gamma < \infty$ such that V (also called a *storage* function) defined as above, is locally bounded. V is the value function of the problem. For the existence and regularity of the value function, we make following assumptions on the dynamics and the running cost.

We assume that there exist $K, c, M \in (0,\infty)$ such that for all $x \in \mathbb{R}^n$,

$$\begin{aligned} &|f(x) - f(y)| \leq K|x-y| \text{ and } f(0) = 0 \\ &(x-y)'\left(f(x) - f(y)\right)| \leq -c|x-y|^2 \\ &|\sigma(x) - \sigma(y)| \leq K|x-y| \\ &|\sigma(x)| \leq M \end{aligned} \tag{3.3}$$

Note that this implies $x'f(x) \leq -c|x|^2$ for all $x \in \mathbb{R}^n$ or the exponential stability of the feedback control.

Regarding the running cost $l(x)$, we assume that there exist $C, \alpha \in (0,\infty)$ such that

$$\begin{aligned} &|l(x) - l(y)| \leq C(1 + |x| + |y|)|x-y| \quad \forall x, y \in \mathbb{R}^n \\ &0 \leq l(x) \leq \alpha|x|^2. \end{aligned} \tag{3.4}$$

This assumes a linear bound on the growth of the Lipschitz constant and restricts l to be zero at the origin.

Only the existence of constants K, C will be important and not the values. Rest of the constants satisfy the following final assumption.

$$\frac{\gamma^2}{2M^2} > \frac{\alpha}{c^2} \tag{3.5}$$

The corresponding HJB PDE is as follows.

$$\begin{aligned} 0 &= -\sup_{u_t\in\mathbb{R}^m}\left\{(f(x)+\sigma(x)u)\cdot\nabla W + l(x) - \frac{\gamma^2}{2}|u_t|^2\right\} \\ &= -\left[f(x)\cdot\nabla W + l(x) + \nabla W'\left(\frac{\sigma(x)\sigma'(x)}{2\gamma^2}\right)\nabla W\right] \\ &\doteq -H(x,\nabla W) \end{aligned} \tag{3.6}$$

The boundary condition is $W(0)=0$.

Such a PDE can have multiple viscosity solutions. Consider a simple one-dimensional problem as an illustration. Let $\dot{\xi}_t = -\xi_t + w_t$, $l(\xi) = \xi^2$ and take $\gamma = 2$. The PDE is then

$$0 = \frac{1}{8}\nabla W - x\nabla W + x^2, \quad \text{and } W(0) = 0.$$

There are two C_∞ solutions, $W^1(x) = (2-\sqrt{2})x^2$ and $W^1(x) = (2+\sqrt{2})x^2$, and an infinite number of viscosity solutions such as

$$W(x) = \begin{cases} (2-\sqrt{2})x^2 & \text{if } x \le 1 \\ (2+\sqrt{2})x^2 - 2\sqrt{2} & \text{if } x > 1 \end{cases}$$

Next section gives an overview of an important earlier work by McEneaney [McE98] which extracts the correct viscosity solution, which is the value function.

3.3 Existence and Uniqueness

Soravia [Sor96] and McEneaney [McE98], [McE07], [McE06] have proved the following important theorems which serve as the foundation.

Theorem 3.3.1. *Assuming (3.3), (3.4) and (3.5), the value function V defined in (3.2), exists and is the unique continuous viscosity solution of the HJB PDE (3.6) in the class*

$$\mathcal{K} \doteq \left\{W \in C(\mathbb{R}^n) : 0 \le W(x) \le c\frac{(\gamma-\delta)^2}{2M^2}|x|^2\right\} \tag{3.7}$$

for some $\delta > 0$. Note that δ depends on the choice of W.

Theorem 3.3.2. *Assuming (3.3), (3.4) and (3.5), if we define a finite horizon value function*

$$U(x,T) \doteq \sup_{u \in L_2^{loc}} \int_0^T l(\xi_t) - \frac{\gamma^2}{2}|u_t|^2 \, dt, \tag{3.8}$$

then for all $x \in \mathbb{R}^n$, $U(x,T)$ is monotonically increasing with T, and

$$\lim_{T \to \infty} U(x,T) = V(x) \tag{3.9}$$

where V is the value function (3.2). The convergence is uniform over the compact subsets of $\mathbb{R}^n$.

Using the semigroup notation introduced in chapter 1, following holds true.

Theorem 3.3.3. *Assuming (3.3), (3.4) and (3.5), for any $T \geq 0$ and for all $x \in \mathbb{R}^n$, V defined in (3.2), also satisfies the following* dynamic programming principle *(DPP).*

$$V(x) = \mathcal{S}_T[V](x) \doteq \sup_{u \in L_2^{loc}} \left[\int_0^T (l(\xi_t) - \frac{\gamma^2}{2}|u_t|^2) \, dt + V(\xi_T) \right] \tag{3.10}$$

Thus V is the fixed point of the operator $\mathcal{S}_T$ for all $T \geq 0$. It is the unique such fixed point in class $\mathcal{K}$ defined in (3.7). Further, given any $W \in \mathcal{K}$, $\lim_{T \to \infty} \mathcal{S}_T[W](x) = V(x)$ *for all $x \in \mathbb{R}^n$. Convergence is uniform on compact sets.*

Note that since $0 \in \mathcal{K}$,

$$\lim_{T \to \infty} \mathcal{S}_T[0](x) = \lim_{T \to \infty} U(x,T) = V(x).$$

Thus the theorem 3.3.2 is a special case of theorem 3.3.3. Also it is known that operators $\mathcal{S}_T$ form a semigroup. Thus for $t_1, t_2 \geq 0$ and $W \in \mathcal{K}$, $\mathcal{S}_{t_1}[\mathcal{S}_{t_1}[W]](x) = \mathcal{S}_{t_1+t_2}(x)$.

This suggests an alternate way to compute the value function, by finding the fixed point of the semigroup or as the limit of iterated semigroup propagations. Starting with an initial guess $W \in \mathcal{K}$, with a given $T > 0$,

$$V(x) = \lim_{t \to \infty} \mathcal{S}_t[W](x) = \lim_{n \to \infty} \mathcal{S}_{nT}[W](x). \tag{3.11}$$

Here $n \in \mathbb{N}$, and $\mathcal{S}_{nT}[W]$ can be constructed iteratively as per $\mathcal{S}_T \cdot \mathcal{S}_{(n-1)T}[W] = \mathcal{S}_T \cdot \mathcal{S}_T[W] \ldots \mathcal{S}_T[W]$.

3.4 Approximation with LQ Hamiltonians

In order to solve the HJB PDE (3.6), one approach is to approximate it by the max-plus summation of linear-quadratic (LQ) Hamiltonians. This section reviews the theoretical development along this approach covered in [McE07] and [McE09]. Specifically, we consider:

$$\begin{aligned} &0 = -H(x, \nabla W) \approx -\widetilde{H}(x, \nabla W) \doteq - \sup_{m \in \{1,2,\ldots,M\}} H^m(x, \nabla W) \\ &W(0) = 0 \end{aligned} \tag{3.12}$$

with each of the constituent H^m is linear-quadratic,

$$H^m(x,p) \doteq \frac{1}{2} x' C^m x + \frac{1}{2} p' \Sigma^m p + (A^m x)' p + x' l_1^m + p' l_2^m + \frac{1}{2}\alpha^m, \tag{3.13}$$

where C^m, Σ^m are $n \times n$ symmetric matrices, $l_1^m, l_2^m \in \mathbb{R}^n$ and $\alpha^m \in \mathbb{R}$.

Note that this approximation using the max-plus sum of linear-quadratic Hamiltonians can approximate any semiconvex Hamiltonian arbitrarily closely. Recall that a function $\phi : \mathbb{R}^n \to \mathbb{R}$ is semiconvex over $\mathbb{R}^n$ if $\exists\, C > 0$ such that $\phi(x) + C|x|^2$ is convex over $x \in \mathbb{R}^n$. This greatly extends the applicability of this method to general nonlinear problems.

If such approximation is exact, then $H = \widetilde{H}$ is associated with an optimal control problem for a switched linear system. For such a problem the dynamics is,

$$\dot{\xi}_t = A^{\mu_t} \xi_t + l_2^{\mu_t} + \sigma^{\mu_t} u_t, \quad \xi_0 = x \tag{3.14}$$

and the corresponding value function is

$$\begin{aligned} \widetilde{V}(x) &= \sup_{u \in L_2^{\text{loc}}} \sup_{\mu \in \mathcal{D}_\infty} \sup_{T < \infty} \widetilde{J}(x, T; u, \mu) \\ &\doteq \sup_{u \in L_2^{\text{loc}}} \sup_{\mu \in \mathcal{D}_\infty} \sup_{T < \infty} \int_0^T l^{\mu_t}(\xi_t) - \frac{\gamma^2}{2} |u_t|^2 \, dt \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} &\mathcal{M} \doteq \{1, 2, \ldots, M\} \text{ and } \sigma^i \text{ are such that } \Sigma^i = \tfrac{\sigma^i \sigma^{i\prime}}{\gamma^2} \text{ for } i \in \mathcal{M} \\ &l^{\mu_t}(\xi_t) \doteq \frac{1}{2} \xi' C^{\mu_t} \xi + \xi' l_1^{\mu_t} + \frac{1}{2} \alpha^{\mu_t}, \\ &\mathcal{D}_\infty \doteq \{\mu : [0, \infty) \to \mathcal{M} : \text{ measurable } \}. \end{aligned}$$

We make following assumptions on this problem for further analysis.

> Assume that the underlying system dynamics and the payoff functions satisfy (3.3), (3.4). Hence the value function V defined by (3.2) is a unique viscosity solution, of $0 = -H(x, \nabla V)$ in class $\mathcal{K}$, defined in (3.7).
> Assume that the approximating Hamiltonian $\widetilde{H}(x,p) = \sup_m H^m(x,p) \leq H(x,p)$ for all $x, p \in \mathbb{R}^n$.
> Assume that there exist $c_A \in (0, \infty)$ such that $x' A_m x \leq -c_A|x|^2$ for all $x \in \mathbb{R}^n$ and all $m \in \mathcal{M}$.
> Assume $H^1(x,p)$ has coefficients satisfying the following: $l_1^1 = l_2^1 = 0$; $\alpha^1 = 0$; there exists $c_{A,1} \in (0, \infty)$ such that $x' A^1 x \leq -c_{A,1}|x|^2$ $\forall x \in \mathbb{R}^n$; C^1 is positive definite, symmetric; and $\gamma^2/c_\sigma^2 > c_D/c_{A,1}^2$, where $c_\sigma \doteq |\sigma^1|$ and c_D is such that $x' D^1 x \leq c_D|x|^2$ $\forall x \in \mathbb{R}^n$. (3.16)
> Assume that the system (3.14) is controllable in the sense that given $x, y \in \mathbb{R}^n$ and $T > 0$, there exist processes $w \in \mathcal{W}$ and μ measurable with range in $\mathcal{M}$, such that $\xi_T = y$ when $\xi_0 = x$ and one applies control w, μ.

Note that the last assumption is satisfied, if for at least one $m \in \mathcal{M}$, $\sigma^m \sigma^{m\prime}$ has n positive eigenvalues.

> Final assumption is that there exist $c_1, c_2 < \infty$ such that for any ϵ-optimal pair, μ^ϵ, w^ϵ for the $\widetilde{H}$ problem, one has $||w^\epsilon||^2_{L_2[0,t]} \leq c_1 + c_2|x|^2$ for all $\epsilon \in (0, 1]$, all $T < \infty$ and all $x \in \mathbb{R}^n$. (3.17)

This behavior was proved in the purely quadratic case (cf. [McE98], [McE07]). But in this general case, we assume it instead.

Now we are ready to review the previous development. Following theorems form the foundations of the numerical methods in the next section, and were proved in [McE09].

Theorem 3.4.1. *Assume (3.16), (3.17), and let V, $\widetilde{V}$ and $\mathcal{K}$ be as defined in (3.15), (3.2) and (3.7), respectively. Then $\widetilde{V}(x)$ exists and $0 \leq \widetilde{V}(x) \leq V(x)$ for all $x \in \mathbb{R}^n$. Consequently, $\widetilde{V} \in \mathcal{K}$.*

$\widetilde{V}$ is also the unique continuous viscosity solution of the PDE,

$$0 = -\widetilde{H}(x, \nabla W)), \quad W(0) = 0$$

in class $\mathcal{K}$. Here $\widetilde{H}$ is defined as per (3.12).

Assuming $\xi_0 = x$ and dynamics (3.15), we define the semigroup,

$$\widetilde{\mathcal{S}}_T[\phi](x) \doteq \sup_{u \in L_2^{\text{loc}}} \sup_{\mu \in \mathcal{D}_\infty} \left[\int_0^T l^{\mu_t}(\xi_t) - \frac{\gamma^2}{2}|u_t|^2 \, dt + \phi(\xi_T) \right]. \tag{3.18}$$

Now we state the dynamic programming principle for this problem, and uniqueness of the $\widetilde{V}$ as a fixed point of the semigroup operation.

Theorem 3.4.2. *Assume (3.16), (3.17). The value function $\widetilde{V}$ as defined in (3.15), is a fixed point of the semigroup operation $\widetilde{\mathcal{S}}_T$ for any $T > 0$.*

$$\widetilde{V}(x) = \sup_{u \in L_2^{loc}} \sup_{\mu \in \mathcal{D}_\infty} \left\{ \int_0^T l^{\mu_t}(\xi_t) - \frac{\gamma^2}{2}|u_t|^2 \, dt + \widetilde{V}(\xi_T) \right\} = \widetilde{\mathcal{S}}_T[\widetilde{V}](x) \tag{3.19}$$

for all $x \in \mathbb{R}^N$. Moreover, it is the unique fixed point in class $\mathcal{K}$, defined in (3.7).

The process μ_t which represents the switching time history, can in general be any measurable function taking values in $\mathcal{M}$. In the next section, we will allow it only to switch at discrete times to motivate the numerical method.

3.5 Discrete Time Approximation

Since we seek to avoid curse-of-dimensionality, we can not discretize over space. Instead discretization will be over time where approximate μ processes will be constant over the length of each time-step.

We define operator $\bar{\mathcal{S}}_\tau$ on $\phi \in \mathcal{K}$ by

$$\begin{aligned} \bar{\mathcal{S}}_\tau[\phi](x) &\doteq \sup_{u \in L_2^{\text{loc}}} \sup_{m \in \mathcal{M}} \left[\int_0^\tau l^m(\xi_t^m) - \frac{\gamma^2}{2}|u_t|^2 \, dt + \phi(\xi_\tau^m) \right] \\ &= \sup_{m \in \mathcal{M}} \mathcal{S}_\tau^m[\phi](x) \end{aligned} \tag{3.20}$$

where ξ^m satisfies dynamics

$$\dot{\xi}_t^m = A^m \xi_t^m + \sigma^m u_t, \quad \xi_0^m = x \tag{3.21}$$

and $\mathcal{S}_\tau^m$ is defined as

$$\mathcal{S}_\tau^m[\phi](x) \doteq \sup_{u \in L_2^{\text{loc}}} \left[\int_0^\tau l^m(\xi_t)^m - \frac{\gamma^2}{2}|u_t|^2 \, dt + \phi(\xi_\tau^m) \right] \tag{3.22}$$

It is useful to note that operators $\bar{\mathcal{S}}_\tau$ do not necessarily form a semigroup, although they do form a sub-semigroup (i.e. $\bar{\mathcal{S}}_{\tau_1+\tau_2}[\phi] \le \bar{\mathcal{S}}_{\tau_1}\bar{\mathcal{S}}_{\tau_2}[\phi]$). Further, one has $\mathcal{S}_\tau^m \le \bar{\mathcal{S}}_\tau \le \widetilde{\mathcal{S}}_\tau$ for all $m \in \mathcal{M}$.

With τ acting as a time-discretization step-size, let us define discretized switching control spaces over infinite and finite horizon.

$$\mathcal{D}_\infty^\tau \doteq \left\{ \mu : [0,\infty) \to \mathcal{M} \;\middle|\; \begin{array}{l} \text{for all } k \in \mathbb{N}, \text{ there exists } m_k \in \mathcal{M} \\ \text{such that } \mu_t = m_k \text{ for all } t \in [(k-1)\tau, k\tau) \end{array} \right\} \tag{3.23}$$

$$\mathcal{D}_T^\tau \doteq \left\{ \mu : [0,T) \to \mathcal{M} \;\middle|\; \begin{array}{l} \text{for all } k \in \mathbb{N}, k \le \lceil \frac{T}{\tau} \rceil, \text{ there exists } m_k \in \mathcal{M} \\ \text{such that } \mu_t = m_k \text{ for all } t \in [(k-1)\tau, \min(k\tau, T)) \end{array} \right\} \tag{3.24}$$

For $N \in \mathbb{N}$, let $\mathcal{M}^n$ be the outer product of $\mathcal{M}$, n times. Define

$$\bar{\bar{\mathcal{S}}}_{N\tau}^\tau[\phi](x) = \sup_{\mu \in \mathcal{D}_\infty^\tau} \sup_{u \in L_2^{\text{loc}}} \int_0^{N\tau} l^{\mu_t}(\xi_t) - \frac{\gamma^2}{2}|u_t|^2 \, dt + \phi(\xi_{N\tau}) \tag{3.25}$$

It is easy to prove that

$$\begin{aligned} \bar{\bar{\mathcal{S}}}_{N\tau}^\tau[\phi](x) &= \max_{m_N \in \mathcal{M}} \{\mathcal{S}_\tau^{m_N}\} \bar{\bar{\mathcal{S}}}_{(N-1)\tau}^\tau[\phi](x) \\ &= \max_{\{m_k\}_{k=1}^N \in \mathcal{M}^N} \left\{ \prod_{k=1}^N \mathcal{S}_\tau^{m_k} \right\} [\phi](x) = \left(\bar{\mathcal{S}}_\tau\right)^N [\phi](x) \end{aligned} \tag{3.26}$$

where the $\prod$ notation indicates operator composition, and the superscript in the last expression indicates repeated application of $\bar{\mathcal{S}}_\tau$, N times.

We will be approximating $\widetilde{V}$ by the fixed point of the semigroup $\bar{\mathcal{S}}_\tau$. This fixed point, or the solution to $W = \bar{\mathcal{S}}_\tau[W]$, will be computed as a limit of semigroup propagations on a starting $\phi \in \mathcal{K}$. Consequently, we need to show that the existence and uniqueness of the fixed point and that it is the limit of semigroup iterations. These have been proved in [McE09] and stated below.

Theorem 3.5.1. *Assume (3.16), (3.17) and the dynamics (3.14). Let* $\widetilde{V}$, $\mathcal{K}$ *and* $\bar{\bar{\mathcal{S}}}^{\tau}_{N\tau}$ *be defined as per (3.15), (3.7) and (3.26) respectively. For all* $x \in \mathbb{R}^n$,

$$\bar{V}_\tau(x) \doteq \lim_{N\to\infty} \bar{\bar{\mathcal{S}}}^{\tau}_{N\tau}[0](x) \tag{3.27}$$

exists and the convergence is uniform on compact sets. $\bar{V}_\tau \in \mathcal{K}$ *and* $0 \le \bar{V}_\tau \le \widetilde{V}$. $\bar{V}_\tau$ *is also the infinite horizon value function for a discrete time switching problem,*

$$\bar{V}_\tau(x) = \sup_{\mu\in\mathcal{D}^\tau_\infty} \sup_{u\in L_2^{loc}} \sup_{T<\infty} \left[\int_0^T l^{\mu_t}(\xi_t) - \frac{\gamma^2}{2}|u_t|^2\, dt\right].$$

Also $\bar{V}_\tau$ *is the fixed point of the semigroup* $\bar{\mathcal{S}}_\tau$. *Thus it is the solution of* $\bar{V}_\tau = \bar{\mathcal{S}}_\tau[\bar{V}_\tau]$ *and* $\bar{V}_\tau(0) = 0$. *Moreover, it is the unique, minimal, nonnegative such solution in class* $\mathcal{K}$.

McEneaney [McE09] also showed that one may approximate $\widetilde{V}$, the solution of $W = \widetilde{\mathcal{S}}_\tau[W]$, to as accurate a level as one desires by solving $W = \bar{\mathcal{S}}_\tau[W]$ for sufficiently small τ. Recall that if $W = \bar{\mathcal{S}}_\tau[W]$, then it satisfies $W = \bar{\bar{\mathcal{S}}}^{\tau}_{N\tau}[W]$ for all $N > 0$. So this is equivalent to introducing a discrete time $\bar{\mu} \in \mathcal{D}^\tau_{N\tau}$ approximation to the μ process in $\widetilde{\mathcal{S}}_{N\tau}$.

Stated below is the main approximation result.

Theorem 3.5.2. *Given* $\epsilon > 0$ *and* $R < \infty$, *there exists* $\tau_{\epsilon,R} > 0$ *such that*

$$\widetilde{V}(x) \le \bar{V}_{\tau_{\epsilon,R}}(x) \le \widetilde{V}(x) + \epsilon \quad \forall x \in \bar{B}(0,R) \tag{3.28}$$

$\bar{V}_{\tau_{\epsilon,R}}(x)$ can itself be computed as the uniformly convergent limit of iterated operations of the semigroup $\bar{\mathcal{S}}_{\tau_{\epsilon,R}}$ on any $\phi \in \mathcal{K}$.

$$\bar{V}_{\tau_{\epsilon,R}}(x) = \lim_{N\to\infty} \bar{\bar{\mathcal{S}}}^{\tau_{\epsilon,R}}_{N\tau_{\epsilon,R}}[\phi](x) = \lim_{N\to\infty} \bar{\mathcal{S}}^{N}_{\tau_{\epsilon,R}}[\phi](x)$$

Specifically one can choose $\phi(x)$ to be a quadratic function.

3.6 Max-Plus Fundamental Solution

In section 3.4, we approximated a nonlinear Hamiltonian by the pointwise maximum of M linear quadratic Hamiltonians in (3.12). Such a Hamiltonian represents a switched linear system switching between M constituent systems with linear dynamics and quadratic payoff. For such a system, in section 3.5, we simplify the switching problem still further, by time discretizing the switching control by (3.20)and (3.25). We can compute arbitrarily accurate solution of the switching problem by decreasing the discretization timestep τ and increasing the time horizon T. The error and the convergence analysis due to such time discretization is carried out in [MK].

Thus we reduce the original problem to that of computing $\mathcal{S}_\tau^m[\phi]$, in which a semigroup for a linear-quadratic system (corresponding to the quadratic Hamiltonian H^m in (3.13), dynamics (3.21) and payoff (3.22)) acts on a quadratic terminal payoff function $\phi(x)$. Fortunately, this is an analytically tractable problem, with a quadratic solution. Due to this, starting with the zero function, ϕ retains the quadratic form throughout the propagation under switching controls and the final value function is simply the max-plus sum of the propagated quadratic forms.

In [McE07], this problem was solved using semiconvex duality. Here we present another form which is more direct. It also turns out to be the max-plus fundamental solution to the underlying problem. This fundamental solution was first proposed for general nonlinear systems in [FM00], and as a special case, the closed form solution for linear time varying systems was derived in [Des] using max-plus algebra. This fundamental solution is more direct and does not use the semiconvex duality. Below is a similar analysis stemming from the same fundamental solution. Here we derive the closed form propagation formulas for computing $\mathcal{S}_\tau^m[\phi]$. For brevity, we shall drop the superscript m from the dynamics and payoff functions, and implicitly understand that they correspond to the m'th constituent linear-quadratic system.

First we shall define the general fundamental solution/ max-plus kernel. Consider a system starting with $\xi_0 = x$ and evolving as per

$$\dot{\xi}_s = A\xi_s + l_2 + \sigma u_s \tag{3.29}$$

and the payoff obtained along a particular trajectory is

$$J(x,u,t) \doteq \int_0^t l(\xi_s) - \frac{1}{2}|u_s|^2\, ds + \phi(\xi_t) \tag{3.30}$$

where

$$l(\xi_s) = \frac{1}{2}\xi_s' C \xi_s + \xi_s' l_1 + \frac{1}{2}\alpha. \tag{3.31}$$

and the value function for the finite horizon problem with terminal payoff ϕ given by the semigroup operation $\mathcal{S}_t^m[\phi]$ is

$$V(x,t) \doteq \mathcal{S}_t^m[\phi](x) = \sup_{u \in L_2^{\text{loc}}} J(x,u,t) \tag{3.32}$$

We shall now define a *max-plus kernel* $I : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$I^t(x,y) \doteq \begin{cases} \sup_{u\in\mathcal{U}^t(x,y)} \int_0^t l(\xi_s) - \frac{1}{2}|u_s|^2\, ds & \text{if } \mathcal{U}^t(x,y) \neq \emptyset \\ -\infty & \text{otherwise} \end{cases} \tag{3.33}$$

where

$$\mathcal{U}^t(x,y) \doteq \{u \in L_2(0,t) : \xi_0 = x, \xi_t = y \text{ under the dynamics (3.29)}\}$$

Note that $I^t = -\infty$ indicates that it is impossible to reach y from x in time interval $(0,\tau)$ using any possible control u. Thus if the system is controllable, then $I^t(x,y) > -\infty$ for all $t > 0$, and $I^0(x,y) = -\infty$ for $x \neq y$, and 0 otherwise.

McEneaney and Fleming proposed the above kernel in [FM00] and [Fle03]. They also proved the following theorem.

Theorem 3.6.1. *With the dynamics (3.29) and the terminal payoff $\phi(x)$, assume that the finite horizon value function $V(x,t) \doteq \mathcal{S}_t^m[\phi]$ as defined by (3.32), exists. Then with I^t defined as per (3.33), we have*

$$\mathcal{S}_t^m[\phi](x) = \sup_{y\in\mathbb{R}^n} (I^t(x,y) + \phi(y)) = \int_{\mathbb{R}^n}^{\oplus} I^t(x,y) \otimes \phi(y)\, dy \tag{3.34}$$

Since I^t depends only on the dynamics (3.29) and the running payoff $l(x_s) - \frac{1}{2}|u_s|^2$, it is independent of the terminal payoff $\phi(x_t)$. Hence it can serve as a *Fundamental solution* and obtain $\mathcal{S}_t^m[\phi](x)$ for any $\phi(x)$ by a kernel operation.

Now we shall find a closed form solution for $I^t(x, y)$. To do this we parametrize $\phi(x)$ by an additional variable z, which along with (3.36) gives us the next lemma. First, we assume the following

1. Let C, P, Σ are $n \times n$ symmetric matrices. Assume that the solution to the differential Riccati equation (DRE)
$$-\dot{P}_t = A'P_t + P_tA + C + P_t\Sigma P_t, \qquad P_T = P$$
exists for $t \in [0, T]$. (3.35)

2. The dynamics (3.29), hence the pair (A, σ) is controllable. That is given $t > 0$, for all $x, y \in \mathbb{R}^n$, starting from $\xi_0 = x$ there exists controls u such that $\xi_t = y$.

Let $\Sigma = \sigma\sigma'$, and let ∇_t and ∇_x denote $\partial/\partial t$ and $\partial/\partial x$ respectively. Consider the following Hamilton-Jacobi-Bellman PDE.

$$\begin{aligned} 0 &= -\nabla_t W(x, t) - H(x, \nabla_x W(x, t)) \\ W(x, T) &= \phi(x) \end{aligned} \tag{3.36}$$

where

$$\begin{aligned} H^m(x, p) &\doteq \sup_{u \in \mathbb{R}^n} \left\{ p'\left(Ax + l_2 + \sigma u\right) + \frac{1}{2}x'Cx + x'l_1 + \frac{1}{2}\alpha - \frac{1}{2}|u|^2 \right\} \\ &= \frac{1}{2}x'Cx + \frac{1}{2}p'\Sigma p + p'Ax + p'l_2 + x'l_1 + \frac{\alpha}{2}. \end{aligned} \tag{3.37}$$

Lemma 3.6.2. *Let the terminal payoff be*

$$\phi(x) = \phi^z(x) = \phi(x, z) = \frac{1}{2}x'Px + x'Sz + \frac{1}{2}z'Qz + x'a + z'b + \frac{\beta}{2}.$$

Assume that S is invertible and P is such that (3.35) holds true. Then there exists a C^∞ solution to (3.36) for all $x \in \mathbb{R}^n$ and $t \in [0, T]$, given by

$$W_z^m(x, t) = \frac{1}{2}x'P_tx + x'S_tz + \frac{1}{2}z'Q_tz + x'a_t + z'b_t + \frac{\beta_t}{2} \tag{3.38}$$

where P_t, S_t, Q_t, a_t, b_t, β_t *satisfy* $P_T = P$, $S_T = S$, $Q_T = Q$, $a_T = a$, $b_T = b$, *and*

$$\begin{aligned}
-\dot{P}_t &= A'P_t + P_tA + C + P_t\Sigma P_t \\
-\dot{S}_t &= (A + \Sigma P_t)'S_t \\
-\dot{Q}_t &= S_t'\Sigma S_t \\
-\dot{a}_t &= (A + \Sigma P_t)'a_t + l_1 + P_t l_2 \\
-\dot{b}_t &= S_t'(l_2 + \Sigma a_t) \\
-\dot{\beta} &= \alpha + a_t'\Sigma a_t + 2a_t' l_2
\end{aligned} \tag{3.39}$$

and S_t *is invertible for all* $t \in [0, T]$.

Proof. Existence of solution $P_t : 0 \le t \le T$ is assumed in (3.35). This combined with local boundedness and continuity of coefficients, guarantees the existence of S_t, a_t, and hence of Q_t, b_t and β_t. The proof that it solves HJB is immediate by substitution in (3.36).

If we define $B_t \doteq -(A + \Sigma P_t)$, then $S_t = \Phi_B(t,T)S_T = \Phi_B(t,T)S$, where Φ_B is the state transition matrix of the system $\dot{\xi}_t = B_t\xi_t$. By Abel-Jacobi-Liouville formula

$$\det \Phi_B(t, 0) = e^{\int_T^t \mathbf{Tr} B(s)\, ds} > 0$$

Since both $\Phi_B(t,T)$ and $S_T = S$ are invertible, $S_t = \Phi_B(t,T)S_T$ is invertible as well. $\square$

Next we have the verification theorem to connect the HJB PDE solution to the control value function.

Theorem 3.6.3. *Assume (3.35). Let* $x, z \in \mathbb{R}^n$ *and* $t \in [0,T]$, $W_z^m(x,t)$ *as per (3.38) and* $V_z^m(x,t)$ *be the value function as defined in (3.32). For all* $t \in [0,T]$ *and* $u \in L_2[0, T-t]$, *one has*

$$W_z^m(x,t) \ge J_z^m(x, u, T-t)$$

and $W_z^m(x,t) = J_z^m(x, \tilde{u}, T-t)$, *where*

$$\tilde{u}_s = \tilde{u}(s, \tilde{\xi}_s) = \sigma' \nabla W_z^m(s, \tilde{\xi}_s) = \sigma' \left(P_s\tilde{\xi}_s + S_s z + a_s \right). \tag{3.40}$$

Thus using (3.38) this implies,

$$V_z^m(x, T-t) = W_z^m(x,t) = \frac{1}{2}x'P_tx + x'S_tz + \frac{1}{2}z'Q_tz + x'a_t + z'b_t + \frac{\beta_t}{2}. \quad (3.41)$$

Proof. Let ∇_t and ∇_x denote $\partial/\partial t$ and $\partial/\partial x$ respectively. Let $u \in L_2[t,T]$, and ξ be the corresponding time shifted trajectory with $\xi_t = x$ and satisfying (3.29).

$$J_z^m(x,u,T-t) = \int_t^T \left(l(\xi_s) - \frac{1}{2}|u_s|^2 + (A\xi_t + l_2 + \sigma u_s)'\nabla W_z^m(\xi_s, s) \right) ds + \phi^z(\xi_T) - \int_t^T (A\xi_s + l_2 + \sigma u_s)'\nabla W_z^m(\xi_s,s)\, ds$$

which by definition of H

$$\leq \int_t^T H(\xi_s, \nabla W_z^m(\xi_s,s))\, ds + \phi^z(\xi_T) - \int_t^T (A\xi_s + l_2 + \sigma u_s)'\nabla W^z(\xi_s,s)\, ds$$

which by (3.36) and (3.29)

$$\begin{aligned} &= \int_t^T \left\{ -\nabla_s W_z(\xi_s,s) - \dot{\xi}_s \nabla W_z(\xi_s,s) \right\} ds + \phi^z(\xi_T) \\ &= -\int_t^T \frac{d}{ds} W_z(\xi_s,s)\, ds + \phi^z(\xi_T) \\ &= W_z^m(x,t) - W_z^m(\xi_T,T) + \phi^z(\xi_T) \\ &= W_z^m(x,t) \end{aligned}$$

by (3.36).

Also note that in the proof, if we substitute $\tilde{u}_s = \sigma(s)'\nabla W_z^m(\tilde{\xi}_s, s) = \sigma(s)'\left(P_s\tilde{\xi}_s + S_sz + a_s\right)$, then we have the equality throughout, and $J_z^m(x,\tilde{u},T-t) = W_z^m(x,t) = V_z^m(x,T-t)$. Hence proved. □

Remark 3.6.4. Since the dynamics (3.29) is time-invariant, we can define

$$\widehat{V}(x,t) \doteq \left\{ \sup_{u \in L_2^{\text{loc}}} \int_t^T l(\xi_s,u_s) - \frac{1}{2}|u_s|^2\, ds + \phi^z(\xi_T) \;\middle|\; \begin{array}{l} \xi_0 = x \\ \xi_t \text{ evolves as per (3.29)} \end{array} \right\}$$

by shifting the time back by t

$$= \left\{ \sup_{u \in L_2^{\text{loc}}} \int_0^{T-t} l(\xi_s, u_s) - \frac{1}{2}|u_s|^2 \, ds + \phi(\xi_{T-t}) \;\middle|\; \begin{array}{l} \xi_t = x \\ \xi_t \text{ evolves as per (3.29)} \end{array} \right\}$$
$$= V(x, T-t) = W(x,t)$$
$$= \frac{1}{2}x'P_t x + x'S_t z + \frac{1}{2}z'Q_t z + x'a_t + z'b_t + \frac{\beta_t}{2} \tag{3.42}$$

Now we prove a useful lemma towards obtaining the max-plus fundamental solution.

Lemma 3.6.5. *Consider the system trajectory $\tilde{\xi}_s$ starting from $\tilde{\xi}_t = x$ and evolving according to (3.29) under the optimal control $\tilde{u}_s = \sigma' \left(P_s \tilde{\xi}_s + S_s z + a_s \right)$ from Theorem 3.6.3. Then for $t \leq t_1 \leq s_2 \leq T$,*

$$S'_{t_1}\tilde{\xi}_{t_1} + Q_{t_1} z + b_{t_1} = S'_{t_2}\tilde{\xi}_{t_2} + Q_{t_2} z + b_{t_2} \tag{3.43}$$

Proof. By linear system theory, for a system evolving as per

$$\begin{aligned} \dot{\tilde{\xi}}_s &= A\tilde{x}_s + l_2 + \sigma \tilde{u}_s \\ &= A\tilde{\xi}_s + l_2 + \sigma\sigma' \left(P_s \tilde{\xi}_s + S_s z + a_s \right) \\ &= (A + \Sigma P_s)\, \tilde{\xi}_s + \Sigma(S_s z + a_s) + l_2 \end{aligned}$$

solution is given as

$$\tilde{\xi}_{t_2} = \Phi_B(t_2, t_1)\tilde{\xi}_{t_1} + \int_{t_1}^{t_2} \Phi_B(t_2, s)(\Sigma S_s z + \Sigma a_s + l_2)\, ds \tag{3.44}$$

where $\Phi_B(t_2, t_1) = U_{t_2} U_{t_1}^{-1}$, where U_s is the solution of differential equation $\dot{U}_s = B(s)U_s$, with $B(s) = A + \Sigma P_s$.

It is well known that the state transition matrix

$$\Phi_{B(s)}(t_2, t_1) = \Phi'_{-B(s)'}(t_1, t_2)$$

now, noting from (3.39) that $\dot{S}_s = -(A + \Sigma P_s)' S_s = -B(s)' S_s$, and since S_{t_2} is invertible, we have

$$\Phi_{B(s)}(t_2, t_1) = \Phi'_{-B(s)'}(t_1, t_2) = \left(S_{t_1} S_{t_2}^{-1} \right)' = S_{t_2}^{-1\prime} S_{t_1}{}' \tag{3.45}$$

Substituting in (3.44), and noting from (3.39) that $\dot{Q}_s = -S_s{}'\Sigma(s)S_s$ and $-\dot{b}_s = S_s{}'(l_2 + \Sigma a_s)$,

$$\begin{aligned}
\tilde{\xi}_{t_2} &= S_{t_2}^{-1\prime} S_{t_1}{}'\tilde{\xi}_{t_1} + S_{t_2}^{-1\prime}\int_{t_1}^{t_2} S_s{}'(\Sigma S_s z + \Sigma a_s + l_2)\, ds \\
&= S_{t_2}^{-1\prime} S_{t_1}{}'\tilde{\xi}_{t_1} + S_{t_2}^{-1\prime}\left(\int_{t_1}^{t_2} S_s{}'\Sigma S_s\, ds\right) z + S_{t_2}^{-1\prime}\left(\int_{t_1}^{t_2} S_s{}'(\Sigma a_s - l_2)\, ds\right) \\
&= S_{t_2}^{-1\prime} S_{t_1}{}'\tilde{x}_{t_1} + S_{t_2}^{-1\prime}(Q_{t_1} - Q_{t_2})\, z + S_{t_2}^{-1\prime}(b_{t_1} - b_{t_2})
\end{aligned}$$

thus we have,

$$S_{t_2}{}'\tilde{\xi}_{t_2} + Q_{t_2} z + b_{t_2} = S_{t_1}{}'\tilde{\xi}_{t_1} + Q_{t_1} z + b_{t_1}$$

□

Remark 3.6.6. Note that $\forall z$, since S_{t_1} and S_{t_2} are invertible, (3.43) suggests a one-one and onto relation between start and end of optimal trajectories, ξ_{t_1} and ξ_{t_2}. Thus $\forall y \in \Re^n$ there exists a $x = S_{t_2}^{-1\prime}(S_{t_1}{}'y + (Q_{t_1} - Q_{t_2})z + b_{t_1} - b_{t_2})$ such that optimal trajectory $\tilde{x}$ starting at $\tilde{x}_{t_1} = x$, ends with y. Thus every $y \in \Re^n$ is an optimal point for some initial condition.

Remark 3.6.7. Note that due to max-plus linearity, if $k \in \Re$,

$$\hat{V}^z(x,t) \doteq \mathcal{S}_t[\phi^z + k](x) = \mathcal{S}_t[\phi^z](x) + k = V^z(x,t) + k$$

Thus while keeping the dynamics and the running payoff the same, adding a constant to the terminal payoff only shifts the value function accordingly. The gradient hence the optimal feedback control remains the same.

$$\hat{u}_t(x) = \sigma\nabla\hat{V}^z(x, T-t) = \sigma\nabla V^z(x, T-t) = \tilde{u}_t(x)$$

Hence the optimal trajectory, which is the solution to $\dot{\hat{x}}_t = A\hat{x}_t + l_2 + \sigma\hat{u}_t(x)$, also stays the same.

Now we shall prove another useful lemma before turning to the main result.

Lemma 3.6.8. *Assume (3.35). Given* $0 \le t_1 < t_2 \le T$*, and* Q_t *evolving according to (3.39) with terminal value* $Q_T = Q$*, then*

$$Q_{t_1} - Q_{t_2} \succ 0$$

Proof. Since we assumed (A, σ) is controllable, the following *controllability grammian* is invertible for any $0 \le t_1 < t_2 \le T$. Here since A is time-invariant, $\Phi_A(t_1, s) = e^{A(t_1 - s)}$.

$$\int_{t_1}^{t_2} \Phi_A(t_1, s)\sigma\sigma'\Phi_A(t_1, s)'\, dt \succ 0 \tag{3.46}$$

Thus for all $x, y \in \mathbb{R}^n$, $\exists$ control $\hat{u}_t$ such that is the trajectory $\dot{\hat{x}} = A\hat{x}_t + l_2 + \sigma\hat{u}_t$ with $\hat{x}_{t_1} = x$ satisfies $\hat{x}_{t_2} = y$.

Now we claim that system $(A + \Sigma P_t, \sigma)$ is also controllable. This is clear because by using control $\bar{u}_t = \hat{u}_t - \sigma' P_t x_t$, we can keep the system trajectory same and reach from x to y.

$$\begin{aligned}
\dot{\hat{x}} &= A\hat{x}_t + l_2 + \sigma\hat{u}_t \\
&= (A + \sigma\sigma' P_t)x_t + l_2 + \sigma\left(\hat{u}_t - \sigma' P_t x_t\right) \\
&= (A + \Sigma P_t)x_t + l_2 + \sigma\bar{u}_t
\end{aligned}$$

Hence similar to (3.46), using $B(t) = A + \Sigma P_t$ and $\sigma\sigma' = \Sigma(t)$, following controllability grammian is invertible.

$$\int_{t_1}^{t_2} \Phi_B(t_1, s)\Sigma\Phi_B(t_1, s)'\, dt \succ 0 \tag{3.47}$$

Substituting $\Phi_B(t_1, s) = S_{t_1}^{-1}{}' S_s{}'$ from (3.45),

$$\begin{aligned}
\int_{t_1}^{t_2} \Phi_B(t_2, s)\Sigma\Phi_B(t_2, s)'\, ds &= \int_{t_1}^{t_2} S_{t_1}^{-1}{}' S_s{}' \Sigma S_s S_{t_1}^{-1}\, ds \\
&= S_{t_1}^{-1}{}' \left\{ \int_{t_1}^{t_2} S_s{}' \Sigma S_s\, ds \right\} S_{t_1}^{-1} \\
&= S_{t_1}^{-1}{}' \left(Q_{t_1} - Q_{t_2}\right) S_{t_1}^{-1}
\end{aligned} \tag{3.48}$$

where in the last equation, we used Q_t evolution from (3.39). Using (3.47) and since S_{t_1} is invertible by Lemma 3.6.2, we have $Q_{t_1} - Q_{t_2} \succ 0$. □

Theorem 3.6.9. *Given $x, y \in \mathbb{R}^n$ and $0 \le t_1 < t_2 \le T$, then if $\delta \doteq t_2 - t_1$,*

$$\inf_{z \in \mathbb{R}^n} \left[\widehat{V}^z(x, t_1) - \widehat{V}^z(y, t_2) \right] = I_\delta(x, y) \tag{3.49}$$

Since by (3.42), $\widehat{V}^z(x,t_1) = \frac{1}{2}x'P_{t_1}x + x'S_{t_1}z + \frac{1}{2}z'Q_{t_1}z + x'a_{t_1} + z'b_{t_1} + \frac{\beta_{t_1}}{2}$ *and* $\widehat{V}^z(x,t_2) = \frac{1}{2}x'P_{t_2}x + x'S_{t_2}z + \frac{1}{2}z'Q_{t_2}z + x'a_{t_2} + z'b_{t_2} + \frac{\beta_{t_2}}{2}$, *the max-plus kernel* $I_\delta(x,y)$ *is also bivariate quadratic.*

$$\begin{aligned}
I_\delta(x,y) &= \frac{1}{2}x'I_\delta^{11}x + x'I_\delta^{12}y + \frac{1}{2}y'I_\delta^{22}y + x'I_\delta^{1} + y'I_\delta^{2} + I_\delta^{0} \qquad \textit{where}\\
I_\delta^{11} &= P_{t_1} - S_{t_1}(Q_{t_1}-Q_{t_2})^{-1}{S_{t_1}}'\\
I_\delta^{12} &= S_{t_1}(Q_{t_1}-Q_{t_2})^{-1}{S_{t_2}}'\\
I_\delta^{22} &= -P_{t_2} - S_{t_2}(Q_{t_1}-Q_{t_2})^{-1}{S_{t_2}}'\\
I_\delta^{1} &= a_{t_1} - S_{t_1}(Q_{t_1}-Q_{t_2})^{-1}(b_{t_1}-b_{t_2})\\
I_\delta^{2} &= -a_{t_2} + S_{t_2}(Q_{t_1}-Q_{t_2})^{-1}(b_{t_1}-b_{t_2})\\
I_\delta^{0} &= \beta_{t_1} - \beta_{t_2} - (b_{t_1}-b_{t_2})'(Q_{t_1}-Q_{t_2})^{-1}(b_{t_1}-b_{t_2})
\end{aligned} \tag{3.50}$$

Proof. Let $\xi_{t_1} = x$. Since $\Sigma \succeq 0$ and S_t is invertible, by (3.39), $-\dot{Q}_s = S_t'\Sigma(t)S_t \succeq 0$, hence $Q_{t_1} - Q_{t_2} \succeq 0$. For any $z \in \mathbb{R}^n$

$$\begin{aligned}
&\widehat{V}^z(x,t_1) - \widehat{V}^z(y,t_2)\\
&\quad= \mathcal{S}_{t_2-t_1}[\widehat{V}^z](x,t_2) - \widehat{V}^z(y,t_2)\\
&\quad= \sup_{u\in L_2[t_1,t_2]}\left\{\int_{t_1}^{t_2} l_t(\xi_t,u_t)\,dt + \widehat{V}^z(\xi_{t_2},t_2) - \widehat{V}^z(y,t_2)\right\}
\end{aligned}$$

substituting for $\widehat{V}^z(\cdot,t_2)$,

$$= \sup_{u\in L_2[t_1,t_2]}\left\{\int_{t_1}^{t_2} l_t(\xi_t,u_t)\,dt + \frac{1}{2}\xi_{t_2}'P_{t_2}\xi_{t_2} - \frac{1}{2}y'P_{t_2}y + (\xi_{t_2}-y)'(S_{t_2}z + a_{t_2})\right\}$$

Since $\mathcal{U}_{t_1}^{t_2}(x,y) = \{u \in L_2(t_1,t_2) : \xi_{t_1} = x, \xi_{t_2} = y\} \subset L_2(t_1,t_2)$, and $\forall u \in \mathcal{U}_{t_1}^{t_2}(x,y)$, $\xi_{t_2} = y$.

$$\begin{aligned}
&\geq \sup_{u\in\mathcal{U}_{t_1}^{t_2}(x,y)}\left\{\int_{t_1}^{t_2} l_t(\xi_t,u_t)\,dt + \frac{1}{2}y'P_{t_2}y - \frac{1}{2}y'P_{t_2}y + (y-y)'(S_{t_2}z + a_{t_2})\right\}\\
&= \sup_{u\in\mathcal{U}_{t_1}^{t_2}(x,y)}\int_{t_1}^{t_2} l_t(\xi_t,u_t)\,dt = I_\delta(x,y)
\end{aligned} \tag{3.51}$$

Taking infimum over all $z \in \mathbb{R}^n$,

$$\inf_{z\in\mathbb{R}^n}\left[\widehat{V}^z(x,t_1) - \widehat{V}^z(y,t_2)\right] \geq I_\delta(x,y) \tag{3.52}$$

Since $Q_{t_1} - Q_{t_2} \succ 0$ by 3.6.8, define $\hat{z} = (Q_{t_1} - Q_{t_2})^{-1}({S_{t_2}}'y - {S_{t_1}}'x + b_{t_2} - b_{t_1})$. Hence

$$ {S_{t_2}}'y + Q_{t_2}\hat{z} + b_{t_2} = {S_{t_1}}'x + Q_{t_1}\hat{z} + b_{t_1} $$

Hence using (3.43) the optimal trajectory $\tilde{x}_t$ starting from $\tilde{x}_{t_1} = x$ and with terminal payoff $\widehat{V}^{\hat{z}}(\cdot, t_2)$, ends at $\tilde{x}_{t_2} = y$. Let the corresponding optimal control be $\tilde{u}_t$. Let us define $k = -\widehat{V}^{\hat{z}}(y, t_2) = -\left(\frac{1}{2}y'P_{t_2}y + y'S_{t_2}\hat{z} + \frac{1}{2}\hat{z}'Q_{t_2}\hat{z} + y'a_{t_2} + \hat{z}'b_{t_2} + \frac{\beta_{t_2}}{2}\right)$ to create a shifted terminal payoff function

$$ \begin{aligned} U^{\hat{z}}_{t_2}(x) &= \widehat{V}^{\hat{z}}(x, t_2) + k = \widehat{V}^{\hat{z}}(x, t_2) - \widehat{V}^{\hat{z}}(y, t_2) \\ &= \frac{1}{2}x'P_{t_2}x - \frac{1}{2}y'P_{t_2}y + (x-y)'(S_{t_2}\hat{z} + a_{t_2}) \end{aligned} \tag{3.53} $$

From remark 3.6.7, $\tilde{u}_t$, $\tilde{\xi}_t$ are also the optimal control and trajectory for the following problem with the terminal payoff $U^z_{t_2}$. Hence

$$ \begin{aligned} \widehat{V}^{\hat{z}}(x, t_1) - \widehat{V}^{\hat{z}}(y, t_2) &= \left\{ \sup_{u \in L_2[t_1, t_2]} \int_{t_1}^{t_2} l_t(\xi_t, u_t)\, dt + \widehat{V}^{\hat{z}}(x, t_2) \right\} - \widehat{V}^{\hat{z}}(y, t_2) \\ &= \sup_{u \in L_2[t_1, t_2]} \left\{ \int_{t_1}^{t_2} l_t(\xi_t, u_t)\, dt + \widehat{V}^{\hat{z}}(x, t_2) - \widehat{V}^{\hat{z}}(y, t_2) \right\} \\ &= \sup_{u \in L_2[t_1, t_2]} \int_{t_1}^{t_2} l_t(\xi_t, u_t)\, dt + U_{t_2}(\xi_{t_2}) \\ &= \int_{t_1}^{t_2} l_t(\tilde{\xi}_t, \tilde{u}_t)\, dt + U^{\hat{z}}_{t_2}(\tilde{\xi}_{t_2}) \end{aligned} $$

since $U^{\hat{z}}_{t_2}(\tilde{\xi}_{t_2}) = U^{\hat{z}}_{t_2}(y) = 0$ from (3.53) and $\tilde{u} \in \mathcal{U}^{t_2}_{t_1}(x, y)$

$$ \leq \sup_{u \in \mathcal{U}^{t_2}_{t_1}(x,y)} \int_{t_1}^{t_2} l_t(\xi_t, u_t)\, dt = I_\delta(x, y) \tag{3.54} $$

Thus we have

$$ \inf_{z} \left[\widehat{V}^z(x, t_1) - \widehat{V}^z(y, t_2) \right] \leq \widehat{V}^{\hat{z}}(x, t_1) - \widehat{V}^{\hat{z}}(y, t_2) \leq I_\delta(x, y) \tag{3.55} $$

Hence (3.52) and (3.55) together give us (3.49) and also the following

$$ \inf_{z \in \mathbb{R}^n} \left[\widehat{V}^z(x, t_1) - \widehat{V}^z(y, t_2) \right] = \widehat{V}^{\hat{z}}(x, t_1) - \widehat{V}^{\hat{z}}(y, t_2) = I_\delta(x, y) \tag{3.56} $$

with $\hat{z} = (Q_{t_1} - Q_{t_2})^{-1}({S_{t_2}}'y - {S_{t_1}}'x + b_{t_2} - b_{t_1})$.

Substituting $\hat{z}$ in (3.56) and expanding, we get (3.50). □

3.7 The Algorithm

Now we begin the discussion of the actual algorithm. From Theorem 3.5.1, $\bar{V} \doteq \lim_{N\to\infty} \bar{\bar{\mathcal{S}}}^{\tau}_{N\tau}[0](x)$. Let $\bar{V}^0 \equiv 0$ and define for $k \geq 1$,

$$\bar{V}^k \doteq \bar{\bar{\mathcal{S}}}^{\tau}_{k\tau}[0](x), \tag{3.57}$$

Then we have $\bar{V} = \lim_{N\to\infty} \bar{V}^N(x)$. In practice, we approximate the infinite horizon by a sufficiently long finite horizon $N\tau$. The propagation from from $\bar{V}^0$ to $\bar{V}^N$ is achieved as follows.

For all $\{m_i\}_{i=1}^k \in \mathcal{M}^k$, define

$$v_{\{m_i\}_{i=1}^k}(x) \doteq \prod_{i=1}^{k} \mathcal{S}^{m_i}_{\tau}[0](x).$$

Then we have

$$\bar{V}^k = \bar{\bar{\mathcal{S}}}^{\tau}_{k\tau}[0] = \max_{\{m_i\}_{i=1}^k \in \mathcal{M}^k} \left\{ \prod_{i=1}^{k} \mathcal{S}^{m_i}_{\tau} \right\} [0](x) = \max_{\{m_i\}_{i=1}^k \in \mathcal{M}^k} v_{\{m_i\}_{i=1}^k}(x).$$

Also by (3.27),

$$\bar{V}^{k+1} \doteq \bar{\mathcal{S}}_{\tau}[\bar{V}^k] = \max_{m_{k+1}\in\mathcal{M}} \mathcal{S}^{m_k}_{\tau} \left[\max_{\{m_i\}_{i=1}^k \in \mathcal{M}^n} v_{\{m_i\}_{i=1}^k} \right] (x)$$

Since $\mathcal{S}^{m_k}_{\tau}$ is max-plus linear,

$$\begin{aligned} &= \max_{\{m_i\}_{i=1}^{k+1} \in \mathcal{M}^{k+1}} \mathcal{S}^{m_k}_{\tau} \left[v_{\{m_i\}_{i=1}^k} \right] (x) \\ &= \max_{\{m_i\}_{i=1}^{k+1} \in \mathcal{M}^{k+1}} v_{\{m_i\}_{i=1}^{k+1}}(x) \end{aligned} \tag{3.58}$$

The algorithm shall consist of the forward propagation of $v_{\{m_i\}_{i=1}^k}$ to $v_{\{m_i\}_{i=1}^{k+1}} = \mathcal{S}^{m_k}_{\tau} \left[v_{\{m_i\}_{i=1}^k} \right]$ for all k-tuple switchings $\{m_i\}_{i=1}^k \in \mathcal{M}^k$ and all $m_{k+1} \in \mathcal{M}$. Such propagation is carried out from $k = 0$ to some termination step $k = N$. Value function is obtained by taking the maximum over all such $v^N_{\{m_i\}_{i=1}^N}$.

It is important to note that the computation of each $v_{\{m_i\}_{i=1}^k}$ is analytical. We will indicate the actual analytical computations in the steps below. Note that many formulae are duplicated here to make this section self-contained.

- First approximate a nonlinear Hamiltonian we wish to solve, with a max-plus summation of linear quadratic Hamiltonians as in (3.12).

$$H(x,p) \approx \widetilde{H}(x,p) \doteq \max_{m\in\{1,2,\ldots,M\}} H^m(x,p)$$

where each H^m is linear-quadratic,

$$H^m(x,p) = \frac{1}{2}x'C^m x + \frac{1}{2}p'\Sigma^m p + (A^m x)'p + x'l_1^m + p'l_2^m + \frac{1}{2}\alpha^m, \qquad (3.59)$$

- Choose time step $\tau \in \mathbb{R}$ and $N \in \mathbb{N}$. Thus we shall approximate the infinite horizon value function $\widetilde{V}$ for the switching problem, as defined in (3.15) by $\bar{V}^N$, the value function of discretized switching problem with horizon $N\tau$ as defined in (3.57).

- For each $m \in \mathcal{M}$ starting with any $(P,S,Q,a,b,\beta)_0$, evolve them forward in time by τ as per (3.39), in which all paparemeters correspond to m'th Hamiltonian. Restating here:

$$\begin{aligned}
-\dot{P}_t^m &= A^{m\prime}P_t^m + P_t^m A^m + C^m + P_t^m\Sigma^m P_t^m \\
-\dot{S}_t^m &= (A^m + \Sigma^m P_t^m)'S_t^m \\
-\dot{Q}_t^m &= S_t^{m\prime}\Sigma^m S_t^m \\
-\dot{a}_t^m &= (A^m + \Sigma^m P_t^m)'a_t^m + l_1^m + P_t^m l_2^m \\
-\dot{b}_t^m &= S_t^{m\prime}(l_2^m + \Sigma^m a_t^m) \\
-\dot{\beta}_t^m &= \alpha^m + a_t^{m\prime}\Sigma^m a_t^m + 2a_t^{m\prime}l_2^m
\end{aligned} \qquad (3.60)$$

Lets denote the solution set by $(P,S,Q,a,b,\beta)_\tau^m$.

- Now for each $m \in \mathcal{M}$, compute the parameters of the max-plus fundamental solution,

$$I_\tau^m(x,y) = \frac{1}{2}x'I_{m,\tau}^{11}x + x'I_{m,\tau}^{12}y + \frac{1}{2}y'I_{m,\tau}^{22}y + x'I_{m,\tau}^1 + y'I_{m,\tau}^2 + I_{m,\tau}^0,$$

as per (3.50). Restating here:

$$\begin{aligned}
I_{m,\tau}^{11} &= P_0 - S_0(Q_0 - Q_\tau^m)^{-1}S_0{}' \\
I_{m,\tau}^{12} &= S_0(Q_0 - Q_\tau^m)^{-1}S_\tau^{m\prime} \\
I_{m,\tau}^{22} &= -P_\tau^m - S_\tau^m(Q_0 - Q_\tau^m)^{-1}S_\tau^{m\prime} \\
I_{m,\tau}^{1} &= a_0 - S_0(Q_0 - Q_\tau^m)^{-1}(b_0 - b_\tau^m) \\
I_{m,\tau}^{2} &= -a_\tau^m + S_\tau^m(Q_0 - Q_\tau^m)^{-1}(b_0 - b_\tau^m) \\
I_{m,\tau}^{0} &= \beta_0 - \beta_\tau^m - (b_0 - b_\tau^m)'(Q_0 - Q_\tau^m)^{-1}(b_0 - b_\tau^m)
\end{aligned} \tag{3.61}$$

Note that as a special case, we can choose $P_0 = Q_0 = 0$, $S_0 = I$, $a_0 = b_0 = 0$ and $\beta_0 = 0$, to simplify above formulae considerably. We note that parameters of I^m need only be computed once, and thus can be precomputed.

- Starting with $v^0(x) = 0$, we shall prove that $v_{\{m_i\}_{i=1}^k}$ retain the quadratic form as k increases by induction. Lets us assume $v_{\{m_i\}_{i=1}^k}$ in the following quadratic form.

$$v_{\{m_i\}_{i=1}^k}(x) = \frac{1}{2}x'\hat{P}_{\{m_i\}_{i=1}^k}x + x'\hat{a}_{\{m_i\}_{i=1}^k} + \frac{1}{2}\hat{\beta}_{\{m_i\}_{i=1}^k} \tag{3.62}$$

Then for each $m_{k+1} \in \mathcal{M}$, by Theorem 3.6.1,

$$\begin{aligned}
v_{\{m_i\}_{i=1}^{k+1}}(x) &= \mathcal{S}_\tau^{m_k}\left[v_{\{m_i\}_{i=1}^k}\right](x) \\
&= \max_{y\in\mathbb{R}^n}\left\{I_\tau^{m_{k+1}}(x,y) + v_{\{m_i\}_{i=1}^k}(y)\right\} \\
&= \frac{1}{2}x'\hat{P}_{\{m_i\}_{i=1}^{k+1}}x + x'\hat{a}_{\{m_i\}_{i=1}^{k+1}} + \frac{1}{2}\hat{\beta}_{\{m_i\}_{i=1}^{k+1}}
\end{aligned} \tag{3.63}$$

Note that maximum exists because otherwise, due to the quadratic form the supremum would be ∞ for all x. This is impossible because, the LHS $\leq \bar{V}(x) < \infty$. Now by (3.62), (3.63) and (3.61),

$$\begin{aligned}
\hat{P}_{\{m_i\}_{i=1}^{k+1}} &= I_{m_{k+1},\tau}^{11} - I_{m_{k+1},\tau}^{12}\left(I_{m_{k+1},\tau}^{22} + \hat{P}_{\{m_i\}_{i=1}^k}\right)^{-1} I_{m_{k+1},\tau}^{12}{}' \\
\hat{a}_{\{m_i\}_{i=1}^{k+1}} &= -I_{m_{k+1},\tau}^{12}\left(I_{m_{k+1},\tau}^{22} + \hat{P}_{\{m_i\}_{i=1}^k}\right)^{-1}\left(\hat{a}_{\{m_i\}_{i=1}^k} + I_{m_{k+1},\tau}^{2}\right) + I_{m_{k+1},\tau}^{1} \\
\hat{\beta}_{\{m_i\}_{i=1}^{k+1}} &= \hat{\beta}_{\{m_i\}_{i=1}^k} + I_{m_{k+1},\tau}^{0} \\
&\quad - \left(\hat{a}_{\{m_i\}_{i=1}^k} + I_{m_{k+1},\tau}^{2}\right)'\left(I_{m_{k+1},\tau}^{22} + \hat{P}_{\{m_i\}_{i=1}^k}\right)^{-1}\left(\hat{a}_{\{m_i\}_{i=1}^k} + I_{m_{k+1},\tau}^{2}\right)
\end{aligned} \tag{3.64}$$

Similar to the max-plus counterpart in [McE07], the errors in the solution are due to :

1. The approximation of $H(x,p)$ by $\widetilde{H}(x,p)$. The error analysis for the max-plus methods for such approximation was carried out in [MD08a] and [MDb].

2. Discretization of the timestep τ and premature termination of the horizon at $N\tau$. Error analysis for this approximation for max-plus methods was carried out in [McE09] and [MK].

Note that the computation of each triplet $(\hat{P},\hat{a},\hat{\beta})_{\{m_i\}_{i=1}^k}$ grows like the cube of the space dimension (due to matrix operations). Thus one avoids the curse-of-dimensionality.

However, the curse of curse-of-dimensionality is replaced by another type of rapid computational cost growth. Here, we refer to this as the *curse-of-complexity* . If $\#\mathcal{M}=1$, then all the computations of our algorithm (except the solution of the Riccati equation) are unnecessary, and we *informally* refer to this as complexity one. When there are $M=\#\mathcal{M}$ such quadratics in the Hamiltonian, $\widetilde{H}$ we say it has complexity M. Note that

$$\#\left\{v_{\{m_i\}_{i=1}^N}|m_i\in\mathcal{M}\quad\forall\, i\in\{1,2,\ldots,N\}\right\}\approx M^N$$

For large N, this number is indeed large. We shall discuss ways to limit this quadratic growth with N in the next section, but it is significant to note that the exponential computational cost growth with space dimension n is limited to cubic growth.

3.7.1 Pruning

The number of quadratics in the solution $\bar{V}^k$ grows exponentially with k. However in practice, relatively few of these actually contribute to $\bar{V}^k$. Thus it is useful to prune the set, to contain the curse-of-complexity. Note that if

$$v_{\{\hat{m}_i\}_{i=1}^k}(x)\leq\bigoplus_{\{\hat{m}_i\}_{i=1}^k\neq\{m_i\}_{i=1}^k}v_{\{m_i\}_{i=1}^k}(x)\qquad\forall x\in\mathbb{R}^n$$

then

$$\bar{\mathcal{S}}_t\left[v_{\{\hat{m}_i\}_{i=1}^k}\right](x) \leq \bar{\mathcal{S}}_t\left[\bigoplus_{\{\hat{m}_i\}_{i=1}^k \neq \{m_i\}_{i=1}^k} v_{\{m_i\}_{i=1}^k}\right](x) = \bar{\mathcal{S}}_t[\bar{V}^k] \qquad \forall x \in \mathbb{R}^n$$

Thus $v_{\{\hat{m}_i\}_{i=1}^k}$ will play no role in the computation of $\bar{V}^{k\tau+t}$, for $t \geq 0$. Thus one may prune such $v_{\{\hat{m}_i\}_{i=1}^k}$ without any loss of accuracy. Pruning methods of various order designed for max-plus methods in [MDG08] can be easily adapted for above problem to check if a quadratic dominates the max-plus sum of the rest, and prune it. Note that such pruning methods incur additional computational burden. But usually it is justifiable in their efficacy in attenuating the exponential curse-of-complexity as the horizon grows. The next chapter covers these methods in more detail.

Chapter 4

Curse-of-Complexity Attenuation in the Curse-of-Dimensionality Free Method for HJB PDEs

4.1 Introduction

In chapters 1 and 3, we discussed the curse-of-dimensionality-free method proposed by McEneaney (cf. [McE07], [McE05], [McE06]) for certain HJB PDEs arising in problems involving objective maximization. This method is based on the max-plus linearity of the underlying semigroup and is free from the curse-of-dimensionality. In fact, the computational growth with dimensionality is cubic. However, there is an exponential computational growth in a certain measure of complexity of the Hamiltonian, which is used to express the HJB PDE. Under this measure, the minimal complexity Hamiltonian is linear/quadratic – solvable using a Riccati equation. If the Hamiltonian is given as a pointwise maximum of M linear/quadratic Hamiltonians, then one could say the complexity of the Hamiltonian is M. Such PDEs can also arise in switched linear systems.

This method constructs the value function as a max-plus sum, i.e., a pointwise maximum, of certain quadratic functions. An infinite time-horizon problem is considered, and as such, the value function is approximated by iterating a finite-horizon semigroup until a large enough propagation horizon is reached. This finite-horizon semigroup itself is approximated as the maximum of a finite number of quadratic forms, or as a semigroup for a system switching between M linear-quadratic systems. The approximate value function at each iteration is stored as a set of quadratic functions. Acting on these quadratic functions with the above semigroup leads to a new approximation, where the number of quadratics grows by a fixed factor at each iteration. This is the curse-of-complexity. To attenuate this computational growth, a pruning method based on semidefinite programming (SDP) is developed here.

4.2 Problem statement and assumptions

To make this chapter self contained, the problem statements and assumptions described in section 3.4 are repeated here.

The HJB PDEs we consider, arise in infinite-horizon nonlinear optimal control problems, and their Hamiltonians are given as (or well-approximated by) pointwise maxima of linear-quadratic functions. Note that pointwise maxima of quadratic forms can approximate, arbitrarily closely, any semiconvex function. More specifically, we consider

$$0 = -\widetilde{H}(x, \nabla V) = -\max_{m\in\{1,2,\dots,M\}} \{H^m(x, \nabla V)\} \tag{4.1}$$

$$V(0) = 0 \tag{4.2}$$

(i.e., with boundary condition $V = 0$ at the origin) where each of the constituent Hamiltonians has the form

$$H^m(x,p) \doteq \frac{1}{2}x'C^m x + \frac{1}{2}p'\Sigma^m p + (A^m x)'p + x'l_1^m + p'l_2^m + \frac{1}{2}\alpha^m, \tag{4.3}$$

where C^m, Σ^m are $n \times n$ symmetric matrices, $l_1^m, l_2^m \in \mathbb{R}^n$ and $\alpha^m \in \mathbb{R}$.

Hamiltonian $\widetilde{H}$ is associated with an optimal control problem for switched linear systems. Let $\mathcal{M} \doteq \{1, 2, \ldots M\}$. The corresponding value function is

$$\begin{aligned}\widetilde{V}(x) &= \sup_{u \in L_2^{\text{loc}}} \sup_{\mu \in \mathcal{D}_\infty} \sup_{T<\infty} \widetilde{J}(x, T; u, \mu) \\ &= \sup_{u \in L_2^{\text{loc}}} \sup_{\mu \in \mathcal{D}_\infty} \sup_{T<\infty} \int_0^T l^{\mu_t}(\xi_t) - \frac{\gamma^2}{2}|u_t|^2 \, dt \end{aligned} \tag{4.4}$$

where $l_t^{\mu}(x) \doteq \frac{1}{2}x^T C^{\mu_t} x + (l_1^{\mu_t})^T x + \frac{1}{2}\alpha^{\mu_t}$, $\mathcal{M} \doteq \{1, 2, \ldots, M\}$, $\mathcal{D}_\infty = \{\mu : [0, \infty) \to \mathcal{M} \; : \; \text{measurable}\}$, $L_2^{\text{loc}} \doteq \{u : [0, \infty) \to \mathbb{R}^n \,|\, u \in L_2[0, T] \quad \forall T < \infty\}$ and σ^i and γ are such that $\Sigma^i = \frac{\sigma^i {\sigma^i}'}{\gamma^2}$ for $i \in \mathcal{M}$.

With μ_t as the switching control in addition to the control u, the state dynamics are given by

$$\dot{\xi} = A^{\mu_t}\xi + l_2^{\mu_t} + \sigma^{\mu_t} u_t, \quad \xi_0 = x \tag{4.5}$$

To motivate the assumptions for this rather general problem class, we consider $\widetilde{H}$ as being constructed so as to resemble the Hamiltonian, H, for the originating control problem of interest. Hence we make the following block of assumptions, earlier seen in section 3.4.

> We assume that the underlying system dynamics and the payoff functions satisfy (3.3), (3.4). Hence the value function V defined by (3.2) is a unique viscosity solution, of $0 = -H(x, \nabla V)$ in class $\mathcal{K}$, defined in (3.7).
> The approximating Hamiltonian $\widetilde{H}(x, p) = \sup_m H^m(x, p) \leq H(x, p)$ for all $x, p \in \mathbb{R}^n$.
> There exists $c_A \in (0, \infty)$ such that $x' A_m x \leq -c_A|x|^2$ for all $x \in \mathbb{R}^n$ and all $m \in \mathcal{M}$. (4.6)
> $H^1(x, p)$ has coefficients satisfying the following: $l_1^1 = l_2^1 = 0$; $\alpha^1 = 0$; there exists $c_{A,1} \in (0, \infty)$ such that $x' A^1 x \leq -c_{A,1}|x|^2 \; \forall x \in \mathbb{R}^n$; D^1 is positive definite, symmetric; and $\gamma^2/c_\sigma^2 > c_D/c_{A,1}^2$, where $c_\sigma \doteq |\sigma^1|$ and c_D is such that $x' D^1 x \leq c_D|x|^2 \; \forall x \in \mathbb{R}^n$.
> System (3.14) is controllable in the sense that given $x, y \in \mathbb{R}^n$ and $T > 0$, there exist processes $w \in \mathcal{W}$ and μ measurable with range in $\mathcal{M}$, such that $\xi_T = y$ when $\xi_0 = x$ and one applies control w, μ.

Note that the last of these assumptions, the controllability assumption, is satisfied if there exists at least one $m \in \mathcal{M}$ such that $\sigma^m(\sigma^m)^T$ (which is $n \times n$) has n positive eigenvalues.

Final assumption is that there exist $c_1, c_2 < \infty$ such that for any ϵ-optimal pair, μ^ϵ, w^ϵ for the $\widetilde{H}$ problem, one has $||w^\epsilon||^2_{L_2[0,t]} \leq c_1 + c_2|x|^2$ for all $\epsilon \in (0, 1]$, all $T < \infty$ and all $x \in \mathbb{R}^n$. (4.7)

Above assertion was proved in the purely quadratic case (cf. [McE98], [McE07]). But in this general case, we assume it instead.

Now, define the operator

$$\widetilde{S}_T[\phi] \doteq \sup_{w \in \mathcal{W}} \sup_{\mu \in \mathcal{D}_T} \int_0^T l^{\mu_t}(\xi_t) - \frac{\gamma^2}{2}|w_t|^2 \, dt + \phi(\xi_T)$$

where $\mathcal{D}_T \doteq \{\mu : [0, T) \to \mathcal{M} \ : \ \text{measurable}\}$. Under the above assumptions, a viscosity solution, $\widetilde{V}$ of (4.1),(4.2) exists, satisfies $0 \leq \widetilde{V} \leq V$ and is given by $\widetilde{V} = \lim_{T \to \infty} \widetilde{S}_T[V_0]$ for any $V_0 \in Q_K$ such that $0 \leq V_0 \leq \widetilde{V}$, [MK], [McE07].

In the max-plus algebra, addition and multiplication are defined as $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$, respectively. It is well known that $\widetilde{S}_T$ forms a max-plus linear semigroup.

4.3 Curse-of-dimensionality-free algorithm

The key steps in the curse-of-dimensionality-free algorithm developed in [McE07] are given below. Since we are interested in understanding how the curse-of-complexity arises in this algorithm, we shall sidestep the theoretical foundations which are well covered in [McE07], [McE06], and focus on the algorithmic flow.

4.3.1 Approximate propagation

Define the constituent-Hamiltonian semigroup operators as

$$\mathcal{S}^m_\tau[\phi] = \sup_{w \in \mathcal{W}} \int_0^\tau L^m(\xi_t) - \frac{\gamma^2}{2}|w_t|^2 \, dt + \phi(\xi_\tau).$$

Importantly, propagation of a quadratic $\phi \in \mathcal{K}$ by an $\mathcal{S}_\tau^m$ operator can be reduced to a solution of a differential Riccati equation. Define the time-indexed operators

$$\bar{S}_\tau[\phi](x) = \max_{m\in\mathcal{M}} S_\tau^m[\phi](x) = \bigoplus_{m\in\mathcal{M}} S_\tau^m[\phi](x).$$

Fix any $T < \infty$. Under the above assumptions, we have (c.f., [MK])

$$\lim_{N\to\infty} \left\{\overline{\mathcal{S}}_{T/N}\right\}^N [\phi] = \widetilde{S}_T[\phi] \text{ and } \lim_{T\to\infty} \widetilde{\mathcal{S}}_T[\phi] = \widetilde{V}$$

where the superscript N represents repeated application of the operator, N times.

Error analysis of this method, carried out in [McE09] guarantees that an arbitrarily accurate solution can be found on a compact set with a small enough timestep τ and long enough time horizon $N\tau$.

In practice, starting with a quadratic $\phi \in \mathcal{K}$, the numerical method uses a discretization timestep τ and a time horizon $N\tau$, and approximates the infinite horizon value function as

$$\begin{aligned}\widetilde{V}(x) \approx \bar{V}^N(x) &\doteq \bar{\mathcal{S}}_{N\tau}[\phi](x) = \left\{\bar{\mathcal{S}}_\tau\right\}^N [\phi](x) \\ &= \left\{\bigoplus_{m\in\mathcal{M}} \mathcal{S}_\tau^m\right\}^N [\phi](x) = \bigoplus_{\{m_i\}_{i=1}^N \in\mathcal{M}^N} \left\{\prod_{i=1}^N \mathcal{S}_\tau^{m_i}\right\} [\phi](x).\end{aligned}$$

For each $k \in \mathbb{N}$, $\{m_i\}_{i=1}^k \in \mathcal{M}^k$, if we define $v_{\{m_i\}_{i=1}^k}(x) = \prod_{i=1}^k \mathcal{S}_\tau^{m_i}[\phi](x)$, then we have

$$\widetilde{V}(x) \approx \bar{V}^N(x) = \bigoplus_{\{m_i\}_{i=1}^N \in\mathcal{M}^N} v_{\{m_i\}_{i=1}^N}(x).$$

Note that $v_{\{m_i\}_{i=1}^k}(x) = \mathcal{S}_\tau^{m_k}[v_{\{m_i\}_{i=1}^{k-1}}](x)$, and if $v_{\{m_i\}_{i=1}^{k-1}}(x)$ is quadratic, so is $v_{\{m_i\}_{i=1}^k}(x)$. Since we start with a quadratic $\phi(x)$, by induction each of the $v_{\{m_i\}_{i=1}^{k-1}}$ is quadratic. Unfortunately, the number of quadratics in the above approximation is M^N, which grows exponentially in complexity, thus leading to the *curse-of-complexity.* Nevertheless, only a few of these contribute to the max-plus sum. The rest can be pruned without affecting the value function. This problem is formulated and partially solved in the rest of this chapter.

4.4 Pruning Algorithms

In the above curse-of-dimensionality-free algorithm, at step k, $\bar{V}^k$ is represented as a max-plus sum of quadratics. Let us index the elements of this sum by integers $i \in \mathcal{I}_k$ (rather than by the sequences $\{m_i\}_{i=1}^k$). That is, we have

$$\bar{V}^k(z) = \bigoplus_{i \in \mathcal{I}^k} v_i^k(z)$$

where we let each v_i^k be given in the form

$$v_i^k(z) = v_i(z) = z^T A_i z + 2b_i^T z + c_i$$

where we delete the superscript k for simplicity of notation here and in the sequel.

Recall that we are reducing computational cost by pruning quadratics (v_i) which do not contribute to the solution approximation (not achieving the maximum at any $z \in \mathbb{R}^n$). Consequently, we want to determine whether the p^{th} quadratic contributes to the pointwise maximum. i.e. whether there is a region where it is greater than all other quadratics. Fix $p \in \mathcal{I}^k$. Thus we want to ensure feasibility of

$$v_p(z) \geq v_i(z) \quad \forall i \neq p. \tag{4.8}$$

Alternatively, we consider the problem:

$$\begin{aligned} &\text{Maximize } G(z,\nu) \doteq \nu \quad \text{ subject to} \\ &v_p(z) - v_i(z) \geq \nu \quad \forall i \neq p. \end{aligned} \tag{4.9}$$

Then, the maximum value of ν, $\bar{\nu}$, is the maximum amount by which the p^{th} quadratic can be lowered before it submerges below the max-plus sum of the rest. If $\bar{\nu} \leq 0$, then p^{th} quadratic does not contribute to the max-plus sum, and hence it can be pruned without consequence. If $\bar{\nu} > 0$,the quadratic contributes to the max-plus sum, and $\bar{\nu}$ can serve as some measure of contribution of the p^{th} quadratic to the value function, enabling us to rank the quadratics. This is useful in over-pruning.

Note that (4.9) implies that the importance metric,

$$\bar{\nu} = \max_{z \in \Re^n} \min_{i \neq p} v_p(z) - v_i(z) \tag{4.10}$$

This ranking scheme is independent of the location of the quadratic. Since in curse-of-dimensionality-free method, solution is grown from near the origin, and region of interest, where we want to construct the value function, is often near the origin, we would like to have a ranking scheme which reflects this bias. Hence we consider following importance metric

$$\nu^0 = \max_{z \in \Re^n} \frac{1}{1+|z|^2} \min_{i \neq p} (v_p(z) - v_i(z)) \tag{4.11}$$

which discounts the quadratics which contribute to the value function far away from origin. The extent of this bias between location and contribution, can be tweaked by a multiplier on the term $|z|^2$. Similar to (4.9), the above metric can be reformulated as

$$\begin{aligned} &\text{Maximize } G(z,\nu) \doteq \nu \quad \text{subject to} \\ &v_p(z) - v_i(z) \geq \nu(1+|z|^2) \;\; \forall i \neq p. \end{aligned} \tag{4.12}$$

and $v_p(z)$ can be pruned if and only if the maximum value $\nu^0 \leq 0$.

4.4.1 Pairwise pruning

Before undertaking the pruning using semidefinite programming, pairwise pruning is used, which checks between all pairs of quadratic basis functions, and prunes those which are completely dominated by another. Let $A = A_i - A_j$, $b = b_i - b_j$, $c = c_i - c_j$, and define $q(z) = z^{'}Az + 2b^{'}z + c$. Then q is nonnegative everywhere if and only if the homogeneous quadratic form, $z^{'}Az + 2tb^T z + ct^2$ is nonnegative for all $z \in \mathbb{R}^n$ and all $t \in \mathbb{R}$ (easily proved using $q(t^{-1}z) \geq 0$ when $t \neq 0$). Latter statement is true if and only if

$$\begin{bmatrix} c & b^{'} \\ b & A \end{bmatrix} \succeq 0. \tag{4.13}$$

If we define for any index p, $Q_p = \begin{bmatrix} c_p & b_p^{'} \\ b_p & A_p \end{bmatrix}$. Using (4.13), in the pairwise comparison between i'th and j'th quadratics, later can be pruned if $Q_i - Q_j \succeq 0$. Pairwise pruning reduces the computational effort of the semidefinite pruning by getting rid of obviously dominated quadratics.

4.4.2 Shor's semidefinite relaxation based pruning

The problem of evaluating an individual quadratic $v_p(z)$ for pruning, (4.12), can be rephrased as below. Let $q_i(z) = v_p(z) - v_i(z)$ for all $i \neq p$. Then, v_p can be pruned if and only if

$$\nu^0 \doteq \max_{z,\nu} \left\{ \nu : \; q_i(z) - \nu(1+|z|^2) \geq 0 \;\; \forall i \neq p \right\} \leq 0. \tag{4.14}$$

Lemma 4.4.1. *With $\lambda \in \Re^{\#\mathcal{I}_k - 1}$ such that $\lambda_i \geq 0$ and $\lambda \neq 0$, $\nu \in \Re$ is an upper bound on ν^0 if following condition is satisfied.*

$$\sum_{i \neq p} \lambda_i (q_i(z) - \nu(1+|z|)^2) \leq 0 \; \forall z \tag{4.15}$$

Proof. From (4.14), $\exists z$ such that, $q_i(z) - v^0(1+|z|^2) \geq 0, \; \forall i \neq p$, therefore,

$$\sum_{i \neq p} \lambda_i \left(q_i(z) - v^0(1+|z|^2) \right) \geq 0 \tag{4.16}$$

Subtracting (4.15) from (4.16)

$$(\nu - \nu^0)(1+|z|^2) \sum_i \lambda_i \geq 0$$

with assumptions on λ, $\sum_{i \neq p} \lambda_i > 0$, so that we can divide by it, to get, $\nu \geq \nu^0$. Hence proved.

Now we will seek to minimize this upper bound ν by varying λ and ν subject to constraint (4.15). Also note that, if (λ, ν) are feasible, so is $(k\lambda, \nu)$ for $k > 0$. using this we can normalize λ by dividing by $\sum \lambda_i$. Which implies, λ lies within a simplex Λ, $\lambda_i \geq 0$, $\sum \lambda_i = 1$.

$$\nu_0 = \min_{\lambda \in \Lambda, \nu \in \Re} \left\{ \nu : \sum_{i \neq p} \lambda_i q_i(z) \leq \nu(1+|z|)^2 \; \forall z \right\} \tag{4.17}$$

Since $q_i(z) = v_p - v_i$, using linear superposition and result (4.13), (4.17) can be reposed with following semidefinite program:

$$\nu_0 = \min_{\lambda \in \Lambda, \nu} \left\{ \nu : \sum_{i \neq p} \lambda_i \left(Q_p - Q_i \right) \preceq \nu I \right\} \tag{4.18}$$

Note that if such a minimal $\nu_0 < 0$ value, the by Lemma 4.4.1, $\nu^0 \leq \nu_0 \leq 0$. Hence as per (4.14), p'th quadratic, $v_p(z)$ can be pruned. Since this gives sufficient condition for pruning, it leads to conservative pruning. If $\nu_0 > 0$, the prunability is not conclusive. Nevertheless, it does give us a working indication of the importance of the quadratic. Since (4.18) can be restated as,

$$\nu_0 = \min_{\lambda \in \Lambda, \nu} \left\{ \nu : Q_p \preceq \nu I + \sum_{i \neq p} \lambda_i Q_i \right\} \tag{4.19}$$

if $\nu_0 > 0$, it indicates that the p^{th} quadratic has to come down by at least $\nu_0(1+|z|^2)$, before it is dominated by the convex hull of the remaining quadratics.

An additional way to develop intuition for result (4.19) is as follows. The above test evaluates the p^{th} quadratic v_p, which can be pruned if $\nu = 0$ satisfies the inequality in (4.19). Thus v_p can be pruned if, $Q_p \preceq \sum_{i \neq p} \lambda_i Q_i$. Thus, if the convex hull of remaining quadratics intersects the semidefinite cone of quadratics greater than v_p for all $z \in \Re^n$, then v_p can be pruned.

4.4.3 Dual of Shor's relaxation based pruning

If $q_i(z) = v_p(z) - v_i(z) = z' \bar{A}_i z + 2\bar{b}_i' z + \bar{c}_i$, and if we define,

$$\bar{Q}_i = \begin{bmatrix} \bar{c}_i & \bar{b}_i' \\ \bar{b}_i & \bar{A}_i \end{bmatrix} = Q_p - Q_i \text{ and } Z(z) = \begin{bmatrix} 1 & z' \\ z & zz' \end{bmatrix}$$

Then value of a quadratic form $q_i(z)$ can be written as Frobenius inner product of $\bar{Q}_i$ and $Z(z)$.

$$\begin{aligned} q_i(z) &= z' \bar{A}_i z + 2\bar{b}_i' z + \bar{c}_i \\ &= \begin{bmatrix} 1 \\ z \end{bmatrix}' \bar{Q}_i \begin{bmatrix} 1 \\ z \end{bmatrix} = \mathrm{Tr}\left(\bar{Q}_i \begin{bmatrix} 1 \\ z \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix}' \right) \\ &= \langle \bar{Q}_i, Z(z) \rangle \end{aligned}$$

Similarly, $q_i(z) - \nu(1 + z^2) = \langle \bar{Q}_i - \nu I, Z(z) \rangle$ Thus from (4.14), v_p can be pruned if and only if

$$\nu^0 = \max_{z \in \Re^n, \nu \in \Re} \left\{ \nu : \ \langle \bar{Q}_i - \nu I, Z(z) \rangle \geq 0 \ \ \forall i \neq p \right\} \leq 0 \tag{4.20}$$

Let $\mathcal{Z}$ be the nonlinear manifold of the set of all symmetric dyadic matrices $Z(z), z \in \Re^n$. All matrices in $\mathcal{Z}$ are positive semidefinite with northwestern entry 1. Let $\bar{\mathcal{Z}}$ be set of all such matrices. Replacing $\mathcal{Z}$ with $\bar{\mathcal{Z}} \supset \mathcal{Z}$, , we get relaxation and an upper bound for the maximum in (4.20).

$$\nu^0 \leq \bar{\nu}^0 \doteq \max_{Z \in \bar{\mathcal{Z}}, \nu \in \Re} \left\{ \nu : \ \langle \bar{Q}_i - \nu I, Z \rangle \geq 0 \ \ \forall i \neq p \right\}$$

If $\bar{\nu}^0 \leq 0$, then $\nu^0 \leq 0$, implying prunability. Thus v_p can be pruned if

$$\max_{Z, \nu \in \Re} \left\{ \nu \ \middle| \ \begin{array}{c} \langle \bar{Q}_i - \nu I, Z \rangle \geq 0 \ \ \forall i \neq p \\ Z \succeq 0, Z_{11} = 1 \end{array} \right\} \leq 0 \tag{4.21}$$

Since $Z \succeq 0$ and $Z_{11} = 1$, $\mathrm{Tr}(Z) > 0$. Also note that $\langle I, Z \rangle = \mathrm{Tr}(Z)$. If we define $Y = Z/\mathrm{Tr}(Z)$, then $Y \succeq 0$, $\mathrm{Tr}(Y) = 1$. Also

$$\langle \bar{Q}_i - \nu I, Z \rangle = \langle \bar{Q}_i, Z \rangle - \nu \mathrm{Tr}(Z) = \mathrm{Tr}(Z) \{ \langle \bar{Q}_i, Y \rangle - \nu \}$$

constraint set in (4.21) can be simplified to following semidefinite program. Thus $v_p(z)$ can be pruned if

$$\max_{Y, \nu \in \Re} \left\{ \nu \ \middle| \ \begin{array}{c} \langle \bar{Q}_i, Y \rangle \geq \nu \ \ \forall i \neq p \\ Y \succeq 0, \mathrm{Tr}(Y) = 1 \end{array} \right\} \leq 0 \tag{4.22}$$

This program is a dual of (4.18).

Intuitively, $L(Q) = \langle Q, Y \rangle$ can be thought as a linear functional over space of $n \times n$ symmetric matrices, taking constant values over hyperplanes normal (in the sense of above inner product) to Y. So we are searching in the space of hyperplane normals $Y \succeq 0, \mathrm{Tr}(Y) = 1$, which is a slice of the cone of semidefinite matrices. If we can find a hyperplane, separating Q_p from convex hull of rest of $Q_i, \forall i \neq p$, then prunability of p'th quadratic is not conclusive. Hence it need not be pruned.

Since above is the sufficient condition for pruning, it leads to conservative pruning. It is not a necessary condition, due to the nonconvexity of $\mathcal{Z}$, manifold of dyadic matrices. To achive better pruning, higher order pruning techniques can be used.

4.5 Computational complexity

Since our aim is to reduce the curse-of-complexity without loosing the freedom from the curse-of-dimensionality, it is worthwhile to discuss the computational overhead involved in these pruning methods. They are polynomial in both dimensionality and the number of quadratic functions. In particular, they are free from the curse-of-dimensionality.

A generic semi-definite program $\mathcal{P}$ is given by

$$P_0 = \min_{\eta \in \mathbb{R}^N} \left\{ c^{'}\eta : A_0 + \sum_{j=1}^{N} \eta_j A_j \geq 0, ||\eta||_2 \leq R \right\}$$

where the A_j are symmetric matrices with $\widetilde{M}$ diagonal blocks of size $k_i \times k_i$, $i = 1, \ldots \widetilde{M}$. We say that η^{ε} is an ε–optimal solution if

$$\|\eta^{\varepsilon}\|_2 \leq R,\ A_0 + \sum_{j=1}^{N} \eta_j^{\varepsilon} A_j \geq -\epsilon I,\ c^{'}\eta^{\varepsilon} \leq P_0 + \epsilon.$$

In [BTN01], the authors derive the computational complexity of obtaining such an η^{ε}.

$$\mathcal{C}(\mathcal{P}, \epsilon) = \mathcal{O}(1) \left(1 + \sum_{1}^{\widetilde{M}} k_i \right)^{1/2} \cdot N \left(N^2 + N \sum_{1}^{\widetilde{M}} k_i^2 + \sum_{1}^{\widetilde{M}} k_i^3 \right) \mathcal{D}(\mathcal{P}, \epsilon) \tag{4.23}$$

where $\mathcal{D}(\mathcal{P}, \epsilon)$ depends on the specific problem data (as indicated by the $\mathcal{P}$ in the argument as well as ϵ). Using this expression, we can obtain an upper bound for the arithmetic complexity of the pruning algorithms. Assuming the worst case scenario where no quadratic gets pruned, we find the complexity of testing one quadratic for pruning as follows.

For the Shor's relaxation pruning of (4.18), we have $\eta = [\nu\ \lambda^{'}]^{'}$. Hence $N = \#\mathcal{I}_k$. In addition to the main LMI, we have $\lambda_i \geq 0$ and an equality $\sum_{\lambda_i} = 1$, which can be split up into two inequalities, giving the number of LMI blocks $\widetilde{M} = \#\mathcal{I}_k + 2$, and $k_1 = n+1$, $k_2 = k_3 = \cdots = k_{\widetilde{M}} = 1$. Substituting these into (4.23), complexity is

$$\mathcal{C}(\mathcal{P},\epsilon) = \mathcal{O}(1)\left(\#\mathcal{I}_k + n + 3\right)^{1/2} \#\mathcal{I}_k^2 \cdot \left[3\#\mathcal{I}_k + 2 + (n+1)^2(n+2)\right] \mathcal{D}(\mathcal{P},\epsilon). \tag{4.24}$$

The complexity grows as $n^{3.5}$ and $\#\mathcal{I}_k^{3.5}$.

For the dual of Shor's relaxation pruning of (4.22), independent variables are Y, which has $(n+1)(n+2)/2$ free variables (corresponding to the upper triangle), and ν. Thus $N = \frac{1}{2}(n^2 + 3n + 4)$. Number of block LMIs is $\widetilde{M} = \#\mathcal{I}_k + 2$, and $k_1 = \cdots = k_{\widetilde{M}-3} = 1$, $k_{\widetilde{M}-2} = n+1$, $k_{\widetilde{M}-1} = k_{\widetilde{M}} = 1$. Substituting these into (4.23), complexity grows as $n^{6.5}$ and $\#\mathcal{I}_k^{1.5}$.

Note that these computational cost growth rates as a function of space dimension, n, are poorer than the cubic growth of the basic algorithm. However, they are still tremendously helpful for real-world problems, and the curse-of-dimensionality is still very far off.

4.6 Importance-based over-pruning

Both the Shor's relaxation and its dual pruning schemes generate importance metric for unpruned quadratics. Over-pruning keeps a limited number of more important quadratics, due to memory/computational constraints. This is useful in containing the complexity growth, while ensuring an accurate solution in a region of interest. At end of k'th iteration, the algorithm sorts v_i^k according to importance metric, and keeps at most $L(k)$ quadratics, pruning the rest.

However, there does not yet exist a theory which allows us to map the importance measure of a quadratic at step k to an error bound in the approximation at the terminal step. Regardless, application of these methods has proven extremely fruitful, as can be seen in the example below.

4.7 Six-dimensional, six-Hamiltonian example

We applied the curse-of-dimensionality-free method to the following problem over $\mathbb{R}^6$ with six constituent quadratic Hamiltonians. Note that the problem was tweaked to exhibit sufficiently complex and interesting behavior, such that there is interaction amongst dimensions, and each operator is important somewhere in the domain. Hence the following data yields a reasonably rich problem.

We shall specify the matrices in terms of the following building blocks for the dynamics:

$$A_a = \begin{bmatrix} -1 & .5 \\ .1 & -1 \end{bmatrix}, \qquad A_b = A_a,$$
$$A_c = A_a, \qquad A_d = \begin{bmatrix} -1 & .5 \\ .3 & -1 \end{bmatrix},$$
$$A_e = A_a, \qquad A_f = \begin{bmatrix} -1 & .5 \\ .1 & -1 \end{bmatrix},$$

$$\Sigma_a = 0.4 \times \begin{bmatrix} 0.27 & -.01 \\ -.01 & 0.27 \end{bmatrix}, \qquad \Sigma_b = 0.4\Sigma_a,$$
$$\Sigma_c = \Sigma_a, \qquad \Sigma_d = 0.4\Sigma_a,$$
$$\Sigma_e = \Sigma_a, \qquad \Sigma_f = 0.4\Sigma_a,$$

and the following building blocks for the payoff functions:

$$D_a = \begin{bmatrix} 1.5 & .2 \\ .2 & 1.5 \end{bmatrix}, \qquad D_b = 1.4 \times D_a,$$
$$D_c = 1.4 \times \begin{bmatrix} 0.2 & 1.5 \\ 1.5 & 0.2 \end{bmatrix}, \qquad D_d = 1.2 \times \begin{bmatrix} 1.6 & 0 \\ 0 & 0 \end{bmatrix},$$
$$D_e = 1.1 \times \begin{bmatrix} 0.3 & 1.5 \\ 1.5 & 0.3 \end{bmatrix}, \qquad D_f = 1.3 \times \begin{bmatrix} 0 & 0 \\ 0 & 1.6 \end{bmatrix}.$$

We will use a parameter to adjust the interaction in the dynamics across the dimensions, and this will be $\gamma = -0.1$. Now we are ready to define each of the Hamiltonians. We need to specify parameters for the dynamics (A, Σ, l_2) and the payoff (D, l_1, α). For the example below, $l_1 = 0$ and $l_2 = 0$ for all the Hamiltonians. The remaining parameters are as follows.

For the first Hamiltonian, H^1, we let

$$A^1 = \begin{bmatrix} A_a & \gamma I & \gamma I \\ \gamma I & A_a & 0 \\ \gamma I & 0 & A_a \end{bmatrix}, \qquad \Sigma^1 = \begin{bmatrix} \Sigma_a & 0 & 0 \\ 0 & \Sigma_a & 0 \\ 0 & 0 & \Sigma_a \end{bmatrix}$$

$$D^1 = \begin{bmatrix} D_a & 0 & 0 \\ 0 & D_a & 0 \\ 0 & 0 & D_a \end{bmatrix}, \quad \alpha_1 = 0\,.$$

For the second Hamiltonian, H^2, we let

$$A^2 = \begin{bmatrix} A_b & \gamma I & \gamma I \\ \gamma I & A_b & 0 \\ \gamma I & 0 & A_b \end{bmatrix}, \qquad \Sigma^2 = \begin{bmatrix} \Sigma_b & 0 & 0 \\ 0 & \Sigma_b & 0 \\ 0 & 0 & \Sigma_b \end{bmatrix}$$

$$D^2 = \begin{bmatrix} D_b & 0 & 0 \\ 0 & D_b & 0 \\ 0 & 0 & D_b \end{bmatrix}, \quad \alpha_2 = -0.4\,.$$

For the third Hamiltonian, H^3, we let

$$A^3 = \begin{bmatrix} A_c & \gamma I & \gamma I \\ \gamma I & A_c & 0 \\ \gamma I & 0 & A_c \end{bmatrix}, \qquad \Sigma^3 = \begin{bmatrix} \Sigma_c & 0 & 0 \\ 0 & \Sigma_c & 0 \\ 0 & 0 & \Sigma_c \end{bmatrix}$$

$$D^3 = \begin{bmatrix} D_c & 0 & 0 \\ 0 & D_c & 0 \\ 0 & 0 & D_c \end{bmatrix}, \quad \alpha_3 = 0.$$

For the fourth Hamiltonian, H^4, we let

$$A^4 = \begin{bmatrix} A_d & \gamma I & \gamma I \\ \gamma I & A_d & 0 \\ \gamma I & 0 & A_d \end{bmatrix}, \qquad \Sigma^4 = \begin{bmatrix} \Sigma_d & 0 & 0 \\ 0 & \Sigma_d & 0 \\ 0 & 0 & \Sigma_d \end{bmatrix}$$

$$D^4 = \begin{bmatrix} D_d & 0 & 0 \\ 0 & D_d & 0 \\ 0 & 0 & D_d \end{bmatrix}, \quad \alpha_4 = -0.4\,.$$

For the fifth Hamiltonian, H^5, we let

$$A^5 = \begin{bmatrix} A_e & \gamma I & \gamma I \\ \gamma I & A_e & 0 \\ \gamma I & 0 & A_e \end{bmatrix}, \qquad \Sigma^5 = \begin{bmatrix} \Sigma_e & 0 & 0 \\ 0 & \Sigma_e & 0 \\ 0 & 0 & \Sigma_e \end{bmatrix}$$

$$D^5 = \begin{bmatrix} D_e & 0 & 0 \\ 0 & D_e & 0 \\ 0 & 0 & D_e \end{bmatrix}, \qquad \alpha_5 = 0.$$

For the sixth Hamiltonian, H^6, we let

$$A^6 = \begin{bmatrix} A_f & \gamma I & \gamma I \\ \gamma I & A_f & 0 \\ \gamma I & 0 & A_f \end{bmatrix}, \qquad \Sigma^6 = \begin{bmatrix} \Sigma_f & 0 & 0 \\ 0 & \Sigma_f & 0 \\ 0 & 0 & \Sigma_f \end{bmatrix}$$

$$D^6 = \begin{bmatrix} D_f & 0 & 0 \\ 0 & D_f & 0 \\ 0 & 0 & D_f \end{bmatrix}, \qquad \alpha_6 = -0.4\,.$$

For this example, we let the time-discretization step-size be $\tau = 0.2$, and propagation was carried out with the Shor's semidefinite relaxation based pruning, implemented using YALMIP preprocessor [Lf04] and SeduMi solver [Stu99]. The overpruning threshold was set heuristically to $L(k) = 20 + 6k$. That is, a maximum of $L(k)$ quadratics, v_i^k, were retained at the k^{th} step. In this test, 25 iteration steps were carried out in 30 minutes, yielding a rather accurate solution in a compact domain in all six dimensions. This computation-time is for an Apple mac desktop, from roughly 2005. Slices of statistics for this value function along the 1-2 axes are shown in the accompanying figures. The backsubstitution error depends on the propagation as well as the time-discretization. The theoretical error bounds in [MK] are of the form $\varepsilon(1 + |x|^2)$ (over the entire space) where $\varepsilon \downarrow 0$ as the number of propagation steps goes to infinity and time-discretization go to zero, with the convergence rates derived in above reference.

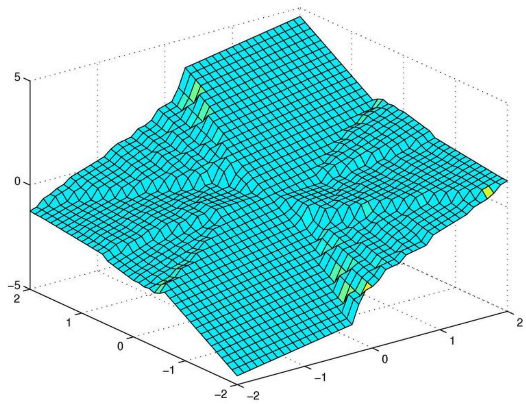


Figure 4.1: x_1 partial on the x_1, x_2 plane

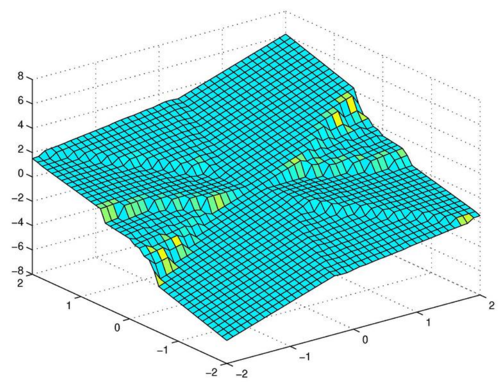


Figure 4.2: x_2 partial on the x_1, x_2 plane

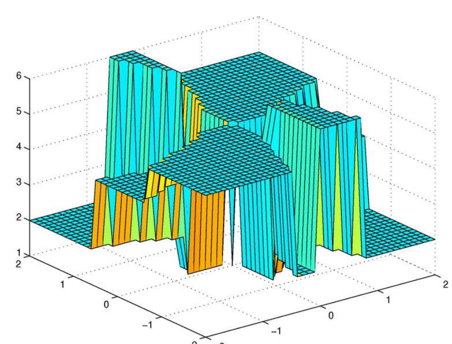


Figure 4.3: Optimal switching policy on the x_1, x_2 plane

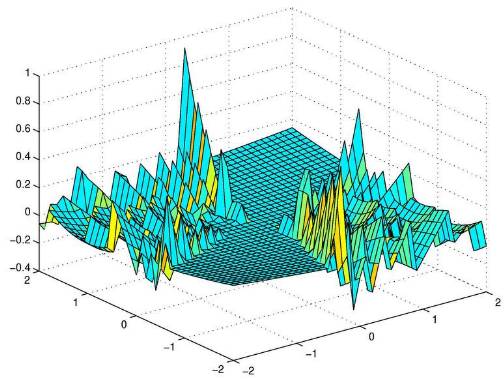


Figure 4.4: Backsubstitution error on the x_1, x_2 plane

4.8 Conclusions

Thus in this paper, two semi-definite programming schemes for pruning the quadratics were proposed for containing the curse-of-complexity in the curse-of-dimensionality-free method. Computational complexity for both is polynomial in space dimension. Both give us an importance metric to rank the quadratics according to the importance, which is a function of contribution to the point-wise maximum and its location. This is useful in case we need to over-prune. These methods have been applied to solve a sample six dimensional, six Hamiltonian problem in a reasonable amount of time.

Chapter 4, in part, is a reprint of the material as it appears in the proceedings of American Control Conference 2008, McEneaney, William; Deshpande, Ameet; Gaubert, Stéphane, IEEE Press, 2008 [MDG08]. The dissertation author was the coauthor of this paper.

Chapter 5

Payoff Suboptimality and Errors in Value Induced by Approximation of the Hamiltonian

5.1 Introduction

Chapter 3 reviewed the curse-of-dimensionality-free method, which is based on the max-plus linearity of underlying semigroup (cf. [MK], [McE07], [McE06]). Chapter 4, based on [MDG08], discussed the curse-of-complexity that arises in the above method and developed the pruning methods based on semidefinite programming, and example problems with high dimensionality were solved.

This approach has, so far, only been developed for steady-state problems over the entire space, although the class could be enlarged. (For other max-plus-based methods developed for larger classes of problems, see [AGL08], [MAL04], [FM00], [McE06], [DM03].) The curse-of-dimensionality-free approach currently handles HJB PDE problems of form

$$0 = -\widetilde{H}(x, \nabla V) \ \ \forall x \in \mathbb{R}^n \setminus \{0\}, \quad V(0) = 0 \tag{5.1}$$

where

$$\widetilde{H}(x, \nabla V) = \max_{m \in \mathcal{M}} \{H^m(x, \nabla V)\}, \tag{5.2}$$

$\mathcal{M} = \{1, 2 \ldots M\}$, and the H^m have computationally simpler forms. In particular, the H^m considered to date have been quadratic forms.

In [McE07], [McE06], the method was developed and the curse-of-dimensionality-free nature was made clear. In [McE09], [MK], the convergence rate for the algorithm was obtained. In particular, it was shown that there were two parameters, τ and $T = N\tau$ such that the errors go to zero as $T = N\tau \to \infty$ and $\tau \downarrow 0$. Further, a required relation between the relative T and τ rates was indicated. The errors in the pre-limit solution approximation are bounded in a form $0 \leq \widetilde{V} - V^a \leq \varepsilon(1 + |x|^2)$ where $\widetilde{V}$ is the true solution and V^a is the computed approximation. Additionally, we had $T = N\tau \propto \varepsilon^{-1}$ and $\tau \propto \varepsilon^2$, and so $N \propto \varepsilon^{-3}$. The computational cost growth with (space dimension) n is only of the order of n^3 (due to some matrix inverses). However, the approach is subject to a curse-of-complexity, where the computational cost can grow like M^N. Attenuation of this curse-of-complexity growth through pruning, using semidefinite programming, is an active area of research [MDG08].

Although the PDEs of (5.1) are certainly nontrivial nonlinear PDEs, we would like to solve more general HJB PDEs. A function, say $F(y)$, is semiconvex if given any $R < \infty$, there exists $C_R < \infty$ such that $F(y) + \frac{C_R}{2}|y|^2$ is convex over $B_R(0)$. (Note that the space of semiconvex functions certainly contains both C^2 and the space of convex functions as subspaces.) It is well known that one can approximate any semiconvex function as the pointwise maximum of quadratic forms. In fact, this is simply a max-plus basis expansion over the max-plus vector space, or moduloid, of semiconvex functions (c.f., [McE06]). With this in mind, we see that we could approximate any semiconvex Hamiltonian by a Hamiltonian, $\widetilde{H}$, of the form (5.2) with quadratic H^m. We could then solve the HJB PDE problem (5.1) with a curse-of-dimensionality-free method, thereby yielding an approximate solution of the HJB PDE with the original semiconvex Hamiltonian. Such a procedure would induce two error sources. The first consists of the errors in the solution of (5.1) generated by the curse-of-dimensionality-free algorithm. These are briefly discussed in the previous paragraph, and fully discussed in [McE09], [MK].

The second source is that induced by the approximation of the original Hamiltonian by $\widetilde{H}$. This latter error source is under discussion here. Although the analysis to follow is specifically oriented toward approximation by $\widetilde{H}$ of the above form, the general concepts may be more widely applicable. Further, in addition to obtaining bounds on the difference between the solution of the original and approximating HJB PDE problems, we also obtain a lower bound on the suboptimality of the controller obtained by use of the solution of (5.1) in the controller computation. This latter question is, of course, of significant practical value.

5.2 Problem Statement and Assumptions

We will consider the HJB PDE problem

$$\begin{aligned} 0 &= -H(x, \nabla V) = -\sup_{w \in \mathbb{R}^k} \left[f'(x,w) \nabla V + l(x,w) \right], \\ V(0) &= 0 \end{aligned} \tag{5.3}$$

where $x \in \mathbb{R}^n$. More specifically, we are seeking the particular viscosity solution of (5.3) which is the value function of the following optimal control problem. The dynamics are given by

$$\dot{\xi}_t = f(\xi_t, w_t) \doteq g(\xi_t) + \sigma(\xi_t) w_t, \tag{5.4}$$

and the running cost is

$$l(\xi_t, w_t) \doteq L(\xi_t) - \frac{\gamma^2}{2} |w_t|^2. \tag{5.5}$$

The value function we seek, maximizing the payoff over controls $w \in \mathcal{W} \doteq L_2([0,\infty); \mathbb{R}^k)$, is

$$\widehat{V}(x) = \sup_{w \in \mathcal{W}} \sup_{T < \infty} \left\{ \int_0^T l(\xi_t, w_t) dt \,\middle|\, \xi_0 = x \right\}. \tag{5.6}$$

We assume, $\exists K, c, d_\sigma, C, \alpha \in (0,\infty)$ such that the following hold. $g(x)$ is globally Lipschitz continuous with constant K, $(x-y)^T(g(x)-g(y)) \le -c|x-y|^2$ for all x, y, and $g(0) = 0$. $\sigma(x)$ is Lipschitz continuous with constant K, and its norm is bounded globally by d_σ. $|L(x)-L(y)| \le C(1+|x|+|y|)|x-y|$ for all x, y, and $0 \le L(x) \le \alpha|x|^2$ for all x. Finally, we assume $\gamma^2/d_\sigma^2 > \alpha/c^2$. $(A.V)$

It is worth noting, that with the above forms for f and l,

$$H(x,p) = g(x)'p + L(x) + \tfrac{1}{2\gamma^2}p'\sigma(x)\sigma'(x)p.$$

In [McE06], it was demonstrated that the above assumptions guarantee the following:

Theorem 5.2.1. $\widehat{V}$ *(given by (5.6)) is a continuous viscosity solution of (5.3), and is the unique such solution within the class*

$$\mathcal{G}_{\bar{\delta}} \doteq \{\phi : \phi \textit{ is semiconvex, } 0 \le \phi(x) \le c\frac{\gamma^2-\bar{\delta}^2}{2d_\sigma^2}|x|^2\} \tag{5.7}$$

for $\bar{\delta} > 0$ *sufficiently small.*

The goal is to approximately compute $\widehat{V}$ by approximating H by an $\widetilde{H}$ taking the form (5.2) with quadratic H^m, and then to solve (5.1) with the curse-of-dimensionality-free method [McE07], [McE06].

In particular, we assume that H and $\widetilde{H}$ are *close* in the following sense. Assume that:

There exists $\theta > 0$ such that, for all $x, p \in \mathbb{R}^n$ such that $\widetilde{H}(x,p) \le 0$, one has $(A.c)$

$$\widetilde{H}(x,p) \le H(x,p) \le \widetilde{H}(x,p) + \theta\left[|x|^2 + |p|^2\right].$$

Note that the coefficient θ parameterizes the degree of closeness between H and $\widetilde{H}$. As we are dealing with max-plus vector spaces, $\widetilde{H}$ approximates H from below (c.f. [McE06]), and so this approximation assumption is one-sided.

Let $D^-V(x)$ denote the sub-differential of V at x, i.e.,

$$D^-V(x) = \left\{ p \in \mathbb{R}^n \,\middle|\, \liminf_{y \to x} \frac{V(y) - V(x) - p \cdot (y - x)}{|y - x|} \geq 0 \right\}.$$

Remark 5.2.2. If $\widetilde{V}$ is a viscosity solution of (5.1), and $p \in D^-\widetilde{V}(x)$, then by the definition of viscosity solutions, $\widetilde{H}(x,p) \leq 0$. Consequently, the inequalities of Assumption $(A.c)$ hold for all x, p such that $p \in D^-\widetilde{V}(x)$.

We will suppose that the H^m are generalized quadratic forms, with parameters meeting certain conditions which guarantee existence and uniqueness within a certain function class. The H^m take the form

$$\begin{aligned} H^m(x,p) = \tfrac{1}{2}x'D^m x + \tfrac{1}{2}p'\Sigma^m p + (A^m x)'p \\ + (l_1^m)'x + (l_2^m)'p + \alpha^m \end{aligned} \tag{5.8}$$

where each $\Sigma^m = (1/\gamma^2)\sigma^m(\sigma^m)'$ for appropriate matrices σ^m. In regards to H^m, we make following assumptions, which ensure existence of a solution meeting the boundary condition at the origin (c.f. [MK]).

> Assume there exists $c_A \in (0, \infty)$ such that $x'A^m x \leq -c_A|x|^2$ for all $x \in \mathbb{R}^n$ and all $m \in \mathcal{M}$.
> Assume $H^1(x,p)$ has coefficients satisfying the following: $l_1^1 = l_2^1 = 0$; $\alpha^1 = 0$; there exists $c_{A,1} \in (0, \infty)$ such that $x'A^1 x \leq -c_{A,1}|x|^2$ $\forall x \in \mathbb{R}^n$; D^1 is positive definite, symmetric; $\Sigma^1 > 0$; and $\gamma^2/c_\sigma^2 > c_D/c_{A,1}^2$, where c_D is such that $x'D^1 x \leq c_D|x|^2$ $\forall x \in \mathbb{R}^n$ and $c_\sigma \doteq |\sigma^1|$. $(A.m)$
> Assume that system $\dot{\xi}^{\mu_t} = A^{\mu_t}\xi^{\mu_t} + l_2^{\mu_t} + \sigma^{\mu_t}w$ is controllable in the sense that given $x, y \in \mathbb{R}^n$ and $T > 0$, there exist processes $w \in \mathcal{W}$ and μ measurable with range in $\mathcal{M}$, such that $\xi_T = y$ when $\xi_0 = x$ and one applies controls w, μ.

The first assumption in $(A.m)$ is not restrictive, as without this nominal stability, sensible problems with positive definite running cost would have unbounded value. The second of the assumptions assures that at least one of the Hamiltonians has a purely quadratic structure, and this one typically "looks like" the H near the origin. The controllability assumption is (currently) needed for technical reasons.

We let

$$\widetilde{V}(x) \doteq \sup_{T<\infty} \sup_{\mu \in \mathcal{D}_\infty} \sup_{w \in \mathcal{W}} \int_0^T L^{\mu_t}(\xi_t) - \frac{\gamma^2}{2}|w_t|^2 \, dt$$

where

$$L^m(x) = \tfrac{1}{2}x' D^m x + (l_1^m)' x + \alpha^m,$$
$$\dot{\xi} = A^{\mu_t}\xi_t + l_2^{\mu_t} + \sigma^{\mu_t} w_t,$$

and

$$\mathcal{D}_\infty = \{\mu : [0,\infty) \to \mathcal{M} \mid \text{ measurable } \}.$$

In [McE06], [McE07], it was shown that:

Theorem 5.2.3. *$\widetilde{V}$ is the unique viscosity solution of (5.1) in the class of continuous functions satisfying $V(x) \in [0, \widehat{V}(x)]$ for all $x \in \mathbb{R}^n$.*

5.3 Preliminaries

The following lemmas will be useful further below. Let $T \in (0,\infty)$, and let W be the finite horizon value function given by

$$W(x,T) = \sup_{w \in \mathcal{W}} \int_0^T l(\xi_t, w_t)dt, \qquad \xi_0 = x, \tag{5.9}$$

where ξ satisfies (5.4). Noting that $\widehat{V} \geq 0$, we see that

$$W(x,T) \leq \sup_{w \in \mathcal{W}} \int_0^T l(\xi_t, w_t)dt + \widehat{V}(\xi_T), \qquad \xi_0 = x.$$

Also, let $\bar{\bar{\delta}} \doteq \gamma^2 - \frac{2d_\sigma^2 \alpha}{c^2}$. With these definitions and [McE06], Ch. 3, we immediately obtain the following two lemmas.

Lemma 5.3.1. *Let w_t^ε be ε–optimal (with $\varepsilon \in (0,1]$) for problem (5.9). Then,*

$$\tfrac{1}{2}\|w^\varepsilon\|_{L_2[0,T]}^2 \leq \frac{\varepsilon}{\bar{\bar{\delta}}} + \frac{1}{\bar{\bar{\delta}}}\left[\frac{c\gamma^2}{2d_\sigma^2}e^{-cT} + \frac{\alpha}{c}\right]|x|^2. \tag{5.10}$$

Lemma 5.3.2. *Let w_t^ε be ε–optimal (with $\varepsilon \in (0,1]$) for problem (5.9), and let ξ_t^ε be the corresponding state process. Then,*

$$\int_0^T |\xi_t^\varepsilon|^2 dt \leq \frac{2\varepsilon}{\bar{\delta}}\frac{d_\sigma^2}{c} + \frac{d_\sigma^2}{\bar{\delta}c}\left[\left(\frac{2\alpha}{c^2} + \frac{\gamma^2}{d_\sigma}\right) + \frac{1}{c}\right]|x|^2. \tag{5.11}$$

5.4 Error in the Value Function

As noted in Theorem 5.2.3, $0 \leq \widetilde{V}(x) \leq \widehat{V}(x)$ for all $x \in \mathbb{R}^n$. Now we obtain an upper bound on $\widehat{V} - \widetilde{V}$. The main result and core of the proof are Theorem 5.4.6 below and its corresponding proof. Prior to this we obtain some technical results.

Lemma 5.4.1. *There exists $K_g < \infty$ such that for any $x \in \mathbb{R}^n$,*

$$|p| \leq K_g|x| \qquad \forall\, p \in D^-\widetilde{V}(x).$$

Proof. By Theorem 5.2.3, Remark 5.2.2, and (5.2), for all $p \in D^-\widetilde{V}(x)$, one has

$$H^1(x,p) \leq \widetilde{H}(x,p) \leq 0.$$

Using (5.8) and Assumption $(A.m)$, this implies

$$x^T D^1 x + p^T \Sigma^1 p + (A^1 x)^T p \leq 0 \qquad \forall\, p \in D^-\widetilde{V}(x).$$

Rearranging this, and dropping superscripts for convenience, yields

$$\left(p + \frac{\Sigma^{-1}Ax}{2}\right)^T \Sigma \left(p + \frac{\Sigma^{-1}Ax}{2}\right) \leq x^T(A\Sigma^{-1}A - D)x.$$

Thus

$$\left|p + \frac{\Sigma^{-1}Ax}{2}\right|^2 \lambda_{\min}[\Sigma] \leq |x|^2 \lambda_{\max}[A\Sigma^{-1}A - D],$$

where, $\lambda_{\min}[X] = \min_i \mathbf{Re}(\lambda_i[X])$ and $\lambda_{\max}[X] = \max_i \mathbf{Re}(\lambda_i[X])$ with $\lambda_i[X]$ being the eigenvalues of X. By Assumption $(A.m)$, $\lambda_{\min}[\Sigma] = \lambda_{\min}[\Sigma^1] > 0$. With a little calculation, this implies the desired result. □

Remark 5.4.2. Using the above proof, a specific value of the bound, K_g, can be explicitly computed as:

$$K_g = \frac{1}{2}\lambda_{\max}[\Sigma^{1^{-1}} A^1] + \sqrt{\frac{\lambda_{\max}[A^1 \Sigma^{1^{-1}} A^1 - D^1]}{\lambda_{\min}[\Sigma^1]}}. \tag{5.12}$$

Fix $R < \infty$, and let $x \in B_R$. Let $\varepsilon \in (0,1]$, and let w^ε be an ε–optimal controller for (5.9). Also, let ξ^ε denote the corresponding state process.

Lemma 5.4.3. *For any $T \in [0,\infty)$, ξ^ε_t is absolutely continuous on $[0,T]$.*

Proof. Fix any $\delta > 0$. Consider any finite set of disjoint subintervals of $[0,T]$, say $\{[s_i,t_i] \,|\, i \in]1,N[\}$, such that $t_i < s_{i+1}$ for all $i \in]1,N-1[$ and such that $\sum_{i\le N} |t_i - s_i| = \delta$. We have

$$\sum_{i=1}^{N} |\xi^\varepsilon_{t_i} - \xi^\varepsilon_{s_i}| = \sum_{i=1}^{N} \left| \int_{s_i}^{t_i} g(\xi^\varepsilon_t) + \sigma(\xi^\varepsilon_t) w^\varepsilon_t \, dt \right|,$$

which by Assumption $(A.V)$

$$\le \sum_{i=1}^{N} \int_{s_i}^{t_i} K|\xi^\varepsilon_t| + d_\sigma |w^\varepsilon_t| \, dt \tag{5.13}$$

for the proper choice of K_1.

From inequality (3.17) from [McE06] (which follows easily from assumptions $(A.V)$), there exists $C_4 < \infty$, independent of T, such that

$$|\xi^\varepsilon_t|^2 \le C_4 \left(1 + |x|^2 + \int_0^t |w^\varepsilon_r|^2 \, dr \right) \qquad \forall\, t \in [0,T]$$

which by Lemma 5.3.1,

$$\le C_5(1+|x|^2) \qquad \forall\, t \in [0,T], \tag{5.14}$$

for the proper choice of $C_5 < \infty$ (independent of $T \in [0,\infty)$ and $\varepsilon \in (0,1]$). Combining (5.13) and (5.14), one finds for the proper choice of $C_6 < \infty$ (independent of $T \in [0,\infty)$ and $\varepsilon \in (0,1]$),

$$\begin{aligned} \sum_{i=1}^{N} |\xi^\varepsilon_{t_i} - \xi^\varepsilon_{s_i}| &\le \sum_{i=1}^{N} \left\{ \int_{s_i}^{t_i} C_6(1+|x|) \, dt + \int_{s_i}^{t_i} d_\sigma |w^\varepsilon_t| \, dt \right\} \\ &= C_6(1+|x|)\delta + \sum_{i=1}^{N} \int_{s_i}^{t_i} d_\sigma |w^\varepsilon_t| \, dt. \end{aligned} \tag{5.15}$$

Define

$$\phi(t) = \begin{cases} 1 & \text{if } t \in [s_i,t_i] \text{ for some } i \\ 0 & \text{otherwise.} \end{cases}$$

With this definition, (5.15) becomes

$$\sum_{i=1}^{N} |\xi^\varepsilon_{t_i} - \xi^\varepsilon_{s_i}| \le C_6(1+|x|)\delta + d_\sigma \int_0^T \phi(t)|w^\varepsilon_t|\, dt.$$

which by the Cauchy-Schwarz inequality

$$\le C_6(1+|x|)\delta + d_\sigma \left[\int_0^T (\phi(t))^2\, dt\right]^{1/2} \|w^\varepsilon_t\|$$
$$\le C_6(1+|x|)\delta + d_\sigma \delta^{1/2}\|w^\varepsilon_t\|$$

which, by Lemma 5.3.1 again

$$\le C_6(1+|x|)\delta + C_7(1+|x|)\delta^{1/2}$$

for the proper choice of $C_7 < \infty$. This last inequality implies absolute continuity. □

Lemma 5.4.4. *For any $T \in [0,\infty)$, $\widetilde{V}(\xi^\varepsilon_t)$ is absolutely continuous on $[0,T]$, and*

$$\widetilde{V}(\xi^\varepsilon_T) - \widetilde{V}(x) = \int_0^T \frac{d}{dt}\widetilde{V}(\xi^\varepsilon_t)\, dt,$$

where the time-derivative exists almost everywhere.

Proof. The semiconvexity of $\widetilde{V}$ (given in Theorem 5.2.3) implies local Lipschitz behavior (c.f., [Fle77]). Further, by the continuity given in Lemma 5.4.3 and finiteness of T, ξ^ε_t remains in a bounded set. Combining the absolute continuity of ξ^ε obtained in Lemma 5.4.3 with the Lipschitz property of $\widetilde{V}$ over the bounded set immediately implies the absolute continuity of $\widetilde{V}(\xi^\varepsilon_t)$. The remaining assertion is a direct result of the absolute continuity. □

Lemma 5.4.5. *For any $T \in [0,\infty)$,*

$$\widetilde{V}(\xi^\varepsilon_T) - \widetilde{V}(x) = \int_0^T \max_{p\in D^-\widetilde{V}(\xi^\varepsilon_t)} p \cdot f(\xi^\varepsilon_t, w^\varepsilon_t)\, dt.$$

Proof. By the semiconvexity of $\widetilde{V}$, the (one-sided) directional derivative, $\widetilde{V}_u(x)$, exists for all $x \in \mathbb{R}^n$ and all $|u| = 1$ in $\mathbb{R}^n$ (c.f., [BCD97], Th. II.4.7). Now,

$$\begin{aligned}\frac{d}{dt}\widetilde{V}(\xi^\varepsilon_t) &= \lim_{\delta\to 0}\frac{1}{\delta}[\widetilde{V}(\xi^\varepsilon_{t+\delta}) - \widetilde{V}(\xi^\varepsilon_t)] \\ &= \lim_{\delta\to 0}\frac{1}{\delta}[\widetilde{V}(\xi^\varepsilon_t + \delta f(\xi^\varepsilon_t, w^\varepsilon_t) + \mathcal{O}(\delta^2)) - \widetilde{V}(\xi^\varepsilon_t)] \\ &= |f(\xi^\varepsilon_t, w^\varepsilon_t)|\widetilde{V}_{u_t}(\xi^\varepsilon_t)\end{aligned}$$

with $u_t = f(\xi_t^\varepsilon, w_t^\varepsilon)/|f(\xi_t^\varepsilon, w_t^\varepsilon)|$ when $f(\xi_t^\varepsilon, w_t^\varepsilon) \neq 0$, and u_t an arbitrary unit vector otherwise. Again applying [BCD97], Th. II.4.7, this yields

$$\begin{aligned}\frac{d}{dt}\widetilde{V}(\xi_t^\varepsilon) &= |f(\xi_t^\varepsilon, w_t^\varepsilon)| \max_{p \in D^-\widetilde{V}(\xi_t^\varepsilon)} p \cdot u_t \\ &= \max_{p \in D^-\widetilde{V}(\xi_t^\varepsilon)} p \cdot f(\xi_t^\varepsilon, w_t^\varepsilon).\end{aligned}$$

□

We now proceed to obtain the main result of the section. For any $t \in [0, T]$, let

$$v_t^\varepsilon \doteq \max_{p \in D^-\widetilde{V}(\xi_t^\varepsilon)} p \cdot f(\xi_t^\varepsilon, w_t^\varepsilon).$$

By the ε-optimality of w^ε, one has

$$W(x,T) \leq \int_0^T \left[l(\xi_t^\varepsilon, w_t^\varepsilon) + v_t^\varepsilon\right] dt - \int_0^T v_t^\varepsilon \, dt + \varepsilon$$

(where existence of the integrals follows from Lemma 5.4.5). Then, by Lemma 5.4.5,

$$\begin{aligned}W(x,T) \leq \widetilde{V}(x) - \widetilde{V}(\xi_T^\varepsilon) \\ + \int_0^T \left[l(\xi_t^\varepsilon, w_t^\varepsilon) + v_t^\varepsilon\right] dt + \varepsilon.\end{aligned} \tag{5.16}$$

For any $t \in [0, T]$, let

$$p_t^\varepsilon \in \operatorname*{argmax}_{p \in D^-\widetilde{V}(\xi_t^\varepsilon)} p \cdot f(\xi_t^\varepsilon, w_t^\varepsilon).$$

Then,

$$l(\xi_t^\varepsilon, w_t^\varepsilon) + v_t^\varepsilon = l(\xi_t^\varepsilon, w_t^\varepsilon) + p_t^\varepsilon \cdot f(\xi_t^\varepsilon, w_t^\varepsilon)$$

which by Assumption $(A.c)$,

$$\leq \widetilde{H}(\xi_t^\varepsilon, p_t^\varepsilon) + \theta(|\xi_t^\varepsilon|^2 + |p_t^\varepsilon|^2). \tag{5.17}$$

However, by the definition of a viscosity solution, and the fact that $p_t^\varepsilon \in D^-\widetilde{V}(\xi_t^\varepsilon)$, $\widetilde{H}(\xi_t^\varepsilon, p_t^\varepsilon) \leq 0$, and so, (5.17) yields

$$l(\xi_t^\varepsilon, w_t^\varepsilon) + v_t^\varepsilon \leq \theta(|\xi_t^\varepsilon|^2 + |p_t^\varepsilon|^2)$$

which by Lemma 5.4.1,

$$\leq \theta(1+K_g^2)|\xi_t^\varepsilon|^2. \tag{5.18}$$

Substituting (5.18) into (5.16), one obtains

$$W(x,T) \leq \widetilde{V}(x) - \widetilde{V}(\xi_T^\varepsilon) + \theta(1+K_g^2)\int_0^T |\xi_t^\varepsilon|^2\,dt + \varepsilon,$$

and noting $\widetilde{V} \geq 0$,

$$\leq \widetilde{V}(x) + \theta(1+K_g^2)\int_0^T |\xi_t^\varepsilon|^2\,dt + \varepsilon,$$

which, by Lemma 5.3.2,

$$\leq \widetilde{V}(x) + \theta(1+K_g^2)[C_1 + C_2|x|^2] + \varepsilon, \tag{5.19}$$

where

$$C_1 = 2d_\sigma^2/(\bar{\delta}c) \quad \text{and} \quad C_2 = \frac{d_\sigma^2}{\bar{\delta}c}\left[\left(\frac{2\alpha}{c^2} + \frac{\gamma^2}{d_\sigma}\right) + \frac{1}{c}\right].$$

Since this is true for all $\varepsilon \in (0,1]$, we have

$$W(x,T) \leq \widetilde{V}(x) + \theta(1+K_g^2)[C_1 + C_2|x|^2]. \tag{5.20}$$

Then, noting (c.f., [McE06]) that $W(x,T) \to \widehat{V}(x)$ as $T \to \infty$, (5.20) yields the value approximation result:

Theorem 5.4.6. *There exists $C_3 < \infty$ such that*

$$\widehat{V}(x) - \theta(1+K_g^2)[C_1 + C_2|x|^2] \leq \widetilde{V}(x) \leq \widehat{V}(x) \qquad \forall\, x \in \mathbb{R}^n, \tag{5.21}$$

where

$$C_1 = 2d_\sigma^2/(\bar{\delta}c), \quad C_2 = \frac{d_\sigma^2}{\bar{\delta}c}\left[\left(\frac{2\alpha}{c^2} + \frac{\gamma^2}{d_\sigma}\right) + \frac{1}{c}\right], \tag{5.22}$$

and θ is as given in Assumption (A.c).

Thus, we see that $\widetilde{V}$ approximates $\widehat{V}$ arbitrarily well if $\widetilde{H}$ is sufficiently close to H, this closeness being parameterized by θ.

5.5 Degree of Suboptimality of the Controller

In the previous section, it was shown that if the approximating Hamiltonian is close to the Hamiltonian of the originating problem in a certain sense, then the corresponding viscosity solutions will be close in an appropriate sense. However, recall that we are specifically concerned with a case where we can efficiently solve the HJB PDE with the approximating Hamiltonian, and would like to use this solution to generate a controller for the originating problem. Consequently, we would like to know whether an (approximate) optimal control generated from the solution of the approximate HJB PDE, will perform well when applied to the true system, which is described by the originating Hamiltonian. We begin with some preparatory results, which are minor variations of the well-known properties of viscosity solutions and semiconvexity. Between Lemma 5.5.4 and Lemma 5.5.6, the optimal control approximation will be introduced. The main development will begin with Theorem 5.5.8.

Lemma 5.5.1. *Suppose V is a semiconvex viscosity solution of $0 = \widehat{H}(x, \nabla V)$, where $\widehat{H}$ is continuous. Let $D^-V(x)$ denote the sub-differential of V at x. For any $x, q \in \mathbb{R}^n$, there exists $\bar{p} \in D^-V(x)$ such that*

$$\bar{p} \cdot q = \max_{p \in D^-V(x)} p \cdot q \tag{5.23}$$

and

$$\widehat{H}(x, \bar{p}) = 0. \tag{5.24}$$

Proof. Let

$$D^*V(x) \doteq \left\{ p \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \{x_n\} \subseteq \mathcal{A}, \text{ such that} \\ x_n \to x \text{ and } p = \lim_{n\to\infty} \nabla V(x_n) \end{array} \right\} \tag{5.25}$$

where $\mathcal{A} = \{a \in \mathbb{R}^n \,|\, \nabla V(x) \,\text{exists}\,\}$. We note that by Rademacher's Theorem (c.f., [Zie89]), the Lebesgue measure of $\mathcal{A}^c$ is zero, due to the fact that V is locally Lipschitz, which follows from the semiconvexity (c.f., [Fle77]). The generalized gradient (c.f., [Cla83]) is the convex hull of $D^*V(x)$, denoted by $\langle D^*V(x) \rangle$. Then, by the semiconvexity of V and [BCD97], Proposition II.4.7,

$$D^-V(x) = \langle D^*V(x) \rangle. \tag{5.26}$$

Obviously, for any $q \in \mathbb{R}^n$, $p \cdot q$ is linear as a function of p, and so it takes its maximum over a convex hull at a point in the generating set. Using this observation and (5.26), we see that

$$\max_{p\in D^-V(x)} p\cdot q = \max_{p\in\langle D^*V(x)\rangle} p\cdot q = \max_{p\in D^*V(x)} p\cdot q. \tag{5.27}$$

Let $\bar p \in \operatorname{argmax}_{p\in D^*V(x)} p\cdot q$. By (5.26) and (5.27), we see that it is an element of D^-V that achieves the maximum in (5.23). Then, by (5.25), there exists $x_n \to x$ with $\nabla V(x_n) \to \bar p$. However, $\widehat H(x_n, \nabla V(x_n)) = 0$ for all n, and so, by the continuity of $\widehat H$, $\widehat H(x,p) = 0$.

□

It will be helpful to make the following definitions. Let

$$\begin{aligned}\mathcal{P}(x) &= \mathcal{P}(x;\widetilde V)\\ &\doteq \operatorname{argmax}\big\{f(x,w)\cdot p + l(x,w)\,|\,(w,p)\in\mathbb{R}^k\times D^-\widetilde V(x)\big\}.\end{aligned}$$

Also, let

$$\mathcal{W}^0(x;\widetilde V) = \operatorname*{argmax}_{w\in\mathbb{R}^k} \max_{p\in D^-\widetilde V(x)} \left[f(x,w)\cdot p + l(x,w)\right],$$

and

$$\begin{aligned}\mathcal{P}^0(x;\widetilde V) &= \operatorname*{argmax}_{p\in D^-\widetilde V(x)} \max_{w\in\mathbb{R}^k} \left[f(x,w)\cdot p + l(x,w)\right]\\ &= \operatorname*{argmax}_{p\in D^-\widetilde V(x)} \left[g(x)\cdot p + \tfrac{1}{2\gamma^2} p'\sigma(x)\sigma'(x)p\right].\end{aligned}$$

It will also be handy to note some simple relations.

Lemma 5.5.2. *If $\hat w \in \mathcal{W}^0(x;\widetilde V)$, then there exists $\hat p \in D^-\widetilde V(x)$ such that $(\hat w,\hat p)\in \mathcal{P}(x)$. On the other hand, $(\hat w,\hat p)\in\mathcal{P}(x)$ implies that $\hat w\in\mathcal{W}^0(x;\widetilde V)$.*

Proof. To simplify the notation, let $G(x,w,p) \doteq f(x,w)\cdot p + l(x,w)$. Suppose $\hat w \in \mathcal{W}^0(x;\widetilde V)$. Then

$$\max_{p\in D^-\widetilde V(x)} G(x,\hat w,p) = \max_{w\in\mathbb{R}^k}\max_{p\in D^-\widetilde V(x)} G(x,w,p),$$

which implies that there exists $\hat p\in D^-\widetilde V(x)$ such that $(\hat w,\hat p)\in\mathcal{P}(x)$.

Alternatively, let $(\hat w,\hat p)\in\mathcal{P}(x)$. Then,

$$G(x,\hat w,\hat p) = \max_{w\in\mathbb{R}^k}\max_{p\in D^-\widetilde V(x)} G(x,w,p),$$

which implies

$$\max_{p \in D^- \widetilde{V}(x)} G(x, \hat{w}, p) \geq \max_{w \in \mathbb{R}^k} \max_{p \in D^- \widetilde{V}(x)} G(x, w, p). \tag{5.28}$$

Now, suppose $\hat{w} \notin \mathcal{W}^0(x; \widetilde{V})$. This implies that

$$\max_{p \in D^- \widetilde{V}(x)} G(x, \hat{w}, p) < \max_{w \in \mathbb{R}^k} \max_{p \in D^- \widetilde{V}(x)} G(x, w, p), \tag{5.29}$$

Which contradicts (5.28).

□

Analogously, one has:

Lemma 5.5.3. *If $\hat{p} \in \mathcal{P}^0(x; \widetilde{V})$, then there exists $\hat{w} \in \mathbb{R}^k$ such that $(\hat{w}, \hat{p}) \in \mathcal{P}(x)$. On the other hand, $(\hat{w}, \hat{p}) \in \mathcal{P}(x)$ implies that $\hat{p} \in \mathcal{P}^0(x; \widetilde{V})$.*

We now get a simple representation for $\hat{w}$, which will be useful in bounding the control effort.

Lemma 5.5.4. *Suppose $\hat{p} \in \mathcal{P}^0(x; \widetilde{V})$, and let $\hat{w} = \hat{w}(x, \hat{p}) = \frac{1}{\gamma^2} \sigma'(x) \hat{p}$. Then, $(\hat{w}, \hat{p}) \in \mathcal{P}(x)$, and $\hat{w} \in \mathcal{W}^0(x; \widetilde{V})$.*

Proof. Using (5.4) and (5.5), we have

$$\begin{aligned} f(x, \hat{w}) \cdot \hat{p} + &l(x, \hat{w}) \\ &= [g(x) + \sigma(x)\hat{w}] \cdot \hat{p} + L(x) - \frac{\gamma^2}{2} |\hat{w}|^2 \end{aligned}$$

which by a simple calculation of the maximum of a quadratic,

$$= \max_{w \in \mathbb{R}^k} [f(x, w) \cdot \hat{p} + l(x, w)],$$

which, since $\hat{p} \in \mathcal{P}^0(x; \widetilde{V})$,

$$= \max_{w \in \mathbb{R}^k} \max_{p \in D^- \widetilde{V}(x)} [f(x, w) \cdot p + l(x, w)],$$

which implies $(\hat{w}, \hat{p}) \in \mathcal{P}(x)$. The second assertion then follows from Lemma 5.5.2.

□

We now deal with a technical issue related to existence of solutions. We will make an assumption, and then indicate a class of systems meeting the assumption. Let

$$F^s(x) \doteq \left\{ g(x) + \frac{1}{\gamma^2}\sigma(x)\sigma'(x)p \,|\, p \in \mathcal{P}^0(x) \right\}$$
$$= \left\{ g(x) + \frac{1}{\gamma^2}\sigma(x)w \,|\, w = \frac{1}{\gamma^2}\sigma'(x)p,\ p \in \mathcal{P}^0(x) \right\}$$

which by Lemma 5.5.4,

$$= \left\{ g(x) + \frac{1}{\gamma^2}\sigma(x)w \,|\, w \in \mathcal{W}^0(x) \right\}.$$

Consider the differential inclusion

$$\dot{\xi} \in F^s(\xi), \qquad \xi_0 = x. \tag{5.30}$$

We assume there exists a a locally Lipschitz solution of (5.30). $\qquad (A.s)$

We denote this solution of (5.30) as $\bar{\xi}_\cdot$. Note that in the case where $\widetilde{V}$ is smooth, (5.30) reduces to an ordinary differential equation, and there is no technical issue. However, in general, existence proofs for differential inclusions are less trivial than those for differential equations. One class of problems where it is known that $(A.s)$ holds is as follows.

Theorem 5.5.5. *Suppose that $-F^s$ is monotone in the sense that $(u-v)\cdot(x-y) \geq 0$ for all $u \in -F^s(x)$, $v \in -F^s(y)$, and all $x, y \in \mathbb{R}^n$. Then for any $x \in \mathbb{R}^n$, there exists a locally Lipschitz solution of (5.30), and further, $\|\xi_t\|$ is monotonically decreasing.*

Proof. With a possible linear rescaling of the time variable, $I - F^s$ is onto (where I indicates the identity mapping). By [AC84], Theorem 3.1.1, this implies that $-F^S$ is maximal monotone. Then, by [AC84], Theorem 3.2.1, one obtains the result. □

Lemma 5.5.6. *For any $T \in [0, \infty)$,*

$$\widetilde{V}(\bar{\xi}_T) - \widetilde{V}(x) = \int_0^T \frac{d}{dt}\widetilde{V}(\bar{\xi}_t)\,dt$$

where $\frac{d}{dt}\widetilde{V}(\bar{\xi}_t)$ exists a.e.

Proof. By the existence and continuity of $\bar{\xi}$ on $[0,T]$ (for any T), there exists $R_T < \infty$ such that $|\bar{\xi}_t| \leq R_T$ for all $t \in [0,T]$. Also, by the Lipschitz continuity of $\widetilde{V}$ (implied by the semiconvexity, c.f., [Fle77]), there exists $K_T < \infty$ such that

$$|\widetilde{V}(x) - \widetilde{V}(y)| \leq K_T(x-y) \qquad \forall\, x, y \in B_{R_T}, \tag{5.31}$$

and this implies $|p| \leq K_T$ for all $p \in D^-\widetilde{V}(x)$ for all $x \in B_{R_T}$. By (5.31) and the Lipschitz behavior of $\bar{\xi}$ on $[0,T]$, $\widetilde{V}(\bar{\xi}_\cdot)$ is Lipschitz on $[0,T]$, which implies absolute continuity. Therefore, $\frac{d}{dt}\widetilde{V}(\bar{\xi}_t)$ exists a.e. on $[0,T]$, and $\widetilde{V}(\bar{\xi}_t) - \widetilde{V}(\bar{\xi}_s) = \int_s^t \frac{d}{dr}\widetilde{V}(\bar{\xi}_r)\,dr$ for all $0 \leq s \leq t \leq T$.

□

As noted in the proof just above, there exists $R_T < \infty$ such that $\bar{\xi}_t \in \bar{B}_{R_T}$ for all $t \in [0,T]$ Then, again using the local Lipschitz nature of $\widetilde{V}$, there exists $K_T < \infty$ such that $D^-\widetilde{V}(x) \subseteq \bar{B}_{K_T}(0)$ for all $x \in \bar{B}_{R_T}(0)$, that is $D^-\widetilde{V}(\bar{B}_{R_T}(0)) \subseteq \bar{B}_{K_T}(0)$. Consequently, $\mathcal{W}^0(\bar{B}_{R_T}(0); \widetilde{V}) \subseteq \frac{d_\sigma}{\gamma^2}\bar{B}_{K_T}(0)$. Then, by for example, [AC84] Corollary 1.14.1, there exists a measurable selection, $\bar{w}_\cdot$, where for a.e. $t \in [0,T]$,

$$\bar{w}_t \in \{\mathcal{W}^0(\bar{\xi}_t; \widetilde{V}) \,|\, g(\bar{\xi}_t) + \sigma(\bar{\xi}_t)\bar{w}_t = \dot{\bar{\xi}}_t\}.$$

Of course, where $\nabla\widetilde{V}(\bar{\xi}_t)$ exists, this reduces to $\bar{w}_t = \frac{1}{\gamma^2}\sigma'(\bar{\xi}_t)\nabla\widetilde{V}(\bar{\xi}_t)$.

The proof of the following lemma is essentially identical to the proof of Lemma 5.4.5, and so we do not repeat it.

Lemma 5.5.7. *For any* $T \in [0,\infty)$*,*

$$\widetilde{V}(\bar{\xi}_T) - \widetilde{V}(x) = \int_0^T \max_{p \in D^-\widetilde{V}(\bar{\xi}_t)} p \cdot f(\bar{\xi}_t, \bar{w}_t)\,dt.$$

It will be necessary to show that solutions driven by our feedback control are well-behaved, i.e., staying bounded and eventually decaying to the origin. This step is comprised of Theorem 5.5.8 to Lemma 5.5.11.

Theorem 5.5.8. *For any* $T \in [0,\infty)$*,*

$$\int_0^T l(\bar{\xi}_t, \bar{w}_t)\,dt \geq \widetilde{V}(x) - \widetilde{V}(\bar{\xi}_T).$$

Proof.

$$\int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt = \int_0^T \left[l(\bar{\xi}_t, \bar{w}_t) + \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} f(\bar{\xi}_t, \bar{w}_t) \cdot p \right] dt$$
$$- \int_0^T \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} f(\bar{\xi}_t, \bar{w}_t) \cdot p \, dt, \tag{5.32}$$

where the integrability follows from Lemma 5.5.7.

Define

$$\mathcal{H}_0(x; \widetilde{H}) \doteq \{ p \in \mathbb{R}^n \mid \widetilde{H}(x, p) = 0 \}.$$

Then, note that

$$l(\bar{\xi}_t, \bar{w}_t) + \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} f(\bar{\xi}_t, \bar{w}_t) \cdot p$$
$$= \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} \left[l(\bar{\xi}_t, \bar{w}_t) + f(\bar{\xi}_t, \bar{w}_t) \cdot p \right],$$

and, since $\bar{w}_t \in \mathcal{W}^0(\bar{\xi}_t; \widetilde{V})$,

$$= \max_{w \in \mathbb{R}^k} \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} \left[l(\bar{\xi}_t, w) + f(\bar{\xi}_t, w) \cdot p \right],$$
$$= \max_{w \in \mathbb{R}^k} \left\{ l(\bar{\xi}_t, w) + \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} \left[f(\bar{\xi}_t, w) \cdot p \right] \right\},$$

which by Lemma 5.5.1,

$$= \max_{w \in \mathbb{R}^k} \left\{ l(\bar{\xi}_t, w) + \max_{p \in D^- \widetilde{V}(\bar{\xi}_t) \cap \mathcal{H}_0(\bar{\xi}_t; \widetilde{H})} \left[f(\bar{\xi}_t, w) \cdot p \right] \right\},$$
$$= \max_{p \in D^- \widetilde{V}(\bar{\xi}_t) \cap \mathcal{H}_0(\bar{\xi}_t; \widetilde{H})} \max_{w \in \mathbb{R}^k} \left[l(\bar{\xi}_t, w) + f(\bar{\xi}_t, w) \cdot p \right],$$
$$= \max_{p \in D^- \widetilde{V}(\bar{\xi}_t) \cap \mathcal{H}_0(\bar{\xi}_t; \widetilde{H})} H(\bar{\xi}_t, p),$$

which by Assumption $(A.c)$,

$$\geq \max_{p \in D^- \widetilde{V}(\bar{\xi}_t) \cap \mathcal{H}_0(\bar{\xi}_t; \widetilde{H})} \widetilde{H}(\bar{\xi}_t, p),$$

which since $p \in \mathcal{H}_0(\bar{\xi}_t; \widetilde{H})$,

$$= 0.$$

Integrating this over time, we see that,

$$\int_0^T \left[l(\bar{\xi}_t, \bar{w}_t) + \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} f(\bar{\xi}_t, \bar{w}_t) \cdot p \right] dt \geq 0. \tag{5.33}$$

Substituting (5.33) into (5.32), one finds

$$\int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt \geq -\int_0^T \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} f(\bar{\xi}_t, \bar{w}_t) \cdot p\, dt,$$

which by Lemma 5.5.7,

$$= \widetilde{V}(x) - \widetilde{V}(\bar{\xi}_T).$$

□

Corollary 5.5.9. *For any $x \in \mathbb{R}^n$, and any $T \in [0, \infty)$,*

$$\int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt + \widehat{V}(\bar{\xi}_T) \geq \widehat{V}(x) - K_x$$

where $K_x \doteq \widehat{V}(x) - \widetilde{V}(x)$.

Proof. The result follows immediately from the theorem by noting that $\widetilde{V}(\bar{\xi}_T) \leq \widehat{V}(\bar{\xi}_T)$. □

From Corollary 5.5.9 and [McE06], Lemma 3.17, one has

Corollary 5.5.10. *For any $R < \infty$, there exists $M_R < \infty$ such that for all $|x| \leq R$ and all $T \in [0, \infty)$,*

$$\int_0^T |\bar{\xi}_t|^2\, dt \leq M_R.$$

Lemma 5.5.11. *Given $\varepsilon \in (0, 1]$, $x \in \mathbb{R}^n$ and $\overline{T} < \infty$, there exists $T > \overline{T}$ such that*

$$0 \leq \widetilde{V}(\bar{\xi}_T) \leq \widehat{V}(\bar{\xi}_T) < \varepsilon.$$

Proof. As the other inequalities are already proven, we prove only the rightmost. Using Corollary 5.5.10, it is easy to show that given $\bar{\varepsilon} > 0$ and $\overline{T} < \infty$, there exists $T \in [\overline{T}, \infty)$ such that

$$|\bar{\xi}_T|^2 < \bar{\varepsilon}. \tag{5.34}$$

From [McE06], Theorems 3.19 and 3.20, there exists $C_V < \infty$ such that $\widehat{V}(x) \leq C_V |x|^2$, and consequently,

$$\widehat{V}(\bar{\xi}_T) \leq C_V |\bar{\xi}_T|^2. \tag{5.35}$$

Combining (5.34) and (5.35) yields the result.

□

We now begin the development leading to the main result of the section. By Corollary 5.5.9 and Lemma 5.5.11, we see that given $\varepsilon \in (0,1]$ and $\overline{T} < \infty$, there exists $T \in [\overline{T}, \infty)$ such that

$$\int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt \geq \widehat{V}(x) - K_x - \varepsilon. \tag{5.36}$$

Recalling the specific form of l given in (5.5) and the growth on L given by Assumption $(A.V)$, we see that (5.36) implies

$$\frac{\gamma^2}{2} \|\bar{w}_\cdot\|^2_{L_2(0,T)} \leq \alpha \|\bar{\xi}\|^2_{L_2(0,T)} + K_x + \varepsilon - \widehat{V}(x)$$

which, by Corollary 5.5.10

$$\leq \alpha M_{|x|} + K_x + \varepsilon - \widehat{V}(x).$$

Since this is true for any $\overline{T} < \infty$,

$$\|\bar{w}_\cdot\|^2_{L_2(0,\infty)} \leq \overline{M}_x \doteq \frac{2}{\gamma^2}\left[\alpha M_{|x|} + K_x + 1 - \widehat{V}(x)\right]. \tag{5.37}$$

Combining (5.37) and Corollary 5.5.10, we see that given $\hat{\varepsilon} > 0$, there exists $\widehat{T} < \infty$ such that

$$\|\bar{\xi}\|^2_{L_2(\widehat{T},\infty)}, \|\bar{w}_\cdot\|^2_{L_2(\widehat{T},\infty)} < \hat{\varepsilon},$$

which implies that given $\hat{\varepsilon} > 0$, there exists $\widetilde{T} < \infty$ such that

$$\int_{\widehat{T}}^{\infty} \left| l(\bar{\xi}_t, \bar{w}_t) \right| dt < \hat{\varepsilon}, \tag{5.38}$$

which implies that $\lim_{T\to\infty} \int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt$ exists. In particular, given $\hat{\varepsilon} > 0$,

$$\left| \int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt - \lim_{T\to\infty} \int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt \right| < \hat{\varepsilon} \tag{5.39}$$

for all $T \geq \widetilde{T}$. By (5.39) and Theorem 5.5.8, given $\hat{\varepsilon} > 0$,

$$\lim_{T\to\infty} \int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt \geq \widetilde{V}(x) - \hat{\varepsilon} - \widetilde{V}(\bar{\xi}_T) \qquad \forall\, T \geq \widetilde{T}. \tag{5.40}$$

Combining (5.40) and Lemma 5.5.11 (with $\overline{T}$ replacing $\widetilde{T}$), one sees that given $\hat{\varepsilon} > 0$,

$$\lim_{T\to\infty} \int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt \geq \widetilde{V}(x) - 2\hat{\varepsilon}.$$

Lastly, since this is true for all $\hat{\varepsilon} > 0$, we obtain:

Theorem 5.5.12.

$$\lim_{T\to\infty}\int_0^T l(\bar{\xi}_t,\bar{w}_t)\,dt \ge \widetilde{V}(x).$$

Combining Theorem 5.5.12 and Theorem 5.4.6, we have:

Theorem 5.5.13. *For any $x\in\mathbb{R}^n$,*

$$\lim_{T\to\infty}\int_0^T l(\bar{\xi}_t,\bar{w}_t)\,dt \ge \widehat{V}(x) - \theta(1+K_g^2)[C_1 + C_2|x|^2],$$

where C_1, C_2 are as given in (5.22), and K_g is indicated in Lemma 5.4.1, with an explicit bound given in Remark 5.4.2.

In other words, the payoff obtained with control $\bar{w}_\cdot$, based on solution of the approximating problem, will be arbitrarily close to the optimal payoff, $\widehat{V}(x)$. Further, the bound on the difference, $\theta C_3(1+|x|^2)$, goes to zero as $\theta\to 0$, where θ parameterizes the closeness of $\widetilde{H}$ to the originating Hamiltonian, H.

5.6 Numerical Examples

We include two examples validating the bounds obtained above. More specifically, we validate the error bounds of Theorems 5.4.6 and 5.5.13.

5.6.1 Simplest example

The simplest possible example is to approximate a quadratic Hamiltonian for a problem in one dimension by another quadratic Hamiltonian. In such a case, one can validate the results analytically. The PDE we choose to approximate is

$$0 = -H(x,\nabla V) = -\left(\frac{1}{2}Dx^2 + \frac{1}{2}\Sigma(\nabla V)^2 + Ax\nabla V\right)$$

with $A=-1$, $D=1$, $\Sigma=0.9$. Without loss of generality, we may let $\sigma=1$, in which case one has $\gamma=\sqrt{\sigma^2/\Sigma}=1.0541$.

The associated value function (PDE solution) is quadratic and given by $\widehat{V}(x) = \frac{1}{2}Px^2$, where P is the smallest nonnegative solution to $A'P+PA+D+P\Sigma P=0$. Hence $P=(-A-\sqrt{A^2-\Sigma D})/\Sigma = 0.7597$.

We approximate the Hamiltonian in our above PDE crudely by another quadratic Hamiltonian. In particular, we take

$$-\widetilde{H}(x,p) = -\left(\frac{1}{2}(D-\epsilon)x^2 + \frac{1}{2}(\Sigma-\epsilon)p^2 + Axp\right).$$

The associated value function is $\widetilde{V}(x) = \frac{1}{2}x'\widetilde{P}x$, where $\widetilde{P}$ is the smallest nonnegative solution of $A'\widetilde{P} + \widetilde{P}A + (D-\epsilon) + \widetilde{P}(\Sigma-\epsilon)\widetilde{P} = 0$. With $\epsilon = 0.2$, one obtains $\widetilde{P} = 0.5$.

In order to validate the bounds in Theorems, 5.4.6 and 5.5.13, we need to compute values for θ, C_1, C_2 and K_g. We will obtain C_1, C_2 and K_g from (5.12) and (5.22). To obtain these values, we note immediately that one has $d_\sigma = ||\sigma||_2 = 1$, $c = -\lambda_{\max}[A] = 1$ and $\alpha = \lambda_{\max}[D] = 1$. (Here, we again use the notation $\lambda_{\max}[X] = \max_i \mathbf{Re}(\lambda_i[X])$ and $\lambda_{\min}[X] = \min_i \mathbf{Re}(\lambda_i[X])$, where the $\lambda_i[X]$ are the eigenvalues of X.) To choose lower bounds, we need highest possible $\bar{\delta}$ satisfying (5.7), that is $\frac{1}{2}Px^2 \le c\frac{\gamma^2-\bar{\delta}^2}{2d_\sigma^2}|x|^2$. Hence we take $\bar{\delta} = \sqrt{\gamma^2 - \frac{1}{c}\lambda_{\max}(P)d_\sigma^2} = 0.3746$. With these coefficient values, we find

$$K_g = \frac{1}{2}\lambda_{\max}[(\Sigma-\epsilon)^{-1}A] + \sqrt{\frac{\lambda_{\max}[A(\Sigma-\epsilon)^{-1}A-(D-\epsilon)]}{\lambda_{\min}[\Sigma-\epsilon]}} = 0.2333,$$

$$C_1 = 2d_\sigma^2/(\bar{\delta}c) = 5.34$$

and

$$C_2 = \frac{d_\sigma^2}{\bar{\delta}c}\left[\left(\frac{2\alpha}{c^2} + \frac{\gamma^2}{d_\sigma}\right) + \frac{1}{c}\right] = 10.41\,.$$

One can also easily verify that

$$\theta \doteq \max_{x,p} \frac{H(x,p) - \widetilde{H}(x,p)}{|x|^2 + |p|^2} = \frac{\epsilon}{2} = 0.1\,.$$

We now define the normalized value function error, V_{err} as

$$V_{err}(x) \doteq \frac{\widehat{V}(x) - \widetilde{V}(x)}{\theta(1+K_g^2)(C_1 + C_2|x|^2)}. \tag{5.41}$$

If the bound in Theorem 5.4.6 is correct, then this should lie in $[0,1]$ for all $x \in \mathbb{R}^n$. Using the values computed above for all the terms on the right-hand side of (5.41), we have

$$V_{err}(x) = \frac{0.2597|x|^2}{0.5630 + 1.098|x|^2},$$

and by inspection, we see that this lies in $[0, 1]$ for all x.

The approximate optimal feedback control is $\hat{w}(x) = \frac{1}{\gamma^2}\sigma'(x)\tilde{p}$, where $\tilde{p} \in \mathcal{P}^0(x; \widetilde{V}) = \operatorname{argmax}_{p \in D^- \widetilde{V}(x)} H(x, p)$. Hence $\hat{w}(x) = \frac{\sigma}{\gamma^2}\widetilde{P}x = 0.556x$. The dynamics corresponding to this approximately optimal control is

$$\dot{\bar{\xi}}_t = f(\bar{\xi}_t, \hat{w}(\bar{\xi}_t)) = g(\bar{\xi}_t) + \sigma(\bar{\xi}_t)\hat{w}(\bar{\xi}_t) = (A + \Sigma\widetilde{P})\bar{\xi},$$

and the solution is simply $\bar{\xi}_t = e^{(A+\Sigma\widetilde{P})t}x_0$. With the above problem data, $A + \Sigma\widetilde{P} = -0.55$, and so this solution is $\bar{\xi}_t = e^{-0.55t}x_0$. The associated running cost is

$$l(\bar{\xi}_t, \bar{w}_t) = L(\bar{\xi}_t) - \frac{\gamma^2}{2}|\bar{w}_t|^2 = L(\bar{\xi}_t) - \frac{\gamma^2}{2}|\hat{w}(\bar{\xi}_t)|^2 = \frac{1}{2}\bar{\xi}_t^T(D - \widetilde{P}\Sigma\widetilde{P})\bar{\xi}_t.$$

Integrating this along the approximate optimal trajectory, we find the approximate optimal payoff for a trajectory starting at $\bar{\xi}_0 = x_0$ to be

$$\begin{aligned} P_{app}(x_0) &\doteq \lim_{T \to \infty} \int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt = \frac{1}{2}x_0^2 \int_0^\infty e^{2(A+\Sigma\widetilde{P})t}\, dt \\ &= \frac{1}{2}x_0^2 \frac{(\widetilde{P}\Sigma\widetilde{P} - D)}{2(A + \Sigma\widetilde{P})} = \frac{1}{2}0.7045|x_0|^2. \end{aligned}$$

We define the normalized payoff error, P_{err}, to be

$$P_{err}(x_0) \doteq \frac{\widehat{V}(x_0) - P_{app}(x_0)}{\theta(1 + K_g^2)(C_1 + C_2|x_0|^2)}. \tag{5.42}$$

By Theorem 5.5.13, this should lie in $[0, 1]$ for all $x \in \mathbb{R}^n$. Substituting in the above data, we find

$$P_{err}(x_0) = \frac{0.0552|x_0|^2}{0.5630 + 1.098|x_0|^2},$$

which clearly lies in $[0, 1]$ for all x.

Note that as proven in Theorem 5.5.12, in this example we found $P_{app} \geq \widetilde{V}$ and hence $P_{err} \geq V_{err}$. That is, the suboptimality of the approximate payoff is less than the error in the approximate value function. This is expected intuitively since in calculating P_{app}, we have access not only to the approximate value function, $\widetilde{V}(x)$, but to the *exact* dynamics and running cost.

5.6.2 Four Hamiltonians and four-dimensional state space

We now verify our bounds in a more complex case, with four constituent quadratic forms in the approximating Hamiltonian. In order to make this verification tractable, we again approximate only a simple PDE with a quadratic Hamiltonian, and consequently, have an analytical expression for the true solution, $\widehat{V}$. Again, it would make no sense to use our methods on a problem with known solution, but we find it useful to do so as a means for verifying the results obtained above. The four-dimensional approximating Hamiltonian will be constructed from the following building blocks. We take

$$A_a = \begin{bmatrix} -1 & .5 \\ .1 & -1 \end{bmatrix}, \qquad A_b = A_a,$$

$$A_c = A_a, \qquad A_d = \begin{bmatrix} -1 & .5 \\ .3 & -1 \end{bmatrix},$$

$$\Sigma_a = \begin{bmatrix} 1.08 & -.04 \\ -.04 & 1.08 \end{bmatrix}, \qquad \Sigma_b = \Sigma_a,$$

$$\Sigma_c = \Sigma_a, \qquad \Sigma_d = \Sigma_a,$$

and

$$D_a = \begin{bmatrix} 1.5 & .2 \\ .2 & 1.5 \end{bmatrix}, \qquad D_b = 1.4 \times D_a,$$

$$D_c = 1.6 \times D_a, \qquad D_d = 1.8 \times D_a.$$

We will use a parameter to adjust the interaction in the dynamics across the dimensions, and this will be $\eta = -0.01$. Now we are ready to define each of the Hamiltonians. The true PDE is given by

$$0 = -H(x, \nabla V) = -\left(\frac{1}{2} x' D x + \frac{1}{2} (\nabla V)' \Sigma \nabla V + (Ax)' \nabla V \right)$$

where the coefficients in the Hamiltonian are

$$A = \begin{bmatrix} A_d & \eta I \\ \eta I & A_d \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \Sigma_d & 0 \\ 0 & \Sigma_d \end{bmatrix}$$

$$D = \begin{bmatrix} D_d & 0 \\ 0 & D_d \end{bmatrix}.$$

Recall that we solve the approximating HJB PDE, $0 = -\widetilde{H}(x, \nabla V)$ where in our example here, the H^m take the specific form

$$H^m(x,p) = \tfrac{1}{2}x' D^m x + \tfrac{1}{2}p'\Sigma^m p + (A^m x)' p + \alpha^m,$$

that is, with $l_1^m, l_2^m = 0$. We take $M = 4$, i.e., $\widetilde{H}(x,p) = \max_{m \in \{1,2,3,4\}} H^m(x,p)$. For the first Hamiltonian, H^1, we let

$$A^1 = \begin{bmatrix} A_a & \eta I \\ \eta I & A_a \end{bmatrix}, \qquad \Sigma^1 = \begin{bmatrix} \Sigma_a & 0 \\ 0 & \Sigma_a \end{bmatrix}$$
$$D^1 = \begin{bmatrix} D_a & 0 \\ 0 & D_a \end{bmatrix}, \qquad \alpha^1 = 0\,.$$

For the second Hamiltonian, H^2, we let

$$A^2 = \begin{bmatrix} A_b & \eta I \\ \eta I & A_b \end{bmatrix}, \qquad \Sigma^2 = \begin{bmatrix} \Sigma_b & 0 \\ 0 & \Sigma_b \end{bmatrix}$$
$$D^2 = \begin{bmatrix} D_b & 0 \\ 0 & D_b \end{bmatrix}, \qquad \alpha^2 = -1\,.$$

For the third Hamiltonian, H^3, we let

$$A^3 = \begin{bmatrix} A_c & \eta I \\ \eta I & A_c \end{bmatrix}, \qquad \Sigma^3 = \begin{bmatrix} \Sigma_c & 0 \\ 0 & \Sigma_c \end{bmatrix}$$
$$D^3 = \begin{bmatrix} D_c & 0 \\ 0 & D_c \end{bmatrix}, \qquad \alpha^3 = -3.$$

For the fourth Hamiltonian, H^4, we let

$$A^4 = \begin{bmatrix} A_d & \eta I \\ \eta I & A_d \end{bmatrix}, \qquad \Sigma^4 = \begin{bmatrix} \Sigma_d & 0 \\ 0 & \Sigma_d \end{bmatrix}$$
$$D^4 = \begin{bmatrix} D_d & 0 \\ 0 & D_d \end{bmatrix}, \qquad \alpha^4 = -8\,.$$

We solve $0 = -\widetilde{H}(x, \nabla V)$ to obtain $\widetilde{V}$, using the curse-of-dimensionality-free algorithm [MDG08]–[McE05]. The time-step size used was $d_t = 0.3$ and the time horizon was taken to be $T = 4$ (c.f. [McE06]). (The computation required approximately 20 minutes on a typical desktop machine.) As the value is a function on $\mathbb{R}^4$, we cannot display it. For illustrative purposes, we plot information on the (x_3, x_4) plane (passing through the origin). Of course, as we are using a numerical method, we are only computing an approximation to $\widetilde{V}$, and we denote the approximate solution by $\bar{V}$. The backsubstitution error is given by $e(x) = \widetilde{H}(x, \nabla\bar{V}(x))$. The backsubstitution error and the optimal controller switching schedule on this plane are plotted in Figures 5.1 and 5.2. One can see that the solution converged within the domain of interest.

A heuristic measure of the error in gradient can developed using Newton's method as follows. Given $x \in \Re^n$, let $\bar{m} = \operatorname{argmax}_m H^m(x, \nabla\bar{V})$. If $\tilde{p} = \nabla\widetilde{V}(x)$ and $\bar{p} = \nabla\bar{V}(x)$ (where we are assuming existence of the gradients), $0 = \widetilde{H}(x, \tilde{p}) \approx \widetilde{H}(x, \bar{p}) + (\tilde{p} - \bar{p})'\nabla_p\widetilde{H}(x, \bar{p}) = e - (\Sigma^{\bar{m}}\bar{p} + A^{\bar{m}}x)$. To obtain our heuristic gauge of accuracy, we choose the nearest p, $\hat{p}$, which is the solution of linearized approximation to the quadratic Hamiltonian, and use this as an approximation for $\tilde{p}$. That is, we take

$$\hat{p} \doteq \operatorname*{argmin}_p \left\{ |p - \bar{p}| : e - (\Sigma^{\bar{m}}\bar{p} + A^{\bar{m}}x)'(p - \bar{p}) = 0 \right\}.$$

This can solved using the pseudo-inverse, yielding

$$\tilde{p} - \bar{p} \approx \hat{p} - \bar{p} = (\Sigma_{\bar{m}}\bar{p} + A_{\bar{m}}x)^{-1}e.$$

The relative error in the gradient, $\frac{|\tilde{p}-\bar{p}|}{\bar{p}}$, is plotted in Figure 5.3, and we see that it is everywhere less than 10%, and generally much smaller. This indicates that our following check on the bound of Theorem 5.4.6 is valid.

As in the previous example, we obtain C_1, C_2 and K_g from (5.12) and (5.22), and obtain $K_g = 7.07$, $C_1 = 1.07$ and $C_2 = 15.03$. We find a value for θ satisfying

$$H(x, p) - \theta(|x|^2 + |p|^2) \leq \widetilde{H}(x, p) \leq H(x, p)$$

by a rough search. In particular, we see that $\theta = 0.66$ satisfies the condition. As in the previous example, we compute V_{err} from (5.41), although in this case, we use the computationally obtained $\bar{V}$ in place of $\widetilde{V}$. Again, Theorem 5.4.6 is validated if $V_{err}(x) \in [0, 1]$ for all x. This was verified, and in particular, we plot the value of V_{err} on the (x_3, x_4) plane in Figure 5.4. We also compute the resulting suboptimality in the payoff (see (5.42)) on the same plane in Figure 5.5. In this case, for the values of $P_{app}(\cdot)$ at each point, x_0, are obtained by integrating along the trajectory, starting at x_0, using a control based on the gradient of the approximating value function. In particular, at each point along the trajectory, this gradient is obtained as the exact gradient of the quadratic which is maximal at that point.

Chapter 5, in part, is a reprint of the material as it appears in the international symposium on Mathematical Theory of Networks and Systems, (MTNS) 2008, McEneaney, William; Deshpande, Ameet, Birkhäuser Press [MD08b]. The same chapter, in full, has been submitted for publication as it may appear in SIAM Journal of Control and Optimization, McEneaney, William; Deshpande, Ameet, SIAM Press [MDb]. The dissertation author was the coauthor of these papers.

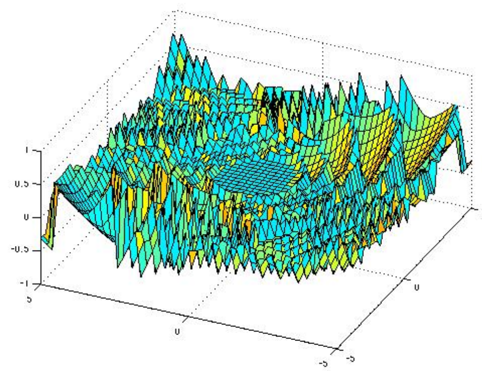


Figure 5.1: backsubstitution error along 3-4 plane

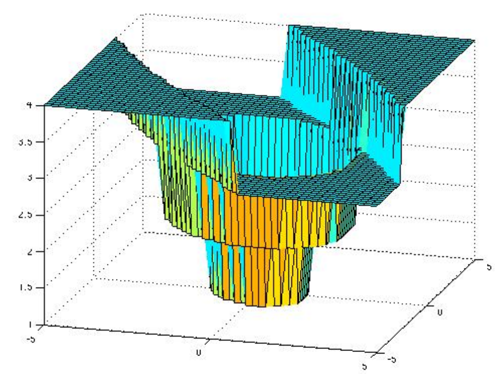


Figure 5.2: optimal switching along 3-4 plane

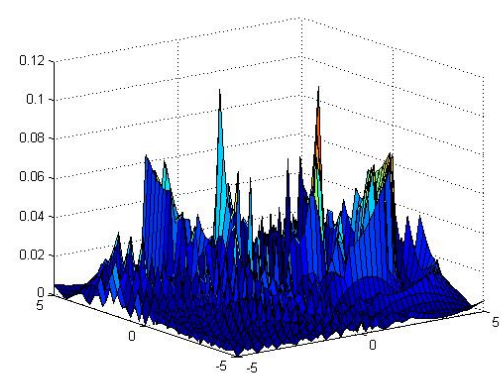


Figure 5.3: relative gradient error along 3-4 plane

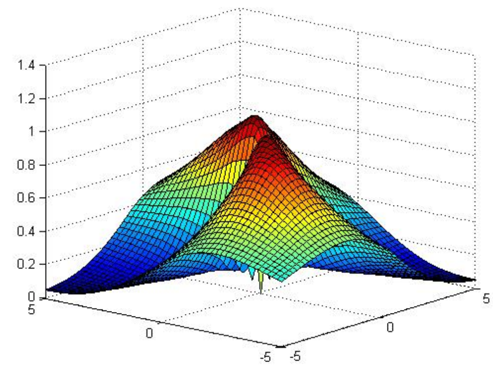


Figure 5.4: Normalized Value function error along 3-4 plane

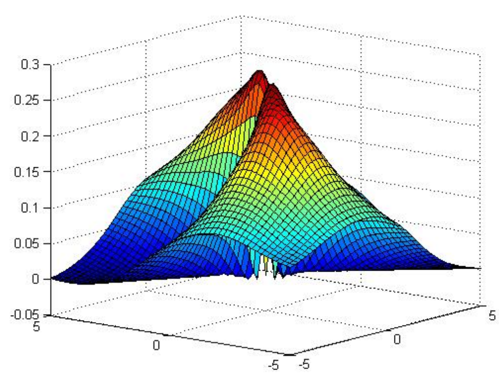


Figure 5.5: Normalized Payoff function error along 3-4 plane

Chapter 6

A Curse-of-Dimensionality-Free Numerical Method based on Min-Plus algebra

6.1 Introduction

In chapter 3, we reviewed the curse-of-dimensionality-free method developed to solve a certain class of semiconvex Hamilton-Jacobi-Bellman (HJB) PDEs (cf. [McE07], [MK], [McE09], [MDG08], [McE06]). These PDEs arise in the infinite horizon problems with stable dynamics and unbounded control and state space, in which we aim to maximize a payoff. In this method, the Hamiltonian is approximated by the max-plus sum (pointwise maximum) of many linear/quadratic Hamiltonians, and the infinite horizon value function is obtained as the limit of finite horizon problem as the time grows. This method surprisingly is not subject to the curse-of-dimensionality. In fact, the computational growth in state-space dimension is of the order of n^3. There is however no *free lunch*, since the curse-of-dimensionality is replaced by the curse-of-complexity. Various pruning methods developed to attenuate this curse were reviewed in chapter 4.

In this chapter, we shall extend the above method to solve a semiconcave HJB PDE, which can be approximated as the min-plus sum (pointwise minimum) of linear/quadratic Hamiltonians. These arise in infinite horizon problems with unbounded control and state space, and in which we aim to minimize an integral cost. In such problems, the nominal dynamics does have a fixed point at origin, but the stability need not be assumed, making it more general.

In the section 6.3, drawing on the previous work in [Lio00] and [BL97], we shall prove the existence and uniqueness of the value function within a certain class of viscosity solutions of the HJB PDE. Section 6.6 shall discuss the min-plus fundamental solution, and derive the same for linear/quadratic PDEs. In section 6.5, we shall prove some important theorems regarding the validity of the discrete time approximation for propagation in finding the value function. The main algorithm will be covered in section 6.7. Lastly we shall examine a sample numerical example to test our algorithm in section 6.8.

6.2 The Problem

We consider the problem with dynamics

$$\begin{aligned} \dot{\xi}_t &= f(\xi_t) + \sigma(\xi_t) u_t \\ \xi_0 &= x \in \mathbb{R}^n \end{aligned} \tag{6.1}$$

where the state is $\xi_t \in \mathbb{R}^n$, the control u lies in $L_2^{\mathrm{loc}} \doteq \{u : [0,\infty) \to \mathbb{R}^m : u \in L_2[0,T] \quad \forall T < \infty\}$, f represents nominal dynamics, and σ is $n \times m$ matrix valued multiplier on the control.

The optimal control objective is to minimize the integral cost

$$V(x) \doteq \inf_{u \in L_2^{\mathrm{loc}}} \sup_{T<\infty} \int_0^T l(\xi_t) + \frac{1}{2}|u_t|^2 \, dt. \tag{6.2}$$

V is the value function of the problem. For the existence and regularity of the value function, we make following assumptions on the dynamics and the running cost. The proof of existence is similar to that of lemma 1 in [CM04]. Hence we shall not reproduce it.

We assume that there exists $c_1, c_2, c_3 \in (0, \infty)$ such that for all $x \in \mathbb{R}^n$,

$$\begin{aligned} &(x-y)'\,(f(x)-f(y))\,| \le c_1|x-y|^2 \\ &|\sigma(x)-\sigma(y)| \le c_1|x-y| \\ &|\sigma(x)| \le c_2 \\ \text{and } &f(0)=0 \end{aligned} \tag{6.3}$$

Regarding the running cost $l(x)$, we assume that

$$\begin{aligned} &l(\cdot) \text{ is nonnegative and continuous in } \mathbb{R}^n \\ &l(0)=0 \quad \text{and} \quad c_3|x|^2 \le l(x) \end{aligned} \tag{6.4}$$

We shall prove that the HJB PDE for this problem which the value function satisfies in the viscosity sense is as follows:

$$\begin{aligned} 0 &= -\inf_{u_t \in \mathbb{R}^m} \left\{ (f(x)+\sigma(x)u)\cdot\nabla V + l(x) + \frac{1}{2}|u_t|^2 \right\} \\ &= -\left[f(x)\cdot\nabla V + l(x) - \nabla V' \left(\frac{\sigma(x)\sigma'(x)}{2} \right) \nabla V \right] \\ &\doteq -H(x, \nabla V) \end{aligned} \tag{6.5}$$

In next section, we prove that it is the unique viscosity solution in a certain class of functions.

6.3 Existence and Uniqueness

As our assumptions (6.3), (6.4) satisfy the assumptions of the theorem 3.4 in Da Lio [Lio00], following theorem holds true.

Theorem 6.3.1. *Assume (6.3) and (6.4). With the control* $u \in L^\infty \doteq L^\infty([0,\infty), \mathbb{R}^n)$*, and with the dynamics same as (6.1), the value function*

$$V^\infty(x) \doteq \inf_{u \in L^\infty} \sup_{T<\infty} \int_0^T l(\xi_t) + \frac{1}{2}|u_t|^2\, dt \tag{6.6}$$

exists and is the unique viscosity solution of the HJB PDE (6.5) in the class

$$\mathcal{K} \doteq \{W \in C(\mathbb{R}^n) : W \text{ is nonnegative, } W(0)=0 \text{ and } ||W||_R < +\infty, \forall R > 0\} \tag{6.7}$$

where $||W||_R \doteq \sup\{|W(x)| + |q| : x \in \bar{B}(0,R), q \in D^- W(x)$. *Also if* $U(x,t)$ *is the solution to the finite horizon Lagrange problem*

$$U^\infty(x,T) \doteq \inf_{u \in L^\infty} \int_0^T l(\xi_t) + \frac{1}{2}|u_t|^2 \, dt, \tag{6.8}$$

then for all $x \in \mathbb{R}^n$, $U^\infty(x,T)$ *increases monotonically with* T *and*

$$\lim_{T \to \infty} U^\infty(x,T) = V^\infty. \tag{6.9}$$

Now, we shall expand the control space to $L_2^{\text{loc}} \supseteq L^\infty([0,\infty), \mathbb{R}^n)$, and prove that the value function remains the same and is thus the unique viscosity solution of (6.5) in the class $\mathcal{K}$. First we shall prove an useful lemma.

Lemma 6.3.2. *Assuming (6.3), (6.4),* $\xi_0 = x$ *and* $u \in L_2^{loc}$, *we have following bounds on the system trajectory.*

$$|\xi_t - x|^2 \le e^{(2c_1 + c_2 + 1)t} \left(|f(x)|^2 t + c_2 \int_0^t |u_s|^2 \, ds \right) \tag{6.10}$$

$$|\xi_t|^2 \le e^{(2c_1 + c_2)t} |x|^2 + c_2 \int_0^t e^{(2c_1 + c_2)(t-s)} |u_s|^2 \, ds \tag{6.11}$$

Proof. By (6.1), ξ_t is an absolutely continuous function of t. Hence if we define $R \doteq \int_0^t |\xi_s - x|^2 \, dt$, then by the fundamental theorem of calculus, for all $t \ge 0$, $\dot{R}(t) = |\xi_t - x|^2$, and we have

$$\begin{aligned}
\dot{R}(t) &= 2 \int_0^t (\xi_s - x)' \left(f(\xi_s) + \sigma(\xi_s) u_s \right) \, ds \\
&= 2 \int_0^t (\xi_s - x)' \left(f(\xi_s) - f(x) + f(x) + \sigma(\xi_s) u_t \right) \, ds
\end{aligned}$$

using (6.3), $(\xi_s - x)'(f(\xi_s) - f(x)) \le c_1 |\xi_s - x|^2$,

$$\le 2c_1 \int_0^t |\xi_s - x|^2 \, ds + 2 \int_0^t |\xi_s - x||f(x)| \, ds + 2c_2 \int_0^t |\xi_s - x||u_s| \, ds$$

using $2ab \le |a|^2 + |b|^2$

$$\begin{aligned}
&\le 2c_1 \int_0^t |\xi_s - x|^2 \, dt + \int_0^t |\xi_s - x|^2 \, ds + |f(x)|^2 t + c_2 \left[\int_0^t |\xi_s - x|^2 \, ds + \int_0^t |u_s|^2 \, ds \right] \\
&\le (2c_1 + c_2 + 1) \int_0^t |\xi_s - x|^2 \, ds + |f(x)|^2 t + c_2 \int_0^t |u_s|^2 \, ds \\
&\le (2c_1 + c_2 + 1) R(t) + |f(x)|^2 t + c_2 \int_0^t |u_s|^2 \, ds
\end{aligned} \tag{6.12}$$

Solving this ODI, we find

$$R(t) \leq \frac{1}{2c_1+c_2+1}\int_0^t \left(e^{(2c_1+c_2+1)(t-s)}-1\right)\left(|f(x)|^2+c_2|u_s|^2\right)\,ds$$
$$\leq \frac{1}{2c_1+c_2+1}\left(e^{(2c_1+c_2+1)t}-1\right)\left(|f(x)|^2 t + c_2\int_0^t |u_s|^2\,ds\right) \tag{6.13}$$

Substituting in (6.12)

$$|\xi_t - x|^2 = \dot{R}(t) \leq e^{(2c_1+c_2+1)t}\left(|f(x)|^2 t + c_2\int_0^t |u_s|^2\,ds\right)$$

which gives us (6.10).

To prove (6.11), we define $Q(t) \doteq \int_0^t |\xi_s|^2\,ds$. Therefore, $\dot{Q}(t) = |\xi_t|^2$, and

$$\dot{Q}(t) = |x|^2 + 2\int_0^t \xi_s'(f(\xi_s)+\sigma(\xi_s)u_s)\,ds$$

using (6.3), $(\xi-0)'(f(\xi)-f(0)) \leq c_1|\xi|^2$, $|\sigma(\xi)| \leq c_2$ and $2\xi' u \leq |\xi|^2 + |u^2|$,

$$\leq |x|^2 + (2c_1+c_2)\int_0^t |\xi_s|^2\,ds + c_2\int_0^t |u_s|^2\,ds$$
$$\leq |x|^2 + (2c_1+c_2)Q(t) + c_2\int_0^t |u_s|^2\,ds \tag{6.14}$$

Solving this ODI, we find

$$Q(t) \leq \frac{c_2}{2c_1+c_2}\int_0^t \left(e^{(2c_1+c_2)(t-s)}-1\right)|u_s|^2\,ds + \frac{|x|^2}{2c_1+c_2}\left(e^{(2c_1+c_2)t}-1\right) \tag{6.15}$$

Substituting in (6.14),

$$|\xi_t|^2 = \dot{Q}(t) \leq e^{(2c_1+c_2)t}|x|^2 + c_2\int_0^t e^{(2c_1+c_2)(t-s)}|u_s|^2\,ds \tag{6.16}$$

which is (6.11). Hence proved.

□

Theorem 6.3.3. *Assuming (6.3)and (6.4), let V and V^∞ be defined as per (6.2) and (6.6) respectively. Then*

$$V(x) = V^\infty(x) \tag{6.17}$$

Hence by Theorem 6.3.1, V is the unique viscosity solution of the HJB PDE (6.5) in the class $\mathcal{K}$ given by (6.7).

Proof. Since $L^\infty \subseteq L_2^{\text{loc}}$, from (6.2) and (6.6), it is clear that

$$V(x) \leq V^\infty. \tag{6.18}$$

Now we shall prove the other inequality.

From the definition of $V(x)$ and non-negativity of the integrand, for any $\epsilon > 0$, there exists control $u^\epsilon \in L_2^{\text{loc}}$ and the corresponding trajectory ξ_t^ϵ, such that

$$V(x) + \epsilon \geq \sup_{T<\infty} \int_0^T l(\xi_t^\epsilon) + \frac{1}{2}|u_t^\epsilon|^2\, dt = \int_0^\infty l(\xi_t^\epsilon) + \frac{1}{2}|u_t^\epsilon|^2\, dt \tag{6.19}$$

Note that by (6.4) $l(\xi_t) \geq 0$. Hence using (6.19), $V(x) + \epsilon \geq \int_0^\infty \frac{1}{2}|u_t^\epsilon|^2\, dt$. Thus $u^\epsilon \in L_2 \doteq L_2([0,\infty), \mathbb{R}^m)$. For such $u^\epsilon \in L_2$ and for all $k \in \mathbb{N}$, define truncated controls

$$u_t^k = u^{\epsilon,k}(t) \doteq \begin{cases} \min\left(1, \frac{k}{|u_t^\epsilon|}\right) u_t^\epsilon & \text{if } u_t^\epsilon \neq 0 \\ 0 & \text{if } u_t^\epsilon = 0 \end{cases}$$

where we dropped the ϵ superscript, for the convenience of notation.

Note that for all $k \in \mathbb{N}$, $u^k \in L^\infty([0,\infty), \mathbb{R}^m)$. Also for all $t \in [0,\infty)$, the series u_t^k is monotonically increasing in k and converge to u_t^ϵ. Thus $\lim_{k\to\infty} u_t^k = u_t^\epsilon$ for all $t \in [0,\infty)$. Note that since $u^\epsilon \in L_2$, $\int_0^T |u_t^k|^2\, dt \leq \int_0^T |u_t^\epsilon|^2\, dt \leq V(x) + \epsilon$, for all k and T. By letting $T \to \infty$ and the Dominated Convergence Theorem, $u^k \to u^\epsilon$ in L_2 norm.

Let $\xi^k(\cdot)$ and $\xi^\epsilon(\cdot)$ be the trajectories corresponding to the controls u^k and u^ϵ respectively. Now, using Lemma 6.3.2, both $|\xi_t^k|$ and $|\xi_t^\epsilon|$ are bounded in $[0,T]$ by a constant, independent of k. Now we prove that ξ_t^k converges to ξ_t^ϵ locally uniformly in $[0,\infty)$. The following estimates hold:

$$\begin{aligned}
|\xi_t^\epsilon - \xi_t^k| &= \left|\int_0^t f(\xi_s^\epsilon) - f(\xi_s^k)\, ds\right| + \left|\int_0^t \sigma(\xi_s^\epsilon)u_s^\epsilon - \sigma(\xi_s^k)u_s^k\, ds\right| \\
&\leq \left|\int_0^t f(\xi_s^\epsilon) - f(\xi_s^k)\, ds\right| + \left|\int_0^t \sigma(\xi_s^\epsilon)u_s^k - \sigma(\xi_s^k)u_s^k\, ds\right| \\
&\quad + \left|\int_0^t \sigma(\xi_s^\epsilon)u_s^\epsilon - \sigma(\xi_s^\epsilon)u_s^k\, ds\right| \\
&\leq \int_0^t c_1(1 + |u_s^k|)|\xi_s^\epsilon - \xi_s^k|\, ds + \int_0^t |\sigma(\xi_s^\epsilon)| \left|u_s^k - u_s^\epsilon\right|\, ds
\end{aligned}$$

Since $(u^\epsilon - u^k) \in L_2$, define $\beta_k \doteq ||u^\epsilon - u^k||_2 \doteq \left(\int_0^\infty \left|u_s^k - u_s^\epsilon\right|^2 ds\right)^{\frac{1}{2}}$. Using $|\sigma(\cdot)| \leq c_2$ from (6.3) and the Holder inequality,

$$\leq \int_0^t c_1(1+|u_s^k|)|\xi_s^\epsilon - \xi_s^k|\, ds + c_2\sqrt{t}\beta_k$$

using Gronwall's inequality and $|u_t^k| \leq |u_t^\epsilon|$ and $t \leq T$

$$\begin{aligned}&\leq c_2\beta_k\left(\sqrt{t} + c_1\int_0^t (1+|u_t^\epsilon|)\exp\left(c_1\int_s^t (1+|u_p^\epsilon|)\,dp\right)\sqrt{s}\,ds\right)\\&\leq c_2\beta_k\left(\sqrt{T} + c_1\exp\left(c_1 T + c_1\sqrt{T}||u^\epsilon||_2\right)\left(\frac{2}{3}T^{3/2} + \frac{T}{\sqrt{2}}||u^\epsilon||_2\right)\right)\end{aligned} \tag{6.20}$$

where $||u^\epsilon||_2 \doteq \left(\int_0^\infty |u_t^\epsilon|^2\, dt\right)^{\frac{1}{2}}$. Note that since $u^\epsilon \in L_2([0,\infty),\mathbb{R}^m)$, $||u^\epsilon||_2 < \infty$. Hence the expression in parentheses in (6.20) is finite. Since $u^k \to u^\epsilon$ in L_2, $\beta_k \to 0$. Thus $|\xi_t^\epsilon - \xi_t^k| \to 0$ uniformly in $[0,T]$ for any $T < \infty$. Now using (6.19), we have:

$$\begin{aligned}V(x) + \epsilon &\geq \int_0^\infty l(\xi_t^\epsilon) + \frac{1}{2}|u_t^\epsilon|^2\, dt\\&\geq \int_0^T l(\xi_t^k) + \frac{1}{2}|u_t^k|^2\, dt - \int_0^T \left|l(\xi_t^k) - l(\xi_t^\epsilon)\right|\, dt\end{aligned}$$

Using $u_t^k \in L^\infty$ and (6.8),

$$\geq U^\infty(x,T) - \int_0^T \left|l(\xi_t^k) - l(\xi_t^\epsilon)\right|\, dt$$

which is true for all $k \in \mathbb{N}$ and $T < \infty$. Since $|\xi_t^\epsilon|$ and $|\xi_t^k|$ are bounded for $t \in [0,T]$, let the bound be C. Since $l(\cdot)$ is continuous over the compact set $\bar{B}(0,C)$, it is uniformly continuous by Heine-Cantor theorem. Thus $\exists\delta > 0$ such that for $y,z \in \bar{B}(0,C)$, if $|y-z| \leq \delta$ then $|l(y) - l(z)| \leq \epsilon/T$. Since, $\xi^k \to \xi^\epsilon$ locally uniformly, we can choose $k = k_T \in \mathbb{N}$ such that $|\xi_t^{k_T} - \xi_t^\epsilon| \leq \delta$ for $t \in [0,T]$. For such k_T, $|l(\xi_t^k) - l(\xi_t^\epsilon)| \leq \epsilon/T$ for all $t \in [0,T]$. Hence,

$$\geq U^\infty(x,T) - \epsilon$$

Finally letting $T \to \infty$ and using (6.9),

$$\geq V^{\infty} - \epsilon \tag{6.21}$$

Since ϵ is arbitrary, (6.18) and (6.21) give us the result.

□

Remark 6.3.4. With $U^{\infty}(x,T)$ as defined in (6.8), if we define

$$U(x,T) \doteq \inf_{u \in L_2^{\text{loc}}} \int_0^T l(\xi_t) + \frac{1}{2}|u_t|^2 \, dt, \tag{6.22}$$

then by a very similar proof as above, we can show that $U(x,T) = U^{\infty}(x,T)$. Thus using (6.22), (6.9) and Theorem 6.3.3, we have

$$\lim_{T \to \infty} U(x,T) = \lim_{T \to \infty} U^{\infty}(x,T) = V^{\infty}(x) = V(x) \tag{6.23}$$

Now, we shall prove the uniqueness of the value function V, as the fixed point of the associated semigroup. First we prove a simple lemma.

Lemma 6.3.5. *Let u^{ϵ} be the ϵ optimal control for the problem (6.2), and ξ^{ϵ} be the corresponding state trajectory. Then*

$$\lim_{t \to \infty} \xi_t^{\epsilon} = 0 \tag{6.24}$$

Proof. Since u^{ϵ} is ϵ optimal,

$$V(x) + \epsilon \geq \int_0^{\infty} l(\xi_t^{\epsilon}) + \frac{1}{2}|u_t^{\epsilon}|^2 \, dt$$

using (6.4),

$$\geq c_3 \int_0^{\infty} |\xi_t^{\epsilon}|^2 \, dt + \frac{1}{2} \int_0^{\infty} |u_t^{\epsilon}|^2 \, dt \tag{6.25}$$

Since $\int_0^{\infty} |\xi_t^{\epsilon}|^2 \, dt < \infty$, we have

$$\liminf_{t \to \infty} |\xi_t^{\epsilon}| = 0. \tag{6.26}$$

Now choose any $\delta > 0$. Since $\int_0^{\infty} |u_t^{\epsilon}|^2 \, dt < \infty$, there exist $T_1 < \infty$ such that $\int_{T_1}^{\infty} |u_t^{\epsilon}|^2 \, dt \leq \frac{\delta}{c_2}$ and since $\int_0^{\infty} |\xi_t^{\epsilon}|^2 \, dt < \infty$, there exists $T_2 < \infty$ such that

$\int_{T_2}^{\infty} |\xi_t^\epsilon|^2\,dt \le \frac{\delta}{2c_1+c_2}$. By (6.26), there exists $T_3 \ge \max(T_1, T_2)$ such that $|\xi_{T_3}^\epsilon| \le \delta$. Hence by shifting time by T_3 in (6.14), for all $t \ge T_3$,

$$\begin{aligned} |\xi_t^\epsilon|^2 &\le |\xi_{T_3}^\epsilon|^2 + (2c_1+c_2)\int_{T_3}^{t} |\xi_s^\epsilon|^2\,ds + c_2\int_{T_3}^{t} |u_s^\epsilon|^2\,ds \\ &\le |\xi_{T_3}^\epsilon|^2 + (2c_1+c_2)\int_{T_3}^{\infty} |\xi_s^\epsilon|^2\,ds + c_2\int_{T_3}^{\infty} |u_s^\epsilon|^2\,ds \\ &\le \delta^2 + 2\delta \end{aligned}$$

Thus $|\xi_t^\epsilon| \le \sqrt{\delta^2+2\delta}$ for all $t > T_3$. Hence $\limsup_{t\to\infty} |\xi_t^\epsilon| \le \sqrt{\delta^2+2\delta}$ for any $\delta > 0$. Since δ is arbitrary and using (6.26),

$$\limsup_{t\to\infty} |\xi_t^\epsilon| = \liminf_{t\to\infty} |\xi_t^\epsilon| = \lim_{t\to\infty} |\xi_t^\epsilon| = 0.$$

Hence proved. □

Now for any $T > 0$, define the semigroup

$$\mathcal{S}_T[\phi](x) \doteq \inf_{u\in L_2^{\mathrm{loc}}} \sup_{T<\infty} \left[\int_0^T l(\xi_t) + \frac{1}{2}|u_t|^2\,dt + \phi(\xi_T)\right] \tag{6.27}$$

where ξ_t evolves as per (6.1), with $\xi_0 = x$. It is well-known ([Mas87] and many others) that this $\mathcal{S}_T$ form a semigroup which is linear under min-plus algebra (in which $a \otimes b \doteq a + b$ and $a \oplus b \doteq \min(a,b)$). That is, if $a, b \in \mathbb{R}$ and $\phi_1(\cdot)$ andf $\phi_2(\cdot)$ are functions over $\mathbb{R}^n$,

$$\begin{aligned} \mathcal{S}_{T_1}\mathcal{S}_{T_2} &= \mathcal{S}_{T_1+T_2} \\ \mathcal{S}_T\left[a\otimes\phi_1 \oplus b\otimes\phi_2\right] &= a\otimes\mathcal{S}_T[\phi_1] \oplus b\otimes\mathcal{S}_T[\phi_2] \end{aligned} \tag{6.28}$$

Using this notation, we have the next theorem:

Theorem 6.3.6. *For any $T \ge 0$ and for all $x \in \mathbb{R}^n$, the value function V as defined in (6.2), satisfies*

$$V(x) = \mathcal{S}_T[V](x) \doteq \inf_{u\in L_2^{loc}} \sup_{T<\infty} \left[\int_0^T l(\xi_t) + \frac{1}{2}|u_t|^2\,dt + V(\xi_T)\right] \tag{6.29}$$

Proof. This is the celebrated *dynamic programming principle*. The proof is entirely standard, hence we do not include it here. □

Theorem 6.3.7. *Assuming (6.3), (6.4), for any $T > 0$, the value function V as defined in (6.2), is the unique solution of*

$$W = \mathcal{S}_T[W] \tag{6.30}$$

in the class $\mathcal{K}$ as defined in (6.7). Further, given any $W \in \mathcal{K}$,

$$\lim_{k\to\infty} \mathcal{S}_{kT}[W](x) = V(x) \tag{6.31}$$

for all $x \in \mathbb{R}^n$ locally uniformly (uniformly on the compact sets).

Proof. Theorem 6.3.6 proved that $V \in \mathcal{K}$ is the solution of (6.30). Now suppose that another function $W \in \mathcal{K}$ solves (6.30). Using semigroup property for all $k \in \mathbb{N}$, we get

$$\mathcal{S}_{kT}[W] = \mathcal{S}_{(k-1)T}\mathcal{S}_T[W] = \mathcal{S}_{(k-1)T}\ldots\mathcal{S}_T[W] = W$$

Taking limit as $k \to \infty$,

$$W(x) = \lim_{n\to\infty} \mathcal{S}_{nT}[W](x)$$

since $W \in \mathcal{K}$, $W(\cdot) \geq 0$,

$$\geq \lim_{n\to\infty} \mathcal{S}_{nT}[0](x)$$

note that from (6.27) and (6.22), $\mathcal{S}_{nT}[0] = U(x, nT)$. Now using (6.23),

$$= \lim_{n\to\infty} U(x, nT) = V(x) \tag{6.32}$$

Now we wish to prove the reverse inequality. Choose any $\epsilon > 0$, and consider ϵ optimal controls u^ϵ, for the problem (6.2) and the corresponding trajectory ξ^ϵ. Note that, since W is continuous and $W(0) = 0$, there exists $\delta > 0$, such that $|W(x)| < \epsilon$ for any $|x| \in B(0,\delta)$. Also using Lemma 6.3.5, $\lim_{t\to\infty} \xi_t^\epsilon = 0$. Thus we can choose $k^\delta \in \mathbb{N}$ such that for all $t \geq k^\delta T$, $|\xi_t^\epsilon| \leq \delta$. Using this, we have

$$\begin{aligned} W(x) = \mathcal{S}_{k^\delta T}[W](x) &\leq \int_0^{k^\delta T} \left(l(\xi_t^\epsilon) + \frac{1}{2}|u_t^\epsilon|^2\, dt \right) + W(\xi_{k^\delta T}^\epsilon) \\ &\leq \int_0^{\infty} \left(l(\xi_t^\epsilon) + \frac{1}{2}|u_t^\epsilon|^2\, dt \right) + \epsilon \\ &\leq V(x) + 2\epsilon \end{aligned} \tag{6.33}$$

Since ϵ is arbitrary, (6.32) and (6.33) give us $V = W$. Thus V is the unique fixed point of (6.30) in class $\mathcal{K}$.

Now we shall prove that V is the unique limit of the semigroup iterations on any function in class $\mathcal{K}$. Let $\phi \in \mathcal{K}$. Given any $\epsilon > 0$, there exists $\delta > 0$ such that $\phi(x) < \epsilon$ for all $x \in B(0, \delta)$. Consider again ϵ optimal controls u^ϵ for the problem (6.2), and the trajectory ξ^ϵ. Since $\xi^\epsilon_t \to 0$, there exists $k_\delta \in \mathbb{N}$ such that for all $t \geq k_\delta T$, $|\xi_t| \leq \delta$, and we have:

$$\begin{aligned}\mathcal{S}_{kT}[\phi](x) &\leq \int_0^{kT} l(\xi^\epsilon_t) + \frac{1}{2}|u^\epsilon_t|^2 \, dt + \phi(\xi^\epsilon_{kT}) \\ &\leq \int_0^{\infty} l(\xi^\epsilon_t) + \frac{1}{2}|u^\epsilon_t|^2 \, dt + \epsilon \\ &\leq V(x) + 2\epsilon\end{aligned}$$

Letting $k \to \infty$, and since ϵ is arbitrary,

$$\limsup_{k \geq k_\delta} \mathcal{S}_{kT} \leq V(x). \tag{6.34}$$

Also since $\phi(x) \geq 0$, $\mathcal{S}_{kT}[\phi] \geq \mathcal{S}_{kT}[0]$. Letting $k \to \infty$ and using (6.23),

$$\liminf_{k \in \mathbb{N}} \mathcal{S}_{kT}[\phi](x) \geq \lim_{k \to \infty} \mathcal{S}_{kT}[0](x) = \lim_{k \to \infty} U(x, kT) = V(x) \tag{6.35}$$

Using (6.34) and (6.35), we have the result (6.31).

The proof of the uniform convergence over compact sets is along similar lines but slightly technical. Given $\epsilon > 0$ and $R > 0$, for any $x \in \bar{B}(0, R)$, let u^ϵ and ξ^ϵ be ϵ optimal control and trajectory for problem (6.2). Due to nonnegativity and continuity of V at 0, we can choose $\epsilon \geq \delta_\epsilon > 0$ such that for all $|x| \leq \delta_\epsilon$, $0 \leq V(x) < \epsilon$. Now define

$$T_\epsilon \doteq \frac{\max_{|x| \leq R} V(x) + \epsilon}{c_3 \delta_\epsilon^2} \tag{6.36}$$

We claim that there exists $T \in [0, T_\epsilon]$ such that $|\xi_T| \leq \delta_\epsilon$. This is because, supposing otherwise $|\xi_t| > \delta_\epsilon$ for all $t \in [0, T_\epsilon]$ and

$$\begin{aligned}\max_{|x| \leq R} V(x) + \epsilon \geq V(x) + \epsilon &\geq \int_0^\infty l(\xi^\epsilon_t) + \frac{1}{2}|u^\epsilon_t|^2 \, dt \\ &\geq \int_0^{T_\epsilon} l(\xi^\epsilon_t) \, dt \geq \int_0^{T_\epsilon} c_3 |\xi^\epsilon_t|^2 \, dt > T_\epsilon c_3 \delta_\epsilon^2 = \max_{|x| \leq R} V(x) + \epsilon\end{aligned}$$

which is a contradiction.

Now note that ϵ optimal controls are also optimal for for any state along the trajectory. Using (6.2) and (6.29)

$$V(x) \leq \int_0^T l(\xi_s^\epsilon) + \frac{1}{2}|u_s^\epsilon|^2\, ds + V(\xi_T^\epsilon) \leq \int_0^\infty l(\xi_s^\epsilon) + \frac{1}{2}|u_s^\epsilon|^2\, ds$$
$$\leq V(x) + \epsilon \leq \int_0^T l(\xi_s^\epsilon) + V(\xi_T^\epsilon) + \epsilon$$

Thus subtracting $\int_0^T l(\xi_s^\epsilon)$ from all terms,

$$V(\xi_T^\epsilon) \leq \int_T^\infty l(\xi_s^\epsilon) + \frac{1}{2}|u_s^\epsilon|^2\, ds \leq V(\xi_T^\epsilon) + \epsilon \tag{6.37}$$

Hence

$$V(\xi_T^\epsilon) + \epsilon \geq \int_T^\infty l(\xi_s^\epsilon) + \frac{1}{2}|u_s^\epsilon|^2\, ds \geq c_3 \int_T^\infty |\xi_s^\epsilon|^2 + \frac{1}{2}\int_T^\infty |u_s^\epsilon|^2\, ds \tag{6.38}$$

Now time shifting (6.12) by T, we have for any $t \geq T_\epsilon \geq T$,

$$|\xi_t^\epsilon|^2 \leq |\xi_T^\epsilon + (2c_1 + c_2)\int_T^t |\xi_s^\epsilon|^2\, ds + c_2 \int_T^t |u_s^\epsilon|^2\, ds$$

using (6.37)and (6.38),

$$\leq \delta_\epsilon^2 + \frac{2c_1 + c_2}{c_3}(V(\xi_T^\epsilon) + \epsilon) + 2c_2(V(\xi_T^\epsilon) + \epsilon)$$

Now using $|\xi_T^\epsilon| \leq \delta_\epsilon \leq \epsilon$ and hence $V(\xi_T^\epsilon) < \epsilon$,

$$\leq \epsilon^2 + 2\epsilon\frac{2c_1 + c_2}{c_3} + 4\epsilon c_2 \doteq \omega_1(\epsilon) \tag{6.39}$$

where ω_1 is a modulus function.

Now lets define

$$\omega_\phi(\epsilon) \doteq \max_{|y| \leq \omega_1(\epsilon)} \phi(y). \tag{6.40}$$

It can be easily seen that this is also a modulus, due to continuity of ϕ at 0, and since $\phi(0) = 0$.

Thus for all $t \geq T_\epsilon$ and all $x \in \bar{B}(0, R)$

$$\mathcal{S}_t[\phi](x) \leq \int_0^t l(\xi_s^\epsilon) + \frac{1}{2}|u_s^\epsilon|^2\, ds + \phi(\xi_t^\epsilon)$$

by (6.39) and (6.40),

$$\begin{aligned}
&\leq \int_0^\infty l(\xi_s^\epsilon) + \frac{1}{2}|u_s^\epsilon|^2\, ds + \max_{|y|\leq\omega_1(\epsilon)} \phi(y) \\
&\leq V(x) + \epsilon + \omega_\phi(\epsilon) \qquad (6.41)
\end{aligned}$$

Now we shall prove reverse inequality in an analogous manner. Let T_ϵ be defined as per (6.36). Let us redefine u^ϵ and ξ^ϵ to be ϵ optimal control and trajectory for the finite horizon problem (6.22) with horizon T_ϵ. Note that using the semigroup notation $U(x, T_\epsilon)$ defined in (6.22), can be re-expressed as $\mathcal{S}_{T_\epsilon}[0](x)$. As before, we claim that for some $T \in [0, T_\epsilon]$, $|\xi_T^\epsilon| \leq \delta_\epsilon$, since if it is not true, then

$$\begin{aligned}
\max_{|x|\leq R} V(x) + \epsilon \geq V(x) + \epsilon \geq \mathcal{S}_{T_\epsilon}[0](x) + \epsilon &\geq \int_0^{T_\epsilon} l(\xi_t^\epsilon) + \frac{1}{2}|u_t^\epsilon|^2\, dt \\
&\geq \int_0^{T_\epsilon} c_3|\xi_t^\epsilon|^2\, dt > T_\epsilon c_3 \delta_\epsilon^2 = \max_{|x|\leq R} V(x) + \epsilon
\end{aligned}$$

which is a contradiction.

Hence for any $t \geq T_\epsilon$ and for all $x \in \bar{B}(0, R)$, using monotonicity of $\mathcal{S}_t[0]$ with t, $T \leq T_\epsilon$ and $|V(\xi_T^\epsilon| \leq \delta_\epsilon$,

$$\begin{aligned}
\mathcal{S}_t[\phi](x) + \epsilon \geq \mathcal{S}_t[0](x) + \epsilon \geq \mathcal{S}_{T_\epsilon}[0](x) + \epsilon &\geq \int_0^{T_\epsilon} l(\xi_s^\epsilon) + \frac{1}{2}|u_s^\epsilon|^2\, ds \\
&\geq \int_0^{T} l(\xi_s^\epsilon) + \frac{1}{2}|u_s^\epsilon|^2\, ds \qquad (6.42) \\
&\geq V(x) - V(\xi_T) \\
&\geq V(x) - \epsilon
\end{aligned}$$

Using (6.41) and (6.42) we have for all $t \geq T_\epsilon$ and for all $x \in \bar{B}(0, R)$,

$$|\mathcal{S}_t[\phi](x) - V(x)| \leq 2\epsilon + \omega_\phi(\epsilon). \qquad (6.43)$$

Thus we have uniform convergence over compact sets. □

6.4 Approximation with LQ Hamiltonians

To solve the HJB PDE (6.5), we shall approximate the Hamiltonian by the min-plus summation of linear-quadratic (LQ) Hamiltonians. Note that this vastly expands the class of approximation, since minimum of LQ functions can approximate any semiconcave Hamiltonian. Specifically, we consider:

$$\begin{aligned} 0 = -H(x, \nabla V) \approx -\widetilde{H}(x, \nabla V) &\doteq - \inf_{m \in \{1,2,\ldots,M\}} H^m(x, \nabla V) \\ V(0) = 0 & \end{aligned} \tag{6.44}$$

with each H^m is linear-quadratic,

$$H^m(x,p) = \frac{1}{2}x'C^m x - \frac{1}{2}p'\Sigma^m p + (A^m x)'p + x' l_1^m + p' l_2^m + \frac{1}{2}\alpha^m, \tag{6.45}$$

where C^m, Σ^m are $n \times n$ symmetric matrices, $l_1^m, l_2^m \in \mathbb{R}^n$ and $\alpha^m \in \mathbb{R}$. $\widetilde{H}$ is associated with an optimal control problem for a switched linear system. For such problem the dynamics is,

$$\dot{\xi}_t = f^{\mu_t}(\xi_t) + \sigma^{\mu_t} u_t, \quad \xi_0 = x \tag{6.46}$$

where $f^{\mu_t} \doteq A^{\mu_t}\xi_t + l_2^{\mu_t}$ and the corresponding value function is

$$\begin{aligned} \widetilde{V}(x) &= \inf_{u \in L_2^{\text{loc}}} \inf_{\mu \in \mathcal{D}_\infty} \sup_{T<\infty} \widetilde{J}(x, T; u, \mu) \\ &\doteq \inf_{u \in L_2^{\text{loc}}} \inf_{\mu \in \mathcal{D}_\infty} \sup_{T<\infty} \int_0^T l^{\mu_t}(\xi_t) + \frac{1}{2}|u_t|^2 \, dt \end{aligned} \tag{6.47}$$

where

$$\begin{aligned} \mathcal{M} &\doteq \{1, 2, \ldots, M\} \text{ and } \sigma^i \text{ are such that } \Sigma^i = \sigma^i \sigma^{i\prime} \text{ for } i \in \mathcal{M} \\ l^{\mu_t}(\xi_t) &\doteq \frac{1}{2}\xi' C^{\mu_t}\xi + \xi' l_1^{\mu_t} + \frac{1}{2}\alpha^{\mu_t}, \\ \mathcal{D}_\infty &\doteq \{\mu : [0, \infty) \to \mathcal{M} : \text{ measurable } \}. \end{aligned}$$

In forthcoming development, we shall see that $\widetilde{V}$ is also the viscosity solution of $0 = \widetilde{H}(x, \nabla \widetilde{V})$. When the approximation (6.44) is exact, then $V = \widetilde{V}$. We make the following assumptions on such an approximation for further analysis.

1. Assume that the underlying system dynamics and the payoff functions satisfy (6.3), (6.4). Hence the value function V defined by (6.2) is a unique viscosity solution, of $0 = -H(x, \nabla V)$ in class $\mathcal{K}$, as defined in (6.7).

2. With $\widetilde{H}$ defined as per (6.44), assume that $\widetilde{H}(x, p) = \inf_m H^m(x, p) \geq H(x, p)$ for all $x, p \in \mathbb{R}^n$.

3. Assume that there exist $\bar{c}_1, \bar{c}_2 \in (0, \infty)$ such that $|A^m| \leq c_1$, $|\sigma^m| \leq c_2$ for all $m \in \mathcal{M} \doteq \{1, 2, \ldots, M\}$. (6.48)

4. Assume that $H^1(x, p)$ has coefficients satisfying the following: $l_1^1 = l_2^1 = 0$; $\alpha_1 = 0$.

5. Assume that system (6.46) is controllable for any $\mu_t = m \in \mathcal{M}$. Thus the pair (A^m, σ^m) is controllable for each $m \in \mathcal{M}$.

Specifically in (6.48).3, if $\#\mathcal{M} < \infty$, we can choose $\bar{c}_1 = \max_{m \in \mathcal{M}} |A^m|$ and $\bar{c}_2 = \max_{m \in \mathcal{M}} |\sigma^m|$. Also note that one consequence of (6.48).1, (6.48).2 , (6.45) and (6.4) is that

$$|l^m(x)| = |H^m(x, 0)| \geq \widetilde{H}(x, 0) \geq H(x, 0) = l(x) \geq c_3 |x|^2 \tag{6.49}$$

Now we *assume* the following proposition and proceed. It runs parallel to the development for max-plus curse-of-dimensionality-free method covered in [McE06], [McE07].

Proposition 6.4.1. *Assuming (6.48), value function* $\widetilde{V}$ *as defined in (6.47), is the viscosity solution to* $0 = -\widetilde{H}(0, \nabla \widetilde{V})$*, where* $\widetilde{H}$ *is defined as per (6.44), (6.45). Further* $\widetilde{V} \in \mathcal{K}$*, where* $\mathcal{K}$ *is defined as per (6.7).*

Now we state the dynamic programming principle for this problem without proof, which is entirely standard.

Theorem 6.4.2. *Assume (6.48) and value function $\widetilde{V}$ as defined in (6.47). Then for all $x \in \mathbb{R}^N$ and $t > 0$,*

$$\widetilde{V}(x) = \inf_{u \in L_2^{loc}} \inf_{\mu \in \mathcal{D}_\infty} \left\{ \int_0^t l^{\mu_t}(\xi_t) + \frac{1}{2}|u_t|^2 \, dt + \widetilde{V}(\xi_t) \right\} \tag{6.50}$$

Now we propose two bounds on the trajectories of such systems.

Lemma 6.4.3. *Assuming (6.48) and the dynamics (6.46), if , we have following bounds on the system trajectory.*

$$|\xi_t - x|^2 \leq e^{(2\bar{c}_1 + \bar{c}_2 + 1)t} \left(\bar{c}_1^2 |x|^2 t + \bar{c}_2 \int_0^t |u_s|^2 \, ds \right) \tag{6.51}$$

$$|\xi_t|^2 \leq e^{(2\bar{c}_1 + \bar{c}_2)t} |x|^2 + \bar{c}_2 \int_0^t e^{(2\bar{c}_1 + \bar{c}_2)(t-s)} |u_s|^2 \, ds \tag{6.52}$$

Proof. For each m, with $f^m(x) \doteq A^m x + l_2^m$ and $\sigma^m(x) \doteq \sigma^m$, it is easy to see that $f^m(\cdot), \sigma^m(\cdot), \bar{c}_1, \bar{c}_2$ satisfy the assumptions (6.3). Now, replacing $f(\xi)$ and $\sigma(\xi)$ with f^{μ_t} and σ^{μ_t} respectively in the proof of lemma 6.3.2 with other minor modifications gives us the above. □

Theorem 6.4.4. *Assume (6.48) and proposition 6.4.1. With underlying dynamics (6.46), define*

$$\widetilde{U}(x, T) \doteq \inf_{\mu \in \mathcal{D}_\infty} \inf_{u \in \mathbb{R}^n} \left\{ \int_0^T l^{\mu_t}(\xi_t) + \frac{1}{2}|u_t|^2 \, dt \right\} \tag{6.53}$$

Then $\lim_{T \to \infty} \widetilde{U}(x, T) = \widetilde{V}(x)$.

Proof. Since $l^{\mu_t}(x) \geq c_3 |x|^2$ by (6.49), it is easy to see that $\widetilde{U}(x, T)$ is monotonically increasing with T, and are bounded above by $\widetilde{V}(x) \in \mathcal{K}$, hence they converge to a limit, $\widetilde{U}(x, \infty) \leq \widetilde{V}(x)$. Now we wish to prove that $\widetilde{V}(x) \leq \widetilde{U}(x, \infty)$ which proves the theorem.

Since $\widetilde{V} \in \mathcal{K}$, $\widetilde{V}$ is continuous and $\widetilde{V}(0) = 0$. Thus for any $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that for any $\xi \in B(0, \delta_\epsilon)$, $|\widetilde{V}(\xi)| < \epsilon$. Since $\widetilde{U}(x, T) \uparrow \widetilde{U}(x, \infty)$, there exists $T_\epsilon > \frac{\widetilde{V}(x) + \epsilon}{c_3 \delta_\epsilon^2}$ such that

$$\widetilde{U}(x, \infty) \geq \widetilde{U}(x, T_\epsilon) \geq \widetilde{U}(x, \infty) - \epsilon. \tag{6.54}$$

Further there exist controls μ^ϵ, u^ϵ and corresponding trajectory ξ^ϵ such that

$$\widetilde{U}(x, T_\epsilon) + \epsilon \geq \int_0^{T_\epsilon} l^{\mu_t^\epsilon}(\xi_t^\epsilon) + \frac{1}{2}|u_t^\epsilon|^2 \, dt \tag{6.55}$$

Since $T_\epsilon > \frac{\widetilde{V}(x)+\epsilon}{c_3\delta_\epsilon^2}$, there exists $\bar{T} \in [0, T_\epsilon]$ such that $|\xi^\epsilon_{\bar{T}}| \le \delta_\epsilon$. This is so because, assuming otherwise, leads to following contradiction.

$$\widetilde{V}(x) + \epsilon \ge \widetilde{U}(x, T_\epsilon) + \epsilon \ge \int_0^{T_\epsilon} l^{\mu^\epsilon_t}(\xi^\epsilon_t) + \frac{1}{2}|u^\epsilon_t|^2 \, dt > \int_0^{\frac{\widetilde{V}(x)+\epsilon}{c_3\delta_\epsilon^2}} c_3\delta_\epsilon^2 \, dt \ge \widetilde{V}(x) + \epsilon$$

Consequently, $\widetilde{V}(\xi^\epsilon_{\bar{T}}) < \epsilon$. Now using the DPP (6.4.2), and (6.54), (6.54), we get

$$\begin{aligned}
\widetilde{V}(x) &\le \int_0^{\bar{T}} l^{\mu^\epsilon_t}(\xi^\epsilon_t) + \frac{1}{2}|u^\epsilon_t|^2 \, dt + \widetilde{V}(\xi^\epsilon_{\bar{T}}) \\
&\le \int_0^{\bar{T}} l^{\mu^\epsilon_t}(\xi^\epsilon_t) + \frac{1}{2}|u^\epsilon_t|^2 \, dt + \epsilon \\
&\le \int_0^{T_\epsilon} l^{\mu^\epsilon_t}(\xi^\epsilon_t) + \frac{1}{2}|u^\epsilon_t|^2 \, dt + \epsilon \\
&\le \widetilde{U}(x, T_\epsilon) + 2\epsilon \\
&\le \widetilde{U}(x, \infty) + 2\epsilon
\end{aligned}$$

Since ϵ is arbitrary, $\widetilde{V}(x) \le \widetilde{U}(x, \infty)$. Hence proved. □

With $\xi_0 = x$ and dynamics (6.46), define the semigroup,

$$\widetilde{\mathcal{S}}_T[\phi](x) \doteq \inf_{u \in L_2^{\text{loc}}} \inf_{\mu \in \mathcal{D}_T} \left[\int_0^T l^{\mu_t}(\xi_t) + \frac{1}{2}|u_t|^2 \, dt + \phi(\xi_T) \right] \tag{6.56}$$

where

$$\mathcal{D}_T \doteq \{\mu : [0, T] \to \mathcal{M} : \text{ measurable}\}. \tag{6.57}$$

Now we need a lemma to prove uniqueness of $\widetilde{V}$ as the unique fixed point of the semigroup operation. Then we have the following.

Theorem 6.4.5. *Assume (6.48) and proposition 6.4.1. With the underlying dynamics (6.46), fix any $T > 0$. Value function $\widetilde{V}$ is the unique continuous solution of*

$$W = \widetilde{\mathcal{S}}_T[W]$$

in the class $\mathcal{K}$. Further given any $W \in \mathcal{K}$, $\lim_{T\to\infty} \widetilde{\mathcal{S}}[W](x) = \widetilde{V}(x)$ for all $x \in \mathbb{R}^n$, uniformly on compact sets.

Proof. Using the trajectory bounds from the Lemma 6.4.3, we can show that the ϵ optimal trajectory for $\widetilde{V}$ defined in (6.47) goes to 0. The proof is very similar to that of Lemma 6.3.5. Using this, and the lemmas 6.4.3, 6.4.4, the proof follows almost verbatim to the proof of the Theorem 6.3.7. Hence we not reproduce it here.

□

6.5 Discrete Time Approximation

Since we seek to avoid curse-of-dimensionality, we can not discretize over space. Instead discretization will be over time where approximate μ processes will be constant over the length of each time-step.

We define operator $\bar{\mathcal{S}}_\tau$ on $\mathcal{K}$ by

$$\begin{aligned} \bar{\mathcal{S}}_\tau &\doteq \inf_{u\in L_2^{\mathrm{loc}}} \inf_{m\in\mathcal{M}} \left[\int_0^\tau l^m(\xi_t^m + \frac{1}{2}|u_t|^2\, dt + \phi(\xi_\tau^m) \right] \\ &= \inf_{m\in\mathcal{M}} \mathcal{S}_\tau^m[\phi](x) \end{aligned} \tag{6.58}$$

where ξ^m satisfies dynamics

$$\dot{\xi}_t^m = A^m \xi_t^m + \sigma^m u_t, \quad \xi_0^m = x \tag{6.59}$$

and $\mathcal{S}_\tau^m$ is defined as

$$\mathcal{S}_\tau^m \doteq \inf_{u\in L_2^{\mathrm{loc}}} \left[\int_0^\tau l^m(\xi_t)^m + \frac{1}{2}|u_t|^2\, dt + \phi(\xi_\tau^m) \right] \tag{6.60}$$

It is useful to note that operators $\bar{\mathcal{S}}_\tau$ do not necessarily form a semigroup, although they do form a super-semigroup (i.e. $\bar{\mathcal{S}}_{\tau_1+\tau_2}[\phi] \geq \bar{\mathcal{S}}_{\tau_1}\bar{\mathcal{S}}_{\tau_2}[\phi]$). Further, one has $\mathcal{S}_\tau^m \geq \bar{\mathcal{S}}_\tau \geq \widetilde{\mathcal{S}}_\tau$ for all $m \in \mathcal{M}$.

With τ acting as a time-discretization step-size, let us define discretized switching control spaces over infinite and finite horizon.

$$\mathcal{D}_\infty^\tau \doteq \left\{ \mu : [0,\infty) \to \mathcal{M} \;\middle|\; \begin{array}{l} \text{for all } k \in \mathbb{N}, \text{ there exists } m_k \in \mathcal{M} \\ \text{such that } \mu_t = m_k \text{ for all } t \in [(k-1)\tau, k\tau) \end{array} \right\} \tag{6.61}$$

$$\mathcal{D}_T^\tau \doteq \left\{ \mu : [0,T) \to \mathcal{M} \;\middle|\; \begin{array}{l} \text{for all } k \in \mathbb{N}, k \le \lceil \frac{T}{\tau} \rceil, \text{ there exists } m_k \in \mathcal{M} \\ \text{such that } \mu_t = m_k \text{ for all } t \in [(k-1)\tau, \min(k\tau, T)) \end{array} \right\} \tag{6.62}$$

For $N \in \mathbb{N}$, let $\mathcal{M}^n$ be the outer product of $\mathcal{M}$, n times, and define

$$\bar{\bar{\mathcal{S}}}_{N\tau}^\tau[\phi](x) = \inf_{\mu \in \mathcal{D}_\infty^\tau} \inf_{u \in L_2^{\text{loc}}} \int_0^{N\tau} l^{\mu_t}(\xi_t) + \frac{1}{2}|u_t|^2 \, dt + \phi(\xi_{N\tau}) \tag{6.63}$$

It is easy to prove that

$$\begin{aligned} \bar{\bar{\mathcal{S}}}_{N\tau}^\tau[\phi](x) &= \min_{m_N \in \mathcal{M}} \{\mathcal{S}_\tau^{m_N}\} \, \bar{\bar{\mathcal{S}}}_{(N-1)\tau}^\tau[\phi](x) \\ &= \min_{\{m_k\}_{k=1}^N \in \mathcal{M}^N} \left\{ \prod_{k=1}^N \mathcal{S}_\tau^{m_k} \right\} [\phi](x) = \left(\bar{\mathcal{S}}_\tau\right)^N [\phi](x) \end{aligned} \tag{6.64}$$

where the $\prod$ notation indicates operator composition, and the superscript in the last expression indicates repeated application of $\bar{\mathcal{S}}_\tau$, N times.

We will be approximating $\widetilde{V}$ by solving $W = \bar{\mathcal{S}}_\tau[W]$. Consequently we will need to show that the existence and uniqueness of the solution to $W = \bar{\mathcal{S}}_\tau[W]$. We begin with existence.

Theorem 6.5.1. *Assume (6.48) and the dynamics (6.46). Let*

$$\bar{V} \doteq \inf_{\mu \in \mathcal{D}_\infty^\tau} \inf_{u \in L_2^{loc}} \int_0^\infty l^{\mu_t}(\xi_t) + \frac{1}{2}|u_t|^2 \, dt \tag{6.65}$$

for all $x \in \mathbb{R}^n$. Then $\bar{V}$ satisfies

$$\bar{V} = \bar{\mathcal{S}}_\tau[\bar{V}] \quad \text{and } \bar{V}(0) = 0 \tag{6.66}$$

Further, $0 \le \widetilde{V} \le \bar{V}$.

Proof. Choose $\mu_t = 1$ for all t. By the controllability of (A^1, σ^1), as per (6.48), there exists controls $\bar{u}_t$ under which the trajectory satisfies $\xi_0 = x$ and $\xi_T = 0$ for some $T > 0$. Let $\bar{u}_t = 0$ for $t > T$. Since $l_1^1 = l_2^1 = 0$ and $\alpha_1 = 0$, as per (6.48), $\xi_t = 0$ for $t > T$ and

$$\bar{V}(x) \leq J(x, \bar{u}, \infty) = \int_0^\infty l^{\mu_t}(\xi_t) + \frac{1}{2}|\bar{u}_t|^2 \, dt < \infty$$

Hence $\bar{V}$ exists and is locally bounded.

Since $\mathcal{D}_\infty^\tau \subseteq \mathcal{D}_\infty$ and $\widetilde{V} \in \mathcal{K}$, we have $0 \leq \widetilde{V} \leq \bar{V}$. Also the dynamic programming principle for the discrete time switching problem, which we state without proof, gives us:

$$\bar{V}(x) \doteq \inf_{\mathcal{D}_\infty^\tau} \inf_{\bar{u} \in L_2^{\text{loc}}} \left\{ \int_0^\tau l^{\mu_t}(\xi_t) + \frac{1}{2}|\bar{u}_t|^2 \, dt + \bar{V}(\xi_\tau) \right\} = \bar{\mathcal{S}}_\tau[\bar{V}](x) \tag{6.67}$$

Finally from the (6.48), at $x = 0$, $0 \leq \widetilde{V}(x) = \bar{V}(0) \leq J(0, \bar{u}, \infty) = 0$. Hence $\bar{V}(0) = 0$.

□

Now we *assume* the following lemma and proceed. Note that it is the discrete time counterpart of lemma 6.55.

Proposition 6.5.2. *Assume (6.48) and the dynamics (6.46). Define*

$$\bar{U}(x, N\tau) \doteq \inf_{\mu \in \mathcal{D}_\infty^\tau} \inf_{u \in \mathbb{R}^n} \left\{ \int_0^{N\tau} l^{\mu_t}(\xi_t) + \frac{1}{2}|u_t|^2 \, dt \right\} \tag{6.68}$$

Then $\lim_{N\to\infty} \bar{U}(x, N\tau) = \bar{V}(x)$.

Theorem 6.5.3. *Assuming proposition 6.5.2, $\bar{V}$ is the unique nonnegative solution to (6.66). Further, given any $W \in \mathcal{K}$, $\lim_{N\to\infty} \bar{\bar{\mathcal{S}}}_{N\tau}^\tau[W](x) = \bar{V}(x)$ for all $x \in \mathbb{R}^n$, uniformly on compact sets.*

Proof. Since $\mathcal{D}_\infty^\tau \subset \mathcal{D}_\infty$, using the trajectory bounds from the Lemma 6.4.3, we can show that the ϵ optimal trajectory for $\bar{V}$ defined in (6.65) goes to 0. The proof is very similar to that of 6.3.5. Using this and the Lemmas 6.4.3, 6.5.2, the proof follows almost verbatim to the proof of the Theorem 6.3.7. Hence we do not reproduce it here. □

We now *assume* that one may approximate $\widetilde{V}$, the solution of $W = \widetilde{\mathcal{S}}_\tau[W]$ to as accurate a level as one desires by solving $W = \bar{\mathcal{S}}_\tau[W]$ for sufficiently small τ. equivalent to introducing a discrete time $\bar{\mu} \in \mathcal{D}^\tau_{N\tau}$ approximation to the μ process in $\widetilde{\mathcal{S}}_{N\tau}$.

Proposition 6.5.4. *Given $\epsilon > 0$ and $R < \infty$, there exists $\tau > 0$ such that*

$$\widetilde{V}(x) \leq \bar{V}(x) \leq \widetilde{V}(x) + \epsilon \quad \forall x \in \bar{B}(0, R) \tag{6.69}$$

6.6 Min-Plus Fundamental Solution

In section 6.4, we approximated a nonlinear Hamiltonian by the pointwise minimum of M linear quadratic Hamiltonians in (6.44). Such Hamiltonian represents a switched linear system switching between M constituent systems with linear dynamics and quadratic payoff. For such a system, in section 6.5, we simplify the switching problem still further, by time discretizing the switching control by (6.58)and (6.63). We can compute arbitrarily accurate solution of the switching problem by decreasing the discretization timestep τ and increasing the time horizon T. Although the complete error or convergence analysis is not yet carried out, it would be similar to that for the max-plus counterpart derived in [MK].

Thus we reduce the original problem to that of computing $\mathcal{S}^m_\tau[\phi]$, in which a semigroup for a linear-quadratic system (corresponding to the quadratic Hamiltonian H^m in (6.45), dynamics (6.59) and cost (6.60)) acts on a quadratic terminal cost function $\phi(x)$. Fortunately this is an analytically tractable problem, with a quadratic solution. Due to this, starting with the zero function, ϕ retains the quadratic form throughout the propagation under switching controls and the final value function is simply the max-plus sum of the propagated quadratic forms.

In [McE07], this problem was solved using semiconvex duality. Here we present another form which is more direct. It also turns out to be the max-plus fundamental solution to the underlying problem. This fundamental solution was first proposed for general nonlinear systems in [FM00], and as a special case, the closed form solution for linear time varying systems was derived in [Des] using max-plus algebra. Below is a similar analysis stemming from the min-plus analysis.

Here we derive the closed form propagation formulae for computing $\mathcal{S}_\tau^m[\phi]$. For brevity, we shall drop the superscript m from the dynamics and cost fuctions, and implicitly understand that they correspond to the m'th constituent linear-quadratic system.

First we shall define the general fundamental solution/ max-plus kernel. Consider a system starting with $\xi_0 = x$ and evolving as per

$$\dot{\xi}_s = A\xi_s + l_2 + \sigma u_s \tag{6.70}$$

and cost incurred along a particular trajectory is

$$J^m(x,u,t) \doteq \int_0^t l(\xi_s) + \frac{1}{2}|u_s|^2\,ds + \phi(\xi_t) \tag{6.71}$$

where

$$l(\xi_s) = \frac{1}{2}\xi_s' C\xi_s + \xi_s' l_1 + \frac{1}{2}\alpha. \tag{6.72}$$

and the value function for the finite horizon problem with terminal cost ϕ given by the semigroup operation $\mathcal{S}_t^m[\phi]$ is

$$V^m(x,t) \doteq \mathcal{S}_t^m[\phi](x) = \inf_{u\in L_2^{\text{loc}}} J^m(x,u,t) \tag{6.73}$$

We shall now define a *min-plus kernel* $I : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$I^t(x,y) \doteq \begin{cases} \inf_{u\in\mathcal{U}^t(x,y)} \int_0^t l(\xi_s) + \frac{1}{2}|u_s|^2\,ds & \text{if } \mathcal{U}^t(x,y) \neq \emptyset \\ \infty & \text{otherwise} \end{cases} \tag{6.74}$$

where

$$\mathcal{U}^t(x,y) \doteq \{u \in L_2(0,t) : \xi_0 = x, \xi_t = y \text{ under the dynamics (6.70)}\}$$

Note that $I^t = \infty$ indicates that it is impossible to reach y from x in time interval $(0,\tau)$ using any possible control u. Thus if the system is controllable, then $I^t(x,y) < \infty$ for all $t > 0$, and $I^0(x,y) = \infty$ for $x \neq y$, and 0 otherwise.

McEneaney and Fleming proposed the above kernel in [FM00] and [Fle03]. They also proved the following theorem.

Theorem 6.6.1. *With the dynamics (6.70) and the terminal cost $\phi(x)$, assume that the finite horizon value function $V^m(x,t) \doteq \mathcal{S}_t^m[\phi]$ as defined by (6.73), exists. Then with I^t defined as per (6.74), we have*

$$\mathcal{S}_t^m[\phi](x) = \inf_{y\in\mathbb{R}^n}(I^t(x,y)+\phi(y)) = \int_{\mathbb{R}^n}^{\oplus} I^t(x,y)\otimes\phi(y)\,dy \tag{6.75}$$

Since I^t depends only on the dynamics (6.70) and the running cost $l(x_s)+\frac{1}{2}|u_s|^2$, it is independent of the terminal payoff $\phi(x_t)$. Hence it can serve as a *Fundamental solution* and obtain $\mathcal{S}_t^m[\phi](x)$ for any $\phi(x)$ by a kernel operation.

Now we shall seek to find a closed form solution for $I^t(x,y)$. To do this we parametrize $\phi(x)$ by an additional variable z, which along with (6.77) gives us the next lemma. But first we assume the following

1. Let C, P, Σ are $n\times n$ symmetric matrices. Assume that the solution to the differential Riccati equation (DRE)

$$-\dot{P}_t = A'P_t + P_tA + C - P_t\Sigma P_t, \qquad P_T = P \tag{6.76}$$

exists for $t\in[0,T]$.

2. The dynamics (6.70), hence the pair (A,σ) is controllable. That is given $t>0$, for all $x,y\in\mathbb{R}^n$, starting from $\xi_0 = x$ there exists controls u such that $\xi_t = y$.

Let $\Sigma = \sigma\sigma'$, and let ∇_t and ∇_x denote $\partial/\partial t$ and $\partial/\partial x$ respectively. Consider the following Hamilton-Jacobi-Bellman PDE.

$$\begin{aligned}\nabla_t W^m(x,t) &= -H^m(x,\nabla_x W^m(x,t))\\ W^m(x,T) &= \phi(x)\end{aligned} \tag{6.77}$$

where

$$\begin{aligned}H^m(x,p) &\doteq \inf_{u\in\mathbb{R}^n}\left\{p'\left(Ax+l_2+\sigma u\right)+\frac{1}{2}x'Cx+x'l_1+\frac{1}{2}\alpha+\frac{1}{2}|u|^2\right\}\\ &= \frac{1}{2}x'Cx-\frac{1}{2}p'\Sigma p+p'Ax+p'l_2+x'l_1+\frac{\alpha}{2}.\end{aligned} \tag{6.78}$$

Lemma 6.6.2. *Let the terminal cost be*

$$\phi(x) = \phi^z(x) = \phi(x, z) = \frac{1}{2}x'Px + x'Sz + \frac{1}{2}z'Qz + x'a + z'b + \frac{\beta}{2}.$$

Assume that S is invertible and P is such that (6.76) holds true. Then there exists a C^∞ solution to (6.77) for all $x \in \mathbb{R}^n$ and $t \in [0, T]$, given by

$$W_z^m(x, t) = \frac{1}{2}x'P_t x + x'S_t z + \frac{1}{2}z'Q_t z + x'a_t + z'b_t + \frac{\beta_t}{2} \tag{6.79}$$

where P_t, S_t, Q_t, a_t, b_t, β_t satisfy $P_T = P$, $S_T = S$, $Q_T = Q$, $a_T = a$, $b_T = b$, and

$$\begin{aligned} -\dot{P}_t &= A'P_t + P_t A + C - P_t \Sigma P_t \\ -\dot{S}_t &= (A - \Sigma P_t)' S_t \\ -\dot{Q}_t &= -S_t' \Sigma S_t \\ -\dot{a}_t &= (A - \Sigma P_t)' a_t + l_1 + P_t l_2 \\ -\dot{b}_t &= S_t'(l_2 - \Sigma a_t) \\ -\dot{\beta} &= \alpha - a_t' \Sigma a_t + 2a_t' l_2 \end{aligned} \tag{6.80}$$

and S_t is invertible for all $t \in [0, T]$.

Proof. Existence of solution $P_t : 0 \le t \le T$ is assumed in (6.76). This combined with local boundedness and continuity of coefficients, guarantees existence of S_t, a_t, and hence of Q_t, b_t and β_t. The proof that it solves HJB is immediate by substitution in (6.77).

If we define $B_t \doteq -(A - \Sigma P_t)$, then $S_t = \Phi_B(t, T)S_T = \Phi_B(t, T)S$, where Φ_B is the state transition matrix of the system $\dot{\xi}_t = B_t \xi_t$. By Abel-Jacobi-Liouville formula

$$\det \Phi_B(t, 0) = e^{\int_T^t \mathbf{Tr} B(s)\, ds} > 0$$

Since both $\Phi_B(t, T)$ and $S_T = S$ are invertible, $S_t = \Phi_B(t, T)S_T$ is invertible as well. □

Next we have the verification theorem to connect HJB PDE solution to the control value function.

Theorem 6.6.3. *Assume (6.76). Let* $x, z \in \mathbb{R}^n$ *and* $t \in [0,T]$, $W_z^m(x,t)$ *as per (6.79) and* $V_z^m(x,t)$ *be the value function as defined in (6.73). For all* $t \in [0,T]$ *and* $u \in L_2[0, T-t]$, *one has*

$$W_z^m(x,t) \le J_z^m(x,u,T-t)$$

and $W_z^m(x,t) = J_z^m(x,\tilde{u},T-t)$, *where*

$$\tilde{u}_s = \tilde{u}(s,\tilde{\xi}_s) = -\sigma'\nabla W_z^m(s,\tilde{\xi}_s) = -\sigma'\left(P_s\tilde{\xi}_s + S_s z + a_s\right). \tag{6.81}$$

Thus using (6.79) this implies,

$$V_z^m(x,T-t) = W_z^m(x,t) = \frac{1}{2}x'P_t x + x'S_t z + \frac{1}{2}z'Q_t z + x'a_t + z'b_t + \frac{\beta_t}{2}. \tag{6.82}$$

Proof. Let ∇_t and ∇_x denote $\partial/\partial t$ and $\partial/\partial x$ respectively. Let $u \in L_2[t,T]$, and ξ be corresponding time shifted trajectory with $\xi_t = x$ and satisfying (6.70).

$$\begin{aligned} J_z^m(x,u,T-t) = &\int_t^T \left(l(\xi_s) + \frac{1}{2}|u_s|^2 + (A\xi_t + l_2 + \sigma u_s)'\nabla W_z^m(\xi_s,s) \right) ds + \phi^z(\xi_T) \\ &- \int_t^T (A\xi_s + l_2 + \sigma u_s)'\nabla W_z^m(\xi_s,s)\, ds \end{aligned}$$

which by definition of H^m

$$\begin{aligned} &\ge \int_t^T H^m(\xi_s, \nabla W_z^m(\xi_s,s))\, ds + \phi^z(\xi_T) \\ &\quad - \int_t^T (A\xi_s + l_2 + \sigma u_s)'\nabla W^z(\xi_s,s)\, ds \end{aligned}$$

which by (6.77) and (6.70)

$$\begin{aligned} &= \int_t^T \left\{ -\nabla_s W_z^m(\xi_s,s) - \dot{\xi}_s \nabla W_z(\xi_s,s) \right\} ds + \phi^z(\xi_T) \\ &= -\int_t^T \frac{d}{ds} W_z^m(\xi_s,s)\, ds + \phi^z(\xi_T) \\ &= W_z^m(x,t) - W_z^m(\xi_T,T) + \phi^z(\xi_T) \\ &= W_z^m(x,t) \end{aligned}$$

by (6.77).

Also note that in the proof, if we substitute $\tilde{u}_s = -\sigma(s)'\nabla W_z^m(\tilde{\xi}_s,s) = -\sigma(s)'\left(P_s\tilde{\xi}_s + S_s z + a_s\right)$, then we have the equality throughout, and $J_z^m(x,\tilde{u},T-t) = W_z^m(x,t) = V_z^m(x,T-t)$. Hence proved. □

Remark 6.6.4. Since the dynamics (6.70) is time-invariant, we can define

$$\widehat{V}^m(x,t) \doteq \left\{ \inf_{u\in L_2^{\text{loc}}} \int_t^T l(\xi_s,u_s) + \frac{1}{2}|u_s|^2\,ds + \phi^z(\xi_T) \;\middle|\; \begin{array}{l} \xi_0 = x \\ \xi_t \text{ evolves as per (6.70)} \end{array} \right\}$$

by shifting the time back by t

$$\begin{aligned} &= \left\{ \inf_{u\in L_2^{\text{loc}}} \int_0^{T-t} l(\xi_s,u_s) + \frac{1}{2}|u_s|^2\,ds + \phi(\xi_{T-t}) \;\middle|\; \begin{array}{l} \xi_t = x \\ \xi_t \text{ evolves as per (6.70)} \end{array} \right\} \\ &= V^m(x,T-t) = W^m(x,t) \\ &= \frac{1}{2}x'P_tx + x'S_tz + \frac{1}{2}z'Q_tz + x'a_t + z'b_t + \frac{\beta_t}{2} \end{aligned} \tag{6.83}$$

Now we prove a useful lemma towards obtaining the min-plus fundamental solution.

Lemma 6.6.5. *Consider the system trajectory $\tilde{\xi}_s$ starting from $\tilde{\xi}_t = x$ and evolving according to (6.70) under the optimal control $\tilde{u}_s = -\sigma'\left(P_s\tilde{\xi}_s + S_sz + a_s\right)$ from Theorem 6.6.3. Then for $t \le t_1 \le s_2 \le T$,*

$$S'_{t_1}\tilde{\xi}_{t_1} + Q_{t_1}z + b_{t_1} = S'_{t_2}\tilde{\xi}_{t_2} + Q_{t_2}z + b_{t_2} \tag{6.84}$$

Proof. By linear system theory, for a system evolving as per

$$\begin{aligned} \dot{\tilde{\xi}}_s &= A\tilde{x}_s + l_2 + \sigma\tilde{u}_s \\ &= A\tilde{\xi}_s + l_2 - \sigma\sigma'\left(P_s\tilde{\xi}_s + S_sz + a_s\right) \\ &= (A - \Sigma P_s)\,\tilde{\xi}_s - \Sigma(S_sz + a_s) + l_2 \end{aligned}$$

solution is given as

$$\tilde{\xi}_{t_2} = \Phi_B(t_2,t_1)\tilde{\xi}_{t_1} + \int_{t_1}^{t_2} \Phi_B(t_2,s)(-\Sigma S_sz - \Sigma a_s + l_2)\,ds \tag{6.85}$$

where $\Phi_B(t_2,t_1) = U_{t_2}U_{t_1}^{-1}$, where U_s is the solution of differential equation $\dot{U}_s = B(s)U_s$, with $B(s) = A - \Sigma P_s$.

It is well known that the state transition matrix

$$\Phi_{B(s)}(t_2,t_1) = \Phi'_{-B(s)'}(t_1,t_2)$$

now, noting from (6.80) that $\dot{S}_s = -(A - \Sigma P_s)' S_s = -B(s)' S_s$, and since S_{t_2} is invertible, we have

$$\Phi_{B(s)}(t_2, t_1) = \Phi'_{-B(s)'}(t_1, t_2) = \left(S_{t_1} S_{t_2}^{-1}\right)' = S_{t_2}^{-1\prime} S_{t_1}{}' \tag{6.86}$$

Substituting in (6.85), and noting from (6.80) that $\dot{Q}_s = S_s{}'\Sigma(s) S_s$ and $-\dot{b}_s = S_s{}'(l_2 - \Sigma a_s)$,

$$\begin{aligned}
\tilde{\xi}_{t_2} &= S_{t_2}^{-1\prime} S_{t_1}{}' \tilde{\xi}_{t_1} + S_{t_2}^{-1\prime} \int_{t_1}^{t_2} S_s{}'(-\Sigma S_s z - \Sigma a_s + l_2)\, ds \\
&= S_{t_2}^{-1\prime} S_{t_1}{}' \tilde{\xi}_{t_1} - S_{t_2}^{-1\prime} \left(\int_{t_1}^{t_2} S_s{}' \Sigma S_s \, ds \right) z - S_{t_2}^{-1\prime} \left(\int_{t_1}^{t_2} S_s{}'(\Sigma a_s - l_2)\, ds \right) \\
&= S_{t_2}^{-1\prime} S_{t_1}{}' \tilde{x}_{t_1} + S_{t_2}^{-1\prime} (Q_{t_1} - Q_{t_2})\, z + S_{t_2}^{-1\prime} (b_{t_1} - b_{t_2})
\end{aligned}$$

thus we have,

$$S_{t_2}{}' \tilde{\xi}_{t_2} + Q_{t_2} z + b_{t_2} = S_{t_1}{}' \tilde{\xi}_{t_1} + Q_{t_1} z + b_{t_1}$$

□

Remark 6.6.6. Note that $\forall z$, since S_{t_1} and S_{t_2} are invertible, (6.84) suggests a one-one and onto relation between start and end of optimal trajectories, ξ_{t_1} and ξ_{t_2}. Thus $\forall y \in \Re^n$ there exists a $x = S_{t_2}^{-1\prime} (S_{t_1}{}' y + (Q_{t_1} - Q_{t_2}) z + b_{t_1} - b_{t_2})$ such that optimal trajectory $\tilde{x}$ starting at $\tilde{x}_{t_1} = x$, ends with y. Thus every $y \in \Re^n$ is an optimal point for some initial condition.

Remark 6.6.7. Note that due to min-plus linearity (6.28), if $k \in \Re$, using (6.29),

$$\hat{V}^z(x, t) = \mathcal{S}_t[\phi^z + k](x) = \mathcal{S}_t[\phi^z](x) + k = V^z(x, t) + k$$

Thus while keeping the dynamics and the running cost the same, adding a constant to the terminal cost only shifts the value function accordingly. The gradient hence the optimal feedback control remains the same.

$$\hat{u}_t(x) = -\sigma \nabla \hat{V}^z(x, T - t) = -\sigma \nabla V^z(x, T - t) = \tilde{u}_t(x)$$

Hence the optimal trajectory, which is the solution to $\dot{\hat{x}}_t = A\hat{x}_t + l_2 + \sigma \hat{u}_t(x)$, also stays the same.

Now we shall prove another useful lemma before turning to the main result.

Lemma 6.6.8. *Assume (6.76). Given $0 \leq t_1 < t_2 \leq T$, and Q_t evolving according to (6.80) with terminal value $Q_T = Q$, then*

$$Q_{t_2} - Q_{t_1} \succ 0$$

Proof. Since we assumed (A, σ) is controllable, the following *controllability grammian* is invertible for any $0 \leq t_1 < t_2 \leq T$. Here since A is time-invariant, $\Phi_A(t_1, s) = e^{A(t_1 - s)}$.

$$\int_{t_1}^{t_2} \Phi_A(t_1, s)\sigma\sigma'\Phi_A(t_1, s)'\, dt \succ 0 \tag{6.87}$$

Thus for all $x, y \in \mathbb{R}^n$, $\exists$ control $\hat{u}_t$ such that is the trajectory $\dot{\hat{x}} = A\hat{x}_t + l_2 + \sigma\hat{u}_t$ with $\hat{x}_{t_1} = x$ satisfies $\hat{x}_{t_2} = y$.

Now we claim that system $(A - \Sigma P_t, \sigma)$ is also controllable. This is clear because by using control $\bar{u}_t = \hat{u}_t + \sigma' P_t x_t$, we can keep the system trajectory same and reach from x to y.

$$\begin{aligned}
\dot{\hat{x}} &= A\hat{x}_t + l_2 + \sigma\hat{u}_t \\
&= (A - \sigma\sigma' P_t)x_t + l_2 + \sigma\left(\hat{u}_t + \sigma' P_t x_t\right) \\
&= (A - \Sigma P_t)x_t + l_2 + \sigma\bar{u}_t
\end{aligned}$$

Hence similar to (6.87), using $B(t) = A - \Sigma P_t$ and $\sigma\sigma' = \Sigma(t)$, following controllability grammian is invertible.

$$\int_{t_1}^{t_2} \Phi_B(t_1, s)\Sigma\Phi_B(t_1, s)'\, dt \succ 0 \tag{6.88}$$

Substituting $\Phi_B(t_1, s) = S_{t_1}^{-1\prime} S_s{}'$ from (6.86),

$$\begin{aligned}
\int_{t_1}^{t_2} \Phi_B(t_2, s)\Sigma\Phi_B(t_2, s)'\, ds &= \int_{t_1}^{t_2} S_{t_1}^{-1\prime} S_s{}' \Sigma S_s S_{t_1}^{-1}\, ds \\
&= S_{t_1}^{-1\prime} \left\{ \int_{t_1}^{t_2} S_s{}' \Sigma S_s\, ds \right\} S_{t_1}^{-1} \\
&= S_{t_1}^{-1\prime} \left(Q_{t_2} - Q_{t_1}\right) S_{t_1}^{-1}
\end{aligned} \tag{6.89}$$

where in last equation, we used Q_t evolution from (6.80). Using (6.88) and since S_{t_1} is invertible by Lemma 6.6.2, we have $Q_{t_2} - Q_{t_1} \succ 0$. □

Theorem 6.6.9. *Given $x, y \in \mathbb{R}^n$ and $0 \le t_1 < t_2 \le T$, then if $\delta \doteq t_2 - t_1$,*

$$\sup_{z\in\mathbb{R}^n}\left[\widehat{V}^z(x,t_1) - \widehat{V}^z(y,t_2)\right] = I_\delta(x,y) \tag{6.90}$$

Since by (6.83), $\widehat{V}^z(x,t_1) = \frac{1}{2}x'P_{t_1}x + x'S_{t_1}z + \frac{1}{2}z'Q_{t_1}z + x'a_{t_1} + z'b_{t_1} + \frac{\beta_{t_1}}{2}$ and $\widehat{V}^z(x,t_2) = \frac{1}{2}x'P_{t_2}x + x'S_{t_2}z + \frac{1}{2}z'Q_{t_2}z + x'a_{t_2} + z'b_{t_2} + \frac{\beta_{t_2}}{2}$, the max-plus kernel $I_\delta(x,y)$ is also bivariate quadratic.

$$\begin{aligned}
I_\delta(x,y) &= \frac{1}{2}x'I_\delta^{11}x + x'I_\delta^{12}y + \frac{1}{2}y'I_\delta^{22}y + x'I_\delta^{1} + y'I_\delta^{2} + I_\delta^{0} \qquad \textit{where}\\
I_\delta^{11} &= P_{t_1} - S_{t_1}(Q_{t_1} - Q_{t_2})^{-1}{S_{t_1}}'\\
I_\delta^{12} &= S_{t_1}(Q_{t_1} - Q_{t_2})^{-1}{S_{t_2}}'\\
I_\delta^{22} &= -P_{t_2} - S_{t_2}(Q_{t_1} - Q_{t_2})^{-1}{S_{t_2}}'\\
I_\delta^{1} &= a_{t_1} - S_{t_1}(Q_{t_1} - Q_{t_2})^{-1}(b_{t_1} - b_{t_2})\\
I_\delta^{2} &= -a_{t_2} + S_{t_2}(Q_{t_1} - Q_{t_2})^{-1}(b_{t_1} - b_{t_2})\\
I_\delta^{0} &= \beta_{t_1} - \beta_{t_2} - (b_{t_1} - b_{t_2})'(Q_{t_1} - Q_{t_2})^{-1}(b_{t_1} - b_{t_2})
\end{aligned} \tag{6.91}$$

Proof. Let $\xi_{t_1} = x$. Since $\Sigma \succeq 0$ and S_t is invertible, by (6.80), $\dot{Q}_s = S_t'\Sigma(t)S_t \succeq 0$, hence $Q_{t_1} - Q_{t_2} \preceq 0$. For any $z \in \mathbb{R}^n$

$$\begin{aligned}
&\widehat{V}^z(x,t_1) - \widehat{V}^z(y,t_2)\\
&\quad = \mathcal{S}_{t_2-t_1}[\widehat{V}^z](x,t_2) - \widehat{V}^z(y,t_2)\\
&\quad = \inf_{u\in L_2[t_1,t_2]}\left\{\int_{t_1}^{t_2} l_t(\xi_t,u_t)\,dt + \widehat{V}^z(\xi_{t_2},t_2) - \widehat{V}^z(y,t_2)\right\}
\end{aligned}$$

substituting for $\widehat{V}^z(\cdot,t_2)$,

$$= \inf_{u\in L_2[t_1,t_2]}\left\{\int_{t_1}^{t_2} l_t(\xi_t,u_t)\,dt + \frac{1}{2}\xi_{t_2}'P_{t_2}\xi_{t_2} - \frac{1}{2}y'P_{t_2}y + (\xi_{t_2} - y)'(S_{t_2}z + a_{t_2})\right\}$$

Since $\mathcal{U}_{t_1}^{t_2}(x,y) = \{u \in L_2(t_1,t_2) : \xi_{t_1} = x, \xi_{t_2} = y\} \subset L_2(t_1,t_2)$, and $\forall u \in \mathcal{U}_{t_1}^{t_2}(x,y)$, $\xi_{t_2} = y$.

$$\begin{aligned}
&\le \inf_{u\in\mathcal{U}_{t_1}^{t_2}(x,y)}\left\{\int_{t_1}^{t_2} l_t(\xi_t,u_t)\,dt + \frac{1}{2}y'P_{t_2}y - \frac{1}{2}y'P_{t_2}y + (y-y)'(S_{t_2}z + a_{t_2})\right\}\\
&= \inf_{u\in\mathcal{U}_{t_1}^{t_2}(x,y)}\int_{t_1}^{t_2} l_t(\xi_t,u_t)\,dt = I_\delta(x,y)
\end{aligned} \tag{6.92}$$

Taking supremum over all $z \in \mathbb{R}^n$,

$$\sup_{z\in\mathbb{R}^n}\left[\widehat{V}^z(x,t_1)-\widehat{V}^z(y,t_2)\right] \le I_\delta(x,y) \tag{6.93}$$

Since $Q_{t_2}-Q_{t_1}\succ 0$ by 6.6.8, define $\hat{z}=(Q_{t_2}-Q_{t_1})^{-1}({S_{t_1}}'x-{S_{t_2}}'y+b_{t_1}-b_{t_2})$. Hence

$${S_{t_2}}'y+Q_{t_2}\hat{z}+b_{t_2}={S_{t_1}}'x+Q_{t_1}\hat{z}+b_{t_1}$$

Hence using (6.84) the optimal trajectory $\tilde{x}_t$ starting from $\tilde{x}_{t_1}=x$ and with terminal payoff $\widehat{V}^{\hat{z}}(\cdot,t_2)$, ends at $\tilde{x}_{t_2}=y$. Let the corresponding optimal control be $\tilde{u}_t$. Let us define $k=-\widehat{V}^{\hat{z}}(y,t_2)=-\left(\frac{1}{2}y'P_{t_2}y+y'S_{t_2}\hat{z}+\frac{1}{2}\hat{z}'Q_{t_2}\hat{z}+y'a_{t_2}+\hat{z}'b_{t_2}+\frac{\beta_{t_2}}{2}\right)$ to create a shifted terminal payoff function

$$\begin{aligned} U^{\hat{z}}_{t_2}(x)&=\widehat{V}^{\hat{z}}(x,t_2)+k=\widehat{V}^{\hat{z}}(x,t_2)-\widehat{V}^{\hat{z}}(y,t_2)\\ &=\frac{1}{2}x'P_{t_2}x-\frac{1}{2}y'P_{t_2}y+(x-y)'(S_{t_2}\hat{z}+a_{t_2})\end{aligned} \tag{6.94}$$

From remark 6.6.7, $\tilde{u}_t$, $\tilde{\xi}_t$ are also the optimal control and trajectory for the following problem with the terminal payoff $U^z_{t_2}$. Hence

$$\begin{aligned}\widehat{V}^{\hat{z}}(x,t_1)-\widehat{V}^{\hat{z}}(y,t_2)&=\left\{\inf_{u\in L_2[t_1,t_2]}\int_{t_1}^{t_2}l_t(\xi_t,u_t)\,dt+\widehat{V}^{\hat{z}}(x,t_2)\right\}-\widehat{V}^{\hat{z}}(y,t_2)\\ &=\inf_{u\in L_2[t_1,t_2]}\left\{\int_{t_1}^{t_2}l_t(\xi_t,u_t)\,dt+\widehat{V}^{\hat{z}}(x,t_2)-\widehat{V}^{\hat{z}}(y,t_2)\right\}\\ &=\inf_{u\in L_2[t_1,t_2]}\int_{t_1}^{t_2}l_t(\xi_t,u_t)\,dt+U_{t_2}(\xi_{t_2})\\ &=\int_{t_1}^{t_2}l_t(\tilde{\xi}_t,\tilde{u}_t)\,dt+U^{\hat{z}}_{t_2}(\tilde{\xi}_{t_2})\end{aligned}$$

since $U^{\hat{z}}_{t_2}(\tilde{\xi}_{t_2})=U^{\hat{z}}_{t_2}(y)=0$ from (6.94) and $\tilde{u}\in\mathcal{U}^{t_2}_{t_1}(x,y)$

$$\ge \inf_{u\in\mathcal{U}^{t_2}_{t_1}(x,y)}\int_{t_1}^{t_2}l_t(\xi_t,u_t)\,dt=I_\delta(x,y) \tag{6.95}$$

Thus we have

$$\sup_z\left[\widehat{V}^z(x,t_1)-\widehat{V}^z(y,t_2)\right]\ge\widehat{V}^{\hat{z}}(x,t_1)-\widehat{V}^{\hat{z}}(y,t_2)\ge I_\delta(x,y) \tag{6.96}$$

Hence (6.93) and (6.96) together give us (6.90) and also the following

$$\sup_{z\in\mathbb{R}^n}\left[\widehat{V}^z(x,t_1)-\widehat{V}^z(y,t_2)\right]=\widehat{V}^{\hat{z}}(x,t_1)-\widehat{V}^{\hat{z}}(y,t_2)=I_\delta(x,y) \tag{6.97}$$

with $\hat{z} = (Q_{t_2} - Q_{t_1})^{-1}(S_{t_1}'y - S_{t_2}'x + b_{t_1} - b_{t_2})$.

Substituting $\hat{z}$ in (6.97) and expanding, we get (6.91). □

6.7 The Algorithm

Now we begin the discussion of the actual algorithm. From Theorem 6.5.1, $\bar{V} \doteq \lim_{N\to\infty} \bar{\bar{\mathcal{S}}}^{\tau}_{N\tau}[0](x)$. Let $\bar{V}^0 \equiv 0$ and define for $k \geq 1$,

$$\bar{V}^k \doteq \bar{\bar{\mathcal{S}}}^{\tau}_{k\tau}[0](x), \tag{6.98}$$

Then we have $\bar{V} = \lim_{N\to\infty} \bar{V}^N(x)$. In practice, we approximate the infinite horizon by a sufficiently long finite horizon $N\tau$. The propagation from from $\bar{V}^0$ to $\bar{V}^N$ is achieved as follows.

For all $\{m_i\}_{i=1}^k \in \mathcal{M}^k$, define

$$v_{\{m_i\}_{i=1}^k}(x) \doteq \prod_{i=1}^{k} \mathcal{S}_\tau^{m_i}[0](x).$$

Then we have

$$\bar{V}^k = \bar{\bar{\mathcal{S}}}^{\tau}_{k\tau}[0] = \min_{\{m_i\}_{i=1}^k \in \mathcal{M}^k} \left\{ \prod_{i=1}^{k} \mathcal{S}_\tau^{m_i} \right\} [0](x) = \min_{\{m_i\}_{i=1}^k \in \mathcal{M}^k} v_{\{m_i\}_{i=1}^k}(x).$$

Also by (6.65),

$$\bar{V}^{k+1} \doteq \bar{\mathcal{S}}_\tau[\bar{V}^k] = \min_{m_{k+1}\in\mathcal{M}} \mathcal{S}_\tau^{m_k} \left[\min_{\{m_i\}_{i=1}^k \in \mathcal{M}^n} v_{\{m_i\}_{i=1}^k} \right](x)$$

Since $\mathcal{S}_\tau^{m_k}$ is min-plus linear,

$$\begin{aligned} &= \min_{\{m_i\}_{i=1}^{k+1} \in \mathcal{M}^{k+1}} \mathcal{S}_\tau^{m_k} \left[v_{\{m_i\}_{i=1}^k} \right](x) \\ &= \min_{\{m_i\}_{i=1}^{k+1} \in \mathcal{M}^{k+1}} v_{\{m_i\}_{i=1}^{k+1}}(x) \end{aligned} \tag{6.99}$$

The algorithm shall consist of the forward propagation of $v_{\{m_i\}_{i=1}^k}$ to $v_{\{m_i\}_{i=1}^{k+1}} = \mathcal{S}_\tau^{m_k}\left[v_{\{m_i\}_{i=1}^k}\right]$ for all k-tuple switchings $\{m_i\}_{i=1}^k \in \mathcal{M}^k$ and all $m_{k+1} \in \mathcal{M}$. Such propagation is carried out from $k = 0$ to some termination step $k = N$. The value function is obtained by taking the minimum over all such $v^N_{\{m_i\}_{i=1}^N}$.

It is important to note that the computation of each $v_{\{m_i\}_{i=1}^k}$ is analytical. We will indicate the actual analytical computations in the steps below. Note that many formulae are duplicated here to make this section self-contained.

- First approximate a nonlinear Hamiltonian we wish to solve, with a min-plus summation of linear quadratic Hamiltonians as in (6.44).

$$H(x,p) \approx \widetilde{H}(x,p) \doteq \min_{m\in\{1,2,\ldots,M\}} H^m(x,p)$$

where each H^m is linear-quadratic,

$$H^m(x,p) = \frac{1}{2}x'C^m x - \frac{1}{2}p'\Sigma^m p + (A^m x)'p + x'l_1^m + p'l_2^m + \frac{1}{2}\alpha^m, \quad (6.100)$$

- Choose time step $\tau \in \mathbb{R}$ and $N \in \mathbb{N}$. Thus we shall approximate the infinite horizon value function $\widetilde{V}$ for the switching problem, as defined in (6.47) by $\bar{V}^N$, the value function of discretized switching problem with horizon $N\tau$ as defined in (6.98).

- For each $m \in \mathcal{M}$ starting with any $(P,S,Q,a,b,\beta)_0$, evolve them forward in time by τ as per (6.80), in which all parameters correspond to m'th Hamiltonian. Restating here:

$$\begin{aligned}
-\dot{P}_t^m &= A^{m\prime}P_t^m + P_t^m A^m + C^m - P_t^m\Sigma^m P_t^m \\
-\dot{S}_t^m &= (A^m - \Sigma^m P_t^m)'S_t^m \\
-\dot{Q}_t^m &= -S_t^{m\prime}\Sigma^m S_t^m \\
-\dot{a}_t^m &= (A^m - \Sigma^m P_t^m)'a_t^m + l_1^m + P_t^m l_2^m \\
-\dot{b}_t^m &= S_t^{m\prime}(l_2^m - \Sigma^m a_t^m) \\
-\dot{\beta}_t^m &= \alpha^m - a_t^{m\prime}\Sigma^m a_t^m + 2a_t^{m\prime}l_2^m
\end{aligned} \quad (6.101)$$

Lets denote the solution set by $(P,S,Q,a,b,\beta)_\tau^m$.

- Now for each $m \in \mathcal{M}$, compute the parameters of the min-plus fundamental solution,

$$I_\tau^m(x,y) = \frac{1}{2}x'I_{m,\tau}^{11}x + x'I_{m,\tau}^{12}y + \frac{1}{2}y'I_{m,\tau}^{22}y + x'I_{m,\tau}^1 + y'I_{m,\tau}^2 + I_{m,\tau}^0,$$

as per (6.91). Restating here:

$$\begin{aligned}
I^{11}_{m,\tau} &= P_0 - S_0(Q_0 - Q^m_\tau)^{-1}S_0{}' \\
I^{12}_{m,\tau} &= S_0(Q_0 - Q^m_\tau)^{-1}S^{m\prime}_\tau \\
I^{22}_{m,\tau} &= -P^m_\tau - S^m_\tau(Q_0 - Q^m_\tau)^{-1}S^{m\prime}_\tau \\
I^{1}_{m,\tau} &= a_0 - S_0(Q_0 - Q^m_\tau)^{-1}(b_0 - b^m_\tau) \\
I^{2}_{m,\tau} &= -a^m_\tau + S^m_\tau(Q_0 - Q^m_\tau)^{-1}(b_0 - b^m_\tau) \\
I^{0}_{m,\tau} &= \beta_0 - \beta^m_\tau - (b_0 - b^m_\tau)'(Q_0 - Q^m_\tau)^{-1}(b_0 - b^m_\tau)
\end{aligned} \tag{6.102}$$

Note that as a special case, we can choose $P_0 = Q_0 = 0$, $S_0 = I$, $a_0 = b_0 = 0$ and $\beta_0 = 0$, to simplify above formulae considerably. We note that parameters of I^m need only be computed once, and thus can be precomputed.

- Starting with $v^0(x) = 0$, we shall prove that $v_{\{m_i\}_{i=1}^k}$ retain the quadratic form as k increases by induction. Lets us assume $v_{\{m_i\}_{i=1}^k}$ in the following quadratic form.

$$v_{\{m_i\}_{i=1}^k}(x) = \frac{1}{2}x'\hat{P}_{\{m_i\}_{i=1}^k}x + x'\hat{a}_{\{m_i\}_{i=1}^k} + \frac{1}{2}\hat{\beta}_{\{m_i\}_{i=1}^k} \tag{6.103}$$

Then for each $m_{k+1} \in \mathcal{M}$, by Theorem 6.6.1,

$$\begin{aligned}
v_{\{m_i\}_{i=1}^{k+1}}(x) &= \mathcal{S}^{m_k}_\tau\left[v_{\{m_i\}_{i=1}^k}\right](x) \\
&= \min_{y\in\mathbb{R}^n}\left\{I^{m_{k+1}}_\tau(x,y) + v_{\{m_i\}_{i=1}^k}(y)\right\} \\
&= \frac{1}{2}x'\hat{P}_{\{m_i\}_{i=1}^{k+1}}x + x'\hat{a}_{\{m_i\}_{i=1}^{k+1}} + \frac{1}{2}\hat{\beta}_{\{m_i\}_{i=1}^{k+1}}
\end{aligned} \tag{6.104}$$

Note that the minimum exists because otherwise, due to the quadratic form the infimum would be $-\infty$ for all x. This is impossible because, the LHS $\geq \bar{V}(x) \geq 0$. Now by (6.103), (6.104) and (6.102),

$$\begin{aligned}
\hat{P}_{\{m_i\}_{i=1}^{k+1}} &= I^{11}_{m_{k+1},\tau} - I^{12}_{m_{k+1},\tau}\left(I^{22}_{m_{k+1},\tau} + \hat{P}_{\{m_i\}_{i=1}^k}\right)^{-1} I^{12}_{m_{k+1},\tau}{}' \\
\hat{a}_{\{m_i\}_{i=1}^{k+1}} &= -I^{12}_{m_{k+1},\tau}\left(I^{22}_{m_{k+1},\tau} + \hat{P}_{\{m_i\}_{i=1}^k}\right)^{-1}\left(\hat{a}_{\{m_i\}_{i=1}^k} + I^{2}_{m_{k+1},\tau}\right) + I^{1}_{m_{k+1},\tau} \\
\hat{\beta}_{\{m_i\}_{i=1}^{k+1}} &= \hat{\beta}_{\{m_i\}_{i=1}^k} + I^{0}_{m_{k+1},\tau} \\
&\quad - \left(\hat{a}_{\{m_i\}_{i=1}^k} + I^{2}_{m_{k+1},\tau}\right)'\left(I^{22}_{m_{k+1},\tau} + \hat{P}_{\{m_i\}_{i=1}^k}\right)^{-1}\left(\hat{a}_{\{m_i\}_{i=1}^k} + I^{2}_{m_{k+1},\tau}\right)
\end{aligned} \tag{6.105}$$

Similar to the max-plus counterpart in [McE07], the errors in the solution are due to :

1. The approximation of $H(x,p)$ by $\widetilde{H}(x,p)$. The error analysis for the max-plus methods for such approximation was carried out in [MD08a] and [MDb]. Analysis for the min-plus method is expected to be similar.

2. Discretization of the timestep τ and premature termination of the horizon at $N\tau$. Errors analysis for this approximation for max-plus methods was carried out in [McE09] and [MK]. Analysis for the min-plus method is expected to be similar.

Note that the computation of each triplet $(\hat{P}, \hat{a}, \hat{\beta})_{\{m_i\}_{i=1}^k}$ grows like the cube of the space dimension (due to matrix operations). Thus one avoids the curse-of-dimensionality.

However, the curse-of-dimensionality is replaced by another type of rapid computational cost growth. Here, we refer to this as the *curse-of-complexity* . If $\#\mathcal{M} = 1$, then all the computations of our algorithm (excepting the solution of the Riccati equation) are unnecessary, and we *informally* refer to this as complexity one. When there are $M = \#\mathcal{M}$ such quadratics in the Hamiltonian, $\widetilde{H}$ we say it has complexity M. Note that

$$\#\left\{v_{\{m_i\}_{i=1}^N} | m_i \in \mathcal{M} \quad \forall\, i \in \{1, 2, \ldots, N\}\right\} \approx M^N$$

For large N, this number is indeed large. We shall discuss ways to limit this quadratic growth with N in the next section, but it is significant to note that the exponential computational cost growth with space dimension n is limited to cubic growth.

6.7.1 Pruning

The number of quadratics in the solution $\bar{V}^k$ grows exponentially with k. however in practice, relatively few of these actually contribute to $\bar{V}^k$. Thus it is useful to prune the set, to contain the curse-of-complexity. Note that if

$$v_{\{\hat{m}_i\}_{i=1}^k}(x) \geq \bigoplus_{\{\hat{m}_i\}_{i=1}^k \neq \{m_i\}_{i=1}^k} v_{\{m_i\}_{i=1}^k}(x) \qquad \forall x \in \mathbb{R}^n$$

then

$$\bar{\mathcal{S}}_t \left[v_{\{\hat{m}_i\}_{i=1}^k} \right](x) \geq \bar{\mathcal{S}}_t \left[\bigoplus_{\{\hat{m}_i\}_{i=1}^k \neq \{m_i\}_{i=1}^k} v_{\{m_i\}_{i=1}^k} \right](x) = \bar{\mathcal{S}}_t[\bar{V}^k] \qquad \forall x \in \mathbb{R}^n$$

Thus $v_{\{\hat{m}_i\}_{i=1}^k}$ will play no role in the computation of $\bar{V}^{k\tau+t}$, for $t \geq 0$. Thus one may prune such $v_{\{\hat{m}_i\}_{i=1}^k}$ without any loss of accuracy. Pruning methods of various order designed for max-plus methods in [MDG08] can be easily adapted for above problem to check if a quadratic dominates the min-plus sum of the rest, and prune it. Note that such pruning methods incur additional computational burden. But usually it is justifiable in their efficacy in attenuating the exponential curse-of-complexity as the horizon grows.

Specifically, in the examples below, we used a second order pruning method based on semi-definite programming developed in [MDG08].

6.8 Numerical Examples

A number of examples were tested on a standard 2004 PC. The algorithm described above was coded in MATLAB using a second order pruning technique described in [MDG08]. Since the plots below require one to compute value functions on a gridded plane in the space, computational time for such plotting was not included in the quoted computational time.

We will discuss examples with the complexity 3, in dimensions 2 and 4. Note that all parameters with the superscript j correspond to the Hamiltonian H^j, as defined in (6.45). Observe that the dynamics is parametrized by A, l_2, Σ, and the payoff by C, l_1, α.

Second order example: Let the parameters of the constituent Hamiltonian be as below.

$$A^1 = \begin{bmatrix} 1 & -0.5 \\ -0.1 & 1 \end{bmatrix}, \quad C^1 = \begin{bmatrix} 1.5 & 0.2 \\ 0.2 & 1.5 \end{bmatrix}, \quad \Sigma^1 = \begin{bmatrix} 0.22 & -0.01 \\ -0.01 & 0.22 \end{bmatrix},$$

$$l_1^1 = l_2^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \alpha^1 = 0,$$

$$A^2 = \begin{bmatrix} -0.35 & 0.7 \\ 0.7 & -0.07 \end{bmatrix}, \quad C^2 = 0.6C^1 \quad \Sigma^2 = \Sigma^1,$$

$$l_1^2 = 0.05 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad l_2^2 = 0.04 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \alpha^2 = 30$$

$$A^3 = 0.5A^2, \quad C^3 = C^2, \quad \Sigma^3 = \Sigma^1, \quad l_1^3 = l_1^2, \quad l_2^3 = l_2^2, \quad \alpha^3 = 45$$

The timestep was chosen to be 0.1 seconds. The solution converges in the area of interest, i.e. a square with side 10 centered at origin in about 14 timesteps which corresponds to the time horizon of 1.4 seconds. No over pruning was used. The growth in number of quadratic functions remaining after each propagation is shown in figure 6.8. Note that at last timestep, using the pruning methods, the number of quadratics is just 160, which is far lesser than $3^{14} = 4782969$ without pruning. The backsubstitution error and the optimal switching regimes are shown in the figure 6.1. Here the switching error plot is rotated for better viewing. Note that the error has converged, and has a distribution centered at zero. Magnitude of the error can be decreased arbitrarily by decreasing the timestep, or by reintegrating the optimal switching histories of the computed quadratics at a smaller timestep. Thus the error, though coarse, represents the value function (figure 6.3 and optimal switching regimes (figure 6.1) fairly accurately. The gradients in x and y directions are shown in figure 6.2. This solution was computed on a 2005 laptop PC with 1.5 GHz speed and 512 MB of memory, in 131 seconds.

A heuristic measure of the error in gradient can developed using Newton's method as follows. Let $\widetilde{V}$ and $\bar{V}$ be the exact and approximate solution of $0 = -\widetilde{H}(x, \nabla(\cdot))$. Given $x \in \Re^n$, let $\bar{m} = \operatorname{argmin}_m H^m(x, \nabla\bar{V})$. If $\tilde{p} = \nabla\widetilde{V}(x)$ and $\bar{p} = \nabla\bar{V}(x)$ (where we are assuming the existence of gradients), $0 =$

$\widetilde{H}(x,\tilde{p}) \approx \widetilde{H}(x,\bar{p}) + (\tilde{p}-\bar{p})'\nabla_p \widetilde{H}(x,\bar{p}) = e - (\Sigma^{\bar{m}}\bar{p} + A^{\bar{m}}x)$, where $e \doteq \widetilde{H}(x,\bar{p})$ is the backsubstitution error. To obtain our heuristic gauge of accuracy, we choose the nearest p, $\hat{p}$, which is the solution of linearized approximation to the quadratic Hamiltonian, and use this as an approximation for $\tilde{p}$. That is, we take

$$\hat{p} \doteq \operatorname*{argmin}_p \left\{ |p-\bar{p}| : e - (\Sigma^{\bar{m}}\bar{p} + A^{\bar{m}}x)'(p-\bar{p}) = 0 \right\}.$$

This can solved using the pseudo-inverse, yielding

$$\tilde{p} - \bar{p} \approx \hat{p} - \bar{p} = (\Sigma_{\bar{m}}\bar{p} + A_{\bar{m}}x)^{-1}e.$$

The relative error in the gradient, $\frac{|\tilde{p}-\bar{p}|}{\bar{p}}$, is plotted in Figure 6.4, and we see that on an average, it is 2%, and generally much smaller, except at the points of switching discontinuities.

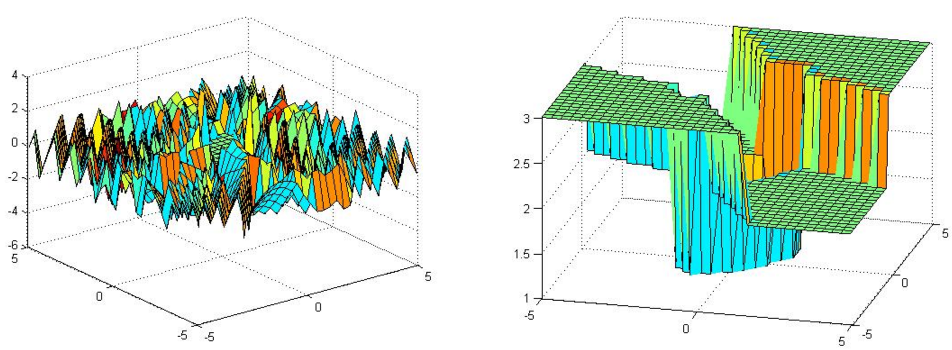


Figure 6.1: Backsubstitution error and optimal switching regimes for the 2D problem

Fourth order example: Let the parameters of the constituent Hamiltonian be defined in terms of the building blocks below.

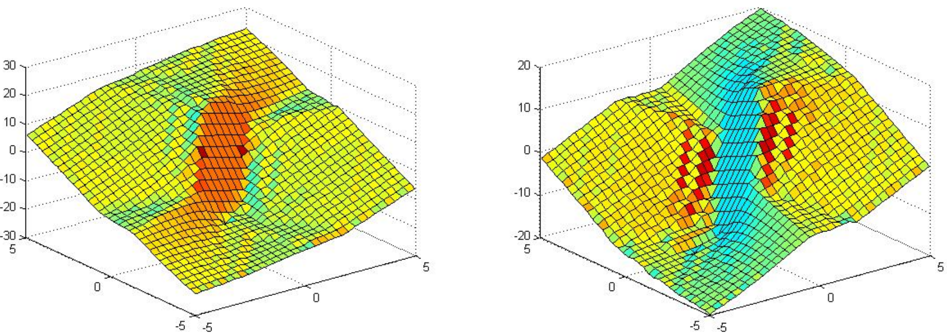


Figure 6.2: First and second partials of the Value for the 2D problem

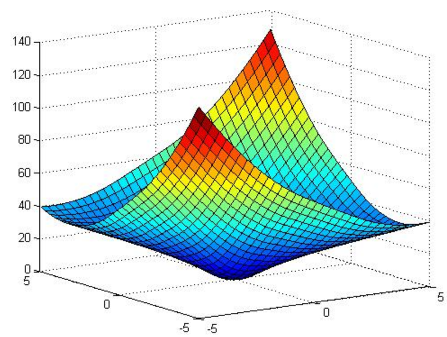


Figure 6.3: Value function for the 2D problem

$$A1 = \begin{bmatrix} 1 & -0.5 \\ -0.1 & 1 \end{bmatrix}, \quad A2 = \begin{bmatrix} -0.35 & 0.7 \\ 0.7 & -0.07 \end{bmatrix}, \quad C = \begin{bmatrix} 1.5 & 0.2 \\ 0.2 & 1.5 \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} 0.22 & -0.01 \\ -0.01 & 0.22 \end{bmatrix}, \quad a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad l_1 = 0.05 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad l_2 = 0.04 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We will use a as the coupling matrix between the dynamics in different

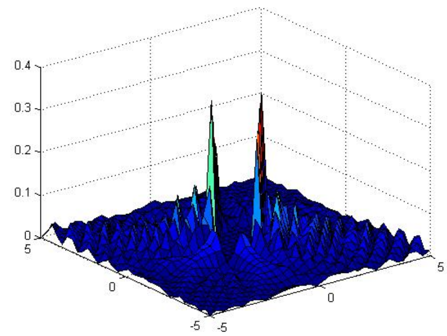


Figure 6.4: Relative gradient error for the 2D problem

dimensions.

$$A^1 = \begin{bmatrix} A1 & a \\ a & A1 \end{bmatrix},\ C^1 = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix},\ \Sigma^1 = \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix},\ l_1^1 = l_2^1 = \mathbf{0}, \quad \alpha^1 = 0,$$

$$A^2 = \begin{bmatrix} A2 & a \\ a & A2 \end{bmatrix},\ C^2 = 0.6C^1, \quad \Sigma^2 = \Sigma^1, \quad l_1^2 = \begin{bmatrix} l_1 \\ l_1 \end{bmatrix},\ l_2^2 = \begin{bmatrix} 1_2 \\ 1_2 \end{bmatrix},\ \alpha^2 = 30$$

$$A^3 = 0.8A^2, \quad C^3 = C^2, \quad \Sigma^3 = \Sigma^1, \quad l_1^3 = l_1^2, \quad l_2^3 = l_2^2, \quad \alpha^3 = 35$$

Again the timestep was chosen to be 0.1 seconds. The solution converges in the area of interest, i.e. a hypercube with side 10 centered at origin in about 14 timesteps which corresponds to the time horizon of 1.4 seconds. No over pruning was used. The growth in number of quadratic functions remaining after each propagation is shown in figure 6.8. Note that at last timestep, using the pruning methods, the number of quadratics is just 198, which is far lesser than $3^{14} = 4782969$. This shows the utility of the pruning methods developed in [MDG08] to attenuate the *curse-of-complexity* . The backsubstitution errors along $1-2$ and $3-4$ planes are shown in the figure 6.5. Note that the error has converged, and has a distribution centered at zero. The magnitude of error can be decreased arbitrarily by decreasing the timestep, or by reintegrating the optimal switching histories of the computed quadratics at a smaller timestep. Thus the error, though coarse, represents the value function and optimal switching regimes (figure 6.7) fairly accurately. First and second partials of the value function for the $1-2$ plane are shown in figure 6.6. This solution was computed on a 2005 laptop PC with 1.5 GHz speed and 512 MB of memory, in 152 seconds.

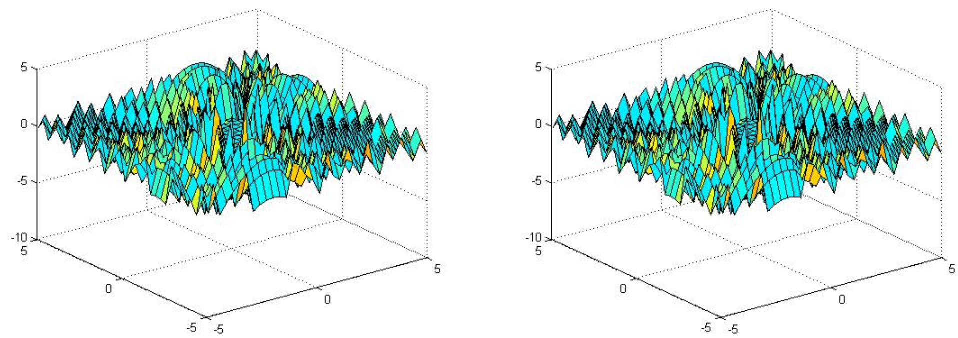


Figure 6.5: Backsubstitution error along $1-2$ and $3-4$ planes for the 4D problem

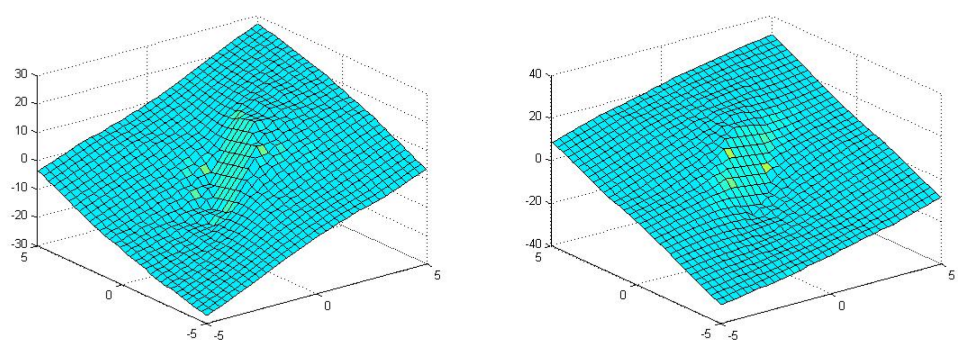


Figure 6.6: First and second partials of the Value in $1-2$ plane for the 4D problem

Chapter 6, in part, is currently being prepared for submission for publication of the material. McEneaney, William; Deshpande, Ameet [MDa]. The dissertation author was the coauthor of this paper.

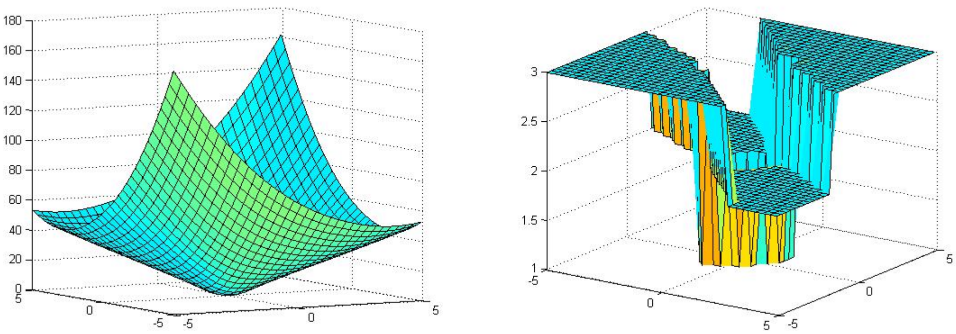


Figure 6.7: Value function and the optimal switching regimes in $1-2$ plane for the 4D problem

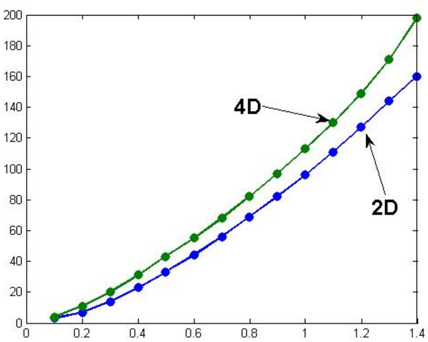


Figure 6.8: Time vs. number of quadratics in the 2D and 4D problems

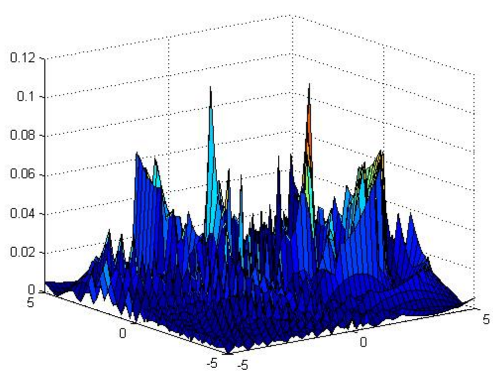


Figure 6.9: relative gradient error along 3-4 plane

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