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# SUBCRITICAL $L^p$ BOUNDS ON SPECTRAL CLUSTERS FOR LIPSCHITZ METRICS

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ABSTRACT. We establish asymptotic bounds on the  $L^p$  norms of spectrally localized functions in the case of two-dimensional Dirichlet forms with coefficients of Lipschitz regularity. These bounds are new for the range  $6 < p < \infty$ . A key step in the proof is bounding the rate at which energy spreads for solutions to hyperbolic equations with Lipschitz coefficients.

## 1. INTRODUCTION

The purpose of this paper is to establish  $L^p$  bounds on eigenfunctions, or more generally spectrally localized functions, associated to Dirichlet forms on a compact manifold. The question of interest is the dependance of the bounds on the Hölder regularity of the coefficients of the form. We consider here the case of Dirichlet forms with Lipschitz coefficients for simplicity, but the proofs can be adapted to the case of  $C^s$  coefficients, where  $0 < s < 2$ . Our work is restricted to the case of two-dimensional manifolds, however.

Consider the eigenvalue problem for a Dirichlet form, where we work on a compact manifold  $M$  without boundary,

$$d^*(a d\phi) + \lambda^2 \rho \phi = 0.$$

Here,  $a$  is a section of real, symmetric quadratic forms on  $T^*(M)$ , with associated linear transforms  $a_x : T_x^*(M) \rightarrow T_x(M)$ , and  $\rho$  is a real valued function on  $M$ . Here,  $d^*$  denotes the adjoint of  $d$  relative to a fixed volume form  $dx$ . We assume both  $a$  and  $\rho$  are strictly positive, with uniform bounds above and below. We note that this setting includes the Laplace-Beltrami operator on a Riemannian manifold. The parameter  $\lambda \geq 0$  is referred to as the frequency of the eigenfunction  $\phi$ .

A *spectral cluster* of frequency  $\lambda$  is a combination of eigenfunctions with frequencies in the range  $[\lambda - 1, \lambda]$ . In the case that  $a$  and  $\rho$  are smooth, Sogge [9] established the following best possible  $L^p$  bounds on spectral clusters,

$$(1) \quad \|f\|_{L^p(M)} \lesssim \begin{cases} \lambda^{\frac{n-1}{2}(\frac{1}{2} - \frac{1}{p})} \|f\|_{L^2(M)}, & 2 \leq p \leq p_n \\ \lambda^{n(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}} \|f\|_{L^2(M)}, & p_n \leq p \leq \infty \end{cases}$$

The critical index is  $p_n = \frac{2(n+1)}{n-1}$ . Semiclassical generalizations were obtained by Koch-Tataru-Zworski [4]. The bounds (1) hold in case  $a$  and  $\rho$  are of regularity  $C^{1,1}$  by [5], but

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based on an observation of Grieser [2], and examples of Smith-Sogge [7] and the authors [3], they fail for coefficients of  $C^s$  regularity if  $s < 2$ .

For metrics of regularity  $C^s$  with  $s < 2$  (or Lipschitz in case  $s = 1$ ) best possible  $L^p$  bounds on spectral clusters have been established on the range  $2 \leq p \leq p_n$ , as well as for  $p = \infty$ ; see [6] for the case  $1 \leq s < 2$ , and [3] for the case  $s < 1$ . This leaves open the *subcritical* case  $p_n < p < \infty$ , where the upper bounds on the exponent of  $\lambda$  that can be obtained from [3] and [6] by interpolation do not match the lower bounds that follow from the examples of [3] and [7].

In this paper we obtain bounds for  $p_n < p < \infty$ , for Lipschitz coefficients and  $n = 2$ , which improve upon the results of [6]. They do not match the exponent displayed by the Rayleigh whispering mode example noted in [2], but the difference is exponentially small as  $p \rightarrow \infty$ . Our results are restricted to  $n = 2$ , but all steps adapt to  $C^s$  coefficients for  $0 < s < 2$ , and improve upon [3] and [6] on this range of  $p$ .

Thus, consider a Dirichlet form on a two-dimensional compact manifold without boundary, with  $a$  and  $\rho$  of Lipschitz regularity. Let

$$\gamma(p) = 2\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}$$

be the exponent occurring in the subcritical estimates (1).

By Theorem 2 of [6], in this case *no-loss estimates* hold on cubes  $Q$  of sidelength  $\lambda^{-\frac{1}{3}}$ ,

$$(2) \quad \|f\|_{L^p(Q)} \lesssim \lambda^{\gamma(p)} \|f\|_{L^2(M)}, \quad 6 \leq p \leq \infty.$$

The Rayleigh whispering mode examples show that if  $p = 6$  the size of  $Q$  cannot be increased without increasing the exponent. The main result of this paper is that, for larger  $p$ , the following *log-loss estimates* hold on cubes  $Q$  of sidelength  $\lambda^{-\frac{1}{3}2^{(6-p)/2}}$

$$(3) \quad \|f\|_{L^p(Q)} \lesssim (\log \lambda)^{p-6} \lambda^{\gamma(p)} \|f\|_{L^2(M)}, \quad p = 6, 8, 10, 12, \dots$$

If  $f$  is conically microlocalized in frequency, then (see [6, (14)-(15)])  $Q$  can be replaced by a thin slab of size  $1 \times \lambda^{-\frac{1}{3}2^{(6-p)/2}}$ . Summing over such slabs, one obtains the following.

**Theorem 1.** *Suppose that  $a$  and  $\rho$  are of Lipschitz regularity, on a two-dimensional compact manifold without boundary. Then, for  $p = 6, 8, 10, 12, \dots$*

$$(4) \quad \|f\|_{L^p(M)} \lesssim (\log \lambda)^{p-6} \lambda^{\sigma(p)} \|f\|_{L^2(M)}, \quad \sigma(p) = \gamma(p) + \frac{1}{3p} 2^{\frac{6-p}{2}}.$$

To place this result in context, the Rayleigh mode and a reflection argument shows that, for Lipschitz coefficients and  $n = 2$ , one cannot establish better estimates than the following

$$(5) \quad \|f\|_{L^p(M)} \lesssim \begin{cases} \lambda^{\frac{2}{3}(\frac{1}{2} - \frac{1}{p})} \|f\|_{L^2(M)}, & 2 \leq p \leq 8 \\ \lambda^{\gamma(p)} \|f\|_{L^2(M)}, & 8 \leq p \leq \infty \end{cases}$$

The results of [6] imply that the bounds (5) hold for general Lipschitz coefficients for the cases  $2 \leq p \leq 6$  and  $p = \infty$ . The exponent  $\sigma(p)$  in (4), which agrees with that of (5) for  $p = 6$ , misses (5) for  $6 < p < \infty$  by a factor which decays exponentially as  $p \rightarrow \infty$ .

We remark that the bounds (5) were established in [8] for smooth Dirichlet forms on two-dimensional manifolds with boundary, with either Dirichlet or Neumann conditions

at the boundary. A manifold with boundary can be thought of as a special case of a Lipschitz metric, by reflecting coefficients normally across the boundary. The example of [7] for Lipschitz metrics and  $n = 2$  is generated by reflecting a Rayleigh whispering mode from the unit disc.

The proof of (3) is inductive. The estimate for  $p + 2$  is derived from the estimate for  $p$ , together with an almost orthogonal decomposition of  $f$  into tubular pieces. Essentially, one can localize  $f$  in frequency to a cone of angle  $\delta$ , and in space to a tube of diameter  $\delta^2$ , and control the energy flow over distance  $\delta$ . For this reason, the diameter of the log-loss cubes for  $p + 2$  is the square root of the diameter of the log-loss cubes for  $p$ . The argument that allows summation over different tubes with a  $(\log \lambda)^2$  loss works only for  $n = 2$ , however. Improving Theorem 1 appears then to hinge on controlling energy flow over longer distances, and improving the summation argument to allow  $n \geq 3$ .

The bounds we establish hold more generally for functions that satisfy a quasimode condition

$$(6) \quad d^*(a df) + \lambda^2 \rho f = d^*g_1 + g_2.$$

If  $f$  is a spectral cluster, then (6) holds on  $M$  with  $g_1 = 0$  and  $\|g_2\|_{L^2} \lesssim \lambda \|f\|_{L^2}$ . Allowing the term  $g_1$  makes localization arguments simpler. In particular (6) holds, with  $\|g_1\|_{L^2} \lesssim \|f\|_{L^2}$  and  $\|g_2\|_{L^2} \lesssim \lambda \|f\|_{L^2}$ , for the product of a spectral cluster  $f$  with a unit size bump function, so we may assume that we work in local coordinates. After rescaling and extending, we may assume that  $a$  and  $\rho$  are globally close on  $\mathbb{R}^2$  to the flat metric,

$$\|a^{ij} - \delta^{ij}\|_{Lip(\mathbb{R}^2)} + \|\rho - 1\|_{Lip(\mathbb{R}^2)} \leq c_0.$$

We establish the estimate (3) by an induction argument, for which the starting point is the localized version of the no-loss estimate (2) for  $p = 6$ . At each step of the induction  $p$  increases by 2, and we establish estimates on cubes of square-root the sidelength of the previous step. A loss of  $(\log \lambda)^2$  is incurred at each step, however. The hypothesis and induction argument are as follows.

**Hypothesis 2.** *Suppose that equation (6) holds on  $Q^*$ , where  $Q^*$  denotes the double of the cube  $Q$ . Then the following inequality holds, where  $\ell(Q)$  denotes the sidelength of  $Q$*

$$\begin{aligned} \|f\|_{L^p(Q)} \leq C_p (\log \lambda)^{p-6} \lambda^{\gamma(p)} & \left( \ell(Q)^{-\frac{1}{2}} \|f\|_{L^2(Q^*)} + \lambda^{-1} \ell(Q)^{-\frac{1}{2}} \|df\|_{L^2(Q^*)} \right. \\ & \left. + \ell(Q)^{\frac{1}{2}} \|g_1\|_{L^2(Q^*)} + \lambda^{-1} \ell(Q)^{\frac{1}{2}} \|g_2\|_{L^2(Q^*)} \right). \end{aligned}$$

**Theorem 3.** *Assume that Hypothesis 2 holds for a given  $p \in [6, \infty)$ , uniformly over cubes  $Q$  of a given sidelength  $\ell(Q) = \delta^2$ , where  $1 \geq \delta \geq \lambda^{-\frac{1}{6}}$ . Then Hypothesis 2 holds with  $p$  replaced by  $p + 2$ , uniformly over cubes  $Q$  of sidelength  $\ell(Q) = \delta$ .*

We remark that the norm on the right hand side in Hypothesis 2 should be thought of as the  $L^2$ -energy of  $f$  on  $Q^*$ . If the functions involved are localized to frequencies  $\xi$  of magnitude  $\lambda$ , and  $\xi$  in a small cone about the  $\xi_1$  axis, then the right hand side is a replacement for  $\|f\|_{L_{x_1}^\infty L_{x_2}^2} + \|Pf\|_{L_{x_1}^1 L_{x_2}^\infty}$ .

The outline of this paper is as follows. In Section 2, we establish the key decomposition of  $f$  as a sum of terms, each supported in a thin geodesic tube of dimensions  $\delta \times \delta^2$ . This decomposition at multiple scales is inspired by the work of Geba-Tataru [1]. In Section

3 we establish  $\ell^q$  bounds on the overlaps of collections of such tubes, which is applied in Section 4 to complete the proof.

For the remainder of this section we carry out some simple initial reductions. Consider a cube  $Q_0$  of sidelength  $\delta$ . Let  $\psi$  be a scaled bump function, supported in  $Q_0^*$  and equal to 1 on  $Q_0$ . Then Hypothesis 2 with  $\ell(Q) = \delta$  is unchanged if we replace  $f$  by  $\psi f$ , hence we may assume  $f$  is supported in  $Q_0^*$ , and in particular use  $\|f\|_{L^2}$  instead of  $\|f\|_{L^2(Q_0^*)}$ .

Next, we split  $f$  into components  $f = f_{<\lambda} + f_\lambda + f_{>\lambda}$ , by localizing respectively to frequencies smaller than  $c^2\lambda$ , comparable to  $\lambda$ , and larger than  $c^{-2}\lambda$ , where  $c$  is a fixed small constant. By the arguments of [6, Corollary 5],

$$\lambda \|f_{<\lambda}\|_{L^2} + \|df_{>\lambda}\|_{L^2} \lesssim \|f\|_{L^2} + \lambda^{-1} \|df\|_{L^2} + \|g_1\|_{L^2} + \lambda^{-1} \|g_2\|_{L^2}.$$

Since  $\lambda^{2(\frac{1}{2}-\frac{1}{p})-1} \leq \lambda^{\gamma(p)} \ell(Q_0)^{\frac{1}{2}}$ , Sobolev embedding yields that Hypothesis 2 holds with  $\|f_{<\lambda}\|_{L^p}$  and  $\|f_{>\lambda}\|_{L^p}$  on the left hand side. Thus we restrict attention to the case that  $f$  is frequency localized to  $|\xi| \approx \lambda$ . By writing  $f_\lambda$  as a finite sum of terms, we may also assume that  $f$  is frequency localized to  $|\xi_2| \leq c\lambda$ .

Define the operator

$$P_{\lambda\delta} = d^* a_{\lambda\delta} d + \lambda^2 \rho_{\lambda\delta},$$

where the coefficients  $a_{\lambda\delta}$  and  $\rho_{\lambda\delta}$  are smoothly truncated in frequency to  $|\xi| \leq c\lambda\delta$ .

Provided  $\ell(Q) \leq \delta$ , then Hypothesis 2 is unchanged if we replace the defining equation by  $P_{\lambda\delta} f = d^* g_1 + g_2$ , since the difference  $(P - P_{\lambda\delta})f$  can be absorbed into  $g_1$  and  $g_2$ , leaving the right hand side of the inequality unchanged up to a constant.

Given a cube  $Q$  and parameters  $\delta, \lambda$ , we set

$$\|f\|_{\lambda,\delta,Q} = \delta^{-\frac{1}{2}} \|f\|_{L^2(Q)} + \lambda^{-1} \delta^{-\frac{1}{2}} \|df\|_{L^2(Q)} + \lambda^{-1} \delta^{\frac{1}{2}} \|P_{\lambda\delta} f\|_{L^2(Q)}.$$

We use  $\|f\|_{\lambda,\delta}$  to denote the norm in case  $Q = \mathbb{R}^2$ .

Since  $f$  is frequency localized to  $|\xi| \approx \lambda$ , as is  $P_{\lambda\delta} f$ , we may absorb the term  $d^* g_1$  into  $g_2$ . Thus, by the preceding comments, we are reduced to the following.

**Theorem 4.** *Suppose that  $f$  is frequency localized to  $|\xi| \approx \lambda$  and  $|\xi_2| \leq c\lambda$ . Then the following holds, uniformly on cubes  $Q$  of sidelength  $\delta$ ,*

$$\|f\|_{L^{p+2}(Q)} \lesssim (\log \lambda)^{p-4} \lambda^{\gamma(p+2)} \|f\|_{\lambda,\delta}$$

*under the assumption that the following holds, uniformly on cubes  $Q$  of sidelength  $\delta^2$ ,*

$$\|f\|_{L^p(Q)} \lesssim (\log \lambda)^{p-6} \lambda^{\gamma(p)} \|f\|_{\lambda,\delta^2,Q^*}$$

## 2. THE TUBE DECOMPOSITION

Let  $f$  be as in Theorem 4, and fix a cube  $Q_0$  of sidelength  $\delta$  and center  $x_0$ . As above, let  $\psi = 1$  on  $Q_0$  and vanish outside  $Q_0^*$ . In this section we produce a decomposition

$$(7) \quad \psi f = \sum_{T \in \mathcal{T}} f_T + f_0,$$

where  $f_0$  is an error term whose  $L^p$  norms can be appropriately bounded by Sobolev embedding. Each  $f_T$  is compactly supported in a tube  $T$ . The index  $T$  varies over a collection  $\mathcal{T}$  of tubes of diameter  $\delta^2$  and length  $\delta$ , associated to bicharacteristic directions

of  $P_{\lambda\delta}$  at angular separation  $\delta$ . Each  $\widehat{f}_T$  is concentrated (in a weighted  $L^2$  sense) in a ball of diameter  $\lambda\delta$ . We further have the bounds,

$$(8) \quad \left( \sum_T \|f_T\|_{\lambda,\delta}^2 \right)^{\frac{1}{2}} \leq C \|f\|_{\lambda,\delta}.$$

Let  $\Gamma$  be the characteristic set of  $P_{\lambda\delta}$  which lies near  $Q_0^* \times \text{support}(\widehat{f})$ ,

$$\Gamma = \{(x, \xi) : \langle a_{\lambda\delta}(x) \xi, \xi \rangle = \lambda^2 \rho_{\lambda\delta}(x)\} \cap Q_0^* \times \{|\xi_2| \leq 2c\lambda\}.$$

Since  $a_{\lambda\delta}$  and  $\rho_{\lambda\delta}$  are pointwise close to the flat metric, the set  $\Gamma$  can be realized as the union of two graphs  $\xi_1 = \gamma_{\pm}(\xi_2)$ . Since  $a_{\lambda\delta}$  and  $\rho_{\lambda\delta}$  are Lipschitz and  $|x - x_0| \leq \delta$ , the characteristic set  $\Gamma$  is contained in a  $\lambda\delta$  neighborhood of  $\Gamma_{x_0}$ . Let  $q_j(\xi)$  be a finite-overlap cover of  $\Gamma$  by  $\approx \delta^{-1}$  smooth bump functions, each supported in a ball of diameter  $\approx \lambda\delta$  centered on  $\Gamma_{x_0}$ , so that  $\phi(\xi) = 1 - \sum_j q_j(\xi)$  vanishes on a  $\lambda\delta$  size neighborhood of  $\Gamma$ . Thus,  $\phi(\xi)P_{\lambda\delta}(x, \xi)^{-1} \lesssim (\lambda^2\delta)^{-1}$  near  $Q_0^* \times \text{support}(\widehat{f})$ . Set

$$\psi f = \sum_j \psi(x) q_j(D) f + \psi(x) \phi(D) f.$$

Let  $q(x, \xi) = \psi(x) \phi(\xi) P_{\lambda\delta}(x, \xi)^{-1}$  smoothly localized in  $\xi$  to  $\{|\xi_1| \approx \lambda, |\xi_2| \leq 2c\lambda\}$ . Then

$$\psi(x) \phi(D) f = q(x, D) P_{\lambda\delta} f + r(x, D) f,$$

where  $r$  is of size  $\lambda^{-1}\delta^{-2}$ . Precisely,  $q$  and  $r$  are supported where  $|\xi| \approx \lambda$ , and

$$\lambda^2 \delta q(\lambda^{-1}\delta^{-1}x, \lambda\delta\xi), \quad \lambda \delta^2 r(\lambda^{-1}\delta^{-1}x, \lambda\delta\xi) \in S_{0,0}^0.$$

It follows that

$$\|\psi(x) \phi(D) f\|_{H^1} \lesssim \delta^{-\frac{3}{2}} \|f\|_{\lambda,\delta} \lesssim \lambda^{\frac{1}{4}} \|f\|_{\lambda,\delta}.$$

Since  $p \geq 6$  then  $\frac{1}{4} \leq \gamma(p+2)$ , and Sobolev embedding yields

$$\|\psi(x) \phi(D) f\|_{L^{p+2}} \lesssim \lambda^{\gamma(p+2)} \|f\|_{\lambda,\delta},$$

hence we may take  $\psi(x) \phi(D) f$  as the term  $f_0$ .

For each fixed  $j$ , consider the term  $\psi(x) q_j(D) f$ , and let  $\xi_j$  be the center of the support of  $q_j(\xi)$ . We can assume that the angular separation satisfies  $\angle(\xi_i, \xi_j) \gtrsim \delta |i - j|$ .

Let  $V_j$  denote the vector

$$V_j = a_{\lambda\delta}(x_0) \xi_j.$$

Take a partition of unity in the  $x_2$  variable, subordinate to a cover by intervals of length  $\delta^2$ , and extend it to a partition of unity in  $(x_1, x_2)$  which is translation invariant under  $V_j$ , then multiply by  $\psi$ . We denote the elements by  $\psi_T(x) : T \in \mathcal{T}_j$ , so that  $\psi = \sum_{T \in \mathcal{T}_j} \psi_T$ . With  $f_T = \psi_T q_j(D) f$ , and  $\mathcal{T} = \cup_j \mathcal{T}_j$ , we have the decomposition (7).

Each  $T = \text{support}(\psi_T)$  is contained in a tube of width  $\delta^2$ , length  $\delta$ , and direction  $V_j$ , where  $V_j$  lies within a small angle of the  $x_1$  axis. We also have

$$(9) \quad |(V_j \cdot d)^k \psi_T| \lesssim \lambda^k \delta^{-k}, \quad |\partial_x^\alpha \psi_T| \lesssim \delta^{-2|\alpha|} \leq \delta^{-2} (\lambda\delta)^{|\alpha|-1} \quad \text{if } |\alpha| \geq 1.$$

We then expand  $P_{\lambda\delta}(\psi_T q_j(D) f)$  as

$$(10) \quad (d^* a_{\lambda\delta} d \psi_T) q_j(D) f + 2 \langle a_{\lambda\delta} d \psi_T, d(q_j(D) f) \rangle + \psi_T [P_{\lambda\delta}, q_j(D)] f + \psi_T q_j(D) P_{\lambda\delta} f,$$

and seek to show that

$$\sum_j \sum_{T \in \mathcal{T}_j} \lambda^{-2} \delta \|P_{\lambda\delta}(\psi_T q_j(D)f)\|_{L^2}^2 \lesssim \|f\|_{\lambda,\delta}^2.$$

For the first term in (10), this follows by the finite overlap of the  $\psi_T$  for  $T \in \mathcal{T}_j$  and the finite overlap of the  $q_j(\xi)$ , together with the pointwise bounds (9). The fourth term is similarly handled by the finite overlap properties.

For the third term in (10), we have the simple commutator bounds

$$\|[P_{\lambda\delta}, q_j(D)]\|_{L^2 \rightarrow L^2} \leq \lambda^2 (\lambda\delta)^{-1} = \lambda\delta^{-1}.$$

Additionally, by the frequency localization of  $a_{\lambda\delta}$  and  $\rho_{\lambda\delta}$ , the commutators have finite overlap as  $j$  varies, yielding square summability over  $j$ .

We expand the brackets in the second term in (10) as

$$\begin{aligned} \langle (a_{\lambda\delta}(x) - a_{\lambda\delta}(x_0))d\psi_T, dq_j(D)f \rangle + i \langle V_j, d\psi_T \rangle q_j(D)f \\ + \langle a_{\lambda\delta}(x_0)d\psi_T, (d - i\xi_j)(q_j(D)f) \rangle. \end{aligned}$$

Each term has  $L^2$  norm bounded by  $\lambda\delta^{-1}\|f\|_{L^2}$ , and finite overlap properties yield square summability as above.

The finite overlap properties similarly yield that

$$\sum_{T \in \mathcal{T}} \|f_T\|_{L^2}^2 + \lambda^{-2} \|df_T\|_{L^2}^2 \lesssim \|f\|_{L^2}^2 + \lambda^{-2} \|df\|_{L^2}^2,$$

completing the verification of (8).  $\square$

We also need a stronger inequality. For  $T \in \mathcal{T}$ , let  $\xi_T$  equal  $\xi_j$  if  $f_T = \psi_T(x)q_j(D)f$ . Thus, the frequencies of  $f_T$  are concentrated in the  $\lambda\delta$ -ball about  $\xi_T$ , and  $|\xi_T| \approx \lambda$ .

**Lemma 5.** *The following bounds hold, for each  $\alpha, \beta$ ,*

$$(11) \quad \left( \sum_T \lambda^{-2|\alpha|} (\lambda\delta)^{-2|\beta|} \|D^\alpha (D - \xi_T)^\beta f_T\|_{L_{x_1}^\infty L_{x_2}^2}^2 \right)^{\frac{1}{2}} \leq C_{\alpha,\beta} \|f\|_{\delta,\lambda}.$$

*Proof.* Observe that we can write

$$\lambda^{-|\alpha|} (\lambda\delta)^{-|\beta|} D^\alpha (D - \xi_T)^\beta \psi_T(x) q_j(D) = \tilde{\psi}_T(x) \tilde{q}_j(D)$$

where  $\tilde{\psi}_T(x)$  and  $\tilde{q}_j(\xi)$  satisfy similar support and derivative bounds as  $\psi_T$  and  $q_j$ . Hence, the proof we present for the case  $\alpha = \beta = 0$  applies to the general case.

Let  $q'_j(\xi)$  be a smooth cutoff to the  $\delta\lambda$ -neighborhood of  $\text{support}(q_j)$ . Then

$$\|(1 - q'_j(D))\psi_T q_j(D)f\|_{L^2} \lesssim \lambda^{-N} \|f\|_{L^2}.$$

Since the number of tubes is bounded by  $\delta^{-2} \ll \lambda$ , Sobolev embedding establishes the desired bounds on these terms. We set  $f'_T = q'_j(D)f_T$ . By commutator arguments as above we have  $\|f'_T\|_{\delta,\lambda} \lesssim \|f_T\|_{\delta,\lambda}$ . The proof will then follow from (8) by showing that

$$(12) \quad \|f'_T\|_{L_{x_1}^\infty L_{x_2}^2} \lesssim \|f'_T\|_{\lambda,\delta}.$$

We establish (12) by energy inequality arguments. Let  $V$  denote the vector field

$$V = 2(\partial_1 f'_T) a_{\lambda\delta} df'_T + (\lambda^2 \rho_{\lambda\delta} f'_T{}^2 - \langle a_{\lambda\delta} df'_T, df'_T \rangle) \vec{e}_1$$

Then

$$d^*V = 2(\partial_1 f'_T) P_{\lambda\delta} f'_T + \lambda^2 (\partial_1 \rho_{\lambda\delta}) f'_T{}^2 - \langle (\partial_1 a_{\lambda\delta}) df'_T, df'_T \rangle$$

Applying the divergence theorem on the set  $x_1 \leq r$  yields

$$\int_{x_1=r} V_1 dx' \lesssim \lambda^2 \|f'_T\|_{L^2}^2 + \delta^{-1} \|df'_T\|_{L^2}^2 + \delta \|P_{\lambda\delta} f'_T\|_{L^2}^2 \leq \lambda^2 \|f'_T\|_{\lambda,\delta}^2$$

Since  $a_{\lambda\delta}$  and  $\rho_{\lambda\delta}$  are pointwise close to the flat metric, we have pointwise that

$$V_1 \geq \frac{3}{4} |\partial_1 f'_T|^2 + \frac{3}{4} \lambda^2 |f'_T|^2 - \frac{3}{2} |\partial_2 f'_T|^2$$

The frequency localization of  $\widehat{f'_T}$  to  $|\xi_2| \leq c\lambda$  yields

$$2 \int_{x_1=r} V_1 dx' \geq \int_{x_1=r} |df'_T|^2 + \lambda^2 |f'_T|^2 dx'. \quad \square$$

### 3. OVERLAP ESTIMATES

In this section we establish simple bounds on the overlap of tubes, and resulting  $\ell^p$  bounds on the counting function.

**Lemma 6.** *Let  $x$  and  $y$  be two points in  $Q_0$ . Then the number of distinct tubes  $T \in \mathcal{T}$  which pass within distance  $4\delta^2$  of both  $x$  and  $y$  is bounded by  $C \min(\delta^{-1}, \frac{\delta}{|x_1 - y_1|})$ .*

*Proof.* For each  $j$ , there is a fixed bound on the number of tubes  $T \in \mathcal{T}_j$  which pass within distance  $4\delta^2$  of  $x$ . It thus suffices to bound the number of distinct  $j$  such that the line through  $x$  in direction  $V_j$  passes within distant  $\sim \delta^2$  of  $y$ . The above bound is then a simple consequence of the fact that  $\angle(V_i, V_j) \gtrsim \delta|i - j|$ .  $\square$

Now consider a collection  $\mathcal{N} \subset \mathcal{T}$  containing  $N$  distinct tubes. We make a decomposition of the cube  $Q_0$  into a  $\delta^{-1} \times \delta^{-1}$  grid  $\mathcal{Q}$  of cubes  $Q$  of sidelength  $\delta^2$ . Let  $n_Q$  denote the number of tubes in  $\mathcal{N}$  which intersect  $Q^*$ ,

$$n_Q = \#\{T \in \mathcal{N} : T \cap Q^* \neq \emptyset\}.$$

By  $\|n_Q\|_{\ell^p \ell^q}$  we mean the  $\ell_{x_1}^p \ell_{x_2}^q(\mathcal{Q})$  norm of the counting function  $n_Q$ , taken over the grid  $\mathcal{Q}$  of cubes.

**Corollary 7.** *The following bounds hold,*

$$\|n_Q\|_{\ell^\infty \ell^1} \lesssim N, \quad \|n_Q\|_{\ell^2 \ell^\infty} \lesssim |\log \delta|^{\frac{1}{2}} \delta^{-\frac{1}{2}} N^{\frac{1}{2}}.$$

Furthermore, for  $q \geq 3$

$$(13) \quad \|n_Q\|_{\ell^q(\mathcal{Q})} \lesssim |\log \delta|^{\frac{1}{q}} \delta^{-\frac{1}{q}} N^{1-\frac{1}{q}}.$$

*Proof.* The first bound is an immediate consequence of the fact that, for each  $T$  and each  $r$ , there is a fixed upper bound on the number of cubes  $Q^*$  centered on the line  $x_1 = r$  that intersect  $T$ . For the second bound, we consider the map

$$W\{c_T\} = \sum_T c_T \chi_T(Q) \quad \text{where} \quad \begin{cases} \chi_T(Q) = 1, & T \cap Q^* \neq \emptyset \\ \chi_T(Q) = 0, & T \cap Q^* = \emptyset \end{cases}$$



It suffices to show that  $W : \ell^2(\mathcal{T}) \rightarrow \ell^2 \ell^\infty(\mathcal{Q})$  with bound  $|\log \delta|^{\frac{1}{2}} \delta^{-\frac{1}{2}}$ . The map  $WW^*$  takes the form

$$WW^*\{c_Q\}(Q') = \sum_Q n(Q', Q)c_Q, \quad n(Q', Q) = \#\{T : T \cap Q^* \neq \emptyset, T \cap Q'^* \neq \emptyset\}.$$

We need to show that  $WW^* : \ell^2 \ell^1(\mathcal{Q}) \rightarrow \ell^2 \ell^\infty(\mathcal{Q})$  with norm  $|\log \delta| \delta^{-1}$ . This is an easy consequence of the bound from Lemma 6,

$$n(Q', Q) \lesssim \min\left(\delta^{-1}, \delta |x_1(Q') - x_1(Q)|^{-1}\right).$$

Applying interpolation now yields the bounds

$$\|n_Q\|_{\ell^q \ell^r} \lesssim |\log \delta|^{\frac{1}{q}} \delta^{-\frac{1}{q}} N^{1-\frac{1}{q}}, \quad \frac{2}{q} + \frac{1}{r} = 1.$$

Note that if  $q \geq 3$  then  $r \leq q$ , yielding (13).  $\square$

#### 4. PROOF OF THEOREM 4

Given  $f$  and the cube  $Q_0$ , we decompose  $\psi f = \sum_{T \in \mathcal{T}} f_T + f_0$  as in Section 2, and control  $\|f_0\|_{L^{p+2}}$  by Sobolev embedding. We make a further decomposition by collecting together tubes for which  $f_T$  is of comparable energy. Precisely, decompose  $\mathcal{T} = \cup_{k \geq -k_0} \mathcal{N}_k$  where  $T \in \mathcal{N}_k$  if

$$2^{-k-1} \|f\|_{\lambda, \delta} < \|f_T\|_{\lambda, \delta} + \sum_{|\alpha|+|\beta| \leq 3} \lambda^{-|\alpha|} (\lambda \delta)^{-|\beta|} \|D^\alpha (D - \xi_T)^\beta f_T\|_{L_{x_1}^\infty L_{x_2}^2} \leq 2^{-k} \|f\|_{\lambda, \delta}.$$

We handle the tubes for  $k \geq 2 \log_2 \lambda$  by the Sobolev bound  $\|f_T\|_{L^\infty} \leq \|D f_T\|_{L_{x_1}^\infty L_{x_2}^2}$ , since there are at most  $\lambda^{\frac{1}{3}}$  tubes in all. This leaves at most  $\approx \log \lambda$  values of  $k$ , which we handle individually. We thus fix some  $\mathcal{N} = \mathcal{N}_k$ , and let  $N$  be the number of tubes in  $\mathcal{N}$ . We multiply  $f$  by a constant so that  $\|f\|_{\lambda, \delta} = 2^k$ , which by (8) and (11) implies  $N^{\frac{1}{2}} \lesssim \|f\|_{\lambda, \delta}$ . We then need to establish the following.

**Theorem 8.** *Suppose that  $f = \sum_{T \in \mathcal{N}} f_T$ , where each  $f_T$  is supported in  $Q_0$ , and*

$$\|f_T\|_{\delta, \lambda} + \sum_{|\alpha|+|\beta| \leq 3} \lambda^{-|\alpha|} (\lambda \delta)^{-|\beta|} \|D^\alpha (D - \xi_T)^\beta f_T\|_{L_{x_1}^\infty L_{x_2}^2} \leq 1.$$

*Let  $N$  denote the cardinality of  $\mathcal{N}$ . Then, under the conditions of Hypothesis 2,*

$$\|f\|_{L^{p+2}} \lesssim (\log \lambda)^{p-5} \lambda^{\gamma(p+2)} N^{\frac{1}{2}}.$$

*Proof.* As above we decompose  $Q_0$  into cubes  $Q$  of size  $\delta^2$ ,  $Q_0 = \cup_Q Q$ . By hypothesis, for each  $Q$  we have

$$\|f\|_{L^p(Q)} \lesssim (\log \lambda)^{p-6} \lambda^{\gamma(p)} \left\| \sum_{T \cap Q^* \neq \emptyset} f_T \right\|_{\delta^2, \lambda, Q^*}.$$

We first show that

$$(14) \quad \left\| \sum_{T \cap Q^* \neq \emptyset} f_T \right\|_{\delta^2, \lambda, Q^*} \lesssim n_Q^{\frac{1}{2}}.$$

For this, note that  $|a_{\lambda\delta} - a_{\lambda\delta^2}| \leq (\lambda\delta^2)^{-1}$ , hence

$$\left\| \sum_{T \cap Q^* \neq \emptyset} f_T \right\|_{\delta^2, \lambda, Q^*} \leq \sum_{|\alpha| \leq 2} \lambda^{-|\alpha|} \left\| \sum_{T \cap Q^* \neq \emptyset} D^\alpha f_T \right\|_{L^\infty_{x_1} L^2_{x_2}} + \lambda^{-1} \delta \left\| \sum_{T \cap Q^* \neq \emptyset} P_{\lambda\delta} f_T \right\|_{L^2}.$$

For each  $j$ , there are a bounded number of tubes  $T$  with  $\xi_T = \xi_j$  for which  $T \cap Q^* \neq \emptyset$ , hence we can assume the different  $\xi_T$  in the above sum are spaced by distance  $\lambda\delta$  in the  $\xi_2$  variable. Thus,

$$\begin{aligned} \lambda^{-|\alpha|} \left\| \sum_{T \cap Q^* \neq \emptyset} D^\alpha f_T \right\|_{L^\infty_{x_1} L^2_{x_2}} &\lesssim \left( \sum_{|\beta| \leq 1} \sum_{T \cap Q^* \neq \emptyset} \lambda^{-2|\alpha|} (\lambda\delta)^{-2|\beta|} \|(D - \xi_T)^\beta D^\alpha f_T\|_{L^\infty_{x_1} L^2_{x_2}}^2 \right)^{\frac{1}{2}} \\ &\lesssim n_Q^{\frac{1}{2}}. \end{aligned}$$

To complete the proof of (14), we use that  $n_Q \leq \delta^{-1}$  to bound

$$\lambda^{-1} \delta \sum_{T \cap Q^* \neq \emptyset} \|P_{\lambda\delta} f_T\|_{L^2} \leq \delta^{\frac{1}{2}} \sum_{T \cap Q^* \neq \emptyset} \|f_T\|_{\delta, \lambda} \leq \delta^{\frac{1}{2}} n_Q \leq n_Q^{\frac{1}{2}}.$$

By (14) and Hypothesis 2, we thus have

$$(15) \quad \|f\|_{L^p(Q)} \lesssim (\log \lambda)^{p-6} \lambda^{\gamma(p)} n_Q^{\frac{1}{2}}.$$

We next note the bound

$$\begin{aligned} \|f_T\|_{L^\infty} &\leq 2 \|(D - \xi_T) f_T\|_{L^\infty_{x_1} L^2_{x_2}}^{\frac{1}{2}} \|f_T\|_{L^\infty_{x_1} L^2_{x_2}}^{\frac{1}{2}} \\ &\leq \lambda^{\frac{1}{2}} \delta^{\frac{1}{2}} \sum_{|\beta| \leq 1} (\lambda\delta)^{-|\beta|} \|(D - \xi_T)^\beta f_T\|_{L^\infty_{x_1} L^2_{x_2}} \\ &\leq \lambda^{\frac{1}{2}} \delta^{\frac{1}{2}}. \end{aligned}$$

Consequently,

$$(16) \quad \|f\|_{L^\infty(Q)} \leq \lambda^{\frac{1}{2}} \delta^{\frac{1}{2}} n_Q.$$

Combining (15)–(16) with (13) for  $q = \frac{1}{2}p$  and  $q = \infty$  respectively, we obtain

$$\begin{aligned} \|f\|_{L^p} &\lesssim (\log \lambda)^{p-6+\frac{1}{p}} \lambda^{\gamma(p)} \delta^{-\frac{1}{p}} N^{\frac{1}{2}-\frac{1}{p}} \\ \|f\|_{L^\infty} &\lesssim \lambda^{\frac{1}{2}} \delta^{\frac{1}{2}} N \end{aligned}$$

Interpolation yields

$$\|f\|_{L^{p+2}} \lesssim (\log \lambda)^{\frac{p(p-6)+1}{p+2}} \lambda^{\gamma(p+2)} N^{\frac{1}{2}}$$

Observing that if  $p \geq 6$  we have  $p(p-6)+1 \leq (p+2)(p-5)$  concludes the proof.  $\square$

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