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Essays in Entrepreneurship, Venture Capital and Innovation

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Essays in Entrepreneurship, Venture Capital and Innovation

by

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Abstract

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In chapter 1, I study a dynamic innovation race, in which firms invest to be the first to attain a breakthrough invention. I examine how delays in monitoring firms' investments affect their ability to achieve a coordinated under-investment equilibrium enforced by the threat of elevated investment by rival firms. When monitoring delays are small, equilibrium investment can be considerably delayed, matching the first-best solution under cooperation. Even with significant delays in monitoring, equilibrium investment is below, and firm values are well above, the no-monitoring competitive outcome. As a result, the regulatory goal of transparency can conflict with the goal of encouraging investment in innovation. In addition I study how changing the number of firms in the race can affect the outcome. As the number of firms increases, the incentive to preempt escalates, and coordinated effort boundaries decline, as does the maximum delay time in which the first best can be achieved with coordination.

In chapter 2, I study the contracting problem between investors (limited partners or LPs) and venture capitalists (general partners or GPs) . In real world, GPs are sometimes paid on a deal-by-deal basis and other times on a whole-portfolio basis. When is one method of payment better than the other? I develop a model to see how the method of compensation's payment can affect the behavior of general partners (GP) in a limited partnership agreement (LPA). I show that when assets (projects or firms) are highly correlated or when GPs have low reputation, whole-portfolio contracting is superior to deal-by-deal contracting. In this case, by bundling payouts together, whole- portfolio contracting enhances incentives for GPs to exert effort. Therefore, it is better suited to alleviate the moral hazard problem which is stronger than the adverse selection problem in the case of high correlation of assets or low reputation of GPs. In contrast, for low correlation of assets or high reputation of GPs, information asymmetry concerns dominate and deal-by-deal contracts become optimal, as they can efficiently weed out bad projects one by one. These results shed light on recent empirical findings on the relationship between investors and venture capitalists.

In the name of God.
To my parents, Mostafa and Maryam, my brother Ali and my amazing wife Fatemeh.

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Chapter 1

Dynamic R&D Investment (Joint with Mark Schroder)

1.1 Introduction

This article models a dynamic innovation race in which firms invest to be the first to achieve a breakthrough invention. As in [41], costly effort toward R&D increases the probability of success. The potential payoff (the present value of future profits or royalties) evolves stochastically over time and is earned entirely by the winner of the race when the breakthrough occurs. Information about the competitors' actions facilitates non-cooperative coordinated equilibria in which reduced competition (reduced R&D effort) is enforced by the threat of elevated effort by the competing firms if any deviation is detected. Unlike other articles in the investment literature, we focus on the effects on equilibria from delays in monitoring each other's efforts. Absent monitoring, strategies are based only on the publicly observed payoff process, and effort is exerted only when the payoff process exceeds a fixed threshold. The common threshold falls (i.e., efforts increase) as more firms enter. At the other extreme, if the monitoring delay is sufficiently small, there always exists a coordinated equilibrium which has the same outcome as the first-best cooperative solution (which maximizes the aggregate market value of the firms), with investment that is increasingly postponed as more firms enter the market (the opposite prediction relative to the no-monitoring case).

When the monitoring delay is sufficiently large, the first-best cooperative solution cannot be achieved in equilibrium because the incentive of each firm to preempt the others is too strong. However, a coordinated equilibrium with effort between the no-monitoring and first-best levels always exists and typically achieves large proportional gains in market value for the firms. As the monitoring delay increases, the coordinated-equilibrium efforts increase. Both the effort levels and value-function gains are very sensitive to the number of firms and our measure of R&D productivity (which determines the expected waiting time to the first breakthrough). For example, with two firms and a relatively low incentive to preempt the first best can be achieved only when the monitoring delay time is less than 4.1 days. However, even when the delay is 100 days, firm values in the (best) coordinated equilibrium are approximately twice those of the no-monitoring competitive level. Considering that the first-best value function is only 2.4 times higher than the competitive value function, it shows that even with a long delay in monitoring, coordination yields substantial benefits. Again with two firms, but a strong incentive to preempt, the ratio of the first-best to competitive value functions is much higher at 5.7 but can be achieved in a coordinated equilibrium only when the delay time equals a small fraction of a day. With a 100-day delay time, the ratio of coordinated to competitive value functions falls to 1.9, which is still a significant gain.

As the number of firms increases, the incentive to preempt escalates, and coordinated effort boundaries decline, as does the maximum delay time in which the first best can be achieved with coordination. But for any given monitoring delay time, the relative value-function improvement from coordinating remains significant for even large delays, and the relative gains are not always monotonic in the number of firms, n (even though the ratio of the first-best to competitive value functions is always increasing in n).

In summary, the incentives to coordinate are powerful, even with strong incentives to preempt (which tends to reduce the benefits of coordination), long monitoring delay times (which also inhibits coordination), and settings with two, three, or four firms. Further, increased transparency (i.e., better monitoring) increases firm values at the expense of curtailing investment in innovation.

We also show some comparative-statics differences between the no-monitoring competitive equilibrium and the coordinated equilibrium with monitoring. Effort boundaries are always decreasing in n in the competitive equilibrium (i.e., per-firm effort increases as competition increases), but when the delay time is sufficiently small that the first best can be achieved, the effort boundary is increasing in n in the coordinated equilibrium. For larger delay times, effort boundaries decrease with n in the coordinated equilibrium. Also when the first best can be achieved, the effort boundaries and value functions are increasing in R&D productivity. Otherwise, both the boundaries and value functions are hump-shaped functions of R&D productivity. Low productivity implies that success is likely to be far in the future, and very high productivity results in intense competition that exhausts resources.

The idea that monitoring rivals' actions can facilitate coordination of strategies dates back to the infinite-horizon prisoner-dilemma problem, in which prisoners are discouraged from defecting from the cooperative solution to avoid future retaliation. The same mechanism is at work in our setting as well. Absent retaliation, each firm would want to cheat and exert effort before the payoff process hits the first-best cooperative effort barrier, giving it the exclusive chance to win the payoff until the other firms begin exerting effort. The threat of retaliation eliminates the incentive to cheat and results in a subgame perfect equilibrium. As the delay in monitoring increases, the gain from deviating increases and the cost of retaliation decreases, necessitating a lower effort boundary to restore equilibrium.

Our coordinated equilibrium has similarities to the joint-adoption equilibrium in [16], although ours is a repeated game in a stochastic setting, and theirs is a deterministic setting in which firms choose a single initial investment time, interpreted as an adoption time for a new technology. Therefore, there is no uncertainty either about the timing of innovation or about the payoff. However, the economic arguments underlying their joint-adoption equilibrium are similar to those in our setting: Each firm withholds initiating R&D effort for fear of provoking imitation by the other firms, making all the firms better off.

As in our model, [47] considers a winner-takes-all competition with a stochastic payoff, but, as in [16], it is a model of initial entry (i.e., technology adoption).[1] Once a firm enters, there is uncertainty about the timing of innovation and who will win the innovation race, but unlike in our model, the firm cannot influence the outcome. But the earlier adopters have a higher probability of winning. In our model, the conditional probability of winning is determined by the costly effort by the firm, and so each firm can continue or discontinue the R&D process in a dynamic way. Moreover the results of both [16] and [47] are restricted to a two-firm setting because of the use of the rent-equalization principle. Our results hold for any number of firms, allowing us to examine the effects of increasing competition.

[39] studies a dynamic game of investment in product quality and shows that if the interest rate is sufficiently small, there is a coordinated under-investment equilibrium characterized by a single immediate investment outlay. [32] (building on [44] and [6]) examines a model of product-market competition in which firms coordinate pricing strategies.[2] In contrast to the above models, ours is a stochastic game, which raises novel issues in modeling the effects of monitoring delays. We also allow for any number of firms (the above models assume two firms).

[1] The deterministic entry times of [16] are replaced with deterministic boundaries, such that entry occurs when first hit by the payoff process.

[2] [32] also considers disclosure delays, but in a two-firm deterministic model of product-market competition.

Empirical work on coordinated strategies had focused on product-market competition, where increased disclosure can have conflicting effects: Price transparency facilitates coordination of firms' actions, but also potentially increases competition by reducing consumer search costs. [32], for example, shows that implementation of online gas-price disclosure in Chile raised profit margins in the retail gas industry.[3] [5] show that when cartel enforcement increases, firms respond by sharing more data about their clients and products in their financial disclosures. Increased disclosure facilitates the coordination of product-market strategies, which improves profitability. We are unaware of empirical work on coordination of investment strategies,[4] but we provide a number of novel predictions about the effects of transparency on the investment strategies of competing firms.

There is also a theoretical literature on innovation races that do not consider coordination. [27] examine an incomplete-information model in which each of two competing firms chooses a single initial investment time, but each is uncertain about the other's cost function.[5] After entry by the first firm, the remaining firm is blocked from entry and cannot react in a meaningful way. In [23], the time to an innovation breakthrough is exogenous (independent across firms and exponentially distributed with a fixed hazard rate), but the waiting time to file for a patent is chosen by the firm. The first to file gets the largest prize, which is a deterministic increasing function of the waiting time. No coordination is possible because firms have no information about other firms' possible breakthroughs until a patent is filed, and then the game ends. [41] also considers a costly effort R&D problem in which effort increases the conditional probability of winning,[6] but prizes are deterministic. Our winner-take-all assumption corresponds to her perfect-patent-protection case, in which she finds that more competition increases R&D effort and speeds up innovation (similar to our no-monitoring equilibrium results).

On the technical side of our paper, [45], henceforth S-S, show that problems can arise in modeling games in continuous time, particularly when reactions can occur instantaneously (as in the case of no monitoring delay) and discontinuously change the information available to the other agents. In the spirit of S-S, in Section 1.6, which examines the case of no delay, we deal with this issue by defining the continuous-time game as a limit of a sequence of discrete-time games, in which periods have random

[3] See [32] for a discussion of other empirical papers on the effects of transparency on product-market competition.

[4] One issue is the difficulty of measuring R&D expenditure.

[5] See also [1], who add some generalizations to their model.

[6] See also [34], who introduce a time-0 unobserved drawing of the success intensity from a binomial distribution (either 0 or $\lambda > 0$). Passage of time without success therefore increases the posterior probability that the intensity is zero.

length. The random length is modeled so that each discrete-time game is time homogeneous and Markovian, and has subgame-perfect equilibria solved in closed form.

In other theoretical models with imperfect monitoring (see, for example, [43], and [4]), actions are partly hidden by Brownian noise, and information arrives continuously.[7] In our applied model, imperfect monitoring is caused by a fixed delay time, and information arrives discontinuously when cheating is discovered.

[3] also discuss the difficulties that arise in defining coordination strategies in a continuous-time setting with discontinuous information (as is the case in our model) and perfect monitoring. They show that the equilibrium in [20] violates subgame perfection because of each firm's incentive to preempt, and the equilibrium is, therefore, a competitive equilibrium but not a coordinated equilibrium.[8] We show that there are coordinated equilibria that are much different from the competitive equilibrium, and these equilibria are characterized by significantly delayed investment and much higher firm values.

This paper is organized as follows. Section 1.2 presents the setting and derives the basic properties of each firm's value function. Section 1.3 solves for the n-firm competitive equilibrium in a setting with no monitoring. Section 1.4 solves for the cooperative solution, with effort strategies chosen to maximize the aggregate value of the firms (as if the n firms had a common owner). This establishes the *first-best* effort strategies and firm values, which we show in Section 1.5 matches the (firm-value-maximizing) coordinated equilibrium when the monitoring delay is sufficiently small. Section 1.5 defines the family of coordinated strategies for each firm, based on maintaining a common effort boundary and all firms retaliating by reverting to the competitive strategy as soon as cheating by any firm is discovered (subject to a fixed delay in monitoring). Then the coordinated equilibrium is derived: it is given by the value-maximizing (which corresponds to the highest effort boundary) common boundary such that cheating is, with certainty, never optimal. Section 1.6 deals with the technical issues associated with zero delay (i.e., perfect monitoring). Finally, Section 7 concludes the paper.

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[7]See also [12] and [13], who examine properties of the continuous-time limits of a class of discrete-time games with imperfect monitoring. Note that the continuous-time strategies are well defined in such imperfect monitoring games.

[8]That is, the equilibrium is open loop (in which the strategies of the competitors are held fixed) but not closed loop (in which firms can react to each other's actions). [3] also show that Grenadier's perfectly competitive outcome, with each firm following a simple net-present-value rule, is a coordinated equilibrium.

1.2 Setting

We assume that n identical risk-neutral firms compete, over an infinite horizon, for the right to invest in a single project. Only the first to achieve success wins the project. This model might apply to several firms competing for a single government contract, or pharmaceutical companies competing to develop a drug or vaccine with perfect patent protection. The assumption that only current effort matters in determining the probability of success is a common assumption in the literature, but is obviously a simplification made for tractability. If the probability of success also depended on the aggregate past effort of all firms (e.g., research findings are quickly made public), the essential trade-offs between adhering to and deviating from the trigger strategy remain, but the effort barrier and value functions would now depend on past aggregate effort.

The probability of winning the project depends on the firms' effort exertions. Firm i exerts a time-t effort rate of e_t^i at a cost rate of $q\left(e_t^i\right)$.[9] The probability of firm i winning the project over the next instant dt, conditional on no firm having won to date, is $e_t^i dt$ (where $e_t^i \in \{0, \theta\}$). However, the existence of competing firms reduces firm i's chance of being the *first* to win, and only the first to win is awarded the payoff. That is, with time-t effort rates $e_t^1, \ldots, e_t^n$, the probability of some firm succeeding over the next instant dt, conditional on no success to date, is $\left(e_t^1 + \cdots + e_t^n\right) dt$, and, conditional on a success in the next instant, the probability of firm i winning is $e_t^i / \left(e_t^1 + \cdots + e_t^n\right)$. The analog is n independent exponential clocks running simultaneously, with clock i running at the time-t intensity rate e_t^i. If clock i rings first, then firm i wins the project. More precisely, we can model the time to first success, τ, as the first time that one of the n independent Poisson counting processes N^i, $i \in \{1, \ldots, n\}$, hits one, where the intensity-rate process of N^i is e^i, and each initial value is $N_0^i = 0$.[10] Then $\tau = \inf\left\{t : N_t^1 + \cdots + N_t^n = 1\right\}$, and the winning firm is the i satisfying $N_\tau^i = 1$.

We assume, for each firm i, a binary effort choice $e_t^i \in \{0, \theta\}$ with instantaneous cost $q\left(e_t^i\right) = Q e_t^i$. We interpret θ as a measure of R&D productivity: Higher θ implies, ceteris paribus, a greater probability of success per unit of time that effort is exerted. Generalizations are considered in Section 1.7, but the qualitative properties are similar.

The payoff uncertainty is modeled by a *price-process* P, which follows geometric

[9] The firm is assumed to have access to unlimited funding to finance effort.

[10] The assumption of Poisson arrival for uncertainty is common. See, for example, [31]; [8]; [17]; and [10]. However, the return to investment in R&D is assumed to be deterministic in these papers.

Brownian motion:

$$\frac{dP_t}{P_t} = \alpha dt + \sigma dB_t, \tag{1.1}$$

where α is the growth rate, σ the volatility, and B is standard Brownian motion. We can interpret P_t as the time-t present value of future royalties if the project is adopted at t. Letting r denote the interest rate, we assume $r > \alpha$ to obtain finite firm values (the same assumption is needed in the standard real-options setting).

If success for the winning firm occurs at time t, the project is undertaken, yielding a time-t payoff $f(P_t)$ for that firm, where f is a non-decreasing function. Typically, we let $f(P_t) = \max(0, P_t - K)$, with K denoting the required investment outlay to initiate the project.

Proposition 1 gives a convenient representation of firm i's value function $v^i(P_t; e^i, e^{-i})$ for a given own effort process e^i and total effort process $e^{-i} = \sum_{j \neq i} e^j$ of the other firms.

Proposition 1 *For any bounded effort processes $(e^1, \ldots, e^n)$, firm i's value function is uniquely given by*

$$v^i\left(P_t; e^i, e^{-i}\right) = E_t\left[\int_t^\infty \{f(P_s) - Q\} e_s^i \frac{\xi_s(e^i + e^{-i})}{\xi_t(e^i + e^{-i})} ds\right], \quad t \geq 0, \tag{1.2}$$

where $e^{-i} = \sum_{j \neq i} e^j$ denotes the total effort of the other $n-1$ firms, and, for any bounded process y,

$$\xi_s(y) = \exp\left(-\int_0^s (r + y_u)\, du\right). \tag{1.3}$$

Proof. See Appendix A. ■

In the expression (1.2), the net payout $f(P_s) - Q$ is multiplied by the probability of success per unit time, as well as a discount factor that accounts for both interest-rate discounting as well as the declining unconditional probability of success at any future date. This unconditional probability of success at some future date t is declining in aggregate effort, both over time $[0, t]$ and across firms.

1.3 The No-Monitoring Competitive Equilibrium

This section derives the equilibrium effort strategies and firm values when firms can neither monitor nor react to each other's actions. We refer to this as the *competitive* *equilibrium*, as distinguished from the coordinated equilibria introduced in Section 1.5. We show that the competitive equilibrium is also one of the possible equilibria in the setting with monitoring (firms resort to the competitive equilibrium when retaliating for deviations in coordinated equilibria).

Define $\mathcal{E}$ as the set of feasible effort processes of each firm: Any $e \in \mathcal{E}$ must be adapted and valued in $\{0, \theta\}$ (all-or-nothing effort).

Definition 1 *The collection of effort processes* $(\hat{e}^1, \ldots, \hat{e}^n) \in \mathcal{E}^n$ *is a competitive Nash equilibrium if*

$$v^i\left(P_0; e^i, \hat{e}^{-i}\right) \leq v^i\left(P_0; \hat{e}^i, \hat{e}^{-i}\right), \quad \text{all } e^i \in \mathcal{E}, \quad i = 1, \ldots, n.$$

The definition characterizes a competitive equilibrium, where each firm's effort process is optimal holding fixed the effort *processes* of the other firms.

Optimal Effort

Let $V^i(P; e^{-i})$ denote firm i's optimal value function given the aggregate effort process e^{-i} exerted by the other firms. The usual dynamic-programming argument implies the HJB equation (using the abbreviation $V^i = V^i(P; e^{-i})$)

$$0 = \max_{e_t^i \in \{0,\theta\}} e_t^i \left\{ f(P_t) - V^i - Q \right\} - \left(r + e_t^{-i}\right) V^i + \alpha P_t V_P^i + \frac{\sigma^2}{2} P_t^2 V_{PP}^i, \tag{1.4}$$

and, therefore, optimal effort satisfies

$$\hat{e}_t^i \in \arg \max_{e_t \in \{0,\theta\}} e_t \left\{ f(P_t) - V_t - Q \right\}, \tag{1.5}$$

with the solution

$$\hat{e}_t^i = \begin{cases} \theta & \text{if} \quad f(P_t) - Q \geq V^i(P; e^{-i}), \\ 0 & \text{otherwise} \end{cases}, \quad i = 1, \ldots, n, \quad t \geq 0. \tag{1.6}$$

Optimality has a simple form: Firm i exerts effort-rate θ when the potential immediate payoff $f(P_t)$, less the effort-cost rate Q, exceeds the present value of the future payoff. Otherwise, it is best to wait and exert no effort. Even under more general Markovian price dynamics, the optimality condition takes the same form, and the boundary P^*, above which effort is exerted, solves $V^i(P^*; e^{-i}) = f(P^*) - Q$. Whether immediate effort is optimal at P depends only on whether the net immediate payout exceeds the value function. Because $V^i(P; e^{-i})$ and $f(P)$ are both increasing in P, a reduction in the value function, resulting from an increase in r, for example,[11] implies a reduction in P^* and, therefore, more effort (i.e., effort exerted over a larger price region). Increased discounting makes an immediate payoff more desirable compared to a possible future payout, inducing higher effort.

This effort-boundary characterization differs from that in the traditional real-options setting, in which the exercise boundary satisfies both value matching and smooth pasting, and the value function equals the intrinsic value above the boundary (the analog in our setting is $V^i(P; e^{-i}) = f(P) - Q$ for $P \geq P^*$, which holds in our model only in the limiting case of $\theta \to \infty$). In our setting, the optimal boundary satisfies value matching, but $V^i(P; e^{-i})$ is strictly below $f(P) - Q$ for P above P^*. The difference in the boundary characterizations is because the firm can choose exactly when to adopt the project in the traditional setting, but in our setting, the firm can only exert effort to increase the likelihood of adopting (or winning) the project.

With our assumed payout $f(P) = \max(0, P - K)$ and our linear cost assumption, the cost parameter Q enters the HJB equation (1.4) only via the term $K + Q$.[12] This follows because the time-t conditional probability of succeeding and, therefore, paying K, is e_t; adding the effort-cost rate Q yields a total cost rate of $(K + Q)\, e_t$.

Solution for the Competitive Nash Equilibrium

The following proposition shows that there is a unique n-firm Nash equilibrium within the space of non-decreasing effort functions,[13] and this equilibrium is symmetric, characterized by a common effort boundary $P^{*\mathrm{cmp}}$ above which effort is exerted by every firm. Let $\mathcal{E}_{inc} \subset \mathcal{E}$ denote the set of feasible effort processes that are non-decreasing in the price (i.e., e_t is a non-decreasing function of P_t for each t).

Proposition 2 *Suppose there are n identical firms.*[14] *Within the space $\mathcal{E}_{inc}^n$ of non-decreasing effort processes, there is a unique Nash equilibrium, and this equilibrium is characterized by a common (and constant over time) effort boundary. Letting P^{*cmp} and $V^{cmp}(P)$ denote the Nash equilibrium effort boundary and value function*

[11] See Lemma 9.

[12] Note that $P \geq K$ always holds in the optimal effort region, and therefore the payoff function could be written as $f(P) = P - K$.

[13] This restriction is relaxed in the proof, where we assume that the aggregate effort of any $n-1$ firms is non-decreasing in price. That assumption is made mainly to simplify the proof and can be further relaxed.

[14] This assumption (in this proposition and the propositions below) can be relaxed slightly to n essentially identical firms, with equal $Q^i + K^i$s across firms.

of each firm,

$$P^{*cmp} = P^{*}(\theta,\delta), \tag{1.7}$$
$$V^{cmp}(P) = V(P;\theta,\delta), \quad \textit{for all } P>0, \tag{1.8}$$

with $\delta=\theta(n-1)$*, and where* $P^{*}(\theta,\delta)$ *and* $V(P;\theta,\delta)$ *are given by*

$$P^{*}(\theta,\delta)=\left(\frac{\theta+r+\delta-\alpha}{\theta+r+\delta}\right)\frac{[\beta^{+}-\beta^{-}(\theta+\delta)](r+\delta)+\beta^{+}\theta}{[\beta^{+}-\beta^{-}(\theta+\delta)](r+\delta-\alpha)+\theta(\beta^{+}-1)}(K+Q), \tag{1.9}$$

$$V(P;\theta,\delta)=\begin{cases}[P^{*}(\theta,\delta)-K-Q]\left(\frac{P}{P^{*}(\theta,\delta)}\right)^{\beta^{+}}, & \textit{if } P\le P^{*}(\theta,\delta),\\ -a(\theta,\delta)+b(\theta,\delta)P+c(\theta,\delta)\left(\frac{P}{P^{*}(\theta,\delta)}\right)^{\beta^{-}(\theta+\delta)}, & \textit{if } P>P^{*}(\theta,\delta),\end{cases} \tag{1.10}$$

where the polynomial roots β^{+} *and* β^{-} *are defined in* $(A.1)$ *(recall the abbreviation* $\beta^{+}(0)=\beta^{+}$*), and where*

$$a(\theta,\delta) = \frac{\theta}{\theta+r+\delta}(K+Q), \quad b(\theta,\delta)=\frac{\theta}{\theta+r+\delta-\alpha}, \tag{1.11}$$
$$c(\theta,\delta) = a(\theta,\delta)\frac{r+\delta-\alpha\beta^{+}}{[\beta^{+}-\beta^{-}(\theta+\delta)](r+\delta-\alpha)+\theta(\beta^{+}-1)}.$$

Each firm's equilibrium effort process is to exert effort θ *when* P_t *exceeds* P^{*cmp}*, and zero effort otherwise:*[15]

$$\hat{e}_t=\theta 1_{\{P_t\ge P^{*cmp}\}}, \quad \textit{for all } t>0.$$

The equilibrium is subgame perfect and is therefore also an equilibrium when firms can monitor and react to each other's efforts.[16]

Proof. See Appendix A. ■

[15] The proof shows that the function $V(P;\theta,\delta)$ is increasing and convex in P and satisfies

$$P-(K+Q)\ge V(P;\theta,\delta) \iff P\ge P^{*}(\theta,\delta). \tag{1.12}$$

Therefore, the immediate net payout from winning exceeds the value function only in the effort region.

[16] That is, the equilibrium is both open loop (in which the effort processes of the other firms are held fixed, as in Definition 1) and closed loop (in which each firm can react to the actions of the other firms). It is not always the case that an open-loop equilibrium is also closed loop: See, for example, [3], who show that the equilibrium in [20] is open loop but not closed loop.

Regardless of the history of the effort strategies played by the agents, the fixed-boundary strategies in the proposition are always a Nash equilibrium, and, therefore, the equilibrium is subgame perfect.

The following lemma presents some comparative statics and limits.

Lemma 1 *The equilibrium effort boundary P^{*cmp} and value function $V^{cmp}(P)$, for all $P > 0$, are both increasing in σ and α and decreasing in r. Both P^{*cmp} and $V^{cmp}(P)$ are decreasing in n and satisfy*

$$\lim_{n\to\infty} P^{*cmp} = K + Q, \quad \lim_{n\to\infty} V^{cmp}(P) = 0, \quad \text{all } P > 0.$$

If $n > 1$, then P^{*cmp} and $V^{cmp}(P)$ are not monotonic in θ and satisfy[17]

$$\lim_{\theta\to 0} P^{*cmp} = \lim_{\theta\to\infty} P^{*cmp} = K + Q, \quad \lim_{\theta\to 0} V^{cmp}(P) = \lim_{\theta\to\infty} V^{cmp}(P) = 0, \quad \text{all } P > 0.$$

Proof. See Lemma 9 in Appendix B for monotonicity in σ, α, r, and n. The limits are obtained from Proposition 2. ■

An increase in r increases impatience and elicits earlier effort. An increase in either the growth rate, α, or the volatility, σ, increases the value to waiting and delays effort.

As the number of firms increases, each firm increases effort by reducing its effort boundary. The effort boundary is reduced because the aggregate competing effort exertion reduces the firm's value function (because each firm is less likely to ultimately win the project), and, therefore, an immediate payoff becomes relatively more attractive than the possibility of a deferred payoff. In the limit, as the number of firms increases to infinity, each firm's effort boundary falls to the total cost, $K+Q$, and each firm's value function falls to zero (see [20], for a similar result in a different setting).

As the common maximum effort level θ increases, there are two opposing effects. For low common productivity (θ small), the dominant effect of increased effort productivity is to reduce the expected time to breakthrough, thereby increasing the

[17] When $n = 1$, the effort boundary and value function are monotonically increasing in θ, and the infinite-θ limits match the effort boundary and value function of the single-firm traditional real-options problem with strike price $K + Q$. Defining $P^{*\text{cmp}}(\infty) = \lim_{\theta\to\infty} P^{*\text{cmp}}$, then $P^{*\text{cmp}}(\infty) = \frac{\beta^+}{\beta^+-1}(K+Q)$ and

$$\lim_{\theta\to\infty} V^{\text{cmp}}(P) = \begin{cases} \left(\frac{1}{\beta^+-1}\right)(K+Q)\left(\frac{P}{P^{*\text{cmp}}(\infty)}\right)^{\beta^+} & \text{if} \quad P \le P^{*\text{cmp}}(\infty), \\ P - (K+Q) & \text{if} \quad P \ge P^{*\text{cmp}}(\infty). \end{cases}$$

value function and, therefore, the effort boundary (i.e., firms concentrate effort on more favorable payoff states). But as the common productivity level becomes large, the incentive to preempt the other firms' effort boundaries increases with θ, and, therefore, the value function and effort boundary decrease. The effort boundary and value function of each firm are highest for intermediate productivity, such that effort exertion has a reasonable chance of yielding success, but not so high that competition drives out profits.

In the traditional real-options problem with multiple firms (the limit as $\theta \to \infty$), each firm chooses a stopping time. If the project is awarded only to the firm who exercises first, there is such a strong incentive to preempt that all the competitive Nash equilibria are zero-profit ones. The symmetric zero-profit equilibrium is the limiting case of $\theta \to \infty$ in Lemma 1, with $Q = 0$. In our setting, the preemption incentive is much weaker: breaking from the competitive equilibrium by exerting effort sooner yields an increase in the chance of winning, but this benefit is outweighed by the cost of the additional effort.

Appendix D presents a simple iterative scheme for finding a Nash equilibrium with *heterogeneous* firms. As in the homogeneous case, the computed fixed boundaries of the effort strategies constitute a Nash equilibrium, regardless of the history of strategies played, and, therefore, the equilibrium is subgame perfect. In any equilibrium, the lower cost firm will exert more effort, and the more productive firm (the one with the higher maximum effort level) will exert effort less frequently (i.e., set a higher effort boundary).

1.4 The Cooperative Solution

In this section, we suppose that the n identical firms can cooperate (e.g., there is a common owner) and choose their efforts to maximize the aggregate value of the firms. The cooperative solution will be the basis for determining the first-best coordinated Nash equilibrium in Section 1.5, which can be achieved for sufficiently small delay times.

We imagine a single manager choosing the effort strategies of the n firms to maximize the total value function. The problem is equivalent to that of a combined firm choosing aggregate effort $\tilde{e}_t \in \{0, \theta, 2\theta, \ldots, n\theta\}$ at a per-unit effort cost Q. Letting V denote the aggregate value of the n firms, the combined firm's problem at each t is

$$0 = \sup_{\tilde{e}_t \in \{0,\theta,2\theta,\ldots,n\theta\}} \tilde{e}_t \{f(P_t) - Q - V(P_t)\} - rV(P_t) + \alpha V_P(P_t) P_t + \frac{\sigma^2}{2} V_{PP}(P_t) P_t^2.$$

Because the cost is linear, we get the corner solution (similar to the optimal effort choice in Williams, 1993):

$$\tilde{e}_t = \begin{cases} \theta n & \text{if} \quad f(P_t) - V(P_t) \geq Q. \\ 0 & \text{otherwise.} \end{cases} \tag{1.13}$$

The optimal boundary and aggregate value function are obtained from the single-firm ($n = 1$) case of Proposition 2 by simply replacing θ with θn.

The results are collected in the following proposition, whose proof follows from the above arguments. The optimal effort and value function are expressed per firm.

Proposition 3 *Suppose there are n identical firms. Letting P^{*co} and $V^{co}(P)$ denote the optimal cooperative effort boundary and value function of each firm, then*

$$P^{*co} = P^*(n\theta, 0), \quad V^{co}(P) = \frac{1}{n} V(P; n\theta, 0),$$

where $P^*(n\theta, 0)$ and $V(P; n\theta, 0)$ are obtained from (1.9) *and* (1.10) *(by letting $\delta = 0$ and replacing maximum effort θ with $n\theta$). The cooperative effort process for each firm is to exert effort θ when P_t exceeds P^{*co}, and zero effort otherwise:*

$$\hat{e}_t = \theta 1_{\{P_t \geq P^{*co}\}}, \quad \textit{for all } t > 0.$$

The sum of the n value functions equals the value function of a single firm with maximum effort intensity $n\theta$ instead of θ. The total firm value satisfies value matching and the property (1.12), but for any individual firm, the net payoff from winning the project will exceed the firm's value function for prices in a region below $P^{*\text{co}}$. That is, without some mechanism to enforce cooperation, each firm has an incentive to cheat and exert more than the cooperative effort level (assuming all other firms maintain the boundary $P^{*\text{co}}$) by choosing an effort boundary that lies somewhere in the interval $(P^{*\text{cmp}}, P^{*\text{co}})$.

Lemma 1 implies the following ordering of the equilibrium effort boundaries (we add an argument to the effort boundary to indicate the number of firms):

$$\frac{\beta^+}{\beta^+ - 1}(K + Q) > P^{*\text{co}}(n+1) > P^{*\text{co}}(n) > P^{*\text{cmp}}(n) > P^{*\text{cmp}}(n+1) > K + Q\ , \ \text{ for all } n > 1.$$

The boundary for the cooperative solution always exceeds the boundary for the competitive equilibrium and is strictly increasing in n, converging toward the traditional real-options boundary $\frac{\beta^+}{\beta^+-1}(K + Q)$ as $n \to \infty$. This contrasts with the competitive equilibrium, which is strictly decreasing in n, converging toward the zero-profit boundary $K+Q$. Therefore, for any n firms, less effort is exerted under cooperation than under the competitive equilibrium, with the gap growing in n.

The cooperative effort boundary is also strictly increasing in θ because higher θ works in the same way as an increase in n. In the competitive case, the boundary is highest for intermediate values of θ and is decreasing for sufficiently large θ.

1.5 Coordinated Equilibria

We assume throughout this section that firms can monitor each other's effort with a strictly positive delay time of D years. The technical issues for the case of no delay ($D = 0$) are handled in Section 1.6. We show in Section 1.5 that when D is below some threshold $D^* > 0$, the first-best cooperative outcome can be achieved in a coordinated equilibrium. In Section 1.5, we show that if D exceeds D^*, the first best cannot be sustained in equilibrium, but large value-function gains can be achieved for some common effort boundary P^* that satisfies $P^* \in (P^{*\text{cmp}}, P^{*\text{co}})$.

For any common effort boundary $P^* \in (P^{*\text{cmp}}, P^{*\text{co}}]$, we define the P^* *coordinated strategy* for each firm as follows:

> If no deviations have ever been observed, each firm exerts effort only when $P \geq P^*$. If, subject to a delay of D, a deviation by any firm has been observed, all firms revert immediately to the competitive equilibrium with effort boundary $P^{*\text{cmp}}$, which they maintain forever.

The P^* effort boundary constitutes a *coordinated equilibrium* if no firm chooses to deviate from the above strategy. The first step is to determine the optimal deviating strategy for any firm. If the value function for this strategy never exceeds (with probability one) the value function for conforming, then the P^* effort boundary is a coordinated equilibrium.

Optimal Cheating Strategy

Let $D > 0$ denote the monitoring delay time (in years) and $P^* \in (P^{*\text{cmp}}, P^{*\text{co}}]$ denote an effort boundary. We suppose all firms except firm i are following the P^* coordinated strategy. Suppose also that firm i has deviated from this strategy in the past and knows that this deviation will trigger all firms to revert to the competitive effort boundary $P^{*\text{cmp}}$ in τ years (where $\tau \in [0, D]$)[18], once the deviation has been

[18]That is, firm i began deviating $D - \tau$ years ago.

detected by the other firms. We characterize firm i's optimal effort strategy and value function.

Let $V^{\mathrm{cht}}(P,\tau)$ denote the value function of the cheating (deviating) firm τ years before retaliation begins. We can apply the general value-function expression (1.2) to get the firm-i value function.[19] Because all firms revert to the competitive equilibrium at $\tau=0$, the value function satisfies the initial condition

$$V^{\mathrm{cht}}(P,0)=V^{\mathrm{cmp}}(P)\quad \text{all } P>0,$$

where $V^{\mathrm{cmp}}(P)$ is given in Proposition 2. The HJB equation for the cheating firm, firm i, is (omitting the arguments of V^{cht} and replacing calendar time with time to retaliation), for $\tau>0$, equivalent to (1.4) with some notational changes:

$$0=\sup_{e_\tau\in\{0,\theta\}} e_\tau\left\{P_\tau-(Q+K)-V^{\mathrm{cht}}\right\}-\left(r+e_\tau^{-i}\right)V^{\mathrm{cht}}-\frac{\partial}{\partial\tau}V^{\mathrm{cht}}+\alpha P_\tau V_P^{\mathrm{cht}}+\frac{\sigma^2}{2}P_\tau^2 V_{PP}^{\mathrm{cht}}, \tag{1.14}$$

and, therefore, firm i's optimal effort process, denoted $\hat{e}^{\mathrm{cht}}$, satisfies (analogous to (1.6), but expressed in terms of time to retaliation)

$$\hat{e}_\tau^{\mathrm{cht}}=\begin{cases}\theta & \text{if} \quad P_\tau-(Q+K)\geq V^{\mathrm{cht}}(P,\tau),\\ 0 & \text{otherwise.}\end{cases} \tag{1.15}$$

The following lemma shows that effort is exerted only above a deterministic price threshold which lies between the competitive and cooperative effort boundaries.

Lemma 2 *Suppose only firm i has deviated from the P^* coordinated strategy, and all firms will revert to the competitive equilibrium in τ years. Then there is a deterministic boundary function $P^{*cht}(\tau)$, $\tau\geq 0$, such that optimal firm-i effort under the cheating strategy is*

$$\hat{e}_\tau^{cht}=\begin{cases}\theta & \textit{if} \quad P_\tau\geq P^{*cht}(\tau).\\ 0 & \textit{otherwise.}\end{cases}$$

The boundary $P^{*cht}(\tau)$ is strictly increasing in τ and satisfies

$$P^{*cht}(0)=P^{*cmp},\quad \textit{and}\quad P^{*cmp}<P^{*cht}(\tau)<P^*,\quad \tau\in(0,D].$$

The value function of firm i is strictly increasing in τ and satisfies

$$V^{cht}(P,0)=V^{cmp}(P),\quad \textit{and}\quad V^{cmp}(P)<V^{cht}(P,\tau),\quad \tau\in(0,D].$$

[19] Expressed in years to retaliation, τ, the aggregate effort of the other firms is $e_\tau^{-i}=(n-1)\,\theta 1_{\{P_\tau\geq P^*\}}$ for all $\tau>0$ and $e_\tau^{-i}=(n-1)\,\theta 1_{\{P_\tau\geq P^{*\mathrm{cmp}}\}}$ for all $\tau\leq 0$.

Notice that we cannot order $V^{\text{cht}}(P,\tau)$ relative to the value function from not deviating (although if the P^* coordinated strategy were a Nash equilibrium, it must be the case that cheating does not increase value for any P). Letting $\hat{e}_\tau^{\text{cmp}}$ and $\hat{e}_\tau^{P^*}$ denote the optimal effort processes in the competitive equilibrium and the P^* coordinated strategy, respectively, the lemma implies that $\hat{e}_\tau^{P^*} < \hat{e}_\tau^{\text{cht}} < \hat{e}_\tau^{\text{cmp}}, \quad \tau > 0.$

Sustainability of the First Best

The main result in this section is Proposition 4, which shows that the cooperative solution can be achieved in a coordinated equilibrium (we call this the *first-best coordinated equilibrium*) if the delay time is below some strictly positive cutoff D^*. We present numerical solutions showing that D^* is sensitive to the parameters of the model, particularly the number of firms, n, and the maximum effort intensity, θ.

The previous section examined the optimal strategy for a firm deviating from the P^* coordinated strategy that is followed by the other firms, where the effort boundary P^* satisfies $P^* \in (P^{*\text{cmp}}, P^{*\text{co}}]$. Throughout this section, we let $P^* = P^{*\text{co}}$ (the cooperative boundary) and determine the maximum delay such that the first-best coordinated strategy is a Nash equilibrium. The longer the monitoring delay time, the greater the benefit to cheating (this is shown in Lemma 2), and if the delay is sufficiently large, the value function from cheating, even accounting for retaliation after D years, will exceed the cooperative value function. In fact, we know that as $D \to \infty$ (i.e., no monitoring), the unique equilibrium is the competitive one given in Proposition 2, which is always associated with strictly lower value functions (when $n > 1$) than in the first-best solution.

The following lemma shows that there is a critical delay time, which we denote D^*, such that the first-best coordinated equilibrium can be sustained for any monitoring delay time D satisfying $D \leq D^*$.

Proposition 4 *There exists a strictly positive D^*, which is a function of the parameters $(\theta, n, \alpha, r, \sigma)$, such that for any delay time $D \leq D^*$, the first-best coordinated equilibrium is sustainable, and for $D > D^*$, it is not.*

Numerically, we compute D^* as the largest delay time such that cheating is not optimal at any price (see the proof of Lemma 4):

$$D^* = \sup\left\{\tau > 0 : V^{\text{cht}}(P,\tau) \leq V^{\text{co}}(P) \text{ all } P \geq 0\right\}.$$

(It is easy to show that in the above supremum, it is sufficient to consider only $P \in [P^{*\text{cmp}}, P^{*\text{co}}]$.) Note that the cheating value function must be (weakly) worse than the cooperative value function to ensure that deviating is suboptimal with probability one (see the proof of Lemma 4).

D^* is very sensitive to effort intensity, θ. Low θ implies a small benefit to deviating from the cooperative effort strategy because of the small per-unit-time probability of winning when firm i cheats and is the only firm exerting effort. For a project with a low chance of success, $\theta = 0.1$, and two competing firms, the maximum delay time under which the first-best coordinated equilibrium is sustained is 306.1 days. The maximum delay falls to 4.1 days for $\theta = 1$ and just under 1 day when $\theta = 2$. In addition D^* is also sensitive to the number of competing firms. A larger number of competitors results in a higher cooperative effort boundary and a larger aggregate effort rate above the boundary; both factors generates a strong incentive to preempt the other firms by cheating. With 4 firms, for example, D^* is below 1 day for any θ above 0.8.

Increasing α, the growth rate of P, reduces D^*. Increasing the growth rate encourages firms to be more patient and raise their effort thresholds (see Lemma 1), which increases the incentive to preempt and cheat. The effect of increasing the interest rate r (results not included here) is the opposite: higher discounting reduces $P^{*\mathrm{co}}$ and the cooperative value functions and, therefore, increases D^*.

Finally D^* is hump shaped in σ, the volatility of P: first increasing in σ but eventually decreasing for sufficiently large σ. This appears to be driven by the fact that (from numerical results) the difference in the cooperative and competitive value function is also hump shaped in σ.[20]

Less-Than-First-Best Coordinated Equilibria

In this section, we determine, for each monitoring delay time D, the coordinated-strategy effort boundary that maximizes the value of each firm. That is, we find the maximum common effort boundary in the interval $[P^{*\mathrm{cmp}}, P^{*\mathrm{co}}]$ such that not cheating (i.e., not deviating) is a Nash equilibrium. The previous section showed that when $D \leq D^*$, the first-best coordinated equilibrium, with effort boundary $P^{*\mathrm{co}}$, can be attained. The critical delay time D^* can be large when effort intensity, θ, is small and the number of firms, n, is small. However, for large θ or n, the maximum delay D^* tends to be small because of the high reward to preemption relative to the cost of future retaliation.

We show that even when the delay time D exceeds D^*, and first best cannot be attained in a Nash equilibrium, large proportional gains in the value function can be achieved in a coordinated equilibrium with a common effort boundary above

[20] It can be shown that $\lim_{\sigma\to\infty} V^{\mathrm{co}}(P) = \lim_{\sigma\to\infty} V^{\mathrm{cmp}}(P)$.

the competitive level, but sufficiently below the cooperative level to eliminate any incentive to cheat (and elicit retaliation).

We denote by $P^{*\max}(D)$ the maximum effort boundary among all the coordinated strategies that generate a subgame-perfect equilibrium. Recall that in this equilibrium, each firm exerts effort only when $P_t \geq P^{*\max}(D)$. Any effort boundary above $P^{*\max}(D)$ will elicit cheating with positive probability (followed by retaliation D years later).

To compute $P^{*\max}(D)$, we need the value function assuming all firms adhere to an arbitrary common effort boundary P^*, with each firm exerting effort only when P_t exceeds P^*. Letting $v(P; P^*)$ denote this value function, its solution is provided in the following proposition. The proposition also shows that $v(P; P^*)$ is strictly quasiconcave in P^*, achieving its maximum at the cooperative effort boundary (i.e., $P^* = P^{*\text{co}}$).

Proposition 5 (value function with a common effort boundary) *Suppose there are n identical firms, and each exerts effort if and only if P_t equals or exceeds the effort boundary P^* where $P^* > K + Q$. Then each firm's value $v(P; P^*)$ satisfies*

$$v(P; P^*) = \frac{1}{n} \cdot \begin{cases} \{-a(n\theta, 0) + b(n\theta, 0) P^* + C(P^*)\} (P/P^*)^{\beta^+} & \text{if} \quad P < P^* \\ -a(n\theta, 0) + b(n\theta, 0) P + C(P^*) (P/P^*)^{\beta^-(n\theta)} & \text{otherwise} \end{cases}$$

where

$$C(P^*) = \frac{b(n\theta, 0)(1 - \beta^+) P^* + a(n\theta, 0)\beta^+}{\beta^+ - \beta^-(n\theta)}.$$

For any $P > 0$, the value function $v(P, P^*)$ is maximized at the effort boundary $P^* = P^{*co}$ and is strictly increasing in P^* for $P^* \in (K + Q, P^{*co})$ and strictly decreasing for $P^* > P^{*co}$.

We compute $P^{*\max}(D)$ numerically as follows (we add the effort-boundary argument P^* to the value function V^{cht} of the cheating firm):

$$P^{*\max}(D) = \sup \left\{ P^* \in [P^{*\text{cmp}}, P^{*\text{co}}] : V^{\text{cht}}(P, D; P^*) \leq v(P; P^*) \text{ all } P > 0 \right\}.$$

we map $P^{*\max}(D)$ to a $[0, 1]$ scale by transforming it to the proportion of its distance from $P^{*\text{cmp}}$ to $P^{*\text{co}}$ and denote this distance by $\lambda(D)$:

$$\lambda(D) = \frac{P^{*\max}(D) - P^{*\text{cmp}}}{P^{*\text{co}} - P^{*\text{cmp}}}.$$

Propositions 5 and 2 imply that for any effort boundary $\bar{P} > K + Q$, the value-function ratio is constant in the price region $\{P \leq P^{*\mathrm{cmp}}\}$ and satisfies

$$\frac{v\left(P;\bar{P}\right)}{V^{\mathrm{cmp}}\left(P\right)} = \frac{v\left(\bar{P},\bar{P}\right)}{nV^{\mathrm{cmp}}\left(P^{*\mathrm{cmp}}\right)}\left(\frac{P^{*\mathrm{cmp}}}{\bar{P}}\right)^{\beta^{+}} > 1, \text{ for } P \leq P^{*\mathrm{cmp}}.$$

As $P \to \infty$, the value-function ratio converges to one because the probability that effort will always be exerted henceforth converges to one in both the coordinated and competitive equilibria.

Longer delay times (larger D's) are associated with lower effort-boundary levels that can be sustained as an equilibrium (i.e., $\lambda(D)$ is decreasing in D), and an increase in the number of firms (which also incentives preemption) results in lower effort boundaries (i.e., $\lambda(D)$ is decreasing in n). The declines become more rapid as we progress. Consistent with Proposition 4, when D is sufficiently small, the first best is achieved, and, therefore, $\lambda(D)$ equals one for $D \leq D^{*}$. As D increases toward infinity, $\lambda(D)$ decreases toward zero, corresponding to the competitive equilibrium.

For the two-firm case and $D \leq 4$ days, $\lambda(D) = 1$ (the first best is achieved), and the corresponding value-function ratio is 2.28 (a 128% improvement over the competitive equilibrium). As D increases, the proportionate gains decrease as $\lambda(D)$ decreases, but even for a delay time of 200 days, the value-function ratio is 1.73 despite a small $\lambda(D)$ value of 0.167. For $n = 4$, the proportionate gains from the cooperative outcome are much larger for small D (equal to 4.02 at $D = 0$), with the ratios for all three values of n tending to converge as D increases to 200 days. The ratios at $D = 200$ are 1.73, 1.81, and 1.76 for $n = 2, 3$, and 4, respectively. The ratio is not always monotonic in n: The $n = 3$ and $n = 4$ ratios cross near $D = 100$ days.

The higher maximum effort intensity and higher growth rate of the price process together increase the incentive to preempt, resulting in lower $\lambda(D)$'s, which converge toward zero faster as D increases. However, the relative firm-value gains under the coordinated equilibria are still large for all three values of n for $D \leq 31$. For $n = 4$, the ratio declines from 6.79 to 1.59 as D increases from 0 to 200, and for $n = 2$, the ratio declines from 3.66 to 1.77.

Finally, with parameters consistent with strong incentives to preempt, the cooperative effort boundaries decline very rapidly toward the competitive boundaries as D increases. For a 10-day delay, for example, we get $\lambda(D)$ values of 0.0779, 0.0361, and 0.0225 for $n = 2$, 3, and 4 firms, respectively. But despite the low values of $\lambda(D)$, the relative value-function gains are again large, with corresponding 10-day-delay value-function ratios of 3.93, 4.25, and 4.10, and 200-day-delay value-function ratios of 1.66, 1.52, and 1.41, for $n = 2, 3$, and 4, respectively.

The large benefit to a small effort-boundary increase above the competitive effort boundary is consistent with the following expression (from the proof of Proposition 5) for the sensitivity of the value function $v\left(P^{*\text{cmp}}; P^*\right)$ to an increase in the effort boundary from the competitive level, particularly when the cooperative boundary is well above the competitive boundary:

$$\frac{d}{dP^*} v\left(P^{*\text{cmp}}; P^*\right)\bigg|_{P^*=P^{*\text{cmp}}} = \frac{\left(\beta^+ - 1\right)\left(1 - \beta^-\left(n\theta\right)\right)}{\beta^+ - \beta^-\left(n\theta\right)} b\left(n\theta, 0\right)\left(\frac{P^{*\text{co}} - P^{*\text{cmp}}}{P^{*\text{cmp}}}\right).$$

The value function $v\left(P^{*\text{cmp}}; P^*\right)$ is maximized when the boundary P^* equals the cooperative boundary $P^{*\text{co}}$ (and hence is flat at that point) and is increasing in P^* (as shown in Proposition 5) for boundaries below $P^{*\text{co}}$. The slope tends to be steep near the competitive boundary in the cases when incentives to preempt are strong.

To summarize, the maximum coordinated effort boundaries consistent with equilibrium (that is, the Nash equilibrium that maximizes firm value) decline rapidly as the monitoring delay time increases, and the decline increases faster as θ, α, and n increase. however, the value-function ratio (coordinated relative to competitive) declines very slowly toward the limiting value of one, demonstrating large proportionate gains to coordination even when the coordinated effort boundary is only slightly above the competitive boundary.

1.6 The Case of Zero Delay

The case of zero monitoring delay, $D = 0$, requires special treatment because, as has been pointed out by S-S (in a deterministic setting), the meaning of an *immediate* response to a deviation in the continuous-time setting is not well defined. We therefore formulate the strategy in a sequence of discrete-time approximations, in which the strategies are well defined. The continuous-time solution is defined as the limit of the discrete-time solutions.

Consistent with Proposition 4, the first best is always achieved with perfect monitoring. In our continuous-time setting, the benefits of deviating are infinitesimal because retaliation is immediate, and, therefore, no discount-rate restriction is needed to sustain the equilibrium. Further, the equilibrium is robust to future renegotiation. Any strictly positive time lag before renegotiating is sufficient to sustain the coordinated equilibrium.

The phenomenon of having a coordinated solution as a competitive outcome in repeated games is well known in the literature. See, for example, [15], with perfect monitoring; [14], with imperfect public monitoring; and [24], with private almost-perfect monitoring in the discrete setting. The imperfect monitoring problem in the continuous-time setting is considered in [43] and [4].

Our problem, however, is not a result of the (continuous-time) cases studied before. The focus in the above-mentioned papers is mainly on the distortion of the (signal) observation via Brownian motion rather than the dependence of the payoff on the external stochastic process (the price process, in our model). In the two-firm case, our result is related to the result of [15], where one agent rationally punishes any deviation by the other. However, we cannot apply the method used in that paper. First, payoffs in our model are not a deterministic function of the players' actions. Second, the game studied here has a stochastic stopping time, and it is not an infinity repeated game. The construction in our setting works as well for $n > 2$ because of the uniqueness of the Nash equilibrium. In the course of doing so, we discretize the model at random times rather than fixed times, a method briefly mentioned in [4]. We hope this approach can have potential applications beyond this paper.

Discrete-Time Price-Process Approximation

To use some of our earlier continuous-time results (in particular, the representation (1.2), the HJB equation, and the verification proof), we formulate a binomial model with random period lengths. We use random (Poisson) times between consecutive jumps to ensure that the price process and value functions are homogeneous Markov processes. We assume each firm's effort rate can be changed only at the beginning of each period, and, therefore, the effort strategies are well defined.

We approximate the diffusion price process (1.1) with a right-continuous pure-jump process, in which jumps arrive with constant Poisson intensity rate κ, and the gross return at each jump time t_i, $i = 1, 2, \ldots$, is J_i, which is i.i.d. across i. That is, the price process is constant, except at jump times, when

$$P_{t_i} = P_{t_i-} \cdot J_i, \quad i = 1, 2, \ldots. \tag{1.16}$$

For simplicity, we will assume a binary jump distribution:

$$J_i = \begin{cases} u & \text{with probability} \quad p, \\ 1/u & \text{with probability} \quad 1 - p. \end{cases} \tag{1.17}$$

Analogous to a common implementation of the binomial option-pricing model, for any κ we calibrate the jump model to the continuous-time model by choosing the following u and p:

$$u = e^{\sigma/\sqrt{\kappa}}, \quad p = \frac{\frac{\alpha}{\kappa} + 1 - \frac{1}{u}}{u - \frac{1}{u}}. \tag{1.18}$$

Lemma 3 *With the parameters* (1.18)*, the pure-jump model, defined by* (1.16) *and* (1.17)*, converges in distribution to the continuous-time price process* (1.1) *as* $\kappa \to \infty$*. If* $\kappa > (\alpha/\sigma)^2$*, then* $p \in (0,1)$*.*

Proof. See Appendix A. ■

We henceforth assume that $\kappa > (\alpha/\sigma)^2$ (and, therefore, $p \in (0,1)$).

Solutions in the Pure-Jump Setting

The jump-model counterparts to the diffusion-model parameters $\beta^{\pm}(\eta)$ are given by $\beta^{\pm}_{\kappa}(\eta)$, which are defined in $(A.3)$. The following proposition shows that the solutions in the pure-jump model are obtained by simply replacing $\beta^{\pm}()$ in the diffusion solutions with $\beta^{\pm}_{\kappa}()$.

Proposition 6 *The pure-jump-model solutions, with intensity* κ*, for the competitive equilibria and the cooperative solution, are given by Propositions 2 and 3, respectively, after replacing* $\beta^{+}()$ *and* $\beta^{-}()$ *with* $\beta^{+}_{\kappa}()$ *and* $\beta^{-}_{\kappa}()$*, respectively, in the solution formulas* (1.9)*,* (1.10)*, and* (1.11)*.*

Proof. See Appendix A. ■

All the solutions in the discrete approximation (for any $\kappa > (\alpha/\sigma)^2$) have the same qualitative properties and comparative statics as in the continuous-time case examined earlier.

Finally, as the jump frequency blows up, $\beta^{\pm}_{\kappa}(\eta)$ converge to the corresponding diffusion values:

Lemma 4 *For any* $\eta \geq 0$*, then* $\beta^{+}_{\kappa}(\eta) > 1$*,* $\beta^{-}_{\kappa}(\eta) < 0$*, and*

$$\lim_{\kappa\to\infty} \beta^{+}_{\kappa}(\eta) = \beta^{+}(\eta) \qquad \lim_{\kappa\to\infty} \beta^{-}_{\kappa}(\eta) = \beta^{-}(\eta).$$

Large-κ Coordinated Equilibria

Denote by $P^{*\text{co}}(\kappa)$ and $P^{*\text{cmp}}(\kappa)$ the jump-model (with intensity κ) effort barriers for the cooperative and competitive solutions, respectively. We consider the following *coordinated strategy* for each firm:

> Maintain the cooperative effort strategy, with boundary $P^{*\text{co}}(\kappa)$, as long as no firms have ever exerted effort below the cooperative boundary. As soon as any firm deviates in this manner, all firms revert at the beginning of the next period to the competitive equilibrium, with effort boundary $P^{*\text{cmp}}(\kappa)$, which they maintain forever.

Proposition 7 *For all sufficiently large κ, the coordinated strategy generates a subgame-perfect competitive equilibrium (which we call the first-best coordinated equilibrium) in which each firm exerts effort only when $P_t \geq P^{*co}(\kappa)$.*

Proof. See Appendix A. ■

The idea of the proof is straightforward. The benefit to firm j from deviating, by exerting effort at a price below $P^{*co}(\kappa)$, gives only firm j a chance of winning the project until the other firms respond at the next price jump. This benefit is only temporary, however, and converges to zero as $\kappa \to \infty$. The cost of deviating is a permanent shift to the more costly higher-effort competitive equilibrium. For large enough κ, therefore, deviating is sub-optimal.

The equilibrium is similar to one in the discrete-time infinite-horizon repeated prisoner's dilemma, in which players cooperate as long as neither player defects, then perpetually defect if either player defects. In our context, however, there is no discount-rate condition. Further, it is robust to future renegotiation. As long as there is some fixed time delay before renegotiation, the limiting result still holds because the benefit to deviating disappears as $\kappa \to \infty$, but the cost is uniformly strictly positive for any κ.

1.7 Extensions

In this section we briefly discuss an extension to the case in which the agent can choose different levels of efforts. Suppose $g : R^+ \to R^+$ is an increasing convex function such that $g(0) = 0$ and $g(1) = 1$. Assume the agent can choose among $m+1$ effort levels: $e_t \in E = \left\{0, \frac{1}{m}, ..., 1\right\}$. We assume $q(\frac{i}{m}) = g(\frac{i}{m})Q$ and effort level $\frac{i}{m}$ corresponds to an intensity rate of $\frac{i}{m}$. The agent's problem is then equation (1.4),

$$0 = \sup_{e_t \in E} e_t \left\{f(P_t) - V(P_t)\right\} - q(e_t) - rV(P_t) + \alpha P_t V_P(P_t) + \frac{\sigma^2}{2} P_t^2 V_{PP}(P_t),$$

and the optimal effort satisfies equation (1.5):

$$\hat{e}(P_t, V_t) \in \arg\max_{e_t \in E} e_t \left\{f(P_t) - V_t\right\} - q(e_t).$$

Because the cost function is convex, by writing the equations like the binary case, we can see there exists

$$0 = P_0 < P_1 \leq P_2 \leq P_3 ... \leq P_m < P_{m+1} = \infty,$$

such that the firm exerts effort $\frac{i}{m}$ for $P \in [P_i, P_{i+1})$ for $0 \leq i \leq m$. If $P_i = P_{i+1}$, it means that it is never optimal to exert effort $\frac{i}{m}$, as in the corner solution in the cooperative problem in which the central decision maker always chooses either 0 or n firms to work. For the sake of completeness, we explain the procedure to compute the value function and $P_1, ..., P_m$ in case $m = 2$. The method can be extended to arbitrary m in the obvious way. Assume we have

$$0 = P_0 < P_1 \leq P_2 < P_3 = \infty,$$

such that the effort $\frac{i}{2}$ is optimal in $[P_i, P_{i+1})$ for $i = 0, 1, 2$. We want to find P_1, P_2, and $V(P)$. In the interval $[0, P_1]$, the value function is $V(P_1)(\frac{P}{P_1})^{\beta^+}$ as we had in the binary case because the ODE that we obtain in the no-effort region is the same as in the binary case. In the interval $[P_1, P_2]$, we have

$$V(P) = a_1 P + b_1 + c_1 P^{\beta_1} + d_1 P^{\beta_1'}.$$

We can compute $a_1, b_1, \beta_1 > 1, \beta_1' < 0$ because $V(P)$ for $P \in [P_1, P_2]$ satisfies equation (1.4) for $e = \frac{1}{2}$. The constants c_1 and d_1 remain to be determined. Similarly, for $P \in [P_2, \infty]$ we have

$$V(P) = a_2 P + b_2 + d_2 P^{\beta_2'},$$

where $\beta_2' < 0$. Therefore, we have five unknowns P_1, P_2, c_1, d_1, d_2, and we need five equations to determine them. Three of the equations come from the fact that V is $\mathcal{C}^1$: differentiability at P_1, as well as continuity and differentiability at P_2. Note that continuity at P_1 is automatically satisfied. The next two equations come from the optimality equation (1.5). At P_1 we have

$$0 = \frac{1}{2}[f(P_1) - V(P_1)] - q\left(\frac{1}{2}\right),$$

and at P_2 we have

$$\frac{1}{2}[f(P_2) - V(P_2)] - q\left(\frac{1}{2}\right) = f(P_2) - V(P_2) - q(1).$$

Solving these equations will give us the desired quantities.

1.8 Conclusion

We examine a dynamic model of R&D, which incorporates both economic and technological uncertainties, as well as competition and monitoring between firms. Without monitoring, the essentially unique equilibrium is the competitive one. With perfect monitoring, there are a multitude of equilibria, including the competitive one, as well a coordinated equilibrium that matches the cooperative equilibrium and is enforced by the credible threat of retaliation if any firm deviates. Thus, transparency in R&D investment can benefit firms by allowing them to coordinate, delaying investment and increasing profits.

We show that projects for which the probability of success for each firm is low (i.e., low θ) can have a substantial delay time in monitoring, especially in a two-firm setting, and yet the first best is still attained in a coordinated equilibrium. The maximum delay time under which the first best is achieved is found to be increasing in the discount rate, decreasing in the growth rate of the payout and the number of firms, and hump shaped in volatility.

Finally, our numerical solutions show that even when the monitoring delay time is too large to sustain the first-best outcome, there always exists a coordinated equilibrium that is superior to the competitive equilibrium, with a common effort boundary in between the competitive and cooperative boundaries. The proportionate firm value gains over the competitive levels can be very large even when preemption incentives and delay times are large.

In our model, competition can increase in two different ways: through an increase in the number of firms or through an increase in productivity (i.e., θ) of each firm. In the competitive equilibrium, an increase in the number of firms increases each firm's effort and reduces firm value, a result consistent with the erosive effect of competition found, for example, in [20] and [21]. But the competitive effort boundaries and firm values are not monotonic in productivity. The effort boundary (and firm value) is smallest for low and high productivity and is maximized at some intermediate productivity. In the first-best coordinated equilibria, however, both an increase in the number of firms and an increase in productivity always result in delayed effort, and the increase in productivity increases firm value.

Various generalizations of the model can be considered. We sketch the case of different levels of effort in Section 1.7. Alternative time-homogeneous Markovian price processes are straightforward. We show that the solutions in our pure-jump price process require only a slight modification to the geometric Brownian motion solutions.

We also use a novel method for solving the coordinated equilibria in the case of perfect monitoring by discretizing the game to intervals of random length to preserve the time-homogeneous-Markovian setting. This method enables us to obtain the continuous-time game as a simple limit of the closed-form discrete-time solutions. This method can have applications to other settings with instantaneous reactions of players in continuous time.

Chapter 2

Optimal Design of LPA

2.1 Introduction

In private equity, the relationship between fund managers (general partners or GPs) and investors (limited partners or LPs) is governed by a "limited partnership agreement" (LPA). These contracts are crucial in determining how GPs behave for the following reasons. First, LPs have limited resources outside of these contracts to discipline GPs. Second, these agreements typically remain in effect for about a decade, and recently up to 15 years, (with little room for renegotiation). Finally, GPs' actions are hard to observe and writing a contract which provides the right incentives for GPs is of critical importance.

In general, there are three main financial components in an LPA. These are the management fee, carried interest, and the method of payments to GPs. While the structure of the management fee and carried interest has been the subject of extensive research, there is virtually no theory on why the method of payment is important and how it effects GPs' performance. Historically LPAs offer two methods for paying carried interest to GPs. The first method is deal-by-deal or "American". This provision allows GPs to earn the interest as soon as each deal is exited. The second method is whole-fund or "European". In this method, LPs receive the entire interest on their investment(s) before GPs get any carried interest.[1]

At first glance, it seems that the European method is more favorable to LPs--in fact, [25] calls it the "LP-friendly contract". In particular, if we assume that the GP does not change her strategy under different types of contracting, then whole-

[1] In [30], there is detailed explanation on different provisions for these methods.

portfolio contracting is preferred to the deal-by-deal method for investors. However, as the GP changes her strategy as the contract changes, it is not clear which method is more efficient for investors. Here is an example which illuminates the difference between these two methods. Suppose a GP has invested in a fund consisting of two firms. Suppose one of the firms exits with a high return but the other one loses money so that in total the return is low. In a deal-by-deal contract, the GP would get some interest on the successful exit. However in the whole-portfolio method, since the low-return investment offsets the high-return one, the GP will receive almost nothing and the whole return will go to the investor. [2]

To fix ideas, consider the following scenario. Suppose there is an LP who wants to invest in a pool of two projects but has no expertise to find profitable investment opportunities. As a result, he hires a GP to do the job. The GP has to exert effort to find good investment opportunities, but even with significant effort she may end up with low quality projects. As is prevalent in this setting, the LP has no control over the GP's actions, nor does he know the quality of the projects, unlike the GP. Thus, the contracting is subject to both moral hazard and adverse selection.

Within the setting outlined above, I investigate the conditions under which each method of payment (deal-by-deal or whole-portfolio) is optimal. As a result, I can explain some empirical findings documented in the literature. First, I show that when projects are highly correlated, whole-portfolio contracting is optimal for the LP. As the correlation declines, the space of portfolios where deal-by-deal contracting is preferred expands. This phenomena has been documented empirically in [33]. The mechanism behind this result comes from the trade-off between the moral hazard about the effort to find good projects versus the information asymmetry about the quality of projects. When projects have high correlation, bundling the performance of projects together can enhance incentives for the GP to exert effort on them. In this case, even when projects are subject to different degrees of adverse selection, the loss of efficiency is still low enough that whole-portfolio contracting is preferred to deal-by-deal contracting which can handle adverse selection efficiently.

Second I show that when the GP is not reputable, it is more likely that the LP should use whole-portfolio contracting compared to when contracting with a rep-

[2] Even in the presence of claw-back provisions which requires GPs to return some of the return at the end to the LP, still the GP gets an interest free loan from the LP in the meantime. Moreover in the sample of contracts considered in [7], only about 26% of contracts had claw-back.

utable agent. [3]This result is in alignment with the findings in [25]. In this paper, the authors propose that when a GP is more reputable, they have more market power and hence can get more favorable contracting terms. In my setting, however, this comes from the fact that for non-reputable agents, whole-portfolio contracting can reduce the chance of making bad investments, hence improving the investment strategy. Therefore, the sorting effect exists in this environment, but indirectly as a result of the change of behavior of the agent due to the terms of the contract.

The model yields other results and predictions. For example, I show that when there is little or no information asymmetry about the quality of projects between investor and agent, whole-portfolio contracting is the dominant form of contracting. This can explain why we see this form of contracting when the underlying assets are public firms. Specifically in the case of hedge funds or mutual funds, the payout to the agent is almost always a function of the performance of the whole portfolio rather than the individual performance of assets in the portfolio. I also predict that investors' information can affect the method of payment. When investors are not fully informed on the structure of an investment, they prefer to have a narrower scope of investment (hence higher correlation) and use the whole-portfolio contracting method.

The main feature of the model which enables me to show these results is the fact that projects are heterogeneous. If different projects are always subject to same degree of moral hazard and information asymmetry, then bundling the payouts together has no efficiency loss and whole-portfolio contracting is the dominant method of contracting, as is the case for many contracts in the real world. This is the dominant assumption in the literature, in the seminal work of [9] and subsequent studies. For example, [28] considers a pool of homogeneous projects and show how investors can design better contracts by the pooling and loosening of limited liability. However, when a typical VC invests in a pool of projects, it is reasonable to assume a high degree of heterogeneity between projects.[4]

The heterogeneity of projects creates a trade off between moral hazard and adverse selection. When a contract is written on the whole-portfolio basis, investors

[3]By non-reputable agent, I mean an agent that investor can not verify her access to investment opportunities hence needs to be distinguished from fly-by-night operators.

[4]VCs invest in projects which are highly innovative with unique business plans with very few assets in place unlike for example banks which give loan to ordinary businesses or mortgages to residential/commercial properties. As a result, we expect much more heterogeneity in VCs invested portfolios.

can more easily persuade agents to exert effort on the projects through bundling the payouts. However, whole-portfolio contracting takes away the flexibility to deal with the different degrees of adverse selection that the projects are subject to. For higher correlation between projects or lower reputation of agents, the priority is to mitigate the more severe moral hazard problem, and the whole-portfolio contracting is therefore preferred. On the contrary, when the correlation between projects is low or the agent is reputable, adverse selection is more severe, and deal-by-deal contracting is better suited to deal with this issue.

This paper relates to the theoretical literature in the area of PE funding. In [2], the authors study the problem of leverage in buyouts and show that a combination of ex-ante pooled financing and ex-post deal-by-deal financing is optimal. In their setting, the timing of the investment on projects is different, while in a lot of limited partnership contracts the GP is required to choose the portfolio firms early in the life-span of the LPA. In another similar work, [11] shows why LPs restrict the investment timing of GPs. In both of these works, the authors abstract away from the moral hazard problem between LPs and GPs, and also consider a pool of similar projects. Because of the homogeneity between projects, when the method of financing is restricted to ex-ante, whole-portfolio financing is always optimal in their setting and they are not able to explain the abundance of the deal-by-deal ex-ante contracting in the PE industry.

This work also contributes to the literature on investment pooling and portfolio contracting. [26] consider the case in which investors faces multiple agents and investment pooling and credit rationing can motivate optimal investment strategy. Their main mechanism relies on the competition among agents, while in my work credit rationing has no bite as investors face only one agent. [18] also consider the case of contracting between an investor and multiple agents and focus on the double moral hazard problem between GPs and entrepreneurs. In contrast, I abstract away from GP/entrepreneur problems and focus on the contracting between GPs and LPs. This paper also relates to the literature on moral hazard with learning, [22] and [37], and experimentation and Bandit problems, [40], as well.

Empirically, the first work which addresses the importance of the method of compensation in VC settings is [30]. She shows that the shift in the timing of compensation can affect the present value of the payment to the GP as much as changing the contracting terms themselves. While the importance of the compensation method is discussed in [30], [25] and [33] study the effects of payment methods on the GP investment strategy and fund's return. All of these papers are empirical and offer little theory on the matter.

More broadly, the first work which studies GP compensation is [19]. The authors explore the cross sectional and time variation of the management fee and carried interest in the contract terms, assuming that contracts have the same method of payment. [36] study a similar problem using an option-pricing framework, and focus more on buyout funds. Unlike these works, [42] have access to cash flow data as well as contracting terms, which links the management payment to the performance using a novel data set containing all the payment from a big institutional investor to GPs.

The paper proceeds as follows. Section 2.2, introduces the models and shows the optimal contracting on one project. In Section 2.3, I solves the problem of optimal whole-portfolio contracting and compare it to the deal-by-deal contract. Section 2.4 consider the same problem for non-reputable GPs and I compare the results to the case of reputable agents. Section 2.5 considers various extensions of the model. Finally Section 2.6 concludes.

2.2 Model

There are three classes of agents in the model: limited partners (investors or LPs), general partners (GPs) and fly-by-night operators (FNOs). All agents are risk-neutral and have access to a safe asset technology with a return which is normalized to zero. There are two types of general partners, reputable and non-reputable. Both types of general partners have access to a pool of projects in which they can invest in. The limited partner has capital which is needed to run projects.[5] FNOs have no access to the pool of risky projects but they can mimic the behavior of a GP. I assume that if the GP is reputable, then the LP can verify that she has access to projects. However, the LP can not distinguish between a non-reputable agent and a FNO. Initially, I focus on reputable GPs and discuss contracting with non-reputable GPs in Section 2.4. Every project needs an investment outlay of I. The GP has no initial money and should raise it from the LP if she decides to invest in the project(s).[6] There are two types of projects, $\theta \in \{G, B\}$. A good project (type G) has guaranteed return R (hence it is always successful) but a bad project (type B) has return R with

[5]Throughout the paper, I use he/him to refer to the LP and she/her to refer to the GP.

[6]The assumption that GP has no initial capital has no effect on the results. We can assume that GP needs extra capital I as long the payout of the contract to the GP is at least as her initial capital.

probability p and return 0 with probability $1-p$. The GP can also opt to not invest in a project and invest the raised capital in a safe asset, therefore receiving the return I. Hence, the possible outcomes are $\{0, I, R, 2I, R+I, 2R\}$ if the GP raises enough capital for two projects ($2I$). Clearly, if the GP raises only I, then possible returns are $\{0, I, R\}$. Type B projects are negative NPV, so I assume

Assumption 1

$$pR < I.$$

The GP can exert effort to increase the chance of getting a good project. Moreover, if the GP exerts no effort for a project, then the project which is chosen is guaranteed to be bad (type B). Otherwise, if the GP exerts a binary effort with cost c, the chance of getting a good project (type G) is λ. I assume that the decision to exert effort is optimal in the following sense

Assumption 2

$$\lambda R + (1-\lambda)I > I + c$$

which can be written as

$$R - \frac{c}{\lambda} > I. \tag{2.1}$$

However, it is possible that the agent exerts effort but does not commit to not invest if the quality is bad. In this case, the return is

$$\lambda R + (1-\lambda)pR$$

which is less than $\lambda R + (1-\lambda)I$ by assumption 1. It can be seen that if there is no agency friction, then when equation (2.1) holds, the agent/investor exerts effort to obtain a good project and invest in the project if he ends up with a type G, otherwise keeping the money in a safe asset. In this case, the profit made from the project is $\lambda(R-I)-c$. In my definition $\frac{1}{\lambda}$ measures the extent of the moral hazard issue. Higher λ means higher chance of obtaining a good project, so the moral hazard problem is less severe. On the other hand, p measures the extent of the information asymmetry between agents, since for higher p it is harder to give incentives for the GP to not invest in a bad project.

The model has three dates $t = 0, 1, 2$ and two periods. At $t = 0$, the contract between the GP and the LP is written and capital is raised. Then between dates 0 and 1, the GP can exert effort to increase the chance of getting a good project. At $t = 1$, the type of projects is revealed to the GP and she makes the investment either in these projects or safe assets. Finally, at $t = 2$, cash flows are realized and agents receive their money based on the contract. In the real world, it is possible that projects exit at different times but if the contract is based on whole-portfolio performance, then money is stored in an escrow account until distributed later when all the projects exit. Hence my assumption on having the same exit time is not unrealistic.

Deal-by-deal contract

In this section, I consider the contracting problem when the contract between the GP and the LP is written in a deal-by-deal way. Since agents are risk-neutral, the optimal deal-by-deal contract consists of two optimal contracts on a single project. Therefore, I only need to study the contracting problem for one project.

In order to fund projects, claims $s_{GP}(x) = s(x)$ and $s_{LP}(x) = x - s_{GP}(x)$ are issued, which determines how much agents will receive when the payout of the project is x. I impose following apriori assumptions on the payout of securities.

- **Limited Liability:** $0 \leq s_{GP}(x), s_{LP}(x)$.
- **Monotonicity:** $s(x)$ and $x - s(x)$ are non-decreasing in x when x is an outcome on the equilibrium path.

This monotonicity assumption is common in the literature on security design–See [38], for example. Sometimes the security is assumed to be monotonic on the whole possible set of payouts. . I will revisit this issue in Section B.2.

The LP can not observe the quality of the chosen project (projects) or if the GP exerts effort or not. However, the LP can observe whether the GP invests in the project or in the safe asset. Also, the cash flow is verifiable at the end of period 2 as well.

For one project, after the issuance of $s(x)$, there are four possible strategies by the GP.

1. Do not invest: The return to GP is $s(I)$.
2. Invest with no effort: The return is $ps(R)$.
3. Exert effort and invest regardless of quality: $\lambda s(R) + (1 - \lambda)ps(R) - c$.
4. Exert effort and invest only in the good project: $\lambda s(R) + (1 - \lambda)s(I) - c$.

Clearly, the optimal strategy is the fourth one if there was no agency friction (I will show later that strategy (4) is also optimal even in the presence of agency friction.) . Assuming this, here is how the LP can implement strategy (4). The scheduled payment's system $(s(I), s(R))$ induce the GP to choose strategy (4) if and only if it satisfies

$$s(R) \geq s(I) + \frac{c}{\lambda} \tag{2.2}$$

$$s(I) \geq ps(R). \tag{2.3}$$

The first condition insures that the agent exerts effort to obtain a good project and the second one insures that the agent does not invest if the quality of the project turns out to be bad. As usual, we have the participation constraint by the LP which is

$$E[s_{LP}(x)] = E[x - s(x)] \geq I. \tag{2.4}$$

The problem faced by the LP can then be written as

$$\max_{s(I),s(R)} E[x - s(x)]$$

where $(s(I), s(R))$ satisfy equations (2.2) and (2.3). Inserting equation (2.3) in equation (2.2) gives

$$s(R) \geq ps(R) + \frac{c}{\lambda}$$

hence $s(R) \geq \frac{c}{\lambda(1-p)}$. As a result, the optimal contract will be $(\frac{c}{\lambda(1-p)}, \frac{pc}{(1-p)\lambda})$. Using equation (2.4), The project is funded if and only if we have

$$\lambda[R - \frac{c}{\lambda(1-p)}] + (1-\lambda)[I - \frac{pc}{(1-p)\lambda}] \geq I. \tag{2.5}$$

Under this contract, the profit made by the GP is

$$\begin{aligned} \Pi_{GP} = \Pi &= \lambda \frac{c}{\lambda(1-p)} + (1-\lambda)\frac{pc}{\lambda(1-p)} - c \\ &= \frac{pc}{\lambda(1-p)} \end{aligned} \tag{2.6}$$

Since the optimal effort/investment strategy is chosen by the GP, we have $\Pi_{LP} + \Pi_{GP} = \lambda(R - I) - c$. So the profit made by the LP from the contract on one project is

$$\Pi_{LP} = \lambda(R - I) - c - \frac{pc}{\lambda(1-p)}. \tag{2.7}$$

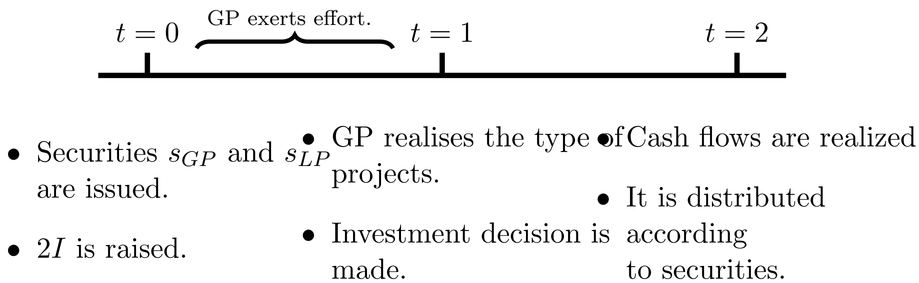

Figure 2.1: Timeline

We can see that for higher p, the LP makes less profit (and therefore the GP makes more). This is because, as mentioned before, higher p is associated with more severe adverse selection and it makes it harder to motivate the GP to invest optimally since the outside option (the bad project) is more appealing. On the contrary, when λ goes up, the profit goes up for the LP (and down for the GP) because the chance of success when exerting effort is higher, so less payment is needed to motivate effort. It is also worth noting that when there is no bad option for investment by the GP, which means $p = 0$, then the LP can get the whole surplus of the project. This case is effectively means that there is no asymmetric information between the LP and the GP. As a result, in a setting with binary effort, contracting alleviates all the friction in the model. In section 2.5, I consider implications when there is more variance for effort in this special important case.

In order to compare the outcome of the results of different strategies induced by the LP, first note that the LP never induces strategy (1) as he personally has access to the safe asset. The second strategy has always negative NPV since the profit by the LP is

$$\begin{aligned} &p(R - s(R)) - I \\ &\leq pR - I < 0 \end{aligned}$$

by assumption 1. Finally to optimally induce strategy (3), note that in this case the LP optimally sets $s(I) = 0$ as I is not the outcome of the induced strategy. To induce effort, the payout should satisfy

$$s(R) \geq ps(R) + \frac{c}{\lambda}$$

which implies $s(R) \geq \frac{c}{\lambda(1-p)}$. Hence the LP issues security $(0, \frac{c}{\lambda(1-p)})$. The profit made by the LP by this contract is

$$(\lambda + (1-\lambda)p)(R - \frac{c}{\lambda(1-p)}) - I.$$

This is less than the profit made by the LP by strategy (4) (equation (2.7)) by assumption 1.

In summary, we have the following for investment on one project.

Proposition 8 (optimal deal-by-deal contract) *The optimal strategy that the LP induces the GP to choose is strategy (4). Moreover, the security* $(s(I), s(R)) = (\frac{c}{\lambda(1-p)}, \frac{pc}{(1-p)\lambda})$ *is issued optimally by the LP to fund the project. The funding is possible if and only if*

$$\Pi_{LP} = \lambda(R - I) - c - \frac{pc}{\lambda(1-p)} \geq 0. \tag{2.8}$$

When there are two projects with parameters (λ_1, p_1) and (λ_2, p_2),in a deal-by-deal contract, the optimal contract for one project is written for each of the projects. Therefore the expected profit by the GP will be

$$\Pi_{GP} = \sum_{i=1}^{2} \frac{p_i c}{\lambda_i(1-p_i)}.$$

In the next section, we see how tying the payouts of the projects together can change the expected payout to the LP (and the GP).

2.3 Optimal Portfolio Contracting

In this section, I analyze the question of optimal contracting when a portfolio of projects is chosen by the GP and the payout can depend on the whole return of the portfolio. Then I compare whole-portfolio contracting with deal-by-deal contracting to see how the investment environment can affect the choice of contract by investors. But first I need to introduce the dependency between projects, which I do in the next part.

Correlation structure of the portfolio

When the GP forms a portfolio of investments, not only the return and quality of each project is important, but the correlation structure between projects is important as well. In [33], the correlation structure in the invested portfolio under different types of contracting is studied, analyzing the correlation between projects in two dimensions of industry and geography.

Here I assume that the correlation between projects is given by a parameter $0 \leq \rho \leq 1$ which means that if the GP exerts effort on both projects, then

$$\mathbb{P}[\text{project 1 and 2 are good} \parallel \text{effort exerted on both projects}]{=}\rho \min(\lambda_1, \lambda_2). \quad (2.9)$$

Here I assume that investors can observe ρ and potentially write a contract conditioned on it. I relax this condition in section 2.5.The cost of exerting effort on both projects is twice that of one project, $2c$. The correlation structure is irrelevant when the contract is written on a deal-by-deal basis. This is because, by risk neutrality, deal-by-deal contracting is equivalent to writing a contract with two different agents. Therefore I only need to study the problem when the payout depends on the payout of both projects.

Before going into detail on the whole-portfolio contracting, let me introduce some preliminary results which are needed later in the discussion. As in the deal-by-deal case, the GP should not invest in the type B project. Hence possible optimal outcomes from the projects are $2I, R+I$ and $2R$. These correspond to cases in which the GP comes up with zero, one or two good projects respectively. As a result, I and R are not possible outcomes if the GP makes optimal investment decisions. Hence to minimize the incentive for these outcomes, it is easy to show that the optimal contract satisfies

$$s(0) = s(I) = s(R) = 0. \quad (2.10)$$

Whole-Portfolio Contract

In this section, I want to see how the optimal contract should be written when the return is a function of the total payouts of the projects. This resembles whole-portfolio contracting. I then compare it to deal-by-deal contracting to find under which parameters each type of contracting is efficient .

When writing the contract on the whole portfolio, as we saw in equation (2.10), we have $s(I) = s(R) = s(0)$. Set

$$(x, y, z) = (s(2R), s(R + I), s(2I)) \quad (2.11)$$

The LP needs to impose some restrictions on the payment to the GP to make sure that the GP only invests in good projects. These conditions are

$$\begin{aligned}
z &\geq \max\{p_1 y, p_2 y, p_1 p_2 x\} \\
y &\geq \max\{z, p_1 x, p_2 x\} \\
x &\geq \max\{z, y\}.
\end{aligned}$$

These inequalities make sure that the GP will invest in good projects and only in good projects (hence withholding money from bad projects). For example, when the agent comes up with two bad projects, the payout for not investing in any bad project (z) is not less than the (expected) payout if the agent invests in one bad project ($p_i y$ for $i = 1, 2$) or invests in two bad projects ($p_1 p_2 x$). A similar explanation applies to $y_i \geq p_{3-i} x$ for $i = 1, 2$. These conditions therefore discourage the GP from making bad investment decisions. Also since $x \geq y \geq z$, the GP will invest in good projects when they are available rather than investing in the safe asset. Note that when the contract satisfies these conditions, the payout to the GP is increasing on the equilibrium as the GP does not invest in a bad project, hence satisfying the monotonicity condition. Set $\lambda_{max} = \max[\lambda_1, \lambda_2]$ and $\lambda_{min} = \min[\lambda_1, \lambda_2]$. In addition, the GP should have incentive to exert effort on both projects. This gives

$$\begin{aligned}
&\rho\lambda_{min} x + [\lambda_1 - 2\rho\lambda_{min} + \lambda_2] y + [1 - \lambda_1 - \lambda_2 + \rho\lambda_{min}] z \\
&\geq \max\{z + 2c, \lambda_{max} y + (1 - \lambda_{max}) z + c\}
\end{aligned}$$

The LHS term is the expected payout to the GP if she exerts effort on both projects. On the RHS we have expected payouts if no effort is exerted or if it is exerted on only one project. For simplicity and without loss of generality, assume that $p_1 \geq p_2$. Note that in the optimum

$$\begin{aligned}
&\rho\lambda_{min} x + [\lambda_1 - 2\rho\lambda_{min} + \lambda_2] y + [1 - \lambda_1 - \lambda_2 + \rho\lambda_{min}] z \\
&= \max\{z + 2c, \lambda_{max} y + (1 - \lambda_{max}) z + c\}.
\end{aligned} \tag{2.12}$$

This is a typical phenomenon when dealing with moral hazard issues. This means that the expected payout to the GP is binding by the condition which induces exerting effort on both projects, otherwise the investor can lower payment in some states of the world without changing GP incentives. More formally, if equation (2.12) does not hold, the transformation $z \to z - \epsilon$ for small enough ϵ should violate the optimality conditions. Otherwise the LP can have a feasible contract with less expected payout to the GP. This means that $z = p_1 y$, hence z can not be reduced. Similarly $y \to y - \epsilon$ should violate the conditions as well, hence one gets $y = p_1 x$. But then in

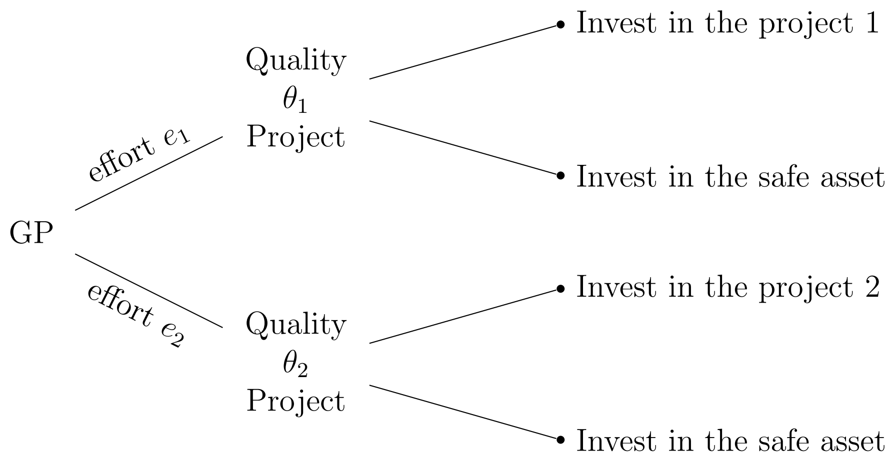

Figure 2.2: GP Problem

any case $x \to x - \epsilon$ is possible because by the last two equalities we have $x > y \geq z$. Therefore the LP problem can be written as

$$\min_{x,y,z} \alpha x + \beta y + \gamma z$$
$$\alpha x + \beta y + \gamma z = \max\{z + 2c, \lambda_{max} y + (1 - \lambda_{max}) z + c\}$$
$$x \geq y \geq z \geq p_1 y \geq p_1^2 x$$

where $(\alpha, \beta, \gamma) = (\rho\lambda_{min}, \lambda_1 - 2\rho\lambda_{min} + \lambda_2, 1 - \lambda_1 - \lambda_2 + \rho\lambda_{min})$. In a similar vein, conditions that induce the choice of optimal investment strategies by the GP are binding as well. Therefore we have the following proposition.

Proposition 9 (Optimal whole-portfolio contract) *The optimal whole-portfolio contract satisfies*

$$z = p_1 y = p_1^2 x \tag{2.13}$$

where

$$z = \frac{2c}{\beta \frac{1-p_1}{p_1} + \alpha \frac{1-p_1^2}{p_1^2}}$$

if

$$\rho \geq \rho^* = \frac{\lambda_{max} - \lambda_{min}}{\lambda_{min}(\frac{1}{p_1} - 1)}$$

otherwise

$$z = \frac{c}{\frac{\alpha(1-p_1^2)}{p_1^2} + \beta \frac{1-p_1}{p_1} - \lambda_{max} \frac{1-p_1}{p_1}}$$

Here is the intuition behind this proposition. When the investor writes the contract, he wants to choose the maximal value for x to give the biggest incentive to the agent to exert effort. However, because of adverse selection, the prize for success cannot be too large, as it leads to inefficient investment decisions by the GP (investing in type B projects). The maximum possible value for x is $\frac{z}{p_1 p_2}$ and maximum attains if and only if equation 2.13 holds. The two different regimes in the proposition correspond to the fact that the expected payout to the GP ($\alpha x + \beta y + \gamma z$), becomes equal to $z + 2c$ or $\lambda_{max} y + (1 - \lambda_{max})z + c$. When the correlation is high, the GP either prefers to exert no effort or to exert effort on both projects because of high dependency between the success in both projects. Hence for high values of ρ,it is needed to pay enough to the GP such that the GP exerts any effort at all. This amount is $s(2I)$ which is the reserve value for the GP. However, when the correlation is low, the payout should compensate for the lower level of inter-dependency between projects. So the payout should be high enough for the case success in both projects so that the GP does not find it beneficial to exert effort on only the easier project (corresponding to λ_{max}).

Note that, as we mentioned, the security as defined here is increasing on the set of possible outcomes on the equilibrium path. However since $s_{GP}(R) = 0$ and $s_{GP}(2I) > 0$, when $R > 2I$, optimal security is not increasing on all possible outcomes. This stems from the fact that the LP wants to push the GP to invest in only good projects and reserve the money if the project is bad. The non-monotonicity of the optimal security has been observed before in the literature, like [35]. The mechanism in [35] which leads to this phenomena is the fact that the contract is written in a way to motivate experimentation by the agent. Hence the principal has to reward for failure so that the agent can take the risk. However, my setting has quite an opposite mechanism–the monotonicity arises because the GP wants to make the LP take less risk. For example, suppose the principal wants the agent to invest in a safe asset, and the agent goes and invests in a bad project instead. So if the payoff is high, it means that the agent deviated from the optimal strategy and as a result she gets punished.

The derivative of $\alpha \frac{1+p_1}{p_1} + \beta$ with respect to ρ equals to

$$\lambda_{min}(\frac{1}{p_1} - 1) > 0$$

hence the payout to the GP decreases as ρ increases. Also note that the total payout of projects (which is $E[s_{GP}] + E[s_{LP}]$) equals to

$$\alpha 2R + \beta(R + I) + \gamma 2I$$

it has derivative (w.r.t ρ)

$$2\lambda_{min}R - 2\lambda_{min}(R+I) + 2\lambda_{min}I = 0$$

so total payout of projects is constant. Therefore we have the following

Proposition 10 (Comparing whole-portfolio with deal-by-deal) *As the correlation ρ increases, the expected payout to the GP decreases and the expected payout to the LP increases. Therefore, for admissible values $(\lambda_1, \lambda_2, p_1, p_s)$, there is $\rho^{**} = \rho(\lambda_1, \lambda_2, p_1, p_s)$,such that for $\rho > \rho^{**}$, whole-portfolio contracting is preferred by the LP and for $\rho < \rho^{**}$ deal-by-deal contracting is preferred by the LP. In addition if $\rho \geq \rho^*$, whole-portfolio contracting is better for the LP (equivalently $\rho^{**} \leq \rho^*$).*

The intuition for the proposition above comes from the fact that when the correlation between projects is higher, it becomes easier to encourage the GP to exert effort on both projects since success in one project increases the chance of success in the other one. As a result, the LP needs to pay less to motivate effort by the GP, hence the LP makes more profit because the total payout of projects is the same for all ρ. Since the deal-by-deal contract is independent from the correlation, from the monotonicity of payout with respect to correlation, we can see that if the LP prefers whole-portfolio contracting to deal-by-deal contracting for a given ρ,then as ρ goes up it is still the case. As I pointed out in the introduction, this result observed empirically in [33]. There the author shows that deal-by-deal compensation induces greater heterogeneity in portfolio investments. So the proposition above rationalizes this finding. In section B.2, I show that this result holds when a strong form of monotonicity is imposed on the security as well.

Whole-portfolio contracting does not depend on p_2. This is because since $p_1 \geq p_2$, the first project has more severe adverse selection problem compared to the second one. Therefore when information asymmetry constraint binds for the first project, it is already alleviate the adverse selection for the second project as well. Mathematically speaking when $z \geq p_1 y$ then already we have $z > p_2 y$ as well. When ρ is large enough, projects are similar to each other and as we saw in Proposition 10, whole-portfolio contracting is more appealing for the LP. This is because, in this case, bundling efforts together gives the LP a big enough benefit that makes up for the loss which comes from having inefficient treatment of adverse selection (in contrast to the deal-by-deal contract which handles this issue efficiently). However for smaller values of ρ, the comparison of benefiting from bundling effort is smaller than the loss of sub-optimal handling of the information asymmetry problem. Recall that the expected payout to the GP in the deal-by-deal case is

$$2c + \Pi_{GP} = 2c + \sum_{i=1}^{2} \frac{p_i c}{\lambda_i (1 - p_i)}$$

which is increasing in p_2. Therefore the discussion above implies the following.

Proposition 11 (Asymmetry of information VS moral hazard) *For given values $(\lambda_1, \lambda_2, p_1, \rho)$, there is $p_2^* = p_2(\lambda_1, \lambda_2, p_1, \rho)$ such that for $p_2 < p_2^*$ deal-by-deal contracting is better for the LP and for $(p_1 \geq) p_2 > p_2^*$ whole-portfolio contracting. When $\rho > \rho^{**}, p_2^* = 0$.*

While Proposition 10 resolves the comparison between whole-portfolio contracting and deal-by-deal in terms of correlation, Proposition 11 helps us to understand the comparison in terms of information asymmetry. Here is the intuition behind this statement. As mentioned before, the investor should take into account the loss of efficient handling of the adverse selection problem. The term $p_1 - p_2$ measures the difference between the adverse selection issues that two projects are subject to. When $p_1 - p_2$ is large (equivalently p_2 is small), the heterogeneity of asymmetry of information between the two projects is large. As a result, it is more efficient to have a deal-by-deal contract for better handling of this issue. Whereas for large p_2 (small $p_1 - p_2$), the loss of efficiency on this issue is negligible, hence whole-portfolio contracting is better.

Whenever $p_2^* = 0$, whole-portfolio contracting is dominant for the set of parameters given. When $\lambda_{max} = \lambda_1$,then $p_2^* > 0$ whenever $\rho < \rho^*$. However when the moral hazard problem is more severe in the first project as well (i.e $\lambda_{max} = \lambda_2$),then for a larger set of ρ, whole-portfolio contracting is dominant. In this case, the investor uses the payout on the second project, which dominates the first in both moral hazard and information asymmetry aspects, as a prize to motivate GP to exert effort on the first project. From the proof of the Propositions 10, the equation that computes p_2^* is given by (when $\rho < \rho^*$)

$$\lambda_{max} \frac{z}{p_1} + (1 - \lambda_{max}) z + c = 2c + \sum_{i=1}^{2} \frac{p_i c}{\lambda_i (1 - p_i)}$$

$$z = \frac{c}{\frac{\alpha(1-p_1^2)}{p_1^2} + \beta \frac{1-p_1}{p_1} - \lambda_{max} \frac{1-p_1}{p_1}}$$

Relative to other variables, we have the following Proposition.

Proposition 12 (Comparative Statics) *In the region $\rho > \rho^*$,expected payout to the GP is increasing in p_1 and is decreasing in λ_{max} and λ_{min}. For $\rho < \rho^*$,expected payout to the GP is increasing with respect to p_1 and λ_{max} and decreasing with respect to λ_{min}.*

Here is the intuition behind Proposition 12. In both regimes of ρ, when p_1 increases, the information asymmetry to be overcome by the LP worsens as the GP finds it more profitable to invest in bad projects. As a result, the expected payout to the GP increases when p_1 increases to compensate for information rent by the GP. With respect to λ_{min}, as λ_{min} increases, it becomes easier for the LP to motivate the GP to exert effort for the harder project, hence the expected payout is decreasing. However with respect to λ_{max}, the relation to the payout depends on which regime ρ is in. In the high correlation regime ($\rho > \rho^*$),as we saw above, the GP has to compensate as much as needed to make the GP exert any effort. As in a classical moral hazard problem, when the task becomes easier the expected payout to the agent decreases. However, in the regime $\rho < \rho^*$, the LP has to compensate the GP for the strategy of exerting effort only on the easier project. This outside option's payout increases as λ_{max} increases, hence the LP has to compensate the GP more for not choosing this strategy.

2.4 Non-Reputable GP

Following [2] and as mentioned in section 2.2, when the GP is not reputable, the LP cannot distinguish the non-reputable GPs from a fly-by-night operator (FNO). In this case the following assumption should be imposed on the securities to discourage FNOs from getting the investment outlay $2I$ and enjoy the managerial fee $s(2I)$ without exerting any effort..

- $s_{GP}(x) = 0$ for $x \leq K$ where K is the committed capital (FNO assumption).

This assumption was first introduced in [2] and has been used in subsequent works (for example [11]). Here I investigate how enforcing this condition can change the contract. Therefore, in essence we have a separating equilibrium in which a contract between the LP and a non-reputable GP satisfies the FNO assumption while a contract between the LP and a well-established (i.e., reputable) GP does not need this condition.

First note that with this assumption, first-best can not be implemented for a single project. This happens since $s_{GP}(I) = 0$, the GP will invest in a project no matter the quality of the project as there is no reward for not investing in a type B

project, hence he implements strategy (3) discussed in section 2.2. The GP exerts effort if

$$s^{FNO}(R) \geq \frac{c}{\lambda(1-p)}. \tag{2.14}$$

It is feasible to fund through this contract if and only if

$$R - \frac{c}{\lambda(1-p)} \geq \frac{I}{\lambda + (1-\lambda)p}. \tag{2.15}$$

Here $\lambda + (1-\lambda)p$ is that chance of having return R. When the project is good (which has probability λ), return R has probability 1 and if the project is bad (with probability $1-\lambda$), the probability of success is p. The profit made by the GP is

$$\begin{aligned}\Pi_{GP}^{FNO} &= (\lambda + (1-\lambda)p)\frac{c}{\lambda(1-p)} - c \\ &= \frac{pc}{\lambda(1-p)}.\end{aligned} \tag{2.16}$$

Comparing with equation (2.6), we can see that both reputable and non-reputable agents make the same profit. This comes from the fact that $s(I) = ps(R)$ for a reputable agent. Therefore a reputable GP gets the same payout as a non-reputable GP in the case of getting of a bad project and investing in the safe asset instead of making a bad investment, which is in the interest of the LP. Not surprisingly, the feasibility condition (2.15) is weaker compared to the case of a reputable agent which is (2.5) since here the investment strategy by the agent is not optimal. In the next part I find the optimal whole-portfolio contract and compare the results with that of section 2.3. If $p = 0$, in a single project's contract, LP gets the whole surplus of the project so for the next part we assume $\max(p_1, p_2) > 0$.

Whole-Portfolio Contracting With FNO

In this part I find the optimal whole-portfolio contracting in the presence of the FNO assumption. By the FNO assumption, the contract satisfies

$$s^{FNO}(I) = s^{FNO}(2I) = 0.$$

Since $s^{FNO}(2I) = 0$, it is impossible to motivate the GP to not invest in any bad project when both projects are bad (as the return to the GP will be zero in this case). So the best strategy that the LP can hope to achieve, similar to [2], is the following

- The GP exerts effort on both projects.
- If at least one of the projects is good, the GP invests in only good projects (optimal choice).
- If both projects are bad, the GP invests in just one bad project.

As we can see, again R is not the outcome of the optimal strategy, so in the optimal contract $s^{FNO}(R) = 0$ holds as well. To induce (constrained) optimal choice of investment after efforts are exerted, assuming $p_1 \geq p_2$, the LP should impose

$$s^{FNO}(2R) \geq s^{FNO}(R+I) \geq p_1 s^{FNO}(2R)$$

Under these conditions, the GP invests only in good projects if any are available and invests in only one bad project if both projects are bad. With this strategy, the total payout to the GP becomes

$$\rho\lambda_{min} s^{FNO}(2R) + (\lambda_1 - 2\rho\lambda_{min} + \lambda_2) s^{FNO}(R+I) + p_1(1 - \lambda_1 - \lambda_2 + \rho\lambda_{min}) s^{FNO}(R+I)$$
$$= \alpha x + \tilde{\beta} y$$

where $(x, y) = (s^{FNO}(2R), s^{FNO}(R+I))$ and $(\alpha, \tilde{\beta}) = (\rho\lambda_{min}, \lambda_1 - 2\rho\lambda_{min} + \lambda_2 + p_1(1 - \lambda_1 - \lambda_2 + \rho\lambda_{min}))$. The only term which is different compared to the reputable agent is the last term. This comes from the fact that in the case of two bad projects, the GP invests in the project corresponded to p_1 and withhold money on the other one. Similar to what we had in Section 2.3, in order to induce effort on both projects, the contract should satisfy

$$\alpha x + \tilde{\beta} y \geq p_1 y + 2c, \theta_i y + c$$

where $\theta_i = \lambda_i + p_1(1 - \lambda_i)$. Therefore, the LP problem can be written as

$$\min_{x,y} \alpha x + \tilde{\beta} y$$
$$x \geq y \geq p_1 x; \quad \alpha x + \tilde{\beta} y \geq \max\{\theta_{max} y + c, p_1 y + 2c\}$$

Like before, in the optimum the moral hazard constraint binds, hence

$$\alpha x + \tilde{\beta} y = \max\{\theta_{max} y + c, p_1 y + 2c\}$$

Similar to the reputable agent, adverse selection binds as well, and we have the following proposition.

Proposition 13 (Optimal whole-portfolio contract for non-reputable GP) *In the optimal whole-portfolio contract we have*

$$y = p_1 x$$

If $\rho \geq \rho^* = \frac{\lambda_{max}-\lambda_{min}}{\lambda_{min}(\frac{1}{p_1}-1)}$ *then*

$$y = \frac{2p_1 c}{\alpha - p_1(p_1 - \tilde{\beta})}$$

otherwise

$$y = \frac{p_1 c}{\alpha - p_1(\theta_{max} - \tilde{\beta})}$$

Moreover $y = s^{FNO}(R+I)$ *coincides with the payout to the GP in the reputable case* $s(R+I) = \frac{z}{p_1}$ *from Proposition 9.*

With the same reasoning as in Proposition 9, two cases are associated with the fact that the expected payout to the GP $(\alpha x + \tilde{\beta} y)$ becomes $\theta_{max} y + c$ or $p_1 y + 2c$. In the non-reputable case, the total payout of both projects is

$$2\alpha R + \tilde{\beta}(R+I) + (1 - \alpha - \tilde{\beta})I.$$

The derivative with respect to ρ of the total payout is

$$\lambda_{min}(p_1 R - I) < 0.$$

As we can see, on one hand the total payout of projects is decreasing with respect to the correlation between projects. Intuitively this is because the only scenario in which the investment decision is not optimal is when both projects are bad and the chance of this scenario is higher for higher correlation. On the other hand, the total payout to the GP is decreasing with respect to correlation as again it gets easier to motivate the GP to exert effort when the correlation goes up. As a result, the total payout to the investor is ambiguous with respect to correlation and depends on the relative magnitude of $p_1 R - I$ and c. Here $p_1 R - I$ measures the inefficiency associated with investing in the bad project with success chance p_1 (hence expected return $p_1 R$) instead of investing in the safe asset (with return I). When c is large enough, since the expected payout to the GP is proportional to c,the total payout to the GP decreases faster compared to the loss of inefficiency which is proportional to $p_1 R - I$. As a result, by increasing ρ the total payout to the LP increases as well. If c is small enough, the reverse phenomenon happens and hence the total payout to the LP is decreasing with respect to ρ. Finally, in the middle range of c, total payout to the GP has an interior optimal correlation. Hence we do not have a monotonic relationship between the LP's payout and the correlation in the non-reputable case. However, we can show the following.

Proposition 14 (Reputable VS Non-reputable agent) *Suppose for parameters* $(\lambda_1, \lambda_2, p_1, p_2, \rho)$,*the investor prefers whole-portfolio contracting when writing contracts with a generic (reputable) GP. This will also be the case when writing a contract with a non-reputable GP. In particular, if* $\rho \geq \rho^* = \frac{\lambda_{max}-\lambda_{min}}{\lambda_{min}(\frac{1}{p_1}-1)}$, *then whole-portfolio contracting is dominant in the non-reputable case as well.*

This proposition comes from the fact that when dealing with non-reputable agents, whole-portfolio contracting can help to improve the investment strategy. Hence the space of parameters in which whole-portfolio contracting is better for the LP is larger compared to the reputable case. As mentioned in the introduction, this result has been observed empirically in [25].

2.5 Extension

In this section, I consider various modifications of the model and how it can affect the results. I make some predictions/observations as well.

No Asymmetric Information

In the special case when $p_1 = p_2 = 0$, as we mentioned in section 2.2, there is no profitable bad option for the GP to invest in. Equivalently, there is no information asymmetry about the quality of projects between the GP and the LP as there is no possible profitable deviation. Under binary effort assumption, by equation (2.6), the whole surplus of every project goes to the LP and hence the method of contracting is irrelevant in this setting. In this case when adverse selection is absent, since effort is binary, contract makes the GP indifferent between exerting effort and not exerting effort and hence the LP can get the first-best outcome. In order to analyze this important special case in depth, here I allow for different levels of effort to see how it affects the contract. Therefore for the purpose of this section, suppose with the variable cost $c_i(\lambda_i)$, the chance of getting a good $i-$project is λ_i. I assume $c_i(0) = c_i'(0) = c_i''(0) = 0$ and $c_i^{(3)} > 0$. I first consider the general case and then restrict to the especial case $p_1 = p_2 = 0$ which we are interested in. As before, the chance of success for a $i-$project of type B is p_i . The first-best effort satisfies

$$\max_{\lambda_i} \lambda_i R + (1-\lambda_i)I - c_i(\lambda_i) - I$$

By FOC the optimal effort satisfies $c_i'(\lambda_i^{FB}) = R - I$. Not surprisingly it is independent of p_i as there is no adverse selection problem. I Assume $c_i'(1) > R - I$ to make sure that $\lambda_i^{FB} < 1$. Now suppose the contract $(s_{GP}(I), s_{GP}(R))$ is offered to the GP. Similar to the binary case, the contract should satisfy

$$s_{GP}(I) \geq p_i s_{GP}(R) \tag{2.17}$$

to make sure that GP does not invest in the bad project. Once offered, the agent chooses effort λ_i which is the solution to the problem

$$\max_{\lambda_i} \lambda_i s_{GP}(R) + (1-\lambda_i) s_{GP}(I) - c_i(\lambda_i)$$

FOC implies

$$s_{GP}(R) - s_{GP}(I) = c_i'(\lambda_i) \tag{2.18}$$

in particular by decreasing $s_{GP}(I)$, the effort increases which is in the favor of LP (both lower payment and higher effort). Hence in the optimal $s_{GP}(I) = p_i s_{GP}(R)$ as contract should satisfies equation (2.17). This gives

$$c_i'(\lambda_i^*) = (1-p_i) s_{GP}(R) \tag{2.19}$$

in the optimum. Once we have this, λ_i is determined by solving

$$\max_{\lambda_i} \lambda_i (R - I) - \frac{\lambda_i c_i'(\lambda_i) + (1-\lambda_i) p_i c_i'(\lambda_i)}{1-p_i}$$

which comes from the LP problem

$$\max_{\lambda_i} \lambda_i [R - s_{GP}(R)] + (1-\lambda_i)[I - p_i s_{GP}(R)]$$

and equation (2.19). Specializing to the case $p_i = 0$,one gets $c_i'(\lambda_i^*) = s_{GP}(R)$. The equation to determine λ_i becomes

$$\max_{\lambda_i} \lambda_i (R - I) - \lambda_i c_i'(\lambda_i)$$

and FOC gives

$$R - I = c_i'(\lambda_i^*) + \lambda_i c"_i(\lambda_i^*).$$

When comparing the second-best effort with first best (equation (2.18)), the extra term $\lambda_i c_i''(\lambda_i)$ measures the moral hazard issue and it reduces the effort by the agent. So in this more flexible setting, first-best is not contactable even without adverse selection. The expected payout from one project to the LP is

$$\lambda_i (R - c_i'(\lambda_i)) + (1-\lambda_i) I - I,$$

which can be written as

$$\lambda_i(R - I - c_i'(\lambda_i))$$
$$= \lambda_i^2 c_i''(\lambda_i).$$

The expected payout to the GP is $\lambda_i c_i'(\lambda_i)$ and $\lambda_i c_i'(\lambda_i) - c_i(\lambda_i) > 0$ is the expected profit for the the GP.

For the whole-portfolio contracting, I consider a simple whole-portfolio contract which pays agent only when the outcome is $2R$. In this case, the GP problem is

$$\max_{\lambda_1,\lambda_2} \lambda_1\lambda_2 s_{GP}(2R) - c_1(\lambda_1) - c_2(\lambda_2)$$

which gives

$$c_2'(\lambda_2) = \lambda_1 s_{GP}(2R)$$
$$c_1'(\lambda_1) = \lambda_2 s_{GP}(2R)$$

Once this, the LP problem is to determine $s_{GP}(2R)$ to maximize the expected revenue which is

$$\max_{s_{GP}(2R)} \lambda_1\lambda_2(2R - s_{GP}(2R)) + [\lambda_1(1-\lambda_2) + \lambda_2(1-\lambda_1)](R+I) + (1-\lambda_1)(1-\lambda_2)2I$$

where λ_1, λ_2 satisfy GP's optimality equations. We have the following proposition which gives answer for a wide class of cost functions.

Proposition 15 (No asymmetric information case) *If $c_1 = a\lambda^m$ and $c_2 = b\lambda^m$ for $m > 2$ and $a, b > 0$, then whole-portfolio contracting is better for the LP compared to deal-by-deal contract.*

The intuition behind the proposition is simple. When there is no asymmetry of information, it is better that contract motivates effort as easily as possible. Tying outcomes together can provide a bigger incentive relative to contracting on projects in deal-by-deal basis. This proposition shed light on the fact that in settings where the investor and agent have same information about the quality of projects, the whole-portfolio contracting is dominant. This includes hedge-funds, mutual-bonds or other contracts on public equities.

Uninformed Investor

In this part, I consider the case in which investor is not informed about the correlation. Other than this, I assume the same setup as in the main model. Since in the deal-by-deal contract correlation has no effect on the outcome, I only focus on the whole-portfolio contracting. Assume that the GP can privately and strategically choose the correlation ρ in the interval $[\rho_1, \rho_2]$. While the interval is common knowledge, the LP does not observe ρ directly. The case of informed investor is a special case when $\rho_1 = \rho_2 = \rho$. Since investment compatibility conditions are independent from ρ, as in the informed case, optimal contract satisfies

$$
\begin{aligned}
&z \geq p_1 y, p_2 y, p_1 p_2 x \\
&y \geq z, p_1 x, p_2 x \\
&x \geq z, y
\end{aligned}
$$

where variables are as in equation (2.11). As before, assume $p_1 \geq p_2$. The expected payout to the GP from choosing ρ is

$$\rho\lambda_{min} x + [\lambda_1 - 2\rho\lambda_{min} + \lambda_2]y + [1 - \lambda_1 - \lambda_2 + \rho\lambda_{min}]z.$$

Derivative with respect to ρ of the expression above is

$$\lambda_{min}(x - 2y + z).$$

There are two possible scenarios for the GP to choose the correlation. If $x > 2y - z$, the payout for two successful exits are relatively high hence GP wants to maximize the chance of having two successful exits. Therefore GP chooses the highest possible correlation i.e $\rho = \rho_2$. In contrary, if y is relatively high ($2y > x - z$), then it is more profitable for GP to have only one successful exit. This event has the highest chance when ρ is smallest which is $\rho = \rho_1$. Now suppose (x, y, z) has the form of $(\frac{z}{p_1^2}, \frac{z}{p_1}, z)$ which is the same as in the optimal contract with informed investor from Proposition 9. In this case, the derivative with respect to ρ of the payout to GP becomes

$$\lambda_{min} z(\frac{1}{p_1^2} - \frac{2}{p_1} + 1) = \lambda_{min} z(\frac{1}{p_1} - 1)^2 > 0.$$

Therefore as argued above, GP chooses the highest value of ρ which is ρ_2. As we saw in the informed problem, when ρ goes up, $E[s_{LP}]$ goes up as well in the optimal informed contract. Therefore LP optimally offers the contract $(\frac{z^*}{p_1^2}, \frac{z^*}{p_1}, z^*)$ where $z^* = z^*(\rho_2)$ is the management fee in the optimal contract for the correlation ρ_2 from Proposition 9. GP optimally chooses ρ_2 as well from discussion above. Therefore if for $\rho = \rho_2$, LP prefers whole-portfolio contracting to deal-by-deal, then the contract above is offered. Otherwise deal-by-deal contract is offered. In summary we have

Proposition 16 (Uninformed Investor) *Suppose GP can privately chooses ρ in the interval $[\rho_1, \rho_2]$. Then*

- *If $\rho_2 < \rho^{**}$, a deal-by-deal contract is offered to GP.*
- *If $\rho_2 \geq \rho^{**}$, a whole-portfolio contract associated to $\rho = \rho_2$ is offered to GP and GP chooses $\rho = \rho_2$ optimally.*

Conditional Contract

In this part, I consider conditional contracting which means that the payout of the contract is a function of the outcome of each project. This definition contains both deal-by-deal contracting as well as whole-portfolio contracting as special cases. When the payout of projects are $2I$ or $2R$, it corresponds uniquely to two bad or good projects respectively. However, there are two possible ways to get the outcome $R + I$. When the first project is type G or when the second one is and the other one is type B. Unlike the whole-portfolio and similar to deal-by-deal, when the contract is fully conditional, total payout to GP can be different in these two cases. Therefore take y_1 and y_2 as possible payouts to GP where y_1 is $s(R + I)$ when the first project is successful and y_2 is that of when the second one is type G. Set $x = s(2R)$ and $z = s(2I)$ as in the whole-portfolio contracting. Also recall that from equation (2.10), we have $s(I) = s(R) = 0$. Similar to whole-portfolio contracting, in the optimal contract, payouts satisfy

$$
\begin{aligned}
&z \geq p_1 p_2 x, p_1 y_1, p_2 y_2 \\
&y_1 \geq z, p_2 x \\
&y_2 \geq z, p_1 x \\
&x \geq z, y_1, y_2
\end{aligned}
$$

When comparing these to analogues inequalities in the whole-portfolio contracting, we see that there is efficiency gain. recall that in the whole-portfolio contracting, z should be bigger than both $p_1 y$ and $p_2 y$ since there is no difference between payouts to GP when the return is $R + I$ and the first project is successful or the return is $R+I$ and the second project is successful. As a result LP should overcompensate GP to cover both cases and this causes some inefficiency when compared to conditional contracting. The same phenomena happens when comparing y and x. y should be bigger than both $p_1 x$ and $p_2 x$ while in the conditional contract there are two different values y_1 and y_2 instead of single payout y. Assume y_{min} and y_{max} are the corresponding payouts for when the project with λ_{min} or λ_{max} succeed respectively.

To motivate GP to exert effort on both projects (moral hazard problem), the contract should satisfy

$$\rho\lambda_{min}x + (\lambda_{max} - \rho\lambda_{min})y_{max} + (1-\rho)\lambda_{min}y_{min} + (1 - \lambda_1 - \lambda_2 + \rho\lambda_{min})z$$
$$\geq z + 2c, \lambda_1 y_1 + (1-\lambda_1)z + c, \lambda_2 y_2 + (1-\lambda_2)z + c$$

So LP problem is

$$\min_{x,y,z} \alpha x + \beta_1 y_1 + \beta_2 y_2 + \gamma z$$
$$\alpha x + \beta_1 y_1 + \beta_2 y_2 + \gamma z \geq z + 2c, \lambda_i y_i + (1-\lambda_i)z + c$$
$$x \geq y_i \geq z \geq p_i y_i \geq p_1 p_2 x$$

where $(\alpha, \beta_1, \beta_2, \gamma) = (\rho\lambda_{min}, \lambda_{max} - \rho\lambda_{min}, (1-\rho)\lambda_{min}, 1 - \lambda_1 - \lambda_2 + \rho\lambda_{min})$. With similar reasoning as in the whole-portfolio contracting, in the optimum

$$\alpha x + \beta_1 y_1 + \beta_2 y_2 + \gamma z = \max\{z + 2c, \lambda_i y_i + (1-\lambda_i)z + c\} \tag{2.20}$$

Similar to Proposition 9 we have the following.

Proposition 17 (Optimal Conditional Contract) *In the optimal contract,*

$$z = p_1 y_1 = p_2 y_2 = p_1 p_2 x \tag{2.21}$$

z is determined such that equation (2.20) is satisfied.

As in the whole-portfolio contracting, the inequalities which are dealing with adverse selection issue are binding as well. It is worth mentioning that when $p_1 = p_2$, we get $y_1 = y_2$ even in the case that $\lambda_1 \neq \lambda_2$,hence the contract reduces to a whole-portfolio contract. This is because the variation in λs affect the moral hazard problem and has no bearing on the asymmetric information issue which rises after exerting effort on projects. In fact when we compare whole-portfolio contracting with the conditional contracting for the loss of efficiency for LP, we have (recall that$p_2 < p_1$)

Proposition 18 (Efficiency loss for whole-portfolio contracting) *The profit made by GP in the conditional contracting decreases as $p_2 (\leq p_1)$ increases. When $p_2 = p_1$,the profit equals the profit made in the whole-portfolio contracting.*

Here is the intuition behind statement above. . As we mentioned in the discussion after proposition 11, $p_1 - p_2$ measures the difference between adverse selection that problems are subject to. When this measure is zero, handling of the problem for one project, efficiently takes care of the other project as well hence whole-portfolio contracting becomes the best conditional contract. As this measure grows, whole-portfolio contracting becomes less and less efficient and GP can extract more rent on the projects.

Bargaining Power

In this part, I consider what happens when GP has the bargaining power for writing the contract. For the moment I assume that the offered contract has to be incentive compatible to induce the GP to choose the optimal strategy (justify it at the end). For reputable GP, the problem becomes

$$\max_{x,y,z} \alpha x + \beta y + \gamma z$$
$$x \geq y \geq z \geq p_1 y \geq p_1^2 x; \quad \alpha x + \beta y + \gamma z \geq z + 2c, \lambda_{max} y + (1 - \lambda_{max}) z + c$$
$$E[s_{LP}] \geq 2I$$

The last inequality is the participation constraint for the LP. Here notation are the same as in subsection 2.3. Not surprisingly, in the optimum, the equality $E[s_{LP}] = 2I$ happens otherwise the contract can be altered in the GP's favor without violating participation constraint by LP (for example by increasing x). Hence unlike investor problem we do not have a unique contract and the contract only needs to satisfy the incentive compatibility equations. Also, as we saw in subsection 2.3 , $E[s_{LP}]+E[s_{GP}]$ is independent of ρ. Therefore, investor breaks even in the optimum and the contract is not unique. To justify the imposing of the incentive compatibility, note that in any feasible contract $E[s_{LP}] \geq 2I$ is required. Also the optimal investment strategy guarantees the maximum possible payout of the projects. Hence imposing them does not reduce the profit by GP.
This result is not surprising as when GP has the market power, since she is the party who takes the action and also observes the quality of the project, she can extract all the rent from projects. So in the presence of the market power by GP, the method of contracting or correlation does not play a role.

When GP is not reputable hence FNO is imposed, take $E[s_{GP}] = \alpha x + \tilde{\beta} y$ as in section 2.4 and then the GP problem becomes

$$\max_{x,y} \alpha x + \tilde{\beta} y$$
$$x \geq y \geq p_1 x; \quad \alpha x + \tilde{\beta} y \geq \max\{\theta_{max} y + c, p_1 y + 2c\}$$
$$E[s_{LP}] \geq 2I$$

Similar to the case of reputable GP, investor breaks even and he is not concerned about the correlation. In addition, if GP can choose the correlation as well, she maximizes the surplus of the project. So as we saw in the section 2.4, total payout is decreasing in ρ so she chooses $\rho = 0$ in this case. Since even for $\rho = 1$, total payout of projects is more than deal-by-deal contract, when GP has market power she chooses whole-portfolio contracting. This stems from the fact that the whole-portfolio contracting persuade GP to have a better investment strategy. Hence in summary we get

Proposition 19 *When GP has the bargaining power, for reputable GP there is no difference between deal-by-deal and whole-portfolio contracting. For non-reputable GP whole-portfolio contracting is preferred for all values of ρ. In addition the contract is not unique and only needs to satisfy the IC by GP as well as PC by LP.*

2.6 Conclusion

In this paper, I proposed a framework to study the scheme of payment in LPA. Unlike usual contracts which only determine the amount of payment for a given return, since GP and LP write a contract on a portfolio, the method of payment is also of vital importance. I compared the main two methods of payments which are the deal-by-deal and the whole-portfolio. Within my setting, I showed that the whole-portfolio contracting is more prevalent when the correlation of investment companies is high or when the reputation of GP is low. Previously documented findings support these result.

In addition, I make some predictions which can guide future studies. For example the informativeness of the investor can also affect the method of contracting and hence the portfolio as well. More informed investors tend to have more deal-by-deal contract and a diverse portfolio while less informed ones have a narrower range of investment and more whole-portfolio contracting. Also when underlying assets are public, whole-portfolio contracting is the typical method of payment which is used.

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Appendix A

Appendix to Chapter 1

A.1 Proofs

To characterize the effort boundary and value function, we define, for any $\eta \geq 0$,

$$\beta^{\pm}(\eta) = \frac{1}{2} - \frac{\alpha}{\sigma^2} \pm \sqrt{\left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2 + \frac{2(r+\eta)}{\sigma^2}}, \tag{A.1}$$

which are the roots of the polynomial

$$p(b) = b^2\sigma^2 + \left(2\alpha - \sigma^2\right)b - 2(r+\eta). \tag{A.2}$$

For the special case when $\eta = 0$, we use the abbreviation $\beta^+(0) = \beta^+$. The following lemma provides bounds and comparative statics for $\beta^+(\eta)$. (The root $\beta^-(\eta)$ is obviously negative.)

Lemma 5 *$\beta^+(\eta)$ is increasing in η and decreasing in α and σ. $\beta^+(\eta) \to (r+\eta)/\alpha$ as $\sigma^2 \to 0$, and $\beta^+(\eta) \to 1$ as $\sigma^2 \to \infty$. Finally,*

$$\beta^+(\eta) \in (1, (r+\eta)/\alpha), \quad \eta \geq 0.$$

Proof. $\beta^+(\delta) > 1$ is easy to confirm. From the form of the polynomial $(A.2)$, we get that $\beta^+(\delta) < (r+\delta)/\alpha$ if and only if the polynomial evaluated at $b = (r+\delta)/\alpha$ is strictly positive, which is true if and only if $r + \delta > \alpha$. The comparative statics with respect to δ and α follow because the polynomial $(A.2)$ is decreasing in δ and increasing in α, for every $b > 0$.

We now show that $\beta^+(\delta)$ is decreasing in σ. Differentiate the implicit equation $p(b) = 0$ to get

$$\frac{db}{d\sigma} = \frac{2\sigma b(1-b)}{2b\sigma^2 + 2\alpha - \sigma^2}.$$

Because $\beta^+(\delta) > 1$ and $\beta^-(\delta) < 0$, the numerator is negative at either root. It is easy to see that the denominator is positive for $\beta^+(\delta)$ and negative for $\beta^-(\delta)$. ■

The jump-model counterparts to the diffusion-model parameters $\beta^\pm(\eta)$ are

$$\beta_\kappa^+(\eta) = \frac{\ln(\psi^+(\eta))}{\ln(u)}, \quad \beta_\kappa^-(\eta) = \frac{\ln(\psi^-(\eta))}{\ln(u)}, \quad \eta \geq 0, \tag{A.3}$$

where, for any $\eta \geq 0$, $\psi^+(\eta)$ and $\psi^-(\eta)$ are the two roots of the polynomial

$$\kappa p \psi^2 - (r + \eta + \kappa)\psi + \kappa(1-p) = 0, \tag{A.4}$$

with $\psi^+(\eta)$ being the larger (the roots are real when $p \in (0,1)$).

Proof of Proposition 1

To simplify notation, we consider only the single-firm ($n = 1$) case (the ideas for the $n > 1$ case are the same). Without loss of generality, we assume that $t = 0$. For any effort process e, the time-0 firm value, conditional on no success to date (i.e., on $\{\tau > t\}$), is the expected discounted payoff:

$$v(P;e) = E_t\left[e^{-r(\tau-t)} f(P_\tau) - \int_t^\tau e^{-r(s-t)} Q e_s ds\right], \quad t \leq \tau, \tag{A.5}$$

where τ is the time to success. We want to show

$$v(P;e) = E\left[\int_0^\infty \{f(P_s) - Q\} e_s \xi_s(e) ds\right],$$

where $\xi_s(e)$ is defined in (1.3). First consider the finite-horizon problem; that is, assume that at some time T, the process is killed. Then Proposition 1 of Duffie, Schroder, and Skiadas (1996) implies that

$$v^T(P;e) = E\left[\int_0^T \{f(P_s) - Q\} e_s \xi_s(e) ds\right] = E[Y_T].$$

Here Y_T represents the integral inside the square brackets. In order to finish the proof, we have to show that the family Y_T of random variables converge. Because e_t is nonnegative and bounded and $|f(P) - Q| \leq P + Q$, this reduces to show that the family

$$X_n = \int_0^n e^{-rs} P_s ds$$

(absolutely) converges. By using Cauchy criteria and the fact that family X_n is increasing, we have to show that

$$\lim_{n\to\infty}\lim_{m\to\infty} P\left(X_{n+m}-X_n\geq\epsilon\right)=0.$$

We have

$$P\left(X_{n+m}-X_n\geq\epsilon\right)\leq\frac{E\left[\int_n^\infty e^{-rs}P_s\right]}{\epsilon}=\frac{\int_n^\infty e^{(\alpha-r)s}ds}{\epsilon}=\frac{\frac{1}{r-\alpha}e^{(\alpha-r)n}}{\epsilon},$$

which goes to zero as n goes to ∞. This completes the proof.

Proof of Proposition 2

Uniqueness in the Two-Firm ($n=2$) Case. Lemma 11 in Appendix B shows that the optimal response to a competitor's effort process $e\left(P\right)$, with e non-decreasing in P, is to choose a fixed boundary P^*, with effort exerted only when the payoff process exceeds P^*. Changing notation for this proof, we let $v\left(P;p\right)$ denote the optimal value function of one of the firms given that the other firm chooses the fixed effort-boundary p. Let $P^*\left(p\right)$ denote the optimal effort-boundary response of one firm to a given effort-boundary p chosen by the other firm.

It follows from the firm-value expression (1.2) that $v\left(P;p\right)$ is strictly increasing in p for each P (because the other firm's effort is decreasing in p) and strictly increasing in P for each p. Lemma 11 implies that $P^*\left(\right)$ is the unique solution to

$$v\left(P^*\left(p\right);p\right)=P^*\left(p\right)-\left(K+Q\right).$$

If $\hat{p}>p$, then $v\left(P^*\left(p\right);\hat{p}\right)>P^*\left(p\right)-\left(K+Q\right)$ together with $P-v\left(P;\hat{p}\right)$ increasing in P (using Lemma 10), imply that $P^*\left(\hat{p}\right)>P^*\left(p\right)$. That is, $P^*\left(\cdot\right)$ is increasing. Continuity of $v\left(P^*\left(p\right);p\right)$ in p implies that $P^*\left(\cdot\right)$ is continuous. Also, $P^*\left(0\right)\in\left(0,P_n^*\right)$, $P_n^*=P^*\left(P_n^*\right)$ and $\lim_{p\to\infty}P^*\left(p\right)=P_1^*$, where P_1^* is the single-firm optimal effort boundary ($p\to\infty$ corresponds to the other firm never exerting effort). Uniqueness of the symmetric Nash equilibrium implies that $P^*\left(\cdot\right)$ only intersects the identity line at P_n^*. Together, these properties imply

$$P^*\left(p\right)>p \text{ if } p<P_n^*, \text{ and } P^*\left(p\right)<p \text{ if } p>P_n^*. \tag{A.6}$$

The response-function properties $(A.6)$ imply a unique equilibrium (whether symmetric or not) because the optimal response to any boundary $\bar{p}<P_n^*$ is a boundary $P^*\left(\bar{p}\right)\in\left(\bar{p},P_n^*\right)$, which induces an optimal response $P^*\left(P^*\left(\bar{p}\right)\right)\in\left(P^*\left(\bar{p}\right),P_n^*\right)$, and so on. That is, we get an increasing monotonic sequence, which must converge to P_n^* (by continuity of $P^*\left(\cdot\right)$ and uniqueness of the symmetric Nash equilibrium). An analogous argument applies to any boundary $\bar{P}>P_n^*$.

Uniqueness in the n-Firm Case. Consider any pair of firms, which we label a and b, and hold fixed the total effort of the other $n-2$ firms. Let $v(P;p)$ denote the optimal value function of either of the pair, given that the other firm chooses the effort-boundary p. The proof in the two-firm case shows that a and b must share a common boundary in the Nash equilibrium effort problem for the pair. It follows that in the n-firm case, there must be a common Nash equilibrium boundary. (Otherwise, if we suppose there is a pair of firms a and b with $P^{*a} \neq P^{*b}$, we get a contradiction.)

Derivation of the Closed-Form Solution. We omit the "cmp" superscript for V and P^*. The HJB equation (1.4) implies that the ODE in the no-effort region is

$$0 = -rV(P) + \alpha P V_P(P) + \frac{\sigma^2}{2} P^2 V_{PP}(P), \quad \text{if } P < P^*, \tag{A.7}$$

which represents the discounted expected value of $V(P^*)$ at the next time the value P_t hits P^*. The well-known solution is

$$V(P) = V(P^*) \left(\frac{P}{P^*}\right)^{\beta^+}, \quad P \leq P^*. \tag{A.8}$$

The optimal-effort equation (1.5) implies that the net potential payoff from exerting effort at the boundary P^* is zero; that is, $P^* - K - Q = V(P^*)$. Together with $(A.8)$, we get the no-effort-region value function (1.10) with $\delta = 0$.

The solution in the effort region, $P \geq P^*$, is obtained from Lemma 8 in Appendix B.

Finally, for the effort and no-effort regions to match the conjectured regions, the solution must satisfy

$$f(P) - V(P) \geq Q \iff P \geq P^*. \tag{A.9}$$

This is also shown in Lemma 8 in Appendix A.2. The verification proof in Appendix A.3 completes the proof of the proposition.

Proof of Lemma 2

From the effort strategy (1.15) and continuity of the value function in price, the cheating boundary satisfies $P^{*\text{cht}}(\tau) = V^{\text{cht}}\left(P^{*\,\text{cht}}(\tau), \tau\right) + Q + K$, for all $\tau \geq 0$. Fixing τ and defining

$$H(p) = p - V^{\text{cht}}(p, t, D) - (Q + K),$$

we show that there is a unique zero of $H(p)$. The time-0 value is $H(0) = -(Q+K)$, and a slight modification of Lemma 10 shows that there exists a $\varepsilon > 0$ such that

$$H'(p) = 1 - \frac{\partial}{\partial p} V^{\text{cht}}(p, \tau) > \varepsilon \ \text{ all } p, \tau \geq 0.$$

A unique $\hat{p}$ satisfying $H(\hat{p}) = 0$ therefore exists, which is the cheating boundary.

Lemma 6 below shows that $V^{\text{cht}}(P, \tau)$ is strictly increasing in τ, and, therefore, (1.15) implies that $P^{*\text{cht}}(\tau)$ is increasing in τ.

The inequality $V^{\text{cht}}(P, \tau) > V^{\text{cmp}}(P)$ for all $P, \tau > 0$ implies, from (1.15) and (1.6), that $\hat{e}_\tau^{\text{cht}} < \hat{e}_\tau^{\text{cmp}}$ for $\tau > 0$ (and $\hat{e}_\tau^{\text{cht}} = \hat{e}_\tau^{\text{cmp}}$ for $\tau = 0$).

The following lemma shows that firm i's value function under the cheating strategy is increasing in delay time (or time to retaliation).

Lemma 6 *Let $P^* \in (P^{*cmp}, P^{*co}]$ and suppose all firms except firm i are following the P^* coordinated strategy. Firm i's value function, $V^{cht}(P, \tau)$, is strictly increasing in τ for any $P > 0$.*

Proof. Without loss of generality, let the current time be zero. For any $\tau > 0$, the total effort process of the other agents under the coordinated strategy is

$$e_t^{-i}(\tau) = (n-1)\,\theta \cdot \begin{cases} 1_{\{P_t \geq P^*\}} & \text{if} \quad t < \tau, \\ 1_{\{P_t \geq P^{*\text{cmp}}\}} & \text{if} \quad t \geq \tau. \end{cases}$$

If $\tau_1 < \tau_2$, then $e_t^{-i}(\tau_1) \geq e_t^{-i}(\tau_2)$ for all t and $\Pr\left(e_t^{-i}(\tau_1) > e_t^{-i}(\tau_2)\right) > 0$ for $t \in [\tau_1, \tau_2)$. The value-function characterization (1.2) implies a strictly higher firm-i value process for τ_2 relative to τ_1. ■

Proof of Proposition 4

We make the following notational change for this proof: We introduce $K + Q$ as an explicit argument in all value functions and let $Q = 0$ to simplify notation. The effort boundary for the cooperative strategy is $P^{*\text{co}}$ throughout the proof.

Lemma 6 shows that $V^{\text{cht}}(P, K, \tau)$ is strictly increasing in τ for any $P > 0$. We also know that $V^{\text{cht}}(P, K, 0) = V^{\text{cmp}}(P, K)$ (immediate retaliation) and that $\lim_{\tau \to \infty} V^{\text{cht}}(P, K, \tau) > V^{\text{co}}(P, K)$ (no threat of retaliation), for any $P > 0$. By continuity and strict monotonicity of $V^{\text{cht}}(P, K, \tau)$ in P, there exists a unique $\tau(P, K)$ such that

$$V^{\text{cht}}(P, K, \tau) \gtreqless V^{\text{co}}(P, K) \;\text{ if } \tau \gtreqless \tau(P, K).$$

Further, degree-1 homogeneity of the value functions in P and K implies that the function τ can instead be expressed in terms of the ratio K/P:

$$V^{\text{cht}}(1, K/P, \tau) \gtreqless V^{\text{co}}(1, K/P) \;\text{ if } \tau \gtreqless \tau(K/P).$$

Finally, define

$$D^* = \inf_{K/P > 0} \tau(K/P).$$

The variable D^* depends on neither the current price P_0 nor the required investment outlay K (nor the effort cost Q). We interpret D^* as the maximum monitoring delay time under which the first-best coordinated equilibrium can be sustained. We justify this interpretation as follows. For any $D < D^*$, the definition of D^* implies $V^{\text{cht}}(P, K, D) < V^{\text{co}}(P, K)$ for all $P, K > 0$, and, therefore, deviating from the cooperative equilibrium with exactly D years remaining before retaliation is suboptimal (regardless of the price). By monotonicity of the cheating value function in τ (Lemma 6), it follows that any future cheating must also be suboptimal.

Conversely, suppose $D > D^*$. Then, for any $K > 0$, there must exist some price $\tilde{P} > 0$ such that $V^{\text{cht}}\left(\tilde{P}, K, D\right) > V^{\text{co}}\left(\tilde{P}, K\right)$. If the current price P_0 equals $\tilde{P}$, then cheating now is optimal. If $P_0 \neq \tilde{P}$, then by continuity, there must exist $\Delta t > 0$ such that $V^{\text{cht}}\left(\tilde{P}, K, D - \Delta t\right) > V^{\text{co}}\left(\tilde{P}, K\right)$, and there is some strictly positive probability that the price could hit the value $\tilde{P}$ in the next Δt units of time. Therefore, if $D > D^*$, then there is a strictly positive probability that cheating will be optimal in the future.

Finally, we now show that $D^* > 0$. Our value-function formulas imply that there is a constant $C_1 > 0$ such that

$$V^{\text{co}}(P, K) > V^{\text{cmp}}(P, K) + C_1, \quad \text{all } K + Q < P < P^{*\text{co}}. \tag{A.10}$$

We next derive the following inequality: Given any $\epsilon > 0$, if $D > 0$ is sufficiently small, then

$$V^{\text{cht}}(P, K, D) < V^{\text{cmp}}(P, K) + \epsilon, \quad \text{all } K + Q < P < P^{*\text{co}}. \tag{A.11}$$

To show $(A.11)$, note that $V^{\text{cht}}(P, K, D) - V^{\text{cmp}}(P, K)$ is bounded above by the discounted expected prize from the cheating firm alone exerting effort in the next D years and allowing that effort to be cost-less. The probability of success in the time interval $[0, D]$ is less than $2\theta D$, and, therefore, the expected prize is less than $2\theta D \cdot E\left[\max_{0<t<D} P_t\right]$. Doob's inequality implies that $E\left[\max_{0<t<D} P_t\right] < 2E[P_D]$, and $E[P_D] < C \cdot P_0$ for some C that can be computed explicitly for GBM. Because we need consider only P in a bounded range, we obtain the uniform bound in $(A.11)$.

To complete the proof that $D^* > 0$, let $\epsilon = C_1$ in $(A.11)$ and combine with $(A.10)$ to get $V^{\text{cht}}(P, K, D) < V^{\text{co}}(P, K)$ (i.e., cheating is suboptimal) for all $K + Q < P < P^{*\text{co}}$ (which is the only price range in which cheating would be considered).

Proof of Proposition 5

Fix an effort boundary P^* satisfying $P^* > K + Q$, and assume that each of the n firms follows the effort strategy $e_t = \theta 1_{\{P_t \geq P^*\}}$, for all $t \geq 0$ (i.e., exerts effort only when P_t exceeds P^*). Let $v(P)$ denote the aggregate value function at price P (we omit the argument P^* for the first part of the proof). The integral representation $(A.5)$ for the value function implies that v is a $\mathcal{C}^1$ function. In the no-effort region, the ODE for the value function is

$$-rv(P) + \alpha P v_P(P) + \frac{\sigma^2}{2} P^2 v_{PP}(P), \quad \text{for } P < P^*,$$

and in the effort region $P > P^*$, we have

$$n\theta \{P - (K + Q) - v(P)\} - rv(P) + \alpha P v_P(P) + \frac{\sigma^2}{2} P^2 v_{PP}(P).$$

From the results in Appendix B, the solution has the form

$$v(P) = \begin{cases} v(P^*) (P/P^*)^{\beta^+} & \text{if} \quad P \leq P^*, \\ -a(n\theta, 0) + b(n\theta, 0) P + C(P^*) (P/P^*)^{\beta^-(n\theta)} & \text{otherwise.} \end{cases}$$

Continuity of $v(P)$ at P^* implies

$$v(P^*) = -a(n\theta, 0) + b(n\theta, 0) P^* + C(P^*), \tag{A.12}$$

and differentiability in P at P^* implies

$$\beta^+ v(P^*)/P^* = b(n\theta, 0) + \beta^-(n\theta) C(P^*)/P^*. \tag{A.13}$$

Substituting $(A.12)$ into $(A.13)$ yields the expression for $C(P^*)$.

To finish the proof, we restore the effort boundary as an argument to v and now denote the boundary by x instead of P^*. To simplify notation, we omit the arguments to $a()$ and $b()$. From $(A.12)$ and the expression for $C(P^*)$, the value function at the effort boundary is

$$v(x; x) = \frac{1}{\beta^+ - \beta^-} \left[a\beta^- + bx \left(1 - \beta^-\right)\right].$$

Fix some $P < x$ and differentiate the value function $v(P; x) = v(x; x)(P/x)^{\beta^+}$ to get

$$\frac{d}{dx} v(P; x) = \frac{1}{(\beta^+ - \beta^-)} \left(\frac{P}{x}\right)^{\beta^+} \left\{ -\frac{a\beta^+\beta^-}{x} - b\left(\beta^+ - 1\right)\left(1 - \beta^-\right)\right\}. \tag{A.14}$$

The derivative is zero at the value

$$\hat{x} = -\frac{a}{b} \frac{\beta^-\beta^+}{(\beta^+ - 1)(1 - \beta^-)}. \tag{A.15}$$

Some tedious calculations confirm that $\hat{x} = P^{*\text{co}}$, the cooperative boundary. Substitute, from $(A.15)$, $a\beta^{+}\left|\beta^{-}\right|/P^{*\text{co}} = b\left(\beta^{+}-1\right)\left(1-\beta^{-}\right)$ into $(A.14)$ to get

$$\frac{d}{dx}v\left(P;x\right) = \left(\frac{P}{x}\right)^{\beta^{+}}\frac{\left(\beta^{+}-1\right)\left(1-\beta^{-}\right)}{\beta^{+}-\beta^{-}}b\left\{\frac{P^{*\text{co}}-x}{x}\right\}, \quad \text{all } P < x.$$

Therefore, $v\left(P;x\right)$ is increasing in x for $x < P^{*\text{co}}$ and decreasing if $x > P^{*\text{co}}$. We obtain the same properties for $\frac{d}{dx}v\left(P;x\right)$ when we consider any $P > x$.

Proof of Lemma 3

The instantaneous expected stock return for the jump process is

$$\frac{\mathcal{D}P_t}{P_t} = \kappa\left\{pu + \left(1-p\right)1/u - 1\right\}.$$

Substituting p in (1.18), this equals α, the instantaneous return in the diffusion case.

Now consider the conditional quadratic variation (the compensator of the quadratic variation) of the log jump price process:

$$\frac{1}{dt}E_t\left(d\ln P_t\right)^2 = \kappa\left\{p\left(\ln u\right)^2 + \left(1-p\right)\left(-\ln u\right)^2\right\} = \kappa\left(\ln u\right)^2.$$

This equals σ^2 for the value of u in (1.18), which matches the conditional quadratic variation in the diffusion case. Convergence follows as in the convergence of the discrete binomial model to geometric Brownian motion (for example, in the Black-Scholes setting). We can use Skorokhod embedding or KMT embedding of a random walk in Brownian motion to prove this. A proof can be found in [29].

Obviously, $p > 0$. The inequality $p < 1$ holds if and only if $\frac{\alpha}{\kappa} + 1 < u$ (i.e., $\ln\left(\frac{\alpha}{\kappa}+1\right) < \sigma/\sqrt{\kappa}$). Our condition $\kappa > \left(\alpha/\sigma\right)^2$ is sufficient for $p < 1$ because $\ln\left(\frac{\alpha}{\kappa}+1\right) \leq \frac{\alpha}{\kappa}$.

Proof of Lemma 4

From the polynomial $(A.4)$, we have

$$\psi^{+}\left(\delta\right) = \frac{r+\delta+\kappa+\sqrt{\left(r+\delta+\kappa\right)^2 - 4\kappa^2 p\left(1-p\right)}}{2\kappa p},$$

or equivalently,

$$\sqrt{\kappa}\left(\psi^{+}-1\right) = \frac{1}{2p}\left[\frac{\left(r+\delta\right)}{\sqrt{\kappa}} + \sqrt{\kappa}\left(1-2p\right)\right] + \frac{1}{2p}\sqrt{\frac{\left(r+\delta\right)^2}{\kappa} + 2\left(r+\delta\right) + \left[\sqrt{\kappa}\left(1-2p\right)\right]^2}. \tag{A.16}$$

Use the Taylor-series expansion $e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$, $x \in \mathbb{R}$, and the expression for p in (1.18) to get

$$\lim_{\kappa\to\infty} \sqrt{\kappa}\,(1-2p) = \frac{\sigma}{2} - \frac{\alpha}{\sigma}.$$

Substitute into $(A.16)$ and together with $\lim_{\kappa\to\infty} p = 1/2$ to get

$$\lim_{\kappa\to\infty} \beta_\kappa^+(\delta) = \frac{1}{\sigma}\lim_{\kappa\to\infty}\sqrt{\kappa}\ln\left(\psi^+(\delta)\right) = \frac{1}{\sigma}\lim_{\kappa\to\infty}\sqrt{\kappa}\left(\psi^+(\delta)-1\right).$$

Finally,

$$\frac{1}{\sigma}\lim_{\kappa\to\infty}\sqrt{\kappa}\left(\psi^+(\delta)-1\right) = \frac{1}{\sigma}\lim_{\kappa\to\infty}\sqrt{\kappa}\,(1-2p) + \frac{1}{\sigma}\lim_{\kappa\to\infty}\sqrt{2(r+\delta)+\left[\sqrt{\kappa}\,(1-2p)\right]^2},$$

which matches the expression for $\beta^+(\delta)$ in $(A.1)$.

We now prove the inequalities for $\beta_\kappa^+(\delta)$ and $\beta_\kappa^-(\delta)$, which are equivalent to $\psi^-(\delta) < 1 < \psi^+(\delta)$. We obtain the latter inequalities by substituting $\psi = 1$ into $(A.4)$ and showing that the left side is strictly negative. Finally, we show that $\alpha < r$ implies that $\psi^+(\delta) > u$. The inequality $u < \psi^+(\delta)$ follows because the left side of $(A.4)$ is negative at $\psi = u$; that is,

$$\kappa\,(pu + (1-p)/u) < r + \delta + \kappa,$$

which follows from (1.18) because $\kappa\{pu + (1-p)\,1/u - 1\} = \alpha$ and $\alpha < r$.

Proof of Proposition 6

We can, without loss of generality, consider the case of a single firm. The strategy is to find a smooth value function $V(P)$ and an optimal-effort boundary P^* such that effort is optimally exerted only when $P_t \geq P^*$. The same value function will apply for the grid of prices corresponding to any starting price level P_0.

Defining

$$p_i = P_0 u^i, \quad j \in \mathcal{Z},$$

where $\mathcal{Z}$ denotes the set of integers, then if, at time t, there have been m jumps, i of which are u jumps, the price will be $P_t = P_0 u^{i-(m-i)} = p_{2i-m}$. The HJB equation (1.4) in this setting is

$$0 = \sup_{e(p_j)\in\{0,\theta\}} e(p_j)\{f(p_j) - V(p_j) - Q\} - rV(p_j) + \kappa\{pV(p_{j+1}) + (1-p)V(p_{j-1}) - V(p_j)\}, \quad j \in \mathcal{Z},$$

and, therefore, optimal effort is $\hat{e}_t = \theta 1_{\{f(p_j)-V(p_j)\geq Q\}}$. Note that between jumps, the price and the value function do not change. Therefore, it is either optimal to exert effort for the entire period between jumps or optimal to exert no effort for the entire period (i.e., it is optimal to keep effort constant between jumps).

The value function in the no-effort region is given by Lemma 7 below. The effort-region difference equation is

$$0 = \theta \left\{ f\left(p_j\right) - V\left(p_j\right) - Q \right\} - rV\left(p_j\right) + \kappa \left\{ pV(p_{j+1}) + (1-p)\, V(p_{j-1}) - V(p_j) \right\}, \quad p_j \geq P^*.$$

Rearrange and substitute $f\left(p_j\right) = p_j - K$ to get

$$V\left(p_j\right) = \frac{\theta}{\theta + r + \kappa} \left\{ p_j - K - Q \right\} + \frac{\kappa}{\theta + r + \kappa} \left\{ pV(p_{j+1}) + (1-p)\, V(p_{j-1}) \right\}, \quad p_j \geq P^*.$$

To find the value function in the effort region, first we find a specific solution to the equation, which we guess takes the form $V(P) = a + bP$. Now we have to match coefficients to find a and b (as in the diffusion case). From the effort-region difference equations, $p_{j+1} = up_j$ and $p_{j-1} = p_j/u$, we get

$$a + bP = \frac{\theta}{\theta + r + \kappa} \{P - K - Q\} + \frac{\kappa}{\theta + r + \kappa} \{p(a + bPu) + (1-p)(a + bP/u)\}.$$

We obtain a by matching the constant terms (implying $a = -\frac{\theta}{\theta + r}\left(K + Q\right)$) and b by matching the coefficients of P,

$$b = \frac{1}{\theta + r + \kappa} \left\{ \theta + \kappa b \left(pu + (1-p) 1/u \right) \right\},$$

and substituting $pu + (1-p)\, 1/u - 1 = \alpha/\kappa$ to get $b = \frac{\theta}{\theta + r - \alpha}$(the same solutions as in the diffusion case).

The general form of the value function (using the same arguments as in the continuous-time case) is

$$a + bP + d \cdot P^{\beta_\kappa^-(\theta)}$$

for some constant d. The rest of the proof, solving for d and P^*, proceeds exactly as in the diffusion case.

To apply the verification result in Proposition 20 (Appendix C), the only modification of the proof is to redefine the drift operator as

$$\mathcal{D}^{e_u} \phi(P) = \kappa \left\{ p\phi(Pu) + (1-p)\, \phi(P/u) - \phi(P) \right\}, \quad P \in \mathbb{R}_{++}.$$

Note that all our main results hold in this setting (in particular, the representation (1.2), the HJB equation, and the verification proof).

Lemma 7 *In the no-effort region,*

$$V(P) = V(P^*)\left(\frac{P}{P^*}\right)^{\beta_\kappa^+(0)}, \quad P \le P^*. \tag{A.17}$$

Proof. The difference equation in the no-effort region is

$$0 = -rV(p_j) + \kappa\left\{pV(p_{j+1}) + (1-p)V(p_{j-1}) - V(p_j)\right\}, \quad p_j < P^*.$$

Rearrange to get

$$V(p_j) = \frac{\kappa}{r+\kappa}\{pV(p_{j+1}) + (1-p)V(p_{j-1})\}. \tag{A.18}$$

It is easy to see that if V_1 and V_2 are solutions, then $aV_1 + bV_2$ is a solution as well, so the space of the solution is linear. Also if $V(p_1) = V(p_0) = 0$, then $V(p_i) = 0$ for all $i \in \mathbb{Z}$, so the space is two-dimensional. We guess that $V(p_j) = \psi^j$ is the solution for some $\psi > 0$. Dividing $(A.18)$ by ψ^{j-1}, we get $\psi = \frac{\kappa}{r+\kappa}(p\psi^2 + 1 - p)$, which is equivalent to $(A.4)$ when $\delta = 0$. The roots are $\psi^+(0)$ and $\psi^-(0)$. Because the value function goes to zero as the price goes to zero, the coefficient of $\psi^-(0)$ in the value function is zero. Therefore, $V(p_i) = k(\psi^+(0))^i$ for some constant j or, equivalently, $V(P) = C(P/P_0)^{\beta_\kappa^+(0)}$ for some constant C. Finally, from continuity of the value function at P^*, we solve for C to get $(A.17)$. ■

Proof of Proposition 7

To simplify notation, throughout the proof, we omit the argument κ from the effort boundary and value function.

For the subgame in which a firm has previously deviated (and, therefore, both firms henceforth play the competitive strategy), we have already shown that the competitive equilibrium is subgame perfect.

For the rest of the proof, we consider the subgame in which no firm has previously deviated. If firm i follows the cooperative effort policy, the proof of Proposition 2 shows that it is suboptimal for firm j to choose a higher boundary (i.e., exert less than the cooperative effort level).

Now consider the case of firm j exerting too much effort. Suppose no firm has yet won the project, and we are at the jump time t_i, with $P_{t_i} < P^{*\text{co}}$, and firm j follows the deviating strategy of exerting effort until the next jump at t_{i+1}. Obviously, deviating cannot be optimal if $f(P_{t_i}) \le Q$, and, therefore, we henceforth assume that $f(P_{t_i}) > Q$. Starting at t_{i+1}, firm i will respond, and the competitive equilibrium will prevail from time t_{i+1} onward. Let $V^{\text{co}}(P)$ and $V(P)$ denote the cooperative and competitive equilibrium value functions for price level P.

Let (e^j, e^{-j}) denote the effort strategies in the competitive equilibrium, and let $(\tilde{e}^j, \tilde{e}^{-j})$ denote firm j's deviating effort strategy and the other firms' responses (cooperative over the next period, and then reverting to competitive). From representation (1.2), the value function from deviating satisfies

$$v^j\left(P_{t_i}; \tilde{e}^j, \tilde{e}^{-j}\right) = \Gamma + E_{t_i}\left[\int_{t_i}^{\infty} \{f(P_s) - Q\} e_s^j \frac{\xi_s(e^j + e^{-j})}{\xi_t(e^j + e^{-i})} ds\right] = \Gamma + V(P_{t_i}),$$

where

$$\Gamma = E_{t_i}\left[\int_{t_i}^{t_{i+1}} \{f(P_{t_i}) - Q\} \frac{\{\tilde{e}_s^j \xi_s(\tilde{e}^j) - e_s^j \xi_s(e^j + e^{-j})\}}{\xi_t(e^j + e^{-i})} ds\right] \leq \{f(P_{t_i}) - Q\}\, \theta E_{t_i}\left[\int_{t_i}^{t_{i+1}} e^{-(r+\theta)(s-t_i)} ds\right].$$

Because the next jump time is exponential,

$$E_{t_i}\left[\int_{t_i}^{t_{i+1}} e^{-(r+\theta)(s-t_i)} ds\right] = \int_0^{\infty} e^{-(r+\theta)t} P(t_{i+1} - t_i > t)\, dt = \frac{1}{r+\theta+\kappa}.$$

Further, because $P_{t_i} < P^{*\mathrm{co}}$, we have, from (1.13), $f(P_{t_i}) - Q < nV^{\mathrm{co}}(P_{t_i})$ and, therefore,

$$\Gamma \leq nV^{\mathrm{co}}(P_{t_i}) \frac{1}{r+\theta+\kappa}.$$

Deviating is sub-optimal if

$$nV^{\mathrm{co}}(P_{t_i}) \frac{1}{r+\theta+\kappa} + V(P_{t_i}) < V^{\mathrm{co}}(P_{t_i}),$$

which is true for sufficiently large κ because $\lim_{\kappa\to\infty} nV_n^{\mathrm{co}}(P_{t_i}) \frac{1}{r+\theta+\kappa} = 0$ and

$$\lim_{\kappa\to\infty} \{V(P_{t_i}) - V^{\mathrm{co}}(P_{t_i})\} = V(P_{t_i}) - V^{\mathrm{co}}(P_{t_i}) < 0.$$

A.2 Auxiliary Results

Note that ρ in the following lemma satisfies $\rho = r + (n-1)\theta$ in the Proposition 2 application.

Lemma 8 *Suppose the value function $V(P)$ is a $\mathcal{C}^1$ function and satisfies equation (A.8), and, for constants $\rho \geq \alpha$ and $\theta > 0$,*

$$0 = \theta\{P - (K+Q) - V(P)\} - \rho V(P) + V_P(P) P\alpha + \frac{1}{2} V_{PP}(P_t) P^2\sigma^2 \quad \text{for } P \geq P^*.$$

Further, suppose the effort boundary P^* satisfies the optimality condition

$$P - (K + Q) \gtreqless V(P) \iff P \gtreqless P^*, \tag{A.19}$$

and assume the boundary condition

$$0 < \lim_{P \to \infty} \frac{V(P)}{P} \leq 1. \tag{A.20}$$

Then, defining $\beta' = \beta'_-$, given by (A.22) below, the value function in the region $\{P \geq P^*\}$ satisfies

$$V(P) = \frac{\theta}{\theta + \rho - \alpha} P + \frac{\theta}{\theta + \rho} (K + Q) \left\{ \left[\frac{\rho - \alpha\beta^+}{(\beta^+ - \beta')(\rho - \alpha) + \theta(\beta^+ - 1)} \right] \left(\frac{P}{P^*} \right)^{\beta'} - 1 \right\},$$

where

$$P^* = \left(\frac{\theta + \rho - \alpha}{\theta + \rho} \right) \frac{(\beta^+ - \beta')\rho + \beta^+\theta}{(\beta^+ - \beta')(\rho - \alpha) + \theta(\beta^+ - 1)} (K + Q). \tag{A.21}$$

(Note that r only enters via β^+.) Finally, V is convex.

Proof. In the region $\{P \leq P^*\}$, the value function has the well-known solution (A.8). In the region $\{P \geq P^*\}$, the general form of the solution is

$$V(P) = a + bP + k_- P^{\beta'_-} + k_+ P^{\beta'_+},$$

where $\beta'_+ > 1 > 0 > \beta'_-$ are the roots of the polynomial $p(y) = -(\theta + \rho) + \alpha y + \frac{1}{2} y (y - 1) \sigma^2$. Therefore,

$$\beta'_\pm = \frac{1}{2} - \frac{\alpha}{\sigma^2} \pm \sqrt{\left(\frac{1}{2} - \frac{\alpha}{\sigma^2} \right)^2 + \frac{2(\theta + \rho)}{\sigma^2}}. \tag{A.22}$$

The boundary condition (A.20) implies that $k_+ = 0$ because $k_+ > 0$ implies the upper bound of (A.20) is violated for sufficiently large P, and $k_+ < 0$ implies the value function would be negative for sufficiently large P, violating the lower bound. Therefore, the value function (in the effort region) satisfies

$$V(P) = a + bP + kP^{\beta'} \quad P \geq P^*, \tag{A.23}$$

where $\beta' = \beta'_-$. Substitute

$$V_P(P) = b + k\beta' P^{\beta'} / P, \quad V_{PP}(P) = k\beta'(\beta' - 1) P^{\beta'} / P^2,$$

into the ODE to get

$$\begin{aligned} 0 &= \theta\left\{P-K-Q\right\}-(\theta+\rho)\left(a+bP+kP^{\beta'}\right) \\ &\quad +\left\{bP+k\beta' P^{\beta'}\right\}\alpha+\frac{1}{2}k\beta'\left(\beta'-1\right)P^{\beta'}\sigma^2. \end{aligned} \tag{A.24}$$

We can then solve for b and a:

$$a=-\frac{\theta\left(K+Q\right)}{\theta+\rho}, \quad b=\frac{\theta}{\theta+\rho-\alpha}. \tag{A.25}$$

Substituting b and a, we have

$$V\left(P\right)=\frac{\theta}{\theta+\rho-\alpha}P-\frac{\theta\left(K+Q\right)}{\theta+\rho}+kP^{\beta'}, \quad \text{for} \quad P\geq P^*. \tag{A.26}$$

We now solve for P^* and k. The optimality condition $(A.19)$ implies that

$$V\left(P^*\right)=P^*-K-Q=a+bP^*+k\left(P^*\right)^{\beta'}, \tag{A.27}$$

and the smoothness of the value function (matching the derivatives of $(A.8)$ and $(A.23)$ at P^*) implies that

$$\beta^+ V\left(P^*\right)\frac{1}{P^*}=b+\beta' k\left(P^*\right)^{\beta'-1}.$$

Combined with $(A.27)$ to solve for $k\left(P^*\right)^{\beta'}$ and substituting a, we get

$$\beta^+\left(P^*-K-Q\right)=bP^*+\beta'\left\{\left(1-b\right)P^*-\frac{\rho\left(K+Q\right)}{\theta+\rho}\right\}.$$

It is immediately apparent that $P^*=\lambda\left(K+Q\right)$ with λ given by

$$\lambda=\frac{1}{\theta+\rho}\frac{\beta^+\left(\theta+\rho\right)-\beta'\rho}{\beta^+-\beta'+b\left(\beta'-1\right)}.$$

Substitute b to get $(A.21)$.

Finally, substitute a into $(A.27)$ to get

$$k\left(P^*\right)^{\beta'}=\left(1-b\right)P^*-\frac{\rho\left(K+Q\right)}{\theta+\rho}, \tag{A.28}$$

and then substitute $(A.28)$ into $(A.26)$ and simplify to get

$$V(P) = \frac{\theta}{\theta+\rho-\alpha}P - \frac{(K+Q)}{\theta+\rho}\left\{\theta + \rho\left(\frac{P}{P^*}\right)^{\beta'}\right\} + \left(\frac{\rho-\alpha}{\theta+\rho-\alpha}\right)P^*\left(\frac{P}{P^*}\right)^{\beta'}.$$

The last step is to confirm the optimality condition $(A.19)$, which follows from $\lim_{P\to\infty} V'(P) = \frac{\theta}{\theta+\rho-\alpha} < 1$ and convexity V, which we now show. Strict convexity of V is equivalent to $k > 0$. From $(A.28)$,

$$k > 0 \iff (1-b)\,P^* > \frac{\rho(K+Q)}{\theta+\rho}.$$

Substituting b and P^* and simplifying, we get

$$k > 0 \iff (\rho-\alpha)\frac{\beta^+\theta + (\beta^+ - \beta')\,\rho}{(\beta^+ - \beta')(\rho-\alpha) + \theta(\beta^+ - 1)} > \rho$$

and finally $k > 0 \iff \beta^+ < \rho/\alpha$. The inequality $\beta^+ < \rho/\alpha$ follows because $\rho \geq r$ and $\beta^+ < r/\alpha$ (from Lemma 5). ■

Lemma 9 *The value function V and optimal boundary P^* in Lemma 8 are strictly increasing in θ (holding fixed ρ), σ, and α, and strictly decreasing in ρ.*

Proof for ρ. The value-matching condition $V(P^*) = P^* - K - Q$ implies that any parameter that increases (decreases) that P^* must also increase (decrease) the value function. From the value-function expression (see (1.2))

$$V_t(e) = E_t\left[\int_t^\infty [f(P_s)\,e_s - q(e_s)] \exp\left(-\int_t^s \left(r + (\rho - r)\,1_{\{P_s > P^*\}} + e_u\right) du\right) ds\right],$$

it is clear that $V(e)$ is decreasing in ρ for any effort policy and P^*. The result for α follows because $V(P)$ is increasing in α for any $P > 0$.

Proof for θ. We use

$$\frac{d\beta'}{d\theta} = -\frac{1}{\sigma^2}\frac{1}{\sqrt{\left(\frac{1}{2}-\frac{\alpha}{\sigma^2}\right)^2 + \frac{2(\theta+\rho)}{\sigma^2}}}$$

and write

$$P^* = \left(\frac{\theta+\rho-\alpha}{\theta+\rho}\right) g(\theta)\,(K+Q), \quad \text{where } g(\theta) = \frac{(\beta^+ - \beta')\,\rho + \beta^+\theta}{(\beta^+ - \beta')(\rho-\alpha) + \theta(\beta^+ - 1)}.$$

Because $\left(\frac{\theta+\rho-\alpha}{\theta+\rho}\right)$ is increasing in θ, it is sufficient to show that g is increasing. Differentiate (note that β does not depend on θ) to get

$$g'(\theta) = \frac{\rho - \alpha\beta^{+}}{\left\{\left(\beta^{+} - \beta'\right)(\rho - \alpha) + \theta\left(\beta^{+} - 1\right)\right\}^{2}} \left[\left(\beta^{+} - \beta'\right) - \frac{\theta}{\sigma^{2}} \frac{1}{\sqrt{\left(\frac{1}{2} - \frac{\alpha}{\sigma^{2}}\right)^{2} + \frac{2(\theta+\rho)}{\sigma^{2}}}}\right].$$

The definitions of β^{+} and β' imply that the square-bracketed term is positive.

Proof for σ. Differentiate P^{*} with respect to σ. To simplify notation, let $f = \beta^{+}$ and $g = \beta^{-}(\delta)$, and denote by primes the derivatives with respect to σ. P^{*} is increasing in σ if and only if

$$\alpha f'g - \alpha g'f + rg' - (r+\theta)f' \geq 0.$$

We know from Lemma 5 that $f < \frac{r}{\alpha}$ and $g' \geq 0$, and, therefore, a sufficient condition is

$$\alpha f'g - (r+\theta)f' \geq 0,$$

which is true because $f', g \leq 0$ (using Lemma 5). ■

Note that the following lemma holds for a more general effort-cost function, and the upper bound is the same for general effort processes as long as each firm's effort is bounded above by θ. Generalizations to unbounded effort are possible.

Lemma 10 *Suppose the total effort of the other $n-1$ firms is not decreasing in price. Then*

$$\frac{d}{dP} V(P) \leq 1 - \frac{r-\alpha}{r+\theta-\alpha}. \tag{A.29}$$

Proof. Let $\bar{e}(P_t)$ denote the total time-t effort exerted by all the other firms, and let $\hat{e}_t$ denote the time-t optimal effort of the firm. We assume that $\bar{e}()$ is a non-decreasing function. The time-0 value function at price P_0 is

$$V(P_0) = E\left[\int_0^{\infty} \left\{(P_s - K)\hat{e}_s - q(\hat{e}_s)\right\} \exp\left(-\int_0^{s} \left(r + \bar{e}_u(P_u) + \hat{e}_u\right) du\right) ds\right]. \tag{A.30}$$

Defining the martingale

$$M_t = \exp\left(-\frac{1}{2}\sigma^2 t + \sigma B_t\right), \quad t > 0,$$

then $P_t = P_0 M_t e^{at}$. Differentiating with respect to P_0, using the envelope theorem to disregard the terms involving $\hat{e}$, and using the $\bar{e}()$ assumption to give an upper bound, we get

$$\frac{d}{dP_0} V(P_0) \leq E\left[\int_0^\infty e^{\alpha s} M_s \hat{e}_s \exp\left(-\int_0^s (r + \bar{e}_u(P_u) + \hat{e}_u)\, du\right) ds\right].$$

Now change the measure to $\tilde{P}$, where $M_t = E_t\left(\frac{d\tilde{P}}{dP}\right)$:

$$\begin{aligned} \frac{d}{dP_0} V_0(P_0) &\leq \tilde{E}\left[\int_0^\infty e^{\alpha s} \hat{e}_s \exp\left(-\int_0^s (r + \bar{e}_u(P_u) + \hat{e}_u)\, du\right) ds\right] \\ &\leq \tilde{E}\left[\int_0^\infty e^{(\alpha - r)s} \hat{e}_s \exp\left(-\int_0^s \hat{e}_u du\right) ds\right]. \end{aligned}$$

Finally, we use integration by parts and $\hat{e}_u \leq \theta$ to get

$$\begin{aligned} \tilde{E}\left[\int_0^\infty e^{(\alpha - r)s} \hat{e}_s \exp\left(-\int_0^s \hat{e}_u du\right) ds\right] &= -\tilde{E}\left[\int_0^\infty e^{(\alpha - r)s} \frac{d}{ds} \exp\left(-\int_0^s \hat{e}_u du\right) ds\right] \\ &= \tilde{E}\left[1 - (r - \alpha)\int_0^\infty e^{(\alpha - r)s} \exp\left(-\int_0^s \hat{e}_u du\right) ds\right] \\ &\leq \tilde{E}\left[1 - (r - \alpha)\int_0^\infty e^{(\alpha - r - \theta)s} ds\right]. \end{aligned}$$

Evaluating the integral on the right side completes the proof. ■

Lemma 11 *Assume the total effort of the other $n-1$ firms is not decreasing in price. Let P^* satisfy*

$$P^* - V(P^*) = K + Q. \tag{A.31}$$

a) Suppose $q(e) = Qe$ and $e \in \{0, \theta\}$. Then $P^* \in (0, \infty)$ exists and is unique, and optimal effort satisfies $\hat{e}(P) = \theta 1_{\{P \geq P^*\}}$.

b) Suppose $q()$ is strictly increasing and convex, $q(0) = 0$, and $q'(0) = Q \geq 0$. Then $P^* \in (0, \infty)$ exists and is unique, and optimal effort satisfies $\hat{e}(P) = (q')^{-1}(P - K - V(P))\, 1_{\{P \geq P^*\}}$, where $(q')^{-1}$ is the inverse function of q'. Optimal effort $\hat{e}(P)$ is, therefore, strictly increasing in P for $P \geq P^*$.

In either case (a) or (b), $V(P)$ is increasing for $P \leq P^*$.

Proof. The agent's optimal effort function $\hat{e}(P)$ satisfies

$$\hat{e}(P) = \arg\max_{e \geq 0} e\{P - K - V(P)\} - q(e).$$

Concavity of the objective function implies either the boundary solution

$$P - K - V(P) - q'(0) < 0 \implies \hat{e}(P) = 0$$

or an "interior" solution, if $P - K - V(P) \geq q'(0)$. If $q(e) = Qe$ and $e \in \{0, \theta\}$, then $P - K - V(P) \geq q'(0)$ implies that $\hat{e}(P) = \theta$. If q is strictly convex, then optimal effort satisfies

$$P - K - V(P) = q'(\hat{e}(P)).$$

That is, the zero-effort region is $\{P : P - K - V(P) < Q\}$, and the positive-effort region (strictly positive on $P - K - V(P) > Q$) is $\{P : P - K - V(P) \geq Q\}$.

The bound $(A.29)$ implies that $P - V(P)$ is strictly increasing in P from 0 toward ∞. Together with continuity of V and $V(0) = 0$, we get that $P^* \in (0, \infty)$ exists and is unique, the zero-effort price region is $[0, P^*)$, and the positive-effort price region is $[P^*, \infty)$.

Choose any $P < P^*$ and $\bar{P} \in (P, P^*)$ and define $\tau = \inf\left\{t : P_t = \bar{P}\right\}$. Because optimal effort is zero on $\{P : P < P^*\}$,

$$V(P) = E\left[\exp\left(-\int_0^{\tau} (r + \bar{e}_u(P_u))\, du\right) V\left(\bar{P}\right)\right],$$

and, therefore, $V(P) < V\left(\bar{P}\right)$, which implies that $V(P)$ is increasing for $P < P^*$. ■

A.3 Verification Proof

In this section, we verify the value functions that were conjectured in the previous sections. The method is standard and follows the one in [46].

Proposition 20 *The function $V^{cmp}(P)$ defined in Proposition 2 is the value function for $v^i(P_t; e^i, e^{-i})$ as defined in equation* (1.2).

Proof. The notation is simpler if we let $n = 1$, in which case $e^{-i} = 0$ and we can omit the e^{-i} argument and drop the i superscripts from $v^i(P_t; e^i, e^{-i})$. We also omit the "cmp" superscript from V. In order to prove this proposition, we need to show that

$$V(p) \geq v(p; e),$$

for any control process e, with equality holding at the optimal effort process $\hat{e}$ in (1.6). By the fact that P_t is Markov process and τ is memory-less, we can see that it is enough to prove the above inequality when e is Markovian (i.e., e_t is a function of P_t only). Also we can assume that $e(p) = 0$ when $p \leq K + Q$ because the expected payout is zero when the price is lower than $K + Q$. Consider any strategy e as above and define the operators

$$\mathcal{D}^{e_u}\phi(x) = \alpha x\phi'(x) + \frac{\sigma^2}{2}x^2\phi''(x), \text{ and } \mathcal{L}^{e_u}\phi(x) = -(r + e_u)\phi(x) + \mathcal{D}^{e_u}\phi(x).$$

By Ito's formula, for any function $\phi(x)$ and any control process e, we have, for $s > t$,

$$\xi_s(e)\phi(P_s) - \xi_t(e)\phi(P_t) = \int_t^s \xi_u(e)\mathcal{L}^{e_u}\phi(P_u)du + \int_t^s \xi_u(e)\phi'(P_u)\sigma P_u dW_u. \quad \text{(A.32)}$$

Now assume that we have initial condition $P_0 = p$ and consider $T > 0$.

By applying formula $(A.32)$ to $\phi = V$, we get

$$V(p) = \xi_T(e)V(P_T) - \int_0^T \xi_u(e)\mathcal{L}^{e_u}V(P_u)du - \int_0^T \xi_u(e)V'(P_u)\sigma P_u dW_u.$$

Now note that $\xi_u(e) \leq e^{-ru}$, $E[P_u] = e^{\alpha u}$, and $||V'||_\infty < C < \infty$ for some constant C (because V is asymptotically linear). These three facts imply that the stochastic integral is a martingale. Also because V solves the HJB equation, we know that

$$\mathcal{L}^{e_u}V(P_u) + [f(P_u)e_u - Qe_u] \leq 0$$

for any feasible e, with equality holding at the optimal effort process $\hat{e}$ defined in (1.6). Taking the expectation, we get

$$V(p) \geq E\left[\xi_T(e)V(P_T) + \int_0^T [f(P_u)e_u - Qe_u]\xi_u(e)du\right],$$

with equality again holding for $\hat{e}$. Letting T go to ∞, the first term goes to zero because V has linear growth. The second term converges to

$$E\left[\int_0^\infty [f(P_u)e_u - Qe_u]\xi_u(e)du\right]$$

by the linear growth condition on f, which completes the proof.

The verification in the case of symmetric equilibrium is exactly the same as the single-firm case when we take $v^i(p; e^i, e^{-i})$ as in (1.2) and $e^{-i} = (n-1)1\{p \geq P^*\}$. ■

A.4 Heterogeneous Firms

Departing from our assumption of identical firms, the following proposition shows that a Nash equilibrium with *heterogeneous firms* can be computed using a simple iterative scheme. An n-firm algorithm could constructed using the same ideas.

Proposition 21 (Construction of two-heterogeneous-firms Nash equilibrium) *Suppose there are two firms, a and b. Assume that each firm's effort-cost function satisfies $q^i(e) = Q^i e$ and $e \in \{0, \theta^i\}$, $i \in \{a, b\}$. Let $P^{*i}(p)$ denote firm i's optimal effort-boundary response to the other firm's effort boundary p.*[1] *Define $p_0^b = \infty$ (i.e., at step 0, firm-b never exerts effort) and the sequence of effort boundaries*

$$p_j^a = P^{*a}\left(p_{j-1}^b\right), \quad p_j^b = P^{*b}\left(p_j^a\right), \quad j = 1, 2, \ldots.$$

Then the sequences $\{p_1^a, p_2^a, \ldots\}$ and $\left\{p_1^b, p_2^b, \ldots\right\}$ are each monotonically decreasing, and their limits,

$$p_\infty^a = \lim_{j \to \infty} p_j^a, \quad p_\infty^b = \lim_{j \to \infty} p_j^b,$$

are a Nash equilibrium effort-boundary pair.

Proof. From the expression (1.2), each firm's value function at any P is decreasing as the other firm's effort increases (equivalently, as the other firm's boundary decreases). Furthermore, from the optimal effort condition (1.5), a decrease in the firm's value function implies an increase in the optimal own-firm effort. The effort-boundary sequences are, therefore, monotonically decreasing and bounded below by zero. It follows that a limit must exist. Continuity of the functions P^{*a} and P^{*b} (see the proof of Proposition 2) implies

$$p_\infty^a = \lim_{j \to \infty} p_j^a = \lim_{j \to \infty} P^{*a}\left(p_{j-1}^b\right) = P^{a*}\left(p_\infty^b\right),$$

and, similarly, $p_\infty^b = P^{*b}(p_\infty^a)$, which proves the Nash equilibrium.

Note that we can compute $p_j^a = P^{*a}\left(p_{j-1}^b\right)$ implicitly from the effort-boundary condition $p_j^a - \left(Q^b + K^a\right) = v^a\left(p_j^a; p_{j-1}^b\right)$, where $v^a\left(p_j^a; p_{j-1}^b\right)$ denotes the value function of firm a with firm-a effort boundary p_j^a and firm-b effort boundary p_{j-1}^b. The computation for $p_j^b = P^{*b}\left(p_j^a\right)$ is analogous. ■

As in the symmetric case, the equilibrium is subgame perfect and is, therefore, both a competitive and coordinated (i.e., closed-loop) equilibrium.

[1] See the proof for a method to compute $P^{*i}(p)$.

The following lemma shows the intuitive result that in any equilibrium with heterogeneous firms, the lower cost firm will exert more effort, and the more productive firm (the one with the higher maximum effort level) will exert effort less frequently (i.e., set a higher effort boundary).

Lemma 12 *Suppose there are two firms, a and b, with $\theta^a \leq \theta^b$, $Q^a + K^a \leq Q^b + K^b$, and at least one inequality is strict. Then the effort boundaries $\left(P^{*a}, P^{*b}\right)$ in any equilibrium satisfy $P^{*a} < P^{*b}$.*

Proof. Consider first the case when $\theta^a = \theta^b$ and $Q^a + K^a < Q^b + K^b$. Suppose, contrary to the statement of the lemma, that $P^{*a} \geq P^{*b}$. Similar to the notation in (1.2), we let $v^a\left(P_t; p^a, p^b\right)$ denote firm a's value function corresponding to current price P_t and firm a and b effort boundaries p^a and p^b. The notation for firm b is analogous. Differencing the optimal boundary conditions $P^{*i} = Q^i + K^i + V^i\left(P^{*i}\right)$, $i \in \{a, b\}$, we get

$$P^{*a} - P^{*b} = \left(Q^a + K^a\right) - \left(Q^b + K^b\right) + v^a\left(P^{*a}; P^{*a}, P^{*b}\right) - v^b\left(P^{*b}; P^{*b}, P^{*a}\right).$$

Now substitute, using Lemma 10,

$$v^a\left(P^{*a}; P^{*a}, P^{*b}\right) \leq v^a\left(P^{*b}; P^{*a}, P^{*b}\right) + (1-\varepsilon)\left(P^{*a} - P^{*b}\right),$$

where $\varepsilon = (r-\alpha)/(r+\theta-\alpha)$. Also, from monotonicity of the value function in the other firm's effort $v^a\left(P^{*b}; P^{*a}, P^{*b}\right) \leq v^a\left(P^{*b}; P^{*a}, P^{*a}\right)$, and, from optimality of P^{*b} for firm b, $v^b\left(P^{*b}; P^{*b}, P^{*a}\right) \geq v^b\left(P^{*b}; P^{*a}, P^{*a}\right)$. Together these inequalities imply

$$\begin{aligned} P^{*a} - P^{*b} \leq\ & \left(Q^a + K^a\right) - \left(Q^b + K^b\right) + v^a\left(P^{*b}; P^{*a}, P^{*a}\right) - v^b\left(P^{*b}; P^{*a}, P^{*a}\right) \\ & + (1-\varepsilon)\left(P^{*a} - P^{*b}\right). \end{aligned} \tag{A.33}$$

Using (1.2) and letting $e_s = \theta 1_{\{P_s \geq P^{*a}\}}$,

$$\begin{aligned} v^a\left(P^{*b}; P^{*a}, P^{*a}\right) - v^b\left(P^{*b}; P^{*a}, P^{*a}\right) &= \frac{1}{2} E\left[\int_0^\infty \left\{\left(Q^b + K^b\right) - \left(Q^a + K^a\right)\right\} 2 e_s \xi_s(2e)\, ds\right] \\ &< \frac{1}{2}\left\{\left(Q^b + K^b\right) - \left(Q^a + K^a\right)\right\} E\left[-\int_0^\infty \frac{d}{ds}\xi_s(2e)\, ds\right] \\ &< \frac{1}{2}\left\{\left(Q^b + K^b\right) - \left(Q^a + K^a\right)\right\}. \end{aligned}$$

Substituting, this implies that

$$\varepsilon\left(P^{*a} - P^{*b}\right) \leq -\frac{1}{2}\left\{\left(Q^b + K^b\right) - \left(Q^a + K^a\right)\right\},$$

which contradicts the supposition that $P^{*a} \geq P^{*b}$.

Consider next the case when $Q^a + K^a = Q^b + K^b$ and $\theta^a < \theta^b$. Suppose to the contrary that $P^{*a} \geq P^{*b}$. Using the same inequalities as above, we obtain $(A.33)$, which implies that

$$\varepsilon\left(P^{*a} - P^{*b}\right) \leq v^a\left(P^{*b}; P^{*a}, P^{*b}\right) - v^b\left(P^{*b}; P^{*a}, P^{*a}\right).$$

But this is a contradiction because $\theta^b > \theta^a$ implies $v^b\left(P^{*b}; P^{*a}, P^{*a}\right) > v^a\left(P^{*b}; P^{*a}, P^{*b}\right)$.

The case when $Q^a + K^a < Q^b + K^b$ and $\theta^a < \theta^b$ follows from the arguments above.

■

Appendix B

Appendix to Chapter 2

B.1 Proofs

Proof of Proposition 17

Let's recall the equations which LP should consider to design the security. We have

$$\begin{aligned} &z \geq p_1 p_2 x, p_1 y_1, p_2 y_2 \\ &y_1 \geq z, p_2 x \\ &y_2 \geq z, p_1 x \\ &x \geq z, y_1, y_2 \end{aligned}$$

where as defined before, $(x, z) = (s(2R), s(2I))$ and y_i is the payout to GP when project i is successful and project $3 - i$ is not. As we saw, LP problem is

$$\begin{aligned} &\min_{x,y,z} \alpha x + \beta_1 y_1 + \beta_2 y_2 + \gamma z \\ &\alpha x + \beta_1 y_1 + \beta_2 y_2 + \gamma z \geq z + 2c, \lambda_i y_i + (1 - \lambda_i) z + c \\ &x \geq y_i \geq z \geq p_i y_i \geq p_1 p_2 x \end{aligned}$$

where $(\alpha, \beta_1, \beta_2, \gamma) = (\lambda_1 \lambda_2, \lambda_1 - \rho\lambda_2, (1 - \rho)\lambda_2, 1 - \lambda_1 - \lambda_2 + \rho\lambda_2)$. In the discussion before the proposition, we showed that

$$\alpha x + \beta_1 y_1 + \beta_2 y_2 + \gamma z = \max\{z + 2c, \lambda_i y_i + (1 - \lambda_i) z + c\} \tag{B.1}$$

Now we claim that, in the optimum either $z = p_1 y_1$ or $z = p_2 y_2$. This happens since if z can be reduced, the coefficient of z on the LHS which is $\gamma = 1 - \lambda_1 - \lambda_2 + \rho\lambda_2$ is less than (or equal to if $\rho = 1$) the minimum of coefficients on the RHS of equation (B.1) which is $\min(1 - \lambda_1, 1 - \lambda_2)$. Having this, assume that $z = p_1 y_1 \geq p_2 y_2$. Consider several cases

- First assume that the maximum on the RHS of equation (B.1) happens at $z+2c$. Then we get

$$\alpha x = z(1 - \frac{\beta_1}{p_1} - \gamma) - \beta_2 y_2 + 2c.$$

After multiplying by $p_1 p_2$ and using the facts that $y_2 \leq \frac{z}{p_2}, x \leq \frac{z}{p_1 p_2}$ we have

$$z[\alpha(1 - p_1 p_2) + p_2 \beta_1 (1 - p_1) + p_1 \beta_2 (1 - p_2)] \\ \geq 2 p_1 p_2 c$$

with strict inequality if either $y_2 < \frac{z}{p_2}$ or $x < \frac{z}{p_1 p_2}$. This gives a lower bound for z. In the optimum the inequality becomes equality (as LP looks for the minimum payment to GP) hence we get $z = p_1 y_1 = p_2 y_2 = p_1 p_2 x$.

- Now assume that maximum happens at $\lambda_1 y_1 + (1 - \lambda_1) z + c$. Then similar to the previous case, we can write

$$\alpha x = \lambda_1 \frac{z}{p_1} + (1 - \lambda_1) z - \beta_1 \frac{z}{p_1} - \beta_2 y_2 - \gamma z + c$$

Again inequalities as in the previous case gives us a lower bound for z which is binding in the optimum hence we get the same relationship between variables.

- Finally assume that maximum happens at $\lambda_2 y_2 + (1 - \lambda_2) z + c$ and assume this is strictly bigger than other two terms ($z + 2c$ and $\lambda_1 y_1 + (1 - \lambda_1) z + c$). If $z = p_1 y_1 = p_2 y_2$, then the same reasoning as in the previous case works. So assume that $z = p_1 y_1 > p_2 y_2 \geq p_1 p_2 x$. Now reduce z by ϵ and y_1 by $\frac{\epsilon}{p_1}$ and change x accordingly to x' such that

$$\alpha x' + \beta_1 (y_1 - \frac{\epsilon}{p_1}) + \beta_2 y_2 + \gamma (z - \epsilon) = \lambda_2 y_2 + (1 - \lambda_2)(z - \epsilon) + c$$

If this change is permissible, then the new contract $(x', y_1 - \frac{\epsilon}{p_1}, y_2, z - \epsilon)$ is strictly better for LP. So it would not be permissible. If $x' > x$,this means that we should have had $p_1 x = y_2$ so that an increase in x is not permissible. However in this case, the original equation for the expected payout to GP, becomes

$$\alpha (x - y_2) = \alpha x (1 - p_1) = z(1 - \frac{\beta_1}{p_1} - \gamma - \lambda_2) + c$$

so

$$\alpha \frac{z}{p_1 p_2} (1 - p_1) \geq z(1 - \frac{\beta}{p_1} - \gamma - \lambda_2) + c$$

and again in the optimum this should be equality hence $z = p_1 p_2 x$ which proves the claim. If $x' < x$, then either $x = y_2$ or $x = y_1$. But these contracts can not be optimal since reducing all the payouts z, y_1 and y_2 and increasing x is allowed here which improves payout to LP.

Proof of Proposition9 and 10

Let's recall the LP problem.

$$\min_{x,y,z} \alpha x + \beta y + \gamma z$$
$$\alpha x + \beta y + \gamma z \geq z + 2c, \lambda_{max} y + (1 - \lambda_{max}) z + c$$
$$x \geq y \geq z \geq p_1 y \geq p_1^2 x$$

where $(\alpha, \beta, \gamma) = (\rho\lambda_{min}, \lambda_1 - 2\rho\lambda_{min} + \lambda_2, 1 - \lambda_1 - \lambda_2 + \rho\lambda_{min})$. As in the case of Proposition 17, we make two preliminary observations.

1. In the optimum, we have $\alpha x + \beta y + \gamma z = \max\{z + 2c, \lambda_{max} y + (1 - \lambda_{max}) z + c\}$. Suppose not. Then the transformation $z \to z - \epsilon$ for small enough ϵ, should violate the conditions of the LP problem otherwise reducing z improves the payout to LP. This means $z = p_1 y$. But then $y \to y - \epsilon$ should violate the conditions so similarly we get $y = p_1 x$ as well. But then $x \to x - \epsilon$ is legitimate because by $z = p_1 y = p_1^2 x$ we have $x > y > z$.

2. In the optimum, $z = p_1 y$. This comes from the fact that in the LP problem $\gamma = 1 - \lambda_1 - \lambda_2 + \rho\lambda_{min} \leq 1 - \lambda_1, 1 - \lambda_2$. So if the move $z \to z - \epsilon$ is allowed, the subtraction on the LHS which is $\gamma\epsilon$ is less than the subtraction on the RHS which is either ϵ or $(1 - \lambda_{max})\epsilon$.

By point 1 above, the expected payout to GP will be $\max\{z + 2c, \lambda_{max} y + (1 - \lambda_{max}) z + c\}$. This function is increasing in z, equals to $z + 2c$ for $z \leq \frac{p_1 c}{\lambda_{max}(1-p_1)}$ (since $z = p_1 y$) and it is $\lambda_{max} y + (1 - \lambda_{max}) z + c$ otherwise. Because the payout is increasing in z, GP searches for a feasible contract with least amount of z. First we see when LP is able to write a contract with $z \leq \frac{p_1 c}{\lambda_{max}(1-p_1)}$. Here is how the contract is designed. As we saw, in this range of z,

$$\alpha x + \beta y + \gamma z = \max\{z + 2c, \lambda_{max} y + (1 - \lambda_1) z + c\} = z + 2c$$

therefore

$$x = \frac{z(1 - \gamma - \frac{\beta}{p_1}) + 2c}{\alpha} \tag{B.2}$$

hence GP should solve

$$\min z$$
$$x \geq \frac{z}{p_1} \geq p_1 x \quad z \leq \frac{p_1 c}{\lambda_{max}(1-p_1)}$$

given x as in the equation (B.2). $p_1 x \geq z$ implies

$$2p_1 c \geq (1-p_1)(\alpha+\beta)z$$

which implies

$$z \leq \frac{2p_1 c}{(\alpha+\beta)(1-p_1)}$$

On the other hand $z \geq p_1^2 x$, implies

$$[\beta\frac{1-p_1}{p_1} + \alpha\frac{1-p_1^2}{p_1^2}]z \geq 2c$$

which gives

$$z \geq \frac{2c}{\beta\frac{1-p_1}{p_1} + \alpha\frac{1-p_1^2}{p_1^2}} \tag{B.3}$$

Since

$$\frac{2c}{\beta\frac{1-p_1}{p_1} + \alpha\frac{1-p_1^2}{p_1^2}} < \frac{2p_1 c}{(\alpha+\beta)(1-p_1)}$$

one needs to compare the lower bound on the z from equation (B.3) with the initial condition $z \leq \frac{p_1 c}{\lambda_{max}(1-p_1)}$. This gives us that there is answer in this region if and only if

$$[\beta\frac{1-p_1}{p_1} + \alpha\frac{1-p_1^2}{p_1^2}] \geq \frac{2\lambda_{max}(1-p_1)}{p_1}$$

which simplifies to

$$\beta + \alpha\frac{1+p_1}{p_1} \geq 2\lambda_{max}$$

Substituting α and β gives us that this happens if and only if

$$\rho \geq \frac{\lambda_{max} - \lambda_{min}}{\lambda_{min}(\frac{1}{p_1} - 1)}$$

In this case the contract is written with z as given by equality in the equation (B.3) which is the minimal z hence $z = p_1 y = p_1^2 x$. Now consider the case $\rho < \frac{\lambda_{max}-\lambda_{min}}{\lambda_{min}(\frac{1}{p_1}-1)}$.

The expected payout to GP then has the form $\lambda_{max}y + (1-\lambda_{max})z + c$ for some $z > \frac{p_1 c}{\lambda_{max}(1-p_1)}$. In this case we get

$$\alpha x + \beta y + \gamma z = \lambda_{max}y + (1-\lambda_{max})z + c = (\frac{\lambda_{max}}{p_1} + 1 - \lambda_{max})z + c$$

hence

$$x = \frac{(\frac{\lambda_{max}}{p_1} + 1 - \lambda_{max} - \frac{\beta}{p_1} - \gamma)z + c}{\alpha}$$

Similar to the previous case, the optimization becomes

$$\min z$$
$$x \geq \frac{z}{p_1} \geq p_1 x \quad z \geq \frac{p_1 c}{\lambda_{max}(1-p_1)}$$

$p_1 x \geq z$ implies

$$[\lambda_{max} + p_1 - p_1\lambda_{max} - \beta - p_1\gamma]z + p_1 c \geq \alpha z$$

which is

$$p_1 c \geq (\alpha + \beta - \lambda_{max})(1-p_1)z$$

We have $\alpha + \beta - \lambda_{max} = \lambda_{min}(1-\rho)$

$$z \leq \frac{p_1 c}{(1-p_1)\lambda_{min}(1-\rho)}$$

Finally $z \geq p_1^2 x$ implies

$$(\frac{\lambda_{max}}{p_1} + 1 - \lambda_{max} - \frac{\beta}{p_1} - \gamma)z + c \leq \frac{\alpha z}{p_1^2}$$

which implies

$$c \leq [\frac{\alpha}{p_1^2} - (\frac{\lambda_{max}}{p_1} + 1 - \lambda_{max} - \frac{\beta}{p_1} - \gamma)]z$$

We can write this as

$$c \leq [\frac{\alpha(1-p_1^2)}{p_1^2} + \beta\frac{1-p_1}{p_1} - \lambda_{max}\frac{1-p_1}{p_1}]z$$

therefore again the contract has the following form

$$z = \frac{c}{\frac{\alpha(1-p_1^2)}{p_1^2} + \beta\frac{1-p_1}{p_1} - \lambda_{max}\frac{1-p_1}{p_1}}$$
$$y = \frac{z}{p_1}$$
$$x = \frac{z}{p_1^2}$$

Finally to check the upper bound $\frac{p_1 c}{(1-p_1)\lambda_{min}(1-\rho)}$ for z, one needs to verify

$$\frac{\alpha(1-p_1^2)}{p_1} + \beta(1-p_1) - \lambda_{max}(1-p_1) \geq \lambda_{min}(1-p_1)(1-\rho)$$

Canceling $1-p_1$,we get

$$\frac{\alpha(1+p_1)}{p_1} + \beta - \lambda_{max} \geq \lambda_{min}(1-\rho)$$

LHS equals to

$$\lambda_{min}(\rho(\frac{1}{p_1} - 1) + 1) \geq \lambda_{min}(1-\rho)$$

which is obvious.

The only part from Proposition 10, which needs proof is that when $\rho \geq \frac{\lambda_{max}-\lambda_{min}}{\lambda_{min}(\frac{1}{p_1}-1)}$, then LP makes more profit by whole-portfolio contracting. This is because, as we saw in the proof above, for these values of ρ,the expected profit by GP with whole-portfolio contract is $z + 2c$ for some $z \leq \frac{p_1 c}{\lambda_{max}(1-p_1)}$ which is less than $\Pi_{GP} = 2c + \sum_{i=1}^{2} \frac{p_i c}{\lambda_i(1-p_i)}$ that GP makes under the deal-by-deal contract.

Proof of Proposition 12

As we saw in the Proof of Proposition 9, when $\rho > \rho^*$, the expected payout to GP is $z + 2c$. The expression for z in this region is

$$z = \frac{2c}{\beta\frac{1-p_1}{p_1} + \alpha\frac{1-p_1^2}{p_1^2}}$$

So it is only needed to look at how denominator changes when parameters change. Denominator is equal to

$$\beta\frac{1-p_1}{p_1} + \alpha\frac{1-p_1^2}{p_1^2}$$
$$= \frac{1-p_1}{p_1}[\alpha\frac{1+p_1}{p_1} + \beta]$$

Derivative with respect to p_1 becomes

$$-\frac{1}{p_1^2}[\alpha\frac{1+p_1}{p_1}+\beta]+\frac{1-p_1}{p_1}\times\alpha\frac{-1}{p_1^2}<0$$

With respect to λ_{min} and λ_{max}, derivatives are $\frac{1-p_1}{p_1}[\rho\frac{1+p_1}{p_1}+1-2\rho]$ and $\frac{1-p_1}{p_1}$ and both are positive. This proves the proposition in the region $\rho>\rho^*$. In the region $\rho<\rho^*$,the expected payout to GP is $\lambda_{max}y+(1-\lambda_{max})z=\frac{\lambda_{max}(1-p_1)+p_1}{p_1}z$. The expression for z in this area is

$$z=\frac{c}{\frac{\alpha(1-p_1^2)}{p_1^2}+\beta\frac{1-p_1}{p_1}-\lambda_{max}\frac{1-p_1}{p_1}}$$

with respect to λ_{min},the derivative of the denominator is same as above. With respect to λ_{max}, the derivative of z is zero. However since the expected payout in this regime is $\frac{\lambda_{max}(1-p_1)+p_1}{p_1}z$,it is increasing. Finally with respect to p_1,the denominator for $\frac{1-p_1}{p_1}z$, is $\frac{\alpha(1+p_1)}{p_1}+\beta-\lambda_{max}$. Derivative with respect to p_1 of this term is $-\frac{\alpha}{p_1^2}<0$ hence denominator is decreasing and the whole term is increasing. This finishes the argument.

Proof of Proposition 13

As we mentioned in the discussion proceeding the Proposition 13, in the optimal contract we have

$$\alpha x+\tilde{\beta}y=\max\{\theta_{max}y+c,p_1y+2c\}$$

By this, LP problem can be written as

$$\min_{x,y}\alpha x+\tilde{\beta}y$$
$$x\geq y\geq p_1x;\quad \alpha x+\tilde{\beta}y=\max\{\theta_{max}y+c,p_1y+2c\}$$

Similar to the proof of Proposition9, we consider two possible cases for y.

- If $y\leq\frac{c}{\theta_{max}-p_1}=\frac{c}{\lambda_{max}(1-p_1)}$, then we $p_1y+2c=\max\{\theta_{max}y+c,p_1y+2c\}$,hence one gets

  $$x=f(y)=\frac{(p_1-\tilde{\beta})y+2c}{\alpha}$$

  by $p_1y+2c=\alpha x+\tilde{\beta}y$. Therefore LP problem becomes

  $$\min y$$
  $$\min\{f(y),\frac{c}{\theta_m-p_1}\}\geq y\geq p_1f(y)$$

$y \geq p_1 f(y)$ implies

$$y \geq \frac{2p_1 c}{\alpha - p_1(p_1 - \tilde{\beta})} \tag{B.4}$$

$x \geq y$ implies

$$(\alpha - (p_1 - \tilde{\beta}))y \leq 2c$$

if $\alpha - (p - \tilde{\beta}) > 0$, then

$$y \leq \frac{2c}{(\alpha - (p_1 - \tilde{\beta}))}$$

otherwise always (B.4) is satisfied. Note that if $\alpha - (p - \tilde{\beta}) > 0$, then

$$\frac{2p_1 c}{\alpha - p_1(p_1 - \tilde{\beta})} \leq \frac{2c}{(\alpha - (p_1 - \tilde{\beta}))}$$

So there is answer satisfying $y \leq \frac{c}{\theta_{max} - p_1}$ if

$$\frac{2p_1 c}{\alpha - p_1(p_1 - \tilde{\beta})} \leq \frac{c}{\theta_{max} - p_1} \tag{B.5}$$

In which case $y = \frac{2p_1 c}{\alpha - p_1(p_1 - \tilde{\beta})}$ and $x = f(y) = \frac{y}{p_1}$. Equation (B.5) holds if

$$\alpha - p_1^2 + p_1\tilde{\beta} \geq 2\lambda_{max} p_1(1 - p_1)$$

which, after canceling and factoring $(1 - p_1)$,becomes

$$\lambda_{min}[(1 - p_1)\rho + p_1] \geq \lambda_{max} p_1$$

which is equivalent to

$$\rho \geq \frac{\lambda_{max} - \lambda_{min}}{\lambda_{min}(\frac{1}{p_1} - 1)}$$

- If $\rho < \frac{\lambda_{max} - \lambda_{min}}{\lambda_{min}(\frac{1}{p_1} - 1)}$, then the contract with $y \leq \frac{c}{\lambda_{max}(1 - p_1)}$ is not possible. In this case,

$$f(y) = x = \frac{(\theta_{max} - \tilde{\beta})y + c}{\alpha}$$

So LP should solve

$$\min y$$
$$f(y) \geq y \geq \max\{p_1 f(y), \frac{c}{\lambda_{max}(1 - p_1)}\}$$

$y \geq p_1 f(y)$ implies

$$y \geq \frac{p_1 c}{\alpha - p_1(\theta_{max} - \tilde{\beta})}$$

$x \geq y$ implies

$$(\alpha - (\theta_{max} - \tilde{\beta}))y \leq c$$

if $\alpha - (\theta_{max} - \tilde{\beta}) > 0$ this means

$$y \leq \frac{c}{(\alpha - (\theta - \tilde{\beta}))}$$

Otherwise always it is satisfied. In any case, in this region, the possible minimum for y is $\frac{p_1 c}{\alpha - p_1(\theta_{max} - \tilde{\beta})}$, in which case $x = \frac{y}{p_1}$.

Finally to show the equality $s^{FNO}(R+I) = s(R+I)$, note that $\tilde{\beta} = \beta + p_1(1-\alpha-\beta)$. Hence in the region $\rho < \frac{\lambda_{max} - \lambda_{min}}{\lambda_{min}(\frac{1}{p_1} - 1)}$, it is enough to show

$$\begin{aligned}
&\alpha(1 - p_1^2) + \beta p_1(1 - p_1) - \lambda_{max} p_1(1 - p_1) = \\
&\alpha - p_1(\lambda_{max} + p_1(1 - \lambda_{max})) + p_1(\beta + p_1(1 - \alpha - \beta))
\end{aligned}$$

which is correct after simplification. In the region $\rho \geq \frac{\lambda_{max} - \lambda_{min}}{\lambda_{min}(\frac{1}{p_1} - 1)}$ similar algebra works.

Proof of Proposition 14

We use the fact that $s^{FNO}(R + I) = s(R + I)$ which was proved in Proposition 13. I claim, expected payout to both types of GPs are the same under whole-portfolio contracting. For $\rho < \rho^*$ both are equal to $p_1 y + 2c = z + 2c$. In the region $\rho > \rho^*$, for non-reputable GP, expected payout is equal to

$$\begin{aligned}
&\theta_{max} y + c = \\
&\lambda_{max} y + (1 - \lambda_{max}) p_1 y + c
\end{aligned}$$

By Proposition 9 is the same as $\lambda_{max} y + (1 - \lambda_{max}) z + c$ which is the expected payout to reputable GP by the same proposition. Also as we saw by equation (2.16) and discussion after it, the expected payouts to both types of GP under deal-by-deal are the same as well. Now in the reputable case, both types of contracting induce the optimal investment strategy of exerting effort and invest only in good project. However for non-reputable GP, whole-portfolio contracting improves total payout of the projects compared to deal-by-deal as we saw in Subsection 2.4. Therefore it can only increase the profit of the LP compared to the reputable case since the expected payout to GP is the same for both types of GPs. This completes the argument.

Proof of Proposition 15

With given cost functions, in the deal-by-deal, the effort λ_1 is determined by

$$R - I = am\lambda_1^{m-1} + am(m-1)\lambda_1^{m-1} = am^2\lambda_1^{m-1}$$

so $\lambda_1 = \sqrt[m-1]{\frac{R-I}{am^2}}$ and the profit from first project is $\lambda_1^2 c"(\lambda_1) = a\lambda_1^2 m(m-1)\lambda_1^{m-2}$. For the second project we need only to change a to b. In the whole-portfolio case, using equations

$$\begin{aligned} bm\lambda_2^{m-1} &= c_2'(\lambda_2) = \lambda_1 s_{GP}(2R) \\ am\lambda_1^{m-1} &= c_1'(\lambda_1) = \lambda_2 s_{GP}(2R) \end{aligned}$$

we have $am\lambda_1^m = bm\lambda_2^m$ which gives $\lambda_2 = C\lambda_1$ where $C = \sqrt[m]{\frac{a}{b}}$. Then second equation above gives

$$\frac{a}{C} m\lambda_1^{m-2} = s_{GP}(2R)$$

Therefore LP problem can be written as ($\lambda_1 = \lambda$)

$$\max_{\lambda} C\lambda^2[2R - s_{GP}(2R)] + [\lambda(1 - C\lambda) + C\lambda(1-\lambda)](R + I) + (1-\lambda)(1 - C\lambda)2I - 2I$$

From above $C\lambda^2 s_{GP}(2R) = am\lambda^m$ and the profit is

$$\begin{aligned} &2C\lambda^2 R - am\lambda^m + \lambda R - C\lambda^2 R + \lambda I - C\lambda^2 I \\ &+ C\lambda R - C\lambda^2 R + C\lambda I - C\lambda^2 I - 2\lambda I - 2C\lambda I + 2C\lambda^2 I \end{aligned}$$

which simplifies to

$$(\lambda + C\lambda)(R - I) - am\lambda^m$$

FOC gives

$$(1 + C)(R - I) = am^2\lambda^{m-1}$$

so

$$\lambda_1 = \lambda = \sqrt[m-1]{\frac{(1+C)(R-I)}{am^2}}$$

Total profit by LP in this case can be written as

$$\lambda(1 + C)(R - I)[1 - \frac{1}{m}]$$

So in order to show that LP makes more money with whole-portfolio compared to deal-by-deal, we have to show

$$\sqrt[m-1]{\frac{(1+C)}{a}}(1 + C) > \sqrt[m-1]{\frac{1}{a}} + \sqrt[m-1]{\frac{1}{b}}$$

since $b = \frac{a}{C^m}$, this simplifies to show

$$\sqrt[m-1]{(1+C)}(1+C) > 1 + \sqrt[m-1]{C^m}$$

which is equivalent to (set $C = d^{m-1}$)

$$(1 + d^{m-1})^m > (1 + d^m)^{m-1}$$

Since the problem is symmetric with respect to a and b we can assume $a \leq b$ hence $d \leq 1$,which makes the inequality above trivial.

B.2 Robustness

Increasing Assumption

The security defined in Proposition 9 is increasing on the set of possible payouts with positive probability (i.e on equilibrium path). However if the payout of good project is not R for sure or GP makes a wrong decision, it is possible to get a payout of size R. To show robustness of our main result on the relation between correlation and security design, we have the following proposition.

Proposition 22 *If the condition $s(R) \geq s(2I)$ is imposed to the security $s_{GP} = s$, the result of Proposition 10 remains unchanged if $\frac{1}{4} \geq p_1 \geq p_2$.*

Proof.

Take $(z, w, x, y) = (s(2I), s(R), s(R + I), s(2R))$. Similar to the case of security with no restriction, in the optimum $s(I) = 0$. Also since R is not outcome of optimal investment strategy, it should be the lowest possible value such that the security remains increasing hence $s(R) = s(2I) = z$. With the same reason as discussed in subsection 2.3, to motivate for optimal investment strategy, contract should satisfy

$$z \geq p_1 p_2 x + [p_1(1 - p_2) + p_2(1 - p_1)]z, p_1 y$$
$$y \geq p_1 x + (1 - p_1)z$$
$$x \geq z, y$$

Also to motivate effort,

$$\rho\lambda_{min}x + [\lambda_1 - 2\rho\lambda_{min} + \lambda_2]y + [1 - \lambda_1 - \lambda_2 + \rho\lambda_{min}]z$$
$$\geq z + 2c, \lambda_{max}y + (1 - \lambda_{max})z + c$$

Therefore LP problem is

$$\min_{x,y,z} \alpha x + \beta y + \gamma z$$

subject to conditions above. Here as before, $(\alpha,\beta,\gamma)=(\rho\lambda_{min},\lambda_1\text{-}2\rho\lambda_{min}+\lambda_2,1\text{-}\lambda_1\text{-}\lambda_2+\rho\lambda_{min})$. With similar argument as in the main case, we have

$$\alpha x + \beta y + \gamma z = \max\{z + 2c, \lambda_{max} y + (1 - \lambda_{max})z + c\}$$
$$z = \max\{p_1 p_2 x + [p_1(1 - p_2) + p_2(1 - p_1)]z, p_1 y\}$$

Based on which terms becomes maximum on the RHS of the first equality, I consider the following two cases

1. First consider the case $z \leq \frac{p_1 c}{\lambda_{max}(1-p_1)}$. In this case the maximum payout will be $z + 2c$ so we have

   $$\alpha x + \beta y + \gamma z = z + 2c$$

   by this

   $$x = \frac{z(1 - \gamma) - \beta y + 2c}{\alpha}$$

   Now divide this case to two sub-cases.

   - $p_1 y = \max\{p_1 p_2 x + [p_1(1 - p_2) + p_2(1 - p_1)]z, p_1 y\} = z$. As the result

     $$x = \frac{z(1 - \gamma - \frac{\beta}{p_1}) + 2c}{\alpha}$$

     hence the problem for LP can be written as

     $$\min z$$
     $$x \geq y = \frac{z}{p_1}$$
     $$y \geq p_1 x + (1 - p_1)z$$
     $$z = p_1 y \geq p_1 p_2 x + [p_1(1 - p_2) + p_2(1 - p_1)]z$$

     The last two inequalities can be written as

     $$z \geq \frac{p_1^2}{1 - p_1 + p_1^2} x = \theta_1 x$$
     $$z \geq \frac{p_1 p_2}{1 - p_1 + 2p_1 p_2 - p_2} x = \theta_2 x$$

RHS of the second inequality is increasing in p_2. So if $p_2 \leq p^*$, for some $p^* \geq 0$, the first inequality is effective otherwise the second one is. In any case, we get

$$\alpha z/\theta_i \geq z(1-\gamma-\frac{\beta}{p_1})+2c$$

so for optimal z we have

$$z=\frac{2c}{\beta\frac{1-p_1}{p_1}+\alpha\frac{1-\theta_{max}}{\theta_{max}}}$$

which is similar to the formula as in Proposition 9. When comparing with the initial inequality, we get

$$\alpha\frac{1-\theta_i}{\theta_i}+\beta\frac{1-p_1}{p_1} \geq \frac{2\lambda_{max}(1-p_1)}{p_1}$$

so similar to the main case, if we have

$$\alpha\frac{1-\theta_i}{\theta_i}+\beta\frac{1-p_1}{p_1}$$

is increasing in ρ, then we have shown the proposition. The derivative with respect to ρ is

$$\lambda_{min}(\frac{1-\theta_i}{\theta_i}-2\frac{1-p_1}{p_1})$$

Maximum value for θ_i is when $p_1 = p_2$ and for this case by $p_1 \leq \frac{1}{4}$ we get $\frac{1-\theta_i}{\theta_i} \geq 2\frac{1-p_1}{p_1}$ hence the derivative above is always positive. This shows that payout to LP is increasing in ρ in this case.

- $p_1p_2x + [p_1(1-p_2)+p_2(1-p_1)]z = \max\{p_1p_2x + [p_1(1-p_2)+p_2(1-p_1)]z, p_1y\} = z$. From this we get

$$x=\frac{1-[p_1(1-p_2)+p_2(1-p_1)]}{p_1p_2}z=qz$$

on the other hand, we have

$$x=\frac{z(1-\gamma)-\beta y+2c}{\alpha}$$

so we have

$$\beta y = 2c - \alpha qz + z(1-\gamma)$$

which gives y in terms of z. So LP problem is

$$\min z$$

where $p_1 y \leq z$ and $y \geq p_1 x + (1-p_1)z = (p_1 q + 1 - p_1)z$. If the coefficient in the later inequality is bigger than $\frac{1}{p_1}$,there is no possible solution. Otherwise, when insert y from equality above in the inequality $p_1 y \leq z$, in the optimal it binds hence $p_1 y = z$ and problem reduces to the previous case.

2. Now consider the case $z \geq \frac{p_1 c}{\lambda_{max}(1-p_1)}$. As saw above this happens for small ρ. In this case we have

$$\begin{aligned}&\alpha x + \beta y + \gamma z = \max\{z + 2c, \lambda_{max} y + (1-\lambda_{max})z + c\}\\&= \lambda_{max} y + (1-\lambda_{max})z + c\end{aligned}$$

like in the previous case, we consider two sub-cases.

- $p_1 y = \max\{p_1 p_2 x + [p_1(1-p_2) + p_2(1-p_1)]z, p_1 y\} = z$. Then the equation for expected payout can be written as

$$x = \frac{(\frac{\lambda_{max}}{p_1} + 1 - \lambda_{max} - \frac{\beta}{p_1} - \gamma)z + c}{\alpha}$$

similar to the previous case, LP problem becomes

$$\begin{aligned}&\min z\\&x \geq y = \frac{z}{p_1}\\&y \geq p_1 x + (1-p_1)z\\&z = p_1 y \geq p_1 p_2 x + [p_1(1-p_2) + p_2(1-p_1)]z\end{aligned}$$

$p_1 x \geq z$ implies that

$$p_1 c \geq (\alpha + \beta - \lambda_{max})(1-p_1)z$$

which in turns implies that

$$z \leq \frac{p_1 c}{(1-p_1)\lambda_{min}(1-\rho)}$$

similar to the previous case, we can write the last two inequalities as

$$\begin{aligned}z &\geq \frac{p_1^2}{1-p_1+p_1^2}x = \theta_1 x\\z &\geq \frac{p_1 p_2}{1-p_1+2p_1p_2-p_2}x = \theta_2 x\end{aligned}$$

In any case this gives us

$$\alpha z/\theta_i \geq (\frac{\lambda_{max}}{p_1} + 1 - \lambda_{max} - \frac{\beta}{p_1} - \gamma)z + c$$

which gives us the optimal z as the bigger term of two inequalities above is binding. Hence we have

$$z = \frac{c}{\frac{\alpha(1-\theta_{max})}{\theta_{max}} + \beta\frac{1-p_1}{p_1} - \lambda_{max}\frac{1-p_1}{p_1}}$$

Again this is similar to the formula we have as in Proposition 9 and with the same reason as above z is decreasing in ρ which shows our claim in this case.

- $p_1p_2x + [p_1(1-p_2) + p_2(1-p_1)]z = \max\{p_1p_2x + [p_1(1-p_2) + p_2(1-p_1)]z, p_1y\} = z$. As previous case, this gives

$$x = \frac{1 - [p_1(1-p_2) + p_2(1-p_1)]}{p_1p_2}z = qz$$

On the other hand we have

$$\alpha x + \beta y + \gamma z = \lambda_{max}y + (1-\lambda_{max})z + c$$

which gives

$$(\beta - \lambda_{max})y = (1 - \lambda_{max} - \gamma - \alpha q)z + c$$

so LP problem is

$$\min z$$

$p_1y \leq z$ and $y \geq p_1x + (1-p_1)z = (p_1q + 1 - p_1)z$. Similar to the case we studied before either this does not have solution or we get $p_1y = z$ in the optimum hence it reduces to the previous case. This finishes the argument.

■

Return Distribution

Here I want to generalize the distribution function for the projects. Let's assume the support of both types of projects G and B are $0, R_1, R_2$. For the G type the chances are $\{0, p, 1-p\}$ respectively and for the B type it is $\{1-p_1-p_2, p_1, p_2\}$. All the other variables, definitions and assumptions are the same as in the main model. First want to see how contract on one project is written. In order to persuade the optimal investment strategy (not investing on bad project), we should have

$$s_{GP}(I) \geq p_1 s_{GP}(R_1) + p_2 s_{GP}(R_2) \tag{B.6}$$

Also in order to motivate effort, we have

$$E[s_{GP}(G)] = p s_{GP}(R_1) + (1-p) s_{GP}(R_2) \geq s_{GP}(I) + \frac{c}{\lambda} \tag{B.7}$$

In the optimal, with the same reasoning as in the binary case, both these inequalities are binding to minimize the expected payout to GP. LP problem is

$$\min_{s_{GP}(R_1), s_{GP}(R_2)} \lambda E[s_{GP}(G)] + (1-\lambda) s_{GP}(I)$$

Conditioned to equations (equities in optimum) (B.6) and (B.7). Because of optimality, LP problem can be written as

$$\min_{x,y} [\lambda p + (1-\lambda) p_1] x + [\lambda(1-p) + (1-\lambda) p_2] y$$

where $(x, y) = (s_{GP}(R_1), s_{GP}(R_2))$. The relation between x and y comes from (B.7) above which can be written as $x = \gamma y + \zeta$ where

$$\begin{aligned} \gamma &= -\frac{1-p-p_2}{p-p_1} \\ \zeta &= \frac{c}{\lambda} \end{aligned} \tag{B.8}$$

So the minimization problem is linear in y and hence in the optimum either we have $x = 0$ or $y = 0$ as none of payouts can be negative. More precisely, the coefficient of y in the LP problem is

$$-[\lambda p + (1-\lambda) p_1] \frac{1-p-p_2}{p-p_1} + [\lambda(1-p) + (1-\lambda) p_2]$$

if this coefficient is positive then we should have $y = 0$ otherwise $x = 0$. Whichever happens, we get the value of the other variable from equation $x = \gamma y + \zeta$ above. If the payout of the projects has n different values, the same conclusion holds as the problem is linear in payouts. In summary we have

Proposition 23 *With setup as above, the optimal contract satisfies either $s(R_1) = 0$ or $s(R_2) = 0$. In particular $s(R_2) = 0$ if and only if*

$$[\lambda(1-p) + (1-\lambda)p_2](p - p_1) > [\lambda p + (1-\lambda)p_1](1 - p - p_2)$$

The other one is computed by the equation $x = \gamma y + \zeta$, where γ, ζ are given in equations (B.8). Finally $s(I)$ is computed from the equation (B.6) when it is equality.

Let's just explain the intuition behind the property that either $s(R_1) = 0$ or $s(R_2) = 0$. LP wants to minimize the incentive for the GP to invest in a bad project. To do this, LP considers the (weighted) difference of probabilities that either the outcome is R_1 or R_2. Then he makes no payment in the state with lower chance. The other outcome associates to higher chance of investing in the good project so GP motivates it in the contract.

Now I look at the whole-portfolio problem. Again we assume that the policy implemented is optimal so we have $s(I) = s(R_1) = s(R_2) = 0$. Possible returns from optimal strategy are $2I$ when two projects are bad, $R_i + I$ when one is good and other is bad and finally $2R_i$ or $R_1 + R_2$ when both are good. I assume parameters for the first projects are λ_1, p, p_1 and p_2. For the second one λ_2, q, q_1 and q_2. I show the payouts to GP by $z = s(2I), y_i = s(R_i + I), x_i = s(2R_i)$ and $x = s(R_1 + R_2)$. I assume correlation ρ between good projects as in the main case and assume that for good projects the realization of the returns are independent. We have

$$\begin{aligned}
&s(2I) \geq \sum p_i s(R_i + I), \sum q_i s(R_i + I), \sum p_i q_j s(R_i + R_j)\\
&ps(R_1 + I) + (1-p)s(R_2 + I) = E_{G_1}[s(R_i + I)] \geq\\
&s(2I), [pq_2 + (1-p)q_1]s(R_1 + R_2) + pq_1 s(2R_1) + (1-p)q_2 s(2R_2)\\
&qs(R_1 + I) + (1-q)s(R_2 + I) = E_{G_2}[s(R_i + I)] \geq\\
&s(2I), [qp_2 + (1-q)p_1]s(R_1 + R_2) + qp_1 s(2R_1) + (1-q)p_2 s(2R_2)\\
&[p(1-q) + q(1-p)]s(R_1 + R_2) + pqs(2R_1) + (1-p)(1-q)s(2R_2)\\
&\ \geq E_{G_j}[s(R_i + I)]
\end{aligned} \tag{B.9}$$

And finally LP should impose the equation which motivates effort. This can be written as

$$\begin{aligned}
&\rho\lambda_{min}E_{G_1,G_2}[s(R_i + R_j)] + (\lambda_{max} - \rho\lambda_{min})E_{G_{max}}[s(R_i + I)]\\
&+ (1-\rho)\lambda_{min}E_{G_{min}}[s(R_i + I)] + (1 - \lambda_1 - \lambda_2 + \rho\lambda_{min})s(2I)\\
&\geq z + 2I, \lambda_{max}E_{G_{max}}[s(R_i + I)] + (1-\lambda_{max})z + c, \lambda_{min}E_{G_{min}}[s(R_i + I)] + (1-\lambda_{min})z + c
\end{aligned}$$

As in the binary case, LP wants to minimize the expected payout to GP (LHS of the last inequality) given constraints above. Similar to the binary case, we can see that $E_{G_1,G_2}[s(R_i + R_j)]$ can be represented with an inequality. So LP problem can be written as

$$\min E_{G_1,G_2}[s(R_i + R_j)]$$

subject to conditions for optimal investment and motivation for effort. Since the general problem seems hard to solve and get good intuition from, I stick to two important especial cases.

1. Suppose either p_1 and q_1 are small or p_2 and q_2 are small. In this case, suppose LP changes compensations for GP in the case of two successful investment, while expected payout remains the same. By this I mean changing x_i and x such that

   $$[p(1-q) + q(1-p)]x + pqx_1 + (1-p)(1-q)x_2$$

   remains fixed. By this change, RHS of the equations for optimal investment in case of one success or no success can be changed (The first three equation in the set of equations (B.9)) . As long as RHS becomes smaller in these equations, the change can be good (or have no effect if conditions are not binding on them). So it is better for GP to consider payouts to minimize RHS of motivating equations. So in this case, in the optimal contract, we get only $x_i > 0$ (and x_j, x are zero) when p_i and q_i are small. Similar reasoning implies only $y_i > 0$ and $y_j = 0$ when p_i and q_i are small. Hence in this case problem reduces effectively to the binary case.

2. Now consider the orthogonal problem to what we discussed in the previous part. So I assume $p_2 = q_1 = 0$ so the first bad project only have return R_1 and the second one only R_2. Equations (B.9) are reduced to

   $$\begin{aligned}
   &s(2I) \geq p_1 s(R_1 + I), q_2 s(R_2 + I), p_1 q_2 s(R_1 + R_2)\\
   &ps(R_1 + I) + (1-p)s(R_2 + I) = E_{G_1}[s(R_i + I)] \geq\\
   &s(2I), pq_2 s(R_1 + R_2) + (1-p)q_2 s(2R_2)\\
   &qs(R_1 + I) + (1-q)s(R_2 + I) = E_{G_2}[s(R_i + I)] \geq\\
   &s(2I), (1-q)p_1 s(R_1 + R_2) + qp_1 s(2R_1)\\
   &[p(1-q) + q(1-p)]s(R_1 + R_2) + pqs(2R_1) + (1-p)(1-q)s(2R_2)\\
   &\geq E_{G_j}[s(R_i + I)]
   \end{aligned}$$

   In this case if we get $s(2I) = z$, then similar reasoning as in binary case,

   $$\begin{aligned}
   z &= p_1 s(R_1 + I) = p_1^2 s(2R_1)\\
   z &= q_2 s(R_2 + I) = q_2^2 s(2R_2)\\
   z &= p_1 q_2 s(R_1 + R_2)
   \end{aligned}$$

which implies that , as in the main case, $E[s_{GP}]$ is decreasing in ρ as well.