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Global Existence of Solutions to Semilinear Klein-Gordon Equations

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Global Existence of Solutions to Semilinear Klein-Gordon Equations

A dissertation submitted in partial satisfaction of the
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by

Nina Pikula

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2019

The dissertation of Nina Pikula is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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DEDICATION

To my family, for their unwavering support and encouragement

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ABSTRACT OF THE DISSERTATION

Global Existence of Solutions to Semilinear Klein-Gordon Equations

by

Nina Pikula

Doctor of Philosophy in Mathematics

University of California San Diego, 2019

Professor Ioan Bejenaru, Chair
Professor Jacob Sterbenz, Co-Chair

In this thesis, we prove two main results on nonlinear Klein-Gordon equations. First, we establish global existence of solutions to general second order semilinear Klein-Gordon equations for small initial data and $n = 3$ spatial dimensions. Then, we prove low regularity well-posedness in $n = 2$ spatial dimensions and higher for a quadratic power-type Klein-Gordon system with different masses satisfying a suitable nonresonance condition.

For the first result, our main tool is the Normal Forms Method of Shatah. The key idea behind this approach is to decompose u into a sum of two functions, U and W, where

W solves a third order system and U is written explicitly as a function of u and its first order derivatives. The explicit form of U and good behavior of solutions to higher order systems allows us to gain control of both U and W, and thus u.

For the multiple mass system, we apply a standard duality argument to reduce our proof of well-posedness to the establishment of a set of trilinear estimates. The proof of these estimates relies heavily on the special properties of our iteration spaces. In particular, using these spaces allows us to readily exploit the absence of resonant terms and extend important bilinear estimates proved for free solutions to more general functions.

Chapter 1

Introduction

Relativistic wave equations such as the Klein-Gordon equation have been of interest to theoretical physicists in various branches of physics [24]. In recent decades, there has been considerable interest among physicists in finding exact solutions to nonlinear Klein-Gordon equations with various vector and scalar potentials (for example, see [13], [3], [17]). Our focus in this thesis will be on proving global existence for a broad class of nonlinear Klein-Gordon equations known as second-order semilinear Klein-Gordon equations. Before we explain what this means in precise terms, we will first present the reader with a brief background of the Klein-Gordon equation. The derivation that follows, along with other relevant material, can be found in Chapters 1 and 9 of [1].

The homogeneous Klein-Gordon equation

$$(\frac{\partial^2}{\partial_t^2} - \Delta + \mu)u = 0 \tag{1.1}$$

is a wave equation introduced by physicists Oskar Klein and Walter Gordon as a relativistic alternative to the Schroedinger equation [15].

From physics we know that the nonrelativistic expression for the energy of a free particle is given by

$$E = \frac{\vec{p} \cdot \vec{p}}{2m}, \tag{1.2}$$

where $\vec{p}$ is the associated momentum and m is the associated mass.

Applying standard Quantum Mechanics theory, we may replace

$$E \to i\hbar \frac{\partial}{\partial t}, \quad \vec{p} \to -i\hbar \nabla \tag{1.3}$$

to obtain the Schroedinger equation for a free particle

$$i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2 \Delta}{2m}\psi, \tag{1.4}$$

where $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ is a function for which the above operations are well-defined.

In order to produce a relativistic version of the above, it is natural to attempt to replace the energy equation, (1.2), with the relativistic energy expression

$$E^2 = \vec{p} \cdot \vec{p} + m^2, \tag{1.5}$$

where we have chosen our units so that c, the speed of light, is 1. Once again we may apply the replacements in (1.3) to obtain

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\psi(x,t) = (-\hbar^2 \Delta + m^2)\psi(x,t).$$

Simplifying the above we get

$$\left(\Box + \frac{m^2}{\hbar^2}\right)\psi(x,t) = 0,$$

where $\Box = \partial_t^2 - \Delta$ is the D'Alembertian operator. We recognize the above as the homogeneous Klein-Gordon equation.

The discussion above has focused exclusively on the case of a free particle. In reality, particles typically interact strongly with other particles and fields. In order to develop a more robust theory, it is useful to study (1.1) with a forcing term, F, added to the right-hand side. For example, a simplified model of the interaction between a spin-zero meson and an electromagnetic field with associated electromagnetic four-potential $A^\alpha = (\phi, \vec{A})$ leads to the following equation ([1], Chapter 9):

$$\left[\left(i\frac{\partial}{\partial x_\mu} - eA^\mu(x,t)\right)^2 - m^2\right]u(x,t) = 0, \tag{1.6}$$

where e is the elementary charge and we have used the summation convention.

We can rewrite (1.6) as

$$(\Box + m^2)u(x,t) = -ie(\partial_\mu A^\mu(x,t) + A^\mu(x,t)\partial_\mu)u(x,t) + e^2A(x,t)^2u(x,t),$$

leading us to our first example of an inhomogeneous Klein-Gordon system. In practice, the four-potential A^μ will often also depend on u and it's space and time derivatives leading to more complicated nonlinearities. In this thesis, we will primarily study inhomogeneous systems of the form

$$(\frac{\partial^2}{\partial t^2} - \Delta + \mu)u = F(u, \partial u, \partial_t u). \tag{1.7}$$

The equation above is an example of a semilinear Klein-Gordon equation because its nonlinearity, F, only depends on derivatives of at most first order. Real world phenomena are often better approximated by nonlinearities that include second order derivatives. Unfortunately, such systems are significantly more difficult to handle so we focus our efforts towards fully understanding the semilinear case.

We are finally in a position to to discuss one of the main subjects of this thesis.

Before we introduce the main theorem, we remind the reader that for $s > 0$ the L^2-based Sobolev space, H^s, is defined by the following norm

$$\|u\|_{H^s(\mathbb{R}^n)} := \|\langle\xi\rangle^s \hat{u}\|_{L^2(\mathbb{R}^n)},$$

where $\langle\cdot\rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$.

We say that a nonlinearity $F = F(u, \partial_t u, \partial u) \in C^\infty$ is of order p for some $p \in \mathbb{Z}_{\geq 0}$ if F is a polynomial whose lowest order term has degree p.

In this thesis we will be focusing primarily on second order nonlinearities. As a physical motivation, consider the Yukawa-coupled Klein-Gordon-Dirac system (see [4] and 10.2 in [1]), given by:

$$(-i\gamma^\mu \partial_\mu + M)\psi = \phi\psi \quad (M > 0),$$

$$(\Box + m^2)\phi = \psi^\dagger \gamma^0 \psi \quad (m > 0),$$

where $\psi : \mathbb{R}^{1+3} \to \mathbb{C}^4$ is the spinor field, $\phi : \mathbb{R}^{1+3} \to \mathbb{C}$ is a scalar field, and the γ^μ, for $\mu = 0, 1, 2, 3$, are the Dirac matrices.

This system is a simple model of a proton-proton (or neutron-neutron) interaction in which one proton is scattered by the meson field produced by another proton. We remark that the constant M here represents the mass of the proton ($M = 938\frac{Mev}{e^2}$) and m typically represents the mass of a π-meson ($m = 140\frac{Mev}{e^2}$ for $\pi^\pm$ and $m = 135\frac{Mev}{e^2}$ for π^0) or a K-meson ($m = 494\frac{Mev}{e^2}$ for $K^\pm$ and $m = 498\frac{Mev}{e^2}$ for K^0). It is therefore reasonable to assume in the above model that the constants m and M satisfy the condition $2M > m > 0$. When we discuss multiple mass second order Klein-Gordon systems later in this thesis, we will need to impose a similar condition on our masses in order to close our argument.

Although this condition may seem contrived from a mathematical perspective, it is often reasonable to assume given the physical motivation behind many of these models.

We now turn our attention to discussing the major results of this thesis. One of our main goals is to prove global well-posedness of the second order three dimensional semilinear Klein-Gordon system in H^s for $s > 10$. That is, we would like to prove the following theorem

Theorem 1.0.1. *Let* $n = 3, s > 10$ *and suppose* $F(u, \partial u, \partial_t u)$ *is of order 2. The, there exists an* $\epsilon > 0$ *such that for initial data*

$$(u_0, u_1) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n), \quad \|(u_0, u_1)\|_{H^s \times H^{s-1}} < \epsilon,$$

the equation

$$(\Box + \mu^2)u = F(u, \partial u, \partial_t u) \tag{1.8}$$

has a global solution in $C(\mathbb{R}, H^s) \cap C^1(\mathbb{R}, H^{s-1})$ *which depends continuously on the initial data* (u_0, u_1).

We remark that the novelty of this result stems from the control of the quadratic terms, at least for initial data belonging to a natural Hilbert space such as H^s_x. Global existence in 3 dimensions has already been established for third order nonlinearities (see [19]).

Previous work on the second order semilinear Klein-Gordon system in H^s_x was conducted by Delort and Fang in [6]. In contrast to our result where global existence was established for general second order nonlinearities, Delort and Fang only proved almost global existence (time of existence has a lower bound $T_\epsilon \geq ce^{1/\epsilon}$) for a small subclass of second order nonlinearities with a special null structure that gives one better control over the solution.

The standard approach to proving existence for higher order systems is to apply apriori energy and dispersive estimates. Unfortunately, if F is second order, the standard techniques fail and a more refined approach must be employed. One such approach, introduced by Shatah in [20], is the Normal Forms method. This method allows us to transform our second order problem into a third order one, for which global existence has long been established.

It is important to note that the value of s, also known as the regularity, in Theorem 1.0.1 is not necessarily optimal and further work could be done in the future to lower this value. For many reasons, lowering the regularity assumptions on the initial data is a goal among those working on existence problems. For example, many key structural features of the solution, such as conservation laws (energy, momentum), are typically associated to low regularities such as L^2 and H^1. Furthermore, the challenge of working at low regularities forces us to exploit structural properties of the equation and develop new techniques that have applications for smooth data.

In Chapter 5, we explore the low-regularity problem in the case where F is a homogeneous quadratic polynomial in u (e.g. $F(u, \partial u, \partial_t u) = F(u) := u^2$). In fact, we consider a more general system of different masses

$$(\Box + m_i^2)u_i = F_i(u_1, ..., u_k) \quad i = 1, ..., k$$

For this problem, we handle the difficult $n = 2$ case and extend our result to $n \geq 2$ dimensions. In particular, we prove the following theorem

Theorem 1.0.2. *Let* $n \geq 2, s \geq \max\left(\frac{1}{2}+, \frac{n-2}{2}\right), k \in \mathbb{N}$*, and let* $F_1, ..., F_k$ *be homogeneous quadratic polynomials and* $m_1, ..., m_k > 0$ *be such that*

$$2 \min\left(\{m_j\}\right) > \max\left(\{m_j\}\right)$$

Then there exists an $\epsilon > 0$ *such that for initial data*

$$(f_i, g_i) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)\,, \quad \|(f_i, g_i)\|_{H^s \times H^{s-1}} < \epsilon$$

the system

$$(\Box + m_i^2) u_i = F_i(u_1, ..., u_k) \quad i = 1, ..., k$$

has a global solution in $C(\mathbb{R}, H^s) \cap C^1(\mathbb{R}, H^{s-1})$ *which depends continuously on the initial data* (f_i, g_i)*.*

We remark that a similar result was obtained by Tobias Schottdorf in [18]. Unfortunately, we have found serious gaps in his proof and we suspect this to be the reason his work has remained unpublished.

A general outline of this thesis is as follows: Chapter 2 is dedicated to introducing notation and establishing standard tools from Analysis and PDEs. In Chapter 3 we discuss general techniques for proving existence results and in Chapter 4 we apply these methods to prove wellposedness of the third-order semi-linear Klein-Gordon System. In Chapter 5 we employ modern machinery to prove low regularity wellposedness for the multiple mass second order system discussed above. Finally, Chapters 6 and 7 are dedicated to proving Theorem 1.0.1.

Chapter 2

Background Material

In this chapter we introduce notation and well-known results from Analysis and PDEs that will be referenced repeatedly throughout this thesis.

2.1 Preliminaries

We denote $A \ll B$ to mean $A \leq dB$ for some absolute constant $0 < d < \frac{1}{N}$ for some large N. We denote $A \lesssim B$ to mean $A \leq CB$ for some absolute constant $C > 0$ and we define $A \sim B$ to mean $\frac{1}{C}A \leq B \leq CB$.

For completeness, we will present many of the following results in the case of a general $\sigma-$finite measure space $(X, \mathcal{M}, \mu)$ where X represents the domain we are working in, μ is the chosen measure, and $\mathcal{M}$ is the set of all $\mu-$measurable functions. We begin with the definition of L^p spaces on $(X, \mathcal{M}, \mu)$.

Definition 2.1.1. *Given $1 \leq p < \infty$ and a measurable function $f : X \to \mathbb{C}$, define*

$$\|f\|_{L^p} = [\int_X |f|^p d\mu]^{\frac{1}{p}},$$

$$\|f\|_{L^\infty} = \inf\{a \geq 0 : \mu(\{x : |f(x)| > a\}) = 0\}.$$

Definition 2.1.2. *For $1 \leq p \leq \infty$ we define the space $L^p(X)$ as*

$$L^p(\mathbb{R}^n) = \{f : X \to \mathbb{C} : f \textit{ is measurable and } \|f\|_{L^p} < \infty\}.$$

It is well-known (see Theorems 6.6 and 6.8 in [8]) that $L^p(\mathbb{X})$ is a Banach space for each $1 \leq p \leq \infty$. We present the following well-known results

Theorem 2.1.1 (Holder's inequality ([8], pp 198))**.** *Suppose $1 \leq p, q, r \leq \infty$ are such that $\frac{1}{p} + \frac{1}{p'} = \frac{1}{r}$. If f and g are measurable functions on X, then*

$$\|fg\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}.$$

Definition 2.1.3. *Given a measure space $(X, \mathcal{M}, \mu)$, we say that a statement holds true for almost every (a.e.) $x \in X$ if the set on which the statement is false has measure 0 in X.*

Theorem 2.1.2 (Minkowski's inequality(Theorem 6.19 in [8]))**.** *Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are $\sigma-$finite measure spaces.If $1 \leq p < \infty, f(\cdot, y) \in L^p(\mu)$ for a.e. $y \in Y$, and the function $y \to \|f(\cdot, y)\|_{L^p}$ is in $L^1(Y)$, then $f(x, \cdot) \in L^1(X)$ for a.e. x, the function $x \to \int f(x, y)d\nu(y)$ is in $L^p(X)$, and*

$$\| \int f(x, y)d\nu(y)\|_{L^p} \leq \int \|f(\cdot, y)\|_{L^p} d\nu(y).$$

Theorem 2.1.3 (The Riesz-Thorin Interpolation Theorem ([8], Theorem 6.27))**.** *Suppose* $(X, \mathcal{M}, \mu)$ *and* $(Y, \mathcal{N}, \nu)$ *are* $\sigma-$*finite measure spaces and* $p_0, p_1, q_0, q_1 \in [1, \infty]$. *For* $0 < t < 1$ *define* p_t *and* q_t *by*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

If T *is a linear map from* $L^{p_0}(X) + L^{p_1}(X)$ *into* $L^{q_0}(Y) + L^{q_1}(Y)$ *such that* $\|Tf\|_{L^{q_0}} \leq M_0\|f\|_{L^{p_0}}$ *for* $f \in L^{p_0}(X)$ *and* $\|Tf\|_{L^{q_1}} \leq M_1\|f\|_{L^{p_1}}$ *for* $f \in L^{p_1}(X)$, *then* $\|Tf\|_{L^{q_t}} \leq M_0^{1-t}M_1^t\|f\|_{L^{p_t}}$ *for* $f \in L^{p_t}(X), 0 < t < 1$.

We remark that the above results also hold when $X = \mathbb{N}$ and μ is the counting measure on $\mathbb{N}$. In this case

$$\left(\int |f|^p d\mu\right)^{1/p} = \left(\sum_{n=1}^{\infty} |f(n)|^p\right)^{1/p}.$$

We now present some important results from Fourier analysis. Recall the definition of the Fourier transform $\mathcal{F}$ on $L^1(\mathbb{R}^n)$:

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(x)dx.$$

We will assume that the reader is familiar with basic properties of the Fourier transform, such as its behavior under translation, dilation, conjugation and differentiation.

The following two results are well-known

Theorem 2.1.4 (The Plancherel Theorem ([8], Theorem 8.29))**.** *If* $f \in L^1 \cap L^2$, *then* $\hat{f} \in L^2$, *and* $\mathcal{F}|(L^1 \cap L^2)$ *extends to a unitary isomorphism on* L^2.

Definition 2.1.4. *For* $1 \leq p \leq \infty$, *we define the Holder-conjugate of* p, *denoted* p' *to be given by* $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 2.1.5 (The Hausdorff-Young Inequality ([8], Theorem 8.30))**.** *Suppose that* $1 \leq p \leq 2$*. If* $f \in L^p(\mathbb{R}^n)$*, then* $\hat{f} \in L^{p'}(\mathbb{R}^n)$ *and* $\|\hat{f}\|_{L^{p'}} \leq \|f\|_{L^p}$*.*

So far, most our results have been related to L^p spaces. We now turn our attention to developing the theory of Sobolev Spaces. Before we do so, we must first familiarize the reader with the definition of a weak derivative.

Definition 2.1.5. *A vector of the form* $\alpha = (\alpha_1, ...\alpha_n)$*, where each* α_i *is a nonnegative integer, is called a multiindex with order*

$$|\alpha| = \sum_{i=1}^{n} \alpha_i.$$

Definition 2.1.6. *We let* $C_c^\infty(\mathbb{R}^n)$ *denote the space of all infinitely differentiable functions on* $\mathbb{R}^n$ *with compact support.*

Definition 2.1.7. *We define* $L^1_{loc}(\mathbb{R}^n)$ *to be the space consisting of all functions* $f : \mathbb{R}^n \to \mathbb{C}$ *with the property*

$$\int_X |f| dx < \infty$$

for every compact subset X *of* $\mathbb{R}^n$*.*

Definition 2.1.8. *Suppose* $f, g \in L^1_{loc}(\mathbb{R}^n)$ *and* α *is a multi-index. We say that* g *is the* $\alpha^{th}-$*weak partial derivative of* f*, denoted*

$$D^\alpha f = g,$$

if for each $\phi \in C_c^\infty(\mathbb{R}^n)$ *we have*

$$\int_{\mathbb{R}^n} f(D^\alpha \phi) dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} g\phi dx.$$

We are finally ready to define Sobolev spaces.

Definition 2.1.9. *Suppose $1 \leq p \leq \infty$ and k is a nonnegative integer. For $f \in L^1_{loc}(\mathbb{R}^n)$, we define the norm*

$$\|f\|_{W^{k,p}} := \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p}^p \right)^{1/p} \qquad \text{if } p < \infty,$$

$$\|f\|_{W^{k,\infty}} := \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty}.$$

Definition 2.1.10. *Given $1 \leq p \leq \infty$ and $k \in \mathbb{Z}_{\geq 0}$, we define the Sobolev space $W^{k,p}(\mathbb{R}^n)$ as*

$$W^{k,p} := \{f \in L^p(\mathbb{R}^n) : \|f\|_{W^{k,p}} < \infty\}.$$

It is common convention to denote $H^k(\mathbb{R}^n) := W^{k,2}(\mathbb{R}^n)$.

We next introduce another important space of functions, known as the Schwartz class

Definition 2.1.11. *Let N be a positive integer and α a multiindex of arbitrary length. Define*

$$\|f\|_{(N,\alpha)} := \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha f(x)|.$$

Definition 2.1.12. *Define the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ as*

$$\mathcal{S}(\mathbb{R}^n) := \{f \in C^\infty : \|f\|_{(N,\alpha)} < \infty \text{ for all } N, \alpha\}.$$

Theorem 2.1.6 ([8], Corollary 8.23)**.** *The Fourier transform, $\mathcal{F}$, maps the Schwartz class, $\mathcal{S}$ continuously into itself.*

Lemma 2.1.1 ([22], Lemma 23)**.** *Suppose $p \in [1, \infty)$ and k is a nonnegative integer. Then the space $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$.*

Corollary 2.1.1. *If* $1 \leq p < \infty$*,* $\mathcal{S}(\mathbb{R}^n)$ *is dense in* $W^{k,p}(\mathbb{R}^n)$*. In other words, every function in* $W^{k,p}(\mathbb{R}^n)$ *is in* $\mathcal{S}(\mathbb{R}^n)$ *or a limit point of a sequence of functions in* $\mathcal{S}(\mathbb{R}^n)$ *with respect to the* $\|\cdot\|_{W^{k,p}(\mathbb{R}^n)}$ *norm.*

Proof. This follows from the above corollary and the fact $C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \subseteq W^{k,p}(\mathbb{R}^n)$. □

The preceding lemma demonstrates that for $1 \leq p < \infty$, $W^{k,p}(\mathbb{R}^n)$ can alternatively be defined as the closure of $\mathcal{S}(\mathbb{R}^n)$ with respect to the $\|\cdot\|_{W^{k,p}(\mathbb{R}^n)}$ norm. This interpretation will be useful for extending $W^{k,p}$ to the case where k is not necessarily a nonnegative integer, but a real number $r > 0$.

Recall from Chapter 1 that $\langle\cdot\rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$.

Definition 2.1.13. *Given* $r \in \mathbb{R}, f \in S(\mathbb{R}^n)$*, we denote by* $\langle D\rangle^r f$ *the following expression*

$$\langle D\rangle^r f(x) := \mathcal{F}^{-1}(\langle\xi\rangle^r \hat{f}(\xi)) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\langle\xi\rangle^r \hat{f}(\xi)d\xi.$$

Definition 2.1.14. *Let* $1 < p < \infty$ *and* r *be a nonnegative real number. We define the norm* $\|\cdot\|_{[r,p]}$ *on* $\mathcal{S}(\mathbb{R}^n)$ *as follows*

$$\|f\|_{[r,p]} := \|\langle D\rangle^r f\|_{L^p}.$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$*.*

It is well-known (see [23], Appendix) that $W^{k,p}$ where $k \in \mathbb{Z}_{\geq 0}$ and $1 < p < \infty$ can alternatively be defined as the closure of the Schwartz space under the $\|\cdot\|_{[k,p]}$ norm. In fact, this definition can be extended to all $k \in \mathbb{R}_{\geq 0}$.

We will use the following two results repeatedly throughout this thesis

Theorem 2.1.7 (Sobolev Embedding ([23], Appendix))**.** *Suppose for a given $s > 0$ we have*

$$1 < p < q < \infty \text{ and } \frac{1}{p} - \frac{1}{q} = \frac{s}{n}.$$

Assume $f \in L^q(\mathbb{R}^n)$, then there exists a constant $C = C(p, q, s, n)$ such that

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{W^{s,p}(\mathbb{R}^n)}.$$

Theorem 2.1.8 (L^∞ Sobolev Embedding ([21], Appendix))**.** *Suppose $f \in L^\infty(\mathbb{R}^n)$, then there exists a $C = C(p, s, n)$ so that*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq C\|f\|_{W^{s,p}(\mathbb{R}^n)},$$

provided $p > \frac{s}{n}$

We now present the final result of this section: The Hardy-Littlewood Fractional Integral Inequality. This set of inequalities will be crucial for establishing important dispersive estimates.

Theorem 2.1.9 (Hardy-Littlewood Fractional Integral Inequality ([21], Appendix))**.** *Fix $0 < \alpha < 1$ and $1 < p < q < \infty$ satisfying*

$$1 - (1/p - 1/q) = \alpha.$$

Let

$$I_\alpha f(t) = \int_{-\infty}^{\infty} f(s)|t - s|^{-\alpha} ds.$$

Then there exists a constant $C = C(\alpha, p, q)$ such that

$$\|I_\alpha f\|_{L^q(\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R})}.$$

We conclude this section by defining the mixed time-space Lebesgue spaces.

Definition 2.1.15. *Given a time interval I, define the mixed norm space $L_t^p(I; L_x^q(\mathbb{R}^n))$ by the norm*

$$\|f\|_{L_t^p(I;L_x^q(\mathbb{R}^n))} = \left(\int_I \|f(\cdot,t)\|^p_{L_x^q(\mathbb{R}^n)} dt\right)^{\frac{1}{p}}.$$

When there is no ambiguity regarding the interval I, we will denote the above by $L_t^p L_x^q$.

We can define $L_t^p W_x^{r,q}$ in a similar manner.

2.2 Littlewood-Paley Theory

It is often useful in our analysis to decompose functions into low, medium, and high frequencies. In order to make this classification rigorous, we introduce what is known as the Littlewood-Paley Theory.

Let $\mathcal{X}$ denote a smooth, nonnegative, even function supported in $\{t : |t| \leq 2\}$ such that $\mathcal{X}(t) = 1$ when $|t| \leq 1$. Define $\psi(t) := \mathcal{X}(t) - \mathcal{X}(2t)$ and $\psi_k := \psi(2^{-k}\cdot)$ for $k \geq 1$. Let $\psi_0 := Id - \sum_{k\geq 1} \psi_k$. For $k \geq 1$, define $\tilde{\psi}_k = \sum_{i=k-1}^{k+1} \psi_i \psi_k$ and $\tilde{\psi}_0 = \sum_{i=0}^{1} \psi_i \psi_0$. Furthermore, define $\psi_{\leq K} = \sum_{0\leq k' \leq k} \psi_{k'}$

Definition 2.2.1. *Given $k \geq 0$ define the Fourier multipliers $P_k, P_{\leq k}, \tilde{P}_k$ by*

$$\widehat{P_k u}(\xi) = \psi_k(\xi)\hat{u}(\xi),$$

$$\widehat{P_{\leq k} u}(\xi) = \psi_{\leq k}(\xi)\hat{u}(\xi),$$

$$\widehat{\tilde{P}_k u}(\xi) = \tilde{\psi}_k(\xi)\hat{u}(\xi).$$

We present the following estimate, known as the Littlewood-Paley inequality, without proof

Proposition 2.2.1 ([23], pp 334)**.** *Given* $p \in (1, \infty)$ *and* $f \in L^p(\mathbb{R}^n)$*, we have*

$$\|f\|_{L^p(\mathbb{R}^n)} \sim_{p,n} \|(\sum_{k\geq 0} |P_k f|^2)^{1/2}\|_{L^p(\mathbb{R}^n)}.$$

Definition 2.2.2. *Let* $r \geq 0$ *and* $p \in [1, \infty]$*. Define the spaces* $L^p(\mathbb{R}^n)[\boldsymbol{k}]$ *and* $W^{r,p}(\mathbb{R}^n)[\boldsymbol{k}]$ *by the norms*

$$\|f\|_{L^p[\boldsymbol{k}]} = \left(\sum_{k\geq 0} 2^{2kr}\|P_k f\|_{L^p}^2\right)^{\frac{1}{2}},$$

$$\|f\|_{W^{r,p}[\boldsymbol{k}]} = \left(\sum_{k\geq 0} 2^{2kr}\|P_k f\|_{L^p}^2\right)^{\frac{1}{2}}.$$

We denote $H^r[\boldsymbol{k}] := W^{r,p}[\boldsymbol{k}]$*.*

Observe that if $r = 0$ then $L^p(\mathbb{R}^n)[\mathbf{k}] = W^{r,p}(\mathbb{R}^n)[\mathbf{k}]$. We also define the analogous spaces for mixed space-time Sobolev Spaces.

Definition 2.2.3. *Define the space* $L_t^p W_x^{r,q}[\boldsymbol{k}]$ *by the norm*

$$\|u\|_{L_t^p W_x^{r,q}[\boldsymbol{k}]} = \left(\sum_{k\geq 0} 2^{2kr}\|P_k u\|_{L_t^p L_x^q}^2\right)^{\frac{1}{2}}.$$

The main result of this section will be the following

Proposition 2.2.2. *Suppose* $r \geq 0$ *.*

If $1 < p \leq 2$*,*

$$\|f\|_{L^p(\mathbb{R}^n)[\boldsymbol{k}]} \lesssim_{p,n} \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.1}$$

If $2 \leq p < \infty$*,*

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim_{p,n} \|f\|_{L^p(\mathbb{R}^n)[\boldsymbol{k}]}. \tag{2.2}$$

Furthermore,

$$\|f\|_{H^r(\mathbb{R}^n)} \sim_{p,n} \|f\|_{H^r(\mathbb{R}^n)[\boldsymbol{k}]}. \tag{2.3}$$

Proof. We first focus on proving (2.2) From the Littlewood-Paley inequality, we know that

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}^n)} &\sim_{p,n} \|(\sum_{k\geq 0} |P_k f|^2)^{1/2}\|_{L^p(\mathbb{R}^n)} \\ &= \|\sum_{k\geq 0} |P_k f|^2\|^{\frac{1}{2}}_{L^q(\mathbb{R}^n)},\end{aligned}$$

provided $q = \frac{p}{2}$. As $q \geq 1$. Minkowski's inequality allows us to bound the above by

$$\begin{aligned}&\leq (\sum_{k\geq 0} \||P_k f|^2\|_{L^q})^{1/2} \\ &= (\sum_{k\geq 0} \|P_k f\|^2_{L^p})^{1/2},\end{aligned}$$

as desired.

We now turn our attention to proving (2.1). Once again, we apply the Littlewood-Paley inequality

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}^n)} &\sim_{p,n} \left[\int (\sum_{k\geq 0} |P_k f|^2)^{p/2}\right]^{\frac{1}{p}} \\ &= \left[\int (\sum_{k\geq 0} |P_k f|^{pq})^{1/q}\right]^{\frac{1}{pq}},\end{aligned}$$

provided $q = \frac{2}{p}$. As $q \geq 1$ we can again apply Minkowski's inequality to show that the above is

$$\begin{aligned}&\geq \left[\sum_{k\geq 0} (\int |P_k f|^p)^q\right]^{\frac{1}{pq}} \\ &= \left[\sum_{k\geq 0} (\int |P_k f|^p)^{2/p}\right]^{\frac{1}{2}} \\ &= (\sum_{k\geq 0} \|P_k f\|^2_{L^p})^{1/2}.\end{aligned}$$

Finally, we focus on proving (2.3). Combining (2.1) and (2.2), we see that

$$\|f\|_{L^2} \sim \|f\|_{L^2[\mathbf{k}]}.$$

Therefore,

$$\begin{aligned}
\|f\|_{H^r} &\sim \|\langle D\rangle^r f\|_{L^2} \\
&\sim \left(\sum_{k\geq 0} \|P_k \langle D\rangle^r f\|_{L^2}^2\right)^{1/2} \\
&\sim \left(\sum_{k\geq 0} \|P_k \langle 2^k\rangle^r f\|_{L^2}^2\right)^{1/2} \\
&\sim \left(\sum_{k\geq 0} 2^2 k \|P_k f\|_{L^2}^2\right)^{1/2}.
\end{aligned}$$

□

We conclude with the following result that will greatly simplify our proofs in later chapters.

Theorem 2.2.1. *Assume* $1 \leq p_i, \tilde{p}_i, q_i, \tilde{q}_i \leq \infty$ *for* $i \in \{1, 2\}$ *are such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{\tilde{p}_1} + \frac{1}{\tilde{p}_2}, \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{\tilde{q}_1} + \frac{1}{\tilde{q}_2}$. *Furthermore, assume* $r, \lambda, \tilde{\lambda} > 0, \sigma, \tilde{\sigma} \geq 0$, *then*

$$\begin{aligned}
\|vw\|_{L_t^p W_x^{r,q}[\boldsymbol{k}]} \lesssim\ & \|v\|_{L_t^{p_1} W_x^{r+\sigma, q_1}[\boldsymbol{k}]} \|w\|_{L_t^{p_2} W_x^{\lambda-\sigma, q_2}[\boldsymbol{k}]} \\
&+ \|v\|_{L_t^{\tilde{p}_1} W_x^{\tilde{\lambda}-\tilde{\sigma}, \tilde{q}_1}[\mathbf{k}]} \|w\|_{L_t^{\tilde{p}_2} W_x^{r+\tilde{\sigma}, \tilde{q}_2}[\boldsymbol{k}]}.
\end{aligned}$$

Proof. Let $C \geq 10$ be a fixed constant, then

$$\begin{aligned}
\|vw\|_{L_t^p W_x^{r,q}[\mathbf{k}]} &= \Big(\sum_{k\geq 0} 2^{2kr} \|P_k(vw)\|^2_{L_t^p L_x^q} \Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{k\geq 0} 2^{2kr} (\sum_{k_1\geq 0} \sum_{k_2 \geq 0} \|P_k(P_{k_1} v P_{k_2} w)\|_{L_t^p L_x^q})^2 \Big)^{\frac{1}{2}} \\
&\lesssim \Big(\sum_{k\geq 0} 2^{2kr} (\sum_{|k_1-k|\leq C} \sum_{k_2\leq k+C} \|P_{k_1} v P_{k_2} w\|_{L_t^p L_x^q})^2 \Big)^{\frac{1}{2}} \\
&+ \Big(\sum_{k\geq 0} 2^{2kr} (\sum_{k_1\geq k-C} \sum_{|k_2-k_1|\leq C} \|P_{k_1} v P_{k_2} w\|_{L_t^p L_x^q})^2 \Big)^{\frac{1}{2}} \\
&+ \Big(\sum_{k\geq 0} 2^{2kr} (\sum_{k_1\leq k+C} \sum_{|k_2-k|+C} \|P_{k_1} v P_{k_2} w\|_{L_t^p L_x^q})^2 \Big)^{\frac{1}{2}} \\
&= I + II + III.
\end{aligned}$$

We first bound (I). By Holder's inequality,

$$\begin{aligned}
(I) &\lesssim \Big(\sum_{k\geq 0} 2^{2kr} (\sum_{|k_1-k|\leq C} \sum_{k_2\leq k+C} \|P_{k_1} v\|_{L_t^{p_1} L_x^{q_1}} \|P_{k_2} w\|_{L_t^{p_2} L_x^{q_2}})^2 \Big)^{\frac{1}{2}} \\
&\lesssim \Big(\sum_{k\geq 0} 2^{2kr} (\sum_{|k_1-k|\leq C} \sum_{k_2\leq k+C} 2^{k_1(\sigma)} \|P_{k_1} v\|_{L_t^{p_1} L_x^{q_1}} 2^{k_2(-\sigma)} \|P_{k_2} w\|_{L_t^{p_2} L_x^{q_2}})^2 \Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{k\geq 0} 2^{2kr} (\sum_{|k_1-k|\leq C} 2^{k_1(\sigma)} \|P_{k_1} v\|_{L_t^{p_1} L_x^{q_1}})^2 \Big)^{\frac{1}{2}} \Big(\sum_{k_2\geq 0} 2^{k_2(-\sigma)} \|P_{k_2} w\|_{L_t^{p_2} L_x^{q_2}} \Big).
\end{aligned}$$

By Young's inequality in k and Cauchy-Schwartz in k_2, we can bound the above by

$$\begin{aligned}
&\leq \Big(\sum_{k\geq 0} 2^{2k(r+\sigma)} \|P_k v\|^2_{L_t^{p_1} L_x^{q_1}} \Big)^{\frac{1}{2}} \Big(\sum_{k_2\geq 0} 2^{2k_2(\lambda-\sigma)} \|P_{k_2} w\|^2_{L_t^{p_2} L_x^{q_2}} \Big)^{\frac{1}{2}} \Big(\sum_{k_2\geq 0} 2^{-2k_2\lambda} \Big)^{\frac{1}{2}} \\
&\lesssim \|v\|_{L_t^{p_1} W_x^{r+\sigma, q_1}[\mathbf{k}]} \|w\|_{L_t^{p_2} W_x^{\lambda-\sigma, q_2}[\mathbf{k}]}.
\end{aligned}$$

We now turn our attention to (II). Again, by Holder's Inequality and Young's

Inequality in k_1, we observe that

$$
\begin{aligned}
(II) &\lesssim \Big(\sum_{k\geq 0} 2^{2kr}\Big(\sum_{k_1\geq k-C}\sum_{|k_2-k_1|\leq C} \|P_{k_1}v\|_{L_t^{p_1}L_x^{q_1}}\|P_{k_2}w\|_{L_t^{p_2}L_x^{q_2}}\Big)^2\Big)^{\frac{1}{2}} \\
&\lesssim \Big(\sum_{k\geq 0} 2^{2kr}\Big(\sum_{k_1\geq k-C}\sum_{|k_2-k_1|\leq C} 2^{k_1(\sigma)}\|P_{k_1}v\|_{L_t^{p_1}L_x^{q_1}}2^{k_2(-\sigma)}\|P_{k_2}w\|_{L_t^{p_2}L_x^{q_2}}\Big)^2\Big)^{\frac{1}{2}} \\
&\lesssim \Big(\sum_{k\geq 0} 2^{2kr}\Big(\sum_{k_1\geq k-C} 2^{k_1(\sigma)}\|P_{k_1}v\|_{L_t^{p_1}L_x^{q_1}}\Big)^2\Big(\sum_{k_2\geq k-C} 2^{k_2(-\sigma)}\|P_{k_2}w\|_{L_t^{p_2}L_x^{q_2}}\Big)^2\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Applying the Cauchy-Schwartz inequality in both k_1 and k_2 independently, we conclude this is

$$
\begin{aligned}
&\lesssim \Big(\sum_{k\geq 0} 2^{2kr}\Big(\sum_{k_1\geq k-C} 2^{-2k_1 r}\Big)\|v\|^2_{L_t^{p_1}W_x^{r+\sigma,q_1}[\mathbf{k}]}\Big(\sum_{k_2\geq k-C} 2^{-2k\lambda}\Big)\|w\|^2_{L_t^{p_2}W_x^{\lambda-\sigma,q_2}[\mathbf{k}]}\Big)^{\frac{1}{2}} \\
&\lesssim \Big(\sum_{k\geq 0} 2^{2k(r-r-\lambda)}\Big)^{\frac{1}{2}}\|v\|_{L_t^{p_1}W_x^{r+\sigma,q_1}[\mathbf{k}]}\|w\|_{L_t^{p_2}W_x^{\lambda-\sigma,q_2}[\mathbf{k}]} \\
&\lesssim \|v\|_{L_t^{p_1}W_x^{r,q_1}[\mathbf{k}]}\|w\|_{L_t^{p_2}W_x^{\lambda,q_2}[\mathbf{k}]}.
\end{aligned}
$$

By interchanging the roles of v and w in the proof of estimate (I), we can conclude that

$$
(III) \lesssim \|v\|_{L_t^{\tilde{p}_1}W_x^{\tilde{\lambda}-\tilde{\sigma},\tilde{q}_1}[\mathbf{k}]}\|w\|_{L_t^{\tilde{p}_2}W_x^{r+\tilde{\sigma},\tilde{q}_2}[\mathbf{k}]}.
$$

□

2.3 Strichartz Estimates

We can take advantage of the dispersive nature of the Klein-Gordon equation, along with interpolation and duality arguments, to obtain a very useful set of mixed norm Sobolev space estimates known as Strichartz estimates. We dedicate this section to proving these estimates.

Suppose $k \geq 0$ and ψ_k is defined as in the previous section, and let $\phi_k := \mathcal{F}^{-1}(\psi_k)$. We present the following set of estimates without proof.

Proposition 2.3.1 ([2], Theorem 3.2, [10], Appendix)**.** *Let $n \geq 2$, then*

$$\|e^{it\langle D\rangle}\phi_0\|_{L_x^\infty(\mathbb{R}^n)} \lesssim \min\{1, |t|^{-\frac{n}{2}}\}, \tag{2.4}$$

$$\|e^{it\langle D\rangle}\phi_k\|_{L_x^\infty(\mathbb{R}^n)} \lesssim 2^{nk}\min\{1, (2^k|t|)^{-\frac{n-1}{2}}\}\min\{1, (2^{-k}|t|)^{-\frac{1}{2}}\} \tag{2.5}$$

for $k \geq 1$.

Theorem 2.3.1 ([16], Lemma 2.1)**.** *Suppose $f \in L_x^2(\mathbb{R}^n)$ and let p, q be such that $2 < p \leq \infty$, $2 \leq q < \frac{2\sigma_i}{\sigma_i - 1}$, and $\frac{1}{p} + \frac{\sigma_i}{q} = \frac{\sigma_i}{2}$, where $\sigma_1 = \frac{n}{2}, \sigma_2 = \frac{n-1}{2}$, and $\sigma_i \geq 1$, then*

$$\|e^{it\langle D\rangle}P_kf\|_{L_t^pL_x^q} \lesssim 2^{k\alpha(q)}\|P_kf\|_{L_x^2} \tag{2.6}$$

,where $\alpha(q) = \lambda_i(\frac{1}{2} - \frac{1}{q})$ for $\lambda_i = \frac{2\sigma_i+2}{2}$.

Proof. Observe that $\hat{\phi}_k = \sum_{j=k-1}^{k+1}\hat{\phi}_j\hat{\psi}_k$ so that $\phi_k * e^{it\langle D\rangle}f = \sum_{j=k-1}^{k+1}e^{it\langle D\rangle}\phi_j * (\phi_k * f)$.

From (2.5), we have

$$\|e^{it\langle D\rangle}\phi_k\|_{L_x^\infty} \lesssim 2^{\lambda_i k}|t|^{-\sigma_i}.$$

Combining these two facts gives us the bounds

$$\begin{aligned}\|\phi_k * (e^{it\langle D\rangle}f)\|_{L_x^\infty} &\leq \sum_{k=j-1}^{j+1}\|e^{it\langle D\rangle}\phi_j\|_{L_x^\infty}\|\phi_k * f\|_{L_x^1} \\ &\lesssim 2^{\lambda_i}|t|^{-\sigma_i}\|P_kf\|_{L_x^1}.\end{aligned}$$

Unitarity of the operator $T := e^{it\langle D\rangle}$ in L^2 gives us

$$\|\phi_k * (e^{it\langle D\rangle}f)\|_{L_x^2} \leq \|P_kf\|_{L_x^2}.$$

Interpolating between these two results yields

$$\|e^{it\langle D\rangle}(P_k f)\|_{L^q_x} \leq 2^{2\alpha(q)k}|t|^{-2\sigma_i(\frac{1}{2}-\frac{1}{q})}\|f_k\|_{L^{q'}_x}$$

for $2 \leq q \leq \infty$. Applying the Hardy-Littlewood-Sobolev theorem of fractional integration in the time variable results in the estimate

$$\|\int_{-\infty}^{\infty} e^{i(t-s)\langle D\rangle} P_k f ds\|_{L^p_t L^q_x} \lesssim 2^{2\alpha(q)k}\|P_k f\|_{L^{p'}_t L^{q'}_x}.$$

We now apply what is known as a TT^* argument.

If we let $T := e^{it\langle D\rangle}$, then the above expression is equivalent to

$$\|TT^* P_k f\|_{L^p_t L^q_x} \lesssim 2^{2\alpha(q)k}\|P_k f\|_{L^{p'}_t L^{q'}_x} \tag{2.7}$$

This gives us our desired result, for

$$\|T^* P_k f\|^2_{L^2_x} = \langle T^* P_k f, T^* P_k f\rangle = \int \langle TT^* P_k f, P_k f\rangle dt$$

$$\lesssim \|TT^* P_k f\|_{L^p_t L^q_x}\|f_k\|_{L^{p'}_t L^{q'}_x}$$

Therefore, by (2.7), we have

$$\|T^* P_k f\|_{L^2_x} \lesssim 2^{\alpha(q)k}\|P_k f\|_{L^{p'}_t L^{q'}_x}, \tag{2.8}$$

which by duality gives us (2.6). □

We introduce the following result without proof.

Lemma 2.3.1. *(Christ-Kiselev([23], Lemma 2.4)) Suppose* $1 \leq q, \tilde{q} \leq \infty$ *and* I *is a time interval. Let* $K \in C(I \times I; B(L^{\tilde{q}}, L^q))$ *be a kernel taking values in the space of bounded linear operators from* $L^{\tilde{q}}$ *to* L^q *and suppose that* $1 \leq \tilde{p} < p \leq \infty$ *are such that*

$$\|\int_I K(t,s)f(s)ds\|_{L^p_t(I;L^q_x)} \leq A\|f\|_{L^{\tilde{p}}_t(I;L^{\tilde{q}}_x)}$$

for all $f \in L_t^{\tilde{p}}(I; L_x^{\tilde{q}})$ *and some* $A > 0$. *Then, one also has*

$$\| \int_{s \in I : s < t} K(t,s) f(s) ds \|_{L_t^p(I; L_x^q)} \lesssim_{\tilde{p}, p} A \|f\|_{L_t^{\tilde{p}}(I; L_x^{\tilde{q}})}.$$

Theorem 2.3.2 ([16], Lemma 2.1)**.** *Let* σ_i, λ_i *for* $i \in \{1,2\}$ *be as before. Suppose* p_1, p_2, q_1, q_2 *are such that* $2 < p_j \leq \infty$, $2 \leq q_j < \frac{2\sigma}{\sigma - 2}$ *for* $j \in \{1,2\}$, *and* $\frac{2}{p_i} + \frac{\sigma}{q_i} = \frac{\sigma}{2}$, *then*

$$\| \int_{-\infty}^{t} e^{i(t-s)\langle D \rangle} P_k G(s) ds \|_{L_t^{p_1} L_x^{q_1}} \lesssim 2^{\lambda_i (\frac{1}{q_2'} - \frac{1}{q_1'})} \|P_k G\|_{L_t^{p_2'} L_x^{q_2'}}. \tag{2.9}$$

Proof. As before, define $T := e^{it\langle D \rangle}$.

From (2.6), we conclude that

$$\|T P_k G(s)\|_{L_t^{p_1} L_x^{q_1}} \leq 2^{\lambda_i (\frac{1}{2} - \frac{1}{q_1})} \|P_k G(s)\|_{L_x^2}$$

and

$$\|T P_k G(s)\|_{L_t^{p_2} L_x^{q_2}} \leq 2^{\lambda_i (\frac{1}{2} - \frac{1}{q_1})} \|P_k G(s)\|_{L_x^2}.$$

By duality, it follows that

$$\begin{aligned} \|T^* P_k G(s)\|_{L^2} &\leq 2^{\lambda_i (\frac{1}{2} - \frac{1}{q_2})} \|P_k G\|_{L_t^{p_2'} L_x^{q_2'}} \\ &= 2^{\lambda_i (\frac{1}{q_2'} - \frac{1}{2})} \|P_k G\|_{L_t^{p_2'} L_x^{q_2'}}. \end{aligned}$$

Combining these two estimates gives

$$\|T T^* P_k G\|_{L_t^{p_1} L_x^{q_1}} \leq 2^{\lambda_i (\frac{1}{q_2'} - \frac{1}{q_1'})} \|G_k\|_{L_t^{p_2'} L_x^{q_2'}},$$

where

$$T T^* P_k G = \int_{-\infty}^{\infty} e^{i(t-s)\omega} P_k G(s) ds.$$

Invoking the Christ-Kiselev lemma from above gives us (2.9). □

Theorem 2.3.3. *If $n \geq 4$, we have the following estimates*

$$\|e^{it\langle D\rangle}P_k f\|_{L_t^2 L_x^\infty} \lesssim 2^{k(\frac{n-1}{2})}\|P_k f\|_{L^2}, \tag{2.10}$$

$$\|\int_0^t e^{i(t-s)\langle D\rangle}P_k G(s)ds\|_{L_t^2 L_x^\infty} \lesssim 2^{k(\frac{n-1}{2})}\|P_k G\|_{L_t^1 L_x^2}. \tag{2.11}$$

Proof. Let $T := e^{it\langle D\rangle}$, then the first estimate above is equivalent to

$$\|T^* f_k\|_{L^2} \lesssim 2^{k(\frac{n-1}{2})}\|f_k\|_{L_t^2 L_x^1}, \tag{2.12}$$

which is equivalent to

$$\|TT^* f_k\|_{L_t^2 L_x^\infty} \lesssim 2^{k(n-1)}\|f_k\|_{L_t^2 L_x^1}. \tag{2.13}$$

By definition,

$$\begin{aligned} TT^* &= \int_0^\infty [\mathcal{F}^{-1}(e^{i(t-s)\langle\xi\rangle}\psi_k(\xi)) * f](x)ds \\ &= [\mathcal{F}^{-1}(e^{it\langle\xi\rangle}\psi_k(\xi)) * f](t,x). \end{aligned}$$

By Young's ineqality, the LHS of (2.13) is bounded by $\|e^{it\langle D\rangle}\phi_k\|_{L_t^1 L_x^\infty}$, so it suffices to prove that $\|e^{it\langle D\rangle}\phi_k\|_{L_t^1 L_x^\infty} \lesssim 2^{k(n-1)}$ for all $k \geq 0$. We observe that

$$\begin{aligned} \|e^{it\langle D\rangle}\phi_k\|_{L_t^1 L_x^\infty} &\leq \int_0^{2^{-k}} \|e^{it\omega}\psi_k\|_{L_x^\infty}dt + \int_{2^{-k}}^{2^k} \|e^{it\omega}\psi_k\|_{L_x^\infty}dt + \int_{2^k}^\infty \|e^{it\omega}\psi_k\|_{L_x^\infty}dt \\ &= I + II + III. \end{aligned}$$

By estimate (2.5),

$$I \lesssim \int_0^{2^{-k}} 2^{nk} dt = 2^{nk} 2^{-k} = 2^{k(n-1)}$$

$$II \lesssim \int_{2^{-k}}^{2^k} 2^{nk} 2^{-\frac{n-1}{2}k} |t|^{-\frac{n-1}{2}} dt$$

$$\lesssim 2^{-\frac{n+1}{2}k} t^{-\frac{n-3}{2}}]_{2^k}^{2^{-k}}$$

$$\lesssim 2^{(n-1)k}$$

$$III \lesssim \int_{2^k}^{\infty} 2^{nk} 2^{-\frac{n+1}{2}k} |t|^{-\frac{n}{2}} dt$$

$$\lesssim 2^{2k}$$

$$\lesssim 2^{(n-1)k}.$$

Combining estimate (2.12) and (2.6) for $p = \infty, q = 2$, we obtain

$$\| \int_0^{\infty} e^{i(t-s)\langle D \rangle} P_k G(s) ds \|_{L_t^2 L_x^{\infty}} \lesssim 2^{k(\frac{n-1}{2})} \| P_k G \|_{L_t^1 L_x^2}.$$

invoking the Christ-Kiselev Lemma as in the proof of the previous Theorem, gives us estimate (2.11). □

Theorem 2.3.4. *If $n = 3$, the following estimates hold true*

$$\| e^{it\langle D \rangle} P_k f \|_{L_t^2 L_x^{\infty}} \lesssim 2^k \langle k \rangle^{\frac{1}{2}} \| P_k f \|_{L_x^2}, \tag{2.14}$$

$$\| \int_0^t e^{i(t-s)\langle D \rangle} P_k G(s) ds \|_{L_t^2 L_x^{\infty}} \lesssim 2^k \langle k \rangle^{\frac{1}{2}} \| P_k G \|_{L_t^1 L_x^2}, \tag{2.15}$$

for $k \geq 0$.

Proof. Define

$$T := e^{it\langle D \rangle}.$$

By the TT^* argument we used before, (2.14) follows from proving

$$\|TT^*P_kf\|_{L_t^2L_x^\infty} \lesssim 2^{2k}\langle k\rangle \|P_kf\|_{L_t^2L_x^1}. \tag{2.16}$$

We may assume that $k \geq 1$ as the estimate for $k = 0$ will follow by a similar argument if one replaces estimate (2.5) with (2.4). We observe that

$$\begin{aligned} T(P_kf)(t,x) &= \int_{\mathbb{R}^3} e^{i(x\cdot\xi + t\langle\xi\rangle)}\tilde{\psi}_k(\xi)\psi_k(\xi)\hat{f}(\xi)d\xi \\ &= \sum_{j=k-1}^{k+1}\int_{\mathbb{R}^3} K_j(t,x-y)P_kf(y)dy, \end{aligned}$$

where

$$K_j(t,x) = \int_{\mathbb{R}^3} e^{ix\cdot\xi}e^{it\langle\xi\rangle}\psi_j(\xi)d\xi.$$

Then,

$$\begin{aligned} TT^*P_kf(t,x) &= \int_{\mathbb{R}} e^{i(t-s)\langle D\rangle}P_kf ds \\ &= \sum_{j=k-1}^{k+1}\int_{\mathbb{R}}\int_{\mathbb{R}^2} K_j(t-s,x-y)P_kf(y)dyds \\ &= \sum_{j=k-1}^{k+1} K_j * P_kf. \end{aligned}$$

By Young's inequality, we have

$$\|TT^*P_kf\|_{L_t^2L_x^\infty} \lesssim \|K_k\|_{L_t^1L_x^\infty}\|P_kf\|_{L_t^2L_x^1},$$

so we are reduced to proving

$$\|K_k\|_{L_t^1L_x^\infty} \lesssim 2^{2k}\langle k\rangle. \tag{2.17}$$

From (2.5), it follows that

$$|K_k(t,x)| \lesssim 2^{3k}(1+2^k|(t,x)|)^{-1}\min(1,(1+2^k|(t,x)|)^{-\frac{1}{2}}2^k). \tag{2.18}$$

Define

$$E_1 = \{|(t,x)| \le 2^{-k}\}, \quad E_2 = \{2^{-k} \le |(t,x)| \le 2^k\}, \quad E_3 = \{|(t,x)| \ge 2^k\},$$

then, the left hand side of (2.17) is bounded by

$$\|K_k\|_{L^1_t L^\infty_x(E_1)} + \|K_k\|_{L^1_t L^\infty_x(E_2)} + \|K_k\|_{L^1_t L^\infty_x(E_3)} = I + II + II.$$

Applying estimate (2.18) to (I), we deduce that

$$\begin{aligned}(I) &\lesssim \|2^{3k}(1+2^k|(t,x)|)^{-\frac{1}{2}}\|_{L^1_t L^\infty_x(E_1)}\\ &\le \|2^{3k}(1+2^k|t|)^{-\frac{1}{2}}\|_{L^1_t(E_1)}\\ &\lesssim \int_0^{2^{-k}} 2^{3k} dt \approx 2^{2k}.\end{aligned}$$

Applying estimate (2.18) to (III), we obtain

$$\begin{aligned}(III) &\lesssim \|2^{3k}(1+2^k|(t,x)|)^{-\frac{3}{2}} 2^k\|_{L^1_t L^\infty_x(E_3)}\\ &\le \|2^{3k}2^{-3k/2}|(t,x)|)^{-\frac{3}{2}} 2^k\|_{L^1_t L^\infty_x(E_3)}\\ &\lesssim \int_{2^k}^\infty 2^{5k/2}|t|^{-3/2} dt \approx 2^{5k/2}2^{-k/2} = 2^{2k}.\end{aligned}$$

We apply estimate (2.18) to (II) to deduce

$$\begin{aligned}(II) &\lesssim \|2^{3k}(1+2^k|(t,x)|)^{-1}\|_{L^1_t L^\infty_x(E_2)}\\ &\lesssim \|2^{2k}|t|^{-1}\|_{L^1_t L^\infty_x(E_2)}\\ &\approx 2^{2k}\int_{2^{-k}}^{2^k} |t|^{-1} dt \approx 2^{2k} k \ln(2) \approx 2^{2k} k.\end{aligned}$$

We conclude that

$$\|K_k\|_{L^1_t L^\infty_x} \lesssim 2^{2k}(1+|k|) \approx 2^{2k}\langle k\rangle,$$

as desired.

Combining the dual estimate of (2.14) and (2.6) for $p = \infty, q = 2$, we obtain

$$\| \int_0^\infty e^{i(t-s)\langle D\rangle} P_k G(s) ds \|_{L_t^2 L_x^\infty} \lesssim 2^k \langle k\rangle^{\frac{1}{2}} \|P_k G\|_{L_t^1 L_x^2}.$$

invoking the Christ-Kiselev Lemma gives us estimate (2.15).

□

Theorem 2.3.5. *Suppose $n = 3$ and $\epsilon > 0$ is given, then we have the estimates*

$$\|e^{it\langle D\rangle} f\|_{L_t^2 L_x^\infty[\boldsymbol{k}]} \lesssim_\epsilon \|f\|_{H^{1+\epsilon}[\boldsymbol{k}]}, \tag{2.19}$$

$$\| \int_0^\infty e^{i(t-s)\langle D\rangle} G(s) \|_{L_t^2 L_x^\infty[\boldsymbol{k}]} \lesssim_\epsilon \|G(s)\|_{L_t^1 H^{1+\epsilon}[\boldsymbol{k}]}. \tag{2.20}$$

Proof. From Theorem 2.3.4, we have

$$\|e^{it\langle D\rangle} P_k f\|_{L_t^2 L_x^\infty} \lesssim 2^k \langle k\rangle^{\frac{1}{2}} \|P_k f\|_{L^2}.$$

As $\|f\|_{H^{1+\epsilon}[\mathbf{k}]} \sim \left(\sum_{k=0}^\infty 2^{k(1+\epsilon)} \|P_k f\|_{L^2}^2\right)^{1/2}$, (2.19) follows from proving

$$2^k \langle k\rangle^{\frac{1}{2}} \leq c(\epsilon) 2^{k(1+\epsilon)}.$$

It suffices to show that

$$\langle k\rangle^{\frac{1}{2}} \leq c(\epsilon) 2^{k\epsilon}.$$

Taking ln of both sides, this is equivalent to proving

$$\frac{1}{2} \ln\langle k\rangle \leq c(\epsilon) k\epsilon \ln 2. \tag{2.21}$$

If $k = 0$, this inequality is obviously valid so assume $k \geq 1$. We may bound the LHS of the above by

$$\frac{1}{2} \ln\langle k\rangle \leq \frac{1}{2} \ln(1 + k) \leq \frac{1}{2} k.$$

Letting $c(\epsilon) = \frac{2}{\epsilon \ln 2}$ gives us the desired result.

We now turn our attention to proving (2.20). From (2.15) we have

$$\| \int_0^t e^{i(t-s)\langle D \rangle} P_k G(s) ds \|_{L_t^2 L_x^\infty} \lesssim 2^k \langle k \rangle^{\frac{1}{2}} \| P_k G \|_{L_t^1 L_x^2}.$$

Estimate (2.20) then follows from (2.21) and the definition of $L_t^1 H^{1+\epsilon}[\mathbf{k}]$.

□

2.4 U^p and V^p spaces

If we assume the size of our initial data is small, we can view nonlinear PDEs as small perturbations of linear ones, at least heuristically speaking. It is therefore useful to study the properties of free solutions, as well as how pairs of free solutions interact with one another. In order to extend our findings to more general functions, we will need to find a space rich enough in structure to "record" the behavior we have seen in free solutions. For this task, we introduce the atomic space U^p, as well as its dual (to be defined precisely later) V^p and review their basic properties. We draw our results from [11] where detailed proofs can be found.

To begin with, let $\mathcal{Z}$ denote the set of infinite partitions $-\infty = t_0 < t_1 < ... < t_k = \infty$ and let $\mathcal{Z}_0$ denote the set of finite partitions $-\infty < t_0 < t_1 < ... < t_k < \infty$. In what follows we will primarily consider functions whose values belong to $L^2 := L^2(\mathbb{R}^d; \mathbb{C})$ but the results can be generalized to an arbitrary Hilbert Space.

Definition 2.4.1. *Let $1 \leq p < \infty$, $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subseteq L_x^2$ be such that $\sum_{k=0}^{K-1} \|\phi_k\|_{L^2}^p = 1$ and $\phi_0 = 0$. We call the function $a : \mathbb{R} \to L^2$, given by*

$a = \sum_{k=1}^{K} \mathcal{X}_{[t_{k-1},t_k)}\phi_{k-1}$ a U^p atom. Furthermore, define the atomic space

$$U^p := \{u = \sum_{j=1}^{\infty} \lambda_j a_j | \, a_j \text{ is a } U^p\text{-atom}, \lambda_j \in \mathbb{C} \text{ such that } \sum_{j=1}^{\infty} |\lambda_j| < \infty, \}$$

with norm,

$$\|u\|_{U^p} := \inf\{u = \sum_{j=1}^{\infty} |\lambda_j| \, | \, u = \sum_{j=1}^{\infty} \lambda_j a_j, \, a_j \text{ is a } U^p\text{-atom}, \lambda_j \in \mathbb{C}\}.$$

Definition 2.4.2. *Let $1 \leq p < \infty$. We define V^p as the normed space of all functions $v : \mathbb{R} \to L^2$ such that $v(\infty) := \lim_{t\to\infty} v(t) = 0$ and $v(-\infty)$ exists and for which the norm*

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^{K} \in \mathcal{Z}} (\sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L^2}^p)^{1/p}$$

is finite. Likewise, let V_-^p denote the normed space of functions $v : \mathbb{R} \to L^2$ such that $v(-\infty) = 0, v(\infty)$ exists, and $\|v\|_{V^p} < \infty$

Definition 2.4.3. *Define the closed subspace $V_{rc}^p(V_{-,rc}^p)$ of all right continuous V^p functions (V_-^p functions).*

The following two results illustrate the duality relationship between U^p and $V^{p'}$

Theorem 2.4.1 ([11], Theorem 2.8)**.** *Let $1 < p < \infty$. We have*

$$(U^p)^* = V^{p'}$$

in the sense that there is a bilinear form B such that the mapping

$$T : V^{p'} \to (U^p)^*, T(v) := B(*, v)$$

is an isometric isomorphism.

Proposition 2.4.1 ([11], Proposition 2.10)**.** *Let $1 < p < \infty, u \in V^1_-$ be absolutely continuous on compact intervals and $v \in V^{p'}$. Then,*

$$B(u,v) = -\int_{-\infty}^{\infty} \langle u'(t), v(t)\rangle dt.$$

Definition 2.4.4. *Define $U^p_\pm$ by the norm $\|f\|_{U^p_\pm} = \|e^{\mp it\langle D\rangle} f\|_{U^p}$ and define $V^p_\pm$ in the analogous way. In what follows $L^2 = L^2(\mathbb{R}^{n+1})$.*

Definition 2.4.5. *Let $M = 2^k$ for some $k \in \mathbb{Z}$, then define the multiplier Q_M by*

$$\mathcal{F}_{\tau,\xi}(Q^\pm_M u)(\tau,\xi) = \psi_M(\tau \mp \langle\xi\rangle)\mathcal{F}_{\tau,\xi}(u)(\tau,\xi),$$

$$\mathcal{F}_{\tau,\xi}(Q^\pm_{\leq M} u)(\tau,\xi) = \psi_{\leq M}(\tau \mp \langle\xi\rangle)\mathcal{F}_{\tau,\xi}(u)(\tau,\xi)$$

Definition 2.4.6. *Let $u \in U^p_\pm$ and $(\tau,\xi) \in \mathbb{R}^{n+1}$, then the value $|\tau \mp \langle\xi\rangle|$ is called the modulation of u at (τ,ξ). We say that the modulation is high if $|\tau \mp \langle\xi\rangle| \geq \frac{|\xi|}{8}$.*

Proposition 2.4.2 ([11], Corollary 2.18)**.** *We have, for $M = 2^k, k \in \mathbb{Z}$,*

$$\|Q^\pm_M u\|_{L^2} \lesssim M^{-\frac{1}{2}}\|u\|_{V^2_\pm},$$

$$\|Q^\pm_{\geq M} u\|_{L^2} \lesssim M^{-\frac{1}{2}}\|u\|_{V^2_\pm},$$

$$\|Q^\pm_{<M} u\|_{V^p_\pm} \lesssim \|u\|_{V^p_\pm} \quad , \quad \|Q^\pm_{\geq M} u\|_{V^p_\pm} \lesssim \|u\|_{V^p_\pm}$$

$$\|Q^\pm_{<M} u\|_{U^p_\pm} \lesssim \|u\|_{U^p_\pm} \quad , \quad \|Q^\pm_{\geq M} u\|_{U^p_\pm} \lesssim \|u\|_{U^p_\pm}$$

Observe that the simplest type of elements in $U^p_\pm$ are free solutions and it is therefore natural to expect that we can extend estimates on free solutions to estimates for more general functions in $U^p_\pm$ and its dual $V^{p'}_\pm$. In order to formalize this idea, we will need the following two results

Proposition 2.4.3 ([11], Proposition 2.19). *Let*

$$T_0 : L^2 \times \cdots \times L^2 \to L^1_{loc}(\mathbb{R}^n; \mathbb{C})$$

be a non-linear operator. Assume that for some $1 \leq p, q \leq \infty$

$$\|T_0(e^{\pm_1 it\langle D\rangle}\phi_1, \dots e^{\pm_m it\langle D\rangle}\phi_m)\|_{L^p_t(\mathbb{R}; L^q_{x,y}(\mathbb{R}^n))} \lesssim \prod_{i=1}^{n} \|\phi_i\|_{L^2},$$

then there exists

$$T : U^p_{\pm_1} \times \cdots \times U^p_{\pm_m} \to L^p_t(\mathbb{R}; L^q_{x,y}(\mathbb{R}^n))$$

satisfying

$$\|T(u_1, \dots, u_m)\|_{L^p_t(\mathbb{R}; L^q_{x,y}(\mathbb{R}^n))} \lesssim \prod_{i=1}^{n} \|u_i\|_{U^p_\pm}$$

such that $T(u_1, \dots, u_m)(t)(x, y) = T_0(u_1(t), \dots, u_m(t))(x, y)$

We remark that the above result implies that we can apply Strichartz estimates to general functions in $U^p_\pm$.

Proposition 2.4.4 ([11], Proposition 2.20). *Let* $q > 1, E$ *be a Banach space and* $T : U^q_\pm \to E$ *be a bounded, linear operator with* $\|Tu\|_E \leq C_q \|u\|_{U^q_\pm}$ *for all* $u \in U^q_\pm$ *. In addition, assume that for some* $1 \leq p < q$ *there exists* $C_p \in (0, C_q]$ *such that the estimate* $\|Tu\|_E \leq C_p \|u\|_{U^p_\pm}$ *Then,* T *satisfies the estimate*

$$\|Tu\|_E \lesssim C_p \log(\frac{C_q}{C_p}) \|u\|_{V^p_\pm}.$$

Chapter 3

General Existence Techniques

Our main goal in this section is to introduce a general approach to proving existence and uniqueness of solutions to

$$(\Box + 1)u = F(u, \partial u, \partial_t u) \tag{3.1}$$

with initial data

$$u(0, x) = u_0, \qquad u_t(0, x) = u_1, \tag{3.2}$$

such that $\|(u_0, u_1)\|_{H^s_x \times H^{s-1}_x} < \delta$ for some sufficiently small $\delta > 0$.

The main techniques discussed in this chapter will be the contraction method and the bootstrap argument. In section 3.1, we outline how to apply the contraction method to prove local existence of solutions to the system presented above. Our argument will rely primarily on the Banach Fixed Point Theorem

Theorem 3.0.1. *Let X be a Banach space and suppose $f : X \to X$ is a map such that*

$$\|f(x) - f(y)\|_X \leq C\|x - y\|_X$$

for some $0 \le C < 1$ *and all* $x, y \in X$*. Then* f *has a unique fixed point in* X*.*

Section 3.2 is dedicated to a discussion of the bootstrap argument. This approach takes advantage of what is known as the Bootstrap Principle to establish global existence. While our discussions will be tailored to the inhomogeneous Klein-Gordon system, we remark that these approaches can be applied to a wide class of PDEs. For a more general discussion of these methods, we refer the reader to Chapters 1 and 3 in [23] and Chapter 9 in [7].

3.1 The Contraction Method

Suppose we are given a "nice enough" function $w : \mathbb{R}^{n+1} \to \mathbb{R}$ (perhaps belonging to some mixed norm Sobolev space). It is not difficult to see that the map $t \to A(w)(t)$, given by

$$A(w) = cos(t\langle D\rangle)u_0 + sin(t\langle D\rangle)u_1 + \int_0^t \frac{\sin(\langle D\rangle(t-s))}{\langle D\rangle} F(w(s), \partial w(s), \partial_t w(s))ds \quad (3.3)$$

solves the system

$$(\Box + 1)u = F(w, \partial w, \partial_t w)$$

with initial data

$$u(0,x) = u_0, \qquad u_t(0,x) = u_1. \quad (3.4)$$

We therefore seek a function u that satisfies $A(u) = u$, also known as a fixed point of the mapping A. From the Banach fixed point theorem, we know that it suffices to find a suitable Banach space in which we can run our contraction argument. Unfortunately, this is more easily said than done.

Let $T > 0$ be fixed and suppose $X_T \subseteq L_t^\infty([0,T]; H_x^s)$ is a Banach space. Given $\delta > 0$ we define

$$S_T(\delta) := \{u \in X_T : \|u\|_{X_T} \leq C_0\delta\},$$

where C_0 is a constant that will be determined later.

As S_T is a a closed subset of the Banach Space X_T, it is also a Banach space. Suppose we want to show that A is a contraction on S_T. We first need to establish that A is well-defined. In particular, given $v \in S_T$, we need to show that

$$\|A(v)\|_{X_T} \leq C_0\delta. \tag{3.5}$$

In order to prove that A is a contraction, it suffices to show that

$$\|A(v) - A(w)\|_{X_T} \leq \frac{1}{2}\|v - w\|_{X_T} \tag{3.6}$$

for all $v, w \in S_T$.

We rewrite equation (3.3) as

$$A(w) = W(u_0, u_1) + L(F(w, \partial w, \partial_t w)).$$

Equations (3.5) and (3.6) therefore follow from proving

$$\|W(u_0, u_1)\|_{S_T} \leq \frac{1}{2}C_0\|(u_0, u_1)\|_{H_x^s \times H_x^{s-1}} \tag{3.7}$$

and

$$\|L(F(w, \partial w, \partial_t w))\|_{S_T} \leq \frac{1}{2}C_0\delta, \tag{3.8}$$

$$\|L(F(v, \partial v, \partial_t v)) - L(F(w, \partial w, \partial_t w))\|_{S_T} \leq \frac{1}{2}\|v - w\|_{X_T} \tag{3.9}$$

for all $v, w \in S_T$.

Rather than prove estimates (3.8) and (3.9) directly, we construct an auxiliary space N_T and instead show that

$$\|L(G)\|_{S_T} \leq C\|G\|_{N_T} \tag{3.10}$$

and

$$\|F(w, \partial w, \partial_t w)\|_{N_T} \leq C'\delta, \tag{3.11}$$

$$\|F(v, \partial v, \partial_t v) - F(w, \partial w, \partial_t w)\|_{N_T} \leq C''\|v - w\|_{X_T} \tag{3.12}$$

where $CC' \leq \frac{1}{2}C_0$ and $CC'' \leq \frac{1}{2}$. It is an easy exercise to see that proving the three estimates above is sufficient to close the argument.

We make a few closing remarks about the solution u obtained by the argument above. First, notice that the Banach fixed point theorem implies not only existence, but uniqueness as well. Furthermore, as we have established that $u \in S_T$ and $S_T \subseteq L_t^\infty H_x^s$, we may conclude that $u \in L_t^\infty H_x^s$. In fact, from the definition of A in equation (3.3) it is easy to see that $u \in C([0,T]; H_x^s)$.

Finally, we claim that the solution map is continuous with respect to the initial data. To see why this is the case, consider two sets of initial data (v_0, v_1) and (w_0, w_1) in $H_x^s \times H_x^{s-1}$ with corresponding solutions v and w. Let $(u_0, u_1) := (v_0 - w_0, v_1 - w_1)$, and suppose $\|(u_0, u_1)\|_{H_x^s \times H_x^{s-1}} < \epsilon$ for some $0 < \epsilon \ll 1$. By the contraction argument above we may conclude $\|u\|_{S_T} = \|v - w\|_{S_T} \leq C_0\epsilon$. Letting $\epsilon \to 0$ implies $\|v - w\|_{S_T} \to 0$, as desired.

3.2 Bootstrap and Continuity Methods

The reader may have noticed that the Banach spaces X_T and S_T introduced in the previous section were associated with a time $T > 0$. Often, the contraction argument we discussed in the previous section will only be valid on time intervals $[0, T]$ where $T < T_0$ for some fixed $T_0 \in \mathbb{R}$. This T_0 will usually have some type of inverse relationship with the size of the initial data $\|(u_0, u_1)\|_{H^s_x \times H^{s-1}_x}$, meaning we can extend the time of existence by lowering the size of the initial data. Unfortunately, this does not help us prove global existence as the size of the initial data must be nonzero. In this case, we must apply a different method to establish global existence. While there are numerous approaches, we will focus on only one: The continuity method.

Suppose we are able to prove local existence on some time interval $[0, T]$ using the contraction argument from the previous section. Repeating the argument using initial data $u(T, x), u_t(T, x)$, we can establish existence on the interval $[T, T + \epsilon_1]$, where ϵ_1 depends on $\|(u(T, x), u_t(T, x))\|_{H^s_x \times H^{s-1}_x}$. Iterating this process, we can extend the time of existence to $[0, T + \sum_{i=1}^{\infty} \epsilon_i]$. Unfortunately, if $t \to \|u(t, \cdot), u_t(t, \cdot)\|_{H^s_x \times H^{s-1}_x}$ approaches ∞ in finite time, the ϵ_i will approach 0 and it is possible that the sum $\sum_{i=1}^{\infty} \epsilon_i$ converges to a finite value.

If we can prove that $t \to \|u(t, \cdot), u_t(t, \cdot)\|_{H^s_x \times H^{s-1}_x}$ is bounded, then it will follow that $\sum_{i=1}^{\infty} \epsilon_i = \infty$, and thus global existence is established. As $\|u(t, \cdot), u_t(t, \cdot)\|_{H^s_x \times H^{s-1}_x} \lesssim \|u\|_{S_{T'}}$, if $t \in [0, T']$, this follows from showing $\|u\|_{S_{T'}} < C$ for all $T' > 0$. For this task we will apply what is known as the Bootstrap Principle.

Proposition 3.2.1 (Abstract Bootstrap Principle ([23], Proposition 1.21))**.** *Let I be a time interval, and for each $t \in I$ suppose we have a "hypothesis" $\mathbf{H(t)}$ and a "conclusion"*

$\mathbf{C}(\mathbf{t})$. *Suppose we can verify the following four assertions:*

(a)(Hypothesis implies conclusion) If $\mathbf{H}(\mathbf{t})$ *is true for some* $t \in I$, *then* $\mathbf{C}(\mathbf{t})$ *is also true for that time* t.

(b)(Conclusion is stronger than hypothesis) If $\mathbf{C}(\mathbf{t})$ *is true for some* $t \in I$, *then* $\mathbf{H}(\mathbf{t})$ *is true for all* $t' \in I$ *in a neighborhood of* t.

(c)(Conclusion is closed) If $t_1, t_2, \ldots$ *is a sequence of times in* I *converging to another time* $t \in I$ *and* $\mathbf{C}(\mathbf{t_n})$ *is true for all* t_n, *then* $\mathbf{C}(\mathbf{t})$ *is true.*

(d)(Base case) $\mathbf{H}(\mathbf{t})$ *is true for at least one time* $t \in I$.

Then, $\mathbf{C}(\mathbf{t})$ *is true for all* $t \in I$.

In what follows, assume $I = [0, \infty)$ and that $M > 0$ is a very large constant. Define the hypothesis, $\mathbf{H}(\mathbf{T})$, to be the assertion that we can find a solution, u, to (3.1)-(3.2) on $[0, T]$ using the contraction argument from the previous section, and that u satisfies

$$\|u\|_{S_T} < M\delta. \tag{3.13}$$

Define the conclusion, $\mathbf{C}(\mathbf{T})$, in the same way with estimate (3.13) replaced by

$$\|u\|_{S_T} < \frac{M}{2}\delta. \tag{3.14}$$

Suppose we have already proven local existence on some time interval $[0, T_0]$ using the contraction argument from the previous section. By the discussion above, global existence will be established if we can show that assertions (a)-(d) in Proposition 3.2.1 hold true for the $\mathbf{C}(\mathbf{T})$ and $\mathbf{H}(\mathbf{T})$ we have defined.

Obviously $\mathbf{H}(\mathbf{T_0})$ is true so (d) follows immediately. While it is not apparent from the abstract set-up presented in section 3.1 that (b) and (c) are true, in practice both will follow readily from the definitions of S_T we use.

It is left to prove assertion (a). Unfortunately, this is a challenging task. In fact, we dedicate a large portion of this thesis to proving (a) for the third order semilinear Klein-Gordon system.

Chapter 4

The Third Order Semilinear Klein-Gordon Equation

We dedicate this chapter to studying the third order semilinear Klein-Gordon equation. In section 4.1 we discuss the simpler power type cubic equation and move on to more general nonlinearities in section 4.2.

4.1 Power-type Nonlinearities

Among the simplest types of semi-linear systems to consider are those with a power-type nonlinearity. That is,

$$(\Box + 1)u = u^p \tag{4.1}$$

with initial data

$$u(0,x) = u_0 \in H^s, \quad \partial_t u(0,x) = u_1 \in H^{s-1}$$

where $p \geq 1$.

We can invoke structural properties of the above equation to provide restrictions on the values of s and p. A valuable heuristic to consider is scale invariance. Scaling heuristics are important because they predict a relationship between the time of existence and the regularity of the initial data. We observe that the wave equation

$$\Box u = u^p$$

is invariant under the transformation

$$u(t,x) \rightarrow u_\lambda(t,x) := \lambda^{\frac{-2}{p-1}} u(\frac{t}{\lambda}, \frac{x}{\lambda}),$$

where $\lambda \in \mathbb{R}$ and $\lambda > 0$.

We also remark that that the Schroedinger equation

$$(i\partial_t + \Delta)u = u^p$$

is invariant under the transformation

$$u(t,x) \rightarrow u_\lambda(t,x) := \lambda^{\frac{-2}{p-1}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda}).$$

Furthermore, we observe that in both cases $\|u\|_{\dot{H}^s_x} = \|u_\lambda\|_{\dot{H}^s_x}$ for $s = s_c := \frac{d}{2} - \frac{2}{p-1}$.

The value s_c is called the critical regularity. In the low spatial frequency regime the Klein-Gordon equation resembles the Schroedinger system, and in the high frequency case, it approximates to the wave equation. It is therefore reasonable to expect that the critical regularity for these two equations plays an important role for the Klein-Gordon as well.

Regularities $s > s_c$ are referred to as subcritical, and regularities $s < s_c$ are called supercritical. Higher regularity data is typically better behaved and we therefore expect subcritical solutions to be better behaved than critical solutions. In general, we expect problems with supercritical data to be ill-posed. For $s > s_c$ we can often trade between the size of the initial data and the time of existence: for example, if we can prove local well-posedness on a fixed time interval, $[0, T]$, for data with small H_x^s norm, then we can also establish local well-posedness for large data on a smaller time scale.

The scale invariance heuristic discussed above primarily produced restrictions on the regularity, s. In general, when p is low (say $p = 2, 3$), it is more difficult to establish global well-posedness in lower dimensions ($n = 2, 3$), especially when the initial data is assumed to have regularity near the critical value.

We will save our discussion on the case $p = 2$ for the next chapter as our proof will rely on some higher level machinery. Our primary focus in this section will be to establish global well-posedness for $p = 3$ and $n = 2, 3$. We also remark that in our argument, the nonlinearity $F(u) = u^3$ can be replaced by $F(u) = u^i|u|^{3-i}$ where $0 \leq i \leq 3$. For convenience, we consider only the case $F(u) = u^3$ as it will soon be apparent that our proof extends readily to the other cases. Our aim is to prove the following result

Theorem 4.1.1. *Suppose $s > 1/2$ and $n \in \{2, 3\}$ There exists an $\epsilon > 0$ such that for initial data*

$$(u_0, u_1) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n) \,, \quad \|(u_0, u_1)\|_{H^s \times H^{s-1}} < \epsilon,$$

the equation

$$(\Box + 1)u = u^3 \tag{4.2}$$

with initial data

$$u(0,x) = u_0 \in H^s, \quad \partial_t u(0,x) = u_1 \in H^{s-1} \tag{4.3}$$

has a unique global solution in $C([0,\infty), H^s)$ *which depends continuously on the initial data* (u_0, u_1).

Before embarking on our proof we remark that the critical regularity for $p = 3$ is $\frac{1}{2}$ for $n = 3$ and 0 for $n = 2$. The above theorem establishes well-posedness for all subcritical regularities in $n = 3$, but falls short in the $n = 2$ case. This is unsurprising given our previous discussion on the difficulties associated with lower spatial dimensions.

Proof. Given $T > 0$, let $X_T := L_t^\infty([0,T]; H_x^s) \cap L_t^4([0,T]; W_x^{s-1/2,4}[\mathbf{k}])$ with the corresponding norm

$$\|u\|_{X_T} = \left(\|u\|_{L_t^\infty H_x^s[\mathbf{k}]} + \|u\|_{L_t^4 W_x^{s-1/2,4}[\mathbf{k}]} \right)$$

We define the space S_T as

$$S_T(\epsilon) := \{u \in X_T : \|u\|_{X_T} \le C_0 \epsilon\}$$

where C_0 is a constant that will be determined later.

Finally, we introduce the space $N_T := L_t^{4/3}([0,T]; W_x^{s-1/2,4/3}[\mathbf{k}])$ with the obvious corresponding norm. From our discussion in Section 3.1, local well-posedness of the system (4.2)-(4.3) on $[0,T]$ will follow from proving that the following four estimates hold true for all $v, w \in S_T$

$$\|W(u_0,u_1)\|_{S_T} \le \frac{1}{2} C_0 \|(u_0,u_1)\|_{H_x^s \times H_x^{s-1}}, \tag{4.4}$$

$$\|L(G)\|_{S_T} \le C \|G\|_{N_T}, \tag{4.5}$$

$$\|w^3\|_{N_T} \leq C'\epsilon, \tag{4.6}$$

$$\|v^3 - w^3\|_{N_T} \leq C''\|v - w\|_{X_T}, \tag{4.7}$$

where we require $CC' \leq \frac{1}{2}C_0$, $CC'' \leq \frac{1}{2}$ and we recall that

$$W(u_0, u_1)(t, \cdot) = e^{it\langle D\rangle}u_0(\cdot) + \frac{e^{it\langle D\rangle}u_1(\cdot)}{\langle D\rangle}$$

and

$$L(G)(t, \cdot) := \int_0^t \frac{\sin(i(t-s)\langle D\rangle)G(s, \cdot)}{\langle D\rangle}ds.$$

We first turn our attention to proving estimate (4.4). From the definition of S_T we have

$$\begin{aligned}\|W(u_0, u_1)\|_{S_T} \leq \|e^{it\langle D\rangle}u_0\|_{L_t^\infty H_x^s} + \|e^{it\langle D\rangle}u_0\|_{L_t^4 W_x^{s-1/2,4}} \\ + \|\frac{e^{it\langle D\rangle}u_1}{\langle D\rangle}\|_{L_t^\infty H_x^s} + \|\frac{e^{it\langle D\rangle}u_1}{\langle D\rangle}\|_{L_t^4 W_x^{s-1/2,4}}.\end{aligned}$$

By Theorem 2.3.1 with $\sigma = 1$ and $p = \infty, q = 2$ for the first term and $p = 4, q = 4$ for the second, we obtain

$$\|e^{it\langle D\rangle}u_0\|_{L_t^\infty H_x^s} \lesssim \|u_0\|_{H_x^s}; \quad \|e^{it\langle D\rangle}u_0\|_{L_t^4 W_x^{s-1/2,4}} \lesssim \|u_0\|_{H_x^s}.$$

Similarly,

$$\|\frac{e^{it\langle D\rangle}u_1}{\langle D\rangle}\|_{L_t^\infty H_x^s} \lesssim \|u_1\|_{H_x^{s-1}}; \quad \|\frac{e^{it\langle D\rangle}u_1}{\langle D\rangle}\|_{L_t^4 W_x^{s-1/2,4}} \lesssim \|u_1\|_{H_x^{s-1}}.$$

We see that (4.4) follows if C_0 is chosen to be sufficiently large.

We now turn our attention to proving estimate (4.5). From the definition of S_T, we see that the LHS of (4.5) is

$$\leq \|\int_0^t e^{i(t-s)\langle D\rangle}G(s, \cdot)ds\|_{L_t^\infty H_x^{s-1}} + \|\int_0^t e^{i(t-s)\langle D\rangle}G(s, \cdot)ds\|_{L_t^4 W_x^{s-3/2,4}}.$$

Applying Theorem 2.3.2 with $\sigma = 1, p_2 = q_2 = 4$ and $p_1 = \infty, q = 2$ for the first term and $p_1 = 4, q_1 = 4$ for the second, we can bound the above by

$$\lesssim \|G\|_{L_t^{4/3} W_x^{s-1/2,4/3}} = \|G\|_{N_T}.$$

Rather than prove estimates (4.6) and (4.7) independently and directly, we will instead show that for all $z_1, z_2, z_3 \in X_T$, we have

$$\|z_1 z_2 z_3\|_{N_T} \le K \|z_1\|_{X_T} \|z_2\|_{X_T} \|z_3\|_{X_T}. \tag{4.8}$$

To see that (4.6) follows from (4.8), assume (4.8) is true. Suppose $w \in S_T$, then by (4.8) and the definition of S_T, we can conclude

$$\|w^3\|_{N_T} \le K(C_0 \epsilon)^3$$

if we assume ϵ is small enough so that $K(C_0\epsilon)^3 \le \frac{1}{2} C_0$.

To see that (4.7) also follows from (4.8), suppose that $v, w \in S_T$. Taking advantage of the inequality $|v^3 - w^3| \le 2(|v - w|(v^2 + w^2))$ and applying estimate (4.8), we obtain

$$\begin{aligned} \|v^3 - w^3\|_{N_T} &\le 2\|(v-w)v^2\|_{N_T} + 2\|(v-w)w^2\|_{N_T} \\ &\le 2K\|v - w\|_{X_T}(\|v\|_{X_T}^2 + \|w\|_{X_T}^2) \\ &\le 4K(C_0\epsilon)^2 \|v - w\|_{X_T}, \end{aligned}$$

where the last inequality comes from the definition of S_T. If we assume ϵ is small enough that $4K(C_0\epsilon)^2 \le C''$, then estimate (4.7) follows.

We now focus on proving (4.8). Applying Theorem 2.2.1 with $(p_1, p_2, q_1, q_2, \lambda, \sigma) = (\tilde{p}_1, \tilde{p}_2, \tilde{q}_1, \tilde{q}_2, \tilde{\lambda}, \tilde{\sigma}) = (4, 2, 4, 2, s - 1/2, 0)$, we obtain

$$\|z_1 z_2 z_3\|_{N_T} \lesssim \|z_1\|_{L_t^4 W_x^{s-1/2,4}[\mathbf{k}]} \|z_2 z_3\|_{L_t^2 H_x^{s-1/2}[\mathbf{k}]}.$$

Another iteration of Theorem 2.2.1 with $(p_1, p_2, q_1, q_2, \lambda, \sigma) = (\tilde{p}_1, \tilde{p}_2, \tilde{q}_1, \tilde{q}_2, \tilde{\lambda}, \tilde{\sigma}) = (4, 4, 4, 4, s - 1/2, 0)$ allows us to bound the above by

$$\|z_1\|_{L_t^4 W_x^{s-1/2,4}[\mathbf{k}]} \|z_2\|_{L_t^4 W_x^{s-1/2,4}[\mathbf{k}]} \|z_3\|_{L_t^4 W_x^{s-1/2,4}[\mathbf{k}]}.$$

Estimate (4.8) therefore follows from the definition of $\| \cdot \|_{X_T}$.

We have so far managed to prove local well-posedness on $[0, T]$. We observe that

$$\|u\|_{L_t^\infty([0,T];H_x^s)} \leq \|u\|_{X_T} \leq C_0 \epsilon.$$

As $T > 0$ was chosen arbitrarily and C_0 does not depend on T, we can conclude that $\|u\|_{L_t^\infty([0,\infty);H_x^s)}$ is bounded. Global well-posedness then follows from the discussion in section 3.2. □

4.2 General Third Order Nonlinearities

Having completed our discussion on third order power-type nonlinearities in the previous section, we now turn our attention towards third order nonlinearities that include first order time and space derivatives. Because such nonlinearities are more difficult to deal with, we will only work in $n = 3$ space dimensions.

The general third order semilinear Klein-Gordon system can be expressed as

$$(\Box + 1)u = \sum_{i,j,l=0} \mathcal{A}_{i,j,l}(\partial_1, \partial_2, \partial_3)[\partial_t^i u][\partial_t^j u][\partial_t^l u]. \tag{4.9}$$

where $\mathcal{A}_{i,j,l}$ is a polynomial of order $1 - i$ (respectively $1 - j, 1 - l$) in ∂_1 (respectively ∂_2, ∂_3).

Global well-posedness for this system has long been established (see [19]). The result in [19] relies on stronger assumptions on the initial data, but also assumes a more general form for F that involves second order time and space derivatives because the author considers a general quasilinear system. In general, such systems are more difficult to deal with and one typically needs extra decay assumptions on the initial data in order to close the argument. We are able to weaken the assumptions on the initial data precisely because our nonlinearity only includes first order derivatives.

Rather than prove global well-posedness for the general nonlinearity presented in (4.9), we will instead work with the special case $F(u, \partial_t u, \partial u) = (\partial_t u)u^2$. It will soon be apparent that we can extend our argument to the general case in a straightforward, albeit notationally tedious, manner. Our goal is to prove the following theorem.

Theorem 4.2.1. *Suppose $s > 2$. There exists an $\epsilon > 0$ such that for initial data*

$$(u_0, u_1) \in H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)\,, \quad \|(u_0, u_1)\|_{H^s \times H^{s-1}} < \epsilon$$

the equation

$$(\Box + 1)u = (\partial_t u)u^2 \tag{4.10}$$

with initial data

$$u(x, 0) = u_0 \in H^s, \quad \partial_t u(0, x) = u_1 \in H^{s-1} \tag{4.11}$$

has a unique global solution in $C([0, \infty), H^s) \cap C^1([0, \infty), H^{s-1})$ which depends continuously on the initial data (u_0, u_1).

Proof. Let $0 < \delta < s - 2$ and define $X_T \subseteq L_t^\infty H_x^s[\mathbf{k}] \cap L_t^2 W_x^{1+\delta,\infty}[\mathbf{k}]$ by the norm

$$\|u\|_{X_T} := \sum_{i=0}^{1} \left(\|\partial_t^i u\|_{L_t^\infty H_x^{s-i}[\mathbf{k}]} + \|\partial_t^i u\|_{L_t^2 W_x^{1+\delta-i,\infty}[\mathbf{k}]} \right).$$

As before, we define the space S_T by

$$S_T(\epsilon) := \{u \in X_T : \|u\|_{X_T} \leq C_0\epsilon\}$$

Finally, define the space $N_T := L_t^\infty H_x^{s-1}[\mathbf{k}]$ with the obvious corresponding norm.

Recall that we defined

$$W(u_0, u_1)(t, \cdot) = e^{it\langle D\rangle}u_0(\cdot) + \frac{e^{it\langle D\rangle}u_1(\cdot)}{\langle D\rangle}$$

and

$$L(G)(t, \cdot) := \int_0^t \frac{\sin((t-s)\langle D\rangle)G(s, \cdot)}{\langle D\rangle}ds.$$

Applying the argument from the previous section, we know that Theorem 4.2.1 follows from proving the following three estimates

$$\|W(u_0, u_1)\|_{S_T} \lesssim \|(u_0, u_1)\|_{H_x^s \times H_x^{s-1}}, \tag{4.12}$$

$$\|L(G)\|_{S_T} \lesssim \|G\|_{N_T}, \tag{4.13}$$

$$\|(\partial_t z_1)z_2z_3\|_{N_T} \lesssim \|z_1\|_{X_T}\|z_2\|_{X_T}\|z_3\|_{X_T}, \tag{4.14}$$

provided ϵ is chosen sufficiently small and C_0 adequately large. From the homogeneous estimate in Theorem 2.3.5, we can conclude that

$$\|W(u_0, u_1)\|_{L_t^2 W_x^{1+\delta,\infty}} \lesssim \|(u_0, u_1)\|_{H_x^s \times H_x^{s-1}}, \quad \|\partial_t W(u_0, u_1)\|_{L_t^2 W_x^{\delta,\infty}} \lesssim \|(u_0, u_1)\|_{H_x^s \times H_x^{s-1}},$$

where we have taken advantage of the fact that $s - 2 - \delta > 0$.

Similarly, it is not difficult to see that

$$\|W(u_0, u_1)\|_{L_t^\infty H_x^s} \lesssim \|(u_0, u_1)\|_{H_x^s \times H_x^{s-1}}, \quad \|\partial_t W(u_0, u_1)\|_{L_t^\infty H_x s-1} \lesssim \|(u_0, u_1)\|_{H_x^s \times H_x^{s-1}}.$$

Equation (4.12) therefore follows.

To prove (4.13), recall that

$$\|L(G)\|_{S_T} = \| \int_0^t \frac{\sin((t-s)\langle D\rangle)G(s,\cdot)}{\langle D\rangle} ds\|_{L_t^\infty H_x^s[\mathbf{k}]} \tag{4.15}$$

$$+ \| \int_0^t \frac{\sin((t-s)\langle D\rangle)G(s,\cdot)}{\langle D\rangle} ds\|_{L_t^2 W_x^{1+\delta,\infty}[\mathbf{k}]} \tag{4.16}$$

$$+ \|\partial_t \int_0^t \frac{\sin((t-s)\langle D\rangle)G(s,\cdot)}{\langle D\rangle} ds\|_{L_t^\infty H_x^{s-1}[\mathbf{k}]} \tag{4.17}$$

$$+ \|\partial_t \int_0^t \frac{\sin((t-s)\langle D\rangle)G(s,\cdot)}{\langle D\rangle} ds\|_{L_t^2 W_x^{1+\delta-1,\infty}[\mathbf{k}]}. \tag{4.18}$$

Applying Theorem 2.3.2 with $(p_1, p_2', q_1, q_2') = (\infty, 1, 2, 2)$, we obtain the estimate

$$\| \int_0^t \frac{\sin((t-s)\langle D\rangle)G(s,\cdot)}{\langle D\rangle} ds\|_{L_t^\infty H_x^s[\mathbf{k}]} \lesssim \| \frac{G}{\langle D\rangle} \|_{L_t^1 H_x^s[\mathbf{k}]} = \|G\|_{L_t^1 H_x^{s-1}[\mathbf{k}]}.$$

Taking advantage of the fact that $s-2-\delta > 0$ and applying the inhomogeneous estimate from Theorem 2.3.5 to (4.16), we see that

$$\| \int_0^t \frac{\sin((t-s)\langle D\rangle)G(s,\cdot)}{\langle D\rangle} ds\|_{L_t^2 W_x^{1+\delta,\infty}[\mathbf{k}]} \lesssim \| \frac{G}{\langle D\rangle} \|_{L_t^1 H_x^s[\mathbf{k}]} = \|G\|_{L_t^1 H_x^{s-1}[\mathbf{k}]}.$$

Before we bound (4.17) and (4.18), observe that

$$\begin{aligned}\partial_t \int_0^t \frac{\sin((t-s)\langle D\rangle)G(s,\cdot)}{\langle D\rangle} ds &= \frac{\sin((t-t)\langle D\rangle)G(t,\cdot)}{\langle D\rangle} + \int_0^t \cos((t-s)\langle D\rangle)G(s,\cdot)ds \\ &= \int_0^t \cos((t-s)\langle D\rangle)G(s,\cdot)ds.\end{aligned}$$

Once again we apply Theorems 2.3.2 and 2.3.5 to conclude that

$$\| \int_0^t \cos((t-s)\langle D\rangle)G(s,\cdot)ds\|_{L_t^\infty H_x^{s-1}[\mathbf{k}]} \lesssim \|G\|_{L_t^1 H_x^{s-1}[\mathbf{k}]}$$

and

$$\| \int_0^t \cos((t-s)\langle D\rangle)G(s,\cdot)ds\|_{L_t^2 W_x^{1+\delta,\infty}[\mathbf{k}]} \lesssim \|G\|_{L_t^1 H_x^{s-1}[\mathbf{k}]}.$$

Completing the proof of (4.13)

Finally, we turn our attention towards (4.14). Applying Theorem 2.2.1 with $(p_1, p_2, q_1, q_2, \lambda, \sigma) = (\infty, 1, 2, \infty, \delta, 0)$ and $(\tilde{p}_1, \tilde{p}_2, \tilde{q}_1, \tilde{q}_2, \tilde{\lambda}, \tilde{\sigma}) = (2, 2, \infty, \infty, \delta, 0)$, we obtain

$$\begin{aligned}\|(\partial_t z_1) z_2 z_3\|_{L_t^1 H_x^{s-1}[\mathbf{k}]} \lesssim \|\partial_t z_1\|_{L_t^\infty H_x^{s-1}} \|z_2 z_3\|_{L_t^1 W_x^{\delta,\infty}[\mathbf{k}]} \\ + \|\partial_t z_1\|_{L_t^2 W_x^{\delta,\infty}[\mathbf{k}]} \|z_2 z_3\|_{L_t^2 H_x^{s-1}[\mathbf{k}]}.\end{aligned}$$

Another iteration of Theorem 2.2.1 allows us to bound the above by

$$\begin{aligned}&\lesssim \|\partial_t z_1\|_{L_t^\infty H_x^{s-1}} \|z_2\|_{L_t^2 W_x^{\delta,\infty}[\mathbf{k}]} \|z_3\|_{L_t^2 W_x^{\delta,\infty}[\mathbf{k}]} \\ &+ \|\partial_t z_1\|_{L_t^2 W_x^{\delta,\infty}[\mathbf{k}]} \|z_2\|_{L_t^2 W_x^{\delta,\infty}[\mathbf{k}]} \|z_3\|_{L_t^\infty H_x^{s-1}[\mathbf{k}]}.\end{aligned}$$

Estimate (4.14) therefore follows from the definition of X_T, concluding our proof.

□

Chapter 5

Quadratic Systems with Different Masses

We study the Cauchy problem in dimensions $n \geq 2$ for the semi-linear Klein-Gordon system

$$(\Box + m_i^2)u_i = F_i(u_1, ..., u_k) \quad i = 1, ..., k$$

with initial data

$$u_i(0,x) = f_i \in H^s(\mathbb{R}^n), \quad \partial_t u_i(0,x) = g_i \in H^{s-1}(\mathbb{R}^n),$$

for $s \geq \max\left(\frac{1}{2}+, \frac{n-2}{2}\right)$, where the masses m_i satisfy a suitable nonresonance condition and the F_i are homogeneous quadratic polynomials.

This particular problem was studied by Tobias Schottdorf in [18], and a scalar version was considered by Vladimir Georgiev and Atanas Stefanov in [9] for the case $n = 2$ and H_x^{1+} initial data. In dimension $n = 2$, Schottdorf was able to prove global existence with smooth dependence on the initial data by employing a contraction argument in a U^p

type space. He also claims to prove this in dimensions $n \geq 3$, but unfortunately there is a flaw in his proof which we suspect is the reason his paper remains unpublished. In order to properly close the argument in higher dimensions we find that we must refine the iteration space.

Our goal then is to provide a complete solution to the Cauchy problem studied in [18]. We present the proof in $n = 2$ given by Schottdorf as well as our original argument for dimensions $n \geq 3$. Our aim is to prove Theorem 1.0.2, which we repeat for convenience below.

Theorem 5.0.1. *Let $n \geq 2, s \geq \max(\frac{1}{2}+, \frac{n-2}{2}), k \in \mathbb{N}$ and let $F_1, ..., F_k$ be homogeneous quadratic polynomials and $m_1, ..., m_k > 0$ be such that*

$$2\min(\{m_j\}) > \max(\{m_j\})$$

Then there exists an $\epsilon > 0$ such that for initial data

$$(f_i, g_i) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n), \quad \|(f_i, g_i)\|_{H^s(\mathbb{R}^n)\times H^{s-1}(\mathbb{R}^n)} < \epsilon$$

the system

$$(\Box + m_i^2)u_i = F_i(u_1, ..., u_k) \quad i = 1, ..., k$$

has a global solution in $C(\mathbb{R}, H^s) \cap C^1(\mathbb{R}, H^{s-1})$ which depends continuously on the initial data (f,g).

As the generalization to a system with different masses does not require much additional work provided the required nonresonance condition is met (see section 5.5), we may reduce to the scalar equation

$$(\Box + 1)u = F(u) \tag{5.1}$$

with initial data

$$u(0,x) = f \in H^s, \quad \partial_t u(0,x) = g \in H^{s-1}$$

As before, we will apply a contraction argument to establish global existence. Rather than run our contraction scheme in a mixed Sobolev space like we did in Chapter 4 we will instead work in U^p based spaces. The advantage of working with such spaces is that their elements enjoy many of the same properties as free solutions, such as bilinear and trilinear estimates. Another benefit to working with these spaces is that they allow us to readily exploit the lack of resonant terms. This is because they provide good information on the Fourier transform properties of their elements.

For both $n = 2$ and $n \geq 3$ dimensions we are able to reduce our contraction argument to a set of trilinear estimates which we prove in section 5.3. In $n = 2$ dimensions, we prove these using Strichartz estimates and a modulation argument. In $n \geq 3$ however, the situation is more difficult and we require a more robust approach.

Fortunately, one can exploit the geometry of the characteristic hypersurface in order to establish a useful set of bilinear estimates for free solutions. Applying some basic properties of U^p spaces, we can translate these estimates to bilinear estimates from $e^{\pm it\langle D\rangle}U^2 \times e^{\pm it\langle D\rangle}U^2 \to L^2$. The error in Shottdorf's proof for $n \geq 3$ lies in his attempt to upgrade this estimate to one of the form $e^{\pm it\langle D\rangle}U^4 \times e^{\pm it\langle D\rangle}U^4 \to L^2$ for certain frequency interactions using a flawed orthogonality argument. Without this estimate his proof of the trilinear estimates discussed above is incomplete. We manage to circumvent the orthogonality issue by refining our iteration space as we discuss in the next section.

5.1 Function Spaces

It is useful to translate (5.1) into a first order system. Observe that

$$\Box + 1 = (\langle D\rangle + i\partial_t)(\langle D\rangle - i\partial_t).$$

Hence, for sufficiently nice u satisfying

$$(\Box + 1)u = F, \quad u(0,x) = f, \quad \partial_t u(0,x) = g,$$

we define

$$u^{\pm} = \frac{\langle D\rangle \mp i\partial_t}{2\langle D\rangle} u.$$

Then, $u^{\pm}$ solves

$$(\langle D\rangle \pm i\partial_t)u^{\pm} = \frac{F}{2\langle D\rangle}, u_0^{\pm} := u^{\pm}(0,x) = \frac{1}{2}(f \mp i\frac{g}{\langle D\rangle})$$

and $u = u^{+} + u^{-}$.

We focus our attention first on the case $n = 2$. Let $X_{\pm}^{s}$ denote the closure of $C(\mathbb{R}, H^s) \cap U^2$ with respect to the following norm:

$$\|u\|_{X_{\pm}^{s}} = (\sum_k 2^{2ks} \|P_k u^{\pm}\|_{U_{\pm}^2}^2)^{1/2},$$

where

$$\|u\|_{U_{\pm}^2} = \|e^{\mp it\langle D\rangle} u\|_{U^2}.$$

Define Y^s as the corresponding space where U^2 is replaced by $V^2 = V_{-,rc}^2$ and let

$$X^s = X_+^s \times X_-^s \quad Y^s = Y_+^s \times Y_-^s.$$

Given $s \geq \frac{1}{2}, u_0^\pm \in B_\epsilon(0) \subseteq H^s$ and $(u^+, u^-) \in X^s$, we wish to solve the operator equation

$$u^\pm = e^{\pm it\langle D\rangle} u_0^\pm \mp iI^\pm(u),$$

where $u = u^+ + u^-$ and

$$I^\pm(u) = \int_0^t e^{\pm i(t-s)\langle D\rangle} \frac{F(u(s))}{2\langle D\rangle} ds.$$

We can solve this equation by running a contraction argument in X^s similar to the analogous one for X_T in Chapter 3 once we have established the following bounds:

$$\|e^{\pm it\langle D\rangle} u_0^\pm\|_{X^s_\pm} \lesssim \|u_0^\pm\|_{H^s}, \qquad \|I^\pm(u)\|_{X^s_\pm} \lesssim \|(u^+, u^-)\|^2_{X^s_\pm}.$$

The linear part is easy to estimate:

$$\begin{aligned}
\|e^{\pm it\langle D\rangle} u_0^\pm\|^2_{X^s_\pm} &= \sum_k 2^{2ks} \|e^{\pm it\langle D\rangle} P_k(u_0^\pm)\|^2_{U^2_\pm} \\
&\lesssim \sum_k 2^{2ks} \|P_k(u_0^\pm)\|^2_{U^2} \\
&\lesssim \|u_0^\pm\|^2_{H^s}.
\end{aligned}$$

It remains to prove the bound on the Duhamel term.

Unfortunately we are unable to close the contraction argument in X^s in higher dimensions. We construct a more refined space $\tilde{X}^s$ in what follows. We remark that the special structure of this new space allows us to fix the flaw in Schottdorf's work.

Let $n \geq 3$, then define $\Xi_k = 2^k \cdot \mathbb{Z}^d$ and let $\gamma^{(1)} : \mathbb{R} \to [0, 1]$ be an even smooth function supported in the interval $[-2/3, 2/3]$ with the property that

$$\sum_{d \in \mathbb{Z}} \gamma^{(1)}(\xi - d) = 1 \text{ for } \xi \in \mathbb{R}.$$

Define $\gamma : \mathbb{R}^n \to [0,1]$ by $\gamma(\xi) = \gamma^{(1)}(\xi_1) \cdot \ldots \cdot \gamma^{(1)}(\xi_n)$. For $d \in \Xi_k$, let

$$\gamma_{d,k} = \gamma((\xi - d)/2^k).$$

It follows that

$$\sum_{d \in \Xi_k} \gamma_{d,k}(\xi) \equiv 1 \text{ for } \xi \in \mathbb{R}^n.$$

Define $\Gamma_{d,k}$ to be the Fourier multiplication operator with symbol $\gamma_{d,k}$, and let $U_k^{p,\pm}$ be the subspace of $L_t^\infty L_x^2$ defined by the following norm:

$$\|u\|_{U_k^{p,\pm}} = (\sum_{d \in \Xi_k} \|\Gamma_{d,k} u\|_{U_\pm^p}^2)^{\frac{1}{2}}.$$

Similarly, for $V_k^{p,\pm}$, set

$$U_k^\pm := U_k^{2,\pm}, \quad V_k^\pm := V_k^{2,\pm}$$

Lemma 5.1.1. *Let $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ be such that $k_1 \leq k_2$. Then, for $p > 1$, we have*

$$\|u\|_{U_{k_1}^{p,\pm}} \lesssim \|u\|_{U_{k_2}^{p,\pm}}, \tag{5.2}$$

$$\|u\|_{V_{k_2}^{p,\pm}} \lesssim \|u\|_{V_{k_1}^{p,\pm}}. \tag{5.3}$$

Proof. We first turn our attention to estimate (5.3). Without changing our argument in any significant way, we can replace $V_{k_1}^{p,\pm}, V_{k_2}^{p,\pm}$ with $V_{k_1}^p, V_{k_2}^p$, where the spaces $V_{k_1}^p, V_{k_2}^p$ are defined in the obvious way.

By definition,

$$\|u\|_{V_{k_2}^p} = (\sum_{d \in \Xi_{k_2}} \|\Gamma_{d,k_2} u\|_{V^p}^2)^{\frac{1}{2}}$$

We see that

$$\begin{aligned} \|\Gamma_{d,k_2} u\|_{V^p} &= \sup_{\{t_m\}_{m=0}^M \in \mathcal{Z}} (\sum_{m=1}^M \|\Gamma_{d,k_2} u(t_m) - \Gamma_{d,k_2} u(t_{m-1})\|_{L^2}^p)^{1/p} \\ &\leq \sup_{\{t_m\}_{m=0}^M \in \mathcal{Z}} (\sum_{m=1}^M (\sum_{d' \in \Xi_{k_1}} \|\Gamma_{d,k_2} \Gamma_{d',k_1} u(t_m) - \Gamma_{d,k_2} \Gamma_{d',k_1} u(t_{m-1})\|_{L^2})^p)^{1/p}. \end{aligned}$$

Observe that the terms in the rightmost sum are nonzero for only finitely many d'. Let $S_{k_1,k_2,d}$ denote the index set containing all such d'. Note that the size of $S_{k_1,k_2,d}$ depends only on k_1 and k_2. Applying Minkowski's inequality in m and d' bounds the above expression by

$$\sup_{\{t_m\}_{m=0}^{M}\in\mathcal{Z}} \sum_{d'\in S_{k_1,k_2,d}} (\sum_{m=1}^{M} \|\Gamma_{d',k_1}u(t_m) - \Gamma_{d',k_1}u(t_{m-1})\|_{L^2}^p)^{1/p}$$
$$\leq \sum_{d'\in S_{k_1,k_2,d}} \|\Gamma_{d',k_1}u\|_{V^p}.$$

We conclude that

$$\|u\|_{V^p_{k_2}} \leq (\sum_{d\in\Xi_{k_2}} (\sum_{d'\in S_{k_1,k_2,d}} \|\Gamma_{d',k_1}u\|_{V^p})^2)^{\frac{1}{2}}$$
$$\lesssim_{k_1,k_2} (\sum_{d'\in\Xi_{k_1}} \|\Gamma_{d',k_1}u\|_{V^p}^2)^{\frac{1}{2}} = \|u\|_{V^p_{k_1}},$$

giving us estimate (5.3). Estimate (5.2) then follows from (5.3) by a straight forward duality argument. □

Remark 5.1.1. *From Lemma 5.1.1, we can conclude that* $\|\cdot\|_{U_0^\pm} \lesssim \|\cdot\|_{U_k^\pm}$ *and* $\|\cdot\|_{V_k^\pm} \lesssim \|\cdot\|_{V_0^\pm}$ *for all* $k \geq 0$*. Furthermore, for compactly supported* $u \in U^2_\pm$*, we have* $\|u\|_{U_0^\pm} \lesssim \|u\|_{U^2_\pm}$ *and* $\|u\|_{V^2_\pm} \lesssim \|u\|_{V_0^\pm}$*. We will use both of these facts repeatedly in our estimates for* $n \geq 3$*.*

Finally, define $\widetilde{X}^s_\pm, \widetilde{Y}^s_\pm$ by the norms

$$\|u\|_{\widetilde{X}^s_\pm} = (\sum_k 2^{2ks}\|P_k u\|_{U_0^\pm}^2)^{\frac{1}{2}} \qquad \|u\|_{\widetilde{Y}^s_\pm} = (\sum_k 2^{2ks}\|P_k u\|_{V_0^\pm}^2)^{\frac{1}{2}}$$

and

$$\widetilde{X}^s = \widetilde{X}^s_+ \times \widetilde{X}^s_- \qquad \widetilde{Y}^s = \widetilde{Y}^s_+ \times \widetilde{Y}^s_-.$$

Similarly to the case $n = 2$, given $s \geq \frac{n-2}{2}$, $u_0^\pm \in B_\epsilon(0) \subseteq H^s$ and $(u^+, u^-) \in X^s$, we wish to solve the operator equation

$$u^\pm = e^{\pm it\langle D\rangle} u_0^\pm \mp i\widetilde{I}^\pm(u),$$

where $u = u^+ + u^-$ and

$$\widetilde{I}^\pm(u) = \int_0^t e^{\pm i(t-s)\langle D\rangle} \frac{N(u(s))}{2\langle D\rangle} ds.$$

We can solve this equation by running a standard contraction argument in $\widetilde{X^s}$ once we have established the following bounds

$$\|e^{\pm it\langle D\rangle} u_0^\pm\|_{\widetilde{X}^s_\pm} \lesssim \|u_0^\pm\|_{H^s} \qquad \|I^\pm(u)\|_{\widetilde{X}^s_\pm} \lesssim \|(u^+, u^-)\|^2_{\widetilde{X}^s_\pm}.$$

As before, the linear part is easy to estimate:

$$\begin{aligned}
\|e^{\pm it\langle D\rangle} u_0^\pm\|^2_{\widetilde{X}^s_\pm} &= \sum_k 2^{2ks} \|e^{\pm it\langle D\rangle} P_k(u_0^\pm)\|^2_{U_0^\pm} \\
&\lesssim \sum_k 2^{2ks} \|e^{\pm it\langle D\rangle} P_k(u_0^\pm)\|^2_{U^2_\pm} \\
&\lesssim \sum_k 2^{2ks} \|P_k(u_0^\pm)\|^2_{U^2} \\
&\lesssim \|u_0^\pm\|^2_{H^s}.
\end{aligned}$$

So, it remains to prove the bound on the Duhamel term. In order to do so we need several results.

5.2 Bilinear estimates and Modulation Analysis

The proof of the main theorems of this chapter will rely heavily on the bilinear estimates presented in this section. Note that bounds in U^p type spaces will follow directly from L^p bounds on free solutions due to Proposition 2.4.3.

Later in this chapter we will repeatedly make use of the $n = 2, p = q = 4$ Strichartz estimate proved in Chapter 2. We present it below for convenience.

Proposition 5.2.1 ([18], Proposition 8)**.** *Let* $u_0 \in L^2(\mathbb{R}^2)$, *then*

$$\|e^{\pm it\langle D\rangle}u_0\|_{L_t^4 L_x^4} \lesssim \|\langle D\rangle^{\frac{1}{2}}u_0\|_{L_x^2}.$$

We will repeatedly make use of the following result

Corollary 5.2.1. *Let* $n = 2$ *and* $u_k \in V_\pm^2$ *be localized at frequency* 2^k, *then we have*

$$\|u_k\|_{L_t^4 L_x^4} \lesssim 2^{\frac{k}{2}}\|u\|_{V_\pm^2}. \tag{5.4}$$

Proof. Let $\phi \in L^2(\mathbb{R}^2)$ and define $T_0(\phi) := P_k(\phi)$. By Proposition 5.2.1, we have

$$\|T_0(e^{\pm it\langle D\rangle}\phi)\|_{L_t^4 L_x^4} \lesssim 2^{\frac{k}{2}}\|\phi\|_{L_x^2}$$

Applying Proposition 2.4.3 to T_0 we obtain

$$\|u_k\|_{L_t^4 L_x^4} \lesssim 2^{\frac{k}{2}}\|u_k\|_{U_\pm^4},$$

for all $u \in U_\pm^4$. As $V^2 \subseteq U^4$ we may replace $U_\pm^4$ with $V_\pm^2$ to get the desired bound. □

For our estimates in dimensions $n \geq 3$, we will need the following key result.

Proposition 5.2.2 ([18], Proposition 7)**.** *Let* $n \geq 3$, O, M, N *be dyadic numbers, and* ϕ_M, ψ_N *functions in* $L^2(\mathbb{R}^n)$ *localized at frequencies* M, N *respectively. For* $\pm_1, \pm_2 \in \{+, -\}$ *define* $u_M = e^{\pm_1 it\langle D\rangle}\phi_M, v_N = e^{\pm_2 it\langle D\rangle}\psi_N$ *and denote* L=*min*(O, M, N), H =*max*(O, M, N). *Then,*

$$\|P_O(u_M v_N)\|_{L^2(\mathbb{R}^{n+1})} \lesssim \begin{cases} H^{\frac{1}{2}}L^{\frac{n-2}{2}}\|\phi_M\|_{L_x^2}\|\psi_N\|_{L_x^2} & \textit{if } M \sim N \\ L^{\frac{n-1}{2}}\|\phi_M\|_{L_x^2}\|\psi_N\|_{L_x^2} & \textit{otherwise} \end{cases}$$

Proof. Found in [18] $\square$

Applying Proposition 2.4.3 to the above result yields

Corollary 5.2.2. *Let $n \geq 3$ and L, H be dyadic integers such that $L \ll H$. Furthermore, assume $u_L \in U^2_{\pm_1}, w_H \in U^2_{\pm_2}$ be localized at frequencies L and H respectively. Then, we have*

$$\|u_L w_H\|_{L^2} \lesssim L^{\frac{n-1}{2}} \|u_L\|_{U^2_{\pm_1}} \|w_H\|_{U^2_{\pm_2}}.$$

The following estimate and corresponding proof can also be found in [18]. We remark that the result we obtain here is stronger than what is possible using just Strichartz estimates and Bernstein's inequality.

Proposition 5.2.3 ([18], Proposition 10)**.** *Let $n \geq 3$ and let $\phi_{M,N}$ have Fourier support in a ball of radius M centered at frequency N where $M \lesssim N$. Then, we have*

$$\|e^{it\langle D\rangle}\phi_{M,N}\|_{L^4} \lesssim N^{\frac{1}{4}} M^{\frac{n-2}{4}} \|\phi_{M,N}\|_{L^2}.$$

Proof. For $\phi := \phi_{N,M}$, it suffices to show

$$\|e^{it\langle D\rangle}\phi e^{-it\langle D\rangle}\bar{\phi}\|_{L^2} \lesssim N^{\frac{1}{2}} M^{\frac{n-2}{2}} \|\phi\|^2_{L^2}.$$

As the Fourier supports of ϕ and $\bar{\phi}$ are symmetric through the origin, we conclude that the sum of the supports lies in a ball of radius $\lesssim M$ centered at 0. Therefore, we may rewrite the inequality above as

$$\|P_M(e^{it\langle D\rangle}\phi e^{-it\langle D\rangle}\bar{\phi})\|_{L^2} \lesssim N^{\frac{1}{2}} M^{\frac{n-2}{2}} \|\phi\|^2_{L^2},$$

and we conclude that the desired bound follows from Proposition 5.2.2 $\square$

Combining the above result with Proposition 2.4.3, we obtain the following

Proposition 5.2.4 ([18], Proposition 11). *Let $n \geq 3$ and let $u_{M,N}$ have Fourier support in a ball of radius M centered at frequency N where $M \lesssim N$. Then, we have*

$$\|u_{M,N}\|_{L^4} \lesssim N^{\frac{1}{4}} M^{\frac{n-2}{4}} \|u_{M,N}\|_{U^4_{\pm}}.$$

Note that we can replace U^4 in the bound above with V^2 as $V^2 \subseteq U^4$.

Corollary 5.2.3. *Let $n \geq 3$ and L, H be dyadic integers such that $L \ll H$. Furthermore, let $u_L \in U^2_{\pm_1}, w_H \in U^2_{\pm_2}$ be localized at frequencies L and H respectively. Then, we have*

$$\|u_L w_H\|_{L^2} \lesssim L^{\frac{n-1}{2}} \log^2 (\frac{H}{L}) \|u_L\|_{V^2_{\pm_1}} \|w_H\|_{V^2_{\pm_2}}.$$

Proof. Define $Tw := u_L P_H(w)$. Then,

$$\|Tw\|_{L^2} \lesssim \|u_L\|_{L^4} \|w_H\|_{L^4}.$$

By Corollary 5.2.4, this is

$$\lesssim L^{\frac{n-1}{4}} \|u_L\|_{U^4_{\pm 1}} H^{\frac{n-1}{4}} \|w_H\|_{U^4_{\pm 2}} \tag{5.5}$$

As $U^2 \subseteq U^4$, we can replace the above with

$$\lesssim (LH)^{\frac{n-1}{4}} \|u_L\|_{U^2_{\pm 1}} \|w\|_{U^4_{\pm 2}}.$$

So, we conclude

$$\|T\|_{U^4_{\pm 2} \to L^2} \lesssim (LH)^{\frac{n-1}{4}} \|u_L\|_{U^2_{\pm 1}}.$$

From Corollary 5.2.2 we see that

$$\|T\|_{U^2_{\pm 2} \to L^2} \lesssim L^{\frac{n-1}{2}} \|u_L\|_{U^2_{\pm 1}}.$$

We apply Proposition 2.4.4 with $p = 2, q = 4$ to obtain

$$\|u_L w_H\|_{L^2} \lesssim L^{\frac{n-1}{2}} \ln(\frac{H}{L}) \|u_L\|_{U^2_{\pm 1}} \|w_H\|_{V^2_{\pm 2}}. \tag{5.6}$$

We iterate the argument by defining

$$Su = P_L(u) w_H.$$

Applying the fact that $V^2 \subseteq U^4$ to (5.5), we obtain

$$\|S\|_{U^4_{\pm 1} \to L^2} \lesssim (LH)^{\frac{n-1}{4}} \|w_H\|_{V^2_{\pm 2}}.$$

From (5.6), we observe that

$$\|S\|_{U^2_{\pm 1} \to L^2} \lesssim L^{\frac{n-1}{2}} \ln(\frac{H}{L}) \|w_H\|_{V^2_{\pm 2}}.$$

Applying Proposition 2.4.4 once more gives us the claim □

A crucial feature we exploit in the nonlinear analysis is the absence of resonant terms. In order to formalize this idea, we must first prove an important modulation bound.

Define $\langle \xi \rangle_m := \sqrt{m^2 + |\xi|^2}$.

Lemma 5.2.1. *Assume $m_1, m_2, m_3 > 0$ are such that $2 \min\{m_i\} > \max\{m_i\}$. Let $\epsilon_1, \epsilon_2 \in \{+, -\}$ and let $\xi_1 + \xi_2 = \xi_3$. Then, we have*

$$\max\{\langle \xi_1 \rangle_{m_1}^{-1}, \langle \xi_2 \rangle_{m_2}^{-1}, \langle \xi_3 \rangle_{m_3}^{-1}\} \lesssim |\langle \xi_1 \rangle_{m_1} + \epsilon_1 \langle \xi_2 \rangle_{m_2} + \epsilon_2 \langle \xi_3 \rangle_{m_3}|. \tag{5.7}$$

Proof. Case 1: $\epsilon_1 = \epsilon_2 = +$

This is obvious.

Case 2: $\epsilon_1 = +, \epsilon_2 = -$

Define

$$\Lambda := \langle \xi_1 \rangle_{m_1} + \langle \xi_2 \rangle_{m_2} + \langle \xi_1 + \xi_2 \rangle_{m_3}.$$

It suffices to show that

$$\frac{1}{\Lambda} \lesssim \langle \xi_1 \rangle_{m_1} + \langle \xi_2 \rangle_{m_2} - \langle \xi_1 + \xi_2 \rangle_{m_3},$$

which is equivalent to showing

$$1 \lesssim (\langle \xi_1 \rangle_{m_1} + \langle \xi_2 \rangle_{m_2} - \langle \xi_1 + \xi_2 \rangle_{m_3}) \cdot \Lambda. \tag{5.8}$$

Expanding the RHS of (5.8), gives us

$$\begin{aligned}
&\langle \xi_1 \rangle_{m_1}^2 + \langle \xi_2 \rangle_{m_2}^2 + 2\langle \xi_1 \rangle_{m_1} \langle \xi_2 \rangle_{m_2} - \langle \xi_1 + \xi_2 \rangle_{m_3}^2 \\
&= m_1^2 + m_2^2 - m_3^2 + |\xi_1|^2 + |\xi_2|^2 - |\xi_1 + \xi_2|^2 + 2\langle \xi_1 \rangle_{m_1} \langle \xi_2 \rangle_{m_2}.
\end{aligned}$$

Applying the law of cosines, this

$$= (m_1 + m_2 - m_3)(m_1 + m_2 + m_3) - 2m_1 m_2 - 2|\xi_1||\xi_2| \cos(< (\xi_1, \xi_2)) + 2\langle \xi_1 \rangle_{m_1} \langle \xi_2 \rangle_{m_2}. \tag{5.9}$$

Define

$$\Gamma := \langle \xi_1 \rangle_{m_1} \langle \xi_2 \rangle_{m_2} - (|\xi_1||\xi_2| + m_1 m_2).$$

I claim that $\Gamma \geq 0$. Indeed, we see that

$$\begin{aligned}
\Gamma &= \frac{(\langle \xi_1 \rangle_{m_1} \langle \xi_2 \rangle_{m_2})^2 - (|\xi_1||\xi_2| + m_1 m_2)^2}{\langle \xi_1 \rangle_{m_1} \langle \xi_2 \rangle_{m_2} + |\xi_1||\xi_2| + m_1 m_2} \\
&= \frac{(m_2|\xi_1| + m_1|\xi_2|)^2}{\langle \xi_1 \rangle_{m_1} \langle \xi_2 \rangle_{m_2} + |\xi_1||\xi_2| + m_1 m_2}.
\end{aligned}$$

Therefore, (5.9) is

$$\geq (m_1 + m_2 - m_3)(m_1 + m_2 + m_3)$$
$$\geq \min\{m_i\}(2\min\{m_i\} - \max\{m_i\})$$
$$\gtrsim 1.$$

giving us the desired bound

case 3: $\epsilon_1 = -, \epsilon_2 = +$ (By interchanging the roles of ξ_1 and ξ_2, we see that the case $\epsilon_1 = \epsilon_2 = -$ is identical)

Let $\eta = -\xi_1$, then the right-hand side of (5.7) can be rewritten as

$$|\langle \eta \rangle_{m_1} - \langle \eta + \xi_3 \rangle_{m_2} + \langle \xi_3 \rangle_{m_3}|,$$

reducing us to case 2. □

Lemma 5.2.2. *Let $\pm_1, \pm_2, \pm_3 \in \{+, -\}$ and L, H, H', M_1, M_2, M_3 be dyadic numbers such that $L \ll H \sim H'$. Furthermore, let*

$$u_L = Q^{\pm_1}_{\leq M_1} u_L, v_H = Q^{\pm_2}_{\leq M_2} v_H, w_{H'} = Q^{\pm_3}_{\leq M_3} w_{H'}$$

then, if $\Lambda := \max(M_1, M_2, M_3) \leq CL^{-1}$ for appropriately chosen C, we will have

$$\int\int u_L v_H w_{H'} dx dt = 0. \tag{5.10}$$

Proof. Observe that

$$\int\int u_L v_{H'} w_H dx dt = (\mathcal{F}_{tx} u_L * \mathcal{F}_{tx} v_{H'} * \mathcal{F}_{tx} w_H)(0,0). \tag{5.11}$$

So we consider only the frequencies satisfying $\tau_1 + \tau_2 + \tau_3 = 0, \xi_1 + \xi_2 + \xi_3 = 0$. By definition of $Q^{\pm}_{\leq}$, we also have $|\tau_i \mp_i \langle \xi_i \rangle| \leq \Lambda$, so that on the nonvanishing set we have

$$3\Lambda \geq |\sum_{i=1}^{3} \tau_i \mp_i \langle \xi_i \rangle| = |\sum_{i=1}^{3} \mp_i \langle \xi_i \rangle| \gtrsim L^{-1},$$

where the last inequality is obvious when the three signs agree and follows from Lemma 5.2.1 otherwise. We conclude that for $\Lambda \lesssim L^{-1}$, expression (5.11) must vanish. $\square$

5.3 Trilinear Estimates

In order to prove the necessary Duhamel bound, we will need to take advantage of the duality relationship between U^2 and V^2. Because the nonlinearity F in the Duhamel term is quadratic, it makes sense that our V^2 based estimates will be trilinear in nature. In order to motivate the precise form of these estimates for dimension $n = 2$, we present the following computation from [18].

Let $G = \frac{F(u)}{2\langle D\rangle}$, then

$$\|P_k I^{\pm}(u)\|_{U^2_{\pm}} = \|P_k e^{\mp it\langle D\rangle} I^{\pm}(u)\|_{U^2} = \|P_k \int_0^t e^{\mp is\langle D\rangle} G(s)ds\|_{U^2}.$$

By Proposition 2.4.1 and Theorem 2.4.1 we have that the above

$$\begin{aligned} &= \sup_{\|v\|_{V^2}=1} |B(P_k \int_0^t e^{\mp is\langle D\rangle} f(s)ds, v)| \\ &= \sup_{\|v\|_{V^2}=1} |\int\int f(t)e^{\pm it\langle D\rangle}\overline{P_k v}dxdt| \\ &= \sup_{\|v\|_{V^2_{\pm}}=1} |\int\int f(t)\overline{P_k v}dxdt|. \end{aligned}$$

It will be evident soon that the estimates we need for the case $n = 2$ are exactly those presented in the following theorem

Theorem 5.3.1. *[[18], Theorem 3](Trilinear Estimates for $n = 2$) Let $C \geq 10$ be a fixed constant and $k_1, k_2, k_3 \in \mathbb{Z}_{\geq 0}$ be such that $|k_2 - k_3| \leq C$. Furthermore, suppose*

$s > \frac{1}{2}, \pm_1, \pm_2, \pm_3 \in \{+, -\}$, *then*

$$\frac{1}{2^{k_3}} \Big| \sum_{k_1 \le k_3 + C} \int\int P_{k_1} u P_{k_2} v P_{k_3} w dxdt \Big| \lesssim \Big(\sum_{k_1 \le k_3 + C} 2^{2k_1 s} \| P_{k_1} u \|^2_{V^2_{\pm_1}} \Big)^{\frac{1}{2}} \| P_{k_2} v \|_{V^2_{\pm_2}} \| P_{k_3} w \|_{V^2_{\pm_3}}. \tag{5.12}$$

Furthermore, we have

$$\Big(\sum_{k_1 \le k_3 + C} 2^{-2k_1} 2^{2k_1 s} \sup_{\| P_{k_1} w \|_{V^2_{\pm_3}} = 1} \Big| \int\int P_{k_2} u P_{k_3} v P_{k_1} w dxdt \Big|^2 \Big)^{\frac{1}{2}} \lesssim 2^{k_2 s} \| P_{k_2} u \|_{V^2_{\pm_1}} 2^{k_3 s} \| P_{k_3} v \|_{V^2_{\pm_2}}. \tag{5.13}$$

Proof. We denote $I(k_1) := \int\int P_{k_1} u P_{k_2} v P_{k_3} w dxdt$, then we can decompose

$$I(k_1) = I_0(k_1) + I_1(k_1) + I_2(k_1)$$

where

$$I_0(k_1) = \sum_{M_1 \gtrsim 2^{-k_1}} \int\int Q^{\pm_1}_{M_1} P_{k_1} u Q^{\pm_2}_{\le M_1} P_{k_2} v Q^{\pm_3}_{\le M_1} P_{k_3} w dxdt$$

$$I_1(k_1) = \sum_{M_2 \gtrsim 2^{-k_1}} \int\int Q^{\pm_1}_{\le M_2} P_{k_1} u Q^{\pm_2}_{M_2} P_{k_2} v Q^{\pm_3}_{\le M_2} P_{k_3} w dxdt$$

$$I_2(k_1) = \sum_{M_3 \gtrsim 2^{-k_1}} \int\int Q^{\pm_1}_{\le M_3} P_{k_1} u Q^{\pm_2}_{\le M_3} P_{k_2} v Q^{\pm_3}_{M_3} P_{k_3} w dxdt.$$

Here, the lower bound in the summands comes from Lemma 5.2.2, as the integral vanishes for terms where $\max(M_1, M_2, M_3) \lesssim \max(2^{-k_1}, 2^{-k_2}, 2^{-k_3}) = 2^{-k_1}$. We can therefore bound the LHS of (5.12) by

$$\frac{1}{2^{k_3}} \sum_{k_1 \le k_3 + C} |I_0(k_1)| + \frac{1}{2^{k_3}} \sum_{k_1 \le k_3 + C} |I_1(k_1)| + \frac{1}{2^{k_3}} \sum_{k_1 \le k_3 + C} |I_2(k_1)| = I + II + III.$$

We first turn our attention to bounding (I). It isn't difficult to see that

$$(I) \le \frac{1}{2^{k_3}} \sum_{k_1 \le k_3 + C} \Big(\sum_{M_1 \gtrsim 2^{-k_1}} \| Q^{\pm_1}_{M_1} P_{k_1} u \|_{L^2} \Big) \sup_{M_1 \gtrsim 2^{-k_1}} \| Q^{\pm_2}_{\le M_1} P_{k_2} v Q^{\pm_3}_{\le M_1} P_{k_3} w \|_{L^2}.$$

Applying the first estimate from Proposition 2.4.2 to the high modulation term, we bound the above by

$$\lesssim \frac{1}{2^{k_3}} \sum_{k_1 \leq k_3+C} \Big(\sum_{M_1 \gtrsim 2^{-k_1}} M_1{}^{-\frac{1}{2}} \|P_{k_1} u\|_{V^2_{\pm_1}} \Big) \sup_{M_1 \gtrsim 2^{-k_1}} \|Q^{\pm_2}_{\leq M_1} P_{k_2} v Q^{\pm_3}_{\leq M_1} P_{k_3} w\|_{L^2}$$

$$\lesssim \frac{1}{2^{k_3}} \sum_{k_1 \leq k_3+C} 2^{\frac{k_1}{2}} \|P_{k_1} u\|_{V^2_{\pm_1}} \sup_{M_1 \gtrsim 2^{-k_1}} \|Q^{\pm_2}_{\leq M_1} P_{k_2} v Q^{\pm_3}_{\leq M_1} P_{k_3} w\|_{L^2}$$

Applying Cauchy-Schwartz in k_1 allows us to bound this by

$$\leq \frac{1}{2^{k_3}} \Big(\sum_{k_1 \leq k_3+C} 2^{2k_1 s} \|P_{k_1} u\|^2_{V^2_{\pm_1}} \Big)^{\frac{1}{2}} \Big(\sum_{k_1 \leq k_3+C} 2^{(1-2s)k_1} \sup_{M_1 \gtrsim 2^{-k_1}} \|Q^{\pm_2}_{\leq M_1} P_{k_2} v Q^{\pm_3}_{\leq M_1} P_{k_3} w\|^2_{L^2} \Big)^{\frac{1}{2}}.$$

We use the fact that $1 - 2s < 1$ to bound

$$\sum_{k_1 \leq k_3+C} 2^{(1-2s)k_1} \sup_{M_1 \gtrsim 2^{-k_1}} \|Q^{\pm_2}_{\leq M_1} P_{k_2} v Q^{\pm_3}_{\leq M_1} P_{k_3} w\|^2_{L^2} \leq \sup_{M_1 \gtrsim 2^{-k_3}} \|Q^{\pm_2}_{\leq M_1} P_{k_2} v Q^{\pm_3}_{\leq M_1} P_{k_3} w\|^2_{L^2}$$

$$\leq \sup_{M_1 \gtrsim 2^{-k_3}} \|Q^{\pm_2}_{\leq M_1} P_{k_2} v\|^2_{L^4} \|Q^{\pm_3}_{\leq M_1} P_{k_3} w\|^2_{L^4}.$$

Using Corollary 5.2.1, we can bound this by

$$\lesssim 2^{2k_3} \sup_{M_1 \gtrsim 2^{-K_3}} \|Q^{\pm_2}_{\leq M_1} P_{k_2} v\|^2_{V^2_{\pm_2}} \|Q^{\pm_3}_{\leq M_1} P_{k_3} w\|^2_{V^2_{\pm_3}}.$$

Applying Proposition 2.4.2, this is

$$\lesssim 2^{2k_3} \|P_{k_2} v\|^2_{V^2_{\pm_2}} \|P_{k_3} w\|^2_{V^2_{\pm_3}},$$

which is the desired bound.

Next we turn our attention to bounding (II) (the argument for bounding (III) will be nearly identical). As before, we place the high modulation term in L^2 to bound (II) by

$$2^{-k_3} \sum_{k_1 \leq k_3+C} \Big(\sum_{M_2 \gtrsim 2^{-k_1}} \|Q^{\pm_2}_{M_2} P_{k_2} v\|_{L^2} \Big) \sup_{M_2 \gtrsim 2^{-k_1}} \|Q^{\pm_1}_{\leq M_2} P_{k_1} u Q^{\pm_3}_{\leq M_2} P_{k_3} w\|_{L^2}.$$

Once again we apply Proposition 2.4.2 and Corollary 5.2.1 to bound this by

$$\lesssim 2^{-k_3} \sum_{k_1 \leq k_3 + C} \Big(\sum_{M_2 \gtrsim 2^{-k_1}} M_2^{-\frac{1}{2}} \|P_{k_2} v\|_{V^2_{\pm_2}} \Big) \sup_{M_2 \gtrsim 2^{-k_1}} \|Q^{\pm_1}_{\leq M_2} P_{k_1} u Q^{\pm_3}_{\leq M_2} P_{k_3} w\|_{L^2}$$

$$\lesssim 2^{-k_3} \sum_{k_1 \leq k_3 + C} 2^{\frac{k_1}{2}} 2^{\frac{k_1}{2}} 2^{\frac{k_3}{2}} \sup_{M_2 \gtrsim 2^{-k_1}} \|P_{k_2} v\|_{V^2_{\pm_2}} \|Q^{\pm_1}_{\leq M_2} P_{k_1} u\|_{V^2_{\pm_1}} \|Q^{\pm_3}_{\leq M_2} P_{k_3} w\|_{V^2_{\pm_3}}.$$

Applying Cauchy-Schwartz in k_1 and Proposition 2.4.2, this is

$$\leq 2^{-\frac{k_3}{2}} \Big(\sum_{k_1 \leq k_3 + C} 2^{2k_1 s} \|P_{k_1} u\|^2_{V^2_{\pm_1}} \Big)^{\frac{1}{2}} \Big(\sum_{k_1 \leq k_3 + C} 2^{(1-2s)k_1} 2^{k_1} \Big)^{\frac{1}{2}} \|P_{k_2} v\|_{V^2_{\pm_2}} \|P_{k_3} w\|_{V^2_{\pm_3}},$$

As $s \geq \frac{1}{2}$, we have

$$\sum_{k_1 \leq k_3 + C} 2^{(1-2s)k_1} 2^{k_1} \lesssim \sum_{k_1 \leq k_3 + C} 2^{k_1} \lesssim 2^{k_3},$$

giving us the desired bound.

We now turn to estimate (5.13). We denote $J(k_1) := \int\int P_{k_2} u P_{k_3} v P_{k_1} w dx dt$, then we can decompose

$$J(k_1) = J_0(k_1) + J_1(k_1) + J_2(k_1),$$

where

$$J_0(k_1) = \sum_{M_1 \gtrsim 2^{-k_1}} \int\int Q^{\pm_1}_{M_1} P_{k_2} u Q^{\pm_2}_{\leq M_1} P_{k_3} v Q^{\pm_3}_{\leq M_1} P_{k_1} w dx dt$$

$$J_1(k_1) = \sum_{M_2 \gtrsim 2^{-k_1}} \int\int Q^{\pm_1}_{\leq M_2} P_{k_2} u Q^{\pm_2}_{M_2} P_{k_3} v Q^{\pm_3}_{\leq M_2} P_{k_1} w dx dt$$

$$J_2(k_1) = \sum_{M_3 \gtrsim 2^{-k_1}} \int\int Q^{\pm_1}_{\leq M_3} P_{k_2} u Q^{\pm_2}_{\leq M_3} P_{k_3} v Q^{\pm_3}_{M_3} P_{k_1} w dx dt.$$

We can therefore bound the LHS of (5.13) by

$$\sum_{i=0}^{2} \Big(\sum_{k_1 \leq k_3 + C} 2^{-2k_1} 2^{2k_1 s} \sup_{\|P_{k_1} w\|_{V^2_{\pm_3}} = 1} |J_i(k_1)|^2 \Big)^{\frac{1}{2}} = I + II + III.$$

We first focus on estimating (III). We see that

$$(III)^2 \leq \sum_{k_1 \leq k_3 + C} 2^{-2k_1} 2^{2k_1 s}$$

$$\sup_{\|P_{k_1} w\|_{V^2_{\pm_3}} = 1} \Big(\sum_{M_3 \gtrsim 2^{-k_1}} \|Q^{\pm_3}_{M_3} P_{k_1} w\|_{L^2} \Big)^2 \Big(\sup_{M_3 \gtrsim 2^{-k_1}} \|Q^{\pm_1}_{\leq M_3} P_{k_2} u Q^{\pm_2}_{\leq M_3} P_{k_3} v\|_{L^2} \Big)^2.$$

Applying Proposition 2.4.2 and Corollary 5.2.1 once again, allows us to bound the above by

$$\lesssim \sum_{k_1 \leq k_3 + C} 2^{-2k_1} 2^{2k_1 s}$$

$$\times \sup_{\|P_{k_1} w\|_{V^2_{\pm_3}} = 1} \Big(\sum_{M_3 \gtrsim 2^{-k_1}} M_3^{-\frac{1}{2}} \|P_{k_1} w\|_{V^2_{\pm_3}} \Big)^2 \sup_{M_3 \gtrsim 2^{-k_1}} \big(2^{k_2} \|Q^{\pm_1}_{\leq M_3} P_{k_2} u\|^2_{V^2_{\pm_1}} 2^{k_3} \|Q^{\pm_2}_{\leq M_3} P_{k_3} v\|^2_{V^2_{\pm_2}} \big)$$

$$\lesssim \sum_{k_1 \leq k_3 + C} 2^{-2k_1} 2^{2sk_1} 2^{k_1} \sup_{M_3 \gtrsim 2^{-k_1}} \big(2^{k_2} \|Q^{\pm_1}_{\leq M_3} P_{k_2} u\|^2_{V^2_{\pm_1}} 2^{k_3} \|Q^{\pm_2}_{\leq M_3} P_{k_3} v\|^2_{V^2_{\pm_2}} \big).$$

Applying Proposition 2.4.2 and summing over k_1, we bound the above by

$$\lesssim 2^{(2s-1)k_3} 2^{k_2} \|P_{k_2} u\|^2_{V^2_{\pm_1}} 2^{k_3} \|P_{k_3} v\|^2_{V^2_{\pm_2}}.$$

As $s \geq \frac{1}{2}$, this is

$$\lesssim 2^{2k_2 s} 2^{2k_3 s} \|P_{k_2} u\|^2_{V^2_{\pm_1}} \|P_{k_3} v\|^2_{V^2_{\pm_2}},$$

as desired.

We now focus on bounding I (the argument for bounding II is the same). We observe that

$$(I)^2 \leq \sum_{k_1 \leq k_3 + C} 2^{2k_1(s-1)} \sup_{\|P_{k_1} w\|_{V^2_{\pm_3}} = 1} \Big(\sum_{M_1 \gtrsim 2^{-k_1}} \|Q^{\pm_1}_{M_1} P_{k_2} u\|_{L^2} \Big)^2 \sup_{M_1 \gtrsim 2^{-k_1}} \|Q^{\pm_2}_{\leq M_1} P_{k_3} v Q^{\pm_3}_{\leq M_1} P_{k_1} w\|^2_{L^2}$$

As before, we apply Proposition 2.4.2 and Corollary 5.2.1 to obtain

$$(I)^2 \leq \sum_{k_1 \leq k_3+C} 2^{2k_1(s-1)} \sup_{\|P_{k_1} w\|_{V^2_{\pm_3}}=1} \Big(\sum_{M_1 \gtrsim 2^{-k_1}} M_1^{-\frac{1}{2}} \|P_{k_2} u\|_{V^2_{\pm_1}}\Big)^2$$

$$\times \sup_{M_1 \gtrsim 2^{-k_1}} 2^{k_3} \|Q^{\pm_2}_{\leq M_1} P_{k_3} v\|^2_{V^2_{\pm_2}} 2^{k_1} \|Q^{\pm_3}_{\leq M_1} P_{k_1} w\|^2_{V^2_{\pm_3}}.$$

Applying Proposition 2.4.2 once again, this is

$$\lesssim \sum_{k_1 \leq k_3+C} 2^{2sk_1} 2^{k_2} \|P_{k_2} u\|^2_{V^2_{\pm_1}} 2^{k_3} \|P_{k_3} v\|^2_{V^2_{\pm_2}},$$

which we can bound by

$$\lesssim 2^{2sk_2} 2^{2sk_3} \|P_{k_2} u\|^2_{V^2_{\pm_1}} \|P_{k_3} v\|^2_{V^2_{\pm_2}},$$

as desired. □

Next, we turn to the higher dimensional case

Theorem 5.3.2. *(Trilinear Estimates for $n \geq 3$) Let $s \geq \frac{n-2}{2}, \pm_1, \pm_2, \pm_3 \in \{+,-\}$, $C \geq 10$ and $k_2, k_3 \in \mathbb{Z}_{\geq 0}$ be such that $|k_2 - k_3| \leq C$, then*

$$\frac{1}{2^{k_3}} \Big|\sum_{k_1 \leq k_3+C} \int\int P_{k_1} u P_{k_2} v P_{k_3} w dx dt\Big| \lesssim \Big(\sum_{k_1 \leq k_3+C} 2^{2k_1 s} \|P_{k_1} u\|^2_{V^2_{\pm_1}}\Big)^{\frac{1}{2}} \|P_{k_2} v\|_{V_0^{\pm_2}} \|P_{k_3} w\|_{V_0^{\pm_3}} \tag{5.14}$$

$$\left(\sum_{k_1 \leq k_3+C} 2^{(2s-2)k_1} \sup_{\|P_{k_1} w\|_{V_0^{\pm_3}}=1} \Big(\Big|\sum_{k_1 \leq k_3+C} \int\int P_{k_2} u P_{k_3} v P_{k_1} w dx dt\Big|^2\Big)\right)^{\frac{1}{2}}$$

$$\lesssim 2^{k_2 s} 2^{k_3 s} \|P_{k_2} u\|_{V_0^{\pm}} \|P_{k_3 v}\|_{V_0^{\pm}}. \tag{5.15}$$

Proof. We can bound the left-hand side of (5.14) by

$$\lesssim \frac{1}{2^{k_3}} \sum_{k_1 \leq k_3+C} \sum_{d \in \Xi_{k_1}} \sum_{d' \in \Xi_{k_1}} \left|\int\int P_{k_1} u \Gamma_{d',k_1} P_{k_2} v \Gamma_{d,k_1} P_{k_3} w dx dt\right|. \tag{5.16}$$

Observe that

$$\int P_{k_1}uP_{k_2}vP_{k_3}wdx = (\widehat{P_{k_1}u} * \widehat{P_{k_2}v} * \widehat{P_{k_3}w})(0)$$
$$= \int \widehat{P_{k_1}u}(-\xi_3)\int_{\xi_1+\xi_2=\xi_3} \widehat{P_{k_2}v}(\xi_1) * \widehat{P_{k_3}w}(\xi_2)$$

As $P_{k_1}u$ is supported at frequency 2^{k_1} we can conclude that

$$P_{k_2}vP_{k_3}w = P_{k_1}(P_{k_2}vP_{k_3}w)$$

Therefore, (5.16) can be bounded by

$$\frac{1}{2^{k_3}}\sum_{k_1\leq k_3+C}\sum_{d\in\Xi_{k_1}}\sum_{|d+d'|\lesssim 2^{k_1}}\left|\int\int P_{k_1}u\Gamma_{d',k_1}P_{k_2}v\Gamma_{d,k_1}P_{k_3}wdxdt\right| \tag{5.17}$$

We denote $I_0(k_1) := \int\int P_{k_1}u\Gamma_{d',k_1}P_{k_2}v\Gamma_{d,k_1}P_{k_3}wdxdt$, then, we decompose

$$I(k_1) = I_{00}(k_1) + I_{01}(k_1) + I_{02}(k_1),$$

where

$$I_0(k_1,d,d') = \sum_{M_1\gtrsim 2^{-k_1}}\int\int Q^{\pm_1}_{M_1}P_{k_1}uQ^{\pm_2}_{\leq M_1}\Gamma_{d',k_1}P_{k_2}vQ^{\pm_3}_{\leq M_1}\Gamma_{d,k_1}P_{k_3}wdxdt$$
$$I_1(k_1,d,d') = \sum_{M_2\gtrsim 2^{-k_1}}\int\int Q^{\pm_1}_{\leq M_2}P_{k_1}uQ^{\pm_2}_{M_2}\Gamma_{d',k_1}P_{k_2}vQ^{\pm_3}_{\leq M_2}\Gamma_{d,k_1}P_{k_3}wdxdt$$
$$I_2(k_1,d,d') = \sum_{M_3\gtrsim 2^{-k_1}}\int\int Q^{\pm_1}_{\leq M_3}P_{k_1}uQ^{\pm_2}_{\leq M_3}\Gamma_{d',k_1}P_{k_2}vQ^{\pm_3}_{M_3}\Gamma_{d,k_1}P_{k_3}wdxdt.$$

We can therefore bound the LHS of (5.14) by

$$\frac{1}{2^{k_3}}\sum_{i=0}^{2}\sum_{k_1\leq k_3+C}\sum_{d\in\Xi_{k_1}}\sum_{|d+d'|\lesssim 2^{k_1}}|I_i|$$

We first turn our attention to bounding (I). We observe that (I) is bounded by

$$\frac{1}{2^{k_3}}\sum_{k_1\leq k_3+C}\sum_{d\in\Xi_{k_1}}\sum_{|d'+d|\lesssim 2^{k_1}}(\sum_{M_1\gtrsim 2^{-k_1}}\|Q^{\pm_1}_{M_1}P_{k_1}u\|_{L^2})\sup_{M_1\gtrsim 2^{-k_1}}\|Q^{\pm_2}_{\leq M_1}\Gamma_{d',k_1}P_{k_2}vQ^{\pm_3}_{\leq M_1}\Gamma_{d,k_1}P_{k_3}w\|_{L^2}.$$

Applying Proposition 2.4.2 allows us to bound the above by

$$\lesssim \frac{1}{2^{k_3}} \sum_{k_1 \le k_3 + C} \Big(\big(\sum_{M_1 \gtrsim 2^{-k_1}} M_1^{-\frac{1}{2}} \|P_{k_1} u\|_{V^2_{\pm 1}} \big)$$

$$\times \sum_{d \in \Xi_{k_1}} \sum_{|d'+d| \lesssim 2^{k_1}} \sup_{M_1 \gtrsim 2^{-k_1}} \|Q^{\pm_2}_{\le M_1} \Gamma_{d',k_1} P_{k_2} v\|_{L^4} \|Q^{\pm_3}_{\le M_1} \Gamma_{d,k_1} P_{k_3} w\|_{L^4} \Big)$$

$$\lesssim \frac{1}{2^{k_3}} \sum_{k_1 \le k_3 + C} (2^{\frac{k_1}{2}} \|P_{k_1} u\|_{V^2_{\pm 1}} \sum_{d \in \Xi_{k_3}} \sum_{|d'+d| \lesssim 2^{k_1}} \sup_{M_1 \gtrsim 2^{-k_1}} \|Q^{\pm_2}_{\le M_1} \Gamma_{d',k_1} P_{k_2} v\|_{L^4} \|Q^{\pm_3}_{\le M_1} \Gamma_{d,k_1} P_{k_3} w\|_{L^4}).$$

Applying the Cauchy-Schwartz inequality, can conclude that the above is

$$\lesssim \frac{1}{2^{k_3}} (\sum_{k_1 \le k_3 + C} 2^{2k_1 s} \|P_{k_1} u\|^2_{V^2_{\pm 1}})^{\frac{1}{2}}$$

$$\times (\sum_{k_1 \le k_3 + C} 2^{k_1(1-2s)} (\sum_{d \in \Xi_{k_1}} \sum_{|d'+d| \lesssim 2^{k_1}} \sup_{M_1 \gtrsim 2^{-k_1}} \|Q^{\pm_2}_{\le M_1} \Gamma_{d',k_1} P_{k_2} v\|_{L^4} \|Q^{\pm_3}_{\le M_1} \Gamma_{d,k_1} P_{k_3} w\|_{L^4})^2)^{\frac{1}{2}}.$$

By Proposition 5.2.4, we can bound the 2nd term in parentheses above by

$$\Big(\sum_{k_1 \le k_3 + C} 2^{(1-2s)k_1} (\sum_{d \in \Xi_{k_1}} \sum_{|d'+d| \lesssim 2^{k_1}} 2^{\frac{(n-2)k_1}{2}} 2^{\frac{k_2}{4}} 2^{\frac{k_3}{4}}$$

$$\times \sup_{M_1 \gtrsim 2^{-k_1}} \|Q^{\pm_2}_{\le M_1} \Gamma_{d',k_1} P_{k_2} v\|_{U^4_{\pm 2}} \|Q^{\pm_3}_{\le M_1} \Gamma_{d,k_1} P_{k_3} w\|_{U^4_{\pm 3}})^2 \Big)^{\frac{1}{2}}.$$

By Proposition 2.4.2 and Holder's inequality in d this is

$$\lesssim (\sum_{k_1 \le k_3 + C} 2^{(1-2s)k_1} 2^{(n-2)k_1} 2^{k_3} (\sum_{d \in \Xi_{k_1}} \|\Gamma_{d,k_1} P_{k_3} w\|^2_{U^4_{\pm 3}}) (\sum_{d \in \Xi_{k_1}} (\sum_{|d'+d| \lesssim 2^{k_1}} \|\Gamma_{d',k_1} P_{k_2} v\|_{U^4_{\pm 2}})^2)^{\frac{1}{2}}.$$

Applying Young's inequality in d to the term on the far right allows us to bound this by

$$\lesssim (\sum_{k_1 \le k_3 + C} 2^{(n-1-2s)k_1} 2^{k_3} \|P_{k_2} v\|^2_{U^{4,\pm 2}_{k_1}} \|P_{k_3} w\|^2_{U^{4,\pm 3}_{k_1}})^{\frac{1}{2}}$$

$$\lesssim (\sum_{k_1 \le k_3 + C} 2^{(n-2s-1)k_1} 2^{k_3})^{\frac{1}{2}} \sup_{k_1 \le k_3 + C} \|P_{k_2} v\|_{V^{\pm 2}_{k_1}} \|P_{k_3} w\|_{V^{\pm 3}_{k_1}}$$

$$\lesssim (\sum_{k_1 \le k_3 + C} 2^{(n-2s-1)k_1} 2^{k_3})^{\frac{1}{2}} \sup_{k_1 \le k_3 + C} \|P_{k_2} v\|_{V^{\pm 2}_0} \|P_{k_3} w\|_{V^{\pm 3}_0},$$

where

$$\sum_{k_1 \leq k_3 + C} 2^{(n-2s-1)k_1} \lesssim 2^{(n-1-2s)k_3} \lesssim 2^{k_3},$$

provided $n - 1 - 2s \leq 1$ or equivalently $s \geq \frac{n-2}{2}$.

We now turn our attention to bounding (II) (the argument for bounding (III) is identical). By placing the high modulation term in L^2 and applying Proposition 2.4.2, we can conclude that

$$(II) \leq \frac{1}{2^{k_3}} \sum_{k_1 \leq k_3+C} \sum_{d \in \Xi_{k_1}} \sum_{|d+d'| \lesssim 2^{k_1}} \Big((\sum_{M_2 \gtrsim 2^{-k_1}} \| Q^{\pm_2}_{M_2} \Gamma_{d',k_1} P_{k_2} v \|_{L^2})$$

$$\times \sup_{M_2 \gtrsim 2^{-k_1}} \| Q^{\pm_3}_{\leq M_2} \Gamma_{d,k_1} P_{k_3} w Q^{\pm_1}_{\leq M_2} P_{k_1} u \|_{L^2} big)$$

$$\lesssim \frac{1}{2^{k_3}} \sum_{k_1 \leq k_3+C} \sum_{d \in \Xi_{k_1}} \sum_{|d+d'| \lesssim 2^{k_1}} \Big((\sum_{M_2 \gtrsim 2^{-k_1}} M_2^{-\frac{1}{2}} \| \Gamma_{d',k_1} P_{k_2} v \|_{V^2_{\pm_2}})$$

$$\times \sup_{M_2 \gtrsim 2^{-k_1}} \| Q^{\pm_3}_{\leq M_2} \Gamma_{d,k_1} P_{k_3} w Q^{\pm_1}_{\leq M_2} P_{k_1} u \|_{L^2} \Big)$$

$$\lesssim \frac{1}{2^{k_3}} \sum_{k_1 \leq k_3+C} \sum_{d \in \Xi_{k_1}} \sum_{|d+d'| \lesssim 2^{k_1}} 2^{\frac{k_1}{2}} \| \Gamma_{d',k_1} P_{k_2} v \|_{V^2_{\pm_2}} \sup_{M_2 \gtrsim 2^{-k_1}} \| Q^{\pm_3}_{\leq M_2} \Gamma_{d,k_1} P_{k_3} w \|_{L^4} \| Q^{\pm_1}_{\leq M_2} P_{k_1} u \|_{L^4}.$$

Applying Proposition 5.2.4 to the two terms on the right and using the fact that $V^2 \subseteq U^4$, we may bound the above by

$$\lesssim \frac{1}{2^{k_3}} \sum_{k_1 \leq k_3+C} \sum_{d \in \Xi_{k_1}} \sum_{|d'+d| \leq 2^{k_1}} 2^{\frac{k_1}{2}} 2^{\frac{(2n-3)k_1}{4}} 2^{\frac{k_3}{4}} \| P_{k_1} u \|_{V^2_{\pm_1}} \| \Gamma_{d',k_1} P_{k_2} v \|_{V^2_{\pm_2}} \| \Gamma_{d,k_1} P_{k_3} w \|_{V^2_{\pm_3}}.$$

Applying the Cauchy-Schwartz inequality in k_1 and d, and applying Young's inequality in d we see that this is

$$\lesssim \frac{1}{2^{k_3}} (\sum_{k_1 \leq k_3+C} 2^{2sk_1} \| P_{k_1} u \|_{V^2_{\pm_1}})^{\frac{1}{2}} (\sum_{k_1 \leq k_3+C} 2^{\frac{(2n-3)k_1}{2}} 2^{\frac{k_3}{2}} 2^{k_1(1-2s)} \| P_{k_2} v \|^2_{V^{2,\pm_2}_{k_1}} \| P_{k_2} w \|^2_{V^{2,\pm_3}_{k_1}})^{\frac{1}{2}}$$

$$\lesssim \frac{1}{2^{k_3}} (\sum_{k_1 \leq k_3+C} 2^{2sk_1} \| P_{k_1} u \|_{V^2_{\pm_1}})^{\frac{1}{2}} (\sum_{k_1 \leq k_3+C} 2^{\frac{(2n-3)k_1}{2}} 2^{\frac{k_3}{2}} 2^{(1-2s)k_1})^{\frac{1}{2}} \| P_{k_2} v \|_{V^{\pm_2}_0} \| P_{k_3} w \|_{V^{\pm_3}_0}.$$

The second term in parenthesis is

$$\lesssim \sum_{k_1 \le k_3 + C} 2^{(n-\frac{1}{2}-2s)k_1} 2^{\frac{k_3}{2}} \lesssim 2^{(n-\frac{1}{2}-2s)k_3} 2^{\frac{k_3}{2}} \lesssim 2^{2k_3}.$$

provided $n - \frac{1}{2} - 2s \le \frac{3}{2}$ or equivalently $s \ge \frac{n-2}{2}$, giving us our desired result.

Our next task is to prove inequality (5.15). We may bound the LHS of (5.15) by

$$(\sum_{k_1 \le k_3 + C} 2^{(2s-2)k_1} \sup_{\|P_{k_1} w\|_{V_0^{\pm_3}}=1} (|\sum_{k_1 \le k_3 + C} \sum_{d \in \Xi_{k_1}} \sum_{|d'+d| \lesssim 2^{k_1}} \int\int \Gamma_{d,k_1} P_{k_2} u \Gamma_{d',k_1} P_{k_3} v P_{k_1} w dxdt|^2)^{\frac{1}{2}}. \tag{5.18}$$

We denote $J(k_1) := \int\int \Gamma_{d,k_1} P_{k_2} u \Gamma_{d',k_3} P_{k_3} v P_{k_1} w dxdt$, then we can decompose

$$J(k_1) = J_0(k_1) + J_1(k_1) + J_2(k_1)$$

where

$$J_0(k_1) = \sum_{M_1 \gtrsim 2^{-k_1}} \int\int Q^{\pm_1}_{M_1} \Gamma_{d',k_1} P_{k_3} v Q^{\pm_2}_{\le M_1} \Gamma_{d,k_1} P_{k_2} u Q^{\pm_w}_{\le M_1} P_{k_1} w dxdt$$

$$J_1(k_1) = \sum_{M_2 \gtrsim L^{-k_1}} \int\int Q^{\pm_1}_{\le M_2} \Gamma_{d',k_1} P_{k_3} v Q^{\pm_2}_{M_2} \Gamma_{d,k_1} P_{k_2} u Q^{\pm_w}_{\le M_2} P_{k_1} w dxdt$$

$$J_2(k_1) = \sum_{M_3 \gtrsim L^{-k_1}} \int\int Q^{\pm_1}_{\le M_3} \Gamma_{d',k_1} P_{k_3} v Q^{\pm_2}_{\le M_3} \Gamma_{d,k_1} P_{k_2} u Q^{\pm_w}_{M_3} P_{k_1} w dxdt.$$

We can therefore bound the LHS of (5.18) by

$$\sum_{i=0}^{2} (\sum_{k_1 \le k_3 + C} 2^{(2s-2)k_1} \sup_{\|P_{k_1} w\|_{V_0^{\pm_3}}=1} |\sum_{n \in \Xi_{k_1}} \sum_{|d'+d| \lesssim 2^{k_1}} J_i(k_1)|^2)^{\frac{1}{2}} = I + II + III.$$

We first focus on estimating (III). We can bound (III) by

$$(\sum_{k_1 \le k_3 + C} 2^{k_1(2s-2)} \sup_{\|P_{k_1} w\|_{V_0^{\pm_3}}=1} (\sum_{M_3 \gtrsim 2^{-k_1}} \|Q^{\pm_3}_{M_3} P_{k_1} w\|_{L^2})^2$$

$$\times (\sum_{d \in \Xi_{k_1}} \sum_{|d'+d| \lesssim 2^{k_1}} \sup_{M_3 \gtrsim 2^{-k_1}} \|Q^{\pm_1}_{\le M_3} \Gamma_{d,k_1} P_{k_2} u\|_{L^4} \|Q^{\pm_2}_{\le M_3} \Gamma_{d',k_1} P_{k_3} v\|_{L^4})^2)^{\frac{1}{2}}.$$

By Proposition 2.4.2, this is

$$\lesssim \Big(\sum_{k_1\le k_3+C} 2^{(2s-2)k_1} \sup_{\|P_{k_1}w\|_{V_0^{\pm_3}}=1} \Big(\sum_{M_3\gtrsim 2^{-k_1}} \|P_{k_1}w\|_{V^2_{\pm_3}}\Big)^2$$

$$\times \Big(\sum_{d\in\Xi_{k_1}} \sum_{|d'+d|\lesssim 2^{k_1}} \sup_{M_3\gtrsim 2^{-k_1}} \|Q^{\pm_1}_{\le M_3}\Gamma_{d',k_1}P_{k_3}v\|_{L^4}\|Q^{\pm_2}_{\le M_3}\Gamma_{d,k_1}P_{k_2}u\|_{L^4}\Big)^2\Big)^{\frac{1}{2}}$$

$$\lesssim \Big(\sum_{k_1\le k_3+C} 2^{(2s-2)k_1} \sup_{\|P_{k_1}w\|_{V_0^{\pm_3}}=1} 2^{k_1}\|P_{k_1}w\|^2_{V^2_{\pm_3}}$$

$$\times \Big(\sum_{d\in\Xi_{k_1}} \sum_{|d'+d|\lesssim 2^{k_1}} \sup_{M_3\gtrsim 2^{-k_1}} \|Q^{\pm_1}_{\le M_3}\Gamma_{d',k_1}P_{k_3}v\|_{L^4}\|Q^{\pm_2}_{\le M_3}\Gamma_{d,k_1}P_{k_2}u\|_{L^4}\Big)^2\Big)^{\frac{1}{2}}.$$

Recall that

$$\|P_{k_1}w\|_{V^2_\pm} \lesssim \|P_{k_1}w\|_{V_0^\pm},$$

so we can bound the expression above by

$$\Big(\sum_{k_1\le k_3+C} 2^{k_1(2s-1)}\Big(\sum_{d\in\Xi_{k_1}} \sum_{|d'+d|\lesssim 2^{k_1}} \sup_{M_3\gtrsim 2^{-k_1}} \|Q^{\pm_1}_{\le M_3}\Gamma_{d,k_1}P_{k_2}u\|_{L^4}\|Q^{\pm_2}_{\le M_3}\Gamma_{d',k_1}P_{k_3}v\|_{L^4}\Big)^2\Big)^{\frac{1}{2}}.$$

Applying Proposition 5.2.4 and Proposition 2.4.2 we see that this is

$$\lesssim \Big(\sum_{k_1\le k_3+C} 2^{k_1(2s-1)}\Big(\sum_{d\in\Xi_{k_1}} \sum_{|d'+d|\lesssim 2^{k_1}} 2^{\frac{(n-2)k_1}{2}} 2^{\frac{k_2}{4}} 2^{\frac{k_3}{4}} \|\Gamma_{d,k_1}P_{k_2}u\|_{V^2_\pm}\|\Gamma_{d',k_1}P_{k_3}v\|_{V^2_\pm}\Big)^2\Big)^{\frac{1}{2}}.$$

By Holder's and Young's inequality in d, this is

$$\lesssim \Big(\sum_{k_1\le k_3+C} 2^{(2s+n-3)k_1} 2^{\frac{k_2}{2}} 2^{\frac{k_3}{2}} \|P_{k_2}u\|^2_{V^\pm_{k_1}} \|P_{k_3}v\|^2_{V^\pm_{k_1}}\Big)^{\frac{1}{2}}$$

$$\lesssim \Big(\sum_{k_1\le k_3+C} 2^{(2s+n-3)k_1} 2^{\frac{k_2}{2}} 2^{\frac{k_3}{2}}\Big)^{\frac{1}{2}} \|P_{k_2}u\|_{V_0^\pm} \|P_{k_3}v\|_{V_0^\pm}.$$

The term in parentheses is $\lesssim 2^{2sk_2}2^{2sk_3}$ provided $s \ge \frac{n-2}{2}$ so we have our desired result.

We now turn our attention to bounding (I) (the argument for bounding (II) is the same). It is not difficult to see that

$$(I) \lesssim (\sum_{k_1 \le k_3 + C} 2^{k_1(2s-2)} \sup_{\|P_{k_1} w\|_{V_0^{\pm_3}}=1} (\sum_{d \in \Xi_{k_1}} \sum_{|d'+d| \le 2^{k_1}} 2^{\frac{k_1}{2}} \|\Gamma_{d,k_1} P_{k_2} u\|_{V^2_{\pm_1}}$$

$$\times \sup_{M_1 \gtrsim 2^{-k_1}} \|Q^{\pm_2}_{\le M_1} \Gamma_{d',k_1} P_{k_3} v\|_{L^4} \|Q^{\pm_3}_{\le M_1} P_{k_1} w\|_{L^4})^2)^{\frac{1}{2}}.$$

Applying Proposition 5.2.4 and Proposition 2.4.2 once again

$$\lesssim \Big(\sum_{k_1 \le k_3 + C} 2^{(2s-2)k_1} \sup_{\|P_{k_1} w\|_{V_0^{\pm_3}}=1} (\sum_{d \in \Xi_{k_1}} \sum_{|d'+d| \lesssim 2^{k_1}} 2^{\frac{k_1}{2}} \|\Gamma_{d,k_1} P_{k_2} u\|_{V^2_{\pm_2}}$$

$$\times 2^{\frac{(2n-3)k_1}{4}} 2^{\frac{k_3}{4}} \|\Gamma_{d',k_1} P_{k_3} v\|_{V^2_{\pm}} \|P_{k_1} w\|_{V^2_{\pm}})^2 \Big)^{\frac{1}{2}}.$$

Again we use the fact that $\|P_{k_1} w\|_{V^2_\pm} \lesssim \|P_{k_1} w\|_{V_0^\pm} = 1$ to bound the expression above by

$$\lesssim (\sum_{k_1 \le k_3 + C} 2^{k_1(2s+n-\frac{5}{2})} 2^{\frac{k_3}{2}} (\sum_{d \in \Xi_{k_1}} \sum_{|d'+d| \le 2^{k_1}} \|\Gamma_{d,k_1} P_{k_2} u\|_{V^2_{\pm_2}} \|\Gamma_{d',k_1} P_{k_3} v\|_{V^2_\pm})^2)^{\frac{1}{2}}.$$

Applying Holder's and Young's inequalities in d, we get

$$\lesssim (\sum_{k_1 \le k_3 + C} 2^{2k_1 s} 2^{(n-2)k_3} \|P_{k_2} u\|^2_{V^\pm_{k_1}} \|P_{k_3} v\|^2_{V^\pm_{k_1}})^{\frac{1}{2}}$$

$$\lesssim (\sum_{k_1 \le k_3 + C} 2^{2k_1 s} 2^{k_2(n-2)})^{\frac{1}{2}} \|P_{k_2} u\|_{V_0^\pm} \|P_{k_3} v\|_{V_0^\pm}.$$

But the sum on the left is $\lesssim 2^{2k_2 s} 2^{2k_3 s}$ provided $s \ge \frac{n-2}{2}$ so we have our desired result.

□

5.4 Proof of Main Theorem

We now apply the trilinear estimates from the previous theorem to prove the bound on the Duhamel term for $n \ge 3$ (the case $n = 2$ is similar):

$$\|I^\pm(u)\|_{\widetilde{X}^s_\pm} \lesssim \|(u^+, u^-)\|^2_{\widetilde{X}^s_\pm}.$$

It is not difficult to see that the bound follows from proving the following theorem.

Theorem 5.4.1. *Let $n \geq 3, s \geq \frac{n-2}{2}$. For any $\pm_1, \pm_2 \in \{+,-\}$ we have*

$$I_{\pm_1,\pm_2} := I : \widetilde{Y}^s \times \widetilde{Y}^s \to \widetilde{X}^s,$$

where

$$I((u^+,u^-),(v^+,v^-)) = (I^+(u^{\pm_1}, v^{\pm_2}), I^-(u^{\pm_1}, v^{\pm_2})),$$

$$I^{\pm}(u,v) = \int_0^t e^{\pm i(t-s)\langle D\rangle} \frac{uv}{2\langle D\rangle} ds.$$

In other words, for a constant $C = C(n)$

$$\|I(u,v)\|_{\widetilde{X}^s} \leq C\|u\|_{\widetilde{Y}^s}\|v\|_{\widetilde{Y}^s}$$

In particular, since $\widetilde{X}^s \subseteq \widetilde{Y}^s$, we also have

$$I : \widetilde{X}^s \times \widetilde{X}^s \to \widetilde{X}^s$$

and

$$I : \widetilde{Y}^s \times \widetilde{Y}^s \to \widetilde{Y}^\cdot$$

Proof. We will only show that

$$\|I^+(u,v)\|_{\widetilde{X}^s_{\pm}} \lesssim \|u\|_{\widetilde{Y}^s_{\pm_1}} \|v\|_{\widetilde{Y}^s_{\pm_2}}$$

as the argument for I^- will be nearly identical.

Let $C \geq 10$ be fixed. We observe that

$$\begin{aligned}\|I^+(u,v)\|_{\widetilde{X}^s_\pm} &= \left(\sum_{k=0}^{\infty} 2^{2ks}\|P_k I^+(u,v)\|^2_{U_0^\pm}\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k=0}^{\infty} 2^{2ks}\|P_k \sum_{k_1\leq k+C}\sum_{|k-k_2|\leq C} I^+(P_{k_1}u, P_{k_2}v)\|^2_{U_0^\pm}\right)^{\frac{1}{2}} \\ &+ \left(\sum_{k=0}^{\infty} 2^{2ks}\|P_k \sum_{k_1\geq k-C}\sum_{|k_2-k_1|\leq C} I^+(P_{k_1}u, P_{k_2}v)\|^2_{U_0^\pm}\right)^{\frac{1}{2}} \\ &\left(\sum_{k=0}^{\infty} 2^{2ks}\|P_k \sum_{k_2\leq k+C}\sum_{|k-k_1|\leq C} I^+(P_{k_1}u, P_{k_2}v)\|^2_{U_0^\pm}\right)^{\frac{1}{2}} \\ &= S_1 + S_2 + S_3.\end{aligned}$$

It suffices to consider only S_1 and S_2.

We first handle the term S_1. By a duality argument and estimate (5.14) from Theorem 5.3.2, we see that

$$\begin{aligned}\|P_k \sum_{k_1\leq k+C}\sum_{|k-k_2|\leq C} I^+(P_{k_1}u, P_{k_2}v)\|_{U_0^\pm} &\lesssim \sum_{|k_2-k|\leq C}\frac{1}{2^k}\sup_{\|P_k w\|_{V_0^\mp}=1}\left|\sum_{k_1\leq k+C}\int\int P_{k_1}uP_{k_2}vP_kw\right| \\ &\sum_{|k_2-k|\leq C}\left(\sum_{k_1\leq k+C} 2^{2k_1 s}\|P_{k_1}u\|^2_{V^2_{\pm_1}}\right)^{\frac{1}{2}}\|P_{k_2}v\|_{V_0^{\pm_2}} \\ &\lesssim \|u\|_{\widetilde{Y}^s_{\pm_1}}\sum_{|k_2-k|\leq C}\|P_{k_2}v\|_{V_0^{\pm_2}}.\end{aligned}$$

It follows that

$$S_1 \lesssim \|u\|_{\widetilde{Y}^s_{\pm_1}}\|v\|_{\widetilde{Y}^s_{\pm_2}}.$$

We now turn our attention towards bounding S_2. We observe that

$$S_2 \leq \sum_{k_1\geq 0}\sum_{|k_2-k_1|\leq C}\left(\sum_{k\leq k_1+C} 2^{2ks}\|P_k I^+(P_{k_1}u, P_{k_2}v)\|\right)^{\frac{1}{2}}$$

By duality and estimate (5.15), we conclude that this is

$$\sum_{k_1\geq 0}\sum_{|k_2-k_1|\leq C} 2^{2k_1 s}2^{2k_2 s}\|P_{k_1}u\|_{V_0^\pm}\|P_{k_2}v\|_{V_0^\pm} \lesssim \|u\|_{\widetilde{Y}^s_\pm}\|v\|_{\widetilde{Y}^s_\pm}.$$

□

With a similar argument, we can prove the analogous result for $n = 2$ given by

Theorem 5.4.2. *Let $n = 2, s > \frac{1}{2}$. For any $\pm_1, \pm_2 \in \{+, -\}$ we have*

$$I_{\pm_1,\pm_2} := I : Y^s \times Y^s \to X^s,$$

where

$$I((u^+, u^-), (v^+, v^-)) = (I^+(u^{\pm_1}, v^{\pm_2}), I^-(u^{\pm_1}, v^{\pm_2})),$$

$$I^\pm(u, v) = \int_0^t e^{\pm i(t-s)\langle D\rangle} \frac{uv}{2\langle D\rangle} ds.$$

In other words, for a constant $C = C(n)$,

$$\|I(u, v)\|_{X^s} \leq C\|u\|_{Y^s}\|v\|_{Y^s}.$$

In particular, since $X^s \subseteq Y^s$, we also have

$$I : X^s \times X^s \to X^s$$

and

$$I : Y^s \times Y^s \to Y^s.$$

We are finally ready to prove Theorem 5.0.1. We first consider the case $n = 2$. Using standard contraction mapping techniques we will find a fixed point of the operator equation

$$T^\pm(u^\pm) = e^{\pm it\langle D\rangle} u_0^\pm \mp I^\pm(u),$$

where $u = u^+ + u^-$ and $u_0^\pm = \frac{1}{2}(f \mp \frac{g}{\langle D\rangle})$.

We run our contraction argument in the closed disk

$$D_\delta = \{u \in X^s : \|u\|_{X^s} \leq \delta\},$$

for appropriately chosen δ.

From Theorem 5.4.2 it is not difficult to see that $T : D_\delta \to D_\delta$ is well-defined. In particular, we have that

$$\|e^{\pm it\langle D\rangle}u_0^\pm + iI^\pm(u)\|_{X^s} \lesssim \epsilon + \delta^2 \leq \delta,$$

for δ small enough.

It is left to show that $T^\pm$ is a contraction. Since the nonlinearity in $I^\pm$ is quadratic, we make use of the factorization $a^2 - b^2 = a(a-b) + b(a-b)$ to conclude that

$$\begin{aligned}\|T^\pm(u) - T^\pm(v)\|_{X^s} = \|I^\pm(u) - I^\pm(v)\|_{X^s} &\lesssim (\|u\|_{X^s} + \|v\|_{X^s})\|u-v\|_{X^s} \\ &\lesssim \delta\|u-v\|_{X^s}.\end{aligned}$$

Therefore $T^\pm$ is a contraction for appropriately chosen δ.

The argument for $n \geq 3$ is identical with X^s replaced by $\widetilde{X}^s$.

5.5 Systems of Different Masses

We would like to extend our results for the scalar equation to the system of multiple masses

$$(\Box + m_i^2)u_i = F_i(u_1, ..., u_k) \quad i = 1, ..., k.$$

Fortunately our results transfer quite readily to this generalized system provided we impose the condition

$$2\min\{m_i\} > \max\{m_i\} \tag{5.19}$$

We must first alter our iteration spaces slightly: Instead of using $U^p_\pm, V^p_\pm$ based spaces, we instead work with the spaces $U^p_{\pm\langle D\rangle_m}, V^p_{\pm\langle D\rangle_m}$ defined by the norms

$$\|u\|_{U^p_{\pm\langle D\rangle_m}} = \|e^{\mp\langle D\rangle_m} u\|_{U^p}, \quad \|u\|_{V^p_{\pm\langle D\rangle_m}} = \|e^{\mp\langle D\rangle_m} u\|_{V^p},$$

where we recall that $\langle\cdot\rangle_m = \sqrt{m^2 + |\cdot|^2}$.

It is straightforward, albeit tedious, to show that our estimates for $U^p_\pm, V^p_\pm$ transfer to our new spaces with implicit constants depending on the m_i.

The remaining part of the argument we are left to deal with is the modulation analysis which guarantees the lack of resonant terms. Fortunately, the generality of Lemma 5.2.1, which imposes condition (5.19), allows us to extend our modulation arguments to the multiple mass system.

Chapter 6

The Normal Forms Method

6.1 Motivation

In Chapter 4 we established global well-posedness of the third order semilinear Klein-Gordon system by using Strichartz estimates to run a standard contraction argument. Unfortunately, this approach fails for second order nonlinearities as no combination of Strichartz exponents will allow us to put a quadratic nonlinearity into the needed function spaces. Furthermore, the U^p, V^p approach discussed in Chapter 5 would result in too much derivative loss to allow one to close the contraction argument. Fortunately, we are able to circumvent these issues by applying the Normal Forms transform introduced by Shatah in [20] to reduce our second order system to a third order one.

The basic idea behind the Normal Forms transform is as follows. Suppose we are given a second order system

$$(\Box + 1)u = F(u, \partial_t u, \partial u).$$

We aim to find a decomposition $u = U+W$ where U is given explicitly in terms of $u, \partial_t u, \partial u$ and W solves a third order or higher system. The explicit form of U and our knowledge on solutions to higher order systems will in turn allow us to gain control over u.

To see how this method works in practice, we consider the ODE

$$\partial_t^2 u + u = \alpha_1 u^2 + \alpha_2 u \partial_t u + \alpha_3 (\partial_t u)^2,$$

where the α_i are arbitrary constants. Let

$$U = au^2 + bu\partial_t u + c(\partial_t u)^2,$$

where a, b, c are constants that will be determined later.

We compute

$$\partial_t U = 2au\partial_t u + b((\partial_t u)^2 + u\partial_t^2 u) + 2c(\partial_t u \partial_t^2 u)$$

and

$$\partial_t^2 U = 2a((\partial_t u)^2 + u\partial_t^2 u) + b(3\partial_t u \partial_t^2 u + u\partial_t^3 u) + 2c((\partial_t^2 u)^2 + \partial_t u \partial_t^3 u).$$

Observe that

$$\partial_t^2 u = \alpha_1 u^2 + \alpha_2 u \partial_t u + \alpha_3 (\partial_t u)^2 - u$$

and

$$\partial_t^3 u = Q(u, \partial_t u) - \partial_t u,$$

where $Q(\cdot, \cdot)$ is a polynomial whose lowest order term is quadratic.

Substituting in these expressions into our formula for $\partial_t^2 U$, we obtain

$$\partial_t^2 U = 2a((\partial_t u)^2 - u^2 + C_1(u, \partial_t u)) + b(-4u\partial_t u + C_2(u, \partial_t u)) + 2c(u^2 - (\partial_t u)^2 + C_3(u, \partial_t u)),$$

where the $C_i(\cdot,\cdot)$ are linear combinations of nonlinearities of cubic order or higher. We conclude that

$$\partial_t^2 U + U = (2c-a)u^2 - 3b(u\partial_t u) + (2a-c)(\partial_t u)^2 + C(u,\partial_t u),$$

where $C := 2aC_1 + bC_2 + 2cC_3$.

Let $W := u - U$. We would like to choose a, b, c so that

$$(\Box + 1)W = (\Box + 1)u - (\Box + 1)U = -C(u,\partial_t u). \tag{6.1}$$

Then, our decomposition $u = U + W$ will have the desired properties.

We compute

$$(\partial_t^2 u + u) - (\partial_t^2 U + U) = (\alpha_1 + a - 2c)u^2 + (\alpha_2 + 3b)(u\partial_t u) + (\alpha_3 + c - 2a)(\partial_t u)^2 - C(u,\partial_t u).$$

Equation (6.1) will hold true provided

$$\alpha_1 + a - 2c = 0,$$

$$\alpha_2 + 3b = 0,$$

$$\alpha_3 + c - 2a = 0.$$

When we apply this method to nonlinear PDEs, our constants a, b, c, α_i will be replaced by distributions involving differential operators that will act on u and its time derivatives.

6.2 The Normal Forms Transform of Shatah

We now turn our attention to the three dimensional case. We will closely follow the construction and computations presented in section 7.8 of [12]. Before we proceed any further, we first introduce some notation. Given $v, w \in \mathcal{S}(\mathbb{R}^3)$, we define $\mathcal{B}(\partial', \partial'')[v][w]$ to be the function whose Fourier transform evaluated at ξ is given by

$$(2\pi)^{-3} \int \mathcal{B}(i\xi - i\eta, i\eta)\hat{v}(\xi - \eta)\hat{w}(\eta)d\eta.$$

It is not difficult to see from the above that ∂' acts by differentiating v, whereas ∂'' acts by differentiating w. With the above definition in mind, we can write the general form of a second order semilinear Klein-Gordon equation as

$$(\Box + 1)u = F(u, u') = \sum_{j,k=0}^{1} \mathcal{A}_{jk}(\partial', \partial'')[\partial_t^j u][\partial_t^k u], \tag{6.2}$$

where A_{jk} is a polynomial of degree at most $1 - j$ in ∂' and $1 - k$ in ∂''.

Our goal is to prove global existence for this system given initial data

$$u(0, x) = f \in H^s, \quad \partial_t u(0, x) = g \in H^{s-1}$$

where

$$\|(f, g)\|_{H^s \times H^{s-1}} < \delta \ll 1,$$

for $s > 10$.

Let $D' := -i\partial'$ and $D'' := -i\partial''$. We would like to construct

$$U := \sum_{j,k=0}^{1} \mathcal{B}_{jk}(D', D'')[\partial_t^j u][\partial_t^k u],$$

where the $\mathcal{B}_{jk}$ are chosen so that if $W = u - U$, then $\Box W = F(u, u') - (\Box + 1)U$ is of third order in u.

In order to appropriately choose the B_{jk}, we first compute $(\Box + 1)U$. We see that

$$(1 - \Delta)U = \sum_{j,k=0}^{1} ((1 + |D'|^2) + (2\langle D', D''\rangle - 1) + (1 + |D''|^2))\mathcal{B}_{jk}(D', D'')[\partial_t^j u][\partial_t^k u],$$

$$\partial_t^2 U = \sum_{j,k=0}^{1} \mathcal{B}_{jk}(D', D'')([\partial_t^{j+2} u][\partial_t^k u] + 2[\partial_t^{j+1} u][\partial_t^{k+1} u] + [\partial_t^j u][\partial_t^{k+1} u])$$

Observe that

$$\partial_t^{j+2} u = \partial_t^j (\Delta - u + F(u, u')) = -(|D|^2 + 1)\partial_t^j u + \partial_t^j F(u, u').$$

Applying this substitution, we can conclude that

$$\begin{aligned}
(\Box + 1)U &= (2\langle D', D''\rangle - 1)\mathcal{B}_{00}(D', D'')[u][u] + 2\mathcal{B}_{00}(D', D'')[\partial_t u][\partial_t u] + \mathcal{R}_1 \\
&+ (2\langle D', D''\rangle - 1)\mathcal{B}_{10}(D', D'')[\partial_t u][u] - 2(1 + |D'|^2)\mathcal{B}_{10}(D', D'')[u][\partial_t u] + \mathcal{R}_2 \\
&+ (2\langle D', D''\rangle - 1)\mathcal{B}_{01}(D', D'')[u][\partial_t u] - 2(1 + |D''|^2)\mathcal{B}_{01}(D', D'')[\partial_t u][u] + \mathcal{R}_3 \\
&+ (2\langle D', D''\rangle - 1)\mathcal{B}_{11}(D', D'')[\partial_t u][\partial_t u] + 2(1 + |D'|^2)(1 + |D''|^2)\mathcal{B}_{11}(D', D'')[u][u] + \mathcal{R}_4,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R}_1 &= \mathcal{B}_{00}(D', D'')([F(u, u')][u] + [u][F(u, u')]) \\
\mathcal{R}_2 &= \mathcal{B}_{10}(D', D'')([\partial_t F(u, u')][u] + 2[F(u, u')][\partial_t u] + [\partial_t u][F(u, u')]) \\
\mathcal{R}_3 &= \mathcal{B}_{01}(D', D'')([u][\partial_t F(u, u')] + 2[\partial_t u][F(u, u')] + [F(u, u')][\partial_t u]) \\
\mathcal{R}_4 &= \mathcal{B}_{11}(D', D'')([\partial_t F(u, u')][\partial_t u] + 2[F(u, u')][F(u, u')] - 2(|D''|^2 + 1)[F(u, u')][u] \\
&\quad - 2(|D'|^2 + 1)[u][F(u, u')] + [\partial_t u][\partial_t F(u, u')]).
\end{aligned}$$

Let $\mathcal{R} := \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4$. We would like to choose $\mathcal{B}_{jk}$ so that $(\Box + 1)U = F(u, u') + \mathcal{R}$. This will imply that $(\Box + 1)W = -\mathcal{R}$ is third order in u. From (6.2), we see that

$$\mathcal{B}_{00}(\xi,\eta)(2\langle\xi,\eta\rangle-1)+2\mathcal{B}_{11}(\xi,\eta)(1+|\xi|^2)(1+|\eta|^2)=\mathcal{A}_{00}(i\xi,i\eta),$$

$$\mathcal{B}_{11}(\xi,\eta)(2\langle\xi,\eta\rangle-1)+2\mathcal{B}_{00}(\xi,\eta)=\mathcal{A}_{11}(i\xi,i\eta),$$

$$\mathcal{B}_{10}(\xi,\eta)(2\langle\xi,\eta\rangle-1)-2\mathcal{B}_{01}(\xi,\eta)(1+|\eta|^2)=\mathcal{A}_{10}(i\xi,i\eta),$$

$$\mathcal{B}_{01}(\xi,\eta)(2\langle\xi,\eta\rangle-1)-2\mathcal{B}_{10}(\xi,\eta)(1+|\xi|^2)=\mathcal{A}_{01}(i\xi,i\eta).$$

Solving this system for $\mathcal{B}_{jk}$, we obtain

$$\mathcal{B}_{00}(\xi,\eta)=((1-2\langle\xi,\eta\rangle)\mathcal{A}_{00}(i\xi,i\eta)+2(1+|\xi|^2)(1+|\eta|^2)\mathcal{A}_{11}(i\xi,i\eta))\mathcal{K}(\xi,\eta),$$

$$\mathcal{B}_{11}(\xi,\eta)=((1-2\langle\xi,\eta\rangle)\mathcal{A}_{11}(i\xi,i\eta)+2\mathcal{A}_{00}(i\xi,i\eta))\mathcal{K}(\xi,\eta),$$

$$\mathcal{B}_{01}(\xi,\eta)=((1-2\langle\xi,\eta\rangle)\mathcal{A}_{01}(i\xi,i\eta)-2(1+|\xi|^2)\mathcal{A}_{10}(i\xi,i\eta))\mathcal{K}(\xi,\eta),$$

$$\mathcal{B}_{10}(\xi,\eta)=((1-2\langle\xi,\eta\rangle)\mathcal{A}_{10}(i\xi,i\eta)-2(1+|\eta|^2)\mathcal{A}_{01}(i\xi,i\eta))\mathcal{K}(\xi,\eta).$$

where

$$\mathcal{K}(\xi,\eta)=(4(|\xi|^2|\eta|^2-\langle\xi,\eta\rangle^2+|\xi|^2+|\eta|^2+\langle\xi,\eta\rangle)+3)^{-1}$$

6.3 The Multiplier Class S(a,b,c,d)

Rather than prove bounds on $\mathcal{B}_{jk}$ directly, we will instead establish estimates for a special class of Fourier multipliers. We define this class as follows

Definition 6.3.1. *Let $a,b,c,d\in\mathbb{Z}$ be given. We define the multiplier class $S(a,b,c,d)$ as follows: Given $m(\xi,\eta)\in C^\infty(\mathbb{R}^{3+3})$, we say $m\in S(a,b,c,d)$ if for every $k_1,k_2\in\mathbb{Z}\geq 0$ we have*

$$\|\mathcal{F}^{-1}(P_{k_1}(\xi)P_{k_2}(\eta)m(\xi,\eta))(x,y)\|_{L^1(\mathbb{R}^{3+3})}\leq C2^{ak_1}2^{bk_2}2^{ck}2^{dk'},\tag{6.3}$$

where $k := \max(k_1, k_2), k' = \min(k_1, k_2)$ *and* C *does not depend on our choice of* k_1 *and* k_2

In order to prove some important results regarding elements of $S(a, b, c, d)$ we will need the following generalized version of Holder's inequality.

Lemma 6.3.1. *([12], section 7.8) Suppose* $P \in L^1(\mathbb{R}^{3+3})$ *and* $\mathcal{P} := \hat{P}$*. Let* $1 \leq p, q, r \leq \infty$ *be such that* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$*, then*

$$\|\mathcal{P}(D', D'')[u][v]\|_{L^r} \lesssim \|P\|_{L^1} \|u\|_{L^p} \|v\|_{L^q},$$

for any $u, v \in \mathcal{S}(\mathbb{R}^3)$

Proof. We will first prove this for two cases: $(p, q, r) = (\infty, \infty, \infty)$ and $(p, q, r) = (p, p', \infty)$ where p' denotes the Holder conjugate of p.

Case 1: $(p, q, r) = (\infty, \infty, \infty)$

We have that

$$|\mathcal{P}(D', D'')[u][v](x)| = \left| \int \int P(y, z) u(x - y) v(x - z) dy dz \right|.$$

We can bound this by

$$\|u\|_{L^\infty} \|v\|_{L^\infty} \int \int |P(y, z)| dy dz, \tag{6.4}$$

giving us the desired bound.

Case 2: $(p, q, r) = (p, p', 1)$

Observe that

$$\left| \int \int P(y, z) u(x - y) v(x - z) dy dz \right| \leq \int \int |P(y, z)| |u(x - y)| |v(x - z)| dy dz.$$

Integrating over x and applying Tonelli's theorem, we see that

$$\|\mathcal{P}(D', D'')[u][v](x)\|_{L^1} \leq \int\int |P(y,z)| \int |u(x-y)||v(x-z)| dxdydz.$$

Applying Holder's inequality, we may bound this by

$$\int\int |P(y,z)| dydz \|u\|_{L^p} \|v\|_{L^{p'}}, \tag{6.5}$$

giving us the desired inequality.

Given $v \in \mathcal{S}(\mathbb{R}^3)$ define $T^v : L^1 + L^\infty \to L^1 + L^\infty$ by $T^v(u) = \mathcal{P}(D', D'')[u][v]$. Then from (6.4) and (6.5) we have

$$\|T^v(u)\|_{L^\infty} \leq \|P\|_{L^1} \|u\|_{L^\infty} \|v\|_{L^\infty}$$

and

$$\|T^v(u)\|_{L^1} \leq \|P\|_{L^1} \|u\|_{L^1} \|v\|_{L^\infty}.$$

Let $1 < p < \infty$, then we can apply the Riesz-Thorin Interpolation theorem and deduce from the bounds above that

$$\|T^v(u)\|_{L^p} \leq \|P\|_{L^1} \|u\|_{L^p} \|v\|_{L^\infty}. \tag{6.6}$$

Given $u \in \mathcal{S}(\mathbb{R}^3)$, define $T^u : L^{p'} + L^\infty \to L^{p'} + L^\infty$ by $T^u(v) = \mathcal{P}(D', D'')[u][v]$. From (6.5) we have the inequality

$$\|T^v(u)\|_{L^1} \leq \|P\|_{L^1} \|u\|_{L^p} \|v\|_{L^{p'}},$$

and from (6.6) we have

$$\|T^u(v)\|_{L^p} \leq \|P\|_{L^1} \|u\|_{L^p} \|v\|_{L^\infty}.$$

Interpolating between these two inequalities we get

$$\|T^u(v)\|_{L^r} \le \|P\|_{L^1}\|u\|_{L^p}\|v\|_{L^q},$$

Where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. □

We apply the preceding Lemma to prove the following set of dyadic estimates.

Lemma 6.3.2. *Let $m \in S(a,b,c,d)$ with $a,b,c,d \in \mathbb{Z}$ and let $\mathcal{M}$ be the operator whose Fourier multiplier is m. Then for all $1 \le p,q,r \le \infty$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ and $k_1,k_2 \in \mathbb{Z}_{\ge 0}$ we have the uniform family of bounds*

$$\|\mathcal{M}(D',D'')[P_{k_1}v][P_{k_2}w]\|_{L^r} \lesssim (2^{ck}2^{dk'})2^{k_1 a}\|P_{k_1}v\|_{L^p}2^{k_2 b}\|P_{k_2}w\|_{L^q}, \tag{6.7}$$

where $k := \max(k_1,k_2), k' = \min(k_1,k_2)$.

Proof. Let $v_{k_1} = P_{k_1}v$, $w_{k_2} = P_{k_2}w$ and fix $x \in \mathbb{R}^3$. Then

$$M(D',D'')[P_{k_1}v][P_{k_2}w](x) = \frac{1}{(2\pi)^6}\int\int e^{ix(\xi+\eta)}m(\xi,\eta)\hat{v}_{k_1}(\xi)\hat{w}_{k_2}(\eta)d\xi d\eta.$$

Define $m_{k_1,k_2}(\xi,\eta) = \tilde{P}_{k_1}(\xi)\tilde{P}_{k_2}(\eta)m(\xi,\eta)$ and $\mathcal{M}_{k_1,k_2} = \mathcal{F}^{-1}(m_{k_1,k_2})$, then we can replace the above expression with

$$\begin{aligned} M(D',D'')[P_{k_1}v][P_{k_2}w](x) &= \frac{1}{(2\pi)^6}\int\int e^{ix(\xi+\eta)}m_{k_1,k_2}(\xi,\eta)\hat{v}_{k_1}(\xi)\hat{w}_{k_2}(\eta)d\xi d\eta \\ &= \mathcal{M}_{k_1,k_2}(D',D'')[P_{k_1}v][P_{k_2}w](x). \end{aligned}$$

From Lemma 6.3.1 we see that (6.7) follows from proving

$$\|\mathcal{M}_{k_1,k_2}(D',D'')\|_{L^1(\mathbb{R}^{3+3})} \le C2^{ak_1}2^{bk_2}2^{ck}2^{dk'},$$

where C does not depend on our choice of k_1 and k_2. But this is an immediate consequence of our definition of $S(a,b,c,d)$.

$\square$

We close this section with an important result that will be used repeatedly to prove our main estimates in Chapter 7. We remark that this result is a generalized version of Theorem 2.2.1.

Proposition 6.3.1. *Assume* $\mathcal{M} \in S(a,b,c,d)$ *and* $1 \le p, p_1, \tilde{p}_i, q, q_i, \tilde{q}_i \le \infty$ *are such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{\tilde{p}_1} + \frac{1}{\tilde{p}_2}, \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{\tilde{q}_1} + \frac{1}{\tilde{q}_2}$. *Furthermore, assume* $r, \lambda, \tilde{\lambda} > 0, \sigma, \tilde{\sigma} \ge 0$, *then*

$$\|\mathcal{M}(D',D'')[v][w]\|_{L_t^p W_x^{r,q}[\boldsymbol{k}]} \lesssim \|v\|_{L_t^{p_1} W_x^{r+\sigma+(a+c),q_1}[\boldsymbol{k}]} \|w\|_{L_t^{p_2} W_x^{\lambda-\sigma+(b+d),q_2}[\boldsymbol{k}]} + \|v\|_{L_t^{\tilde{p}_1} W_x^{\tilde{\lambda}-\tilde{\sigma}+(a+d),\tilde{q}_1}[\boldsymbol{k}]} \|w\|_{L_t^{\tilde{p}_2} W_x^{r+\tilde{\sigma}+(b+c),\tilde{q}_2}[\boldsymbol{k}]}.$$

Proof. Let $C \ge 10$ be a fixed constant.

$$\begin{aligned}
\|\mathcal{M}(D',D'')[v][w]\|_{L_t^p W_x^{r,q}[\mathbf{k}]} &= \Big(\sum_{k\ge 0} 2^{2kr} \|P_k \mathcal{M}(D',D'')[v][w]\|^2_{L_t^p L_x^q}\Big)^{\frac{1}{2}} \\
&\le \Big(\sum_{k\ge 0} 2^{2kr} (\sum_{k_1\ge 0}\sum_{k_2\ge 0} \|P_k \mathcal{M}(D',D'')[P_{k_1}v][P_{k_2}w]\|_{L_t^p L_x^q})^2\Big)^{\frac{1}{2}} \\
&\lesssim \Big(\sum_{k\ge 0} 2^{2kr} (\sum_{|k_1-k|\le C}\sum_{k_2\le k+C} \|\mathcal{M}(D',D'')[P_{k_1}v][P_{k_2}w]\|_{L_t^p L_x^q})^2\Big)^{\frac{1}{2}} \\
&+ \Big(\sum_{k\ge 0} 2^{2kr} (\sum_{k_1\ge k-C}\sum_{|k_2-k_1|\le C} \|\mathcal{M}(D',D'')[P_{k_1}v][P_{k_2}w]\|_{L_t^p L_x^q})^2\Big)^{\frac{1}{2}} \\
&+ \Big(\sum_{k\ge 0} 2^{2kr} (\sum_{k_1\le k+C}\sum_{|k_2-k|+C} \|\mathcal{M}(D',D'')[P_{k_1}v][P_{k_2}w]\|_{L_t^p L_x^q})^2\Big)^{\frac{1}{2}} \\
&= I + II + III.
\end{aligned}$$

We first bound (I). By Lemma 6.3.2

$$
\begin{aligned}
(I) &\lesssim \Big(\sum_{k\geq 0} 2^{2kr} (\sum_{|k_1-k|\leq C} \sum_{k_2 \leq k+C} 2^{k_1(a+c)} \|P_{k_1} v\|_{L_t^{p_1} L_x^{q_1}} 2^{k_2(b+d)} \|P_{k_2} w\|_{L_t^{p_2} L_x^{q_2}})^2 \Big)^{\frac{1}{2}} \\
&\lesssim \Big(\sum_{k\geq 0} 2^{2kr} (\sum_{|k_1-k|\leq C} \sum_{k_2 \leq k+C} 2^{k_1(a+c+\sigma)} \|P_{k_1} v\|_{L_t^{p_1} L_x^{q_1}} 2^{k_2(b+d-\sigma)} \|P_{k_2} w\|_{L_t^{p_2} L_x^{q_2}})^2 \Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{k\geq 0} 2^{2kr} (\sum_{|k_1-k|\leq C} 2^{k_1(a+c+\sigma)} \|P_{k_1} v\|_{L_t^{p_1} L_x^{q_1}})^2 \Big)^{\frac{1}{2}} \Big(\sum_{k_2\geq 0} 2^{k_2(b+d-\sigma)} \|P_{k_2} w\|_{L_t^{p_2} L_x^{q_2}} \Big).
\end{aligned}
$$

By Young's inequality in k and Cauchy-Schwartz in k_2 we can bound the above by

$$
\begin{aligned}
&\leq \Big(\sum_{k\geq 0} 2^{2k(r+a+c+\sigma)} \|P_k v\|^2_{L_t^{p_1} L_x^{q_1}} \Big)^{\frac{1}{2}} \Big(\sum_{k_2\geq 0} 2^{2k_2(\lambda-\sigma+b+d)} \|P_{k_2} w\|^2_{L_t^{p_2} L_x^{q_2}} \Big)^{\frac{1}{2}} \Big(\sum_{k_2\geq 0} 2^{-2k_2\lambda} \Big)^{\frac{1}{2}} \\
&\lesssim \|v\|_{L_t^{p_1} W_x^{r+a+c+\sigma, q_1}[\mathbf{k}]} \|w\|_{L_t^{p_2} W_x^{\lambda-\sigma+b+d, q_2}[\mathbf{k}]}.
\end{aligned}
$$

We now turn our attention to (II). Once again, by Lemma 6.3.2 and Young's inequality in k_1, we see that

$$
\begin{aligned}
(II) &\lesssim \Big(\sum_{k\geq 0} 2^{2kr} \Big(\sum_{k_1\geq k-C} \sum_{|k_2-k_1|\leq C} 2^{k_1(a+c)} \|P_{k_1} v\|_{L_t^{p_1} L_x^{q_1}} 2^{k_2(b+d)} \|P_{k_2} w\|_{L_t^{p_2} L_x^{q_2}} \Big)^2 \Big)^{\frac{1}{2}} \\
&\lesssim \Big(\sum_{k\geq 0} 2^{2kr} \Big(\sum_{k_1\geq k-C} \sum_{|k_2-k_1|\leq C} 2^{k_1(a+c+\sigma)} \|P_{k_1} v\|_{L_t^{p_1} L_x^{q_1}} 2^{k_2(b+d-\sigma)} \|P_{k_2} w\|_{L_t^{p_2} L_x^{q_2}} \Big)^2 \Big)^{\frac{1}{2}} \\
&\lesssim \Big(\sum_{k\geq 0} 2^{2kr} \Big(\sum_{k_1\geq k-C} 2^{k_1(a+c+\sigma)} \|P_{k_1} v\|_{L_t^{p_1} L_x^{q_1}} \Big)^2 \Big(\sum_{k_2\geq k-C} 2^{k_2(b+d-\sigma)} \|P_{k_2} w\|_{L_t^{p_2} L_x^{q_2}} \Big)^2 \Big)^{\frac{1}{2}}.
\end{aligned}
$$

Applying the Cauchy-Schwartz inequality in both k_1 and k_2 independently, we conclude that the above is

$$
\begin{aligned}
&\lesssim \Big(\sum_{k\geq 0} 2^{2kr} \Big(\sum_{k_1\geq k-C} 2^{-2k_1 r} \Big) \|v\|^2_{L_t^{p_1} W_x^{r+\sigma a+c, q_1}[\mathbf{k}]} \Big(\sum_{k_2\geq k-C} 2^{-2k\lambda} \Big) \|w\|^2_{L_t^{p_2} W_x^{\lambda-\sigma+b+d, q_2}[\mathbf{k}]} \Big)^{\frac{1}{2}} \\
&\lesssim \Big(\sum_{k\geq 0} 2^{2k(r-r-\lambda)} \Big)^{\frac{1}{2}} \|v\|_{L_t^{p_1} W_x^{r+a+c+\sigma, q_1}[\mathbf{k}]} \|w\|_{L_t^{p_2} W_x^{\lambda-\sigma+b+d, q_2}[\mathbf{k}]} \\
&\lesssim \|v\|_{L_t^{p_1} W_x^{r+a+c, q_1}[\mathbf{k}]} \|w\|_{L_t^{p_2} W_x^{\lambda+b+d, q_2}[\mathbf{k}]}.
\end{aligned}
$$

By interchanging the roles of v and w in the proof of estimate (I), we can conclude that

$$(III) \lesssim \|v\|_{L_t^{\tilde{p}_1} W_x^{\tilde{\lambda}-\tilde{\sigma}+a+d,\tilde{q}_1}[\mathbf{k}]} \|w\|_{L_t^{\tilde{p}_2} W_x^{r+b+c+\tilde{\sigma},\tilde{q}_2}[\mathbf{k}]}.$$

□

6.4 Dyadic Kernel Bounds

Recall that

$$\begin{aligned}
\mathcal{B}_{00}(\xi,\eta) =& ((1-2\langle\xi,\eta\rangle)\mathcal{A}_{00}(i\xi,i\eta) + 2(1+|\xi|^2)(1+|\eta|^2)\mathcal{A}_{11}(i\xi,i\eta))\mathcal{K}(\xi,\eta) \\
\mathcal{B}_{11}(\xi,\eta) =& ((1-2\langle\xi,\eta\rangle)\mathcal{A}_{11}(i\xi,i\eta) + 2\mathcal{A}_{00}(i\xi,i\eta))\mathcal{K}(\xi,\eta) \\
\mathcal{B}_{01}(\xi,\eta) =& ((1-2\langle\xi,\eta\rangle)\mathcal{A}_{01}(i\xi,i\eta) - 2(1+|\xi|^2)\mathcal{A}_{10}(i\xi,i\eta))\mathcal{K}(\xi,\eta) \\
\mathcal{B}_{10}(\xi,\eta) =& ((1-2\langle\xi,\eta\rangle)\mathcal{A}_{10}(i\xi,i\eta) - 2(1+|\eta|^2)\mathcal{A}_{01}(i\xi,i\eta))\mathcal{K}(\xi,\eta),
\end{aligned}$$

where $\mathcal{A}_{ij}(\xi,\eta)$ is a polynomial of degree at most $1-i$ in ξ and $1-j$ in η and

$$\mathcal{K}(\xi,\eta) = (4(|\xi|^2|\eta|^2 - \langle\xi,\eta\rangle^2 + |\xi|^2 + |\eta|^2 + \langle\xi,\eta\rangle) + 3)^{-1} := \frac{1}{H(\xi,\eta)}.$$

Our main goal in this section is to prove the following result.

Proposition 6.4.1. *$\mathcal{A}_{jk} \in S(1-j,1-k,0,0)$ and $\mathcal{B}_{jk} \in S(2-j,2-k,-2,6)$*

It is not difficult to see that if $f \in S(a_1,b_1,c_1,d_1)$ and $g \in S(a_2,b_2,c_2,d_2)$, then $fg \in S(a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2)$. It therefore suffices to show that $\mathcal{K} \in S(0,0,-2,6)$, and that if $P_{ij}(\xi,\eta)$ is a polynomial of degree i in ξ and j in η, then we have $P_{ij} \in S(i,j,0,0)$.

In order to streamline our argument, we aim to find a condition that guarantees that an operator $\mathcal{M}$ with symbol $m(\xi,\eta)$ is in $S(N_1,N_2,N_3,N_4)$. Let

$$\mathcal{M}_{k_1,k_2}(D',D'')(x,y) = (\frac{1}{2\pi})^6 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i(x\cdot\xi+y\cdot\eta)} m(\xi,\eta) P_{k_1}(\xi) P_{k_2}(\eta) d\xi d\eta.$$

Then, by definition of $S(\cdot,\cdot,\cdot,\cdot)$ it suffices to find a condition that guarantees the estimate

$$\|\mathcal{M}_{k_1,k_2}(D',D'')(x,y)\|_{L^1(\mathbb{R}^{3+3})} \leq C2^{N_1k_1}2^{N_2k_2}2^{N_3k}2^{N_4k'}, \tag{6.8}$$

where $k=\max\{k_1,k_2\}, k'=\min\{k_1,k_2\}$ and C does not depend on our choice of k_1,k_2.

We present the following result.

Proposition 6.4.2. *Let $k_1,k_2 \in \mathbb{Z}_{\geq 0}$ and $l,p \in \{1,2,3\}$. Define*

$$m^{(p,l)}_{(a,b)}(\xi,\eta) := \partial^b_{\eta_l}\partial^a_{\xi_p} m(\xi,\eta).$$

If

$$\|\tilde{P}_{k_1}(\xi)\tilde{P}_{k_2}(\eta)m^{(p,l)}_{(a,b)}(\xi,\eta)\|_{L^1(\mathbb{R}^{3+3})} \lesssim 2^{(N_1-a)k_1}2^{(N_2-b)k_2}2^{N_3k}2^{N_4k'}2^{3k_1}2^{3k_2}, \tag{6.9}$$

for all $a,b \in \{0,1,2,3,4\}$ and $p,l \in \{1,2,3\}$, then estimate (6.8) holds.

Proof. We will decompose $\mathbb{R}^{3+3}$ as follows: define the sets

$$E_1 := \{(x,y) \in \mathbb{R}^{3+3} : |x| \leq 2^{-k_1}, |y| \leq 2^{-k_2}\},$$

$$E_2 := \{(x,y) \in \mathbb{R}^{3+3} : |x| \geq 2^{-k_1}\}, \quad E_3 := \{(x,y) \in \mathbb{R}^{3+3} : |y| \geq 2^{-k_2}\}.$$

By symmetry, we only need to show

$$\|\mathcal{M}_{k_1,k_2}(D',D'')(x,y)\|_{L^1(E_1)} \lesssim 2^{N_1k_1}2^{N_2k_2}2^{N_3k}2^{N_4k'},$$

$$\|\mathcal{M}_{k_1,k_2}(D',D'')(x,y)\|_{L^1(E_2)} \lesssim 2^{N_1k_1}2^{N_2k_2}2^{N_3k}2^{N_4k'}$$

Observe that

$$\|\mathcal{M}_{k_1,k_2}(x,y)\|_{L^1(E_1)} \leq |E_1|\|\mathcal{M}_{k_1,k_2}(x,y)\|_{L^\infty}.$$

From the definition of E_1, we see that

$$|E_1| \lesssim 2^{-3k_1}2^{-3k_2}$$

Furthermore,

$$\begin{aligned}\|\mathcal{M}_{k_1,k_2}(D',D'')(x,y)\|_{L^\infty} &\lesssim \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |m(\xi,\eta)P_{k_1}(\xi)P_{k_2}(\eta)|d\xi d\eta \\ &\lesssim \|\tilde{P}_{k_1}(\xi)\tilde{P}_{k_2}(\eta)m_{(0,0)}(\xi,\eta)\|_{L^1(\mathbb{R}^{3+3})} \\ &\lesssim 2^{N_1k_1}2^{N_2k_2}2^{N_3k}2^{N_4k'}2^{3k_1}2^{3k_2},\end{aligned}$$

where the last inequality comes from (6.9).

Our next task is to bound $\|\mathcal{M}_{k_1,k_2}(D',D'')(x,y)\|_{L^1(E_2)}$. Observe that

$$\begin{aligned}&\int_{E_2} |\mathcal{M}_{k_1,k_2}(D',D'')(x,y)|dxdy \\ &\qquad = \int_{E_2}\left|\frac{(1+2^{4k_2}|y|^4)}{(1+2^{4k_2}|y|^4)}\frac{(1+2^{4k_1}|x|^4)}{(1+2^{4k_1}|x|^4)}\mathcal{M}_{k_1,k_2}(D',D'')(x,y)\right|dxdy.\end{aligned}$$

Applying the fact that $2^{k_1}|x| \geq 1$ on E_2 and Holder's inequality, we can conclude the above expression is bounded by

$$\begin{aligned}&\lesssim \|(1+2^{4k_2}|y|^4)(2^{4k_1}|x|^4)\mathcal{M}_{k_1,k_2}(D',D'')(x,y)\|_{L^\infty(E_2)} \\ &\quad\times \int_{\mathbb{R}^{3+3}} \frac{1}{(1+2^{4k_2}|y|^4)}\frac{1}{(1+2^{4k_1}|x|^4)}dxdy \\ &\lesssim 2^{-3k_1}2^{-3k_2}\|(1+2^{4k_2}|y|^4)(2^{4k_1}|x|^4)\mathcal{M}_{k_1,k_2}(D',D'')(x,y)\|_{L^\infty(E_2)},\end{aligned}$$

so it remains to show that

$$\|(1+2^{4k_2}|y|^4)(2^{4k_1}|x|^4)\mathcal{M}_{k_1,k_2}(D',D'')(x,y)\|_{L^\infty(E_2)} \lesssim 2^{N_1k_1}2^{N_2k_2}2^{N_3k}2^{N_4k'}2^{3k_1}2^{3k_2}. \tag{6.10}$$

We can bound the LHS of the above expression by

$$\begin{aligned}&\lesssim \|(1+2^{4k_2}\sum_{l=1}^{3} y_l^4)(2^{4k_1}\sum_{p=1}^{3} x_p^4)\mathcal{M}_{k_1,k_2}(D',D'')(x,y)\|_{L^\infty(E_2)} \\ &\lesssim \sum_{p=1}^{3}\|2^{4k_1}x_p^4\mathcal{M}_{k_1,k_2}(D',D'')(x,y)\|_{L^\infty} + \sum_{p,l=1}^{3}\|2^{4k_2}2^{4k_1}y_l^4x_p^4\mathcal{M}_{k_1,k_2}(D',D'')(x,y)\|_{L^\infty},\end{aligned}$$

so we are left with showing

$$\|2^{4k_1}x_p^4\mathcal{M}_{k_1,k_2}(D',D'')(x,y)\|_{L^\infty} \lesssim 2^{N_1k_1}2^{N_2k_2}2^{N_3k}2^{N_4k'}2^{3k_1}2^{3k_2}, \tag{6.11}$$

$$\|2^{4k_2}2^{4k_1}y_l^4x_p^4\mathcal{M}_{k_1,k_2}(D',D'')(x,y)\|_{L^\infty} \lesssim 2^{N_1k_1}2^{N_2k_2}2^{N_3k}2^{N_4k'}2^{3k_1}2^{3k_2}, \tag{6.12}$$

for all $l,p \in \{1,2,3\}$.

We will mainly focus on proving estimate (6.12) as the proof of (6.11) is similar. By applying basic properties of the Fourier transform, we can conclude tht

$$\begin{aligned}
&2^{4k_2}y_l2^{4k_1}x_p^4\mathcal{M}_{k_1,k_2}(D',D'')(x,y)\\
&\qquad = C2^{4k_1}2^{4k_2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i(x\cdot\xi+y\cdot\eta)}\partial_{\eta_l}^4\partial_{\xi_p}^4[m(\xi,\eta)P_{k_1}(\xi)P_{k_2}(\eta)]d\xi d\eta,
\end{aligned}$$

so,

$$\begin{aligned}
&\|2^{4k_2}y_l2^{4k_1}x_p^4\mathcal{M}_{k_1,k_2}(D',D'')(x,y)\|_{L^\infty}\\
&\qquad \lesssim 2^{4k_1}2^{4k_2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\tilde{P}_{k_1}(\xi)\tilde{P}_{k_2}(\eta)\left|\partial_{\eta_l}^4\partial_{\xi_p}^4\Big(m(\xi,\eta)P_{k_1}(\xi)P_{k_2}(\eta)\Big)\right|d\xi d\eta. \qquad (6.13)
\end{aligned}$$

We observe that

$$\begin{aligned}
\left|\partial_{\eta_l}^4\partial_{\xi_p}^4[m(\xi,\eta)P_{k_1}(\xi)P_{k_2}(\eta)]\right| &\lesssim \sum_{a,b=0}^{4}\left|m_{(a,b)}^{(p,l)}(\xi,\eta)\partial_{\xi_p}^{4-a}P_{k_1}(\xi)\partial_{\eta_l}^{4-b}P_{k_2}(\eta)\right|,\\
&\leq \sum_{a,b=0}^{4}\left|m_{(a,b)}^{(p,l)}(\xi,\eta)\right|\|\partial_{\xi_p}^{4-a}P_{k_1}(\xi)\|_{L^\infty}\|\partial_{\eta_l}^{4-b}P_{k_2}(\eta)\|_{L^\infty}. \qquad (6.14)
\end{aligned}$$

As $P_{k_1}(\xi) = P_0(\frac{\xi}{2^{k_1}})$, we can conclude that

$$\begin{aligned}
\|\partial_{\xi_p}^M P_{k_1}(\xi)\|_{L^\infty} &\leq 2^{-Mk_1}\|\partial_{\xi_p}^M P_0(\xi)\|_{L^\infty}\\
&\lesssim 2^{-Mk_1}.
\end{aligned}$$

Similarly, we have

$$\|\partial_{\eta_l}^M P_{k_2}(\eta)\|_{L^\infty} \lesssim 2^{-Mk_2}.$$

Therefore, we can bound the RHS of equation (6.14) by

$$\lesssim \sum_{a,b=0}^{4} 2^{(a-4)k_1} 2^{(b-4)k_2} \left| m^{(p,l)}_{(a,b)}(\xi,\eta) \right|.$$

This allows us to bound the RHS of (6.13) by

$$\lesssim \sum_{a,b=0}^{4} 2^{ak_1} 2^{bk_2} \| \tilde{P}_{k_1}(\xi) \tilde{P}_{k_2}(\eta) m^{(p,l)}_{(a,b)}(\xi,\eta) \|_{L^1(\mathbb{R}^{3+3})} \lesssim 2^{N_1 k_1} 2^{N_2 k_2} 2^{N_3 k} 2^{N_4 k'} 2^{3k_1} 2^{3k_2},$$

where the last inequality comes from (6.9). This concludes the proof of (6.12). □

Corollary 6.4.1. *If $P_{ij}(\xi,\eta)$ is a polynomial of degree $\leq i$ in ξ and $\leq j$ in η for $i,j \in \mathbb{Z}_{\geq 0}$ then $P_{ij} \in S(i,j,0,0)$.*

Proof. Fix $i,j \in \mathbb{Z}_{\geq 0}$, then for all $a,b \in \mathbb{Z}_{\geq 0}$ it is easy to see that $(P_{ij})^{(p,l)}_{(a,b)}$ is a polynomial of degree $\leq i-a$ in ξ and $\leq j-b$ in η, so we have

$$\begin{aligned} \| \tilde{P}_{k_1}(\xi) \tilde{P}_{k_2}(\eta) (P_{ij})^{(p,l)}_{(a,b)}(\xi,\eta) \|_{L^1(\mathbb{R}^{3+3})} &\lesssim 2^{(i-a)k_1} 2^{(j-b)k_2} \| \tilde{P}_{k_1}(\xi) \tilde{P}_{k_2}(\eta) \|_{L^1(\mathbb{R}^{3+3})} \\ &\lesssim 2^{(i-a)k_1} 2^{(j-b)k_2} 2^{3k_1} 2^{3k_2}. \end{aligned}$$

By Proposition 6.4.2 it follows that $P_{ij} \in S(i,j,0,0)$. □

We now focus our attention on proving $\mathcal{K}(\xi,\eta) \in S(0,0,-2,6)$. By Proposition 6.4.2, this will follow from proving

$$\| \tilde{P}_{k_1}(\xi) \tilde{P}_{k_2}(\eta) \mathcal{K}^{(p,l)}_{(a,b)}(\xi,\eta) \|_{L^1(\mathbb{R}^{3+3})} \lesssim 2^{-ak_1} 2^{-bk_2} 2^{-2k} 2^{6k'} 2^{3k_1} 2^{3k_2}, \tag{6.15}$$

for all $a,b \in \{0,1,2,3,4\}, p,l \in \{1,2,3\}$, where we recall that $k = \max\{k_1,k_2\}, k' = \min\{k_1,k_2\}$ and $\mathcal{K}^{(p,l)}_{(a,b)}(\xi,\eta) = \partial^b_{\eta_l} \partial^a_{\xi_p} \mathcal{K}(\xi,\eta)$ for some fixed $l,p \in \{1,2,3\}$.

Because $\mathcal{K}(\xi,\eta)$ is symmetric in ξ and η we can assume without loss of generality that $k_1 \geq k_2$ so that $k = k_1$ and $k' = k_2$. For convenience we fix $p,l \in \{1,2,3\}$ and write

$\mathcal{K}^{(p,l)}_{(a,b)} = \mathcal{K}_{(a,b)}$ and $H^{(p,l)}_{(a,b)} = H_{(a,b)}$ in which case we can rewrite (6.15) as

$$\|\tilde{P}_{k_1}(\xi)\tilde{P}_{k_2}(\eta)\mathcal{K}_{(a,b)}(\xi,\eta)\|_{L^1(\mathbb{R}^{3+3})} \lesssim 2^{-(2+a)k}2^{(6-b)k'}2^{3k}2^{3k'}. \tag{6.16}$$

Fix $\eta \in \mathbb{R}^3$ with $|\eta| \sim 2^{k'}$. Then for any $\xi \in \mathbb{R}^3$ with $|\xi| \sim 2^k$ we denote $\theta_\eta(\xi) = \theta$ to be the angle between η and ξ. We observe that (6.16) follows from showing

$$\int_{|\eta|\sim 2^{k'}}\int_{(A_+)_\eta} |\mathcal{K}_{(a,b)}(\xi,\eta)| d\xi d\eta \lesssim 2^{-(2+a)k}2^{(6-b)k'}2^{3k_1}2^{3k_2}, \tag{6.17}$$

$$\int_{|\eta|\sim 2^{k'}}\int_{(A_-)_\eta} |\mathcal{K}_{(a,b)}(\xi,\eta)| d\xi d\eta \lesssim 2^{-(2+a)k}2^{(6-b)k'}2^{3k_1}2^{3k_2}, \tag{6.18}$$

$$\int_{|\eta|\sim 2^{k'}}\int_{(B_+)_\eta} |\mathcal{K}_{(a,b)}(\xi,\eta)| d\xi d\eta \lesssim 2^{-(2+a)k}2^{(6-b)k'}2^{3k_1}2^{3k_2}, \tag{6.19}$$

$$\int_{|\eta|\sim 2^{k'}}\int_{(B_-)_\eta} |\mathcal{K}_{(a,b)}(\xi,\eta)| d\xi d\eta \lesssim 2^{-(2+a)k}2^{(6-b)k'}2^{3k_1}2^{3k_2}, \tag{6.20}$$

where

$$(A_+)_\eta = \{\xi \in \mathbb{R}^3 | |\theta_\eta(\xi)| \leq 2^{-k'}\} \cap \{|\xi| \sim 2^k\},$$

$$(A_-)_\eta = \{\xi \in \mathbb{R}^3 | |\pi - \theta_\eta(\xi)| \leq 2^{-k'}\} \cap \{|\xi| \sim 2^k\},$$

$$(B_+)_\eta = \{\xi \in \mathbb{R}^3 | 2^{-k'} \leq |\theta_\eta(\xi)| \leq \frac{\pi}{2}\} \cap \{|\xi| \sim 2^k\},$$

$$(B_-)_\eta = \{\xi \in \mathbb{R}^3 | 2^{-k'} \leq |\pi - \theta_\eta(\xi)| \leq \frac{\pi}{2}\} \cap \{|\xi| \sim 2^k\}.$$

We will only focus on proving estimates (6.17) and (6.19) as (6.18) and (6.20) can be proved using a nearly identical argument. Furthermore, we can replace A_+ and B_+ with $A_+ \cap \{\theta \geq 0\}$ and $B_+ \cap \{\theta \geq 0\}$ respectively. We will first need two results:

Given $a, b \in \mathbb{Z}_{\geq 0}$ with $a + b \geq 1$, define $D(a,b)$ to be the set consisting of all $(a+b)$-tuples whose entries are $0, 1$, or 2 and recall that

$$H(\xi,\eta) := \frac{1}{\mathcal{K}(\xi,\eta)} = (4(|\xi|^2|\eta|^2 - \langle \xi,\eta\rangle^2 + |\xi|^2 + |\eta|^2 + \langle \xi,\eta\rangle) + 3). \tag{6.21}$$

The following result holds true:

Lemma 6.4.1. *Let $a, b \in \mathbb{Z}_{\geq 0}$ with $a + b \geq 1$ and $D(a, b)$ be defined as above, then we have*

$$\mathcal{K}_{(a,b)} = \frac{1}{H^{a+b+1}} \sum_{\substack{(\alpha,\beta)\in D(a,b)^2 \\ |\alpha|=a \\ |\beta|=b}} C(\alpha, \beta) \prod_{i=1}^{a+b} H_{(\alpha_i,\beta_i)} \tag{6.22}$$

where $C(\alpha, \beta) \in \mathbb{R}$.

Proof. We split the proof up into 3 cases.

Case 1: $a \geq 1, b = 0$.

We proceed by induction. The base case is $a = 1, b = 0$. In the case we have

$$\mathcal{K}_{(1,0)} = -\frac{H_{(1,0)}}{H^2},$$

and so estimate (6.22) holds.

Next, assume $\mathcal{K}_{(a',0)}$ satisfies estimate (6.22) for some $a' \geq 1$. We aim to show that $\mathcal{K}_{(a'+1,0)}$ satisfies estimate (6.22). In particular, we would like to show

$$\mathcal{K}_{(a'+1,0)} = \frac{1}{H^{a'+2}} \sum_{\substack{\alpha\in D(a'+1,0) \\ |\alpha|=a'+1}} C(\alpha) \prod_{i=1}^{a'+1} H_{(\alpha_i,0)}. \tag{6.23}$$

We observe that

$$\begin{aligned}
\mathcal{K}_{(a'+1,0)} &= \partial_{\xi_p} \mathcal{K}_{(a',0)} \\
&= \left(\partial_{\xi_p} \frac{1}{H^{a'+1}}\right) \sum_{\substack{\alpha\in D(a',0) \\ |\alpha|=a'}} C(\alpha) \prod_{i=1}^{a'} H_{(\alpha_i,0)} \\
&\quad + \frac{1}{H^{a'+1}} \partial_{\xi_p} \left(\sum_{\substack{\alpha\in D(a',0) \\ |\alpha|=a'}} C(\alpha) \prod_{i=1}^{a'} H_{(\alpha_i,0)} \right) \\
&= I + II.
\end{aligned}$$

Observe that $\partial_{\xi_p}\left(\frac{1}{H^{a'+1}}\right) = -\frac{H_{(1,0)}}{H^{a'+2}}$, therefore

$$\begin{aligned}(I) &= \frac{1}{H^{a'+2}} \sum_{\substack{\alpha \in D(a',0) \\ |\alpha| = a'}} C(\alpha)(-H_{(1,0)}) \prod_{i=1}^{a'} H_{(\alpha_i,0)} \\ &= \frac{1}{H^{a'+2}} \sum_{\substack{\tilde{\alpha} \in D(a'+1,0) \\ |\tilde{\alpha}| = a'+1}} \tilde{C}(\tilde{\alpha}) \prod_{i=1}^{a'+1} H_{(\tilde{\alpha}_i,0)},\end{aligned}$$

where $\tilde{C}(\tilde{\alpha}) = 0$, if $\tilde{\alpha}_{a'+1} \neq 1$.

We now focus our attention on (II). We rewrite (II) as

$$\begin{aligned}(II) &= \frac{1}{H^{a'+1}} \sum_{\substack{\alpha \in D(a',0) \\ |\alpha| = a'}} C(\alpha)\partial_{\xi_p}\left(\prod_{i=1}^{a'} H_{(\alpha_i,0)}\right) \\ &= \frac{1}{H^{a'+2}} \sum_{\substack{\alpha \in D(a',0) \\ |\alpha| = a'}} C(\alpha)H_{(0,0)}\partial_{\xi_p}\left(\prod_{i=1}^{a'} H_{(\alpha_i,0)}\right).\end{aligned}$$

Observe that

$$\partial_{\xi_p}\left(\prod_{i=1}^{a'} H_{(\alpha_i,0)}\right) = \sum_{\substack{\alpha' \in D(a',0) \\ |\alpha'| = a'+1}} C(\alpha,\alpha') \prod_{i=1}^{a'} H_{(\alpha'_i,0)},$$

where many of the $C(\alpha,\alpha')$ are 0. We can therefore conclude that

$$\begin{aligned}(II) &= \frac{1}{H^{a'+2}} \sum_{\substack{\alpha \in D(a',0) \\ |\alpha| = a'}} C(\alpha)H_{(0,0)}\left(\sum_{\substack{\alpha' \in D(a',0) \\ |\alpha'| = a'+1}} C(\alpha,\alpha') \prod_{i=1}^{a'} H_{(\alpha'_i,0)}\right) \\ &= \frac{1}{H^{a'+2}} \sum_{\substack{\tilde{\alpha} \in D(a'+1,0) \\ |\tilde{\alpha}| = a'+1}} \tilde{C}(\tilde{\alpha}) \prod_{i=1}^{a'+1} H_{(\tilde{\alpha}_i,0)},\end{aligned}$$

where $\tilde{C}(\tilde{\alpha}) = 0$ if $\tilde{\alpha}_{a'+1} \neq 0$.

Combining (I) and (II) gives us equation (6.23), completing the proof for case 1.

Case 2: $a \geq 1, b \geq 0$

We will fix $a \geq 1$ and induct on the value of b. The base case $b = 0$ follows from case 1. Suppose $\mathcal{K}_{(a,b')}$ satisfies estimate (6.22) for some $b' \geq 0$. We would like to prove that $\mathcal{K}_{(a,b'+1)}$ satisfies estimate (6.22). Our goal then is to show that

$$\mathcal{K}_{(a,b'+1)} = \frac{1}{H^{a+b'+2}} \sum_{\substack{(\alpha,\beta)\in D(a,b'+1)^2 \\ |\alpha|=a \\ |\beta|=b'+1}} C(\alpha,\beta) \prod_{i=1}^{a+b'+1} H_{(\alpha_i,\beta_i)}, \tag{6.24}$$

where $C(\alpha, \beta) \in \mathbb{R}$

We observe that

$$\begin{aligned}
\mathcal{K}_{(a,b'+1)} &= \partial_{\eta_l} \mathcal{K}_{(a,b')} \\
&= \left(\partial_{\eta_l} \frac{1}{H^{a+b'+1}}\right) \sum_{\substack{\alpha,\beta\in D(a,b') \\ |\alpha|=a \\ |\beta|=b'}} C(\alpha,\beta) \prod_{i=1}^{a+b'} H_{(\alpha_i,\beta_i)} \\
&+ \frac{1}{H^{a+b'+1}} \partial_{\eta_l} \left(\sum_{\substack{\alpha,\beta\in D(a,b') \\ |\alpha|=a \\ |\beta|=b'}} C(\alpha,\beta) \prod_{i=1}^{a+b'} H_{(\alpha_i,\beta_i)} \right) \\
&= I + II.
\end{aligned}$$

Notice that $\partial_{\xi_p} \left(\frac{1}{H^{a+b'+1}}\right) = -\frac{H_{(0,1)}}{H^{a+b'+2}}$, therefore

$$\begin{aligned}
(I) &= \frac{1}{H^{a+b'+2}} \sum_{\substack{\alpha,\beta\in D(a,b') \\ |\alpha|=a \\ |\beta|=b'}} C(\alpha,\beta)(-H_{(0,1)}) \prod_{i=1}^{a+b'} H_{(\alpha_i,\beta_i)} \\
&= \frac{1}{H^{a+b'+2}} \sum_{\substack{\alpha,\tilde{\beta},\in D(a,b'+1) \\ |\alpha|=a \\ |\tilde{\beta}|=b'+1}} \tilde{C}(\alpha,\tilde{\beta}) \prod_{i=1}^{a+b'+1} H_{(\alpha_i,\tilde{\beta}_i)},
\end{aligned}$$

where $\tilde{C}(\alpha, \tilde{\beta}) = 0$ if $\tilde{\beta}_{b'+1} \neq 1$.

We now focus our attention on (II). We rewrite (II) as

$$(II) = \frac{1}{H^{a+b'+1}} \sum_{\substack{\alpha,\beta \in D(a,b') \\ |\alpha|=a \\ |\beta|=b'}} C(\alpha,\beta)\partial_{\eta_l}\left(\prod_{i=1}^{a+b'} H_{(\alpha_i,\beta_i)}\right)$$

$$= \frac{1}{H^{a+b'+2}} \sum_{\substack{\alpha \in D(a,b') \\ |\alpha|=a \\ |\beta|=b'}} C(\alpha,\beta)H_{(0,0)}\partial_{\eta_l}\left(\prod_{i=1}^{a+b'} H_{(\alpha_i,\beta_i)}\right).$$

Observe that

$$\partial_{\eta_l}\left(\prod_{i=1}^{a+b'} H_{(\alpha_i,\beta_i)}\right) = \sum_{\substack{\alpha,\beta' \in D(a,b') \\ |\alpha|=a \\ |\beta'|=b'+1}} C(\alpha,\beta,\beta') \prod_{i=1}^{a+b'} H_{(\alpha_i,\beta'_i)},$$

where many of the $C(\alpha,\beta,\beta')$ are 0. So we can conclude that

$$(II) = \frac{1}{H^{a+b'+2}} \sum_{\substack{\alpha,\beta \in D(a,b') \\ |\alpha|=a \\ |\beta|=b'}} C(\alpha)H_{(0,0)}\left(\sum_{\substack{\alpha,\beta' \in D(a',0) \\ |\alpha|=a \\ |\beta'|=b'+1}} C(\alpha,\alpha') \prod_{i=1}^{a+b'} H_{(\alpha_i,\beta'_i)}\right)$$

$$= \frac{1}{H^{a+b'+2}} \sum_{\substack{\alpha,\tilde{\beta} \in D(a,b'+1) \\ |\alpha|=a \\ |\beta|=b'+1}} \tilde{C}(\alpha,\tilde{\beta}) \prod_{i=1}^{a'+b'1} H_{(\alpha_i,\tilde{\beta}_i)},$$

where $\tilde{C}(\alpha,\tilde{\beta}) = 0$ if $\tilde{\beta}_{b'+1} \neq 0$. Combining (I) and (II) gives us equation (6.24) completing the proof for case 2.

Case 3: $a = 0, b \geq 1$

As $\mathcal{K}$ is symmetric with respect to ξ and η this follows from case 1, completing our proof. □

Lemma 6.4.2. *Fix* $\eta \in \{|\eta| \sim 2^{k'}\}$*, then on* $[(A_+ \cup B_+)_\eta \cap \{\theta_\eta(\xi) \geq 0\}]$ *we have the* *following bounds*

$$\left|\frac{1}{H_{(0,0)}(\xi,\eta)}\right| \lesssim \max\{2^{-2k}, (2^{2k}2^{2k'}\theta^2)^{-1}\},$$

$$\left|H_{(1,0)}(\xi,\eta)\right| \lesssim \max\{2^{2k'}2^{k}\theta, 2^{k}\},$$

$$\left|H_{(0,1)}(\xi,\eta)\right| \lesssim \max\{2^{k'}2^{2k}\theta, 2^{k}\},$$

$$\left|H_{(1,1)}(\xi,\eta)\right| \lesssim \max\{2^{k'}2^{k}\theta, 1\},$$

$$\left|H_{(2,0)}(\xi,\eta)\right| \lesssim 2^{2k'},$$

$$\left|H_{(0,2)}(\xi,\eta)\right| \lesssim 2^{2k},$$

$$\left|H_{(2,1)}(\xi,\eta)\right| \lesssim 2^{k'},$$

$$\left|H_{(1,2)}(\xi,\eta)\right| \lesssim 2^{k},$$

$$\left|H_{(2,2)}(\xi,\eta)\right| \lesssim 1.$$

Proof. Recall that

$$H_{(0,0)} = 4(|\xi|^2|\eta|^2 - \langle\xi,\eta\rangle^2 + |\xi|^2 + |\eta|^2 + \langle\xi,\eta\rangle) + 3.$$

After several straightforward computations we obtain

$$H_{(1,0)} = 4[2\xi_p|\eta|^2 - 2\eta_p\langle\xi,\eta\rangle + 2\xi_p + \eta_p],$$

$$H_{(2,0)} = 4[2|\eta|^2 - 2\eta_p^2 + 2],$$

$$H_{(1,1)} = 4[4\xi_p\eta_l - 4\eta_p\xi_l + \delta_{pl}],$$

$$H_{(2,1)} = 4[4\eta_l - 4\delta_{pl}\eta_p],$$

$$H_{(2,2)} = 16[1 - \delta_l].$$

It is easy to see that $|H_{(2,0)}| \lesssim 2^{2k'}, |H_{(2,1)}| \lesssim 2^{k'}, |H_{(2,2)}| \lesssim 1$ and by symmetry $|H_{(0,2)}| \lesssim 2^{2k}, |H_{(1,2)}| \lesssim 2^k$. It remains to prove the estimates on $|\frac{1}{H_{(0,0)}}|, |H_{(0,1)}|, |H_{(1,0)}|$, and $|H_{(1,1)}|$.

We can rewrite $H_{(0,0)}$ as

$$H_{(0,0)} = 4(|\xi|^2|\eta|^2 \sin^2\theta + |\xi|^2 + |\eta|^2 + \langle \xi, \eta \rangle) + 3.$$

As $\theta^2 \lesssim \sin^2\theta$ on $[0, \frac{\pi}{2}]$ and $|\xi|^2 + |\eta|^2 + \langle \xi, \eta \rangle \gtrsim \max(|\xi|^2, |\eta|^2)$, we see that

$$\min\{2^{2k}, 2^{2k}2^{2k'}\theta^2\} \lesssim |\xi|^2|\eta|^2\theta^2 + \max(|\xi|^2, |\eta|^2) \lesssim |H_{(0,0)}|.$$

We conclude that

$$\left|\frac{1}{H_{(0,0)}}\right| \lesssim \frac{1}{\min\{2^{2k}, 2^{2k}2^{2k'}\theta^2\}} = \max\{2^{-2k}, (2^{2k}2^{2k'}\theta^2)^{-1}\}.$$

We next attempt to bound $|H_{(1,0)}|$. Recall that

$$H_{(1,0)} = 4[2\xi_p|\eta|^2 - 2\eta_p \langle \xi, \eta \rangle + 2\xi_p + \eta_p],$$

and observe that

$$|\xi_p|\eta|^2 - \eta_p \langle \xi, \eta \rangle | = |\xi_p|\eta|^2 - \eta_p|\xi||\eta| \cos\theta|$$

Define $\tilde{\xi} := \frac{\xi}{|\xi|}$ and $\tilde{\eta} := \frac{\eta}{|\eta|}$. Then we can rewrite the above expression as

$$\begin{aligned} |\xi_p|\eta|^2 - \eta_p \langle \xi, \eta \rangle | &= ||\xi||\eta|^2(\tilde{\xi}_p - \tilde{\eta}_p \cos\theta)| \\ &\leq |\xi||\eta|^2[|\tilde{\xi}_p - \tilde{\eta}_p| + (1 - \cos\theta)|\tilde{\eta}_p|] \\ &\leq |\xi||\eta|^2[|\tilde{\xi}_p - \tilde{\eta}_p| + (1 - \cos\theta)]. \end{aligned}$$

Observe that

$$\begin{aligned} |\tilde{\xi}_p - \tilde{\eta}_p|^2 \leq |\tilde{\xi} - \tilde{\eta}|^2 &= |\tilde{\xi}|^2 + |\tilde{\eta}|^2 - 2|\tilde{\xi}||\tilde{\eta}| \cos\theta \\ &= 2(1 - \cos\theta), \end{aligned}$$

so we conclude that

$$
\begin{aligned}
|H_{(1,0)}| &\lesssim |\xi||\eta|^2[(1-\cos\theta)^{1/2} + (1-\cos\theta)]| + 2^k \\
&\lesssim |\xi||\eta|^2|\theta| + 2^k \\
&\lesssim \max(2^k 2^{2k'}|\theta|, 2^k).
\end{aligned}
$$

By symmetry, we also have

$$|H_{(0,1)}| \lesssim \max(2^{2k}2^{k'}|\theta|, 2^k).$$

We now attempt to bound $|H_{(1,1)}|$. Recall that

$$H_{(1,1)} = 4[4\xi_p\eta_l - 4\eta_p\xi_l + \delta_{pl}].$$

Observe that

$$
\begin{aligned}
|\xi_p\eta_l - \eta_p\xi_l| &\leq |\xi \times \eta| \\
&= |\xi||\eta||\sin\theta| \\
&\sim |\xi||\eta||\theta|,
\end{aligned}
$$

Allowing us to deduce that

$$|H_{(1,1)}| \lesssim \max\{2^{k'}2^k\theta, 1\}$$

As desired. □

We are finally ready to prove estimates (6.17) and (6.19).

Proposition 6.4.3. *The following holds true:*

$$\int_{|\eta|\sim 2^{k'}} \int_{(A_+\cap\{\theta\geq 0\})_\eta} |\mathcal{K}_{(a,b)}(\xi,\eta)| d\xi d\eta \lesssim 2^{-(2+a)k}2^{(6-b)k'}2^{3k}2^{3k'}. \tag{6.25}$$

Proof. From Lemma 6.4.1, we know we can bound $|\mathcal{K}_{(a,b)}|$ on $A_+ \cap \{\theta \geq 0\}$ by a sum of terms of the form

$$\left| \frac{H_{(1,0)}^{n_1} H_{(0,1)}^{n_2} H_{(1,1)}^{n_3} H_{(2,0)}^{n_4} H_{(0,2)}^{n_5} H_{(2,1)}^{n_6} H_{(1,2)}^{n_7} H_{(2,2)}^{n_8}}{H_{(0,0)}^{a+b+1-n_0}} \right|, \tag{6.26}$$

where we have

$$\sum_{i=0}^{8} n_i = a + b, \tag{6.27}$$

$$a = n_1 + n_3 + 2n_4 + 2n_6 + n_7 + 2n_8, \tag{6.28}$$

$$b = n_2 + n_3 + 2n_5 + n_6 + 2n_7 + 2n_8. \tag{6.29}$$

From Lemma 6.4.2 and the fact that $0 \leq \theta \leq 2^{-k'}$ on $A_+ \cap \{\theta \geq 0\}$ we know we can bound (6.26) by a product of multiples of 2^k and $2^{k'}$.

We first sum up the powers of 2^k in the numerator. Using Lemma 6.4.2 we see that

The exponent on 2^k in the numerator $\leq n_1 + 2n_2 + n_3 + 2n_5 + n_7.$

The exponent on 2^k in the denminator $\geq 2(a + b + 1 - n_0).$

Combining these two facts, we see that the total power of 2^k is

$$\leq n_1 + 2n_2 + n_3 + 2n_5 + n_7 - 2(a + b + 1 - n_0).$$

By equation (6.27), this

$$= n_1 + 2n_2 + n_3 + 2n_5 + n_7 - 2(1 + \sum_{i=1}^{8} n_i)$$

$$= -(n_1 + n_3 + 2n_4 + 2n_6 + n_7 + 2n_8 + 2)$$

$$= -(a + 2),$$

where the last equality comes from equation (6.28).

We now sum up the powers of $2^{k'}$. As $\theta \lesssim 2^{-k'}$ on A_+ we can replace the first four bounds in Lemma 6.4.2 by

$$\left|\frac{1}{H_{(0,0)}(\xi,\eta)}\right| \lesssim 2^{-2k}$$

$$\left|H_{(1,0)}(\xi,\eta)\right| \lesssim 2^{k'}2^{k}$$

$$\left|H_{(0,1)}(\xi,\eta)\right| \lesssim 2^{2k}$$

$$\left|H_{(1,1)}(\xi,\eta)\right| \lesssim 2^{k},$$

allowing us to conclude

$$\begin{matrix}\text{The exponent on } 2^{k'} \\ \text{in the numerator}\end{matrix} \quad \leq n_1 + 2n_4 + n_6.$$

$$\begin{matrix}\text{The exponent on } 2^{k'} \\ \text{in the denominator}\end{matrix} \quad \geq 0.$$

Combining these two facts, we see that the total power of $2^{k'}$ is

$$\begin{aligned} &\leq n_1 + 2n_4 + n_6 \\ &\leq a \\ &= a + b - b \\ &\leq 8 - b, \end{aligned}$$

where the last inequality comes from our assumption that $a, b \leq 4$.

So far we have shown that $|\mathcal{K}_{(a,b)}(\xi,\eta)| \lesssim 2^{(-2-a)k}2^{(8-b)k'}$ on

$(A_+ \cap \{\theta \geq 0\})_\eta$. Observe that $|(A_+ \cap \{\theta \geq 0\})_\eta| \lesssim 2^{3k}2^{-2k'}$. It follows that

$$\int_{|\eta|\sim 2^{k'}} \int_{(A_+\cap\{\theta\geq 0\})_\eta} |\mathcal{K}(\xi,\eta)| d\xi d\eta \lesssim \int_{|\eta|\sim 2^{k'}} 2^{(-2-a)k}2^{(8-b)k'}|(A_+ \cap \{\theta \geq 0\})_\eta| d\eta$$
$$\lesssim 2^{-(2+a)k}2^{(6-b)k'}2^{3k}2^{3k'},$$

as desired. □

Proposition 6.4.4. *The following holds true:*

$$\int_{|\eta|\sim 2^{k'}} \int_{(B_+\cap\{\theta\geq 0\})_\eta} |\mathcal{K}_{(a,b)}(\xi,\eta)| d\xi d\eta \lesssim 2^{-(2+a)k}2^{(6-b)k'}2^{3k}2^{3k'}. \tag{6.30}$$

Proof. Once again we bound $|\mathcal{K}(a,b)|$ by a sum of terms of the form

$$\left| \frac{H^{n_1}_{(1,0)} H^{n_2}_{(0,1)} H^{n_3}_{(1,1)} H^{n_4}_{(2,0)} H^{n_5}_{(0,2)} H^{n_6}_{(2,1)} H^{n_7}_{(1,2)} H^{n_8}_{(2,2)}}{H^{a+b+1-n_0}_{(0,0)}} \right|,$$

where (6.27), (6.28), and (6.29) are still valid. From Lemma 6.4.2 we know we can bound this by a product of multiples of $2^k, 2^{k'}$ and θ. We first sum up the powers of 2^k in the numerator. Using Lemma 6.4.2 we see that

$$\begin{matrix}\text{The exponent on } 2^k \\ \text{in the numerator}\end{matrix} \leq n_1 + 2n_2 + n_3 + 2n_5 + n_7.$$

$$\begin{matrix}\text{The exponent on } 2^k \\ \text{in the denminator}\end{matrix} \geq 2(a+b+1-n_0).$$

Combining these two facts, we see that the total power of 2^k is

$$\leq n_1 + 2n_2 + n_3 + 2n_5 + n_7 - 2(a+b+1-n_0).$$

By equation (6.27), this

$$= n_1 + 2n_2 + n_3 + 2n_5 + n_7 - 2(1 + \sum_{i=1}^{8} n_i)$$

$$= -(n_1 + n_3 + 2n_4 + 2n_6 + n_7 + 2n_8 + 2)$$

$$= -(a+2).$$

where the last equality comes from equation (6.28).

We now sum up the powers of $2^{k'}$.

$$\begin{array}{l}\text{The exponent on } 2^{k'} \\ \text{in the numerator}\end{array} \quad \leq 2n_1 + n_2 + +n_3 + 2n_4 + n_6.$$

$$\begin{array}{l}\text{The exponent on } 2^{k'} \\ \text{in the denminator}\end{array} \quad = 2(a + b + 1 - n_0).$$

Combining these two facts we see that the total power of $2^{k'}$ is

$$\begin{aligned}&\leq 2n_1 + n_2 + n_3 + 2n_4 + n_6 - 2(a + b + 1 - n_0)\\ &= -(n_2 + n_3 + 2n_5 + n_6 + 2n_7 + 2n_8 + 2).\\ &= -(b + 2)\end{aligned}$$

Finally, we compute a bound for the total exponent on θ. Once again we use Lemma 6.4.2 to deduce

$$\begin{array}{l}\text{The exponent on } \theta \\ \text{in the numerator}\end{array} \quad = n_1 + n_2 + n_3.$$

$$\begin{array}{l}\text{The exponent on } \theta \\ \text{in the denominator}\end{array} \quad = 2(a + b + 1 - n_0),$$

so the total exponent is

$$\begin{aligned}&= n_1 + n_2 + n_3 - 2(a + b + 1 - n_0)\\ &:= \alpha,\end{aligned}$$

where we note that $\alpha < 0$. From equations (6.27)-(6.29) we have

$$a + b = \sum_{i=0}^{8} n_i$$

and

$$a + b = n_1 + n_2 + 2n_3 + 2n_4 + 2n_5 + 3n_6 + 3n_7 + 4n_8.$$

Solving for n_0, we obtain

$$n_0 = n_3 + n_4 + n_5 + 2n_6 + 2n_7 + 3n_8$$

so that

$$\begin{aligned}\alpha &= n_1 + n_2 + 3n_3 + 2n_4 + 2n_5 + 4n_6 + 4n_7 + 6n_8 - 2(a + b + 1)\\ &= n_3 + n_6 + n_7 + 2n_8 + (a + b) - 2(a + b + 1)\\ &= n_3 + n_6 + n_7 + 2n_8 - (a + b + 2).\end{aligned}$$

It follows that

$$-(\alpha + 2) \leq a + b \leq 8.$$

We can therefore conclude that

$$\begin{aligned}\int_{|\eta|\sim 2^{k'}} \int_{B_+\cap\{\theta\geq 0\}} |\mathcal{K}(\xi, \eta)| d\xi d\eta &\lesssim 2^{3k} 2^{3k'} 2^{-(a+2)k} 2^{-(b+2)k'} \int_{2^{-k'}}^{\frac{\pi}{2}} \theta^{\alpha} \sin\theta d\theta\\ &\lesssim 2^{3k} 2^{3k'} 2^{-(a+2)k} 2^{-(b+2)k'} \int_{2^{-k'}}^{\frac{\pi}{2}} \theta^{\alpha+1} d\theta\\ &\lesssim 2^{3k} 2^{3k'} 2^{-(a+2)k} 2^{-(b+2)k'} (-\theta^{\alpha+2})\Big|_{2^{-k'}}^{\frac{\pi}{2}}\\ &\lesssim 2^{3k} 2^{3k'} 2^{-(a+2)k} 2^{-(b+2)k'} 2^{-k'(\alpha+2)}\\ &\lesssim 2^{3k} 2^{3k'} 2^{-(a+2)k} 2^{-(b+2)k'} 2^{8k'}\\ &= 2^{3k} 2^{3k'} 2^{-ak} 2^{-bk'} 2^{-2k} 2^{6k'}.\end{aligned}$$

□

Chapter 7

The Second Order Semilinear Klein-Gordon Equation

7.1 Local Existence

We dedicate this chapter to proving well-posedness of the second order semilinear Klein-Gordon system in $H^s(\mathbb{R}^3)$ for $s > 10$. Our proof will rely on the bootstrap argument outlined in section 3.2. The reader may recall that this method requires an established local theory. For this reason, we dedicate this section to the relatively straightforward task of proving local existence. In particular, we will employ the contraction method outlined in section 3.1 to prove the following theorem

Theorem 7.1.1. *Given* $s > 2$ *and* $0 < T \leq 1$, *there exists* $\delta > 0$ *such that if* $(u_0, u_1) \in H^s \times H^{s-1}$ *with* $\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} < \delta$, *then there exists a unique* $u \in C([0,T); H^s_x)$

satisfying

$$(\Box + 1)u = \sum_{j,k=0}^{1} \mathcal{A}_{jk}(\partial', \partial'')[\partial_t^j u][\partial_t^k u], \tag{7.1}$$

with initial data

$$u(0,x) = u_0 \in H^s(\mathbb{R}^3), \quad \partial_t u(0,x) = u_1 \in H^{s-1}(\mathbb{R}^3).$$

Furthermore, the map $u_0 \longmapsto u$ is Lipschitz continuous.

Proof. Fix $s > 2$, $1 < s' < s - 1$, $0 < T \leq 1$, and define X_T, $S_T(\delta)$, and N_T by the norms

$$\|u\|_{X_T} := \sum_{i=0}^{1} \Big(\|\partial_t^i u\|_{L_t^\infty([0,T], H_x^{s-i})[\mathbf{k}]} + \|\partial_t^i u\|_{L_t^2([0,T], W_x^{s'-i,\infty})[\mathbf{k}]} \Big),$$

$$\|u\|_{S_T} := \{u \in X_T : \|u\|_{X_T} \leq C_0 \delta\},$$

$$\|u\|_{N_T} := \|u\|_{L_t^1([0,T], H_x^{s-1})[\mathbf{k}]},$$

where C_0 is chosen to be sufficiently large.

From our discussion in Chapter 3, local well-posedness on $[0,T]$ will follow from proving the following four estimates

$$\|W(u_0, u_1)\|_{S_T} \lesssim \|(u_0, u_1)\|_{H_x^s \times H_x^{s-1}}, \tag{7.2}$$

$$\|L(G)\|_{S_T} \lesssim \|G\|_{N_T}, \tag{7.3}$$

$$\| \sum_{j,k=0}^{1} \mathcal{A}_{jk}[\partial_t^j w][\partial_t^k w]\|_{N_T}, \lesssim \|w\|_{X_T}^2 \tag{7.4}$$

$$\| \sum_{j,k=0}^{1} (\mathcal{A}_{jk}[\partial_t^j v][\partial_t^k v] - \mathcal{A}_{jk}[\partial_t^j w][\partial_t^k w])\|_{N_T} \lesssim_\delta \|v - w\|_{X_T}, \tag{7.5}$$

for all $v, w \in S_T$. We recall that

$$W(u_0, u_1)(t, \cdot) = e^{it\langle D\rangle} u_0(\cdot) + \frac{e^{it\langle D\rangle} u_1(\cdot)}{\langle D\rangle}$$

and

$$L(G)(t,\cdot) := \int_0^t \frac{\sin(i(t-s)\langle D\rangle)G(s,\cdot)}{\langle D\rangle} ds.$$

Estimates (7.2) and (7.3) follow from nearly identical arguments to the ones used for the analogous estimates in section 4.2. It is not difficult to see that estimate (7.4) follows from proving

$$\|\mathcal{A}_{jk}[\partial_t^j v][\partial_t^k w]\|_{N_T} \lesssim \|v\|_{X_T}\|w\|_{X_T}, \tag{7.6}$$

for all $j,k \in \{0,1\}$ and $v,w \in X_T$.

As $\mathcal{A}_{jk} \in S(1-j,1-k,0,0)$ we can apply 6.3.1 with $r = s-1, \sigma = \tilde{\sigma} = 0, \lambda = \tilde{\lambda} = s'-1, p_1 = \tilde{p}_2 = q_2 = \tilde{q}_1 = \infty, p_2 = \tilde{p}_1 = 1, q_1 = \tilde{q}_2 = 2$ to get

$$\begin{aligned}
\|\mathcal{A}_{jk}[\partial_t^j v][\partial_t^k w]\|_{L_t^1 H_x^{s-1}[\mathbf{k}]} &\lesssim \|\partial_t^j v\|_{L_t^\infty H_x^{s-j}} \|\partial_t^k w\|_{L_t^1 W_x^{s'-k,\infty}[\mathbf{k}]} \\
&\quad + \|\partial_t^j v\|_{L_t^1 W_x^{s'-j,\infty}[\mathbf{k}]} \|\partial_t^k w\|_{L_t^\infty H_x^{s-k}[\mathbf{k}]} \\
&\lesssim \|v\|_{X_T} T^{1/2} \|\partial_t^k w\|_{L_t^2 W_x^{s'-k,\infty}[\mathbf{k}]} + T^{1/2}\|\partial_t^j v\|_{L_t^2 W_x^{s'-j,\infty}[\mathbf{k}]} \|w\|_{X_T} \\
&\lesssim T^{1/2}\|v\|_{X_T}\|w\|_{X_T} \\
&\leq \|v\|_{X_T}\|w\|_{X_T},
\end{aligned}$$

where we used our assumption that $T \leq 1$.

We now turn our attention to estimate (7.5). Let

$$J(v,w) := \sum_{j,k=0}^{1} \mathcal{A}_{jk}[\partial_t^j v][\partial_t^k w].$$

As each $\mathcal{A}_{jk}$ is bilinear, it follows that J is as well so we can conclude

$$\begin{aligned}\|J(v,v)-J(w,w)\|_{X_T} &\leq \|J(v,v)-J(v,w)\|_{X_T}+\|J(v,w)-J(w,w)\|_{X_T}\\ &\leq C\|v\|_{X_T}\|v-w\|_{X_T}+C\|w\|_{X_T}\|v-w\|_{X_T}\\ &\leq 2CC_0\delta\|v-w\|_{X_T},\end{aligned}$$

where the second inequality follows from the bilinearity of J and equation (7.6) and the last inequality follows from the definition of S_T. □

7.2 Main Estimates

Now that the local theory has been established, we are finally in a position to prove global well-posedness. Our goal is to prove the following theorem

Theorem 7.2.1. *Let $s \geq 10+\epsilon$ for a given $\epsilon > 0$. There exists $\delta > 0$ such that if $(u_0,u_1) \in H^s(\mathbb{R}^3)\times H^{s-1}(\mathbb{R}^3)$ with $\|u_0\|_{H^s}+\|u_1\|_{H^{s-1}} < \delta$, then there exists a unique $u \in C_0([0,\infty);H^s_x)$ satisfying*

$$(\Box+1)u = F(u,u') = \sum_{j,k=0}^{1} \mathcal{A}_{jk}(\partial^{'},\partial^{''})[\partial_t^j u][\partial_t^k u], \tag{7.7}$$

Furthermore, the map $u_0 \longmapsto u$ is Lipschitz continuous.

We will take advantage of the Normal forms decomposition $u = U + W$ introduced in the previous chapter in order to close the bootstrap argument discussed in section 3.2.

Let S_T be defined by the norm

$$\|u\|_{S_T} = \sum_{i=0}^{1}\Big(\|\partial_t^i u\|_{L_t^\infty([0,T],H_x^{s-i})[\mathbf{k}]}+\|\partial_t^i u\|_{L_t^2([0,T],W_x^{9-i+\frac{\epsilon}{2},\infty})[\mathbf{k}]}\Big),$$

and assume

$$\|u\|_{S_T} \le M\delta,$$

for some sufficiently large constant M independent of T. By the bootstrap argument it suffices to show that

$$\|u\|_{S_T} \le \frac{M}{2}\delta.$$

As $u = U + W$, this follows from showing

$$\|W\|_{S_T} \lesssim \delta + \|u\|_{S_T}^3 + \|u\|_{S_T}^4 \tag{7.8}$$

and

$$\|U\|_{S_T} \lesssim \|u\|_{S_T}^2 + \|u\|_{S_T}^3, \tag{7.9}$$

provided δ is chosen sufficiently small and M sufficiently large.

Recall that

$$(\Box + 1)W = -\mathcal{R},$$

where

$$\begin{aligned}
\mathcal{R} = {} & \mathcal{B}_{00}(D', D'')([F(u,u')][u] + [u][F(u,u')]) \\
& + \mathcal{B}_{10}(D', D'')([\partial_t F(u,u')][u] + 2[F(u,u')][\partial_t u] + [\partial_t u][F(u,u')]) \\
& + \mathcal{B}_{01}(D', D'')([u][\partial_t F(u,u')] + 2[\partial_t u][F(u,u')] + [F(u,u')][\partial_t u]) \\
& + \mathcal{B}_{11}(D', D'')([\partial_t F(u,u')][\partial_t u] + 2[F(u,u')][F(u,u')] \\
& - 2(|D''|^2 + 1)[F(u,u')][u] - 2(|D'|^2 + 1)[u][F(u,u')] + [\partial_t u][\partial_t F(u,u')]),
\end{aligned}$$

and that

$$W(0) = u(0) - U(0), \qquad \partial_t W(0) = u_t(0) - U_t(0).$$

We can therefore conclude that

$$W(t) = cos(t\langle D\rangle)W(0) + \frac{sin(t\langle D\rangle)}{\langle D\rangle}W_t(0) - \int_0^t \frac{sin((t-s)\langle D\rangle)}{\langle D\rangle}\mathcal{R}ds$$

and

$$\begin{aligned}\|W\|_{S_T} &\lesssim \sum_{i=0}^{1}\left(\|e^{it\langle D\rangle}\partial_t^i W(0)\|_{L_t^\infty H_x^{s-i}[\mathbf{k}]} + \|\partial_t^i W(0)\|_{L_t^2 W_x^{9-i+\frac{\epsilon}{2},\infty}[\mathbf{k}]}\right) + \|\mathcal{R}\|_{L_t^1 H_x^{s-1}[\mathbf{k}]} \\ &\lesssim \|(u(0),u_t(0))\|_{H_x^s\times H_x^{s-1}} + \|(U(0),U_t(0))\|_{H_x^s\times H_x^{s-1}} + \|\mathcal{R}\|_{L_t^1 H_x^{s-1}[\mathbf{k}]} \\ &\lesssim \delta + \|(U(0),U_t(0))\|_{H_x^s\times H_x^{s-1}} + \|\mathcal{R}\|_{L_t^1 H_x^{s-1}[\mathbf{k}]},\end{aligned}$$

so (7.8) follows from proving

$$\|(U(0),U_t(0))\|_{H_x^s\times H_x^{s-1}} + \|\mathcal{R}\|_{L_t^1 H_x^{s-1}} \lesssim \delta + \|u\|_{S_T}^3 + \|u\|_{S_T}^4.$$

I claim that $\|\mathcal{R}(u,u')\|_{L_t^1 H_x^{s-1}[\mathbf{k}]} \lesssim \|u\|_{S_T}^3 + \|u\|_{S_T}^4$. By symmetry, this follows from proving the following:

Proposition 7.2.1. *Let $i,j \in \{0,1\}$, then*

$$\|\mathcal{B}_{ij}(D',D'')[\partial_t^i F(u,u')][\partial_t^j u]\|_{L_t^1 H_x^{s-1}[\boldsymbol{k}]} \lesssim \|u\|_{S_T}^3 + \|u\|_{S_T}^4, \tag{7.10}$$

$$\|\mathcal{B}_{10}(D',D'')[F(u,u')][\partial_t u]\|_{L_t^1 H_x^{s-1}[\boldsymbol{k}]} \lesssim \|u\|_{S_T}^3 + \|u\|_{S_T}^4, \tag{7.11}$$

$$\|\mathcal{B}_{11}(D',D'')[F(u,u')][F(u,u')]\|_{L_t^1 H_x^{s-1}[\boldsymbol{k}]} \lesssim \|u\|_{S_T}^4, \tag{7.12}$$

$$\|\mathcal{B}_{11}(D',D'')[F(u,u')][(|D''|^2+1)u]\|_{L_t^1 H_x^{s-1}[\boldsymbol{k}]} \lesssim \|u\|_{S_T}^3. \tag{7.13}$$

Proof. As $\mathcal{B}_{ij} \in S(2-i, 2-j, -2, 6)$, we may apply Proposition 6.3.1 to (7.10) with $a = 2-i, b = 2-j, c = -2, d = 6, r = s-1, \lambda = \tilde{\lambda} = \frac{\epsilon}{2}, \sigma = \tilde{\sigma} = 0, p_1 = p_2 = 2, q_1 = 2, q_2 = \infty, \tilde{p}_1 = 1, \tilde{p}_2 = \infty, \tilde{q}_1 = \infty, \tilde{q}_2 = 2$ to get

$$\begin{aligned}\text{LHS}(7.10) \lesssim\ & \|\partial_t^i F(u,u')\|_{L_t^2 H_x^{s-1-i}[\mathbf{k}]} \|\partial_t^j u\|_{L_t^2 W_x^{8+\frac{\epsilon}{2}-j,\infty}[\mathbf{k}]} \\ & + \|\partial_t^i F(u,u')\|_{L_t^1 W_x^{8+\frac{\epsilon}{2}-i,\infty}[\mathbf{k}]} \|\partial_t^j u\|_{L_t^\infty H_x^{s-1-j}[\mathbf{k}]}.\end{aligned}$$

Observe that

$$\|\partial_t^j u\|_{L_t^2 W_x^{8+\frac{\epsilon}{2}-j,\infty}[\mathbf{k}]} \lesssim \|\partial_t^j u\|_{L_t^2 W_x^{9+\frac{\epsilon}{2}-j,\infty}[\mathbf{k}]} \lesssim \|u\|_{S_T}$$

and

$$\|\partial_t^j u\|_{L_t^\infty H_x^{s-1-j}[\mathbf{k}]} \lesssim \|\partial_t^j u\|_{L_t^\infty H_x^{s-j}[\mathbf{k}]} \lesssim \|u\|_{S_T}.$$

So, it's left to prove

$$\|\partial_t^i F(u,u')\|_{L_t^2 H_x^{s-1-i}[\mathbf{k}]} + \|\partial_t^i F(u,u')\|_{L_t^1 W_x^{8+\frac{\epsilon}{2}-i,\infty}[\mathbf{k}]} \lesssim \|u\|_{S_T}^2 + \|u\|_{S_T}^3. \tag{7.14}$$

We break this down into two cases.

case 1: $i = 0$

In this case $\partial_t^i F(u,u') = F(u,u') = \sum_{j,k=0}^1 \mathcal{A}_{jk}[\partial_t^j u][\partial_t^k u]$. It therefore suffices to show that, for all $i, j \in \{0,1\}$

$$\|\mathcal{A}_{jk}[\partial_t^j u][\partial_t^k u]\|_{L_t^2 H_x^{s-1}[\mathbf{k}]} + \|\mathcal{A}_{jk}[\partial_t^j u][\partial_t^k u]\|_{L_t^1 W_x^{8+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \lesssim \|u\|_{S_T}^2. \tag{7.15}$$

As $\mathcal{A}_{jk} \in S(1-j, 1-k, 0, 0)$ we can apply Proposition 6.3.1 with $a = 1-j, b = 1-k, c = d = 0, r = s-1, \lambda = \tilde{\lambda} = \frac{\epsilon}{2}, \sigma = \tilde{\sigma} = 0, p_1 = \tilde{p}_2 = \infty, p_2 = \tilde{p}_1 = 2, q_1 = \tilde{q}_2 = 2, q_2 = \tilde{q}_1 = \infty$

to obtain

$$
\begin{aligned}
\|\mathcal{A}_{jk}[\partial_t^j u][\partial_t^k u]\|_{L_t^2 H_x^{s-1}[\mathbf{k}]} &\lesssim \|\partial_t^j u\|_{L_t^\infty H_x^{s-j}[\mathbf{k}]} \|\partial_t^k u\|_{L_t^2 W_x^{\frac{\epsilon}{2}+1-k,\infty}[\mathbf{k}]} \\
&+ \|\partial_t^j u\|_{L_t^2 W_x^{\frac{\epsilon}{2}+1-j,\infty}[\mathbf{k}]} \|\partial_t^k u\|_{L_t^\infty H_x^{s-k}[\mathbf{k}]} \\
&\lesssim \|u\|_{S_T}^2.
\end{aligned}
$$

We now turn to estimating $\|\mathcal{A}_{jk}[\partial_t^j u][\partial_t^k u]\|_{L_t^1 W_x^{8+\frac{\epsilon}{2},\infty}[\mathbf{k}]}$. Applying Proposition 6.3.1 with $a = 1-j, b = 1-k, c = d = 0, \sigma = \tilde{\sigma} = 0, \lambda = \tilde{\lambda} = \frac{\epsilon}{2}, p_1 = p_2 = \tilde{p}_1 = \tilde{p}_2 = 2$, we see that

$$
\begin{aligned}
\|\mathcal{A}_{jk}[\partial_t^j u][\partial_t^k u]\|_{L_t^1 W_x^{8+\frac{\epsilon}{2},\infty}[\mathbf{k}]} &\lesssim \|\partial_t^j u\|_{L_t^2 W_x^{9-j+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \|\partial_t^k u\|_{L_t^2 W_x^{1-k+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \\
&+ \|\partial_t^j u\|_{L_t^2 W_x^{1-j+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \|\partial_t^k u\|_{L_t^2 W_x^{9-k+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \\
&\lesssim \|u\|_{S_T}^2.
\end{aligned}
$$

case 2: $i = 1$

In this case, $\partial_t^i F(u, u') = \partial_t F(u, u') = \sum_{j,k=0}^1 \mathcal{A}_{jk}([\partial_t^{j+1} u][\partial_t^k u] + [\partial_t^j u][\partial_t^{k+1} u])$. By symmetry we only need to consider the terms $\mathcal{A}_{j0}[\partial_t^j u][\partial_t u]$ and $\mathcal{A}_{1k}[\partial_t^2 u][\partial_t^k u]$ for $j, k \in \{0, 1\}$. We first consider $\mathcal{A}_{j0}[\partial_t^j u][\partial_t u]$. We aim to show that

$$
\|\mathcal{A}_{j0}[\partial_t^j u][\partial_t u]\|_{L_t^2 H_x^{s-2}[\mathbf{k}]} + \|\mathcal{A}_{j0}[\partial_t^j u][\partial_t u]\|_{L_t^1 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \lesssim \|u\|_{S_T}^2.
$$

As $\mathcal{A}_{j0} \in S(1-j, 1, 0, 0)$, we can apply Proposition 6.3.1 with $\lambda = \tilde{\lambda} = \frac{\epsilon}{2}, \sigma = \tilde{\sigma} = 0, p_1 = \tilde{p}_2 = \infty, p_2 = \tilde{p}_1 = 2, q_1 = \tilde{q}_2 = 2, q_2 = \tilde{q}_1 = \infty$ to obtain

$$
\begin{aligned}
\|\mathcal{A}_{j0}[\partial_t^j u][\partial_t u]\|_{L_t^2 H_x^{s-2}[\mathbf{k}]} &\lesssim \|\partial_t^j u\|_{L_t^\infty H_x^{s-1-j}[\mathbf{k}]} \|\partial_t u\|_{L_t^2 W_x^{1+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \\
&+ \|\partial_t^j u\|_{L_t^2 W_x^{\frac{\epsilon}{2}+1-j,\infty}[\mathbf{k}]} \|\partial_t u\|_{L_t^\infty H_x^{s-1}[\mathbf{k}]} \\
&\lesssim \|u\|_{S_T}^2.
\end{aligned}
$$

To bound $\|\mathcal{A}_{j0}[\partial_t^j u][\partial_t u]\|_{L_t^1 W_x^{\frac{\epsilon}{2}+7,\infty}[\mathbf{k}]}$ we once again apply Proposition 6.3.1 with $a = 1 - j, b = 1, c = d = 0, \lambda = \tilde{\lambda} = \frac{\epsilon}{2}, \sigma = \tilde{\sigma} = 0, p_1 = p_2 = \tilde{p}_1 = \tilde{p}_2 = 2, q_1 = q_2 = \tilde{q}_1 = \tilde{q}_2 = \infty$ to get

$$\begin{aligned}\|\mathcal{A}_{j0}[\partial_t^j u][\partial_t u]\|_{L_t^1 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]} &\lesssim \|\partial_t^j u\|_{L_t^2 W_x^{8-j,\infty+\frac{\epsilon}{2}}[\mathbf{k}]} \|\partial_t u\|_{L_t^2 W_x^{1+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \\ &+ \|\partial_t^j u\|_{L_t^2 W_x^{1-j+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \|\partial_t u\|_{L_t^2 W_x^{8+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \\ &\lesssim \|u\|_{S_T}^2.\end{aligned}$$

We next consider the term $\mathcal{A}_{1k}[\partial_t^2 u][\partial_t^k u]$. In particular, we aim to show that

$$\|\mathcal{A}_{1k}[\partial_t^2 u][\partial_t^k u]\|_{L_t^2 H_x^{s-2}[\mathbf{k}]} + \|\mathcal{A}_{1k}[\partial_t^2 u][\partial_t^k u]\|_{L_t^1 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \lesssim \|u\|_{S_T}^2. \tag{7.16}$$

As $\mathcal{A}_{1k} \in S(0, 1-k, 0, 0)$, we can apply Proposition 6.3.1 with $\lambda = \tilde{\lambda} = \frac{\epsilon}{2}, \sigma = 0, \tilde{\sigma} = 0, p_1 = \tilde{p}_2 = \infty, p_2 = \tilde{p}_1 = 2, q_1 = \tilde{q}_2 = 2, q_2 = \tilde{q}_1 = \infty$ to obtain

$$\begin{aligned}\|\mathcal{A}_{1k}[\partial_t^2 u][\partial_t^k u]\|_{L_t^2 H_x^{s-2}[\mathbf{k}]} &\lesssim \|\partial_t^2 u\|_{L_t^\infty H_x^{s-2}[\mathbf{k}]} \|\partial_t^k u\|_{L_t^2 W_x^{\frac{\epsilon}{2}+1-k,\infty}[\mathbf{k}]} \\ &+ \|\partial_t^2 u\|_{L_t^2 W_x^{\frac{\epsilon}{2},\infty}[\mathbf{k}]} \|\partial_t^k u\|_{L_t^2 H_x^{s-1-k}[\mathbf{k}]} \\ &\lesssim \Big(\|\partial_t^2 u\|_{L_t^\infty H_x^{s-2}[\mathbf{k}]} + \|\partial_t^2 u\|_{L_t^2 W_x^{\frac{\epsilon}{2},\infty}[\mathbf{k}]}\Big) \|u\|_{S_T}.\end{aligned}$$

It remains to prove the bound

$$\|\partial_t^2 u\|_{L_t^\infty H_x^{s-2}[\mathbf{k}]} + \|\partial_t^2 u\|_{L_t^2 W_x^{\frac{\epsilon}{2},\infty}[\mathbf{k}]} \lesssim \|u\|_{S_T} + \|u\|_{S_T}^2. \tag{7.17}$$

For the term $\|\mathcal{A}_{1k}[\partial_t^2 u][\partial_t^k u]\|_{L_t^1 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]}$ we once again apply Proposition 6.3.1 with $\sigma =$

$\tilde{\sigma} = 0, \lambda = \tilde{\lambda} = \frac{\epsilon}{2}, p_1 = p_2 = \tilde{p}_1 = \tilde{p}_2 = 2, q_1 = q_2 = \tilde{q}_1 = \tilde{q}_2 = \infty$ to get

$$\begin{aligned}\|\mathcal{A}_{1k}[\partial_t^2 u][\partial_t^k u]\|_{L_t^1 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]} &\lesssim \|\partial_t^2 u\|_{L_t^2 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \|\partial_t^k u\|_{L_t^2 W_x^{1-k+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \\ &+ \|\partial_t^2 u\|_{L_t^2 W_x^{\frac{\epsilon}{2},\infty}[\mathbf{k}]} \|\partial_t^k u\|_{L_t^2 W_x^{8-k+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \\ &\lesssim \|\partial_t^2 u\|_{L_t^2 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \|u\|_{S_T}.\end{aligned}$$

Combining the above estimate with equation (7.17), we see that (7.16) follows from

$$\|\partial_t^2 u\|_{L_t^\infty H_x^{s-2}[\mathbf{k}]} + \|\partial_t^2 u\|_{L_t^2 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \lesssim \|u\|_{S_T} + \|u\|_{S_T}^2. \tag{7.18}$$

Recall that

$$\partial_t^2 u = (1 + |D_x|^2)u + F(u, u')$$

The first component is easy to handle as

$$\|(1 + |D_x|^2)u\|_{L_t^\infty H_x^{s-2}[\mathbf{k}]} \lesssim \|u\|_{L_t^\infty H_x^s[\mathbf{k}]} \lesssim \|u\|_{S_T}$$

and

$$\|(1 + |D_x|^2)u\|_{L_t^2 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \lesssim \|u\|_{L_t^2 W_x^{9+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \lesssim \|u\|_{S_T}.$$

So it remains to show that

$$\|F(u, u')\|_{L_t^\infty H_x^{s-2}[\mathbf{k}]} + \|F(u, u')\|_{L_t^2 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \lesssim \|u\|_{S_T}^2.$$

As $F(u, u') = \sum_{j,k=0}^{1} \mathcal{A}_{jk}[\partial_t^j][\partial_t^k]$ this follows from proving the bound

$$\|\mathcal{A}_{jk}[\partial_t^j u][\partial_t^k u]\|_{L_t^\infty H_x^{s-2}[\mathbf{k}]} + \|\mathcal{A}_{jk}[\partial_t^j u][\partial_t^k u]\|_{L_t^2 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \lesssim \|u\|_{S_T}^2, \tag{7.19}$$

for all $j, k \in \{0, 1\}$

Because $\mathcal{A}_{jk} \in S(1-j, 1-k, 0, 0)$, we can apply Proposition 6.3.1 with $\sigma = \tilde{\sigma} = 0, \lambda = \tilde{\lambda} = \epsilon, p_1 = p_2 = \tilde{p}_1 = \tilde{p}_2 = \infty, q_1 = \tilde{q}_2 = 2, q_2 = \tilde{q}_1 = \infty$ to get

$$\|\mathcal{A}_{jk}[\partial_t^j u][\partial_t^k u]\|_{L_t^\infty H_x^{s-2}[\mathbf{k}]} \lesssim \|\partial_t^j u\|_{L_t^\infty H_x^{s-1-j}[\mathbf{k}]} \|\partial_t^k u\|_{L_t^\infty W_x^{1-k+\epsilon,\infty}[\mathbf{k}]} + \|\partial_t^j u\|_{L_t^\infty W_x^{1-j+\epsilon,\infty}[\mathbf{k}]} \|\partial_t^k u\|_{L_t^\infty H_x^{s-1-k}[\mathbf{k}]}.$$

As $s - (1+\epsilon) > \frac{n}{2} = \frac{3}{2}$ we can apply Sobolev embedding to bound the expression above by

$$\begin{aligned}
&\lesssim \|\partial_t^j u\|_{L_t^\infty H_x^{s-1-j}[\mathbf{k}]} \|\partial_t^k\|_{L_t^\infty H_x^{s-k}[\mathbf{k}]} \\
&\quad + \|\partial_t^j u\|_{L_t^\infty H_x^{s-j}[\mathbf{k}]} \|\partial_t^k u\|_{L_t^\infty H_x^{s-1-k}[\mathbf{k}]} \\
&\lesssim \|u\|_{S_T}^2,
\end{aligned}$$

where the last step follows from the definition of $\|\cdot\|_{S_T}$.

Finally, we turn our attention towards the term $\|\mathcal{A}_{jk}[\partial_t^j u][\partial_t^k u]\|_{L_t^2 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]}$. Applying Proposition 6.3.1 with $\sigma = \tilde{\sigma} = 0, \lambda = \tilde{\lambda} = \frac{\epsilon}{2}, p_1 = \tilde{p}_2 = 2, p_2 = \tilde{p}_1 = \infty, q_1 = q_2 = \tilde{q}_1 = \tilde{q}_2 = \infty$, we get

$$\|\mathcal{A}_{jk}[\partial_t^j u][\partial_t^k u]\|_{L_t^2 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \lesssim \|\partial_t^j u\|_{L_t^2 W_x^{8-j+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \|\partial_t^k u\|_{L_t^\infty W_x^{1-k+\frac{\epsilon}{2},\infty}[\mathbf{k}]} + \|\partial_t^j u\|_{L_t^\infty W_x^{1-j+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \|\partial_t^k u\|_{L_t^2 W_x^{8-k+\frac{\epsilon}{2},\infty}[\mathbf{k}]}.$$

We once again apply Sobolev embedding to bound the expression above by

$$\begin{aligned}
&\lesssim \|\partial_t^j u\|_{L_t^2 W_x^{8-j+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \|\partial_t^k u\|_{L_t^\infty H_x^{s-k}[\mathbf{k}]} \\
&+ \|\partial_t^j u\|_{L_t^\infty H_x^{s-j}[\mathbf{k}]} \|\partial_t^k u\|_{L_t^2 W_x^{8-k+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \\
&\leq \|u\|_{S_T}^2.
\end{aligned}$$

This completes the proof of estimate (7.10).

We now focus on proving (7.11). As $\mathcal{B}_{10} \in S(1,2,-2,6)$ we may apply Proposition 6.3.1 to (7.11) with $a=1, b=2, c=-2, d=6, r=s-1, \lambda=\tilde{\lambda}=\frac{\epsilon}{2}, \sigma=\tilde{\sigma}=0, p_1=p_2=2, \tilde{p}_1=1, \tilde{p}_2=\infty, q_1=\tilde{q}_2=2, q_2=\tilde{q}_1=\infty$ to obtain

$$\mathrm{LHS}(7.11) \lesssim \|F(u,u')\|_{L_t^2 H_x^{s-2}[\mathbf{k}]} \|\partial_t u\|_{L_t^2 W_x^{8+\frac{\epsilon}{2},\infty}[\mathbf{k}]} + \|F(u,u')\|_{L_t^1 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \|\partial_t u\|_{L_t^\infty H_x^{s-1}[\mathbf{k}]}.$$

Observe that

$$\|\partial_t u\|_{L_t^2 W_x^{8+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \lesssim \|u\|_{S_T}$$

and

$$\|\partial_t u\|_{L_t^\infty H_x^{s-1}[\mathbf{k}]} \lesssim \|u\|_{S_T},$$

so we are left to show

$$\|F(u,u')\|_{L_t^2 H_x^{s-2}[\mathbf{k}]} + \|F(u,u')\|_{L_t^1 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \lesssim \|u\|_{S_T}^2. \tag{7.20}$$

But the above follows from estimate (7.14).

We now turn our attention to proving estimate (7.12). As $\mathcal{B}_{11} \in S(1,1,-2,6)$, we may apply Proposition 6.3.1 to the LHS of (7.12) with $a=b=1, c=-2, d=6, \sigma=\tilde{\sigma}=1, \lambda=\tilde{\lambda}=\frac{\epsilon}{2}, p_1=p_2=\tilde{p}_1=\tilde{p}_2=2, q_1=\tilde{q}_2=2, q_2=\tilde{q}_1=\infty$ to conclude

$$\mathrm{LHS}(7.12) \lesssim \|F(u,u')\|_{L_t^2 H_x^{s-1}[\mathbf{k}]} \|F(u,u')\|_{L_t^2 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]}.$$

Recall that $F(u,u') = \sum_{j,k=0}^{1} \mathcal{A}_{jk}[\partial_t^j u][\partial_t^k u]$. So it suffices to show that given $i,j \in \{0,1\}$ we have

$$\|\mathcal{A}_{jk}[\partial_t^j u][\partial_t^k u]\|_{L_t^2 H_x^{s-1}[\mathbf{k}]} + \|\mathcal{A}_{jk}[\partial_t^j u][\partial_t^k u]\|_{L_t^2 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \lesssim \|u\|_{S_T}^2.$$

But this follows from estimates (7.15) and (7.19), completing the proof of (7.12).

Finally, we turn our attention towards estimate (7.13). As $\mathcal{B}_{11} \in S(1,1,-2,6)$, we may apply Proposition 6.3.1 to the LHS of (7.13) with $a = b = 1, c = -2, d = 6, \sigma = 1, \tilde{\sigma} = 0, \lambda = \tilde{\lambda} = \frac{\epsilon}{2}, p_1 = p_2 = 2, \tilde{p}_1 = 1, \tilde{p}_2 = \infty, q_1 = \tilde{q}_2 = 2, q_2 = \tilde{q}_1 = \infty$ to obtain

$$\begin{aligned}\text{LHS}(7.13) \lesssim\ & \|F(u,u')\|_{L_t^2 H_x^{s-1}[\mathbf{k}]} \|(|D''|^2+1)u\|_{L_t^2 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]} \\ & + \|F(u,u')\|_{L_t^1 W_x^{7+\frac{\epsilon}{2}+1,\infty}[\mathbf{k}]} \|(|D''|^2+1)u\|_{L_t^\infty H_x^{s-2}[\mathbf{k}]}.\end{aligned}$$

As $F(u,u') = \sum_{j,k=0}^{1} \mathcal{A}_{jk}[\partial_t^j u][\partial_t^k u]$ and we know from (7.15) that

$$\|\mathcal{A}_{jk}[\partial_t^j u][\partial_t^k u]\|_{L_t^2 H_x^{s-1}[\mathbf{k}]} + \|\mathcal{A}_{jk}[\partial_t^j u][\partial_t^k u]\|_{L_t^1 W_x^{7+\frac{\epsilon}{2}+1,\infty}[\mathbf{k}]} \lesssim \|u\|_{S_T}^2.$$

We only need to show that

$$\|(|D''|^2+1)u\|_{L_t^2 W_x^{7+\frac{\epsilon}{2},\infty}[\mathbf{k}]} + \|(|D''|^2+1)u\|_{L_t^\infty H_x^{s-2}[\mathbf{k}]} \lesssim \|u\|_{S_T},$$

but this is obvious from the definition of $\|\cdot\|_{S_T}$. □

In order to complete our proof of estimates (7.8) and (7.9), it's left to prove

Proposition 7.2.2. *The following holds true*

$$\|U(0)\|_{H_x^s} \lesssim \|u(0)\|_{H_x^s}^2 + \|u(0)\|_{H_x^s}^3 \tag{7.21}$$

$$\|U_t(0)\|_{H_x^{s-1}} \lesssim \|u_t(0)\|_{H_x^{s-1}}^2 + \|u_t(0)\|_{H_x^{s-1}}^3 \tag{7.22}$$

$$\|U(t)\|_{S_T} \lesssim \|u(t)\|_{S_T}^2 + \|u(t)\|_{S_T}^3. \tag{7.23}$$

Proof. Recall that $U = \sum_{i,j=0}^{1} \mathcal{B}_{ij}[\partial_t^i u][\partial_t^j u]$ and so

$$\|U\|_{L_t^\infty H_x^s} \lesssim \sum_{i,j=0}^{1} \|\mathcal{B}_{ij}[\partial_t^i u][\partial_t^j u]\|_{L_t^\infty H_x^s}. \tag{7.24}$$

As $\mathcal{B}_{ij} \in S(2-i, 2-j, -2, 6)$ we can apply Proposition 6.3.1 with $\lambda = \tilde{\lambda} = \epsilon, \sigma = \tilde{\sigma} = 0, p_1 = p_2 = \tilde{p}_1 = \tilde{p}_2 = \infty, q_1 = \tilde{q}_2 = 2, q_2 = \tilde{q}_1 = \infty$ to get

$$\|\mathcal{B}_{ij}[\partial_t^i u][\partial_t^j u]\|_{L_t^\infty H_x^s} \lesssim \|\partial_t^i u\|_{L_t^\infty H_x^{s-i}} \|\partial_t^j u\|_{L_t^\infty W_x^{8+\epsilon-j,\infty}} + \|\partial_t^i u\|_{L_t^\infty W_x^{8+\epsilon-i,\infty}} \|\partial_t^j u\|_{L_t^\infty H_x^{s-j}}.$$

As $s - (8+\epsilon) > \frac{3}{2} = \frac{n}{2}$ we can apply Sobolev embedding to bound the above by

$$\begin{aligned} &\lesssim \|\partial_t^i u\|_{L_t^\infty H_x^{s-i}} \|\partial_t^j u\|_{L_t^\infty H_x^{s-j}} \\ &\lesssim \|u\|_{S_T}^2. \end{aligned}$$

We remark that the above proof also implies (7.21).

We now turn our attention towards bounding $\|U\|_{L_t^2 W_x^{9+\frac{\epsilon}{2},\infty}}$. It suffices to show that

$$\|\mathcal{B}_{ij}[\partial_t^i u][\partial_t^j u]\|_{L_t^2 W_x^{9+\frac{\epsilon}{2},\infty}} \lesssim \|u\|_{S_T}^2,$$

for all $i, j \in \{0, 1\}$. Applying Proposition 6.3.1 with $a = 2-i, b = 2-j, c = -2, d = 6, \lambda = \tilde{\lambda} = \epsilon, \sigma = \tilde{\sigma} = 0, p_1 = \tilde{p}_2 = 2, p_2 = \tilde{p}_1 = q_1 = q_2 = \tilde{q}_1 = \tilde{q}_2 = \infty$ we see that

$$\begin{aligned} \|\mathcal{B}_{ij}[\partial_t^i u][\partial_t^j u]\|_{L_t^2 W_x^{9+\frac{\epsilon}{2},\infty}} &\lesssim \|\partial_t^i u\|_{L_t^2 W_x^{9+\frac{\epsilon}{2}-i,\infty}} \|\partial_t^j u\|_{L_t^\infty W_x^{8+\epsilon-j,\infty}} \\ &+ \|\partial_t^i u\|_{L_t^\infty W_x^{8+\epsilon-i,\infty}} \|\partial_t^j u\|_{L_t^2 W_x^{9+\frac{\epsilon}{2}-j,\infty}} \\ &\lesssim \|\partial_t^i u\|_{L_t^2 W_x^{9+\frac{\epsilon}{2}-i,\infty}} \|\partial_t^j u\|_{L_t^\infty H_x^{s-j}} \\ &+ \|\partial_t^i u\|_{L_t^\infty H_x^{s-i}} \|\partial_t^j u\|_{L_t^2 W_x^{9+\frac{\epsilon}{2}-j,\infty}} \\ &\lesssim \|u\|_{S_T}^2. \end{aligned}$$

Our next task is to bound $\|\partial_t U\|_{L_t^\infty H_x^{s-1}} + \|\partial_t U\|_{L_t^2 W_x^{8+\frac{\epsilon}{2},\infty}}$. Observe that

$$\partial_t U = \sum_{i,j=0}^{1} \mathcal{B}_{ij}[\partial_t^{i+1} u][\partial_t^j u] + \mathcal{B}_{ij}[\partial_t^i u][\partial_t^{j+1} u].$$

By symmetry, it suffices to show that

$$\|\mathcal{B}_{ij}[\partial_t^{i+1}u][\partial_t^j u]\|_{L_t^\infty H_x^{s-1}} + \|\mathcal{B}_{ij}[\partial_t^{i+1}u][\partial_t^j u]\|_{L_t^2 W_x^{8+\frac{\epsilon}{2},\infty}} \lesssim \|u\|_{S_T}^2 + \|u\|_{S_T}^3.$$

Applying Proposition 6.3.1 with $a = 2-i, b = 2-j, c = -2, d = 6, \lambda = \tilde{\lambda} = \epsilon, \sigma = 0, \tilde{\sigma} = 1, p_1 = p_2 = \tilde{p}_1 = \tilde{p}_2 = \infty, q_1 = \tilde{q}_2 = 2, q_2 = \tilde{q}_1 = \infty$ we deduce

$$\begin{aligned}\|\mathcal{B}_{ij}[\partial_t^{i+1}u][\partial_t^j u]\|_{L_t^\infty H_x^{s-1}} &\lesssim \|\partial_t^{i+1}u\|_{L_t^\infty H_x^{s-1-i}}\|\partial_t^j u\|_{L_t^\infty W_x^{8+\epsilon-j,\infty}} \\ &\quad + \|\partial_t^{i+1}u\|_{L_t^\infty W_x^{7+\epsilon-i,\infty}}\|\partial_t^j u\|_{L_t^\infty H_x^{s-j}}.\end{aligned}$$

We once again use the fact that $s-(8+\epsilon) > \frac{3}{2}$ to apply Sobolev embedding and bound the above by

$$\begin{aligned}&\lesssim \|\partial_t^{i+1}u\|_{L_t^\infty H_x^{s-1-i}}\|\partial_t^j u\|_{L_t^\infty H_x^{s-j}} \\ &\lesssim \|\partial_t^{i+1}u\|_{L_t^\infty H_x^{s-1-i}}\|u\|_{S_T}.\end{aligned}$$

It remains to show that

$$\|\partial_t^{i+1}u\|_{L_t^\infty H_x^{s-1-i}} \lesssim \|u\|_{S_T} + \|u\|_{S_T}^2. \tag{7.25}$$

When $i = 0$ this is obvious from the definition of $\|\cdot\|_{S_T}$ so we only need to consider the case $i = 1$. In this case the desired bound follows from (7.18). We remark that the above proof also implies (7.22).

In order to bound $\|\mathcal{B}_{ij}[\partial_t^{i+1}u][\partial_t^j u]\|_{L_t^2 W_x^{8+\frac{\epsilon}{2},\infty}}$, we once again apply Proposition 6.3.1 with $a = 2-i, b = 2-j, c = -2, d = 6, \lambda = \tilde{\lambda} = \frac{\epsilon}{2}, \sigma = \tilde{\sigma} = 0, p_1 = \tilde{p}_1 = 2, p_2 = \tilde{p}_2 = q_1 =$

$q_2 = \tilde{q}_1 = \tilde{q}_2 = \infty$ to obtain

$$\begin{aligned}
\|\mathcal{B}_{ij}[\partial_t^{i+1}u][\partial_t^j u]\|_{L_t^2 W_x^{8+\frac{\epsilon}{2},\infty}} &\lesssim \|\partial_t^{i+1}u\|_{L_t^2 W_x^{8+\frac{\epsilon}{2}-i,\infty}} \|\partial_t^j u\|_{L_t^\infty W_x^{8+\frac{\epsilon}{2}-j,\infty}} \\
&\lesssim \|\partial_t^{i+1}u\|_{L_t^2 W_x^{8+\frac{\epsilon}{2}-i,\infty}} \|\partial_t^j u\|_{L_t^\infty H_x^{s-j}} \\
&\lesssim \|\partial_t^{i+1}u\|_{L_t^2 W_x^{8+\frac{\epsilon}{2}-i,\infty}} \|\partial_t^j u\|_{S_T}.
\end{aligned}$$

So, it's left to show that

$$\|\partial_t^{i+1}u\|_{L_t^2 W_x^{8+\frac{\epsilon}{2}-i,\infty}} \lesssim \|u\|_{S_T} + \|u\|_{S_T}^2.$$

When $i = 0$ this is obvious from the definition of $\|\cdot\|_{S_T}$ so it remains to consider the case $i = 1$. In this case the desired bound again follows from (7.18). □

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