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A Steinberg Type Decomposition Theorem for Higher Level Demazure Modules

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A Steinberg Type Decomposition Theorem for Higher Level Demazure Modules

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ABSTRACT OF THE DISSERTATION

A Steinberg Type Decomposition Theorem for Higher Level Demazure Modules

by

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We study Demazure modules which occur in a level ℓ irreducible integrable representation of an affine Lie algebra. We also assume that they are stable under the action of the standard maximal parabolic subalgebra of the affine Lie algebra. We prove that such a module is isomorphic to the fusion product of "prime" Demazure modules, where the prime factors are indexed by dominant integral weights which are either a multiple of ℓ or take value less than ℓ on all simple coroots. Our proof depends on a technical result which we prove in all the classical cases and G_2. Calculations with mathematica show that this result is correct for small values of the level. Using our result, we show that there exist generalizations of Q–systems to pairs of weights where one of the weights is not necessarily rectangular and is of a different level. Our results also allow us to compare the multiplicities of an irreducible representation occurring in the tensor product of certain pairs of irreducible representations, i.e., we establish a version of Schur positivity for such pairs of irreducible modules for a simple Lie algebra. We also present a more refined presentation of a larger family of modules which include Demazure modules.

Contents

Chapter 1

Introduction

Demazure modules associated to simple Lie algebra or more generally a Kac–Moody Lie algebra $\mathfrak{g}$ have been studied intensively since their introduction in [14]. These modules, which are actually modules for a Borel subalgebra of the Lie algebra, are indexed by a dominant integral weight Λ and an element w of the Weyl group. In this paper we shall be concerned with affine Lie algebras and a particular family of Demazure modules: namely those which are preserved by a maximal parabolic subalgebra containing the Borel. More precisely, let $\mathfrak{g}$ be a simple finite–dimensional complex Lie algebra and $\widehat{\mathfrak{g}}$ the corresponding affine Lie algebra. Then the maximal parabolic subalgebra of interest is the current algebra $\mathfrak{g}[t]$ which is the algebra of polynomial maps $\mathbf{C} \to \mathfrak{g}$ with the obvious pointwise bracket. The $\mathfrak{g}[t]$–stable Demazure modules are indexed by a pair (ℓ, λ), where ℓ is the level of the integrable representation of $\widehat{\mathfrak{g}}$ and λ is a dominant integral weight of $\mathfrak{g}$ and we denote the corresponding module by $D(\ell, \lambda)$. In the case when $\ell = 1$, these modules are interesting for a variety of reasons, including the connection with Macdonald polynomials established in [36] for $\mathfrak{sl}_{r+1}$ and in [23] in general.

Our interest in these modules arise from their relationship with the representation theory of quantum affine algebras. This connection was originally developed in [4], [10], [12] where it was shown that the classical limit of certain irreducible representations of the quantum affine algebra can be viewed as graded representations of $\mathfrak{g}[t]$. The classical limits were first related to the $\mathfrak{g}[t]$–stable Demazure modules in level one representations of affine Lie algebras in [8] for $\mathfrak{sl}_{r+1}$. In that paper, the connection was also made between these modules and the fusion product defined in [16] of representations of $\mathfrak{g}[t]$. In [12] it was shown that a Kirillov–Reshetikhin module for a quantum affine algebra is similarly related to a Demazure module when $\mathfrak{g}$ is of classical type.

In [17] and [18] the authors worked with arbitrary untwisted affine Lie algebras and with particular classes of $\mathfrak{g}[t]$–stable Demazure module. In the simply–laced case for instance, they studied the modules $D(\ell, \ell\mu)$ where μ is a dominant integral weight of $\mathfrak{g}$. They proved that such modules were the fusion product of the classical limit of the Kirillov–Reshetikhin modules defined in [12]. (The definition of fusion products of $\mathfrak{g}[t]$–modules is recalled in Section 2 of this paper, for the moment it suffices to say that it is a procedure which defines a cyclic graded $\mathfrak{g}[t]$–module structure on a tensor product of finite–dimensional $\mathfrak{g}$–modules. In particular, the underlying $\mathfrak{g}$ module structure is unchanged, where we are regarding $\mathfrak{g}$ as the subalgebra $\mathfrak{g}[t]$ consisting of constant maps).

A completely obvious question is: what is the analog of the results of [17] and [18] for the module $D(\ell, \ell\mu + \lambda)$ where λ is an arbitrary dominant integral weight. A much less obvious, but very interesting reason to study this question is the following: when $\ell = 2$ and in the case of $\mathfrak{sl}_{n+1}$, these modules are related to the modules for the quantum affine algebra which occur in the work of Hernandez–Leclerc (see [22]). This relationship is made precise in [1].

Recall that Steinberg's tensor product theorem asserts that a simple module $L(\lambda)$ of an algebraic group over characteristic p is isomorphic to a tensor product

$L(p\lambda_1) \otimes L(\lambda_0)$ where $\lambda_0(h_i) \leq p$ for all simple coroots. Our first result establishes an analog of this replacing p by ℓ and the tensor product by fusion product, i.e.,

$$D(\ell, \ell\mu + \lambda) \cong D(\ell, \ell\mu) * D(\ell, \lambda),$$

for all positive integers ℓ and dominant integral weights μ and λ and if $\mathfrak{g}$ is of classical type or G_2. The main obstruction to proving this result in general is a technical proposition (Proposition 2.5) on the affine Weyl group which is problematic for E_8 and F_4. However, computer calculations show that this result is true for small values of ℓ and all λ and μ.

To continue the connection with the work of [22], we define and study the notion of prime representations of $\mathfrak{g}[t]$–modules: namely a module which is not isomorphic to a fusion product of non–trivial $\mathfrak{g}[t]$–modules. We prove that the modules $D(\ell, \ell\omega_i)$ where ω_i is a fundamental weight and $D(\ell, \lambda)$ where $\lambda(h_i) \leq \ell$ for all simple coroots, are prime if $\mathfrak{g}$ is simply–laced. In fact we show that the underlying $\mathfrak{g}$–module is not a tensor product of non–trivial $\mathfrak{g}$–modules. In the case when $\mathfrak{g}$ is of type type A or D we show that any Demazure module is a fusion product of prime Demazure modules.

We use our main result to study generalizations of Q–systems (see [20] for details, [27] for a more recent discussion and [21] , [32] for the quantum analog). In the case of $\mathfrak{sl}_{n+1}$, the Q–system is a classical identity of Schur functions associated to rectangular weights of a fixed height. Equivalently, the Q–system is a short exact sequence

$$0 \to \bigotimes_{\{j:a_{i,j}=-1\}} V(\ell\omega_j) \to V(\ell\omega_i) \otimes V(\ell\omega_i) \to V(\ell+1)\omega_i) \otimes V((\ell-1)\omega_i) \to 0,$$

where $V(r\omega_i)$ is the irreducible representation of sl_{n+1} with highest weight $r\omega_i$. In Theorem 4.2 of this paper, we write down an analogous short exact sequence for the pair $V(\ell\omega_i) \otimes V(\lambda)$ for λ satisfying the restriction that $\lambda(h_i) \leq \ell$ for all simple coroots. In fact we show that we can replace the tensor product of $\mathfrak{sl}_{n+1}$–modules by fusion products of $\mathfrak{sl}_{n+1}[t]$–modules so that all the maps are completely canonical. It is interesting to note that the kernel is in general not a tensor or fusion product of irreducible representations of $\mathfrak{sl}_{n+1}$, but is a fusion product of prime Demazure modules.

1.1 Notation

Throughout the paper $\mathbb{C}$ denotes the field of complex numbers, $\mathbb{Z}$ the set of integers and $\mathbf{Z}_+$, $\mathbf{N}$ the set of non–negative and positive integers respectively. Given any complex Lie algebra $\mathfrak{a}$ we let $\mathbf{U}(\mathfrak{a})$ be the universal enveloping algebra of $\mathfrak{a}$. Also, if t is any indeterminate we let $\mathfrak{a}[t]$ be the Lie algebra of polynomial maps from $\mathbb{C}$ to $\mathfrak{a}$ with the obvious pointwise Lie bracket:

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg, \quad x, y \in \mathfrak{a}, \quad f, g \in \mathbf{C}[t].$$

Let $\operatorname{ev}_0 : \mathfrak{a}[t] \to \mathfrak{a}$ be the map of Lie algebras given by setting $t = 0$. The Lie algebra $\mathfrak{a}[t]$ and its universal enveloping algebra inherit a grading from the degree grading of $\mathbf{C}[t]$, thus an element $a_1 \otimes t^{r_1} \cdots a_s \otimes t^{r_s}$, $a_j \in \mathfrak{a}$, $r_j \in \mathbf{Z}_+$ for $1 \le j \le s$ will have grade $r_1 + \cdots + r_s$. We shall be interested in $\mathbf{Z}$–graded modules for $\mathfrak{a}[t]$. By this we mean a $\mathbf{Z}$–graded vector space $V = \oplus_{s \in \mathbf{Z}} V[s]$ which admits a compatible $\mathfrak{a}[t]$–action,

$$(\mathfrak{a} \otimes t^r) V[s] \subset V[r + s].$$

A morphism of graded $\mathfrak{a}[t]$–modules is just a degree zero map of $\mathfrak{a}[t]$–modules. Given $r \in \mathbf{Z}$ and a graded vector space V, we let $\tau_r^* V$ be the r–th graded shift of V. Clearly the pull–back of any $\mathfrak{a}$–module V by ev_0 defines the structure of a graded $\mathfrak{a}[t]$–module on V and we denote this module by $\operatorname{ev}_0^* V$.

1.2 Lie Algebra Terminology

From now on $\mathfrak{g}$ will be a simple complex Lie algebra of rank n and $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}$. Let R be the corresponding set of roots, α_i, $1 \le i \le n$ be a set of simple roots and R^+ the corresponding set of positive roots and let θ be the highest root of R^+. For $\alpha \in R^+$, we set $d_\alpha = 1$ if α is long and $d_\alpha = 2$ if α is short and $\mathfrak{g}$ is not of type G_2. If $\mathfrak{g}$ is of type G_2, then we set $d_\alpha = 3$ if α is short. The Weyl group W of R is generated by simple reflections $\mathbf{s}_i$, $1 \le i \le n$ and w_0 denotes the unique longest element of W.

Let $x_\alpha^\pm$, $\alpha \in R^+$, h_i, $1 \le i \le n$ be a Chevalley basis for $\mathfrak{g}$. We have

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathfrak{h} = \bigoplus_{i=1}^{n} \mathbb{C}h_i, \quad \mathfrak{n}^\pm = \bigoplus_{\alpha \in R^+} \mathbb{C}x_\alpha^\pm.$$

The fundamental weights $\omega_i \in \mathfrak{h}^*$, $1 \le i \le n$ are defined by setting $\omega_i(h_j) = \delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker delta symbol. The weight lattice P (resp. P^+) is the $\mathbb{Z}$–span (resp. $\mathbb{Z}_+$ span) of the fundamental weights. The root lattice Q and the subset Q^+ are defined in the obvious way using the simple roots. The co–weight lattice L is the sublattice of P spanned by the elements $d_i\omega_i$, $1 \le i \le n$ and the co–root lattice M is defined analogously. The subsets L^+ and M^+ are defined in the obvious way. Let $\mathbf{Z}[P]$ be the integral group ring of P with basis $e(\lambda)$, $\lambda \in P$.

1.3 $\mathfrak{g}$ character

For $\lambda \in P^+$, denote by $V(\lambda)$ the simple finite–dimensional $\mathfrak{g}$–module generated by an element v_λ with defining relations

$$\mathfrak{n}^+ v_\lambda = 0, \quad h_i v_\lambda = \lambda(h_i) v_\lambda, \quad (x_{\alpha_i}^-)^{\lambda(h_i)+1} v_\lambda = 0, \quad 1 \le i \le n.$$

It is well–known that $V(\lambda) \cong V(\mu)$ iff $\lambda = \mu$ and that any finite–dimensional $\mathfrak{g}$–module is isomorphic to a direct sum of modules $V(\lambda)$, $\lambda \in P^+$. If V is a $\mathfrak{h}$–semisimple $\mathfrak{g}$–module (in particular if $\dim V < \infty$), we have

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu, \quad V_\mu = \{v \in V : hv = \mu(h)v, \quad h \in \mathfrak{h}\},$$

and we set $\operatorname{wt} V = \{\mu \in \mathfrak{h}^* : V_\mu \neq 0\}$. If $\dim V_\mu < \infty$ for all $\mu \in \operatorname{wt} V$, then we define $\operatorname{ch}_{\mathfrak{h}} V : \mathfrak{h}^* \to \mathbf{Z}_+$, by sending $\mu \to \dim V_\mu$. If $\operatorname{wt} V$ is a finite set, then

$$\operatorname{ch}_{\mathfrak{h}} V = \sum_{\mu \in \mathfrak{h}^*} \dim V_\mu e(\mu) \in \mathbf{Z}[P].$$

1.4 Untwisted Affine Lie Algebra

We now define the untwisted affine Lie algebra associated to $\mathfrak{g}$ and some related terminology (see [25] for details). The affine Lie algebra $\hat{\mathfrak{g}}$ is given by

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

where c is the canonical central element, and d acts as the derivation $t\frac{d}{dt}$ and commutator

$$[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s} + r\delta_{r,-s}(x, y)c,$$

where $(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \to \mathbf{C}$ is a symmetric nondegenerate invariant bilinear form on $\mathfrak{g}$ normalized so that the square length of the long root is two. The Cartan subalgebra of the affine Lie algebra is

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Regard $\mathfrak{h}^*$ as a subspace of $\hat{\mathfrak{h}}^*$ by setting $\mu(c) = \mu(d) = 0$ for all $\mu \in \mathfrak{h}^*$. Let $\delta, \Lambda_0 \in \hat{\mathfrak{h}}^*$ be given by

$$\delta(d) = 1, \quad \delta(\mathfrak{h} \oplus \mathbf{C}c) = 0, \quad \Lambda_0(c) = 1, \quad \Lambda_0(\mathfrak{h} \oplus \mathbf{C}d) = 0.$$

Extend the non–degenerate form on $\mathfrak{h}^*$ to a non–degenerate form on $\hat{\mathfrak{h}}^*$ by setting,

$$(\delta,\delta) \;=\; (\Lambda_0,\Lambda_0) = 0, \quad (\Lambda_0,\delta) = 1.$$

The elements α_i, $0 \le i \le n$ where $\alpha_0 = -\theta + \delta$ are a set of simple roots for the set of roots of $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$. Let $\hat{R}^+$ be the corresponding set of positive roots,

$$\hat{R}^+ = \{\alpha + r\delta : \alpha \in R,\ r \in \mathbf{N}\} \ \cup R^+ \ \cup \ \{r\delta : r \in \mathbf{N}\}.$$

Set $\hat{\mathfrak{b}} = \hat{\mathfrak{h}} \bigoplus_{\alpha\in\hat{R}^+} \hat{\mathfrak{g}}_\alpha$ and note that

$$\mathfrak{g}[t] \oplus \mathbf{C} \oplus \mathbf{C}d = \mathfrak{n}^- \oplus \hat{\mathfrak{b}}, \qquad \mathfrak{g}[t] = \mathfrak{n}^- \oplus \mathfrak{h} \bigoplus_{\alpha\in\hat{R}^+} \hat{\mathfrak{g}}_\alpha.$$

1.5 Affine Weyl Group

For $1 \le i \le n$, set $\Lambda_i = \omega_i + \omega_i(h_\theta)\Lambda_0 \in \hat{\mathfrak{h}}^*$. The set $\hat{P}^+$ of dominant integral affine weights is defined to be the $\mathbf{Z}_+$–span of the elements $\Lambda_i + \mathbf{Z}\delta$, $0 \le i \le n$ and $\hat{P}$ is defined similarly. The root lattice $\hat{Q}$ is the $\mathbf{Z}$–span of the simple roots α_i, $0 \le i \le n$ and $\hat{Q}^+$ is defined in the obvious way.

The affine Weyl group $\widehat{W}$ acts on $\hat{\mathfrak{h}}^*$ via reflections corresponding to the affine simple roots, in particular $w\delta = \delta$ for all $w \in \widehat{W}$. An equivalent way to define the affine Weyl group is as follows. The finite Weyl group W acts on the co–root lattice M by restricting its action on $\mathfrak{h}^*$ and we have

$$\widehat{W} \cong W \ltimes t_M.$$

The extended Weyl group $\widetilde{W}$ is the semi–direct product of $\widehat{W}$ with the group of affine diagram automorphisms, denoted Σ, and

$$\widetilde{W} \cong W \ltimes t_L$$

where L is the co-weight lattice. Given $\mu \in M$ (resp. L) , we denote by t_μ the corresponding element of $\widehat{W}$ (resp. $\widetilde{W}$). Then,

$$t_\mu(\lambda) = \lambda - (\lambda, \mu)\delta, \quad \lambda \in \mathfrak{h}^* \oplus \mathbf{C}\delta, \qquad t_\mu(\Lambda_0) = \Lambda_0 + \mu - \frac{1}{2}(\mu, \mu)\delta. \tag{1.5.1}$$

Let $\mathbf{Z}[\hat{P}]$ be the integral group ring of $\hat{P}$ with basis $e(\Lambda)$ and let I_δ be the ideal of $\mathbf{Z}[\hat{P}]$ obtained by setting $e(\delta) = 1$. Since we have identified $\mathfrak{h}^*$ with a subspace of $\hat{\mathfrak{h}}^*$, the group ring $\mathbf{Z}[P]$ is isomorphic to a subring of $\mathbf{Z}[\hat{P}]$ and the composite morphism

$$\mathbf{Z}[P] \hookrightarrow \mathbf{Z}[\hat{P}] \longrightarrow \mathbf{Z}[\hat{P}]/I_\delta,$$

is injective. Clearly, the action of $\widetilde{W}$ on $\hat{P}$ induces an action on $\mathbf{Z}[\hat{P}]$ and $\mathbf{Z}[\hat{P}]/I_\delta$ as well.

1.6 Demazure Module

For $\Lambda \in \widehat{P}^+$ let $V(\Lambda)$ be the highest weight, irreducible, integrable $\hat{\mathfrak{g}}$-module with highest weight Λ and highest weight vector v_Λ. Then,

$$V(\Lambda) = \bigoplus_{\eta \in \hat{Q}^+} V(\Lambda)_{\Lambda - \eta}, \quad V(\Lambda)_{\Lambda-\eta} = \{v \in V(\Lambda) : hv = (\Lambda - \eta)(h)v, \quad h \in \hat{\mathfrak{h}}^*\}.$$

For $w \in \widehat{W}$, we have $\dim V(\Lambda)_{w\Lambda} = 1$ and the corresponding Demazure module is,

$$V_w(\Lambda) = \mathbf{U}(\hat{\mathfrak{b}})V(\Lambda)_{w\Lambda}.$$

More generally, given, $\sigma \in \Sigma$ and $w \in \widehat{W}$, set $V_{w\sigma}(\Lambda) = V_w(\sigma\Lambda)$. Since $V(\Lambda)_{\Lambda-\eta+r\delta} = 0$ for all $r \in \mathbf{N}$, it follows that $\dim V_{\sigma w}(\Lambda) < \infty$. In the special case when $w\Lambda|_{\mathfrak{h}} \in -P^+$, the Demazure module $V_w(\Lambda)$ is $\mathfrak{g}$–stable, in other words it is a finite–dimensional module for $\mathfrak{g}[t]$. The action of d defines a grading on $V_w(\Lambda)$ which is compatible with the $\mathbf{Z}$–grading on $\mathfrak{g}[t]$. Finally, note that for $w \in \widetilde{W}$, the function $\mathrm{ch}_{\hat{\mathfrak{h}}} V_w(\Lambda) : \hat{P} \to \mathbf{Z}$ is the mapping $\Lambda' \to \dim V_w(\Lambda)_{\Lambda'}$ and is an element of $\mathbf{Z}[\hat{P}]$.

1.7 Fusion Product

We recall the notion of fusion products of representations of $\mathfrak{g}[t]$ introduced in [16]. Let V be a finite–dimensional cyclic $\mathfrak{g}[t]$ module generated by an element v and for $r \in \mathbf{Z}_+$ set

$$F^r V = \left(\bigoplus_{0 \leq s \leq r} \mathbf{U}(\mathfrak{g}[t])[s] \right) .v$$

Clearly $F^r V$ is a $\mathfrak{g}$–submodule of V and we have a finite $\mathfrak{g}$–module filtration

$$0 \subset F^0 V \subset F^1 V \subset \cdots \subset F^p V = V,$$

for some $p \in \mathbf{Z}_+$. The associated graded vector space $\operatorname{gr} V$ acquires a graded $\mathfrak{g}[t]$–module structure in a natural way and is generated by the image of v in $\operatorname{gr} V$.

Given a $\mathfrak{g}[t]$ module V and $z \in \mathbf{C}$, let V^z be the $\mathfrak{g}[t]$–module with action

$$(x \otimes t^r)w = (x \otimes (t+z)^r)w, \quad x \in \mathfrak{g}, \quad r \in \mathbf{Z}_+, w \in V.$$

If V_s, $1 \leq s \leq k$ are cyclic finite–dimensional $\mathfrak{g}[t]$–modules with cyclic vectors v_s, $1 \leq s \leq k$ and $z_1, \cdots, z_k$ are distinct complex numbers then, the fusion product $V_1^{z_1} * \cdots * V_k^{z_k}$ is defined to be $\operatorname{gr} \mathbf{V}(\mathbf{z})$, where $\mathbf{V}(\mathbf{z})$ is the tensor product

$$\mathbf{V}(\mathbf{z}) = V_1^{z_1} \otimes \cdots \otimes V_k^{z_k}.$$

It was proved in [16] that in fact $\mathbf{V}(\mathbf{z})$ is cyclic and generated by $v_1 \otimes \cdots \otimes v_m$ and hence the fusion product is cyclic on the image $v_1 * \cdots * v_m$ of this element. Clearly the definition of the fusion product depends on the parameters z_s, $1 \leq s \leq k$. However it is conjectured in [16] and (proved in certain cases by various people, [8], [15], [16] [18], [26] for instance) that under suitable conditions on V_s and v_s, the fusion product is independent of the choice of the complex numbers. For ease of notation we shall often suppress the dependence on the complex numbers and write $V_1 * \cdots * V_k$ for $V_1^{z_1} * \cdots * V_k^{z_k}$.

1.8 Length of Weyl Element

We conclude this section with a technical result which will be needed in the proof of Theorem 1. Given $w \in \widehat{W}$, let $\ell(w)$ be the length of a reduced expression of w. Clearly $\ell(w_1w_2) \leq \ell(w_1) + \ell(w_2)$ for all $w_1, w_2 \in \widehat{W}$. An alternative characterization of $\ell(w)$ is

$$\ell(w) = \#\{\alpha \in \hat{R}^+ : w\alpha \in -\hat{R}^+\}. \tag{1.8.1}$$

It is convenient to define the length of an element in the extended Weyl group as well, by

$$\ell(\sigma w) = \ell(w), \quad \text{for } w \in \widehat{W} \text{ and } \sigma \in \Sigma.$$

For $w \in \widetilde{W}$ set $\hat{R}^+_w = \{\alpha \in \hat{R}^+ : w\alpha \in -\hat{R}^+\}$. Since Σ is the group of automorphisms of the Dynkin diagram of $\widehat{\mathfrak{g}}$ it follows that $\ell(w) = \#\hat{R}^+_w$ as well. Note also that for all $w \in \widehat{W}$ and $\sigma \in \Sigma$ we have $\ell(\sigma w \sigma^{-1}) = \ell(w)$ and hence $\ell(w\sigma) = \ell(w)$.

Proposition. (i) Let $w_1, w_2 \in \widetilde{W}$ be such that $\hat{R}^+_{w_2} \subset \hat{R}^+_{w_1w_2}$. Then $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$.

(ii) For $\lambda, \mu \in P^+$ and $w \in W$ we have $\ell(t_{-\mu}t_{-\lambda}w) = \ell(t_{-\mu}) + \ell(t_{-\lambda}w)$.

Proof. Write $w_s = \sigma_s w'_s$ for some $\sigma_s \in \Sigma$ and $w'_s \in \widehat{W}$ for $s = 1, 2$. Hence we get

$$\ell(w_1w_2) = \ell(w'_1\sigma_2w'_2) = \ell((\sigma_2^{-1}w'_1\sigma_2)w'_2) \leq \ell(\sigma_2^{-1}w'_1\sigma_2) + \ell(w'_2) = \ell(w_1) + \ell(w_2).$$

It remains to prove the reverse inequality. For this it is enough to prove that

$$\hat{R}^+_{w_2} \cup w_2^{-1}\hat{R}^+_{w_1} \subset \hat{R}^+_{w_1w_2} \qquad \hat{R}^+_{w_2} \cap w_2^{-1}\hat{R}^+_{w_1} = \emptyset. \tag{1.8.2}$$

To prove the inclusion, we only need to show that $w_2^{-1}\hat{R}^+_{w_1} \subset \hat{R}^+$. For this, note that if $\beta \in \hat{R}^+$, we have

$$-\beta \in w_2^{-1}\hat{R}^+_{w_1} \implies w_2\beta \in -\hat{R}^+_{w_1} \subset -\hat{R}^+ \implies w_1w_2\beta \in -\hat{R}^+,$$

by our hypothesis. On the other hand we also have

$$-\beta \in w_2^{-1}\hat{R}^+_{w_1} \implies -w_2\beta \in \hat{R}^+_{w_1} \implies -w_1w_2\beta \in -\hat{R}^+,$$

which is clearly absurd. The second assertion in (1.8.2) follows from

$$\alpha \in w_2^{-1}\hat{R}^+_{w_1} \implies w_2\alpha \in \hat{R}^+_{w_1} \subset \hat{R}^+ \implies \alpha \notin \hat{R}^+_{w_2},$$

and part (i) of the proposition is established.

For (ii) we see that using part (i), it suffices to prove that if

$$\alpha + p\delta \in \hat{R}^+, \ \ t_{-\lambda}w(\alpha + p\delta) \in -\hat{R}^+ \implies t_{-\mu}t_{-\lambda}w(\alpha + p\delta) \in -\hat{R}^+.$$

Since $\mu \in P^+$ it follows from the explicit formulae for the translations that $t_{-\mu}$ preserves $-(R^+ + \mathbf{Z}_+\delta)$. Hence it suffices to show that

$$\alpha + p\delta \in \hat{R}^+, \ \ t_{-\lambda}w(\alpha + p\delta) \in \hat{R}^- \implies t_{-\lambda}w(\alpha + p\delta) \subset -(R^+ + \mathbf{Z}_+\delta),$$

i.e., that $w\alpha \in -R^+$. But this is again clear from the formulae because $\lambda \in P^+$

□

Chapter 2

Main Results

We begin this section by giving an alternate presentation of the $\mathfrak{g}$–stable Demazure modules and then state our main result in Section 2.4. We then discuss applications of our results, the notion of prime modules and also a generalization of the Q–systems of [20].

2.1 $\mathfrak{g}$-stable Demazure Modules: $D(\ell, \lambda)$

We introduce a family of graded modules for $\mathfrak{g}[t]$. These are indexed by a pair $(\ell, \lambda) \in \mathbf{N} \times P^+$ and the corresponding module is denoted $D(\ell, \lambda)$. For $\alpha \in R^+$, set $s_\alpha, m_\alpha \in \mathbf{N}$ by

$$\lambda(h_\alpha) = d_\alpha \ell(s_\alpha - 1) + m_\alpha, \quad 0 < m_\alpha \le d_\alpha \ell.$$

Then, $D(\ell, \lambda)$ is the $\mathfrak{g}[t]$–module generated by an element w_λ with defining relations:

$$\mathfrak{n}^+[t]w_\lambda = 0, \quad (h_i \otimes t^s)w_\lambda = \delta_{s,0}\lambda(h_i)w_\lambda, \quad (x_{\alpha_i}^-)^{\lambda(h_i)+1}w_\lambda = 0, \quad 1 \le i \le n, \tag{2.1.1}$$

$$(x_\alpha^- \otimes t^{s_\alpha})w_\lambda = 0, \tag{2.1.2}$$

$$(x_\alpha^- \otimes t^{s_\alpha - 1})^{m_\alpha + 1}w_\lambda = 0, \quad \text{if} \quad m_\alpha < d_\alpha \ell. \tag{2.1.3}$$

Remark. The relations in (2.1.1) guarantee that the module $D(\ell, \lambda)$ is finite–dimensional (a more detailed discussion of this can be found in [10]). In particular this gives,

$$(x_\alpha^- \otimes 1)^{\lambda(h_\alpha)+1} w_\lambda = 0,$$

for all $\alpha \in R^+$.

2.2 Properties of $D(\ell, \lambda)$

The defining relations of $D(\ell, \lambda)$ are graded, it follows that $D(\ell, \lambda)$ is a graded $\mathfrak{g}[t]$–module once we declare the grade of w_λ to be zero. Clearly for $s \in \mathbf{Z}$, the graded shift $\tau_s^* D(\ell, \lambda)$ is defined by letting w_λ have grade s. It is elementary to check that $\operatorname{ev}_0^* V(\lambda)$ is the unique irreducible graded quotient of $D(\ell, \lambda)$ and moreover that,

$$D(\ell, \lambda) \cong \operatorname{ev}_0^* V(\lambda), \quad \text{if} \quad \lambda(h_\alpha) \leq d_\alpha \ell, \quad \text{for all} \quad \alpha \in R^+. \tag{2.2.1}$$

It is sometimes necessary to consider simultaneously, the different level Demazure modules associated to a given weight λ, in which case we shall denote the generator of $D(\ell, \lambda)$ by $w_{\lambda,\ell}$ and the integers s_α and m_α by $s_{\alpha,\ell}$ and $m_{\alpha,\ell}$ respectively.

Lemma. For all $(\ell, \lambda) \in \mathbf{N} \times P^+$, we have,

$$\operatorname{Hom}_{\mathfrak{g}[t]}(D(\ell, \lambda), D(\ell + 1, \lambda)) = \mathbf{C}.$$

Moreover any non–zero map is surjective.

Proof. It is clear that any element $\varphi \in \operatorname{Hom}_{\mathfrak{g}[t]}(D(\ell, \lambda), D(\ell + 1, \lambda))$ must send $w_{\lambda,\ell}$ to a scalar multiple of $w_{\lambda,\ell+1}$ and hence the space of homomorphisms is at most one–dimensional. To prove that it is exactly one we must show that $w_{\lambda,\ell+1}$ satisfies the relations of $w_{\lambda,\ell}$. Write

$$\lambda(h_\alpha) = d_\alpha \ell(s_{\alpha,\ell} - 1) + m_{\alpha,\ell} = d_\alpha(\ell + 1)(s_{\alpha,\ell+1} - 1) + m_{\alpha,\ell+1},$$

with $0 < m_{\alpha,\ell} \le d_\alpha \ell$ and $0 < m_{\alpha,\ell+1} \le d_\alpha(\ell+1)$ and using the uniqueness of $s_{\alpha,\ell}$ and $m_{\alpha,\ell}$, we get that either

$$s_{\alpha,\ell} = s_{\alpha,\ell+1}, \quad m_{\alpha,\ell} = m_{\alpha,\ell+1} + d_\alpha(s_{\alpha,\ell+1} - 1) \ge m_{\alpha,\ell+1},$$

or $s_{\alpha,\ell} > s_{\alpha,\ell+1}$. In either case the assertion follows. $\square$

2.3 Connection of $D(\ell, \lambda)$ with Demazure Modules

The following result which is a combination of [18, Section 2.3, Corollary 1], [33, Proposition 3.6] and [11, Theorem 2] explains the connection with Demazure modules.

Proposition. Let $(\ell, \lambda) \in \mathbf{N} \times P^+$ and suppose that $w \in \widehat{W}$, $\sigma \in \Sigma$, $\Lambda \in \hat{P}^+$ are such that

$$w\sigma\Lambda = w_0\lambda + \ell\Lambda_0.$$

Then we have an isomorphism

$$D(\ell, \lambda) \cong V_w(\sigma\Lambda),$$

of $\mathfrak{g}[t]$–modules and hence, for all $\mu \in P$, we have

$$\dim D(\ell, \lambda)_\mu = \sum_{s \in \mathbf{Z}_{\ge 0}} \dim V_w(\sigma\Lambda)_{\ell\Lambda_0 + \mu + s\delta}. \tag{2.3.1}$$

$\square$

2.4 Steinberg-like Fusion Decomposition

The main result of this paper is the following theorem.

Theorem 1. Assume that $\mathfrak{g}$ is of classical type or of type G_2. Let $\lambda \in P^+$ and $k, \ell \in \mathbf{N}$ and write

$$\lambda = \ell \left(\sum_{s=1}^{k} \lambda^s \right) + \lambda^0, \qquad \lambda^s \in L^+, \;\; 1 \le s \le k, \;\; \lambda^0 \in P^+.$$

We have an isomorphism of graded $\mathfrak{g}[t]$–modules,

$$D(\ell,\lambda) \cong D(\ell,\lambda^0)^{z_0} * D(\ell,\ell\lambda^1)^{z_1} * \cdots * D(\ell,\ell\lambda^k)^{z_k},$$

where $z_0, \cdots, z_k$ are distinct complex numbers. In particular, the fusion product on the right hand side is independent of the choice of parameters.

2.5 Affine Weyl Group Result

In the case when $\lambda^0 = 0$ the result was first proved in [18] and a different proof was given in [11]. As in these papers, the proof of our theorem uses the theory of Demazure operators and the following additional key result proved in Chapter 3.

Proposition. Assume that $\mathfrak{g}$ is of classical type or of type G_2. Let $\lambda \in P^+$ and $\ell \in \mathbf{N}$ be such that $\lambda(h_i) \leq d_i\ell$ for all $1 \leq i \leq n$. There exists $\mu \in L^+$ and $w \in W$ such that $wt_\mu(\ell\Lambda_0 + w_0\lambda) \in \hat{P}^+$.

Remark. The restriction on $\mathfrak{g}$ in the main theorem is purely a consequence of the fact that we are able to prove Proposition 2.5 only in the case when $\mathfrak{g}$ is of classical type or of type G_2. Computer calculations for small values of ℓ show that the proposition is true for such ℓ for the other exceptional Lie algebras as well. However a proof for arbitrary ℓ seems difficult for E_8 and F_4.

2.6 Applications

For the rest of the section, we discuss applications of our result. We begin by noting the following corollary of our theorem.

Proposition. Let $\ell \in \mathbf{N}$, $\lambda_1 \in L^+$, and $\lambda_2 \in P^+$. There exists a canonical surjective map of $\mathfrak{g}[t]$–modules

$$D(\ell, \ell\lambda_1) * D(\ell, \lambda_2) \to D(\ell+1, (\ell+1)\mu_1) * D(\ell+1, \mu_2) \to 0$$

for all $\mu_1 \in L^+$, $\mu_2 \in P^+$ with $(\ell+1)\mu_1 + \mu_2 = \ell\lambda_1 + \lambda_2$.

Proof. By Theorem 1 we see that the proposition amounts to proving that

$$\operatorname{Hom}_{\mathfrak{g}[t]}(D(\ell, \ell\lambda_1 + \lambda_2), D(\ell+1, \ell\lambda_1 + \lambda_2)) \neq 0.$$

But this is precisely the statement of Lemma 2.2. □

Corollary. Let $1 \le i \le n$ be such that $\omega_i(h_\alpha) \le 1$ for all $\alpha \in R^+$. For all $\mu, \nu \in P^+$ and $\ell \in \mathbf{N}$ such that $\ell - d_i \ge \max\{\mu(h_\alpha) : \alpha \in R^+\}$ we have,

$$\dim \operatorname{Hom}_{\mathfrak{g}}(V(\nu), V(d_i(\ell+1)\omega_i) \otimes V(\mu)) \le \dim \operatorname{Hom}_{\mathfrak{g}}(V(\nu), V(d_i\ell\omega_i) \otimes V(\mu + d_i\omega_i)).$$

Proof. We apply the proposition by taking $\lambda_1 = d_i\omega_i$ and $\mu + d_i\omega_i = \lambda_2$. The conditions on i and μ imply that $(\mu + d_i\omega_i)(h_\alpha) \le \ell \le d_\alpha\ell$ and $\ell\omega_i(h_\alpha) \le \ell$ for all $\alpha \in R^+$. Equation (2.2.1) now shows that all the Demazure modules involved in the proposition are actually evaluation modules and the result follows. □

Remark. The preceding corollary generalizes Theorem 1(ii) of [6] where the case when μ is also a multiple of ω_i was proved by entirely different methods.

2.7 Q-Systems

We discuss now the kernel of the map defined in Proposition 2.6 and whether it too, can be described in terms of Demazure modules. This question can be related to the notion of Q–systems introduced and studied in [20] for arbitrary simple Lie algebras and for a pair (i,m) where i is a node of the Dynkin diagram and $m \in \mathbf{N}$. Analogs of this system exist for the quantum affine algebras. We refer the reader to [20], [21], [32] for further information. In our discussion here, we restrict ourselves to the simply–laced case and assume that i is such that ω_i is miniscule. For $(i,m) \in I \times \mathbf{N}$ the Q–system is a short exact sequence of $\mathfrak{g}$–modules

$$0 \to \bigotimes_{j:i\sim j} V(m\omega_j) \to V(m\omega_i) \otimes V(m\omega_i) \to V((m+1)\omega_i) \otimes V((m-1)\omega_i) \to 0,$$

where we say that $i \sim j$ if $i \neq j$ and the nodes i and j are connected in the Dynkin diagram. For current algebras, it was proved in [11] that each of the modules in the short exact sequence is a Demazure module for $\mathfrak{g}[t]$ of level m. In fact, a stronger statement was established: that replacing the tensor product of $\mathfrak{g}$–modules by the fusion product of $\mathfrak{g}[t]$–modules gives rise to a canonical short exact sequence of $\mathfrak{g}[t]$–modules.

A natural question to ask is if there is an analog of Q–systems associated to an arbitrary pair of dominant integral weights. In [19], a start was made on this question where they proved that if $\ell \geq m$, then there exists a surjective map of $\mathfrak{g}$–modules

$$V(\ell\omega_i) \otimes V(m\omega_i) \to V((\ell+1)\omega_i) \otimes V((m-1)\omega_i) \to 0,$$

but their methods do not allow them to determine the kernel of this map when $\ell > m$. Our next theorem, has the result of [19] as a special case (by taking $\lambda = m\omega_i$). Moreover, the short exact sequences of $\mathfrak{g}[t]$–modules are seen (by taking $\lambda = \ell\omega_i$) to be generalizations of Q–systems. It also determines the kernel of the map defined in Proposition 2.6 when $\lambda_1 = \omega_i$.

Theorem 2. Assume that $\mathfrak{g}$ is of type A or D and let $1 \leq i \leq n$ be such that $\omega_i(h_\alpha) \leq 1$ for all $\alpha \in R^+$. Choose $(\ell, \lambda) \in \mathbf{N} \times P^+$ such that

$$\lambda(h_i) \geq 1, \qquad \ell \geq \max\{\lambda(h_\alpha) : \alpha \in R^+\}.$$

Let $\nu = \ell\omega_i + \lambda - \lambda(h_i)\alpha_i$ and write $\nu = \ell\nu^1 + \nu^0$ for some $\nu^0 \in P^+$, $\nu^1 \in L^+$. There exists a canonical short exact sequence of $\mathfrak{g}[t]$–modules:

$$0 \to \tau^*_{\lambda(h_i)} \left(D(\ell, \ell\nu^1) * D(\ell, \nu^0)\right) \to D(\ell, \ell\omega_i) * D(\ell, \lambda)$$
$$\to D(\ell + 1, (\ell + 1)\omega_i) * D(\ell + 1, \lambda - \omega_i) \to 0.$$

2.8 Prime $D(\ell, \lambda)$ Modules

The study of graded representations of current algebras was originally motivated by the representation theory of quantum affine algebras. In this theory it is completely natural and interesting to talk about the prime irreducible representations: namely an irreducible representation which is not isomorphic to the tensor product of non–trivial irreducible representations (see [9], [13], [22]). An important family of prime irreducible representations are the Kirillov–Reshetikhin modules. Using the work of several authors ([10], [4],[21], [32], [26]) together with [12] shows that the $\mathfrak{g}[t]$–module $D(\ell, \ell\omega_i)$ is the “limit”of the corresponding Kirillov–Reshetikhin modules. Other examples of prime representations can be found in [7], [12], [22]. In all these examples one actually proves that the underlying $\mathfrak{g}$–module is prime which motivates the following definition.

Definition. We say that a $\mathfrak{g}$–module V is prime if it is not isomorphic to the tensor product of a non-trivial pair of $\mathfrak{g}$–modules.

It is not hard to see that any irreducible finite–dimensional $\mathfrak{g}$–module is prime. It is also trivial to construct examples of prime representations of $\mathfrak{g}$ which are reducible. For instance, in the $\mathfrak{sl}_2$ case the direct sum of the natural and the adjoint representation is obviously prime. In the case when $\dim V < \infty$ it is clear that any $\mathfrak{g}$–module has a prime factorization: in other words, is isomorphic to a tensor product of non–trivial prime modules. However, it is not known in general if such a decomposition is unique. The uniqueness of a tensor product of simple $\mathfrak{g}$–modules was proved fairly recently in [35], [38]. Notice that a $\mathfrak{g}[t]$–module V which is prime is necessarily prime with respect to the fusion product as well.

2.9 Prime Decomposition

Our final result shows that if $\mathfrak{g}$ is of type A or D, then any Demazure module is a fusion product of prime Demazure modules.

Proposition. Let $(\ell, \lambda) \in \mathbf{N} \times P^+$ and let $\mathfrak{g}$ be any simply–laced simple Lie algebra. The module $D(\ell, \lambda)$ is prime if $\lambda = \ell\omega_i$ for some $i \in I$ or $\lambda(h_i) < \ell$ for all $1 \leq i \leq n$. More generally, if $\lambda = \lambda^0 + \sum_{i\in} m_i \ell\omega_i$ where $0 \leq \lambda^0(h_i) < \ell$ for all $1 \leq i \leq n$, and $\mathfrak{g}$ is of type A or D, then the isomorphism

$$D(\ell, \lambda) \cong_{\mathfrak{g}[t]} D(\ell, \ell\omega_1)^{*m_1} * \cdots * D(\ell, \ell\omega_n)^{*m_n} * D(\ell, \lambda^0), \tag{2.9.1}$$

is a prime factorization of $D(\ell, \lambda)$.

Remark. In [1] the relationship of these prime Demazure modules to prime representations of quantum affine algebras is studied.

In Chapter 5, we investigate the $\mathfrak{g}$ character for some prime Demazure modules in type A_n.

Chapter 3

Affine Weyl Group

For $w \in W$ and $\lambda, \mu \in P^+$, we have

$$wt_\mu(\ell\Lambda_0 + w_0\lambda) = \ell\Lambda_0 + w(\ell\mu + w_0\lambda) + A\delta$$

for some $A \in \mathbf{Z}$. Hence, $wt_\mu(\ell\Lambda_0 + w_0\lambda) \in \hat{P}^+$ iff $w \in W$ is such that

$$w(\ell\mu + w_0\lambda) \in P^+ \quad \text{and} \qquad w(\ell\mu + w_0\lambda)(h_\theta) \leq \ell.$$

This shows that Proposition 2.5 is an immediate consequence of the following,

Lemma. Given $(\ell, \lambda) \in \mathbf{N} \times \mathfrak{h}^*$ with $0 \leq \lambda(h_i) \leq d_i\ell$ (equivalently that $0 \leq (\lambda, \alpha_i) \leq \ell$)) for $1 \leq i \leq n$, there exists $\mu \in L^+$ such that

$$|(\ell\mu - \lambda, \alpha)| \leq \ell, \tag{3.0.1}$$

for all $\alpha \in R^+$.

The Lemma is proved in the rest of the section. The strategy for proving the Lemma is as follows. We give an inductive construction of μ in the case of $\mathfrak{g} = C_n$ and use elementary results on root systems to deduce the existence of μ in the other classical cases. In the case of G_2, we write down explicit solutions of μ. *From now on, we will* *assume that (ℓ, λ) are fixed and satisfy the conditions of the Lemma. We remind the reader that we are working with the form on $\mathfrak{h}^*$ which has been normalized so that the square length of a long root is two.*

3.1 Type C

Lemma. Assume that $\mathfrak{g}$ is of type C_n and that α_n is the unique long simple root. There exists $\mu = 2\sum_{i=1}^{n-1} s_i\omega_i$ with $s_i \in \{0,1\}$ satisfying $|(\ell\mu - \lambda, \alpha)| \leq \ell$ for all $\alpha \in R^+$.

Proof. Any short root $\alpha \in R$ is one half the difference of two long roots and hence it suffices to find μ such that $|(\ell\mu - \lambda, \alpha)| \leq \ell$ holds for the long roots.

We proceed by induction on n, with induction beginning at $n = 1$ where we can take $\mu = 0$. For the inductive step assume that the result is proved for the C_{n-1}–subdiagram of C_n defined by the simple roots $\{\alpha_2, \cdots \alpha_n\}$ of C_n. Let $\mu' = 2\sum_{j=2}^{n-1} s_i\omega_i \in L^+$, with $s_i \in \{0,1\}$ such that

$$|(\ell\mu' - \lambda, \alpha)| \leq \ell,$$

for all roots α of C_{n-1}. The only additional long root in C_n is the highest root θ. Moreover, $\theta - 2\alpha_1$ is a root of C_{n-1} and so we take

$$\mu = \begin{cases} \mu' \ \text{ if } |(\lambda, \theta) - \ell(\mu', \theta - 2\alpha_1)| \leq \ell, \\ \\ 2\omega_1 + \mu', \quad \text{otherwise.} \end{cases}$$

A simple calculation completes the proof. □

3.2 Type A

The diagram subalgebra of C_n generated by the root vectors $x_i^{\pm}$, $1 \leq i \leq n-1$ is isomorphic to A_{n-1} and the restriction of the fundamental weights ω_i, $1 \leq i \leq n-1$

of C_n to A_{n-1} gives a set fundamental weights for A_{n-1}. There is one important thing to note here however. The restriction of the normalized form $(\ ,\ )$ of C_n to the A_{n-1} subdiagram is one half of the normalized form on A_{n-1}. This means that if λ is any element in the real span of ω_i, $1 \leq i \leq n-1$ satisfying the conditions of Lemma 3 of A_{n-1} with respect to its normalized form, then the element 2λ regarded as an element of C_n satisfies $0 \leq (2\lambda, \alpha_i) \leq \ell$ for all $1 \leq i \leq n$ with respect to the normalized form on C_n. Hence we can find $\mu = \sum_{i=1}^{n-1} s_i\omega_i$, with $s_i \in \{0, 1\}$ such that

$$|(2\lambda - 2\ell\mu, \alpha)| \leq \ell,$$

for all short roots α of C_n and hence for all roots of A_{n-1}. This gives that μ satisfies (3.0.1) for λ with respect to the form on A_{n-1} and the Lemma is established in this case.

3.3 Type D

To prove the Lemma for D_n, we observe that the subset of short roots of C_n form a root system of type D_n. Notice again that the restriction of the normalized form on C_n to D_n is one half the normalized form of D_n. The simple system for D_n is the set $\{\alpha_i : 1 \leq i \leq n-1\} \cup \{\alpha_{n-1} + \alpha_n\}$ and the set of fundamental weights is $\{\omega_i : 1 \leq i \leq n-2\} \cup \{\omega_{n-1} - \frac{1}{2}\omega_n,\ \ \frac{1}{2}\omega_n\}$. In particular this means that if λ is in the real span of the fundamental weights for D_n satisfying the hypothesis of Lemma 3, then, either 2λ or $2\lambda\sigma$ (here σ is the diagram automorphism of D_n which switches the spin nodes and leaves the others fixed) satisfy the conditions for C_n. Hence we can choose a dominant integral weight for C_n of the form 2μ where $\mu = \sum_{i=1}^{n-1} s_i\omega_i$, $s_i \in \{0, 1\}$, $1 \leq i \leq n-1$ such that

$$|2(\ell\mu - \lambda), \alpha)| \leq \ell \ \ (\text{resp. } \ |2(\ell\mu - \lambda\sigma), \alpha)| \leq \ell)$$

for all short roots α of C_n, i.e., for all roots of D_n. Since μ and $\mu\sigma$ are dominant integral weights of D_n, Lemma 3 follows for the element λ with μ or $\mu\sigma$ and the normalized form of D_n, according as 2λ or $2\lambda\sigma$ is dominant for C_n. *We remark here that the element μ when regarded as an element of D_n is such that it is either not supported on the spin nodes or it is supported on both spin nodes. This is because either $s_{n-1} = 0$ in which case it is not supported on the spin nodes or $s_{n-1} = 1$ and we have*

$$\mu = \sum_{i=1}^{n-2} s_i\omega_i + (\omega_{n-1} - \frac{1}{2}\omega_n) + \frac{1}{2}\omega_n$$

3.4 Type B

To prove the result for B_n we first observe that it is enough to prove that there exists $\mu \in L^+$ such that (3.0.1) is satisfied for the long roots. This is because any short root is half the difference of two long roots. Recall that B_n can be regarded as a subalgebra of D_{n+1} by folding: namely it is the fixed points of the automorphism σ which interchanges the spin nodes and leaves the others fixed. If α_i, $1 \leq i \leq n+1$ are the simple roots of D_{n+1}, then the simple roots of B_n are α_i, $1 \leq i \leq n-1$ and $\frac{1}{2}(\alpha_n + \alpha_{n+1})$. It is easily seen that any long root of B_n is a root of D_{n+1}.

The restriction of the normalized form of D_{n+1} to B_n is the normalized form of B_n. The set of dominant integral weights for B_n is ω_i, $1 \leq i \leq n-1$, and $\frac{1}{2}(\omega_n + \omega_{n+1})$. Given $\lambda = \sum_{i=1}^{n-1} r_i\omega_i + r_n\frac{1}{2}(\omega_n + \omega_{n+1})$, one sees that if λ satisfies the conditions of Lemma 3.0.1 for B_n, then we have that $r_n \leq 2\ell$ and hence λ also satisfies the conditions for D_{n+1}. Choose $\mu = \sum_{i=1}^{n+1} s_i\omega_i$ as in Section 3.3 such $s_i \in \{0, 1\}$ satisfies (3.0.1) for

D_{n+1}. Since either $s_n = s_{n+1} = 0$ or $s_n = s_{n+1} = 1$, we see that μ is in the lattice L^+ for B_n and hence Lemma 3 follows for B_n.

3.5 Type G_2

If $\mathfrak{g}$ is of type G_2, we assume that α_2 is the simple short root. We note that it is enough to prove that there exists a $\mu \in L^+$, which satisfies (3.0.1) only on long roots. This is because any non-simple short root can be written as either a half or a third of the sum of two long roots. Next, we observe that we have,

$$(\omega_1, \alpha_1) = 1, \quad (\omega_2, \alpha_2) = 1/3.$$

Let μ be the following weight in L^+,

$$\mu = \begin{cases} 0, & \text{if } (\lambda, 2\alpha_1 + 3\alpha_2) \le \ell \\ \omega_1, & \text{if } \ell < (\lambda, 2\alpha_1 + 3\alpha_2) \le 3\ell \text{ and } (\lambda, \alpha_1 + 3\alpha_2) \le 2\ell \\ 3\omega_2, & \text{if } 2\ell < (\lambda, 2\alpha_1 + 3\alpha_2) \le 4\ell \text{ and } (\lambda, \alpha_1 + 3\alpha_2) > 2\ell \\ \omega_1 + 3\omega_2, & \text{if } 4\ell < (\lambda, 2\alpha_1 + 3\alpha_2) \le 5\ell \end{cases}$$

where we note that the last condition $4\ell < (\lambda, 2\alpha_1 + 3\alpha_2)$ implies that $(\lambda, \alpha_1 + 3\alpha_2) > 3\ell$. Therefore, one can check easily that the condition $|(\ell\mu - \lambda, \alpha)| \le \ell$ is satisfied for all positive long roots, and hence all positive roots.

3.6 The case of E and F_4

It is clear that it suffices to prove Proposition 2.5 for E_8 and F_4. The methods of this section do not appear to generalize to these cases. However, it is possible to check using mathematica that Proposition 2.5 is true for ℓ at least five. In the tables in the appendix, we associate to the ordered pair $(a_1, \cdots, a_n)$ the weight $\nu = \sum_{i=1}^{n} a_i\omega_i$.

For $\ell = 2$, we provide one solution for every λ with $\lambda(h_i) \leq 1$ for all $1 \leq i \leq n$.

Chapter 4

Proof of Main Results

4.1 Proof of Steinberg Type Factorization

In this section we shall assume Proposition 2.5 and prove Theorem 1. As in [17] and [37], the proof uses the Demazure operators and the Demazure character formula in a crucial way. We recollect these concepts briefly and refer the interested reader to [14], [17], [29] and [31] for a more detailed discussion.

4.1.1

There are two main ingredients in the proof of the Theorem. The first is the following proposition which was proved in [37] but we include a very brief sketch of the proof for the reader's convenience.

Proposition. Let $(\ell, \lambda) \in \mathbb{N} \times P^+$. Let $(p_j, \mu_j) \in \mathbf{N} \times L^+$ for $1 \leq j \leq m$ be such that there exists $\mu \in P^+$ with

$$\ell\mu = p_1\mu_1 + \cdots + p_m\mu_m, \quad \mu(h_\alpha) \geq \sum_{j=1}^{m} \mu_j(h_\alpha), \quad \text{for all} \quad \alpha \in R^+.$$

There exists a non-zero surjective map of graded $\mathfrak{g}[t]$-modules,

$$D(\ell, \ell\mu + \lambda) \longrightarrow D(p_1, p_1\mu_1) * \cdots * D(p_m, p_m\mu_m) * D(\ell, \lambda) \to 0.$$

Proof. For $\alpha \in R^+$, and $1 \le j \le m$, write

$$\lambda(h_\alpha) = d_\alpha \ell(r_\alpha - 1) + m_\alpha, \quad 0 < m_\alpha \le d_\alpha \ell, \qquad \mu(h_\alpha) = d_\alpha s_\alpha, \;\; \mu_j(h_\alpha) = d_\alpha s_\alpha^j.$$

For $1 \le j \le m$ set $v_j = w_{p_j\mu_j}$ and recall that

$$(x_\alpha^- \otimes t^{s_\alpha^j})v_j = 0, \quad (x_\alpha^- \otimes t^{r_\alpha})w_\lambda = 0, \;\; (x_\alpha^- \otimes t^{r_\alpha - 1})^{m_\alpha + 1}w_\lambda = 0.$$

Let $\mathbf{w}$ be the image of $v_1 \otimes \cdots \otimes v_m \otimes w_\lambda$ in $D(p_1, p_1\mu_1) * \cdots * D(p_m, p_m\mu_m) * D(\ell, \lambda)$. The proposition follows if we show that for $\alpha \in R^+$,

$$(x_\alpha^- \otimes t^{s_\alpha + r_\alpha})\mathbf{w} = 0, \qquad \text{and} \;\; (x_\alpha^- \otimes t^{s_\alpha + r_\alpha - 1})^{m_\alpha + 1}\mathbf{w} = 0, \;\; \text{if} \;\; m_\alpha < d_\alpha \ell. \qquad (4.1.1)$$

Set $b_\alpha = s_\alpha - \sum_j s_\alpha^j$ and note that our assumptions imply that $b_\alpha \ge 0$. For $z_1, \cdots, z_{m+1}$ be the distinct complex numbers which define the fusion product. This means that in the corresponding tensor product, we have

$$\begin{aligned}(x_\alpha^- \otimes t^{b_\alpha}(t - z_1)^{s_\alpha^1} \cdots (t - z_m)^{s_\alpha^m}(t - z_{m+1})^{r_\alpha})(v_1 \otimes \cdots \otimes v_m \otimes v_{m+1}) \\ = \sum_{j=1}^{m+1} \Big(v_1 \otimes \cdots \otimes (x_\alpha^- \otimes t^{s_\alpha^j} g_j(t) v_j) \otimes \cdots \otimes v_{m+1} \Big) = 0,\end{aligned}$$

where $v_{m+1} = w_\lambda$ and $g_j(t) = \prod_{r \ne j}(t - z_r + z_j)^{s_\alpha^r}$. It is now immediate that $(x_\alpha^- \otimes t^{s_\alpha + r_\alpha})\mathbf{w} = 0$. The proof of the second equality in (4.1.1) is identical and we omit the details.

□

4.1.2

The second result that we need is the following.

Proposition. For $(\ell, \lambda) \in \mathbb{N} \times P^+$ and $(\ell, \mu) \in \mathbf{N} \times L^+$, we have,

$$\dim D(\ell, \ell\mu + \lambda) = \dim D(\ell, \lambda) \dim D(\ell, \ell\mu).$$

Assuming Proposition 4.1.2 the proof of Theorem 1 is completed as follows. It was proved in [18] that if $\mu_s \in L^+$ for $1 \leq s \leq m$, then

$$\dim D(\ell, \ell\mu) = \prod_{s=1}^{m} \dim D(\ell, \ell\mu_s),$$

where $\mu = \sum_{s=1}^{m} \mu_s$. Using Proposition 4.1.2, we get

$$\dim D(\ell, \ell\mu + \lambda) = \dim \left(D(\ell, \ell\mu_1) * \cdots * D(\ell, \ell\mu_m) * D(\ell, \lambda)\right).$$

Taking $p_1 = \cdots p_m = \ell$ in Proposition 4.1.1 now establishes Theorem 1.

4.1.3

The rest of the section is devoted to the proof of Proposition 4.1.2. Recall from Section 1.5 that the composite map

$$\mathbf{Z}[P] \hookrightarrow \mathbf{Z}[\hat{P}] \longrightarrow \mathbf{Z}[\hat{P}]/I_\delta,$$

is injective. Given two elements χ, χ' of $\mathbf{Z}[\hat{P}]$, we write $\chi \equiv \chi'$ if they have the same image in $\mathbf{Z}[\hat{P}]/I_\delta$.

Lemma. Let $w \in \widehat{W}$, $\sigma \in \Sigma$, $\Lambda \in \hat{P}^+$ and $(\ell, \lambda) \in \mathbf{N} \times P^+$ be such that $w\sigma\Lambda = w_0\lambda + \ell\Lambda_0$. Then $\mathrm{ch}_{\mathfrak{h}} D(\ell, \lambda) = \sum_{\mu \in P} \dim D(\ell, \lambda)_\mu e(\mu) \in \mathbf{Z}[P]$ is invariant under the action of W on P and we have

$$\mathrm{ch}_{\hat{\mathfrak{h}}} V_w(\sigma\Lambda) \equiv e(\ell\Lambda_0) \mathrm{ch}_{\mathfrak{h}} D(\ell, \lambda).$$

Proof. The fact that $\mathrm{ch}_{\mathfrak{h}} D(\ell, \lambda)$ is W–invariant is immediate since $D(\ell, \lambda)$ is a finite–dimensional $\mathfrak{g}$–module. Recall that,

$$\mathrm{ch}_{\hat{\mathfrak{h}}} V_w(\sigma\Lambda) = \sum_{\Lambda' \in \hat{P}} \dim(V_w(\sigma\Lambda)_{\Lambda'}) e(\Lambda').$$

Since $\Lambda(c)=\ell$, we may assume that the sum is over elements of $\hat{P}$ of the form $\ell\Lambda_0+\mu+s\delta$ for $\mu\in P$ and $s\in\mathbf{Z}_{\geq 0}$. Going mod I_δ, we get that

$$\mathrm{ch}_{\hat{\mathfrak{h}}}V_w(\sigma\Lambda)\equiv e(\ell\Lambda_0)\sum_{\mu\in P}\left(\sum_{s\in\mathbf{Z}_{\geq 0}}\dim V_w(\sigma\Lambda)_{\ell\Lambda_0+\mu+s\delta}\right)e(\mu)=e(\ell\Lambda_0)\mathrm{ch}_{\mathfrak{h}}D(\ell,\lambda),$$

where the last equality follows from (2.3.1).

□

4.1.4

For $0\leq i\leq n$, the Demazure operator $D_i:\mathbf{Z}[\hat{P}]\to\mathbf{Z}[\hat{P}]$ is defined by,

$$D_i(e(\Lambda))=\frac{e(\Lambda)-e(\mathbf{s}_i(\Lambda)-\alpha_i)}{1-e(-\alpha_i)}.$$

Here for $1\leq i\leq n$ we identify the generator $\mathbf{s}_i$ of W with the element $(\mathbf{s}_i,0)$ of $\widehat{W}$ and $\mathbf{s}_0=(s_\theta,t_\theta)$. Given a reduced expression $w=\mathbf{s}_{i_1}\cdots\mathbf{s}_{i_r}$ for an element $w\in\widehat{W}$, set $D_w=D_{i_1}\cdots D_{i_r}$, and note that D_w is independent of the choice of reduced expression for w (see [28], Corollary 8.2.10). For $\sigma\in\Sigma$, and $w\in\widehat{W}$, set $D_{w\sigma}(e(\Lambda))=D_w(e(\sigma(\Lambda))$. Since $D_i(e(\delta))=e(\delta)$, it follows that for all $w\in\widetilde{W}$, the operator D_w descends to $\mathbf{Z}[\hat{P}]/I_\delta$.

The following result is proved in [17, Lemma 6, Lemma 7, Section 3].

Lemma. Let $\chi\in\mathbf{Z}[P]$ be a W–invariant element of $\mathbf{Z}[P]$. Then $D_w(\chi)\equiv\chi$ for all $w\in\widetilde{W}$. Moreover, for all $\Lambda\in\hat{P}$, we have

$$D_w(e(\Lambda)\chi)\equiv\chi D_w(e(\Lambda)).$$

□

Along with Lemma 4.1.3, we get

$$D_w(e(\ell\Lambda_0)\mathrm{ch}_{\mathfrak{h}}D(\ell,\lambda)) \equiv D_w(e(\ell\Lambda_0))\mathrm{ch}_{\mathfrak{h}}D(\ell,\lambda), \tag{4.1.2}$$

for all $(\ell,\lambda) \in \mathbf{N} \times P^+$ and $w \in \widetilde{W}$.

4.1.5

The following result may be found in [29, Theorem 3.5] and [28, Theorem 8.2.9].

Theorem 3. For $w \in \widehat{W}$, $\sigma \in \Sigma$, and $\Lambda \in \widehat{P}^+$ we have

$$\mathrm{ch}_{\hat{\mathfrak{h}}}V_w(\sigma\Lambda) = D_{w\sigma}(e(\Lambda)).$$

$\square$

Lemma 4.1.3 and Theorem 3 now gives,

$$D_{w\sigma}(e(\Lambda)) \equiv e(\ell\Lambda_0)\mathrm{ch}_{\mathfrak{h}}D(\ell,\lambda), \tag{4.1.3}$$

for all $\sigma \in \Sigma$ and $w \in \widehat{W}$ such that $w\sigma\Lambda = w_0\lambda + \ell\Lambda_0$.

4.1.6

The next result makes crucial use of Proposition 2.5.

Lemma. Let $\ell \in \mathbf{N}$ and $\lambda \in P^+$ be such that $\lambda = \ell\lambda_1 + \lambda_2$ where $\lambda_1 \in L^+$ and $\lambda_2 \in P^+$ satisfies $\lambda_2(h_i) \leq d_i\ell$ for all $1 \leq i \leq n$. Then,

$$\mathrm{ch}_{\mathfrak{h}}D(\ell,\lambda) = \mathrm{ch}_{\mathfrak{h}}D(\ell,\ell\lambda_1)\mathrm{ch}_{\mathfrak{h}}D(\ell,\lambda_2).$$

Proof. By Proposition 2.5 we can choose $\nu \in L^+$ and $w \in W$ such that

$$\Lambda = w^{-1}t_\nu(\ell\Lambda_0 + w_0\lambda_2) \in \hat{P}^+.$$

Since $t_{w_0\lambda_1}t_{-\nu}w(\Lambda) = \ell\Lambda_0 + w_0\lambda + m\delta$ for some $m \in \mathbf{Z}$, it follows from (4.1.3) that

$$e(\ell\Lambda_0)\mathrm{ch}_{\mathfrak{h}}D(\ell,\lambda) \equiv D_{t_{w_0\lambda_1}t_{-\nu}w}(e(\Lambda)).$$

Proposition 1.8 gives

$$\ell(t_{w_0\lambda_1}t_{-\nu}w) = \ell(t_{w_0\lambda_1}) + \ell(t_{-\nu}w),$$

and hence using the properties of Demazure operators we get,

$$D_{t_{w_0\lambda_1}t_{-\nu}w}(e(\Lambda)) = D_{t_{w_0\lambda_1}}D_{t_{-\nu}w}(e(\Lambda)).$$

Using (4.1.3) we get

$$D_{t_{w_0\lambda_1}}D_{t_{-\nu}w}(e(\Lambda)) \equiv D_{t_{w_0\lambda_1}}(e(\ell\Lambda_0)\mathrm{ch}_{\mathfrak{h}}D(\ell,\lambda_2)).$$

Using (4.1.2) and a further application of (4.1.3) gives,

$$\begin{aligned} D_{t_{w_0\lambda_1}}(e(\ell\Lambda_0)\mathrm{ch}_{\mathfrak{h}}D(\ell,\lambda_2)) &\equiv D_{t_{w_0\lambda_1}}(e(\ell\Lambda_0))\mathrm{ch}_{\mathfrak{h}}D(\ell,\lambda_2) \\ &\equiv e(\ell\Lambda_0)\mathrm{ch}_{\mathfrak{h}}D(\ell,\ell\lambda_1)\mathrm{ch}_{\mathfrak{h}}D(\ell,\lambda_2). \end{aligned}$$

Hence we get

$$\mathrm{ch}_{\mathfrak{h}}D(\ell,\lambda) \equiv \mathrm{ch}_{\mathfrak{h}}D(\ell,\ell\lambda_1)\mathrm{ch}_{\mathfrak{h}}D(\ell,\lambda_2)$$

and the Lemma follows since the map $\mathbf{Z}[P] \to \mathbf{Z}[\hat{P}]/I_\delta$ is injective. □

4.1.7

Proposition 4.1.2 follows if we prove that for all $\lambda \in P^+$ and $\mu \in L^+$, we have

$$D(\ell,\ell\mu+\lambda) \cong_{\mathfrak{g}} D(\ell,\ell\mu) \otimes D(\ell,\lambda).$$

Since finite–dimensional $\mathfrak{g}$–modules are determined by their characters, it suffices to prove that

$$\mathrm{ch}_{\mathfrak{h}}D(\ell,\ell\mu+\lambda) = \mathrm{ch}_{\mathfrak{h}}D(\ell,\ell\mu)\mathrm{ch}_{\mathfrak{h}}D(\ell,\lambda).$$

Write $\lambda = \ell\lambda_1 + \lambda_2$ where $\lambda_1 \in L^+$ and $\lambda_2 \in P^+$ satisfies $\lambda_2(h_i) < d_i\ell$ for all $1 \leq i \leq n$. By Lemma 4.1.6, we get

$$D(\ell, \ell\mu + \lambda) \cong_{\mathfrak{g}} D(\ell, \ell\mu + \ell\lambda_1) \otimes D(\ell, \lambda_2)$$

$$\cong_{\mathfrak{g}} D(\ell, \ell\mu) \otimes D(\ell, \ell\lambda_1) \otimes D(\ell, \lambda_2)$$

$$\cong_{\mathfrak{g}} D(\ell, \ell\mu) \otimes D(\ell, \lambda),$$

where the second and the the third isomorphisms are a further application of Lemma 4.1.6.

4.2 Proof of Generalized Q-System

Throughout this section $\mathfrak{g}$ is simply–laced and $i \in I$ is such that $\omega_i(h_\alpha) \leq 1$ for all $\alpha \in R^+$. In particular, this means that the multiplicity of α_i in any positive root is at most one. We also fix $(\ell, \lambda) \in \mathbf{N} \times P^+$ with $\lambda(h_\alpha) \leq \ell$ for all $\alpha \in R^+$, and write

$$(\ell\omega_i + \lambda)(h_\alpha) = \ell(s_{\alpha,\ell} - 1) + m_{\alpha,\ell}, \quad 0 < m_{\alpha,\ell} \leq \ell \quad \alpha \in R^+.$$

For $\alpha = \sum_{j=1}^{n} r_j\alpha_j$, set

$$\operatorname{supp}\alpha = \{j \in I : r_j > 0\}.$$

4.2.1

Proposition. The defining relation, (2.1.3), of $D(\ell, \ell\omega_i + \lambda)$ is a consequence of (2.1.1), (2.1.2) and the single additional relation,

$$(x_{\alpha_i}^- \otimes t)^{\lambda(h_i)+1} w_{\ell\omega_i+\lambda} = 0. \tag{4.2.1}$$

Proof. A simple calculation shows that either $s_{\alpha_i,\ell} = 1$ and $\lambda(h_i) = 0$ or $s_{\alpha_i,\ell} = 2$ and $m_{\alpha_i,\ell} = \lambda(h_i)$. In the first case, the relation (2.1.2) and in the second case the relation (2.1.3) shows that the relation (4.2.1) does hold in $D(\ell, \ell\omega_i + \lambda)$.

If $\omega_i(h_\alpha)=0$, then $s_{\alpha,\ell}=1$ and $m_{\alpha,\ell}=(\ell\omega_i+\lambda)(h_\alpha)=\lambda(h_\alpha)$. For such α the relation (2.1.3) is $(x_\alpha^-\otimes 1)^{(\ell\omega_i+\lambda)(h_\alpha)+1}w_{\ell\omega_i+\lambda}=0$ which is the content of Remark 2.1. It remains to consider the case when $\omega_i(h_\alpha)=1$ and $\alpha\neq\alpha_i$. If $\lambda(h_\alpha)=0$, then $m_{\alpha,\ell}=\ell$ and there is nothing to check. Otherwise, $\lambda(h_\alpha)>0$ and $s_{\alpha,\ell}=2$, $m_{\alpha,\ell}=\lambda(h_\alpha)$. We proceed by induction on $\operatorname{ht}\alpha$ with induction obviously beginning with $\alpha=\alpha_i$. Writing $\alpha=\beta+\gamma$ for some positive roots β and γ, we assume without loss of generality that $i\notin\operatorname{supp}\gamma$. Since $\alpha(h_\alpha)=2$, and we are in the simply laced case, it follows that

$$(\alpha,\beta)=(\alpha,\gamma)=1,\quad \beta-\gamma\notin R,\quad \beta+\alpha\notin R.$$

By the inductive hypotheses we have

$$(x_\beta^-\otimes t)^{\lambda(h_\beta)+1}w_{\ell\omega_i+\lambda}=0. \tag{4.2.2}$$

Suppose for a contradiction that

$$(x_\alpha^-\otimes t)^{\lambda(h_\alpha)+1}w_{\ell\omega_i+\lambda}\neq 0.$$

Since

$$(\ell\omega_i+\lambda-(\lambda(h_\alpha)+1)\alpha)(h_\gamma)=(\lambda-(\lambda(h_\alpha)+1)\alpha)(h_\gamma)=-\lambda(h_\beta)-1<0,$$

we get by applying the representation theory of $\mathfrak{sl}_2$ to $x_\gamma^\pm, h_\gamma$ that

$$(x_\gamma^+)^{\lambda(h_\beta)+1}(x_\alpha^-\otimes t)^{\lambda(h_\alpha)+1}w_{\ell\omega_i+\lambda}\neq 0.$$

Since

$$[x_\gamma^+,x_\alpha^-]=Ax_\beta^-,\quad [x_\alpha^-,x_\beta^-]=0\quad [x_\beta^-,x_\gamma^+]=0,$$

for some non–zero constant A, it follows by using the first two relations in (2.1.1) that

$$(x_\alpha^-\otimes t)^{\lambda(h_\gamma)}(x_\beta^-\otimes t)^{\lambda(h_\beta)+1}w_{\ell\omega_i+\lambda}\neq 0,$$

which contradicts (4.2.2) and completes the proof.

$\square$

4.2.2

We now prove,

Lemma. Suppose that $\lambda(h_i) > 0$ and $(\ell, \lambda) \in \mathbf{N} \times P^+$. There exists a surjective map of graded $\mathfrak{g}[t]$–modules

$$\pi : D(\ell, \ell\omega_i + \lambda) \to D(\ell + 1, \ell\omega_i + \lambda) \to 0,$$

with

$$\ker \pi = \mathbf{U}(\mathfrak{g}[t])(x_{\alpha_i}^- \otimes t)^{\lambda(h_i)} w_{\ell\omega_i+\lambda}.$$

Proof. The existence of a non–zero map $\pi : D(\ell, \ell\omega_i + \lambda) \to D(\ell + 1, \ell\omega_i + \lambda) \to 0$, is guaranteed by Lemma 2.2. Since $\ell\omega_i + \lambda = (\ell + 1)\omega_i + (\lambda - \omega_i)$ and $\lambda - \omega_i \in P^+$, it follows that Proposition 4.2.1 applies to both $D(\ell, \ell\omega_i + \lambda)$ and to $D(\ell + 1, \ell\omega_i + \lambda)$. In particular, (4.2.1) shows that

$$(x_{\alpha_i}^- \otimes t)^{\lambda(h_i)} w_{\ell\omega_i+\lambda} \in \ker \pi.$$

To prove that it generates the kernel, notice first that $w_{\ell\omega_i+\lambda}$ and $\pi(w_{\ell\omega_i+\lambda})$ both satisfy all the relations in (2.1.1). The Lemma follows if we prove that $(x_\alpha^- \otimes t^{s_{\alpha,\ell}})w_{\ell\omega_i+\lambda}$ is in the $\mathfrak{g}[t]$—submodule of $D(\ell, \ell\omega_i + \lambda)$ generated by $(x_{\alpha_i}^- \otimes t)^{\lambda(h_i)} w_{\ell\omega_i+\lambda}$, where

$$(\ell\omega_i + \lambda)(h_\alpha) = \ell(s_{\alpha,\ell} - 1) + m_{\alpha,\ell} = (\ell + 1)(s_{\alpha,\ell+1} - 1) + m_{\alpha,\ell+1}.$$

If $i \notin \operatorname{supp} \alpha$, then $s_{\alpha,\ell} = s_{\alpha,\ell+1} = 1$ and so $(x_\alpha^- \otimes t^{s_{\alpha,\ell+1}})w_{\ell\omega_i+\lambda} = 0$ and there is nothing to prove. If $i \in \operatorname{supp} \alpha$ and $\lambda(h_\alpha) > 1$ then $(\lambda - \omega_i)(h_\alpha) > 0$ and so $s_{\alpha,\ell} = s_{\alpha,\ell+1} = 2$ and we are done. It remains to consider the case when $\lambda(h_\alpha) = \omega_i(h_\alpha) = 1$. In this case

$$s_{\alpha,\ell} = 2, \quad m_{\alpha,\ell} = 1, \quad s_{\alpha,\ell+1} = 1, \quad m_{\alpha,\ell+1} = \ell + 1 \tag{4.2.3}$$

and the only thing to check is that $(x_\alpha^- \otimes t)w_{\ell\omega_i+\lambda}$ is in the $\mathfrak{g}[t]$–submodule of $D(\ell, \ell\omega_i + \lambda)$ generated by $(x_{\alpha_i}^- \otimes t)w_{\ell\omega_i+\lambda}$. For this we proceed by induction on $\operatorname{ht} \alpha$. If $\operatorname{ht} \alpha = 1$, then $\alpha = \alpha_i$ and hence induction begins. Write $\alpha = \beta + \gamma$ with $i \in \operatorname{supp} \beta$ in which case $i \notin \operatorname{supp} \gamma$. Notice that

$$\lambda(h_\alpha) = 1 \implies \lambda(h_\beta) = 1, \ \ (\ell\omega_i + \lambda)(h_\gamma) = 0.$$

Hence using the induction hypothesis for β and the third equality in (2.1.1) for γ, we get

$$(x_\alpha^- \otimes t)w_{\ell\omega_i+\lambda} = x_\gamma^-(x_\beta^- \otimes t)w_{\ell\omega_i+\lambda} \in \mathbf{U}(\mathfrak{g}[t])(x_{\alpha_i}^- \otimes t)w_{\ell\omega_i+\lambda}.$$

This completes the proof of the Lemma.

□

4.2.3

The following Lemma now clearly completes the proof of Theorem 2.

Lemma. Suppose that $\lambda(h_i) > 0$ and $(\ell, \lambda) \in \mathbf{N} \times P^+$ and let $\mu = \ell\omega_i + \lambda - \lambda(h_i)\alpha_i$. The assignment $w_\mu \to (x_i^- \otimes t)^{\lambda(h_i)} w_{\lambda+\ell\omega_i}$ defines an injective map of $\mathfrak{g}[t]$–modules

$$\iota : \tau^*_{\lambda(h_i)} D(\ell, \mu) \to D(\ell, \lambda + \ell\omega_i).$$

Proof. Choose $\Lambda \in \hat{P}^+$ such that $w\Lambda = w_0(\ell\omega_i + \lambda) + \ell\Lambda_0$ for some $w \in \widehat{W}$. Then,

$$D(\ell, \ell\omega_i + \lambda) \cong_{\mathfrak{g}[t]} V_w(\Lambda).$$

The element $w_{\ell\omega_i+\lambda}$ maps to a non–zero element $v_{w_0 w\Lambda} \in (V_w(\Lambda))_{w_0 w\Lambda}$. Since

$$(w_0 w\Lambda, -\alpha_i + \delta) = (\ell\omega_i + \lambda + \ell\Lambda_0, -\alpha_i + \delta) = -(\lambda, \alpha_i) < 0,$$

it follows from the representation theory of the $\mathfrak{sl}_2$ associated to the root $-\alpha_i + \delta$ that

$$0 \neq (x_i^- \otimes t)^{\lambda(h_i)} v_{w_0 w\Lambda} \in V_w(\Lambda)_{\mathbf{s}_{\alpha_i - \delta} w_0 w\Lambda},$$

where $\mathbf{s}_{\alpha_i-\delta}$ is the reflection in $\widehat{W}$ corresponding to the root $\alpha_i - \delta$. In particular,

$$(x_i^- \otimes t)^{\lambda(h_i)} w_{\ell\omega_i+\lambda} \neq 0.$$

Since $V_w(\Lambda)$ is a $\mathfrak{g}$–stable Demazure module, it follows that the $\mathfrak{g}$–module through $(x_i^- \otimes t)^{\lambda(h_i)} v_{w_0 w\Lambda}$ is contained in it and hence we get that

$$V(\Lambda)_{w_0\mathbf{s}_{\alpha_i-\delta}w_0 w\Lambda} \subset V_w(\Lambda).$$

This means that we have an inclusion of Demazure modules $V_{w_0\mathbf{s}_{\alpha_i-\delta}w_0 w\Lambda}(\Lambda) \hookrightarrow V_w(\Lambda)$. A straightforward calculation now shows that

$$V_{w_0\mathbf{s}_{\alpha_i-\delta}w_0 w\Lambda}(\Lambda) \cong_{\mathfrak{g}[t]} \tau^*_{\lambda(h_i)} D(\ell, \mu)$$

which completes the proof.

□

4.3 Proof of Prime Demazure Modules

To prove Proposition 2.9 we must show that if $(\ell, \lambda) \in \mathbf{N} \times P^+$ is such that $\lambda(h_i) \leq \ell$, then $D(\ell, \lambda)$ is prime. We shall prove this in the rest of the section *assuming that* $\mathfrak{g}$ *is simply–laced, including the algebras of type E.*

4.3.1

The first step in proving Proposition 2.9 is,

Lemma. Let V be a finite–dimensional $\mathfrak{g}$–module such that:

$$\dim V_\lambda = 1, \qquad \operatorname{wt} V \subset \lambda - Q^+.$$

Suppose that $V \cong V_1 \otimes V_2$, where V_j, $j = 1, 2$ are non–trivial finite–dimensional $\mathfrak{g}$–modules. There exists a unique pair of non–zero elements $\mu_j \in \operatorname{wt} V_j \cap P^+$ such that

$$\mu_1 + \mu_2 = \lambda, \qquad \dim \operatorname{Hom}_{\mathfrak{g}}(V(\mu_j), V_j) = 1,$$

and an injective map $V(\mu_1) \otimes V(\mu_2) \to V$ of $\mathfrak{g}$–modules.

Proof. The existence of $\mu_j \in \operatorname{wt} V_j$, $j = 1, 2$, such that $\mu_1 + \mu_2 = \lambda$ is a consequence of the fact that $\dim V_\lambda > 0$ while the uniqueness of these elements is a consequence of the fact that $\dim V_\lambda = 1$. Notice that this also proves that $\dim(V_j)_{\mu_j} = 1$ for $j = 1, 2$. Since $\operatorname{wt} V \subset \lambda - Q^+$ we get $\operatorname{wt} V_j \subset \mu_j - Q^+$ and hence

$$\dim \operatorname{Hom}_{\mathfrak{g}}(V(\mu_j), V_j) = 1, \quad j = 1, 2.$$

If $\mu_1 = 0$ then the argument proves that V_1 is the one–dimensional trivial representation of $\mathfrak{g}$ contradicting our assumptions. This completes the proof of the Lemma. $\square$

4.3.2

For the rest of the section we fix $(\ell, \lambda) \in \mathbf{N} \times P^+$ and an isomorphism

$$D(\ell, \lambda) \cong_{\mathfrak{g}} V_1 \otimes V_2,$$

for some finite–dimensional $\mathfrak{g}$–modules V_1 and V_2. Since $D(\ell, \lambda)$ satisfies the conditions of Lemma 4.3.1 we choose μ_1 and μ_2 as in Lemma 4.3.1 and Proposition 2.9 follows if we prove that either $\mu_1 = 0$ or $\mu_2 = 0$.

4.3.3

We need some additional notation. Given any connected subset $J \subset \{1, \cdots, n\}$ of the Dynkin diagram of $\mathfrak{g}$, set

$$R_J^+ = R^+ \cap \sum_{j \in J} \mathbf{Z}\alpha_j, \quad P_J^+ = P^+ \cap \sum_{j \in J} \mathbf{Z}\omega_j, \quad Q_J^+ = Q^+ \cap \sum_{j \in J} \mathbf{Z}\alpha_j.$$

Let $\mathfrak{g}_J$ be the subalgebra of $\mathfrak{g}$ generated by the elements $x_i^\pm$, $i \in J$ and let $\mathfrak{n}_J^\pm$, $\mathfrak{h}_J$ be defined in the obvious way. Then R_J^+ is the set of positive roots of $\mathfrak{g}_J$ with respect to $\mathfrak{h}_J$ and P_J and Q_J are the corresponding weight and root lattice respectively. Finally, we regard the algebra $\mathfrak{g}_J[t]$ as a subalgebra of $\mathfrak{g}[t]$ in the natural way.

Given $\mu \in P^+$ set

$$V_J(\mu) = \mathbf{U}(\mathfrak{g}_J)v_\mu \subset V(\mu), \qquad D_J(\ell,\mu) = \mathbf{U}(\mathfrak{g}_J[t]))w_\mu \subset D(\ell,\mu).$$

Then $V_J(\mu)$ is the irreducible $\mathfrak{g}_J$–module with highest weight μ_J which is the restriction of μ to $\mathfrak{h}_J$. The module $D_J(\ell,\mu)$ is a quotient of the Demazure module for $\mathfrak{g}_J[t]$ associated to the pair (ℓ,μ_J).

The following is elementary and will be used repeatedly.

Lemma. (i) Suppose that $\mu,\mu' \in P^+$ and $\eta \in Q_J^+$ is such that $\nu = \mu + \mu' - \eta \in P^+$. Then

$$\operatorname{Hom}_{\mathfrak{g}_J+\mathfrak{h}}(V_J(\nu), V_J(\mu') \otimes V_J(\mu)) \cong \operatorname{Hom}_{\mathfrak{g}}(V(\nu), V(\mu') \otimes V(\mu)).$$

(ii) Suppose that $\mu,\nu \in P^+$ are such that $\mu - \nu \in Q_J^+$. Then,

$$\dim \operatorname{Hom}_{\mathfrak{g}_J+\mathfrak{h}}(V_J(\nu), D_J(\ell,\mu)) = \dim \operatorname{Hom}_{\mathfrak{g}}(V(\nu), D(\ell,\mu)). \tag{4.3.1}$$

Proof. Let $\mu|_J$ be the restriction of a weight μ to $\mathfrak{h}_J$. Then, it is clear that

$$V_J(\mu|_J) \otimes V_J(\mu'|_J) \cong_{\mathfrak{g}_J} V_J(\mu) \otimes V_J(\mu')$$

$$0 \to V_J(\mu) \otimes V_J(\mu') \to V(\mu) \otimes V(\mu').$$

Where the injective map is as $\mathfrak{g}_J + \mathfrak{h}$ modules. Next, one can extend the $\mathfrak{g}_J$ module $V_J(\mu|_J) \otimes V_J(\mu'|_J)$ to a $\mathfrak{g}_J + \mathfrak{h}$ module. First, we identify $\mathfrak{g}_J + \mathfrak{h} \cong \mathfrak{g}_J \oplus c_J$ where $[\mathfrak{g}_J, c_J] = 0$. And then we define the action of c_J on the tensor product to be the scalar action by $(\mu+\mu')|_{c_J}$, which is well defined since $\mu,\mu' \in P^+$. $\square$

4.3.4

For $\mu \in P^+$, set $\operatorname{supp}\mu = \{i \in I : \mu(h_i) > 0\}$.

Lemma. Let $(\ell, \lambda) \in \mathbf{N} \times P^+$ with $\lambda(h_i) \leq \ell$ for all $1 \leq i \leq n$. With the notation of Section 4.3.2, we have

$$\operatorname{supp}\mu_1 \cap \operatorname{supp}\mu_2 = \emptyset.$$

In particular, if $\lambda = m\omega_i$ for some $0 \leq m \leq \ell$ and we are in the simply laced case, then $D(\ell, \lambda)$ is prime.

Proof. Suppose for a contradiction that $i \in \operatorname{supp}\mu_1 \cap \operatorname{supp}\mu_2$ for some $1 \leq i \leq n$ and set $J = \{i\}$. Then $\mathfrak{g}_J \cong \mathfrak{sl}_2$ and hence using the Clebsch–Gordon formula and Proposition 4.3.3, we get

$$\operatorname{Hom}_{\mathfrak{g}}(V(\lambda - \alpha_i), V(\mu_1) \otimes V(\mu_2)) = \operatorname{Hom}_{\mathfrak{g}}(V(\mu_1 + \mu_2 - \alpha_i), V(\mu_1) \otimes V(\mu_2)) \neq 0. \quad (4.3.2)$$

Using Lemma 4.3.1 this implies that

$$\operatorname{Hom}_{\mathfrak{g}}(V(\lambda - \alpha_i), D(\ell, \lambda)) \neq 0. \quad (4.3.3)$$

On the other hand since $\lambda(h_i) \leq \ell$, we have that the element $w_\lambda \in D(\ell, \lambda)$ satisfies the defining relation $(x_i^- \otimes t)w_\lambda = 0$ and hence

$$\mathbf{U}(\mathfrak{g}_J[t])w_\lambda \cong \mathbf{U}(\mathfrak{g}_J)w_\lambda \cong V_J(\lambda_J).$$

Using (4.3.1) we get

$$\operatorname{Hom}_{\mathfrak{g}}(V(\lambda - \alpha_i), D(\ell, \lambda)) = 0,$$

which contradicts (4.3.3). This proves the Lemma. □

4.3.5

Lemma. Suppose that $\nu_1, \nu_2 \in P^+$ are such that

$$\operatorname{supp}\nu_1 \cap \operatorname{supp}\nu_2 = \emptyset.$$

There exists a connected subset $J \subset I$ with $\mathfrak{g}_J$ isomorphic to $\mathfrak{sl}_{r+1}$ for some $r \in \mathbf{N}$ and

$$|J \cap \operatorname{supp}\nu_j| = \begin{cases} 1, & \nu_j \neq 0, \\ \\ 0, & \nu_j = 0, \end{cases}, \quad j = 1, 2.$$

Proof. If $\nu_1 = \nu_2 = 0$, we take J to be the empty set while if $\nu_1 = 0$ and $\nu_2 \neq 0$ we take $J = \{i\}$ for some $i \in \operatorname{supp}\nu_2$. Assume now that ν_1 and ν_2 are non–zero. If $\mathfrak{g}$ is of type A_n, assume without loss of generality that $\operatorname{supp}\nu_2$ contains the maximal element in the union $\operatorname{supp}\nu_1 \cup \operatorname{supp}\nu_2$. Choose i_1 to be the maximal element in $\operatorname{supp}\nu_1$ and $i_2 \in \operatorname{supp}\nu_2$ minimal so that $i_2 > i_1$. The minimal connected subset J of I containing i_1 and i_2 satisfies the conditions of the Lemma.

If $\mathfrak{g}$ is of type D or E we let i_0 be the trivalent node and let I_r, $r = 1, 2, 3$ be the three legs of the Dynkin diagram through i_0 and assume without loss of generality that $I_1 = \{i_0, i_1\}$. Assume that $i_1 \notin \operatorname{supp}\nu_2$. Then,

$$\nu_1' = \nu_1 - \nu_1(h_{i_1})\omega_{i_1} \in P^+ \quad \operatorname{supp}\nu_1' \cap \operatorname{supp}\nu_2 = \emptyset, \quad i_1 \notin \operatorname{supp}\nu_1'.$$

If $\nu_1' = 0$ take J to be the connected closure of $\{i_1, i_2\}$ for some $i_2 \in \operatorname{supp}\nu_2$. If $\nu_1' \neq 0$, then the connected closure of $\operatorname{supp}\nu_1' \cup \operatorname{supp}\nu_2$ is contained in $I_2 \cup I_3$ and is of type A. Now, we can use the result for A to find $J \subset I \setminus \{i_1\}$ with the required properties for the pair ν_1', ν_2. But this set also has the desired properties for the pair ν_1, ν_2 and the proof is complete.

□

4.3.6

We return to the notation of Section 4.3.2. Using Lemma 4.3.4 we see that we can choose J as in Lemma 4.3.5 for the pair μ_1, μ_2. Let $\theta_J \in R_J^+$ be the highest root of $\mathfrak{g}_J$ and notice that $\lambda_J = \lambda(h_{i_1})\omega_{i_1} + \lambda(h_{i_2})\omega_{i_2}$. If we assume in addition that $\lambda(h_i) < \ell$ for all $i \in I$, then we see that: $\lambda(h_\alpha) < \ell$ for all $\alpha \in R_J^+$ with $\alpha \neq \theta_J$ and $\lambda(h_{\theta_J}) < 2\ell$.

Hence the following relations hold in $D(\ell, \lambda)$

$$(x_\alpha^- \otimes t)w_\lambda = 0, \quad \alpha \in R_J^+, \quad \alpha \neq \theta_J, \quad (x_{\theta_J}^- \otimes t^2)w_\lambda = 0,$$

$$(x_{\theta_J}^- \otimes t)^r = 0, \quad r > p = \max\{0, \lambda(h_{\theta_J}) - \ell\}.$$

It is again a standard fact that the elements $(x_{\theta_J}^- \otimes t)^s w_\lambda$ are non–zero if $0 \leq s \leq p$. Using the Poincare–Birkhoff–Witt theorem, one sees that

$$\mathbf{U}(\mathfrak{g}_J[t])w_\lambda = \sum_{s=0}^{p} \mathbf{U}(\mathfrak{g}_J)(x_{\theta_J}^- \otimes t)^s w_\lambda.$$

Moreover, a simple calculation shows that $(x_{\theta_J}^- \otimes t)^s w_\lambda$, $s \in \mathbf{Z}_+$ are $\mathfrak{n}^+$–invariant vectors in $D(\ell, \lambda)$ and we have

$$\mathbf{U}(\mathfrak{g}_J[t])w_\lambda \cong_{\mathfrak{g}_J} \bigoplus_{s=0}^{p} \mathbf{U}(\mathfrak{g}_J)(x_{\theta_J}^- \otimes t)^s w_\lambda \cong_{\mathfrak{g}_J} \bigoplus_{s=0}^{p} V_J(\lambda_J - s\theta_J)^{m_s}.$$

Applying (4.3.1), now gives

$$\operatorname{Hom}_{\mathfrak{g}}(V(\lambda - s\theta_J), D(\ell, \lambda)) = 0, \quad s > p. \tag{4.3.4}$$

On the other hand, it is well–known and in any case easily proved that

$$\dim \operatorname{Hom}_{\mathfrak{g}}(V(\mu_1 + \mu_2 - s\theta_J), V(\mu_1) \otimes V(\mu_2)) \neq 0 \text{ if } 0 \leq s \leq \min\{\mu_1(h_{\theta_J}), \mu_2(h_{\theta_J})\}.$$

Since $V(\mu_1) \otimes V(\mu_2)$ is isomorphic to a $\mathfrak{g}$–submodule of $D(\ell, \lambda)$, it follows that

$$\dim \operatorname{Hom}_{\mathfrak{g}}(V(\lambda - s\theta_J), D(\ell, \lambda)) \neq 0 \text{ if } 0 \leq s \leq \min\{\mu_1(h_{\theta_J}), \mu_2(h_{\theta_J})\}. \tag{4.3.5}$$

Since

$$p = \max\{0, \lambda(h_{\theta_J}) - \ell\} = \max\{0, \mu_1(h_{\theta_J}) + \mu_2(h_{\theta_J}) - \ell\} < \min\{\mu_1(h_{\theta_J}), \mu_2(h_{\theta_J})\},$$

we see that (4.3.5) contradicts (4.3.4). The proof of Proposition 2.9 is complete.

Chapter 5

A Character Decomposition

Recall Proposition 2.9 states that in types A and D, given $\ell \in \mathbb{N}$, if we write

$$\lambda = \lambda^0 + \sum_{i \in I} m_i \ell \omega_i$$

for $\lambda \in P^+$ and $0 \leq \lambda^0(h_i) < \ell$ for all $1 \leq i \leq n$, then

$$D(\ell, \lambda) \cong_{\mathfrak{g}[t]} \operatorname{ev}_0^* V(\ell\omega_1)^{*m_1} * \cdots * \operatorname{ev}_0^* V(\ell\omega_n)^{*m_n} * D(\ell, \lambda^0).$$

In fact if $\lambda^0(h_\theta) \leq \ell$, then $D(\ell, \lambda^0) \cong_{\mathfrak{g}[t]} \operatorname{ev}_0^* V(\lambda^0)$. So we assume in the subsequent chapter that $\lambda^0(h_\theta) > \ell$. Our aim is to understand the $\mathfrak{g}$ module structure of the prime Demazure module $D(\ell, \lambda^0)$ when $\lambda^0(h_\theta) > 0$.

5.1 Decomposition of prime modules over $\mathfrak{sl}_3$

Throughout this section we fix the following notation: let $\ell \in \mathbb{N}, k_1, k_2 \in \mathbb{Z}_+$ such that $0 \leq k_1, k_2 < \ell$, and $k_1 + k_2 = \ell + r$, for some $0 < r < \ell$. We write

$$\lambda^0 = k_1\omega_1 + k_2\omega_2 \text{ and } \mu = (\ell - k_2)\omega_1 + (\ell - k_1)\omega_1.$$

We remind the reader that for any $D(\ell, \lambda)$ we denote its generator by $w_{\ell,\lambda}$. In particular, we prove the following proposition in the subsequent sections.

Proposition. There exists the following short exact sequence of $\mathfrak{g}[t]$-modules

$$0 \to \tau_r D(\ell, \mu) \xrightarrow{\phi_1} D(\ell, \lambda^0) \xrightarrow{\phi_2} D(\ell+1, \lambda^0) \to 0.$$

In addition, $D(\ell, \mu) \cong \mathrm{ev}_0^* V(\mu)$.

We will prove Proposition 5.1 in the following sections. But first, we remark that a repeated application of Proposition 5.1 gives us the following immediate consequence.

Corollary. The module $D(\ell, \lambda^0)$ has the Jordan decomposition

$$D(\ell, \lambda^0) \cong_{\mathfrak{g}} V(\lambda^0) \bigoplus_{0 \le i \le r-1} V((\ell + i - k_2)\omega_1 + (\ell + i - k_1)\omega_2)$$

5.1.1

Lemma. There is a surjection

$$\phi_2 : D(\ell, \lambda^0) \to D(\ell+1, \lambda^0)$$

whose kernel is generated by $(x_\theta^- \otimes t)^r w_{\ell,\lambda^0}$.

Proof. By Lemma 2.2 up to scalars there is a unique non-zero surjection. We need only to calculate the kernel of the map which sends w_{ℓ,λ^0} to $w_{\ell+1,\lambda^0}$. Both Demazure modules are quotients of the local Weyl module $W_{\mathrm{loc}}(\lambda^0)$. Hence, we consider the extra relations given by each Demazure module:

$$(x_i^- \otimes t) w_{\ell,\lambda^0} = 0 = (x_i^- \otimes t) w_{\ell+1,\lambda^0}$$

$$(x_\theta^- \otimes t^2) w_{\ell,\lambda^0} = 0 = (x_\theta^- \otimes t^2) w_{\ell+1,\lambda^0}$$

while,

$$(x_\theta^- \otimes t)^{r+1} w_{\ell,\lambda^0} = 0 = (x_\theta^- \otimes t)^r w_{\ell+1,\lambda^0}$$

And hence it is clear that the kernel of ϕ_2 is generated by $(x_\theta^- \otimes t)^r w_{\ell,\lambda^0}$. □

5.1.2

Lemma. The assignment of $w_{\ell,\mu} \mapsto (x_\theta^- \otimes t)^r w_{\ell,\lambda^0}$ defines a well-defined injection of modules

$$\phi_1 : \tau_r D(\ell,\mu) \to D(\ell,\lambda^0).$$

Proof. Since $\mu(h_\theta) = 2\ell - k_1 - k_2 < 2\ell - \ell - r = \ell - r$ then $D(\ell,\mu) \cong_{\mathfrak{g}[t]} \mathrm{ev}_0^* V(\mu)$, if the assignment is well defined, then the $\ker \phi_1 = 0$ since the map is nonzero. We check the map is well-defined. Since $r = k_1 + k_2 - \ell$, then $\lambda^0 - r\theta = \mu$ and the $\mathfrak{h}[t]$ action is preserved. It is also easy to check that $x_i^+(x_\theta^- \otimes t)^r w_{\ell,\lambda^0} = 0$ and hence $(x_\theta^- \otimes t)^r w_{\ell,\lambda^0}$ is a highest weight vector. Lastly, we have seen that $(x_\alpha^- \otimes t)(x_\theta^- \otimes t)^r w_{\ell,\lambda^0} = 0$ for all $\alpha \in R^+$. This completes the proof. □

5.2 Prime modules over $\mathfrak{sl}_{n+1}$

Given $\ell \in \mathbb{N}$, we fix $\lambda = k_1\omega_{i_1} + \cdots + k_r\omega_{i_r}$, such that $\lambda(h_\theta) = \ell + 1$ and let $\gamma = \alpha_{i_1} + \alpha_{i_1+1} + \cdots + \alpha_{i_r-1} + \alpha_{i_r}$.

Theorem 4. There exists a short exact sequence

$$0 \to \tau_1^* D(\ell, \lambda - \gamma) \to D(\ell,\lambda) \to D(\ell+1,\lambda) \to 0$$

An immediate corollary is

Corollary. $D(\ell,\lambda)$ has a Jordan Holder series decomposition

$$D(\ell,\lambda) \cong_{\mathfrak{g}} \bigoplus_{s=0}^{\min\{i_1, n-i_r\}} V(\omega_{i_1-s} + (k_1-1)\omega_{i_1} + k_2\omega_{i_2} + \cdots + k_{r-1}\omega_{i_{r-1}} + (k_r-1)\omega_{i_r} + \omega_{i_r+s})$$

In particular, $D(n-1,\rho) \cong_{\mathfrak{g}} V(\rho) \oplus V(\rho - \omega_1 - \omega_n)$.

5.2.1

Lemma. There exists a surjection

$$D(\ell,\lambda) \to D(\ell+1,\lambda) \to 0$$

with kernel generated by $(x_\gamma^- \otimes t)w_{\ell,\lambda}$.

Proof. By lemma in CSVW the map $w_{\ell,\lambda} \mapsto w_{\ell+1,\lambda}$ is a non-zero surjection.

And hence we only need to show that ϕ is generated by $(x_\gamma^- \otimes t)w_{\ell,\lambda}$. It is clear that $(x_\gamma^- \otimes t)w_{\ell,\lambda} \in \ker\phi$. Suppose that $Xw_{\ell,\lambda} \in \ker\phi$. Then we can write $X = Y + Z$ where Y is in the left ideal of $U(\mathfrak{g}[t])$ generated by $\{x_\alpha^- \otimes t : \lambda(h_\alpha) \le \ell\}$ and Z is in the left ideal generated by $\{x_\alpha^- \otimes t : \lambda(h_\alpha) = \ell+1\}$. Since $Yw_{\ell,\lambda} = 0$, we can simply consider $Zw_{\ell,\lambda} \in \ker\phi$. Hence, consider α such that $\lambda(h_\alpha) = \ell+1$ and write $\alpha = \beta_1 + \gamma + \beta_2$ where $\beta_1 = \alpha_j + \cdots + \alpha_{i_1-1}$ and $\beta_2 = \alpha_{i_r+1} + \cdots + \alpha_k$, $j < k$. Therefore,

$$\begin{aligned}(x_\alpha^- \otimes t)w_{\ell,\lambda} &= [x_{\beta_1}^- \otimes 1, [x_{\beta_2}^- \otimes 1, x_\gamma^- \otimes t]]w_{\ell,\lambda} \\ &= (x_{\beta_1}^- \otimes 1)(x_{\beta_2}^- \otimes 1)(x_\gamma^- \otimes t)w_{\ell,\lambda} \\ &\in U(\mathfrak{n}^-)(x_\gamma^- \otimes t)w_{\ell,\lambda}\end{aligned}$$

Hence, $\ker\phi = U(\mathfrak{g})(x_\gamma^- \otimes t)w_{\ell,\lambda}$. □

5.2.2

Lemma. The assignment $w_{\ell,\lambda-\gamma} \to (x_\gamma^- \otimes t)w_{\ell,\lambda}$ defines an injective map of $\mathfrak{g}[t]$-modules

$$\psi : \tau_1^* D(\ell,\lambda-\gamma) \to D(\ell,\lambda)$$

Proof. The proof follows as it does in Lemma 4.2.3. Choose $\Lambda \in \widehat{P}^+$ such that $w\Lambda = w_0\lambda + \ell\Lambda_0$ for some $w \in \widehat{W}$. Then,

$$D(\ell,\lambda) \cong_{\mathfrak{g}[t]} V_w(\Lambda).$$

Thus, under this isomorphism, we know that $w_{\ell,\lambda}$ is mapped to some nonzero element, $v_{w_0 w\Lambda}$, of $V_w(\Lambda)_{w_0 w\Lambda}$. Also, since

$$(w_0 w\Lambda, -\gamma + \delta) = -1 < 0$$

then by the $\mathfrak{sl}_2$ representation theory associated to the root $-\gamma + \delta$ we have that

$$0 \neq (x_\gamma^- \otimes t) v_{w_0 w\Lambda} \in V_w(\Lambda)_{s_{\gamma-\delta} w_o w\Lambda},$$

where $s_{\gamma-\delta}$ is the reflection in $\widehat{W}$ along the root $\gamma - \delta$. Hence, passing through the isomorphism again, we see that

$$0 \neq (x_\gamma^- \otimes t) w_{\ell,\lambda} \in D(\ell, \lambda).$$

Recall that $V_w(\Lambda)$ is a $\mathfrak{g}$ stable Demazure module, in particular the $\mathfrak{g}$ module through the element $(x_\gamma^- \otimes t) v_{w_0 w\Lambda}$ is contained in $V_w(\Lambda)$ and therefore

$$V(\Lambda)_{w_0 s_{\gamma-\delta} w_0 w\Lambda} \subset V_w(\Lambda).$$

In particular, we have the inclusion of Demazure modules

$$V_{w_0 s_{\gamma-\delta} w_0 w}(\Lambda) \hookrightarrow V_w(\Lambda)$$

where $V_{w_0 s_{\gamma-\delta} w_0 w}(\Lambda) \cong_{\mathfrak{g}[t]} \tau_1^* D(\ell, \lambda - \gamma)$. $\square$

Chapter 6

A Refined Presentation

In this chapter we discuss a family of $\mathfrak{g}[t]$ modules denoted $V(\boldsymbol{\xi})$ which are associated to a R^+-tuple of partitions. In fact, for certain tuples of partitions it is known that these modules are isomorphic to a level ℓ Demazure module. The modules $V(\boldsymbol{\xi})$ were introduced in [11] as quotients of local Weyl modules. In [11] they provided three alternate presentations of the $V(\boldsymbol{\xi})$ modules. In addition, under suitable conditions on the partitions, the $V(\xi)$ modules are isomorphic to some $D(\ell, \lambda)$. In this chapter, I provide a fourth presentation with a reduced, finite set of relations. First, we develop the notation necessary to prove our result.

6.1 Modules associated to λ-compatible partitions

Given a dominant integral weight $\lambda \in P^+$, we say that $\boldsymbol{\xi} = (\xi^\alpha)_{\alpha \in R^+}$ is λ-compatible, if

$$\xi^\alpha = \xi_1^\alpha \geq \xi_2^\alpha \geq \cdots \xi_r^\alpha \geq 0$$

and

$$|\xi^\alpha| = \sum_{i \geq 1} \xi_i^\alpha = \lambda(h_\alpha).$$

In the subsequent sections we will also use the following notation for partitions. If $i_1 \geq i_2 \geq i_3 \geq \cdots \geq i_r \geq 0$ are the distinct parts of the partition where part i_k occurs s_k times, then we will denote this partition by $(i_1^{s_1}, i_2^{s_2}, \cdots, i_r^{s_r})$. If a partition only has two distinct parts it is called a fat hook. In particular, if a fat hook is of the form $(i^a, (i-1)^b)$, then I shall call it a consecutive fat hook.

In [11] they define the modules $V(\boldsymbol{\xi})$ to be the graded quotient of $W_{\text{loc}}(\lambda)$ by the submodule generated by the elements

$$\{(x_\alpha^+ \otimes t)^s (x_\alpha^- \otimes 1)^{s+r} w_\lambda : \alpha \in R^+,\ s, r \in \mathbb{N},\ s + r \geq 1 + rk + \sum_{j \geq k+1} \xi_j^\alpha, \text{ for some } k \in \mathbb{N}\}. \tag{6.1.1}$$

In particular, Proposition 2.4 in [11] states that the modules $V(\boldsymbol{\xi})$ are non-zero, indecomposable $\mathfrak{g}[t]$-modules and under certain conditions are isomorphic to $\operatorname{ev}_0 V(\lambda)$. In particular, if we denote $\{\lambda\}$ to be the R^+-tuple of partitions where each partition $\xi^\alpha = \lambda(h_\alpha)$ and hence has at most one part, then we have the following proposition (in [11]).

Proposition. Let $\lambda \in P^+$.

(i) The module $\operatorname{ev}_0 V(\lambda)$ is the unique irreducible quotient of $V(\boldsymbol{\xi})$ and hence $V(\boldsymbol{\xi})$ is a nonzero, indecomposable $\mathfrak{g}[t]$ module.

(ii) We have an isomorphism,

$$\operatorname{ev}_0 V(\lambda) \cong_{\mathfrak{g}[t]} V(\{\lambda\})$$

6.2 Alternate Presentations

6.2.1 Presentation 2

In this section we give the alternate presentations of the $V(\boldsymbol{\xi})$ modules established in [11]. First, we need notation developed in [11].

Definition. Let $s, r \in \mathbb{Z}_{\geq 0}$,

(i) We define the following set of sequences

$$\mathbf{S}(r,s) = \left\{ (b_p)_{p\geq 0} : b_p \in \mathbb{Z}_{\geq 0},\ \sum_{p\geq 0} b_p = r,\ \sum_{p\geq 0} pb_p = s \right\}$$

(ii) Given $x \in \mathfrak{g}$ we define the elements $\mathbf{x}(r,s) \in \mathbf{U}(\mathfrak{g}[t])$ by

$$\mathbf{x}(r,s) = \sum_{(b_p)_{p\geq 0}\in \mathbf{S}(r,s)} (x \otimes 1)^{b_0} (x \otimes t)^{b_1} \cdots (x \otimes t^s)^{(b_s)}$$

First, note that if $(b_p)_{p\geq 0} \in \mathbf{S}(r,s)$ then $b_p = 0$ for all $p > s$, and so (ii) is well-defined. Also, for any $p \in \mathbb{Z},\ x \in \mathfrak{g}$, we denote $x^{(p)} = x^p/p!$.

Then, by Garland's identity, one can show that relation 6.1.1 is equivalent to

$$\mathbf{x}_\alpha^-(r,s)v_{\boldsymbol{\xi}} = 0,\ \text{if } s + r \geq 1 + rk + \sum_{j\geq k+1} \xi_j^\alpha,\ \alpha \in R^+ \tag{6.2.1}$$

Hence providing a second presentation.

6.2.2 Presentation 3

The third presentation requires defining yet another description of $\mathbf{S}(r,s)$ given in Section 6.2.1. Here, for any $k \in \mathbb{Z}_{\geq 0}$ we let $\mathbf{S}(r,s)_k$ (respectively ${}_k\mathbf{S}(r,s)$) be the subset where the elements $(b_p)_{p\geq 0}$ must also satisfy

$$b_p = 0,\ p \geq k \text{ (respectively, } b_p = 0,\ p < k).$$

In addition, if we let $\ell, m \in \mathbb{Z}_{\geq 0}$, such that both sets $\mathbf{S}(r-\ell, s-m)_k$ and ${}_k\mathbf{S}(\ell, m)$ are nonempty, then the function

$$\mathbf{S}(r-\ell, s-m)_k \times_k \mathbf{S}(\ell, m) \longrightarrow S(r,s)$$

$$((b_p)_{p\geq 0}, (c_p)_{p\geq 0}) \mapsto (b_0, b_1, \cdots, b_{k-1}, c_k, \cdots)$$

is one to one, and we denote its image simply by $\mathbf{S}(r-\ell, s-m)_k \times_k \mathbf{S}(\ell, m)$.

Next, we can define the elements $\mathbf{x}(r,s)_k$ and ${}_k\mathbf{x}(r,s)$ in the obvious way by setting

$$\mathbf{x}(r,s)_k = \sum_{(b_p)_{p\geq 0} \in \mathbf{S}(r,s)_k} (x \otimes 1)^{(b_0)} \cdots (x \otimes t^{k-1})^{(b_{k-1})}$$

$${}_k\mathbf{x}(r,s) = \sum_{(b_p)_{p\geq 0} \in {}_k\mathbf{S}(r,s)} (x \otimes t^k)^{(b_k)} \cdots (x \otimes t^s)^{(b_s)}$$

The following lemma will be useful in Section 6.3 and is a combination of results found in [11] Section 2.5.

Lemma. Let $k, r, s, \in \mathbb{Z}_{\geq 0}$ and $x \in \mathfrak{g}$. Then

(i) $\mathbf{S}(r,s) =_k \mathbf{S}(r,s) \bigcup_{r',s' \in \mathbb{Z}_{\geq 0}} (\mathbf{S}(r-r', s-s')_k \times_k S(r', s'))$

(ii) If in addition, $s + r \geq kr + K$ for some $K \in \mathbb{Z}_{\geq 0}$. Then

$$\mathbf{x}(r,s) =_k \mathbf{x}(r,s) + \sum \mathbf{x}(r-r', s-s')_k \times_k \mathbf{x}(r', s'),$$

where the sum is over all $r', s' \in \mathbb{Z}_{\geq 0}$ satisfying $r' < r, s' < s$ and $s' + r' \geq r'k + K$.

Finally, the last and third presentation given in [11] is that the module $V(\boldsymbol{\xi})$ is generated by the element $v_{\boldsymbol{\xi}}$ with the defining relations of the local Weyl module and the additional relation

$${}_k\mathbf{x}(r,s)v_{\boldsymbol{\xi}} = 0,\ \alpha \in R^+,\ s, r, k \in \mathbb{N},\ s + r \geq 1 + kr + \sum_{j \geq k+1} \xi_j^\alpha$$

In particular it is noted that for all $\alpha \in R^+$ with $r, k \in \mathbb{N}$ which satisfy $r \geq 1+\sum_{j\geq k+1} \xi_j^\alpha$ we have that

$$(x_\alpha^- \otimes t^k)^r v_{\boldsymbol{\xi}} = 0.$$

6.3 A Fourth Presentation

In the fourth presentation that I prove in this thesis, we see that there are only finitely many k in 6.2.2 that are needed in the presentation of $V(\xi)$. In particular, the proof illustrates that in the case that every partition is a consecutive fat hook, the relations are greatly simplified to a single monomial for every positive root.

Theorem 5. Let $\boldsymbol{\xi} = (\xi^\alpha)_{\alpha\in R^+}$ be a λ–compatible $|R^+|$–tuple of partitions where we write $\xi^\alpha = \xi_1^\alpha \geq ... \geq \xi_j^\alpha \geq ... \geq \xi_{s_\alpha}^\alpha$. We let p_i^α denote the number of times the i-th *distinct* part of ξ^α occurs, and m_α the total number of *distinct* parts in ξ^α. Then, $V(\boldsymbol{\xi})$ is isomorphic to the quotient of $W_{\text{loc}}(\lambda)$ by the submodule generated by the elements,

$$\{(x_\alpha^- \otimes t^{s_\alpha})w_\lambda : \alpha \in R^+\} \bigcup$$

$$\{{}_{k_{\alpha,i}}\mathbf{x}_\alpha^-(r,s)w_\lambda : s+r \geq 1 + rk_{\alpha,i} + \sum_{j\geq k_{\alpha,i}+1} \xi_j^\alpha, 1 \leq i \leq m_\alpha, \text{ if } \xi^\alpha \text{ is not a consecutive fat hook } \}$$

where $k_{\alpha,i} = p_1^\alpha + ... + p_i^\alpha$ and $k_{\alpha,n} = s_\alpha$ is the total number of parts.

Proof. Let U be the submodule of $W_{\text{loc}}(\lambda)$ generated by the elements given in the statement of the theorem. Let $\tilde{V}(\boldsymbol{\xi})$ be the corresponding quotient of $W_{\text{loc}}(\lambda)$. We see immediately that $V(\boldsymbol{\xi})$ is a quotient of $\tilde{V}(\boldsymbol{\xi})$ since we can take $k = k_{\alpha,i}$

To prove that they are isomorphic, we must show that: for $\alpha \in R^+$ and $k, r, s \in \mathbf{N}$, either

$$s + r \geq 1 + rk + \sum_{j\geq k+1} \xi_j^\alpha \implies (x_\alpha^+ \otimes t)^s (x_\alpha^- \otimes 1)^{s+r} w_\lambda \in U, \tag{6.3.1}$$

or

$$s+r \geq 1+rk+\sum_{j\geq k+1}\xi_j^\alpha \implies \mathbf{x}_\alpha^-(r,s)w_\lambda \in U. \tag{6.3.2}$$

If $r \geq \xi_1^\alpha$, then

$$s+r \geq 1+k\xi_1^\alpha+\sum_{j\geq k+1}\xi_j^\alpha \geq 1+\sum_{j\geq 1}\xi_j^\alpha = |\xi^\alpha|+1.$$

By local Weyl module relations, we know $(x_\alpha^- \otimes 1)^{s+r}w_\lambda = 0$ and so equation 6.3.1 is proved in this case.

It remains to check what happens if $r \leq \xi_{s_\alpha}^\alpha$ or if $\xi_i \geq r \geq \xi_{i+1}$. Before we proceed, we make note of the following. Since we have that

$$(x_\alpha^- \otimes t^{s_\alpha})w_\lambda \in U$$

then it follows that, $(x_\alpha^- \otimes t^m)w_\lambda \in U$ for all $m \geq s_\alpha$. Second, it follows that if $(b_p)_{p\geq 0} \in S(r,s)$ is such that $b_m > 0$ for some $m \geq s_\alpha$ then

$$((x_\alpha^- \otimes 1)^{(b_0)}...(x_\alpha^- \otimes t^m)^{(b_m)}...)w_\lambda) \in U \tag{6.3.3}$$

So,

$$(\mathbf{x}_\alpha^-(r,s) - \mathbf{x}_\alpha^-(r,s)_{s_\alpha})w_\lambda \in U \tag{6.3.4}$$

Hence, for our remaining cases it will suffice to show that $\mathbf{x}_\alpha^-(r,s)_{s_\alpha}w_\lambda \in U$.

Now, suppose that $\xi_{s_\alpha}^\alpha \geq r$. We claim that,

$$s+r \geq 1+kr+\sum_{j\geq k+1}\xi_j^\alpha \implies s+r \geq 1+s_\alpha r. \tag{6.3.5}$$

For the claim, notice there is nothing to prove if $k \geq s_\alpha$, and if $k < s_\alpha$, then

$$s+r \geq 1+kr+\sum_{j\geq k+1}\xi_j^\alpha \geq 1+kr+(s_\alpha-k)\xi_{s_\alpha}^\alpha \geq 1+kr+(s_\alpha-k)r.$$

This means that if $(b_p)_{p\geq 0} \in \mathbf{S}(r,s)$, then we must have $b_m > 0$ for some $m \geq s_\alpha$, since otherwise we would have $s = \sum_{p<s_\alpha} pb_p \leq r(s_\alpha - 1)$. In particular, we get $\mathbf{x}_\alpha^-(r,s)_{s_\alpha} = 0$

and equation (6.3.4) now proves that $\mathbf{x}_{\alpha}^{-}(r,s)w_{\lambda} \in U$. This also completes the proof when ξ^{α} is a consecutive fat hook.

Lastly, suppose that $\xi_i^{\alpha} \geq r \geq \xi_{i+1}^{\alpha}$. We claim the following inequality holds

$$\sum_{j \geq k_{\alpha,i}} (j-(k_{\alpha,i}-1))b_j \geq 1 + r(k-k_{\alpha,i}) + \sum_{j \geq k+1} \xi_j^{\alpha} \tag{6.3.6}$$

and hence

$$\sum_{j \geq k_{\alpha,i}} (j-(k_{\alpha,i}-1))b_j \geq 1 + \sum_{j \geq k_{\alpha,i}+1} \xi_j^{\alpha} \tag{6.3.7}$$

Inequality 6.3.6 follows from

$$(s_{\alpha}-1)b_{s_{\alpha}-1} + ... + k_{\alpha,i}b_{k_{\alpha,i}} + (k_{\alpha,i}-1)(r - \sum_{j=k_{\alpha,i}}^{s_{\alpha}-1} b_j) \geq s \geq 1 + r(k-1) + \sum_{j \geq k+1} \xi_j^{\alpha}$$

where we are assuming $(b_p)_{p \geq 0} \in S(r,s)_{s_{\alpha}}$ otherwise we are done by 6.3.3.

To prove 6.3.7 first suppose that $k > k_{\alpha,i}$. By our assumption that $r \geq \xi_{i+1}^{\alpha}$, it follows that $r \geq \xi_j^{\alpha}$ for all $j \geq k_{\alpha,i}+1$. Then by 6.3.6,

$$\begin{aligned}
\sum_{j \geq k_{\alpha,i}} (j-(k_{\alpha,i}-1))b_j &\geq 1 + \sum_{j=k_{\alpha,i}+1}^{k} r + \sum_{j \geq k+1} \xi_j^{\alpha} \\
&\geq 1 + \sum_{k_{\alpha,i}+1}^{k} \xi_j^{\alpha} + \sum_{j \geq k+1} \xi_j^{\alpha} \\
&\geq 1 + \sum_{j \geq k_{\alpha,i}+1} \xi_j^{\alpha}
\end{aligned}$$

On the other hand, suppose that $k \leq k_{\alpha,i}$. By our assumption that $r \leq \xi_i^{\alpha}$ it follows that $\xi_j^{\alpha} - r \geq 0$ for all $j \leq k_{\alpha,i}$. Hence, from 6.3.6

$$\begin{aligned}
\sum_{j \geq k_{\alpha,i}} (j-(k_{\alpha,i}-1))b_j &\geq 1 + \sum_{j=k+1}^{k_{\alpha,i}} -r + \sum_{j \geq k+1} \xi_j^{\alpha} \\
&= 1 + \sum_{j=k+1}^{k_{\alpha,i}} (\xi_j^{\alpha} - r) + \sum_{j \geq k_{\alpha,i}+1} \xi_j^{\alpha} \\
&\geq 1 + \sum_{j \geq k_{\alpha,i}+1} \xi_j^{\alpha}
\end{aligned}$$

which proves the claim.

Next, note that by 6.3.7 we can write

$$\begin{aligned} s + r - k_{\alpha,i}r &= s + r(1 - k_{\alpha,i}) \\ &= \sum_{j\geq 0} jb_j + (1 - k_{\alpha,i})\sum_{j\geq 0} b_j \\ &= \sum_{0\leq j\leq k_{\alpha,i}-1} (j - (k_{\alpha,i} - 1))b_j + \sum_{j\geq k_{\alpha,i}} (j - (k_{\alpha,i} - 1))b_j \\ &\geq \sum_{0\leq j\leq k_{\alpha,i}-1} (j - (k_{\alpha,i} - 1))b_j + \left(1 + \sum_{j\geq k_{\alpha,i}+1} \xi_j^\alpha\right) \end{aligned}$$

In other words, we have the following inequality:

$$s + r - k_{\alpha,i}r \geq \left(1 + \sum_{j\geq k_{\alpha,i}+1} \xi_j^\alpha\right) + \left(\sum_{j=0}^{k_{\alpha,i}-1} (j - (k_{\alpha,i} - 1))b_j\right) \tag{6.3.8}$$

By [CV, Lemma 2.5] we can write

$$\mathbf{x}_\alpha^-(r,s)_{s_\alpha} w_\lambda = {}_{k_{\alpha,i}}\mathbf{x}_\alpha^-(r,s)_{s_\alpha} w_\lambda + \sum_{\substack{r'<r \\ s'\leq s}} \mathbf{x}_\alpha^-(r-r', s-s')_{k_{\alpha,i}}\, {}_{k\alpha,i}\mathbf{x}_\alpha^-(r',s')_{s_\alpha} w_\lambda.$$

By 6.3.8 if $(b_p)_{p\geq 0} \in {}_{k\alpha,i}S(r,s)_{s_\alpha}$, then $s + r \geq 1 + rk_{\alpha,i} + \sum_{j\geq k_{\alpha,i}+1} \xi_j^\alpha$ and so ${}_{k_{\alpha,i}}\mathbf{x}_\alpha^-(r,s)_{s_\alpha} w_\lambda \in U$. While, if $(b_p)_{p\geq 0} \in S(r-r', s-s')_{k_{\alpha,i}} \times {}_{k\alpha,i}S(r',s')_{s_\alpha}$, then again by 6.3.8 we have:

$$\begin{aligned} &(s - s') + s' + (r - r') + r' - k_{\alpha,i}(r - r') - k_{\alpha,i}r' \\ &\geq \left(1 + \sum_{j\geq k_{\alpha,i}+1} \xi_j^\alpha\right) + \sum_{j=0}^{k_{\alpha,i}-1} jb_j - k_{\alpha,i}\sum_{j=0}^{k_{\alpha,i}-1} b_j + \sum_{j=0}^{k_{\alpha,i}-1} b_j \\ &= 1 + \sum_{j\geq k_{\alpha,i}+1} \xi_j^\alpha + (s - s') - k_{\alpha,i}(r - r') + (r - r') \end{aligned}$$

Therefore, $s' + r' \geq 1 + k_{\alpha,i}r' + \sum_{j\geq k_{\alpha,i}+1} \xi_j^\alpha$ and ${}_{k\alpha,i}\mathbf{x}_\alpha^-(r',s')_{s_\alpha} w_\lambda \in U$. And hence, $\mathbf{x}_\alpha^-(r,s)_{s_\alpha} w_\lambda \in U$ which completes the proof.

□

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Table .1: Data for F_4 and $\ell = 2$.

λ	μ	λ	μ
(0,0,0,0)	(0,0,0,0)	(1,0,0,0)	(0,0,0,0)
(0,0,0,1)	(0,0,0,0)	(1,0,0,1)	(1,0,0,0)
(0,0,0,2)	(0,0,0,0)	(1,0,0,2)	(0,0,0,2)
(0,0,0,3)	(0,0,0,2)	(1,0,0,3)	(0,0,0,2)
(0,0,1,0)	(0,0,0,0)	(1,0,1,0)	(1,0,0,0)
(0,0,1,1)	(0,0,0,2)	(1,0,1,1)	(0,0,0,2)
(0,0,1,2)	(0,0,0,2)	(1,0,1,2)	(0,0,0,2)
(0,0,1,3)	(0,0,0,2)	(1,0,1,3)	(1,0,0,2)
(0,0,2,0)	(0,0,0,2)	(1,0,2,0)	(0,1,0,0)
(0,0,2,1)	(0,0,0,2)	(1,0,2,1)	(0,0,2,0)
(0,0,2,2)	(0,0,0,2)	(1,0,2,2)	(0,0,2,0)
(0,0,2,3)	(0,0,2,0)	(1,0,2,3)	(0,0,2,0)
(0,0,3,0)	(0,0,2,0)	(1,0,3,0)	(0,0,2,0)
(0,0,3,1)	(0,0,2,0)	(1,0,3,1)	(0,0,2,0)
(0,0,3,2)	(0,0,2,0)	(1,0,3,2)	(0,0,2,0)
(0,0,3,3)	(0,0,2,0)	(1,0,3,3)	(0,0,2,2)
(0,1,0,0)	(1,0,0,0)	(1,1,0,0)	(0,1,0,0)
(0,1,0,1)	(0,0,0,2)	(1,1,0,1)	(0,1,0,0)
(0,1,0,2)	(0,0,0,2)	(1,1,0,2)	(0,1,0,0)
(0,1,0,3)	(0,0,0,2)	(1,1,0,3)	(1,0,0,2)
(0,1,1,0)	(0,1,0,0)	(1,1,1,0)	(0,1,0,0)
(0,1,1,1)	(0,1,0,0)	(1,1,1,1)	(0,0,2,0)
(0,1,1,2)	(0,0,2,0)	(1,1,1,2)	(0,0,2,0)
(0,1,1,3)	(0,0,2,0)	(1,1,1,3)	(0,1,0,2)
(0,1,2,0)	(0,0,2,0)	(1,1,2,0)	(0,0,2,0)
(0,1,2,1)	(0,0,2,0)	(1,1,2,1)	(0,0,2,0)
(0,1,2,2)	(0,0,2,0)	(1,1,2,2)	(0,1,0,2)
(0,1,2,3)	(0,1,0,2)	(1,1,2,3)	(0,0,2,2)
(0,1,3,0)	(0,0,2,0)	(1,1,3,0)	(1,0,2,0)
(0,1,3,1)	(0,0,2,0)	(1,1,3,1)	(1,0,2,0)
(0,1,3,2)	(0,0,2,2)	(1,1,3,2)	(0,0,2,2)
(0,1,3,3)	(0,0,2,2)	(1,1,3,3)	(0,0,2,2)

Table .2: Data for E_8 and $\ell = 2$.

λ	μ	λ	μ
(0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0)	(0,0,0,1,0,1,0,0)	(0,0,0,0,1,0,0,0)
(1,0,0,0,0,0,0,0)	(1,0,0,0,0,0,0,0)	(1,0,0,1,0,1,0,0)	(0,0,0,1,0,0,0,0)
(0,1,0,0,0,0,0,0)	(0,0,0,0,0,0,0,1)	(0,1,0,1,0,1,0,0)	(0,0,0,1,0,0,0,0)
(1,1,0,0,0,0,0,0)	(0,1,0,0,0,0,0,0)	(1,1,0,1,0,1,0,0)	(0,0,0,1,0,0,0,1)
(0,0,1,0,0,0,0,0)	(1,0,0,0,0,0,0,0)	(0,0,1,1,0,1,0,0)	(0,0,0,1,0,0,0,0)
(1,0,1,0,0,0,0,0)	(0,0,1,0,0,0,0,0)	(1,0,1,1,0,1,0,0)	(0,0,1,0,0,1,0,0)
(0,1,1,0,0,0,0,0)	(0,0,1,0,0,0,0,0)	(0,1,1,1,0,1,0,0)	(0,0,0,1,0,0,1,0)
(1,1,1,0,0,0,0,0)	(0,0,1,0,0,0,0,0)	(1,1,1,1,0,1,0,0)	(0,0,0,1,0,1,0,0)
(0,0,0,1,0,0,0,0)	(0,1,0,0,0,0,0,0)	(0,0,0,0,1,1,0,0)	(0,0,0,0,0,1,0,0)
(1,0,0,1,0,0,0,0)	(0,0,1,0,0,0,0,0)	(1,0,0,0,1,1,0,0)	(0,0,0,0,1,0,0,0)
(0,1,0,1,0,0,0,0)	(0,0,0,0,1,0,0,0)	(0,1,0,0,1,1,0,0)	(0,0,0,0,1,0,0,0)
(1,1,0,1,0,0,0,0)	(0,0,0,1,0,0,0,0)	(1,1,0,0,1,1,0,0)	(0,1,0,0,0,1,0,0)
(0,0,1,1,0,0,0,0)	(0,0,0,0,1,0,0,0)	(0,0,1,0,1,1,0,0)	(0,0,0,0,1,0,0,1)
(1,0,1,1,0,0,0,0)	(0,0,0,1,0,0,0,0)	(1,0,1,0,1,1,0,0)	(0,0,1,0,0,1,0,0)
(0,1,1,1,0,0,0,0)	(0,0,0,1,0,0,0,0)	(0,1,1,0,1,1,0,0)	(0,0,1,0,0,1,0,0)
(1,1,1,1,0,0,0,0)	(0,1,1,0,0,0,0,0)	(1,1,1,0,1,1,0,0)	(0,0,1,0,1,0,0,0)
(0,0,0,0,1,0,0,0)	(0,0,0,0,0,0,1,0)	(0,0,0,1,1,1,0,0)	(0,0,0,0,1,0,1,0)
(1,0,0,0,1,0,0,0)	(0,0,0,0,0,1,0,0)	(1,0,0,1,1,1,0,0)	(0,0,0,0,1,1,0,0)
(0,1,0,0,1,0,0,0)	(0,0,0,0,0,1,0,0)	(0,1,0,1,1,1,0,0)	(0,0,0,0,1,1,0,0)
(1,1,0,0,1,0,0,0)	(0,0,0,0,1,0,0,0)	(1,1,0,1,1,1,0,0)	(0,0,0,1,0,1,0,0)
(0,0,1,0,1,0,0,0)	(0,0,0,0,1,0,0,0)	(0,0,1,1,1,1,0,0)	(0,0,0,1,0,1,0,0)
(1,0,1,0,1,0,0,0)	(0,0,0,0,1,0,0,0)	(1,0,1,1,1,1,0,0)	(0,0,0,1,0,1,0,0)
(0,1,1,0,1,0,0,0)	(0,0,0,1,0,0,0,0)	(0,1,1,1,1,1,0,0)	(0,0,0,1,1,0,0,0)
(1,1,1,0,1,0,0,0)	(0,1,1,0,0,0,0,0)	(1,1,1,1,1,1,0,0)	(0,1,1,0,1,0,0,0)
(0,0,0,1,1,0,0,0)	(0,0,0,0,1,0,0,0)	(0,0,0,0,0,0,1,0)	(0,0,0,0,0,0,0,1)
(1,0,0,1,1,0,0,0)	(0,0,0,1,0,0,0,0)	(1,0,0,0,0,0,1,0)	(0,0,0,0,0,0,1,0)
(0,1,0,1,1,0,0,0)	(0,0,0,1,0,0,0,0)	(0,1,0,0,0,0,1,0)	(0,0,0,0,0,0,1,0)
(1,1,0,1,1,0,0,0)	(0,1,0,0,1,0,0,0)	(1,1,0,0,0,0,1,0)	(0,0,0,0,0,1,0,0)
(0,0,1,1,1,0,0,0)	(0,0,0,1,0,0,0,1)	(0,0,1,0,0,0,1,0)	(0,0,0,0,0,0,1,0)
(1,0,1,1,1,0,0,0)	(0,0,1,0,1,0,0,0)	(1,0,1,0,0,0,1,0)	(0,0,1,0,0,0,0,0)
(0,1,1,1,1,0,0,0)	(0,0,1,0,1,0,0,0)	(0,1,1,0,0,0,1,0)	(0,0,0,0,1,0,0,0)
(1,1,1,1,1,0,0,0)	(0,0,1,1,0,0,0,0)	(1,1,1,0,0,0,1,0)	(0,0,0,1,0,0,0,0)
(0,0,0,0,0,1,0,0)	(0,0,0,0,0,0,0,1)	(0,0,0,1,0,0,1,0)	(0,0,0,0,0,1,0,0)
(1,0,0,0,0,1,0,0)	(0,0,0,0,0,0,1,0)	(1,0,0,1,0,0,1,0)	(0,0,0,0,1,0,0,0)
(0,1,0,0,0,1,0,0)	(0,0,0,0,0,1,0,0)	(0,1,0,1,0,0,1,0)	(0,0,0,1,0,0,0,0)
(1,1,0,0,0,1,0,0)	(0,0,0,0,0,1,0,0)	(1,1,0,1,0,0,1,0)	(0,0,0,1,0,0,0,0)
(0,0,1,0,0,1,0,0)	(0,0,0,0,0,1,0,0)	(0,0,1,1,0,0,1,0)	(0,0,0,1,0,0,0,0)
(1,0,1,0,0,1,0,0)	(0,0,0,0,1,0,0,0)	(1,0,1,1,0,0,1,0)	(0,0,0,1,0,0,1,0)
(0,1,1,0,0,1,0,0)	(0,0,0,0,1,0,0,0)	(0,1,1,1,0,0,1,0)	(0,0,0,1,0,0,0,1)

Table .3: Data for E_8 and $\ell = 2$ continued.

λ	μ	λ	μ
(1,1,1,0,0,1,0,0)	(0,0,0,1,0,0,0,0)	(1,1,1,1,0,0,1,0)	(0,0,0,1,0,0,1,0)
(0,0,0,0,1,0,1,0)	(0,0,0,0,0,1,0,0)	(0,0,0,1,1,1,1,0)	(0,0,0,0,1,1,0,0)
(1,0,0,0,1,0,1,0)	(0,0,0,0,1,0,0,0)	(1,0,0,1,1,1,1,0)	(0,0,0,1,0,1,0,0)
(0,1,0,0,1,0,1,0)	(0,0,0,0,1,0,0,0)	(0,1,0,1,1,1,1,0)	(0,0,0,1,0,1,0,0)
(1,1,0,0,1,0,1,0)	(0,0,0,0,1,0,0,1)	(1,1,0,1,1,1,1,0)	(0,0,0,1,0,1,0,1)
(0,0,1,0,1,0,1,0)	(0,0,0,0,1,0,0,0)	(0,0,1,1,1,1,1,0)	(0,0,0,1,0,1,0,0)
(1,0,1,0,1,0,1,0)	(0,0,1,0,0,0,1,0)	(1,0,1,1,1,1,1,0)	(0,0,1,0,1,0,1,0)
(0,1,1,0,1,0,1,0)	(0,0,0,0,1,0,1,0)	(0,1,1,1,1,1,1,0)	(0,0,0,1,0,1,1,0)
(1,1,1,0,1,0,1,0)	(0,0,0,1,0,0,1,0)	(1,1,1,1,1,1,1,0)	(0,0,0,1,1,0,1,0)
(0,0,0,1,1,0,1,0)	(0,0,0,0,1,0,0,1)	(0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0)
(1,0,0,1,1,0,1,0)	(0,0,0,0,1,0,1,0)	(1,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,1)
(0,1,0,1,1,0,1,0)	(0,0,0,1,0,0,1,0)	(0,1,0,0,0,0,0,1)	(0,0,0,0,0,0,0,1)
(1,1,0,1,1,0,1,0)	(0,0,0,1,0,0,1,0)	(1,1,0,0,0,0,0,1)	(0,1,0,0,0,0,0,0)
(0,0,1,1,1,0,1,0)	(0,0,0,1,0,0,1,0)	(0,0,1,0,0,0,0,1)	(0,0,0,0,0,0,1,0)
(1,0,1,1,1,0,1,0)	(0,0,0,1,0,1,0,0)	(1,0,1,0,0,0,0,1)	(0,0,1,0,0,0,0,0)
(0,1,1,1,1,0,1,0)	(0,0,0,1,0,1,0,0)	(0,0,0,1,0,0,0,1)	(0,0,0,0,0,1,0,0)
(1,1,1,1,1,0,1,0)	(0,0,0,1,1,0,0,0)	(1,0,0,1,0,0,0,1)	(0,0,0,0,1,0,0,0)
(0,0,0,0,0,1,1,0)	(0,0,0,0,0,0,1,0)	(0,1,0,1,0,0,0,1)	(0,0,0,0,1,0,0,0)
(1,0,0,0,0,1,1,0)	(0,0,0,0,0,1,0,0)	(1,1,0,1,0,0,0,1)	(0,0,0,1,0,0,0,0)
(0,1,0,0,0,1,1,0)	(0,0,0,0,0,1,0,0)	(0,0,1,1,0,0,0,1)	(0,0,0,1,0,0,0,0)
(1,1,0,0,0,1,1,0)	(0,1,0,0,0,0,1,0)	(1,0,1,1,0,0,0,1)	(0,0,0,1,0,0,0,0)
(0,0,1,0,0,1,1,0)	(0,0,0,0,0,1,0,1)	(0,1,1,1,0,0,0,1)	(0,0,0,1,0,0,0,1)
(1,0,1,0,0,1,1,0)	(0,0,1,0,0,0,1,0)	(1,1,1,1,0,0,0,1)	(0,1,1,0,0,0,0,1)
(0,1,1,0,0,1,1,0)	(0,0,1,0,0,0,1,0)	(0,0,0,0,1,0,0,1)	(0,0,0,0,0,0,1,0)
(1,1,1,0,0,1,1,0)	(0,0,1,0,0,1,0,0)	(1,0,0,0,1,0,0,1)	(0,0,0,0,0,1,0,0)
(0,0,0,1,0,1,1,0)	(0,0,0,0,0,1,1,0)	(0,1,0,0,1,0,0,1)	(0,0,0,0,1,0,0,0)
(1,0,0,1,0,1,1,0)	(0,0,0,0,1,0,1,0)	(1,1,0,0,1,0,0,1)	(0,0,0,0,1,0,0,0)
(0,1,0,1,0,1,1,0)	(0,0,0,0,1,0,1,0)	(0,0,1,0,1,0,0,1)	(0,0,0,0,1,0,0,0)
(1,1,0,1,0,1,1,0)	(0,0,0,1,0,0,1,0)	(1,0,1,0,1,0,0,1)	(0,0,0,0,1,0,0,1)
(0,0,1,1,0,1,1,0)	(0,0,0,1,0,0,1,0)	(0,1,1,0,1,0,0,1)	(0,0,0,0,1,0,0,1)
(1,0,1,1,0,1,1,0)	(0,0,0,1,0,0,1,0)	(1,1,1,0,1,0,0,1)	(0,0,0,1,0,0,0,1)
(0,1,1,1,0,1,1,0)	(0,0,0,1,0,1,0,0)	(0,0,0,1,1,0,0,1)	(0,0,0,0,1,0,0,1)
(1,1,1,1,0,1,1,0)	(0,1,1,0,0,1,0,0)	(1,0,0,1,1,0,0,1)	(0,0,0,1,0,0,0,1)
(0,0,0,0,1,1,1,0)	(0,0,0,0,0,1,0,1)	(0,1,0,1,1,0,0,1)	(0,0,0,1,0,0,0,1)
(1,0,0,0,1,1,1,0)	(0,0,0,0,0,1,1,0)	(1,1,0,1,1,0,0,1)	(0,0,0,1,0,0,1,0)
(0,1,0,0,1,1,1,0)	(0,0,0,0,1,0,1,0)	(0,0,1,1,1,0,0,1)	(0,0,0,1,0,0,0,1)
(1,1,0,0,1,1,1,0)	(0,0,0,0,1,0,1,0)	(1,0,1,1,1,0,0,1)	(0,0,1,0,1,0,0,0)
(0,0,1,0,1,1,1,0)	(0,0,0,0,1,0,1,0)	(0,1,1,1,1,0,0,1)	(0,0,0,1,0,1,0,0)
(1,0,1,0,1,1,1,0)	(0,0,0,0,1,1,0,0)	(1,1,1,1,1,0,0,1)	(0,0,0,1,1,0,0,0)
(0,1,1,0,1,1,1,0)	(0,0,0,0,1,1,0,0)	(0,0,0,0,0,1,0,1)	(0,0,0,0,0,0,1,0)
(1,1,1,0,1,1,1,0)	(0,0,0,1,0,1,0,0)	(1,0,0,0,0,1,0,1)	(0,0,0,0,0,1,0,0)

Table .4: Data for E_8 and $\ell = 2$ continued.

λ	μ	λ	μ
(0,1,0,0,0,1,0,1)	(0,0,0,0,0,1,0,0)	(0,0,0,1,0,0,1,1)	(0,0,0,0,0,1,0,1)
(0,1,0,0,0,1,0,1)	(0,0,0,0,0,1,0,0)	(1,0,0,1,0,0,1,1)	(0,0,0,0,1,0,0,1)
(1,1,0,0,0,1,0,1)	(0,0,0,0,0,1,0,1)	(0,1,0,1,0,0,1,1)	(0,0,0,0,1,0,0,1)
(0,0,1,0,0,1,0,1)	(0,0,0,0,0,1,0,0)	(1,1,0,1,0,0,1,1)	(0,0,0,1,0,0,0,1)
(1,0,1,0,0,1,0,1)	(0,0,1,0,0,0,0,1)	(0,0,1,1,0,0,1,1)	(0,0,0,1,0,0,0,1)
(0,1,1,0,0,1,0,1)	(0,0,0,0,1,0,0,1)	(1,0,1,1,0,0,1,1)	(0,0,0,1,0,0,0,1)
(1,1,1,0,0,1,0,1)	(0,0,0,1,0,0,0,1)	(0,1,1,1,0,0,1,1)	(0,0,0,1,0,0,1,0)
(0,0,0,1,0,1,0,1)	(0,0,0,0,0,1,0,1)	(1,1,1,1,0,0,1,1)	(0,1,1,0,0,0,1,0)
(0,0,0,1,0,1,0,1)	(0,0,0,0,0,1,0,1)	(0,0,0,0,1,0,1,1)	(0,0,0,0,0,0,1,1)
(1,0,0,1,0,1,0,1)	(0,0,0,0,1,0,0,1)	(1,0,0,0,1,0,1,1)	(0,0,0,0,0,1,0,1)
(0,1,0,1,0,1,0,1)	(0,0,0,1,0,0,0,1)	(0,1,0,0,1,0,1,1)	(0,0,0,0,1,0,0,1)
(1,1,0,1,0,1,0,1)	(0,0,0,1,0,0,0,1)	(1,1,0,0,1,0,1,1)	(0,0,0,0,1,0,0,1)
(0,0,1,1,0,1,0,1)	(0,0,0,1,0,0,0,1)	(0,0,1,0,1,0,1,1)	(0,0,0,0,1,0,0,1)
(1,0,1,1,0,1,0,1)	(0,0,0,1,0,0,1,0)	(1,0,1,0,1,0,1,1)	(0,0,0,0,1,0,1,0)
(0,1,1,1,0,1,0,1)	(0,0,0,1,0,0,1,0)	(0,1,1,0,1,0,1,1)	(0,0,0,0,1,0,1,0)
(1,1,1,1,0,1,0,1)	(0,0,0,1,0,1,0,0)	(1,1,1,0,1,0,1,1)	(0,0,0,1,0,0,1,0)
(0,0,0,0,1,1,0,1)	(0,0,0,0,0,1,0,1)	(0,0,0,1,1,0,1,1)	(0,0,0,0,1,0,1,0)
(1,0,0,0,1,1,0,1)	(0,0,0,0,1,0,0,1)	(1,0,0,1,1,0,1,1)	(0,0,0,1,0,0,1,0)
(0,1,0,0,1,1,0,1)	(0,0,0,0,1,0,0,1)	(0,1,0,1,1,0,1,1)	(0,0,0,1,0,0,1,0)
(1,1,0,0,1,1,0,1)	(0,0,0,0,1,0,1,0)	(1,1,0,1,1,0,1,1)	(0,0,0,1,0,0,1,1)
(0,0,1,0,1,1,0,1)	(0,0,0,0,1,0,0,1)	(0,0,1,1,1,0,1,1)	(0,0,0,1,0,0,1,0)
(1,0,1,0,1,1,0,1)	(0,0,1,0,0,1,0,0)	(1,0,1,1,1,0,1,1)	(0,0,1,0,1,0,0,1)
(0,1,1,0,1,1,0,1)	(0,0,0,0,1,1,0,0)	(0,1,1,1,1,0,1,1)	(0,0,0,1,0,1,0,1)
(1,1,1,0,1,1,0,1)	(0,0,0,1,0,1,0,0)	(1,1,1,1,1,0,1,1)	(0,0,0,1,1,0,0,1)
(0,0,0,1,1,1,0,1)	(0,0,0,0,1,0,1,0)	(0,0,0,0,0,1,1,1)	(0,0,0,0,0,0,1,1)
(1,0,0,1,1,1,0,1)	(0,0,0,0,1,1,0,0)	(1,0,0,0,0,1,1,1)	(0,0,0,0,0,1,0,1)
(0,1,0,1,1,1,0,1)	(0,0,0,1,0,1,0,0)	(0,1,0,0,0,1,1,1)	(0,0,0,0,0,1,0,1)
(1,1,0,1,1,1,0,1)	(0,0,0,1,0,1,0,0)	(1,1,0,0,0,1,1,1)	(0,0,0,0,0,1,1,0)
(0,0,1,1,1,1,0,1)	(0,0,0,1,0,1,0,0)	(0,0,1,0,0,1,1,1)	(0,0,0,0,0,1,0,1)
(1,0,1,1,1,1,0,1)	(0,0,0,1,0,1,0,1)	(1,0,1,0,0,1,1,1)	(0,0,1,0,0,0,1,0)
(0,1,1,1,1,1,0,1)	(0,0,0,1,0,1,0,1)	(0,1,1,0,0,1,1,1)	(0,0,0,0,1,0,1,0)
(1,1,1,1,1,1,0,1)	(0,0,0,1,1,0,0,1)	(1,1,1,0,0,1,1,1)	(0,0,0,1,0,0,1,0)
(0,0,0,0,0,0,1,1)	(0,0,0,0,0,0,0,1)	(0,0,0,1,0,1,1,1)	(0,0,0,0,0,1,1,0)
(1,0,0,0,0,0,1,1)	(0,0,0,0,0,0,1,0)	(1,0,0,1,0,1,1,1)	(0,0,0,0,1,0,1,0)
(0,1,0,0,0,0,1,1)	(0,0,0,0,0,0,1,0)	(0,1,0,1,0,1,1,1)	(0,0,0,1,0,0,1,0)
(1,1,0,0,0,0,1,1)	(0,1,0,0,0,0,0,1)	(1,1,0,1,0,1,1,1)	(0,0,0,1,0,0,1,0)
(0,0,1,0,0,0,1,1)	(0,0,0,0,0,0,1,1)	(0,0,1,1,0,1,1,1)	(0,0,0,1,0,0,1,0)
(1,0,1,0,0,0,1,1)	(0,0,1,0,0,0,0,1)	(1,0,1,1,0,1,1,1)	(0,0,0,1,0,0,1,1)
(0,1,1,0,0,0,1,1)	(0,0,1,0,0,0,0,1)	(0,1,1,1,0,1,1,1)	(0,0,0,1,0,0,1,1)
(1,1,1,0,0,0,1,1)	(0,0,1,0,0,0,1,0)	(1,1,1,1,0,1,1,1)	(0,0,0,1,0,1,0,1)

Table .5: Data for E_8 and $\ell = 2$ continued.

λ	μ	λ	μ
(0,0,0,0,1,1,1,1)	(0,0,0,0,0,1,1,0)	(0,0,0,1,1,1,1,1)	(0,0,0,0,1,0,1,1)
(1,0,0,0,1,1,1,1)	(0,0,0,0,1,0,1,0)	(1,0,0,1,1,1,1,1)	(0,0,0,0,1,1,0,1)
(0,1,0,0,1,1,1,1)	(0,0,0,0,1,0,1,0)	(0,1,0,1,1,1,1,1)	(0,0,0,1,0,1,0,1)
(1,1,0,0,1,1,1,1)	(0,0,0,0,1,0,1,1)	(1,1,0,1,1,1,1,1)	(0,0,0,1,0,1,0,1)
(0,0,1,0,1,1,1,1)	(0,0,0,0,1,0,1,0)	(0,0,1,1,1,1,1,1)	(0,0,0,1,0,1,0,1)
(1,0,1,0,1,1,1,1)	(0,0,1,0,0,1,0,1)	(1,0,1,1,1,1,1,1)	(0,0,0,1,0,1,1,0)
(0,1,1,0,1,1,1,1)	(0,0,0,0,1,1,0,1)	(0,1,1,1,1,1,1,1)	(0,0,0,1,0,1,1,0)
(1,1,1,0,1,1,1,1)	(0,0,0,1,0,1,0,1)	(1,1,1,1,1,1,1,1)	(0,0,0,1,1,0,1,0)