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Publication Date
1995-05-01

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California PATH Working Paper
UCB-ITS-PWP-95-6

This work was performed **as** part of the California PATH Program of the University of California, in cooperation with the State of California Business, Transportation, and Housing Agency, Department of Transportation; and the United States Department Transportation, Federal Highway Administration.

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May 1995

ISSN 1055-1417

A Link-Based Variational Inequality Model for Dynamic Departure Time/Route Choice

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June 18, 1993
Revised February 11, 1995

PATH Working Paper No.D94-13

Executive Summary

The dynamic user-optimal (DUO) departure time and route choice problem is to determine travelers' best departure times and route choices at each instant of time. In a previous paper, we presented a route-based two-level optimal control model for the DUO departure time/route choice problem. However, this model is not appropriate for large scale transportation networks because some degree of route enumeration is necessary to solve the model. In this paper, we present a link-based variational inequality (VI) formulation for the DUO departure time/route choice problem so that route enumeration can be avoided in both the formulation and the solution procedure. The model extends our previous VI model for the DUO route choice problem to the case where both departure time and route over a general road network must be chosen simultaneously. By proving the necessity and sufficiency of this VI, we establish the equivalence of the VI formulation and the link-based DUO departure time/route choice conditions.

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1 Introduction

Advanced traveler information systems (ATIS) can provide real-time information regarding incidents and traffic congestion to assist travelers in selecting their best departure times and routes. The exploration of dynamic travel choice models has been motivated by the development of such ATIS systems. The objective of such models is to investigate the optimal strategies for choosing departure times, modes and routes in real-time.

In this paper, we consider an ideal situation where all travelers are equipped with navigation devices and fully comply with the dynamic user-optimal criterion when choosing routes and departure times based on predictive traffic information. We present a dynamic, user-optimal departure time and route choice model for a general network with multiple origin-destination pairs. We model this choice problem by specifying that a given number of travelers are ready for departure between each origin-destination pair at time zero. However, their departure times may be delayed to reduce their overall travel costs.

The choice of departure time has been addressed by several researchers, including Hendrickson and Plank (1984), who developed work trip scheduling models. De Palma et al (1983) and Ben-Akiva et al **(1984)** modeled departure time choice over a simple network with one bottleneck using the general continuous logit model. Mahmassani and Herman (1984) used a traffic flow model to derive the equilibrium joint departure time and route choice pattern over a parallel route network. Mahmassani and Chang (1987) further developed the concept of equilibrium departure time choice and presented the boundedly-rational user-equilibrium concept under which all drivers in the system are satisfied with their current travel choices, and thus feel no need to improve their outcome by changing decisions.

The above effort has been limited to solving departure time choice problems for simple networks. In order to tackle departure time choice problems for larger networks, several models have recently been proposed by various researchers using different approaches on dynamic traffic networks. Janson (1992) formulated a dynamic user-equilibrium traffic assignment model in which trips have variable departure times and scheduled arrival times. Ran et al (1992) formulated a two-level optimal control program for the dynamic user-optimal (DUO) departure time/route choice problem for a multiple origin-destination network. Friesz et al (1993) presented a joint departure time and route choice model using the variational inequality approach. Smith and Ghali (1992) also considered this problem using microscopic representation of vehicle streams.

Similar to static transportation network formulations, the variational inequality (VI) approach could provide general formulations for dynamic transportation network problems, compared to mathematical program and optimal control approaches. The earliest variational inequality problem was a static user-optimal route choice problem, which was formulated by Smith (1979). Later on, Dafermos (1980) developed an elastic demand model with disutility functions using the variational inequality approach. An elastic demand model with demand functions was introduced by Dafermos and Nagurney (1984b). Fisk and Boyce (1983) also presented a set of alternative VI formulations for network equilibrium travel choice problems. Nagurney (1993) summarized the modeling and algorithmic aspects of VI models for static traffic assignment problems. Recently, Friesz et al (1993) formulated a VI model for the simultaneous departure time/route choice problem. Smith (1993) also presented a route-based VI formulation using the packet representation of vehicle groups. Both dynamic models are route-based, which need explicit route enumeration in both formulation and solution.

Because the dynamic traffic flow does not have constant flow rate during propagation over links and routes, the route-based VI can not be transformed into a link-based VI. Thus, it is very difficult to develop a solution algorithm for a route-based VI without explicit route enumeration. Figure 1 shows a 5×5 one-way square grid network with $N = 25$ nodes and $L = 40$ links. We count the routes from node 1 to node 25 (both are on the diagonal line) and the total number of routes is 70.

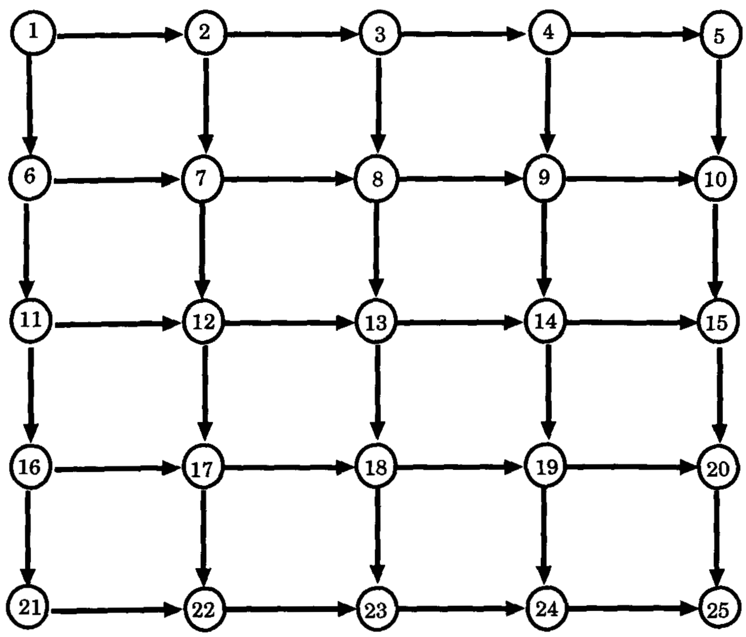

Figure 1: An Example Grid Network

Table 1 illustrates the increase of links and routes with the increase of nodes in such a grid network. Basically, the number of links increases linearly with the increase of nodes. However, the number of routes increases exponentially with the increase of nodes. For example, when there are only $N = 100$ nodes, the number of routes is 48,620. When there are $N = 400$ nodes, the number of routes is over 3.5×10^{10}. We note that the routes in these one-way grid networks are efficient routes in terms of Dial's definition (Dial, 1971), i.e., any node on the route takes the vehicles further away from the origin and closer to the destination. From Table 1, we can conclude that explicit route enumeration is infeasible for large networks.

Table 1: Number of Nodes, Links and Routes

Number of Nodes N	4	9	16	25	36	49	64	100	400
Number of Links L	4	12	24	40	60	84	112	180	760
. Number of Routes	2	6	20	70	252	924	3432	48,620	3.5×10^{10}

Recently, Ran and Boyce (1995) presented a link-based VI model for ideal DUO route choice problem so that explicit route enumeration can be avoided in both the formulation and the solution procedure. This approach allows the dynamic VI route choice model to be applied to realistic transportation networks. Using a similar approach, we extend the dynamic route choice model to include departure time choice as well. A link-based ideal dynamic user-optimal (DUO) departure time/route choice model is presented for a network with multiple origin-destination pairs in this paper. Since this VI model is link-based, it has computational advantages over the route-based models.

In Section 1, the network constraints for the dynamic traffic network model are first introduced. In Section 2, we present the definition of DUO and its corresponding DUO departure time/route choice conditions. The dynamic traffic network constraints are summarized in Section 3. Then, a general link-based variational inequality formulation of the DUO departure time/route choice problem is proposed. Proofs of necessity and sufficiency are given to establish the equivalence of the VI model and the link-based DUO departure time/route choice conditions. Finally, discussion on the VI departure time/route choice model is presented and some future studies are proposed.

2 Dynamic Network Constraints

Here, we consider a network with multiple origins and destinations. The traffic network is represented by a directed graph with nodes and directed links. A node can represent either an origin or a destination, or simply an intersection. The index r denotes an origin node and the index s denotes a destination node.

Consider a fixed time period $[0,T]$ where T is the time sufficient for all persons departing during the peak period to complete their trips. We define

$x_a(t)$ = number of vehicles traveling on link a at time t;

$x_a^{rs}(t)$ = number of vehicles traveling on link a with origin r and destination s at time t.

All variables with superscripts rs denote the variables with origin r and destination s. We have by definition that

$$\sum_{rs} x_a^{rs}(t) = x_a(t) \qquad \forall a. \tag{1}$$

Let $u_a(t)$ denote the inflow rate into link a at time t and $v_a(t)$ denote the exit flow rate from link a at time t. The inflows and exit flows, $u_a(t)$ and $v_a(t)$, are both control variables. The state variable for link a is the number of vehicles $x_a(t)$ on link a. The state equation for link a can then be written as

$$\frac{dx_a^{rs}(t)}{dt} = u_a^{rs}(t) - v_a^{rs}(t) \qquad \forall a, r, s. \tag{2}$$

We assume that the number of vehicles on link a at initial time $t = 0$ equals zero:

$$x_a^{rs}(0) = 0, \qquad \forall a, r, s. \tag{3}$$

Thus, the number of vehicles on link a at any time t is

$$x_a^{rs}(t) = \int_0^t [u_a^{rs}(\omega) - v_a^{rs}(\omega)]d\omega \qquad \forall a, r, s. \tag{4}$$

We require that all variables are nonnegative at all times:

$$x_a^{rs}(t) \geq 0, \quad u_a^{rs}(t) \geq 0, \quad v_a^{rs}(t) \geq 0, \qquad \forall a, r, s. \tag{5}$$

Denote the departure rate from origin node r toward destination node s at time t as $f^{rs}(t)$, which is a function of time; $f_p^{rs}(t)$ denotes the departure rate on route p. $f_p^{rs}(t)$ and $f^{rs}(t)$ are control variables to be determined according to the actual travel time between the origin and the destination. The flow conservation at node j $(j \neq r, \mathrm{s})$ for each O-D pair requires that the flow exiting from links pointing into node j at time t equals the flow entering links which leave node j at time t. Thus, the flow conservation equations can be expressed **as**

$$\sum_{a \in B(j)} v_a^{rs}(t) = \sum_{a \in A(j)} u_a^{rs}(t) \qquad \forall j \neq r, s, \tag{6}$$

where $A(j)$ is the set of links exiting node j, and $B(j)$ is the set of links entering node j.

Assume there are P routes from origin r to destination s (these can be generated as needed). Denote the indicator parameters δ_{ap}^{rs} as

$$\delta_{ap}^{rs} = \begin{cases} 1 & \text{if link } a \text{ is on route } p \text{ between O-D pair } (r, \mathrm{s}) \\ 0 & \text{otherwise.} \end{cases}$$

Flow conservation at origin node r relates the departure rates $(f^{rs}(t)$ and $f_p^{rs}(t))$ to the flow entering each link emanating from the origin. These flow conservation equations for origin r can be expressed as

$$\sum_P \delta_{ap}^{rs} f_p^{rs}(t) = u_a^{rs}(t) \qquad \forall r, s; r \neq s; a \mathbf{E} A(r); \tag{7}$$

$$\sum_{p} f_p^{rs}(t) = f^{rs}(t) \qquad \forall r, s; r \neq s. \tag{8}$$

Denote the *cumulative* number of vehicles departing from origin r to destination s from time 0 to t as the state variable $F^{rs}(t)$. In our joint departure time and route choice problem, a given number of vehicles are scheduled to depart from each origin r to each destination s at the initial time 0. But some may delay their departure times. Thus, we have additional boundary conditions as follows:

$$\int_0^T f^{rs}(t)dt = F^{rs}(T) \qquad \text{is given} \qquad \forall r, s. \tag{9}$$

Also, denote $F_p^{rs}(t)$ as the cumulative number of vehicles that have departed from origin r toward destination s along route p. Then, we have an additional state equation for each origin r

$$\frac{dF_p^{rs}(t)}{dt} = f_p^{rs}(t) \qquad \forall p, r \neq s, s. \tag{10}$$

Also, at initial time $t = 0$,

$$F_p^{rs}(0) = 0, \qquad \forall p, r, s. \tag{11}$$

Denote the arrival flow rate at destination node s from origin node r at time t as $e^{rs}(t)$, which is also a control variable. The control variable $e_p^{rs}(t)$ denotes the arrival rate on route p. Flow conservation at destination node s relates the arriving flow ($e^{rs}(t)$ and $e_p^{rs}(t)$) to the flow exiting each link leading to destination s at time t. Thus, the flow conservation equations for destination s can be expressed as

$$\sum_{p} \delta_{ap}^{rs} e_p^{rs}(t) = v_a^{rs}(t) \qquad \forall r, s; s \neq r; a \in B(s); \tag{12}$$

$$\sum_{p} e_p^{rs}(t) = e^{rs}(t) \qquad Vr,s; s \neq r. \tag{13}$$

Denote the *cumulative* number of vehicles arriving at destination s from origin r at time t by the state variable $E^{rs}(t)$; $E_p^{rs}(t)$ denotes the cumulative number of vehicles arriving at destination s from origin r along route p at time t. Thus, we have an additional state equation for each destination s

$$\frac{dE_p^{rs}(t)}{dt} = e_p^{rs}(t) \qquad \forall p, r, s \neq r. \tag{14}$$

At the initial time $t = 0$,

$$E_p^{rs}(0) = 0 \qquad \forall p, r, s. \tag{15}$$

These variables must be nonnegative at all times:

$$E_p^{rs}(t) \geq 0 \qquad F_p^{rs}(t) \geq 0 \qquad e_p^{rs}(t) \geq 0 \qquad f_p^{rs}(t) \geq 0 \qquad \forall p, r, s. \tag{16}$$

Finally, we define flow propagation constraints to ensure that entering and exiting flows as well as vehicles remaining on links, are consistent with the actual link travel times. Define $\tau_a(t)$ as the actual travel time over link a for vehicles entering link a at time t. $\tau_a(t)$ is assumed to be dependent on the number of vehicles $x_a(t)$, the inflow $u_a(t)$ and the exit flow $v_a(t)$ on link a at time t. Let $x_{ap}^{rs}(t)$ denote the number of vehicles on link a using route p between O-D pair rs at time t. By definition,

$$\sum_{rsp} x_{ap}^{rs}(t) = x_a(t) \qquad \forall a. \tag{17}$$

For any intermediate node j # r on route p, denote a subroute $\tilde{p}$ as the section of route p from node j to destination s. For any link a E $B(j)$, vehicles on link a using route p at any time t must result in either:

1. extra vehicles on downstream links on subroute $\tilde{p}$ at time $t + \tau_a(t)$, or
2. increased exiting vehicles at the destination at time $t + \tau_a(t)$.

It follows that

$$x_{ap}^{rs}(t) = \sum_{b \in \tilde{p}} \{x_{bp}^{rs}[t+\tau_a(t)] - x_{bp}^{rs}(t)\} + \{E_p^{rs}[t+\tau_a(t)] - E_p^{rs}(t)\} \qquad \forall r, s, p, j, a; j \neq r; a \in B(j). \tag{18}$$

We refer the reader to Ran et al (1993) for more details.

Furthermore, two other formulations of the flow propagation constraints (18) could be provided to help readers understand the properties of these constraints. For link a, the summation of the cumulative number of vehicles entering link a during time period $[0,t]$ and the number of existing vehicles at initial time 0 must equal to the cumulative number of vehicles exiting link a during time period $[0,t+\tau_a(t)]$. It follows that

$$\int_0^t u_a^{rs}(\omega)d\omega + x_a^{rs}(0) = \int_0^{t+\tau_a(t)} v_a^{rs}(\omega)d\omega \qquad \forall a, r, s. \tag{19}$$

Taking derivative of the above equation with respect to time t, it follows that

$$u_a^{rs}(t) = \left[1+\frac{\partial\tau_a(t)}{\partial t}\right] v_a^{rs}[t+\tau_a(t)] \qquad \forall a, r, s. \tag{20}$$

Similarly, the number of vehicles on link a at time t must equal to the cumulative number of vehicles exiting link a during time period $[t,t+\tau_a(t)]$. It follows that

$$x_a^{rs}(t) = \int_t^{t+\tau_a(t)} v_a^{rs}(\omega)d\omega \qquad \forall a, r, s. \tag{21}$$

Equations (20) and (21) are two other types of link flow propagation constraints, which are equivalent to (18). The merit of (20) is that this constraint depends only on in-flow and exit flow, and the link flow conservation constraint (2) is combined in this constraint. Thus, the variable $x_a^{rs}(t)$ and any constraint associated with $x_a^{rs}(t)$ can be avoided in the formulation. The merit of (21) is to show that the $x_a^{rs}(t)$ are always nonnegative as long as the $v_a^{rs}(t)$ are nonnegative, allowing thereby to skip the nonnegativity constraint for $x_a^{rs}(t)$. However, in order to maintain consistency with our other dynamic network models, the flow propagation constraint (18) is used in this paper. Table 2 summarizes our notation, which is identical to that in Ran et al (1993), except for the addition of the variables $f^{rs}(t)$ and $F^{rs}(t)$.

Table 2: Summary of Decision Variables

$x_a(t)$	number of vehicles traveling on link a at time t *
$u_a(t)$	inflow rate into link a at time t **
$v_a(t)$	exit flow rate from link a at time t **
$f^{rs}(t)$	departure rate from origin r to destination s at time t **
$F^{rs}(t)$	cumulative number of vehicles departing from origin r to destination s at time t *
$e^{rs}(t)$	arrival rate at destination s from origin r at time t **
$E^{rs}(t)$	cumulative number of vehicles arriving at destination s from origin r at time t *
$\tau_a(t)$	actual travel time over link a for vehicles entering link a at time t

*: state variable

**: control variable

3 The Combined Departure Time/Route Choice Problem

A number of vehicles are ready to depart at the initial time 0, but these drivers may prefer to delay their departure times in order to reduce their driving time. Drivers are assumed to make their departure time choices so as to minimize their individual disutility functions defined on travel time and pre-trip delay. Of course, the change of departure flow rates will change the traffic in the network so that the travel times for other travelers could change.

In reality, drivers' choices of departure time and route are interrelated decisions. Given a desired arrival time, say at the workplace, choice of departure time depends on the driver's estimate of en route travel time. Likewise, choice of route depends on the travel times of alternative routes, which also may vary by time of day.

3.1 Departure Time Choice

We first consider the departure time choice problem. **A** disutility function $\mathcal{U}^{rs}(t)$ based on departure times is defined for travelers departing from origin r to destination s at time t. This disutility function represents a weighted sum of

1. waiting time at the origin node;
2. driving time during the trip;
3. a "bonus" for early arrival or a penalty for late arrival.

Denote $\tau^{rs}(t)$ as the minimal travel time experienced by vehicles departing from origin r to destination s at time t. $\tau^{rs}(t)$ is a functional of all link flow variables at time t, i.e., $\tau^{rs}(t) = \tau^{rs}[u(\omega), v(\omega), x(\omega), t]$ where $w \geq t$. This functional is neither a state variable nor a control variable, and it is not fixed; moreover, it is not available in closed form. Nevertheless, it can be evaluated when $u(\omega)$, $v(\omega)$ and $x(\omega)$ are temporarily fixed (as in a Frank-Wolfe algorithm), which is all that is required for solving the model.

We define one unit of disutility to equal one unit of in-vehicle driving time, and one unit of waiting time prior to departure to be equivalent to a units of disutility $(a \leq 1)$. a could be negative since staying at home has positive utility. Since all travelers are able to depart at time 0, at is the disutility for a departure at time t due to waiting. Furthermore, we assume there is a desired arrival time interval $[t^*_{rs} - \Delta_{rs}, t^*_{rs} + \Delta_{rs}]$ for travelers at each destination s, where t^*_{rs} is the center of the required arrival time interval (e.g. work start time) associated with travelers departing from origin r toward destination s. Δ_{rs} represents the arrival time flexibility at destination s for travelers departing from origin r toward destination s.

We also define the disutility for early or late arrival as follows

$$\mathcal{V}^{rs}[t, \tau^{rs}(t); t^*_{rs}] =$$

$$
\begin{cases} \gamma_1[t+\tau^{rs}(t)-t^*_{rs}+\Delta^*_{rs}]^2 & \text{if } t+\tau^{rs}(t) < t^*_{rs}-\Delta^*_{rs} \qquad \text{(Early arrival)} \\ 0 & \text{if } |t+\tau^{rs}(t)-t^*_{rs}| \leq \Delta^*_{rs} \\ \gamma_2[t+\tau^{rs}(t)-t^*_{rs}-\Delta^*_{rs}]^2 & \text{if } t+\tau^{rs}(t) > t^*_{rs}+\Delta^*_{rs} \qquad \text{(Late arrival)} \end{cases}
$$

where t is the departure time of travelers and γ_1, γ_2 are parameters $(\gamma_2 \gg a)$. γ_1 is negative because early arrival should be encouraged instead of discouraged. This arrival time disutility function is shown in Figure 2. Thus, the disutility function for the joint departure time and route choice problem is constructed as

$$
\mathcal{U}^{rs}(t) = \alpha t + \tau^{rs}(t) + \mathcal{V}^{rs}[t, \tau^{rs}(t); t^*_{rs}] \qquad \forall r, s, \tag{22}
$$

where t is the departure time of travelers. Note that the arrival time disutility function $\mathcal{V}^{rs}$ can be dropped if no desired arrival time interval is prespecified.

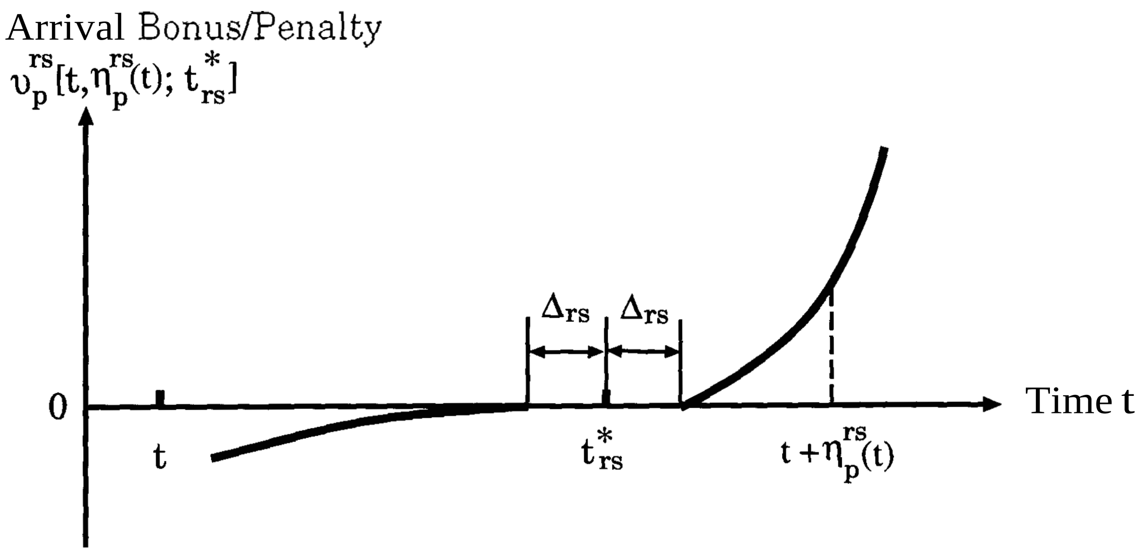

Figure 2: Arrival Time Disutility

The dynamic user-optimal departure time choice conditions require that for each O-D pair rs at any time t, if there is a positive departure flow $f^{rs}(t) > 0$, the disutility $\mathcal{U}^{rs}(t)$ must equal the minimal rs disutility $\mathcal{U}^{rs}_{min}$ over time t. Furthermore, if the departure flow $f^{rs}(t)$ equals zero at time t, the disutility $\mathcal{U}^{rs}(t)$ at time t must be greater than or equal to the minimal rs disutility $\mathcal{U}^{rs}_{min}$. The DUO departure time choice conditions can be written as

$$\mathcal{U}^{rs^*}(t) - \mathcal{U}^{rs}_{min} \geq 0 \qquad \forall r, s; \tag{23}$$

$$f^{rs^*}(t)\ \{\mathcal{U}^{rs^*}(t) - \mathcal{U}^{rs}_{min}\} = 0 \qquad \forall r, s; \tag{24}$$

$$f^{rs}(t) \geq 0 \qquad \forall r, s. \tag{25}$$

where the asterisk denotes that the travel disutility is computed using DUO departure flows.

3.2 Ideal DUO Route Choice

We then consider the route choice problem. The actual travel time $\tau_a[x_a(t), u_a(t), v_a(t)]$, or simply $\tau_a(t)$, over link a is assumed to be dependent on the number of vehicles $x_a(t)$, the inflow $u_a(t)$ and the exit flow $v_a(t)$ on link a at time t. We assume the travel time $\tau_a(t)$ on link a is the sum of two components: 1) a flow-dependent cruise time $g_{1a}[x_a(t), u_a(t)]$ over the uncongested part of link a and 2) a queuing delay $g_{2a}[x_a(t), v_a(t)]$ at the end of link a. It follows that

$$\tau_a(t) = g_{1a}[x_a(t), u_a(t)] + g_{2a}[x_a(t), v_a(t)]. \tag{26}$$

The two components $g_{1a}[x_a(t), u_a(t)]$ and $g_{2a}[x_a(t), v_a(t)]$ of the time-dependent link travel time function $\tau_a[x_a(t), u_a(t), v_a(t)]$ are assumed to be nonnegative and differentiable with respect to $x_a(t), u_a(t)$ and $x_a(t)$, t~,@), respectively.

Since we are considering a continuous time problem and assuming a link travel time function with queuing delay, the flow propagation constraints presented in Section 1 automatically guarantee that the first-in first-out (FIFO) requirement can be satisfied. We note that the FIFO requirement may be violated in a discrete time situation. In Ran (1993), it was suggested to define the time interval lengths and link lengths appropriately so that the FIFO constraint can also be satisfied in discrete models. We note that the traditional BPR functions are not applicable in a dynamic traffic network problem where time-dependent queuing and spillback problems occur. A set of time-dependent link travel time functions for signalized arterial links has been proposed by Ran et al (1992). Those link travel time functions are similar to the above general link travel time functions. It is our intention to employ realistic link travel time functions when our VI departure time/route choice model is implemented on realistic transportation networks.

Consider the flow which originates at node r at time t and is destined for node s. There is a set of routes $\{p\}$ between O-D pair (r, s). Define $\tau_p^{rs}(t)$ as the travel time actually experienced over route p by vehicles departing origin r toward destination s at time t. We use a recursive formula to compute the route travel time $\tau_p^{rs}(t)$ for all allowable routes. Assume route p consists of nodes $(r, 1, \cdots, i, j, \cdots, s)$. Denote $\tau_p^{rj}(t)$ as the travel time actually experienced over route p from origin r to node j by vehicles departing origin r at time t. Then, a recursive formula for route travel time $\tau_p^{rs}(t)$ is:

$$\tau_p^{rj}(t) = \tau_p^{ri}(t) + \tau_a[t + \tau_p^{ri}(t)] \qquad \forall p, r, j; j = 1, 2, \cdots, s;$$

where link $a = (i, j)$. Note that the actual link travel time $\tau_a(t)$ is determined by the present link flow variables, whereas the actual route travel time $\tau_p^{rj}(t)$ would depend on the future flow variables of downstream links as well.

We propose a definition of DUO that reflects the *ideal* route choice behavior of travelers as in Ran et al (1992). The formulation of the ideal DUO route choice problem will be based on the underlying choice criterion that each traveler uses the route that minimizes his/her actual travel time when departing from the origin or any intermediate node to his/her destination.

Ideal DUO: *If, for each O-D pair at each instant of time, the actual travel* *times experienced by travelers departing at the same time are equal and minimal, the dynamic traffic flow over the network is in an* ideal *dynamic user-optimal state.*

The ideal DUO is sometimes termed *predictive* **DUO**. Because it is associated with a predictive optimum state of the network traffic flow. Unlike in the previous dynamic route choice models, we now write the equivalent mathematical inequalities for the ideal **DUO** definition using link and node variables, as suggested by Ran and Boyce (1995).

Define $\tau^{ri^*}(t)$ as the minimal travel time actually experienced by vehicles departing origin r to node i at time t, the asterisk denoting that the travel time is computed using ideal **DUO** traffic flows. For link $a = (i,j)$, the minimal travel time $\tau^{rj^*}(t)$ from origin r to j should be equal to or less than the minimal travel time $\tau^{ri^*}(t)$ from origin r to i plus the actual link travel time $\tau_a[t + \tau^{ri^*}(t)]$ at time instant $[t + \tau^{ri^*}(t)]$. Furthermore, for each O-D pair rs, if any departure flow from origin r at time t enters link $a = (i,j)$ at the earliest clock time $[t + \tau^{ri^*}(t)]$, or $u_a^{rs}[t + \tau^{ri^*}(t)] > 0$, the ideal **DUO** route choice conditions require that the minimal travel time $\tau^{rj^*}(t)$ for vehicles departing origin r toward node j at time t should equal the minimal travel time $\tau^{ri^*}(t)$ for vehicles departing from origin r to i plus the actual link travel time $\tau_a[t + \tau^{ri^*}(t)]$ at time instant $[t + \tau^{ri^*}(t)]$. The link-based ideal **DUO** route choice conditions can be summarized as follows:

$$\tau^{ri^*}(t) + \tau_a[t + \tau^{ri^*}(t)] \geq \tau^{rj^*}(t) \qquad \forall a = (i,j), r; \tag{27}$$

$$\left[\tau^{ri^*}(t) + \tau_a[t + \tau^{ri^*}(t)] - \tau^{rj^*}(t)\right] \ u_a^{rs^*}[t + \tau^{ri^*}(t)] = 0 \qquad vu = (i,j), r, s; \tag{28}$$

$$u_a^{rs}[t + \tau^{ri^*}(t)] \geq 0 \qquad \forall a = (i,j), r, \text{s}. \tag{29}$$

We note that a similar set of link-based ideal DUO route choice conditions were proposed by Kuwahara and Akamatsu **(1993).** In their formulation, they use a different representation of departure/arrival times for traffic flows.

4 A Link-Based Variational Inequality Formulation

Define U as the set of link inflows $u_a^{rs}(\cdot)$ and departure flows $f^{rs}(\cdot)$, i.e., $U = (u, f)$. The constraint set K on U for our dynamic, user-optimal departure time/route choice problem is summarized as two parts:

The Flow Constraints H:

Relationships between state and control variables:

$$\frac{dx_a^{rs}}{dt} = u_a^{rs}(t) - v_a^{rs}(t) \qquad \forall a, r, s; \tag{30}$$

$$\frac{dE_p^{rs}(t)}{dt} = e_p^{rs}(t) \qquad \forall r, s, p; \tag{31}$$

$$\frac{dF_p^{rs}(t)}{dt} = f_p^{rs}(t) \qquad \forall r, s, p; \tag{32}$$

Flow conservation constraints:

$$\sum_p \delta_{ap}^{rs} f_p^{rs}(t) = u_a^{rs}(t) \qquad \forall r, s; a \in A(r); \tag{33}$$

$$\sum_p \delta_{ap}^{rs} e_p^{rs}(t) = v_a^{rs}(t) \qquad \forall r, s; a \in B(s); \tag{34}$$

$$\sum_{a \in B(j)} v_a^{rs}(t) = \sum_{a \in A(j)} u_a^{rs}(t) \qquad \forall r, s, j; j \neq r, s; \tag{35}$$

Definitional constraints:

$$\sum_{rs} u_a^{rs}(t) = u_a(t), \qquad \sum_{rs} v_a^{rs}(t) = v_a(t), \qquad \forall r, s; \tag{36}$$

$$\sum_p x_{ap}^{rs}(t) = x_a^{rs}(t), \qquad \sum_{rsp} x_{ap}^{rs}(t) = x_a(t), \qquad \sum_{rs} x_a^{rs}(t) = x_a(t), \qquad \forall r, s; \tag{37}$$

$$\sum_{P} E_p^{rs}(t) = E^{rs}(t), \qquad \sum_{P} F_p^{rs}(t) = F^{rs}(t), \qquad \forall r,s; \tag{38}$$

$$\sum_{P} f_p^{rs}(t) = f^{rs}(t), \qquad \sum_{P} e_p^{rs}(t) = e^{rs}(t), \qquad \forall r,s; \tag{39}$$

Nonnegativity conditions:

$$x_a^{rs}(t) \geq 0, \quad u_a^{rs}(t) \geq 0, \quad v_a^{rs}(t) \geq 0 \qquad \forall a,r,s; \tag{40}$$

$$e_p^{rs}(t) \geq 0, \quad f_p^{rs}(t) \geq 0, \quad E_p^{rs}(t) \geq 0, \quad F_p^{rs}(t) \geq 0 \qquad \forall p,r,s; \tag{41}$$

$$f^{rs}(t) \geq 0, \qquad F^{rs}(t) \geq 0 \qquad \forall r,s; \tag{42}$$

Boundary conditions:

$$F^{rs}(T) \qquad \text{given} \qquad \forall r,s; \tag{43}$$

$$E_p^{rs}(0) = 0, \quad F_p^{rs}(0) = 0 \quad \forall p,r,s; \qquad x_a^{rs}(0) = 0, \quad \forall a,r,s. \tag{44}$$

The Propagation Constraints P:

$$x_{ap}^{rs}(t) = \sum_{b \in \tilde{p}} \{x_{bp}^{rs}[t+\tau_a(t)] - x_{bp}^{rs}(t)\} + \{E_p^{rs}[t+\tau_a(t)] - E_p^{rs}(t)\} \qquad \forall r,s,p,j; a \in B(j); j \neq r; \tag{45}$$

The flow constraints H define a fixed positive cone, whereas the propagation constraint P consists of a linear relationship depending nonlinearly on the actual travel times $\tau_a(t)$, which are themselves dependent on some flow. In the flow constraints H, the first three constraints **(30)-(32)** are state equations for each link a and for cumulative effects at origins and destinations. Equations **(33)-(35)** are flow conservation constraints at each node including origins and destinations. Other constraints include definitional constraints, nonnegativity, and boundary conditions. In summary, the control variables are $f^{rs}(t)$, $u_a^{rs}(t)$, $v_a^{rs}(t)$, $e_p^{rs}(t)$, and $f_p^{rs}(t)$; the state variables are $F^{rs}(t)$, $x_a^{rs}(t)$, $E_p^{rs}(t)$, and $F_p^{rs}(t)$; the functionals are $\tau^{rs}(t)$.

For vehicles departing from origin r at time t, denote $\Omega_a^{rj^*}(t)$ as the difference between the minimal travel time from r to j and the travel time from origin r to node j via the minimal travel time route from origin r to node i and link a. It follows that

$$\Omega_a^{rj^*}(t) = \tau^{ri^*}(t) + \tau_a[t + \tau^{ri^*}(t)] - \tau^{rj^*}(t) \qquad \forall a, r; a = (i, j). \tag{46}$$

In order to simplify the presentation, we rewrite the combined link-based DUO departure time/route choice conditions as follows:

$$\Omega_a^{rj^*}(t) \geq 0 \qquad \forall a = (i, j), r; \tag{47}$$

$$u_a^{rs^*}[t + \tau^{ri^*}(t)]\, \Omega_a^{rj^*}(t) = 0 \qquad \forall a = (i, j), r, s; \tag{48}$$

$$u_a^{rs}[t + \tau^{ri^*}(t)] \geq 0 \qquad \forall a = (i, j), r, s; \tag{49}$$

$$\mathcal{U}^{rs^*}(t) - \mathcal{U}_{min}^{rs} \geq 0 \qquad \forall r, s; \tag{50}$$

$$f^{rs^*}(t)\{\mathcal{U}^{rs^*}(t) - \mathcal{U}_{min}^{rs}\} = 0 \qquad \forall r, s; \tag{51}$$

$$f^{rs}(t) \geq 0 \qquad \forall r, s. \tag{52}$$

where $\mathcal{U}_{min}^{rs}$ is the minimal rs disutility over time t. Then, the equivalent link-based variational inequality formulation of DUO departure time/route choice conditions (47)-(52) may be stated as follows.

Theorem 1. The dynamic traffic flow $U^* = (u^*, f^*)$ satisfying constraints (30)-(45) is in a DUO departure time/route choice state if and only if it satisfies the variational inequality:

$$\int_0^T \Big\{ \sum_{rs} \sum_a \Omega_a^{rj^*}(t) \big\{ u_a^{rs}[t + \tau^{ri^*}(t)] - u_a^{rs^*}[t + \tau^{ri^*}(t)] \big\}$$
$$+ \sum_{rs} \mathcal{U}^{rs^*}(t) \, \big\{ f^{rs}(t) - f^{rs^*}(t) \big\} \Big\} \, dt \geq 0 \tag{53}$$

$\forall\, U = (u, f)$ satisfying constraints (30)-(45).

Proof of Necessity.

We need to prove that DUO departure time/route choice conditions (47)-(52) imply the variational inequality (53). We first discuss the ideal DUO route choice conditions (47)-(49). Multiplying inequalities (47) and (49), we have

$$u_a^{rs}[t+\tau^{ri^*}(t)]\Omega_a^{rj^*}(t) \geq 0 \qquad \forall a, r, s; a=(i,j). \tag{54}$$

Subtracting equation (48) from inequality (54), we obtain

$$\left\{u_a^{rs}[t+\tau^{ri^*}(t)] - u_a^{rs^*}[t+\tau^{ri^*}(t)]\right\} \Omega_a^{rj^*}(t) \geq 0 \qquad \forall a, r, s; a=(i,j). \tag{55}$$

Summing inequality (55) for all links a and all O-D pairs rs, it follows that

$$\sum_{rs}\sum_{a}\left\{u_a^{rs}[t+\tau^{ri^*}(t)] - u_a^{rs^*}[t+\tau^{ri^*}(t)]\right\} \Omega_a^{rj^*}(t) \geq 0 \qquad \text{where } a=(i,j). \tag{56}$$

Integrating the above inequality (56) from time zero to T, we have

$$\int_0^T \sum_{rs}\sum_{a}\left\{u_a^{rs}[t+\tau^{ri^*}(t)] - u_a^{rs^*}[t+\tau^{ri^*}(t)]\right\} \Omega_a^{rj^*}(t)dt \geq 0 \tag{57}$$

We then discuss DUO departure time choice conditions (50)-(52). Multiplying inequalities (50) and (52), we have

$$f^{rs}(t)\,\{\mathcal{U}^{rs^*}(t) - \mathcal{U}_{min}^{rs}\} \geq 0 \qquad \forall r, s. \tag{58}$$

We subtract equation (51) from inequality (58) and obtain

$$\mathcal{U}^{rs^*}(t)\,\{f^{rs}(t) - f^{rs^*}(t)\} - \mathcal{U}_{min}^{rs}\,\{f^{rs}(t) - f^{rs^*}(t)\} \geq 0 \qquad \forall r, s. \tag{59}$$

Summing inequality (59) for all O-D pairs rs, it follows that

$$\sum_{rs}\mathcal{U}^{rs^*}(t)\,\{f^{rs}(t) - f^{rs^*}(t)\} - \sum_{rs}\mathcal{U}_{min}^{rs}\,\{f^{rs}(t) - f^{rs^*}(t)\} \geq 0 \tag{60}$$

Integrating the above inequality (60) from time zero to T, we have

$$\int_0^T \sum_{rs} \mathcal{U}^{rs^*}(t)\, \{f^{rs}(t) - f^{rs^*}(t)\}\, dt - \int_0^T \sum_{rs} \mathcal{U}^{rs}_{min}\, \{f^{rs}(t) - f^{rs^*}(t)\}\, dt \geq 0 \tag{61}$$

or

$$\int_0^T \sum_{rs} \mathcal{U}^{rs^*}(t)\, \{f^{rs}(t) - f^{rs^*}(t)\}\, dt - \sum_{rs} \mathcal{U}^{rs}_{min} \int_0^T \{f^{rs}(t) - f^{rs^*}(t)\}\, dt \geq 0 \tag{62}$$

By the definition of departure flows, we have

$$\int_0^T f^{rs}(t)\, dt = F^{rs}(T) = \int_0^T f^{rs^*}(t)\, dt$$

Thus, the second term of inequality (62) is zero and inequality (62) becomes

$$\int_0^T \sum_{rs} \mathcal{U}^{rs^*}(t)\{f^{rs}(t) - f^{rs^*}(t)\}\, dt \geq 0 \tag{63}$$

Combining inequalities (57) and (63), we obtain the variational inequality (53).

Proof of Sufficiency.

We need to prove that any solutions $u_a^{rs^*}[t + \tau^{ri^*}(t)]$ and $f^{rs^*}(t)$ to the variational inequality (53) satisfy the DUO departure time/route choice conditions (47)-(52). We know that the first and third ideal DUO route choice conditions **(47)** and (49) hold by definition. The fourth and sixth DUO departure time choice conditions (50) and (52) also hold by definition. Thus, we only need to prove that the second ideal DUO route choice condition (48) and the fifth DUO departure time choice condition (51) also hold.

We first prove that the second ideal DUO route choice condition (48) always holds for all times t. Now let $U^* = (u^*, f^*)$ be a solution for the variational inequality ***(53).*** For each O-D pair rs, we can always find one minimal travel time route $\mathbb{K}$ for vehicles departing origin r at time t, which was evaluated under the optimal flow pattern $\{u_a^{rs^*}[t + \tau^{ri^*}(t)]\}$. For this route $\mathbb{K}$, the first ideal DUO route choice condition **(47)** becomes equality by definition. It follows that

$$\Omega_a^{rj^*}(t) = \tau^{ri^*}(t) + \tau_a[t + \tau^{ri^*}(t)] - \tau^{rj^*}(t) = 0 \qquad \forall a, r, s; a = (i,j); a \in \boldsymbol{k}. \quad (64)$$

Next, we need to find a set of feasible inflows $u_a^{rs}[t + \tau^{ri^*}(t)]$ so that the following equation

$$u_a^{rs}[t + \tau^{ri^*}(t)]\, \Omega_a^{rj^*}(t) = 0 \qquad \forall a, r, s; a = (2, j) \quad (65)$$

always holds. We choose the feasible departure flows $f^{rs}(t)$ to equal the optimal departure flows $f^{rs^*}(t)$ for all O-D pairs rs at each instant of time. Thus, the second term in (53) will vanish. It follows that

$$\int_0^T \sum_{rs} \mathcal{U}^{rs^*}(t)\left\{f^{rs}(t) - f^{rs^*}(t)\right\} dt = 0 \quad (66)$$

We also need to re-route all feasible departure flows $f^{rs}(t)$ for all O-D pairs at each instant of time. For each O-D pair rs, we assign the feasible O-D departure flow $f^{rs}(t)$ to the minimal travel time route $\boldsymbol{k}$, which was evaluated under the optimal flow patterns $\{u_a^{rs^*}[t + \tau^{ri^*}(t)]\}$ This will generate a set of feasible inflow pattern $\{u_a^{rs}[t + \tau^{ri^*}(t)]\}$ which always satisfy equation (65) (because either $\Omega_a^{rj^*}(t) = 0$ for links on route $\boldsymbol{k}$ or $u_a^{rs}[t + \tau^{ri^*}(t)] = 0$ since no flow is routed onto those links which are not on route k.) Summing equations (65) for all links a and all O-D pairs rs, it follows that

$$\sum_{rs} \sum_a u_a^{rs}[t + \tau^{ri^*}(t)]\, \Omega_a^{rj^*}(t) = 0 \qquad \text{where } a = (i,j). \quad (67)$$

Integrating the above equation, we have

$$\int_0^T \sum_{rs} \sum_a \Omega_a^{rj^*}(t)\; u_a^{rs}[t + \tau^{ri^*}(t)]\, dt = 0 \quad (68)$$

Substituting equations (66) and *(68)* into the variational inequality (53), it follows that

$$\int_0^T \sum_{rs} \sum_a \Omega_a^{rj^*}(t) u_a^{rs^*}[t + \tau^{ri^*}(t)] dt \leq 0 \quad (69)$$

Since $\Omega_a^{rj^*}(t)$ and $u_a^{rs^*}[t+\tau^{ri^*}(t)]$ are nonnegative, it follows that

$$u_a^{rs^*}[t+\tau^{ri^*}(t)]\,\Omega_a^{rj^*}(t)=0 \qquad \forall a=(i,j),r,s; \tag{70}$$

for (nearly) all t. The above equation is the same as the second ideal DUO route choice condition (48). In other words, the second ideal DUO route choice condition (48) always hold for any solutions $u_a^{rs^*}[t+\tau^{ri^*}(t)]$ and $f^{rs^*}(t)$ to the variational inequality (53).

Next, we prove that the fifth DUO departure time choice condition (51) always hold as well. For any O-D pair pq, it might be possible that $\mathcal{U}^{pq^*}(t)=\mathcal{U}^{pq}_{min}$ for a single time instant t inside time interval $[d-\delta,d+\delta]$ where $[d-\delta,d+\delta]$ E $[0,T]$, and $\mathcal{U}^{pq^*}(t)>\mathcal{U}^{pq}_{min}$ for a time instant t outside this time interval. Let $\mathcal{U}^{pq}_{max}$ be defined as:

$$\mathcal{U}^{pq}_{max} \stackrel{\text{def}}{=} \sup_{t|f^{pq^*}(t)>0} \mathcal{U}^{pq^*}(t) \tag{71}$$

and let ϵ^{pq} be defined as:

$$\epsilon^{pq}=(\mathcal{U}^{pq}_{max}-\mathcal{U}^{pq}_{min})/4 \tag{72}$$

It follows that ϵ^{pq} is different from 0 if and only if the fifth DUO departure time choice condition (51) is not satisfied by O-D pair pq. Define then a set of feasible departure flows $f^{pq}(t)$ according to the follows:

$$\begin{aligned}
f^{pq}(t) &= 0, \quad (\forall\, t|\, \mathcal{U}^{pq^*}(t)>\mathcal{U}^{pq}_{max}-\epsilon^{pq})\\
f^{pq}(t) &= f^{pq^*}(t)+\varphi^{pq}(t), \quad (\forall\, t|\, \mathcal{U}^{pq^*}(t)<\mathcal{U}^{pq}_{min}+\epsilon^{pq})\\
f^{pq}(t) &= f^{pq^*}(t), \quad (\forall\, t|\, \mathcal{U}^{pq}_{max}-\epsilon^{pq}\geq \mathcal{U}^{pq^*}(t)\geq \mathcal{U}^{pq}_{min}+\epsilon^{pq})
\end{aligned} \tag{73}$$

The flow $\varphi^{pq}(t)$ is the flow displaced from instants t when $\mathcal{U}^{pq^*}(t)$ admits high values to instants when $\mathcal{U}^{pq^*}(t)$ admits low values. This flow needs only be positive and satisfy the generation constraint:

$$\int_{\{\mathcal{U}^{pq^*}(t)<\mathcal{U}^{pq}_{min}+\epsilon^{pq}\}} \varphi^{pq}(t)dt=\int_{\{\mathcal{U}^{pq^*}(t)\geq\mathcal{U}^{pq}_{max}-\epsilon^{pq}\}} f^{pq^*}(t)dt \stackrel{\text{def}}{=} \Phi^{pq} \tag{74}$$

in order that:

$$\int_0^T f^{pq} dt = F^{pq}(T) = \int_0^T f^{pq^*}(t) \tag{75}$$

Using definition (73), it follows that

$$\begin{aligned}
&\int_0^T \sum_{pq} \mathcal{U}^{pq^*}(t) \left\{ f^{rs}(t) - f^{rs^*}(t) \right\} \, dt \\
=\ & \sum_{pq} \left\{ \int_{t|\{\mathcal{U}^{pq^*}(t) < \mathcal{U}^{pq}_{min} + \epsilon^{pq}\}} \mathcal{U}^{pq^*}(t) \cdot \varphi^{pq}(t) dt - \int_{t|\{\mathcal{U}^{pq^*}(t) \geq \mathcal{U}^{pq}_{max} - \epsilon^{pq}\}} \mathcal{U}^{pq^*}(t) \cdot f^{pq^*}(t) dt \right\} \\
\leq\ & \sum_{pq} \left\{ \int_{t|\{\mathcal{U}^{pq^*}(t) < \mathcal{U}^{pq}_{min} + \epsilon^{pq}\}} (\mathcal{U}^{pq}_{min} + \epsilon^{pq}) \cdot \varphi^{pq}(t) dt - \int_{t|\{\mathcal{U}^{pq^*}(t) \geq \mathcal{U}^{pq}_{max} - \epsilon^{pq}\}} (\mathcal{U}^{pq}_{max} - \epsilon^{pq}) \cdot f^{pq^*}(t) dt \right\} \\
=\ & \sum_{pq} \left\{ (\mathcal{U}^{pq}_{min} + \epsilon^{pq}) \cdot \Phi^{pq} - (\mathcal{U}^{pq}_{max} - \epsilon^{pq}) \cdot \Phi^{pq} \right\} \\
=\ & -\sum_{pq} 2\epsilon^{pq} \Phi^{pq}
\end{aligned} \tag{76}$$

Following the above displacement of feasible departure flows defined in (73), the link inflows should be adjusted accordingly so as to be feasible. For each O-D pair pq, we can always find one minimal travel time route k for vehicles departing origin p at time t, which was evaluated under the optimal flow pattern $\{u_a^{pq^*}[t + \tau^{pi^*}(t)]\}$. For this route k, the first ideal DUO route choice condition **(47)** becomes equality by definition. It follows that

$$\Omega_a^{pj^*}(t) = \tau^{pi^*}(t) + \tau_a[t + \tau^{pi^*}(t)] - \tau^{qj^*}(t) = 0 \qquad \forall a, p, q; a = (i, j); a \in k. \tag{77}$$

Next, we need to find a set of feasible inflows $u_a^{pq}[t + \tau^{pi^*}(t)]$ so that the following equation

$$u_a^{pq}[t + \tau^{pi^*}(t)] \, \Omega_a^{pj^*}(t) = 0 \qquad \forall a, p, q; a = (i, j) \tag{78}$$

always hold. For each O-D pair pq at each time instant t, we assign the feasible O-D departure flow $f^{pq}(t)$ to the minimal travel time route k only, which was evaluated under the optimal flow pattern $\{u_a^{pq^*}[t + \tau^{pi^*}(t)]\}$. This will generate a set of feasible inflow pattern $\{u_a^{pq}[t+\tau^{pi^*}(t)]\}$ which always satisfy equation (78). Summing equations (78) for all links a and all O-D pairs pq, it follows that

$$\sum_{pq}\sum_{a} u_a^{pq}[t+\tau^{pi^*}(t)]\ \Omega_a^{pj^*}(t)=0 \qquad \text{where } a=(i,j). \tag{79}$$

Summing equations (48) for all links a and all O-D pairs pq, it follows that

$$\sum_{pq}\sum_{a} u_a^{pq^*}[t+\tau^{pi^*}(t)]\ \Omega_a^{pj^*}(t)=0 \qquad \text{where } a=(i,j). \tag{80}$$

Subtracting equation (80) from equation (79) and integrating the resulted equation, we have

$$\int_0^T \sum_{pq}\sum_{a} \Omega_a^{pj^*}(t)\left\{u_a^{pq}[t+\tau^{pi^*}(t)] - u_a^{pq^*}[t+\tau^{pi^*}(t)]\right\} dt = 0 \tag{81}$$

Substituting equation (81) into the variational inequality (53), it follows that

$$\int_0^T \sum_{pq} \mathcal{U}^{pq^*}(t)\ \left\{f^{pq}(t) - f^{pq^*}(t)\right\}\ dt\ \geq\ 0 \tag{82}$$

Combining inequalities (76) and (82), we obtain

$$-\sum_{pq} 2\epsilon^{pq}\Phi^{pq} \geq 0 \tag{83}$$

Since $\epsilon^{pq} \geq 0$ and $\Phi^{pq} \geq 0$, $\epsilon^{pq} = 0$ must hold. Thus, the fifth DUO departure time choice condition (51) is satisfied for all O-D pairs pq.

Therefore, any optimal solutions $\{u_a^{rs^*}[t+\tau^{ri^*}(t)]\}$ and $\{f^{rs^*}(t)\}$ to the variational inequality (53) will satisfy both the second ideal DUO route choice condition (48) and the fifth DUO departure time choice condition (51). Since we have proved the necessity and sufficiency of the variational inequality (53) in the above, we state that (53) is equivalent to the DUO departure time/route choice conditions (47)-(52). The proof is complete.

5 Concluding Remarks

In this paper, a link-based VI model for DUO departure time/route choice is presented. The necessity and sufficiency proofs of the VI model demonstrate that this model is consistent with the link-based DUO departure time/route choice conditions. Using a link-based VI formulation, explicit route enumeration can be avoided in computation. This feature allows our model to be applied to large-scale dynamic transportation networks with general link travel time functions.

Two major constraints prevent us from applying the existing dynamic transportation network models to ATIS systems. The first concern is the accurate representation of travelers' choice behavior. In future extensions, utility functions instead of pure travel times should be used in route choice problems. Different perceptions and compliance with information must be investigated by stratifying travelers into multiple groups. The second concern is the accurate representation of traffic dynamics on each street link. Since the link traffic dynamics might be very complicated as pointed out by Newell (1993) and Daganzo (1994), a set of appropriate closed-form link travel time functions might involve the interactions of neighboring link flows. This point was also demonstrated in the proposed dynamic link travel time functions for arterial links by Ran et al. (1992). This feature prevents formulating an appropriate optimization model for a realistic departure time/route choice problem. Thus, the general VI formulation approach was proposed for such applications. However, the VI models would require more computational capability than the optimization models.

The proposed link-based VI model for DUO departure time/route choice can be extended to include arrival time choice, destination choice and mode choice as well. Our next step is to develop efficient solution algorithms for the DUO departure time/route choice VI model. We expect that the Frank-Wolfe and diagonalization techniques proposed by Boyce et al. (1995) and Ran et al. (1992) can be applied to solve this model. Other solution algorithms, such as the projection algorithm, implemented by Nagurney (1986) for VI models for static network equilibrium problems, are also extendable for our dynamic VI problem. We note that the solution algorithm for our DUO departure time/route choice VI model has to be implemented on an expanded time-space network proposed in Boyce et al (1995). Other important problems, such as incident related dynamic route choice problems and dynamic congestion pricing problems, will be studied as extensions of this VI model.

Acknowledgments

The constructive comments from two anonymous referees are greatly appreciated. Partial support of this research was provided by the PATH Program, Institute of Transportation Studies, University of California at Berkeley.

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