Submitted to Physical Review Letters

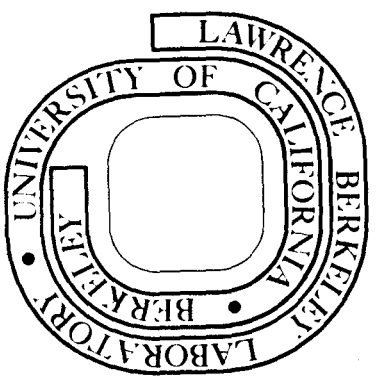
LBL-3079
Preprint C. 1

VACUUM POLARIZATION IN HEAVY ION COLLISIONS

Miklos Gyulassy

July 10, 1974

Prepared for the U. S. Atomic Energy Commission
under Contract W-7405-ENG-48



DISCLAIMER

VACUUM POLARIZATION IN HEAVY ION COLLISIONS*

Miklos Gyulassy

Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

July 10, 1974

ABSTRACT

The results of a study on vacuum polarization, orders $\alpha(Z\alpha)^n$, $n \geq 3$, for large Z systems encountered in heavy ion collisions are presented here. It is shown that the higher order vacuum polarization cannot prevent the $1S_{\frac{1}{2}}$ state from reaching the lower continuum, $E = -m_e c^2$, for some critical charge $Z_{cr} \sim 170$. In addition, the stability and localization of a helium-like system for $Z > Z_{cr}$ is demonstrated.

An interesting application of heavy ion collisions is to the study of quantum electrodynamics of strong fields. For short times, at least, systems with large effective charge Z will be formed with $Z\alpha > 1$. In the strong fields of such systems, highly relativistic electronic bound states are expected to occur with binding energies B exceeding the electron rest mass m_e, and for some critical charge, $Z_{cr} \sim 170$, the $1S_{\frac{1}{2}}$ state is expected to reach the lower continuum with $B = 2m_e$.[1] For $Z > Z_{cr}$, it has been predicted[2,3] that spontaneous e^+e^- pair production will occur with the subsequent capture of two electrons into a tightly bound helium-like state and the ejection of two positrons into continuum states. These predictions

have been based on solutions of the Dirac equation for finite size nuclei in which radiative corrections such as vacuum polarization (VP) and the Lamb shift are ignored. For these predictions to be applicable to heavy ion collisions, it is essential to show that radiative corrections are indeed negligible for large $Z \sim Z_{cr}$ and thus cannot prevent the $1S_{\frac{1}{2}}$ state from reaching the lower continuum. Several qualitative arguments[2,3] and a model calculation based on effective limiting field Lagrangians[4] suggest that the effect of VP does remain small up to Z_{cr}. Yet, a complete quantum electrodynamic calculation had not been carried out to all orders in $Z\alpha$. The purpose of this Letter, then, is to present the results of a numerical calculation of the higher order VP, orders $\alpha(Z\alpha)^n$, $n \geq 3$, based on the methods of Wichmann and Kroll.[5] The results confirm that VP remains a small perturbation even up to Z_{cr}.

Although the effect of higher order VP is always much less than that of the first order (Uehling) potential in atoms with $Z \leq 100$, the results of Wichmann and Kroll[5] for a point nucleus show that the size of the higher order VP increases sharply near $Z\alpha = 1$. If the VP charge accumulated at the origin for orders $\alpha(Z\alpha)^n$, $n \geq 3$, is denoted by Q_{WK}^{3+}, then while Q_{WK}^{3+} is finite and much smaller than the nuclear charge when $Z\alpha = 1$ ($|Q_{WK}^{3+}| \approx 0.05|e| \ll 137|e|$), $dQ_{WK}^{3+}/dZ = -\infty$ at $Z\alpha = 1$. Furthermore, the infinite slope of Q_{WK}^{3+} can be seen to come from the lowest angular momentum $\left(k = \pm(j + \frac{1}{2}) = \pm 1\right)$ contribution to the VP density. The higher angular momentum ($|k| \geq 2$) contribution to Q_{WK}^{3+} is seen to vary smoothly past $Z\alpha = 1$ until $Z\alpha = 2$, where the $|k| = 2$ contribution becomes singular. Of course, these singularities in the VP charge density arise because of the assumed point structure of the

nucleus. Nevertheless, these results indicate that we may expect a rapid increase in the higher order VP for $Z\alpha > 1$ when the finite size of the nucleus is taken into account. Furthermore, the fastest growing part should be that due to the lowest angular momentum ($|k| = 1$) electron loops. Another important property that can be deduced from the results of Ref. 5 is the relative size of the $|k| = 1$ to the $|k| \geq 2$ contributions to the VP charge density. The $|k| \geq 2$ contribution was found to be always less than 10% of the $|k| = 1$ contribution for orders $\alpha(Z\alpha)^n$, $n \geq 3$, for all $Z\alpha \leq 1$.[6,7] Therefore, a good approximation in the calculation of higher order VP, relevant for heavy ion collisions, is obtained by calculating the $|k| = 1$ contribution exactly for finite size nuclei while neglecting the $|k| \geq 2$ contribution. In this way tne dominant and fastest growing part of the VP density is calculated exactly while leaving an overall uncertainty of less than 10% in the final results due to neglect of the $|k| \geq 2$ contribution.

The calculation of the VP density, ρ_{VP}, involves an energy contour integral of the trace of the Green's function, TrG, for the Dirac equation.[5] For $Z < Z_{cr}$, the choice of this contour is clear and is given by C_0 in Fig. 1. With this contour, ρ_{VP} is equal to the vacuum expectation value of the Heisenberg current operator, $\langle 0|J_0(x)|0\rangle$. Thus to first order in α and to all orders in $Z\alpha$, ρ_{VP} can be written formally as

$$\rho_{VP} = \frac{|e|}{2}\left\{\sum_{+}\psi_{+}^{2} - \sum_{-}\psi_{-}^{2}\right\}, \qquad (1)$$

where $\psi_{\pm}$ refer to the positive and negative energy eigenfunctions of the Dirac equation.[8] Here positive energy refers to all eigenvalues

greater than $-m_e$. In particular, the $1S_{\frac{1}{2}}$ state is contained in the first sum. The helium-like charge density, ρ_{He}, is defined as the expectation value of the current operator for a state of two electrons in the $1S_{\frac{1}{2}}$ state: $\langle 2e^-(1S)|J_0(x)|2e^-(1S)\rangle$. This density is related to ρ_{VP} by

$$\rho_{He} = -2|e|\psi^2_{1S_{\frac{1}{2}}} + \rho_{VP} \quad , \tag{2}$$

which is equivalent to the addition of a counter-clockwise contour around the $1S$ pole to C_0 in Fig. 1. The sum of these contours may then be deformed to C_{He}. For $Z < Z_{cr}$, then, ρ_{VP} and ρ_{He} are computed via a contour integral along paths C_0 and C_{He} respectively (in units of $|e|$).

For $Z > Z_{cr}$, the $1S_{\frac{1}{2}}$ pole moves off the physical sheet through the branch point of the lower continuum. Since the vacuum around the bare nucleus can then decay into a helium-like state plus two free positrons,[2,3] it is natural to redefine the vacuum to correspond to the helium-like state.[9] The VP density, ρ_{VP}, is thus defined to equal ρ_{He} for $Z > Z_{cr}$. While Eq. (2) is no longer meaningful for $Z > Z_{cr}$, the contour integral representation for ρ_{He} around path C_{He} in Fig. 1 is still well defined. The charge distribution of the overcritical vacuum is thus calculated with contour C_{He}.

The higher order VP density, ρ^{3+}, for $k = \pm 1$ and $Z < Z_{cr}$ is calculated by expanding TrG in terms of radial Green's functions, TrG_k, and removing the first order Green's functions, TrG_k^1. Isolating the terms corresponding to $k = \pm 1$, the contour integral over path C_0 is performed. In practice, C_0 is deformed to the imaginary axis I, picking up the negative of the residues of all poles of $TrG_{k=\pm 1}$ which lie between zero and $-m_e$. The negative of

those residues are, of course, the modulus squared of the normalized bound state wavefunctions. The calculation of ρ_{He} involves adding to the contour integral along I all but the $1S_{\frac{1}{2}}$ wave functions squared with $m_e < B < 2m_e$.

In addition to ρ^{3+}, the third order, $\alpha(Z\alpha)^3$, VP density, ρ^3, is calculated for $k = \pm 1$ and $Z < Z_{cr}$ to provide a check on internal consistency and to estimate the dependence of ρ^{3+} on different nuclear charge densities. The calculation of ρ^3 involves the contour integral of the third order Green's function, TrG_k^3, for $k = \pm 1$ along contour I. Two models for the nuclear charge densities were used for calculating ρ^3: (I) a shell density, $\rho_{Nuc} = \delta(r - R)/4\pi R^2$ and (II) a uniform density, $\rho_{Nuc} = \theta(R - r)/(4\pi R^3/3)$. The nuclear radius was chosen to be 10 fm in both models. The densities ρ^{3+} and ρ_{He} were calculated with model I.

The construction of TrG_k, TrG_k^1, and TrG_k^3 is the same here as in Ref. 6 and will be discussed in more detail in a subsequent paper.

The following tests check the numerical accuracy of the constructed Green's functions. First, the location of the $1S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ poles were computed as a function of Z. The values of $(Z\alpha)_{cr}$ were determined for model I nuclei for R = 8, 10, and 12 fm, with $(Z\alpha)_{cr} = 1.25189$, 1.27459, and 1.29530 respectively for the $1S_{\frac{1}{2}}$ state, and $(Z\alpha)_{cr} = 1.383$ for R = 10 fm, for the $2P_{\frac{1}{2}}$ state, in agreement with Ref. 3. Furthermore, in agreement with Ref. 2, dB/dZ at $Z = Z_{cr}$ was calculated to be 27 keV for the $1S_{\frac{1}{2}}$ state and 35 keV for the $2P_{\frac{1}{2}}$ state. Secondly, the residues at those poles were calculated in order to check that the normalized bound state wave functions were given correctly. All $S_{\frac{1}{2}}$ and $P_{\frac{1}{2}}$ wavefunctions

calculated thereby were normalized to unity to better than one part per 10^5. The $R \to 0$ limit was taken numerically for $Z\alpha < 1$ to check that these wavefunctions reduce to the point nucleus form. In addition, the $\langle 1/r \rangle$ was computed for these wavefunctions since $dB/dZ \approx \alpha m_e c^2 \langle 1/r \rangle$ which can be compared to the values computed above; at $Z = Z_{cr}$, e.g., $dB/dZ = 28$ keV and 37 keV for the $1S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ states respectively, in good agreement with the above values. Thirdly, for the special case of $Z = 82$, extensive checks on the charge densities were made showing that the limit $R \to 0$ and the finite nuclear size effect agreed with other calculations.[6] Finally, the total space integral Q_{He} of ρ_{He} was computed for the range $0.6 \leq Z\alpha \leq 1.38$ since the extent to which $Q_{He} = -2|e|$ measures the accuracy of the numerical contour integration along I. The computed values of Q_{He} were equal to $-2|e|$ to better than 1 part per 10^4 over the entire range. The accuracy of the computed ρ^{3+} for $Z < Z_{cr}$ is less due to cancellations necessary to insure that the total charge of the vacuum vanish. This accuracy is estimated from the magnitude of the ratio of the integral of ρ^{3+} over all space ($r \lesssim 13\lambda\!\!\!^{-}_e$) to the integral over the range where ρ^{3+} is negative ($r \lesssim 100$ fm). This ratio was found to be $\lesssim 0.01$, indicating a numerical accuracy on the order of one percent.

The computed $1S_{\frac{1}{2}}$ energy shifts for $Z < Z_{cr}$ due to higher order VP are listed in Table I. These energy shifts should be compared to the shift due to the Uehling potential. The Uehling potential is attractive and increases the binding energy of the $1S_{\frac{1}{2}}$ state by approximately 10 keV at Z_{cr}.[1,10] The higher order VP is repulsive but is seen in Table I to reduce the binding energy by only 1 keV at Z_{cr}.

The last two lines, in particular, in Table I indicate the absence of any singularities of ΔE^{3+} at Z_{cr}. Thus, even though the shift ΔE^{3+} due to higher order VP increases rapidly for $Z\alpha > 1$, it remains too small to prevent the $1S_{\frac{1}{2}}$ state from reacning the lower continuum. Furthermore, the results for third order indicate that the dependence of ΔE^{3+} on the specific nuclear charge density is a 10% effect, i.e., on the same order as the uncertainty in ΔE^{3+} due to neglect of the $|k| \geq 2$ contribution to the VP density.

Since the results for $Z < Z_{cr}$ show that VP remains a small perturbation up to Z_{cr}, the use of the unperturbed Green's function, TrG_k and TrG_k^1, in computing $\rho^{3+} \equiv \rho_{He}$ for $Z > Z_{cr}$ will not lead to large errors. In Fig. 2(a), ρ_{He} is plotted for several values of $Z\alpha$ around $(Z\alpha)_{cr} = 1.27459$. The continuity of ρ_{He} at Z_{cr} was examined by calculating ρ_{He} for $Z\alpha = 1.2732$, 1.27445, 1.27545, and 1.28 and checking point by point in the range $0 \leq r \lesssim 13\lambda\!\!\!^{-}_e$ that the values of ρ_{He} for the different Z can be smoothly connected. The increased localization and continuity of ρ_{He} as a function of Z is illustrated in Fig. 2(b), where the average $\langle 1/r \rangle$ for ρ_{He} is plotted. These results demonstrate that the helium-like system is stable and well behaved around Z_{cr} and that the charge density of the overcritical vacuum is indeed highly localized.

It should be noted that for $Z\alpha = 1.383$, the $2P_{\frac{1}{2}}$ state reaches the lower continuum and the helium-like system will decay to a beryllium-like system plus two free positrons.[2,3] The charge density of the beryllium-like system can be studied in the same way as ρ_{He}, simply by shifting the contour C_{He} to the right of the $2P_{\frac{1}{2}}$ pole.

In this Letter the effect of VP in very high Z atoms has been considered. The effect of the Lamb shift has been estimated by other workers,[10] but agreement on the size of that effect has not yet been reached. More work is needed on that problem.

The author gratefully acknowledges helpful conversations with Dr. P. J. Mohr, Dr. W. Greiner, Dr. W. J. Swiatecki, and Dr. E. H. Wichmann.

FOOTNOTES AND REFERENCES

* This work was supported by the U. S. Atomic Energy Commission.

1. W. Pieper and W. Greiner, Z. Physik 218, 327 (1969).
2. B. Muller, J. Rafelski, and W. Greiner, Z. Physik 257, 62 (1972) and Z. Physik 257, 183 (1972).
3. Ya. B. Zel'Dovich and V. S. Popov, Sov. Phys.-Usp. 14, 673 (1972).
4. J. Rafelski, W. Greiner, and L. P. Fulcher, Nuovo Cimento 13, 135 (1973).
5. E. H. Wichmann and N. M. Kroll, Phys. Rev. 101, 843 (1956).
6. M. Gyulassy, Phys. Rev. Letters 32, 1393 (1974).
7. L. S. Brown, R. N. Cahn, and L. D. McLerran, University of Washington preprint, RLO-1388-677 (1974).
8. See, e.g., Ref. 3, p. 687.
9. L. Fulcher and A. Klein, Phys. Rev. D8, 2455 (1973).
10. G. Soff, B. Muller, and J. Rafelski, University of Pennsylvania preprint, UPR-0033N (1974).

Table I. The $1S_{\frac{1}{2}}$ energy shifts in eV due to vacuum polarization orders $\alpha(Z\alpha)^n$, with ΔE^{3+} for $n \geq 3$, and $\Delta E^3_{I,II}$ for $n = 3$, model I and II nuclei. $E_{1S_{\frac{1}{2}}}$ and $E_{2P_{\frac{1}{2}}}$ locate the bound state poles in units of m_e. The nuclear charge density for all but the last column was taken to be model I with $R = 10$ fm.

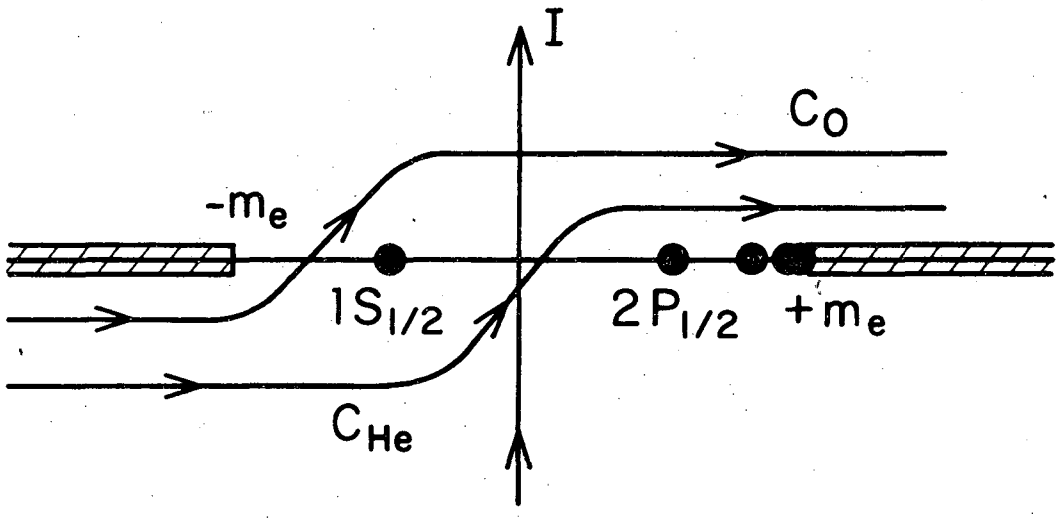
Z	$E_{1S_{\frac{1}{2}}}$	$E_{2P_{\frac{1}{2}}}$	ΔE^{3+}	ΔE_I^3	ΔE_{II}^3
0.95	0.362	0.817	6.26×10	4.70×10	4.92×10
1.12	0.137	0.570	3.07×10^2	1.97×10^2	2.11×10^2
1.205	-0.550	0.265	6.41×10^2	3.68×10^2	3.99×10^2
1.2732	-0.990	-0.118	1.14×10^3	5.66×10^2	6.20×10^2
1.27445	-0.999	-0.126	1.15×10^3	5.70×10^2	6.24×10^2

FIGURE CAPTIONS

Fig. 1. Singularities of the Green's function in the complex energy plane and contours C_0, C_{He}, and I giving the VP and helium-like charge densities in units of $|e|$.

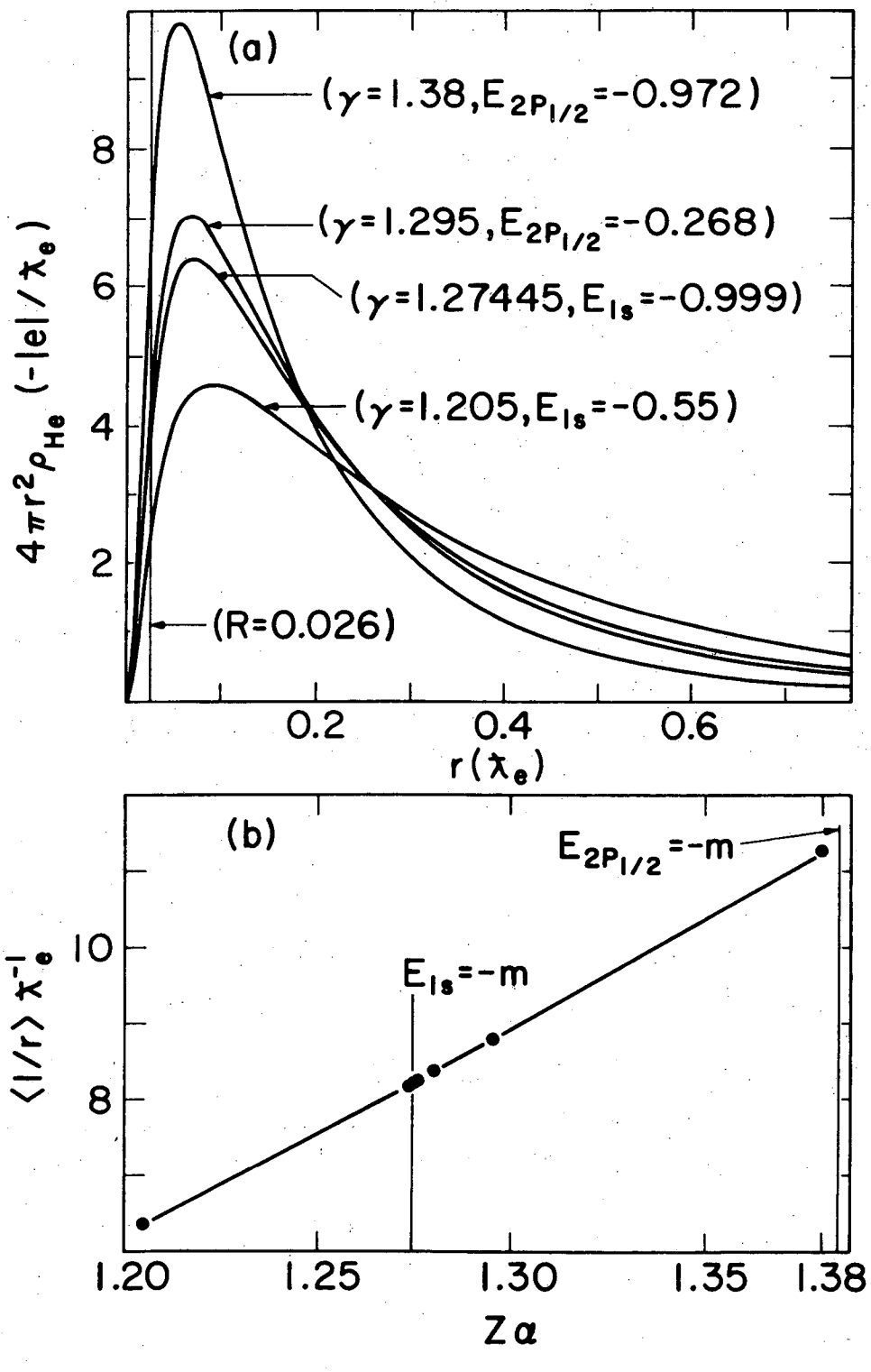
Fig. 2. (a) The helium-like charge density for several values of $\gamma \equiv Z\alpha$ around $(Z\alpha)_{cr} = 1.27459$ with a model I, R = 10 fm nucleus.

(b) The average $\langle 1/r \rangle$ for ρ_{He} as a function of $Z\alpha$.



XBL 747-3588

Fig. 1.



XBL 747-3589

Fig. 2.

TECHNICAL INFORMATION DIVISION
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720