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VECTOR ANALYSIS

1. HISTORICAL SKETCH: (The following is a resume of the Historical Introduction to C. E. Weatherburn's "Elementary Vector Analysis".)

2. The method of subjecting vector quantities to scalar algebra by resolution into three components is due to Descartes (1596-1650). The need of a calculus for dealing directly with vectors has long been recognized; Leibnitz made an attempt, but with little success (1679). In 1806, Argand showed how a geometrical representation could be given to the complex number. This was important to the theory of complex variables; but gave the wrong impression that the theory of real vectors is necessarily dependent upon that of complex numbers. This idea has not entirely disappeared. In 1826, Möbius, one of Gauss' pupils, published his "Barycentrisches Calcul". This was a forerunner of more general analysis by Grassmann. In 1832, Bellavitis published "Calcolo delle Equipollenze", which deals systematically with geometric addition and equality of vectors. In 1834-44, Hamilton produced his "Quaternions", and at the same time, Grassmann produced his "Ausdehnungslehre". These authors, working independently, and along different lines, developed similar analyses.

3. The quaternion is a sort of "sum", or complex of a scalar and a vector, although originally defined as the "quotient" of two vectors. The "Ausdehnungslehre" is an algebra of geometric forms.

4. Tait, (1831-1901, Scotch), was a friend of Hamilton, and a strong exponent of quaternions. He wrote a text, (1867), "Elementary Treatise on Quaternions".

5. Neither Hamilton's nor Grassmann's system met the needs of physicists or applied mathematicians, being too general and too complex for the requirements of ordinary calculations. "The ideas involved in the scalar and vector quantities of mechanics and physics are much simpler than those of Hamilton's theory, in which imaginaries play a large part, and vectors and scalars appear as degenerate quaternions rather than in their own right."

6. The feeling became general that a simpler system was needed. Various men, in different countries, developed identical analyses, as far as elements and functions are concerned. The differences lay only in terminology and notation. In Germany, the men who made noteworthy contributions were Föppl, Abraham, Bucherer, Fischer, Ignatowsky, and Gans: in England, Oliver Heaviside: and in the USA, Willard Gibbs.

7. Gibbs was familiar with the work of Grassmann and Hamilton, and, realizing the need of a simpler system, developed what he called the Vector Analysis. He published privately, for his students, a pamphlet entitled "Elements of Vector Analysis". This was not formally published until 20 years later, when he reluctantly consented to the publication of a fairly complete treatise. He was reluctant because he considered it to be only a special adaptation of the work of others. This was true of some parts of the book, but he contributed largely in systematizing the subject, and advanced it in certain fields.

8. Oliver Heaviside, in England, independently developed an almost identical system, for use in his work in Electromagnetic Theory.

9. Tait, in Scotland, objected violently, and a controversy between the quaternionists and vector analysisists was started, which lasted for several years. Vector Analysis won out, simply because it is the more usable system.

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10. Dr. Edwin Bidwell Wilson, then of Yale, later of M.I.T., a former pupil of Gibbs, wrote the first full exposition of Gibbs' work, "Vector Analysis" (1901, Yale University Press, 2nd Edition, 1909).

11. The Italians have done a great deal of work on this subject. Outstanding are Prof. R. Marcolongo, of the University of Naples, and Prof. C. Bureli-Forti, of the Military Academy of Turin. (Elements de Calcul Vectoriel", Paris, 1910; and a larger work, "Analyse Vectorielle Générale", Paris, 1912.)

12. The order of development of Vector Analysis has been the opposite of what might have been expected. To quote Heaviside (Electromagnetic Theory, vol. 1, p. 136):

13. "Suppose a sufficiently competent mathematician desired to find out from the Cartesian methematics what vector algebra was like, and its laws. He could do so by careful inspection and comparison of the Cartesian formulae. He would find certain combinations of symbols and quantities occurring again and again, usually in systems of threes. He might introduce tentatively an abbreviated notation for these combinations. After a little practice he would perceive the laws according to which these combinations arose and how they operated. Finally, he would come to a very compact system in which vectors themselves and certain simple funtions of vectors appeared, and would be delighted to find that the rules for the multiplication and the general manipulation of these vectors were, considering the complexity of the Cartesian mathematics out of which he had discovered them, of an almost incredible simplicity. But there would be no sign of a quaternion in his result, for one thing; and, for another, there would be no metaphysics or abstruse reasoning required to establish the rules of manipulation of his vectors." This is the logical way for Vector Analysis to have developed, but it is not the way it happened. Its manner of origin has counted against it, but this prejudice has largely disappeared, and the analysis has become very popular. It is almost indispensible in three dimentional work in almost every branch of mathematical physics. The main difficulties left in the way of universal use are the differences in notations used by different authors, but even these difficulties are being wiped out.

14. DEFINITIONS:

14.1 A SCALAR QUANTITY has magnitude, but is not related to any definite direction in space. To specify a scalar, a unit quantity of the same type is needed, as well as the ratio the given quantity bears to that unit, so that it may be expressed as a multiple of that unit. EXAMPLES OF SCALARS: Mass, Volume, density, speed, temperature, work (energy), quantity of heat, electric charge, potential.

14.2 A VECTOR QUANTITY has magnitude, and is related to a certain definite direction in space. To specify a vector is needed not only a unit quantity of the same kind considered apart from direction, and a number which is the measure of the original quantity in terms of this unit, but also a statement of its direction. EXAMPLES OF VECTOR QUANTITIES: Displacement, velocity, acceleration, momentum, force, electric and magnetic intensities.

A Vector quantity can be represented by a straight line, proportional in length to the magnitude of the vector quantity, and drawn in the proper direction. Such a straight line is commonly called a VECTOR.

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14.3 The MODULE of a Vector is the positive number which is the measure of its length.

14.4 A UNIT VECTOR is one whose module is unity.

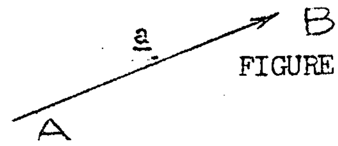


FIGURE 14.4

14.5 Conventions for representing vectors may be illustrated by the accompanying figure, where the vector might be indicated by $\underline{AB}$, $\overline{AB}$, $\bar{a}$, or $\underline{a}$. In print, heavy type is sometimes used to indicate a vector. The unit vector of $\underline{a}$ is usually represented by $\hat{a}$, the magnitude of $\underline{a}$ by a. Thus,

$$\underline{a} = a\,\hat{a}\,.$$

For ease of typing, the letter representing a vector will be underlined, in this paper.

14.6 EQUAL VECTORS. If $\underline{a} = \underline{b}$, then $\hat{a} = \hat{b}$, which means that their directions are identical; and that $a = b$, which indicates equality of magnitude.

14.7 A ZERO or NULL VECTOR is one whose module is zero. All null vectors are necessarily equal.

14.8 The vector which has the same module as $\underline{a}$, but the opposite direction, is defined as the <u>negative</u> of $\underline{a}$, and is denoted by $-\underline{a}$.

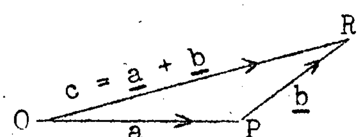
14.9 The value of a vector depends upon its length and direction, but is independent of position, the vector not being localized in any definite line. A single vector cannot, therefore, completely represent the effect of a localized vector quantity, such as a force acting on a rigid body. This effect depends upon the line of action of the force; and it will be shown later that two vectors are necessary for its specification.

15. ADDITION and SUBTRACTION OF VECTORS.

15.1 ADDITION. If three points, O, P, R, are chosen so that $\underline{OP} = \underline{a}$ and $\underline{PR} = \underline{b}$, then the vector $\underline{OR}$ is called the (vector) sum or resultant of $\underline{a}$ and $\underline{b}$. Denoting $\underline{OR}$ by $\underline{c}$ gives

$$\underline{c} = \underline{a} + \underline{b}.$$

FIG. 15.11



By figure 15.12 it may be seen also, that $\underline{c} = \underline{b} + \underline{a}$

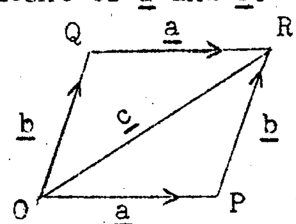


FIG. 15.12

By following this line of reasoning, we find that the sum of any number of vectors is independent of the order and grouping of the terms.

15.2 SUBTRACTION. The subtraction of $\underline{b}$ from $\underline{a}$ is understood to be the addition of $-\underline{b}$ to $\underline{a}$. That is, $\underline{a} - \underline{b} = \underline{a} + (-\underline{b})$. From the figure, where $\underline{QO} = \underline{RP} = \underline{b}$, and $\underline{QR} = \overline{OP} = \underline{a}$,

(See Page 4)

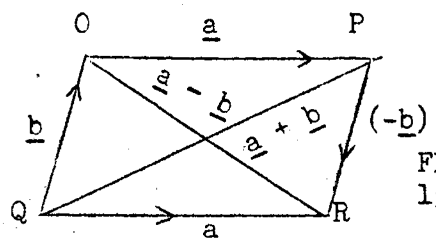


FIG. 15.2

$$\underline{a} + \underline{b} = \underline{QR} + \underline{RP} = \underline{QO} + \underline{OP} = \underline{QP} .$$

and $$\underline{a} - \underline{b} = \underline{OQ} + \underline{QR} = \underline{OP} + \underline{PR} = \underline{OR} .$$

16. MULTIPLICATION BY A NUMBER. If m is any positive real number, $m\underline{a}$ indicates the vector in the same direction as $\underline{a}$, but of m times its length. For example, Newton's Second Law may be written $\underline{F} = m\underline{a}$, where $\underline{F}$, the force, has the same direction as $\underline{a}$, the acceleration, but is m times as great, where m is the mass, a scalar.

17. COMPONENTS OF A VECTOR.

17.1 Three or more vectors are said to be COPLANAR when a plane can be drawn parallel to all of them, otherwise, they are NONCOPLANAR. Any vector, $\underline{r}$, can be expressed as the sum of three others, parallel to any three non-coplanar vectors. Let $\underline{a}$, $\underline{b}$, $\underline{c}$, be unit vectors in the three given noncoplanar directions. With any point O as origin take $\underline{OP} = \underline{r}$, and on OP as diagonal, construct a parallelopiped with edges OA, OB, OC, parallel to $\underline{a}$, $\underline{b}$, $\underline{c}$, respectively. Then, if x, y, z, are the measures of the lengths of its edges, $\underline{r}$ is expressible as the sum

$$\underline{r} = \underline{OA} + \underline{AF} + \underline{FP} = \underline{OA} + \underline{OB} + \underline{OC}$$

$$= x\underline{a} + y\underline{b} + z\underline{c} .$$

Then $\underline{r}$ is the resultant of the three vectors, $x\underline{a}$, $y\underline{b}$, $z\underline{c}$, which are called the components of $\underline{r}$ in the given directions. (x, y, z may be either positive or negative.) If two vectors are equal, their components, parallel to the same axes, are respectively equal.

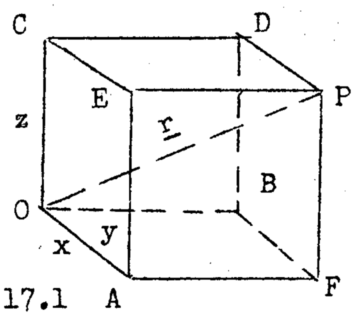


FIGURE 17.1

17.2 UNIT VECTORS, $\underline{i}$, $\underline{j}$, $\underline{k}$. Usually vectors are resolved into components parallel to the Cartesian axes, x, y, z. The unit vectors along the x, y, and z axes are $\underline{i}$, $\underline{j}$, and $\underline{k}$, respectively. (Note that neither i nor j in this notation is the symbol for $\sqrt{-1}$) Thus, a vector, $\underline{r}$ may be represented by $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$. The sum of several vectors, $\underline{r}_1$, $\underline{r}_2$, $\underline{r}_3$, etc., would then be

$$(x_1\underline{i} + y_1\underline{j} + z_1\underline{k}) + (x_2\underline{i} + y_2\underline{j} + z_2\underline{k}) + \ldots\ldots$$

$$= (x_1 + x_2 + x_3 + \ldots) \underline{i} + (y_1 + y_2 + y_3 + \ldots) \underline{j} + (z_1 + z_2 + z_3 + \ldots) \underline{k}$$

17.3 Cartesian analysis deals with vectors and vector quantities by resolving them into rectangular components. In vector analysis, the quantities are treated, as far as possible, without resolution.

18. CENTROIDS.

18.1 Position Vector. When a vector $\underline{OP}$ is used to specify the position of a point P relative to another point O, it is called the position vector of P for the origin O.

Example: If p, q, r, ... are n real numbers, the point G whose position vector is

$$\underline{OG} = (p\underline{a} + q\underline{b} + r\underline{c} + \dots) / (p + q + r + \dots)$$

is called the centroid of the given points with associated numbers p, q, r, ... respectively. (The centroid is independent of the origin of vectors.) Then the center of mass of a set of particles, whose associated numbers are their respective masses, m_1, m_2, m_3, ... at points $\underline{r}_1$, $\underline{r}_2$, $\underline{r}_3$, ... is

$$\underline{\bar{r}} = (m_1\underline{r} + m_2\underline{r} + \dots) / (m_1 + m_2 + \dots) ,$$

$$\underline{\bar{r}} = \sum m\underline{r} / \sum m .$$

18.2 If the number of particles approaches infinity, (the case of a solid body), the limiting position of G is the center of mass of continuous distribution.

19. GEOMETRY.

Equations of some of the simple geometric forms are given here.

19.1 Equation of a STRAIGHT LINE. The vector equation of a straight line through a given point A, parallel to a given vector, $\underline{b}$, is

$$\underline{r} = \underline{OP} = \underline{OA} + \underline{AP}$$

$$= \underline{a} + t\underline{b}.$$

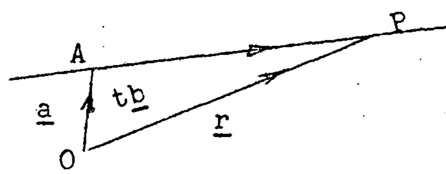


FIGURE 19.11

where $\underline{a}$ is the position vector of the point A, t is a variable scalar, and $\underline{r}$ is the position vector of point P, any point on the required line.

The vector equation of a line through two given points A and B is:

$$\underline{r} = \underline{a} + t(\underline{b} - \underline{a})$$

$$= (1 - t) \underline{a} + t\underline{b}$$

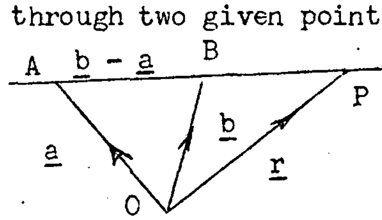


FIGURE 19.12

19.2 Equation of a PLANE. The equation of the plane through the origin, parallel to $\underline{a}$ and $\underline{b}$ is

$$\underline{r} = s\underline{a} + t\underline{b} .$$

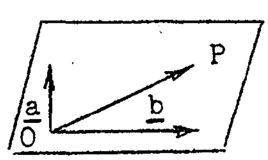


FIGURE 19.21

where s and t are variable scalars.

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The equation of the plane through point C, parallel to $\underline{a}$ and $\underline{b}$, is

$$\underline{r} = \underline{c} + s\underline{a} + t\underline{b}$$

The equation of the plane through three points A,B,C, is:

$$\underline{r} = \underline{a} + s(\underline{b} - \underline{a}) + t(\underline{c} - \underline{a})$$

$$= (1-s-t)\,\underline{a} + s\underline{b} + t\underline{c}$$

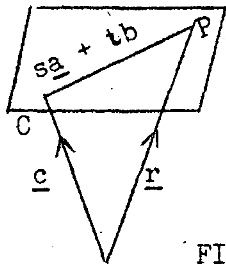


FIG. 19.22

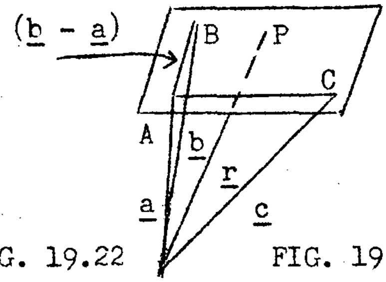


FIG. 19.23

This form is not the only equation of the plane, nor is it the most convenient. This will be discussed more in detail later.

19.3 VECTOR AREAS. An area may be represented by a vector, perpendicular to its surface, of a magnitude proportional to that of the area. The direction of the vector is determined by the direction in which the boundary of the figure is described, the convention being: the normal vector PP' bears to this direction of rotation the same relation as the translation to the direction of rotation of a right-handed screw.

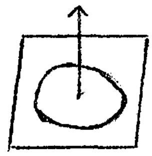


FIG. 19.3

20. PRODUCTS OF TWO VECTORS.

20.1 From the nature of a vector, it is impossible to say what the product of two vectors ought to be. But examination of the ways in which two vectors enter into combinations, in Physics and Mathematics, brings about the definition of two distinct kinds of products, one a scalar, the other a vector.

20.2 SCALAR or DOT PRODUCT. The scalar product of two vectors $\underline{a}$ and $\underline{b}$, whose directions are inclined at an angle θ, is the real number (scalar), $ab \cos \theta$, and is written $\underline{a} \cdot \underline{b} = ab \cos \theta$. It is commonly called the dot product because of the widely accepted method of indication.

The following equations are mathematical statements of the rules governing this type of multiplication.

$$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \quad , \quad \underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c} \quad ;$$

If $\underline{a}$ is perpendicular to $\underline{b}$, $\underline{a} \cdot \underline{b} = 0$, since $\cos \theta = 0$.
If $\underline{a}$ has the same direction as $\underline{b}$, $\underline{a} \cdot \underline{b} = ab$, since $\cos \theta = 1$.
If $\underline{a}$ and $\underline{b}$ have opposite directions, $a \cdot b = -ab$, since $\cos \theta = -1$

$\underline{a} \cdot \underline{a} = a^2$, and is commonly indicated as $\underline{a}^2$.

The square of any unit vector is unity. Thus, $\underline{i}^2 = \underline{j}^2 = \underline{k}^2 = 1$.

But since $\underline{i}$, $\underline{j}$, and $\underline{k}$ are mutually perpendicular, $i \cdot j = j \cdot k = k \cdot i = 0$

$$(n\underline{a}) \cdot \underline{b} = nab \cos \theta = a \cdot (n\underline{b}).$$

Since the scalar product is a number, it may occur as the numerical coefficient of a vector. Thus, $a \cdot bc = (a \cdot b)c$ is a vector in the direction of $\underline{c}$ with module $\underline{a} \cdot \underline{bc}$. Similarly, $\underline{a} \cdot \underline{bc} \cdot \underline{d} = (\underline{a} \cdot \underline{b})\,(\underline{c} \cdot \underline{d})$, a scalar.

20.3 VECTOR OR CROSS PRODUCT. The vector product of two vectors $\underline{a}$ and $\underline{b}$, whose directions are inclined at an angle θ, is the vector whose module is $(ab \sin \theta)$, and whose direction is perpendicular to both $\underline{a}$ and $\underline{b}$, being positive relative to a rotation from $\underline{a}$ to $\underline{b}$. It is indicated by $(\underline{a} \times \underline{b})$, hence the name. The following equations represent laws governing this kind of multiplication. Note that in this multiplication it makes a difference which vector is mentioned first.

$\underline{a} \times \underline{b} = ab \sin \theta \, \underline{n}$, where $\underline{n}$ is unit vector perpendicular to both $\underline{a}$ and $\underline{b}$, in proper direction.

$\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$. $\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = 0$, since $\sin \theta = 0$.

$\underline{i} \times \underline{j} = \underline{k} = -\underline{j} \times \underline{i}$, $\underline{j} \times \underline{k} = \underline{i} = -\underline{k} \times \underline{j}$, $\underline{k} \times \underline{i} = \underline{j} = \underline{i} \times \underline{k}$.

$(m\underline{a}) \times \underline{b} = mab \sin \theta \, \underline{n} = \underline{a} \times (m\underline{b})$. $\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$.

$(\underline{a} + \underline{b} + ---) \times (\underline{l} + \underline{m} + ---) = \underline{a} \times \underline{l} + \underline{a} \times \underline{m} + --- + \underline{b} \times \underline{l} + \underline{b} \times \underline{m} + ---$

$$\underline{a} \times \underline{b} = (a_1\underline{i} + a_2\underline{j} + a_3\underline{k}) \times (b_1\underline{i} + b_2\underline{j} + b_3\underline{k})$$

$$= a_1b_1\underline{i} \times \underline{i} + a_1b_2\underline{i} \times \underline{j} + a_1b_3\underline{i} \times \underline{k} + a_2b_1\underline{j} \times \underline{i} + a_2b_2\underline{j} \times \underline{j} +$$

$$+ a_2b_3\underline{j} \times \underline{k} + a_3b_1\underline{k} \times \underline{i} + a_3b_2\underline{k} \times \underline{j} + a_3b_3\underline{k} \times k \; .$$

But since $\underline{i} \times i = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = 0$, and $\underline{i} \times \underline{j} = -\underline{j} \times \underline{i}$, etc., and

$\underline{i} \times \underline{j} = \underline{k}$, etc.,

$$\underline{a} \times \underline{b} = (a_2b_3 - a_3b_2)\,\underline{i} + (a_3b_1 - a_1b_3)\,\underline{j} + (a_1b_2 - a_2b_1)\,\underline{k} \; ,$$

or $$\underline{a} \times \underline{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \underline{i} & \underline{j} & \underline{k} \end{vmatrix}$$ in determinant form.

21. EQUATION OF THE PLANE. Let p be the length of the perpendicular ON from the origin O to the given plane, and $\hat{n}$ the unit vector normal to the plane, having the direction O to N. Then $\underline{ON} = p\hat{n}$. If $\underline{r}$ is the position vector of any point P on the plane, $\underline{r} \cdot n$ is the projection of OP on ON, and is therefore equal to p. Thus the equation,

$\underline{r} \cdot \hat{n} = p$, is the equation of the plane.

This may be rewritten as

$\underline{r} \cdot \underline{n} = np = q$ (say), and this is taken as the standard form for the equation of a plane.

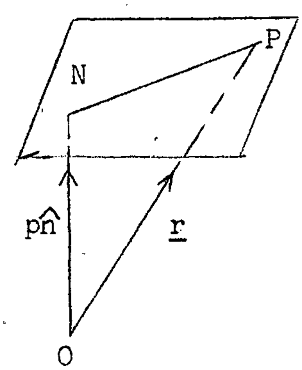


FIGURE 21

The angle between two planes whose equations are $\underline{r}\cdot\underline{n} = q$, and $\underline{r}\cdot\underline{n}' = q'$ is the angle between their normals. But

$$n \cdot n' = nn' \cos\theta \quad , \text{ so that } \theta = \cos^{-1}\frac{\underline{n}\cdot\underline{n}'}{nn'} .$$

22. APPLICATIONS TO MECHANICS.

22.1 Work done by a force.

If $\underline{F}$ = a force, $\underline{d}$ = displacement, and W = work done by force through displacement $\underline{d}$, and θ is the angle between $\underline{F}$ and $\underline{d}$, then

$$W = \underline{F} \cdot \underline{d} = Fd\cos\theta .$$

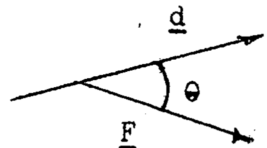


FIGURE 22.11

Torque of a force (Moment).

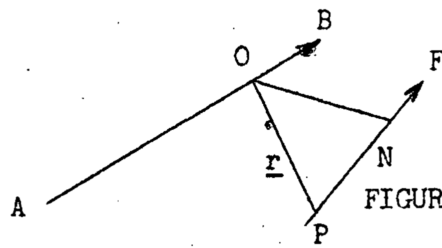


FIGURE 22.12

The moment $\underline{M}$ of a force $\underline{F}$ about a point O is related to an axis through O, perpendicular to the plane common to O and F.

$\underline{M} = \underline{r} \times \underline{F}$, where $\underline{r}$ is the position vector of any point on the line of action of $\underline{F}$ & $\underline{M}$ is represented by a vector perpendicular to $\underline{r}$ and $\underline{F}$, whose magnitude is rF sin OPN = Fr sin OPN = F (ON), where ON is perpendicular to $\underline{F}$.

If there are several forces, $\underline{F}_1$, $\underline{F}_2$, $\underline{F}_3$,--- acting through the same point P, they have a resultant, $\underline{R}$, where $\underline{R} = \sum \underline{F}$.

Then, $\underline{r} \times \underline{R} = \underline{r} \times (\underline{F}_1 + \underline{F}_2 + ---) = \underline{r} \times \underline{F}_1 + \underline{r} \times \underline{F}_2 + ---$

To express numerically the moment of force $\underline{F}$ about O, write $\underline{F} = X\underline{i} + Y\underline{j} + Z\underline{k}$ and $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$. Then

$$\underline{M} = \underline{r} \times \underline{F} = (yZ - zY)\underline{i} + (zX - xZ)\underline{j} + (xY - yX)\underline{k}.$$

In this expression, the coefficients of $\underline{i}$, $\underline{j}$, and $\underline{k}$ are the ordinary scalar moments of the force about the coordinate axes. It follows that the ordinary moment of a force $\underline{F}$ about any straight line through O is the resolved part, along this line, of the vector moment of F about O. In the above case, $M_x = \underline{M} \cdot \underline{i} = yZ - zY$, $M_y = \underline{M} \cdot \underline{j} = zX - xZ$, and $M_z = \underline{M} \cdot \underline{k} = xY - yX$.

If there are several concurrent forces, it follows from the above that the scalar moment of the resultant about any axis through O is equal to the sum of the scalar moments of the several forces about that axis.

22.2 ANGULAR VELOCITY OF A RIGID BODY ABOUT A FIXED AXIS.

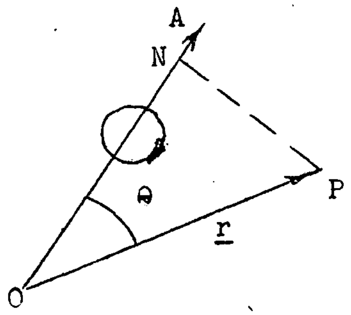


FIGURE 22.2

Consider the rotation of a rigid body, rotating about an axis ON, at the rate of ω radians per second. The angular velocity of the body is specified by a vector $\underline{A}$, whose module is ω, and whose direction is parallel to the axis, in the positive direction relative to the rotation. The velocity, $\underline{v}$, of any point P in the body is given by

$$\underline{v} = \underline{A} \times \underline{r} = (\omega\ r \sin\theta)\ \underline{A}$$

$$= (\omega \cdot NP)\ A \ .$$

23. PRODUCTS OF THREE VECTORS.

23.1 The <u>scalar triple product</u> of three vectors $\underline{a}$, $\underline{b}$, and $\underline{c}$, is the scalar product of $\underline{a}$ and $\underline{b} \times \underline{c}$. It is the measure of the volume of the parallelopiped whose edges are determined by three vectors. The value of this product is unaltered by changing the order of the factors, as long as the cyclic order of the factors is unaltered, or by interchanging the dot and cross. That is,

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{b} \cdot (\underline{c} \times \underline{a}) = \underline{c} \cdot (\underline{a} \times \underline{b}) =$$

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = (\underline{b} \times \underline{c}) \cdot \underline{a} = (\underline{c} \times \underline{a}) \cdot \underline{b} \quad ,$$

but $\underline{a} \cdot (\underline{b} \times \underline{c}) = -\underline{b} \cdot (\underline{a} \times \underline{c})$.

23.2 This product is usually written $(\underline{abc})$, since only the cyclic order of the factors is important. Then $(\underline{abc}) = -(\underline{acb})$.

23.3 The value of the product is given by the determinant

$$(\underline{abc}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

23.4 The <u>Vector Triple Product</u> $\underline{a} \times (\underline{b} \times \underline{c})$ is the vector product of $\underline{a}$ into $(\underline{b} \times \underline{c})$. It is a vector in the plane of $\underline{b}$ and $\underline{a}$, and its value is given by

$$\underline{a} \times (\underline{b} \times \underline{c}) = \underline{a} \cdot \underline{cb} - \underline{a} \cdot \underline{bc} \ .$$

The position of the brackets in this product is not arbitrary; for $(\underline{a} \times \underline{b}) \times \underline{c}$ is a vector in the plane of $\underline{a}$ and $\underline{b}$, and its value is

$$(\underline{a} \times \underline{b}) \times \underline{c} = \underline{a} \cdot \underline{cb} - \underline{b} \cdot \underline{ca} \ .$$

Neither can the order of the factors be changed at pleasure.

23.5 (All of the preceding discussion has been taken from Weatherburn's "Elementary Vector Analysis". This book goes into the geometry of the plane, straight line, the sphere very thoroughly. The following discussion of differentiation of vectors is taken also from Weatherburn.)

24. DIFFERENTIATION.

24.1 If $\underline{r}$ is a function of a scalar variable, t, and $\delta \underline{r}$ is the increment in $\underline{r}$ corresponding to the increment δt in t, then the limiting value of the quotient $\delta \underline{r}/\delta t$ as δt tends to zero is called the derivative of $\underline{r}$ with respect to t. We use the notation

$$\underset{\delta t \to 0}{\mathrm{Lt}} \quad \frac{\delta \underline{r}}{\delta t} = \frac{d\underline{r}}{dt}$$

The derivative of this function is called the second derivative, and so on.

24.2 The rules for differentiating <u>sums</u> <u>and</u> <u>products</u> of vectors are similar to those for algebraic sums and products. Thus

$$\frac{d}{dt}(\underline{r} + \underline{s} + \dots) = \frac{d\underline{r}}{dt} + \frac{d\underline{s}}{dt} + \dots ,$$

$$\frac{d}{dt}(\underline{r} \cdot \underline{s}) = \frac{d\underline{r}}{dt} \cdot \underline{s} + \underline{r} \cdot \frac{d\underline{s}}{dt} ,$$

$$\frac{d}{dt}(\underline{r} \times \underline{s}) = \frac{d\underline{r}}{dt} \times \underline{s} + \underline{r} \times \frac{d\underline{s}}{dt}$$

Differentiating both sides of the equality $\underline{r}^2 = r^2$, and using the second of these formulae, we obtain

$$\underline{r} \cdot d\underline{r}/dt = r \cdot dr/dt .$$

Also, if $\underline{a}$ is a vector of constant length, $\underline{a} \cdot d\underline{a}/dt = 0$, showing that $\underline{a}$ is perpendicular to its derivative.

24.3 Other examples of differentiations:

$$\frac{d}{dt}\left(\underline{r} \times \frac{d\underline{r}}{dt}\right) = \underline{r} \times \frac{d^2\underline{r}}{dt^2}$$

$$\frac{d\underline{r}}{dt} = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k}$$

$$\frac{d}{dt}(\underline{a}\ \underline{b}\ \underline{c}) = \left(\frac{d\underline{a}}{dt}\ \underline{b}\ \underline{c}\right) + \left(\underline{a}\ \frac{d\underline{b}}{dt}\ \underline{c}\right) + \left(\underline{a}\ \underline{b}\ \frac{d\underline{c}}{dt}\right)$$

25. INTEGRATION. Integration is the reverse process to differentiation. The vector $\underline{F}$, whose derivative with respect to t is equal to $\underline{r}$, is called the integral of $\underline{r}$, and is written

$$\underline{F} = \int \underline{r}\, dt$$

A constant of integration may be introduced as in algebraic calculus. Thus

$$2\int \underline{r} \cdot \frac{d\underline{r}}{dt}\, dt = \underline{r}^2 + C = r^2 + C$$

$$\int \underline{r} \times \frac{d^2\underline{r}}{dt^2}\, dt = \underline{r} \times \frac{d\underline{r}}{dt} + C$$

The equation $\frac{d^2\underline{r}}{dt^2} = -n^2\underline{r}$ may be integrated after scalar multiplication of both members with $2\frac{d\underline{r}}{dt}$. We then obtain $\left(\frac{d\underline{r}}{dt}\right)^2 = C - n^2\underline{r}^2$.

A definite integral is defined as in ordinary calculus.

26. RIGID KINEMATICS

26.1 The motion of a rigid body about a fixed axis is at any instant one of rotation about a definite axis through that point, called the instantaneous axis. The angular velocity can then be represented by a vector $\underline{A}$ parallel to this axis. The velocity of the particle at the point $\underline{r}$ is $\underline{v} = \underline{A} \times \underline{r}$, the fixed point being taken as origin.

26.2 When no point in the body is fixed, take the position of any particle as origin, and let $\underline{v}$ be the velocity of that particle. Then the velocity of any other particle whose position vector is $\underline{r}$ is $\underline{V} = \underline{v} + \underline{A} \times \underline{r}$. The vector $\underline{A}$ is independent of the origin, and is called the angular velocity of the body.

26.3 Any motion of a rigid body is equivalent to a screw motion. The axis of the screw is parallel to $\underline{A}$; and the velocity of any particle on the axis is along the axis; being the same for all such particles. The two invariants of the motion are $\underline{A}^2$ and $\Gamma = \underline{v} \cdot \underline{A}$, where $\underline{v}$ is the velocity of any particle. The pitch of the screw is

$$p = \Gamma / \underline{A}^2 .$$

26.4 Simultaneous angular velocities about a fixed point are compounded by vector addition. Simultaneous angular velocities about parallel axes are compounded like parallel forces. Any simultaneous motions corresponding to velocities $\underline{v}_1$, $\underline{v}_2$, ... of a particle chosen as origin, and angular velocities $\underline{A}_1$, $\underline{A}_2$, of the body about that point, are compounded by vector addition of the velocities of the origin, and vector addition of the angular velocities.

VECTOR ANALYSIS

27. No attempt has been made to give any justification of statements made in this paper, and no attempt has been made to be thorough in this necessarily short summary.

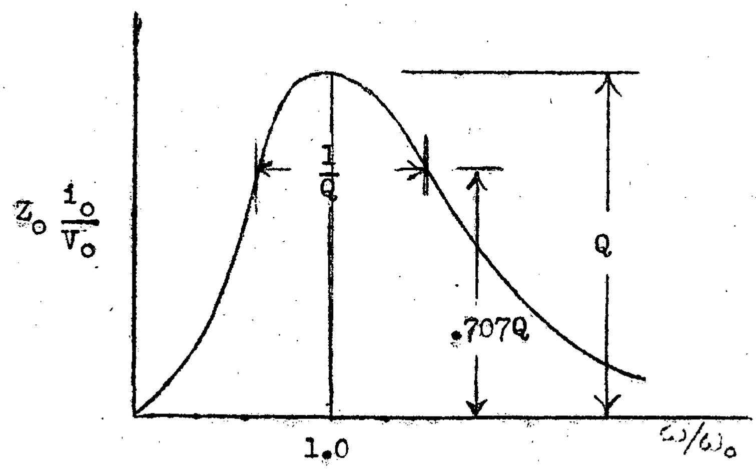
28. "Elementary Vector Analysis", by C. E. Weatherburn was used almost exclusively in this discussion. Other elementary books are J. G. Coffin's "Vector Analysis", and Brands's "Vectorial Mechanics". For more advanced work see Weatherburn's "Advanced Vector Analysis", Haas' "Theoretical Physics" and other books. There are many books in Vector Analysis in the main library at the University of California.

Brand	Vectorial Mechanics	QA805	B7	1930
	(Also in Eng. library	375	B817)	
Coffin	Vector Analysis	QA261	C6	1911
Weatherburn	Elementary V.A.	QA261	W42	
"	Advanced V.A.	QA261	W4	

(Above notes prepared by A. Hailey, 1931)

FLM-1/26/51

COMPUTATION AND MEASUREMENT OF Q AND R_s



Z_o = Characteristic Impedance

i_o = Current at peak of cycle

V_o = Voltage at peak of cycle

ω = Frequency (Angular)

ω_o = Frequency at resonance (Angular)

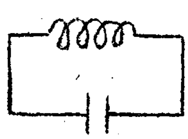
If the resonance curve of a circuit is plotted as above, it was shown that the width of the resonance curve at 0.707 of the height of the curve is equal to $\frac{1}{Q}$

It was also shown that

$$R_s = \frac{V_o^2}{2P} \qquad (2\text{-}1)$$

Where $R_s = \sqrt{L/C} \cdot Q = Z_o Q$

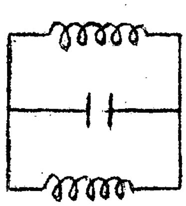
To get a high voltage for a given amount of power, both L/C and Q should be large. How, then, shall we go about it to get these values larger?

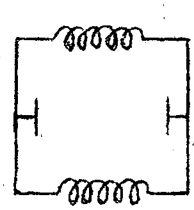


In a circuit of this type, since $Q = \frac{\sqrt{L/C}}{R}$,

Q will be increased by decreasing R

The most obvious way to decrease R is to increase the amount of copper as by putting another inductance in parallel.



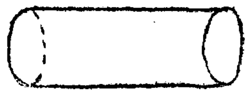


How to increase the value of L/C

This can be done by decreasing C

The plates of the condenser can be moved apart.

Carrying these two processes to the limit will result in a cavity



having the capacity element consisting of the two end plates and the inductance that due to the length of the cylinders.

Another way to look at this is consider the circuit as shown here.

The two end plates have a capacity and the connecting line has an inductance. The resistance of the connecting wire can be reduced by adding more in parallel, until the limit is reached of entirely surrounding the space with a cylindrical shell.

Now $$Q = \frac{2\pi \text{ Energy Stored}}{\text{Power loss/cycle}} = \frac{\omega_o U}{P} \qquad (2\text{-}2)$$

Also from (1-35)

$$R_s = \frac{V_o^2}{2P} = \sqrt{L/C} \cdot Q \qquad (2\text{-}3)$$

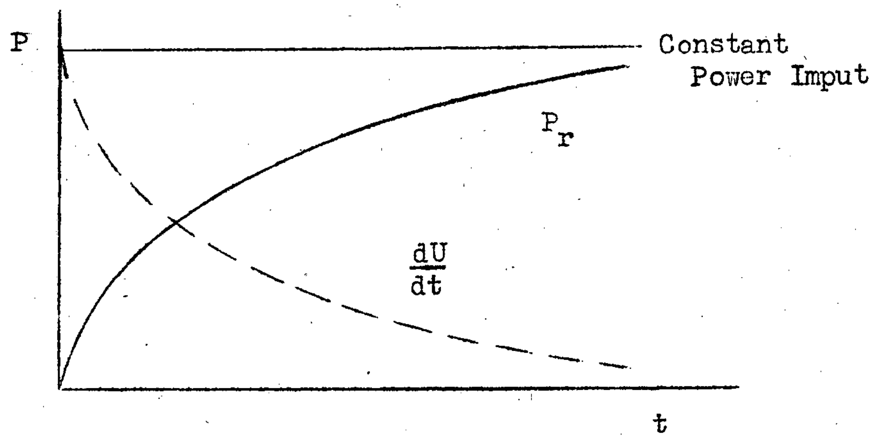
L/C is a function of the geometry of the system only. This is difficult to measure directly, since for a cavity the inductance and capacity elements are mixed together. It is difficult to measure V, altho P can be measured readily. Q is easy to measure by measuring the width of the resonance curve.

The values of R_s and Q can be computed, knowing the geometry of the cavity and the conductivity of the copper, and it is evident from (2-2) and (2-3) that for a given power loss per cycle Q and R_s both vary in the same way. A measurement made of Q will give a correction factor for the computed value, and this same factor will then apply to correct the computed value of the shunt impedance R_s.

Dynamic Behavior of Circuits

The discussion so far has considered steady state conditions, that is the conditions after oscillations have continued long enuf so that each oscillation is the same as the preceding one. Actually the circuit has to start from rest, and since energy has to be built up in the system to reach the steady state condition this will take time. How long will it take to build up?

After equilibrium is established all the power input goes into losses. The process of building up to this point is called excitation. If the total power input is constant the conditions with respect to time may be represented thus



At the start the current is small, and the power loss due to resistance, P_R is low. For a constant power imput the remaining power

$$(P - P_R)$$

goes into storage. This power is evidently equal to the rate of change of energy stored with respect to time or

$$\frac{dU}{dt} = (P - P_R)$$

The full line above represents the way the power loss due to resistance varies, and the dotted line represents the variation of $\frac{dU}{dt}$

An order-of-magnitude estimate of the time required may be made by noting that at equilibrium

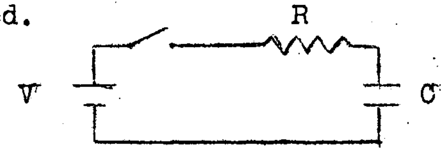
$$Q = \frac{2\pi \text{ Energy Stored}}{\text{Power loss per cycle}}$$

For a constant power input the energy that is put in per cycle is therefore

$$\frac{2\pi}{Q} \text{ fraction of the final energy U.}$$

It will therefore take, as the general order of magnitude, $\frac{Q}{2\pi}$ cycles to reach equilibrium. A system with high Q will therefore take some time to build up to final voltage.

To describe this mathematically the transient conditions must be considered.



Consider a circuit consisting of capacity and resistance, to which a source of voltage, such as a battery may be connected. When the switch is closed the voltage of the battery does not appear across the condenser immediately. The voltage is divided across the resistance and the condenser as long as the current is flowing as follows

$$V = IR + q/C \qquad (2\text{-}4)$$

LECTURE 2

At the start, q is zero, and the voltage drop across the resistance will be V. The current will start out equal to $\frac{V}{R}$. As the current flows, the charge q increases, and the voltage across the condenser increases proportionately. The available drop across the resistance is correspondingly reduced, and the current decreases.

Since $i = \frac{dq}{dt}$ (2-4) may be written

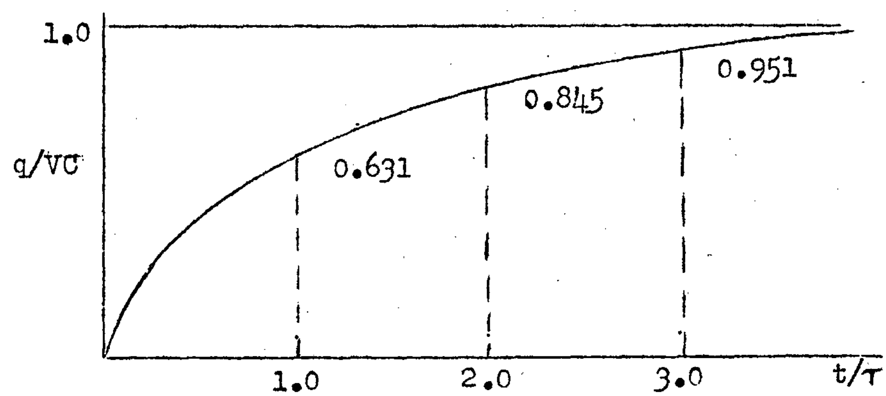
$$V = \frac{dq}{dt} R + \frac{q}{C}$$

(The complex form just cannot be used here, since this only applies where the voltage is varying sinusoidally.)

(2-5) has the following solution

$$q = CV \left(1 - e^{-t/RC} \right) \qquad (2\text{-}6)$$

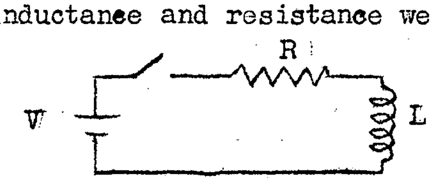
We may plot this as follows:



= RC is the time constant of the circuit and the curve is plotted in terms of the number of units of time equal to the time constant.

This is a transient solution. It will be noted that equilibrium will only be reached exactly when $\tau = \infty$. Actually for practical purposes we will consider equilibrium is reached when the voltage reaches the battery voltage to within a tolerance that is small enuf to be neglected for the purpose say 1% or 5%. In the curve above the 5% tolerance value will be reached when $t/\tau = 3.0$.

For a circuit consisting of inductance and resistance we have



For this case

$$V = iR + L\frac{di}{dt} \qquad (2\text{-}7)$$

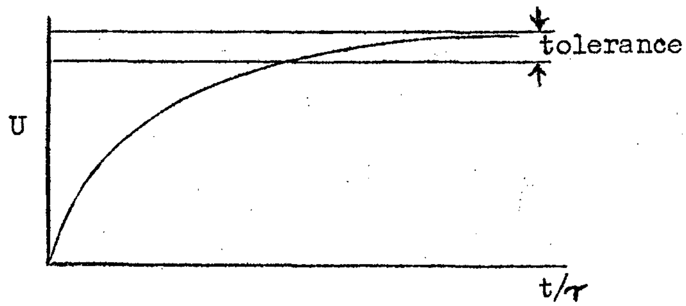
Whence

$$i = \frac{V}{R}\left(1 - e^{-\frac{t}{L/R}}\right) \qquad (2\text{-}8)$$

The same may be used to represent the variation if τ is taken to be L/R for this case.

From the above, it is evident that the higher the inductance or the lower the resistance, the longer will it take to build up the current to the steady state.

Time to put power into a resonant circuit.



Suppose we have a pulsed accelerator (where the power is applied in pulses, with intervals between.) The voltage will build up as shown above. It may be questioned, why not put higher power in at the start, and thereby reduce the time? The answer to this is, if higher power is available, why not use it all the time rather than only at the start? This leads to the conclusion that the maximum available power input should be used at all times, to decrease the build up time.

We may write $$P = \frac{dU}{dt} + P_r \qquad (2\text{-}9)$$

Power Input	Rate of Change of Stored Energy	Power losses due to resistance

Since $$Q = \frac{U}{P_r/\omega} = 2\pi\frac{\text{Energy stored}}{\text{Power loss/cycle}}$$

$$P_r = \frac{\omega U}{Q}$$

(2-9) becomes

$$P = \frac{dU}{dt} + \frac{\omega U}{Q} \qquad (2\text{-}10)$$

This has the solution (see appendix for derivation)

$$U = \frac{PQ}{\omega}(1-e^{-\omega t/Q}) \qquad (2\text{-}11)$$

The same curve for build up will apply again, by letting the time constant $\tau = \omega/Q$ and using as ordinate

$$\frac{U \text{ at any time}}{U \text{ final}}$$

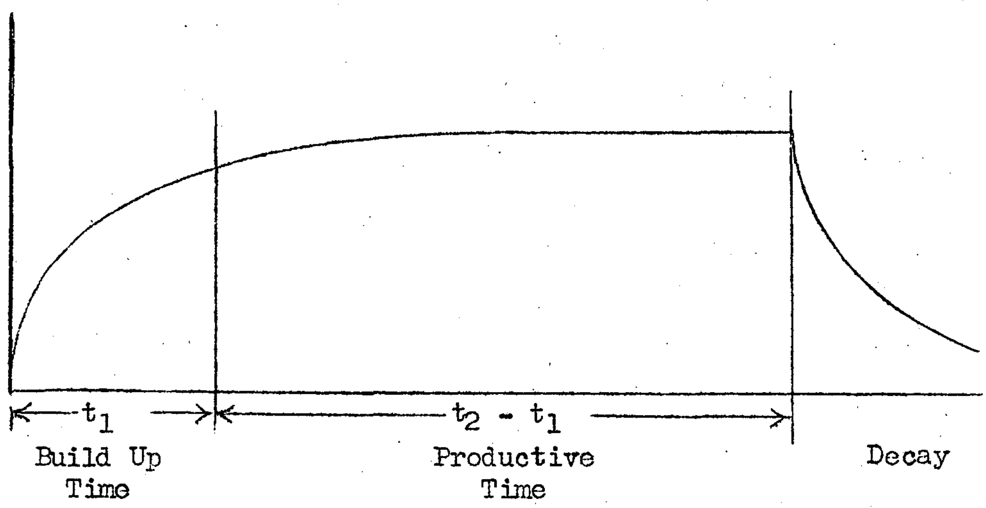
If the tolerance is 5%, $\tau = 3.0$ so the build up time will be

$$t = \frac{3Q}{2\pi} \text{ cycles}$$

As an example, if $Q = 200{,}000$

$$\frac{3 \times Q}{2\pi} \sim 100{,}000 \text{ cycles}$$

The pulse length must take account of this time to build up



For $\omega =$ 12 megacycles and $\tau = 3.0$

$$t_1 = \frac{100{,}000}{12{,}000{,}000} = \frac{1}{120} = 0.0083 \text{ seconds}$$

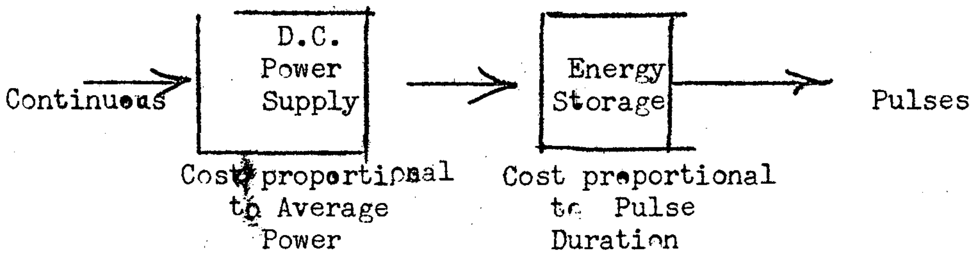
The efficiency of the pulse will then be

$$\frac{t_2 - t_1}{t_2}$$

The efficiency of the pulse can be increased by making the productive time longer, so the question is raised, how do we benefit from pulsing?

The answer to this is a question of economy. If a high power level is required for the desired particle acceleration this can be obtained more economically by pulsed operation, since smaller power equipment can be used if it has provision for storing energy at a low constant rate and releasing it in short pulses at a high rate. Furthermore, some equipment, such as the oscillator tubes, can be run on pulsed operation at very much higher rates than their continuous rating, since the limiting condition is often one of cooling the tube.

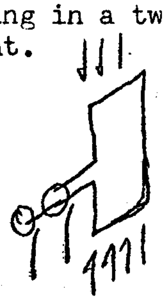
This may be illustrated thus



(In the equation for variation of voltage for an alternating current

$$V = V_o \cos \omega t$$

(ω has been referred to in the notes on the first lecture as the frequency. In this sense ω is the number of radians per second that a loop rotating in a two pole magnetic field would turn to produce the alternating current.



Since one revolution requires 2ω radians the number of cycles per second will be $f = \omega/2\pi$, and the time for one cycle will be $t = \frac{1}{f} = \frac{2\pi}{\omega}$)

Energy Released on Occurrence of a Fault

The energy stored in a resonant circuit in the steady state is, from (2-2)

$$U = \frac{\omega_o{}^{Q}}{P_R}$$

where P_R is the power loss due to resistance, which in the steady state is equal to the power input. The energy storage is a matter of concern in connection with protective devices. The power dissipation may become very high in a circuit with high energy content. Consider, for example the power dissipation if the energy of the circuit is released in a single cycle, as by the occurrence of a short. The time for one cycle is $2\pi/\omega$ so the power dissipated in one cycle will be

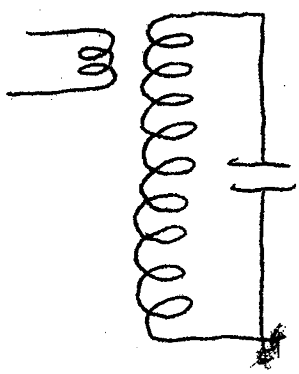
$$U/\frac{2\pi}{\omega}$$

The energy storage in the 40 ft. linear accelerator at the U. C. Radiation Laboratory is about 400 watt seconds. If this power is discharged in 1 microsecond, the instantaneous power would be 400 megawatts. For a frequency of 150 megacycles, a discharge in one cycle would take only $\frac{1}{150}$ second and the power would be 60,000 megawatts.

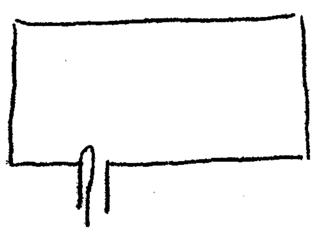
This is all that will be said about LC circuits.

Putting Power into a Resonant Cavity

Power may be put into a resonant cavity to excite it by electromagnetic or electrostatic means. To put it in by electromagnetic means we may in effect make a transformer by linking the power source into a portion of the inductive circuit of the circuit.



Actually this can be done by putting a loop into the cavity, feeding it thru a hole in the side, thus



(actually the loop plane is normal to the axis)

Since it is desirable to have a voltage multiplication, and the secondary (which is the cavity) has in effect only one turn, this can be done by having the loop enclose only a fraction of the field of the cavity. This makes a transformer with a ratio of turns n_1/n_2. Now when one circuit is connected to another by a transformer the impedance of the transformer on the input side will depend on the impedance on the output side times the square of the turns ratio, or

$$R_{input} = \left(\frac{n_1}{n_2}\right)^2 R_s$$

Suppose it is desired to develope 100,000,000 volts in a cavity, and only 10^4 volts can be used in the primary. This means that

$$\frac{n_1}{n_2} = \frac{10^4}{10^8} = 10^{-4}$$

This means that the input impedance will be

$$R_{input} = \left(\frac{n_1}{n_2}\right)^2 R_{output} = (10^{-4})^2 R_{output} = 10^{-8} R_{output}$$

For such a cavity the measured shunt impedance will be

$$R_s = Q\sqrt{L/C} \sim 3 \times 10^8 \text{ ohms}$$

This means that the input impedance R_{input} will be $3 \cdot 10^8 \cdot 10^{-8} \sim 3$ ohms

Let us see what this means in the primary circuit. The power in the primary is

$$P = \frac{V^2}{R_{input}}$$

If the voltage in the primary is limited to 10^4 Volts and R_{input} is 3 ohms,

$$P = \frac{(10^4)^2}{3} = 3{\cdot}3{\cdot}10^7 \text{ watts.}$$

or 33 megawatts

This is impracticable at the present time.

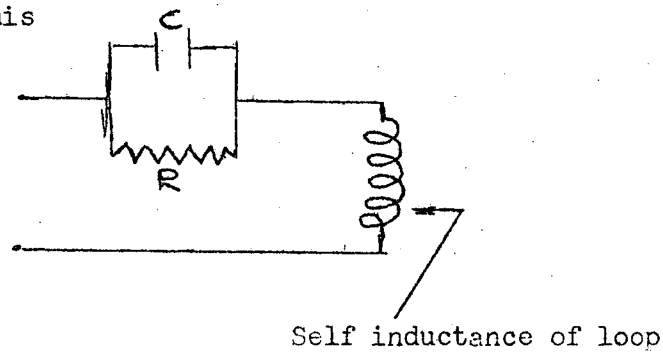
The maximum continuous power rating for the largest oscillator now available is 0.5 megawatt.

The answer for this situation is to use multiple power sources to supply the power. If 10 such power sources are used, $\left(\frac{n_1}{n_2}\right)^2$ must be 10 times as large or the impedance of the primary would be 10 x 3 ohms $= 30$ ohms and $\left(\frac{n_1}{n_2}\right)^2 = 10^{-7}$ instead of 10^{-8} making $\frac{n_1}{n_2} \sim \sqrt{10} \cdot 10^{-4} \sim 3.16 \cdot 10^{-4}$

Suppose we already have such a system and a new power source 10 times as large is developed. The primary must then have to be made smaller in the ratio of $1/\sqrt{\text{number of power sources}}$. This requires a smaller loop for higher power input.

This illustrates one of the difficulties of driving a large cavity from a single source

At resonance the reactance of an LC circuit acts as a pure resistance and the formula above would make it appear that the input impedance on the primary side would also appear as a pure resistive load. Actually all of the field of the primary will not link into all of the turns of the secondary, and the loop will still appear to have some self inductance. This is leakage reactance. The primary should have a low self impedance to minimize the input impedance. The circuit carrying power to the loop will be equivalent to this



To keep the input impedance low, with respect to R, the loop must be designed with thick wide sheets, rather than a round wire.

LECTURE 2

Appendix

Derivation of Solution of Equation (2-10)

$$P = \frac{dU}{dt} + \frac{\omega U}{Q} \tag{2-10}$$

Let $$z = P - \frac{\omega U}{Q} \tag{a}$$

$$\frac{dz}{dt} = -\frac{\omega}{Q}\frac{dU}{dt} \tag{b}$$

Substitute in (2-10)

$$z = \frac{Q}{\omega}\frac{dz}{dt} \tag{c}$$

$$\frac{dz}{z} = -\frac{\omega\, dt}{Q} \tag{d}$$

$$\ln z = -\frac{\omega}{Q}t + C \tag{e}$$

This may put in the form

$$\ln\left(\frac{z}{c_1}\right) = -\frac{\omega t}{Q} \tag{f}$$

from (a) $$z\Big|_{t=0} = P \text{ , since } U = 0 \text{ at } t = 0 \tag{g}$$

from (f) at t = 0

$$\ln\left(\frac{z}{C_1}\right)\Big|_{t=0} = 0 \text{ , from which and (g)}$$

$$C_1 = P$$

So (e) becomes

$$\ln\left(\frac{z}{P}\right) = -\frac{\omega t}{Q} \tag{h}$$

whence $$z = Pe^{-\frac{\omega t}{Q}} \tag{j}$$

substituting (j) in (a)

$$Pe^{-\frac{\omega t}{Q}} = P - \frac{\omega U}{Q} \tag{k}$$

Solving for U

$$U = \frac{PQ}{\omega}\left(1 - e^{-\frac{\omega t}{Q}}\right) \tag{l}$$

That this is the solution may be verified by differentiating and substituting back in (2-10).

FLM 12/5/50

ERRATA

LECTURE 2

Page 2 — Next to last line on the page

power input is constant the conditions . . . . .

Page 3 — On the diagram

Constant Power Input

Page 4 — Line below diagram

τ = RC is the time constant of the circuit

Fourth line below diagram

only be reached exactly when $t/\tau = \infty$.

Page 6 — Line 4

constant $\tau = Q/\omega$

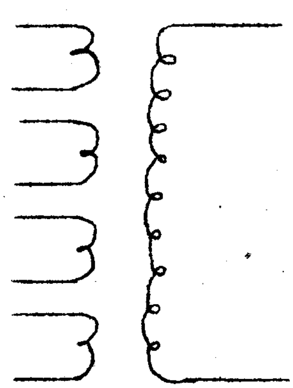
Page 7 — Line below sketch

Since one revolution requires 2π radians

Lower equation

$$U = \frac{Q P_R}{\omega}$$

Page 9 — Added diagram -



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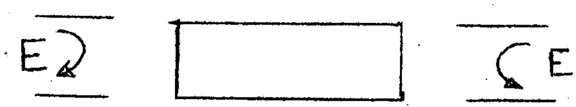
LECTURE 3

By Dr. Andrew Longacre
12/18/50

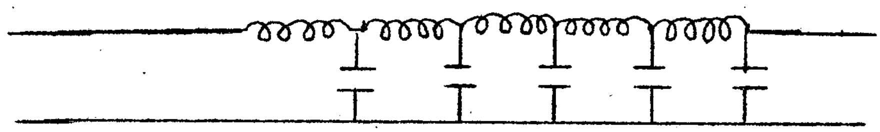
TRANSMISSION LINES

A transmission line is for the purpose of transmitting alternating currents from a source to a point of use. The most general treatment is to consider a disturbance in a field which is transmitted thru the field. This is inserted in Maxwell's Equations, together with the boundary conditions, and a solution is sought. For some cases this can be done but in other cases difficulties arise.

Another method, as discussed in Slater's "Micro-Wave Transmission Lines" is to consider the line as a series of 4-Terminal Networks



Still another technique is to consider the line as a series of lumped constants



We will discuss first an elementary physical picture of what happens in a transmission line and then express this more fully and more mathematically.

It will be convenient since the line may be of considerable length to consider the various characteristics of the line in quantities per unit length.

The line has a characteristic Inductance/Unit Length = L

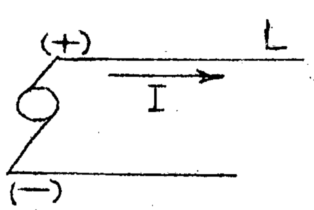
a " Resistance/Unit Length = R

a " Capacity/Unit Length = C

and a " Transconductance or Leakage / Unit Length = G

The last item concerns the current which leaks across the insulation from one conductor to the other and does not go all the way down the line.

For a simple picture, consider a DC generator connected momentarily to a transmission line. A current starts to flow into the line. Since the line has inductance per unit length, a counter



FLM 12/27/50

emf is built up, because the inductance tends to resist a change in the current. (This is analogous to the reaction that the inertia of mass presents to a force tending to accelerate it.) The value of the back emf is

$$V = \frac{d(\phi x)}{dt} \qquad (3\text{-}1)$$

where ϕ = flux per unit length

x = length

(By flux is meant the total strength of the magnetic field. Flux per unit length is the total magnetic field perpendicular to the plane thru the two conductors per unit length along the transmission line. The flux at any point is

$$\phi = LI \qquad (3\text{-}2)$$

For this example it will be assumed that current starts to flow abruptly on contact and remains at a constant value until the contact is interrupted. This is a "square wave." Under these conditions the flux does not vary with time from front to back of the wave so (3-1) can be written

$$V = \phi \frac{dx}{dt} \qquad (3\text{-}3)$$

Now $\frac{dx}{dt}$ is the rate at which the wave moves, or $v = \frac{dx}{dt}$

$$V = \phi v = LIv$$

The current also establishes a charge

$$Q = CV \text{ per unit length}$$

between the opposite portions of the two conductors where C is the capacity per unit length. The current is equal to the rate of change of the charge, or

$$I = \frac{dQ}{dt} = \frac{d(VCx)}{dt} = VCv \qquad (3\text{-}5)$$

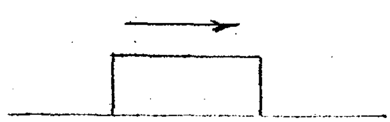
Combining equations (3-4) and 3-5)

$$\frac{V}{I} = Lv = \frac{1}{Cv} \qquad (3\text{-}6)$$

whence $v^2 = \frac{1}{LC}$

$$\text{or } v = \frac{1}{\sqrt{LC}} \qquad (3\text{-}7)$$

If the generator has been disconnected shortly after it was connected, the current flow would represent a square pulse. The question arises, what keeps the rear end moving forward instead of dying out in place? When the source is disconnected the applied voltage drops to zero.



The magnetic field starts to die out, but as it does so it generates a voltage opposing the change. That is, the induced emf at the rear of the pulse attempts to keep the current flowing. This keeps the rear end of the pulse moving forward.

In equation (3-6) we had $\frac{V}{I} = Lv$

and in (3-7) $v = \frac{1}{\sqrt{LC}}$

Combining these

$$\frac{V}{I} = \frac{L}{\sqrt{LC}} = \sqrt{L/C} = Z_c \qquad (3\text{-}8)$$

Z_c is the CHARACTERISTIC IMPEDANCE of the line, and is measured in ohms.

As the pulse passes down the line, energy is being stored in the magnetic and electric fields that are established, and this energy is restored to the line at the rear end of the pulse.

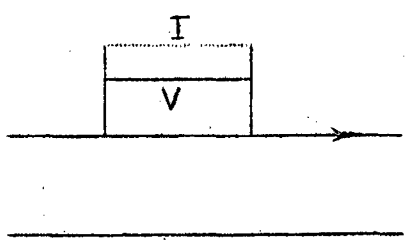
We may express the velocity as follows:

$$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu \epsilon}} = 3 \times 10^{10} \text{ cm/sec} \qquad (3\text{-}9)$$

In the above μ is the inductance of free space and ϵ is the dialectric capacity of free space. When these are expressed in appropriate units the answer is the velocity of light in free space.

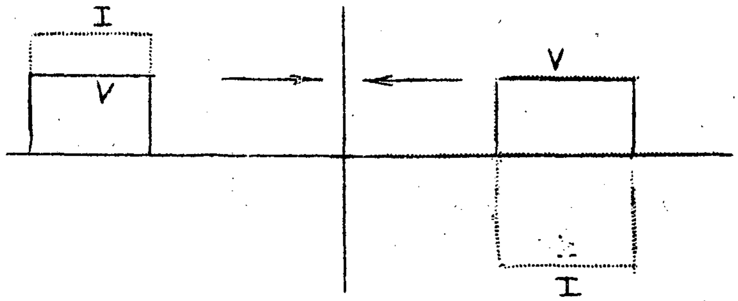
In the development above no account was taken of the resistance of the line. In most cases the resistance of transmission lines used in connection with particle accelerators is low, so that computation of their electrical behavior can be made with sufficient accuracy by omitting the resistance. This considerably simplifies the algebra. However, even though the resistance terms are omitted from the electrical computations, there may be large amounts of heat generated by the heavy current flows and cooling must be provided to take the heat away.

We have considered how a pulse travels down the line. Now the question arises, what happens at the end of the line? This depends on what is at the end of the line. In general we can expect some thing to come back.



Consider a rectangular pulse moving down a line. This will have a voltage and a current, as shown above. There are two extreme cases that may be considered first. The terminous may be open, or it may be shortcircuited. For the open terminus the current at the end must be zero. How can we picture what happens?

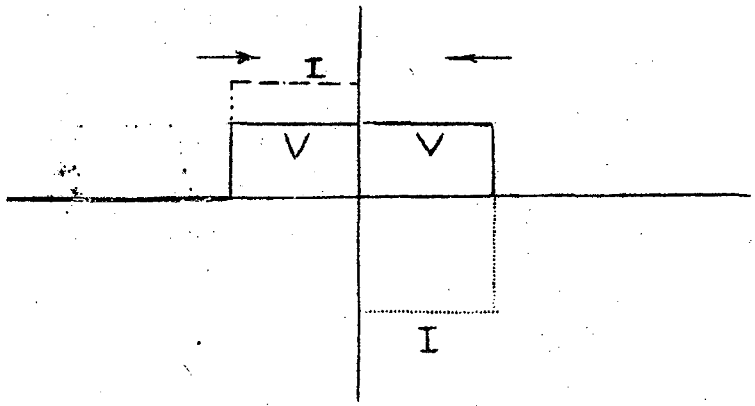
Suppose we consider an imaginary extension of the line with an imaginary current and voltage the same as the real pulse, but moving in the opposite direction



(a)

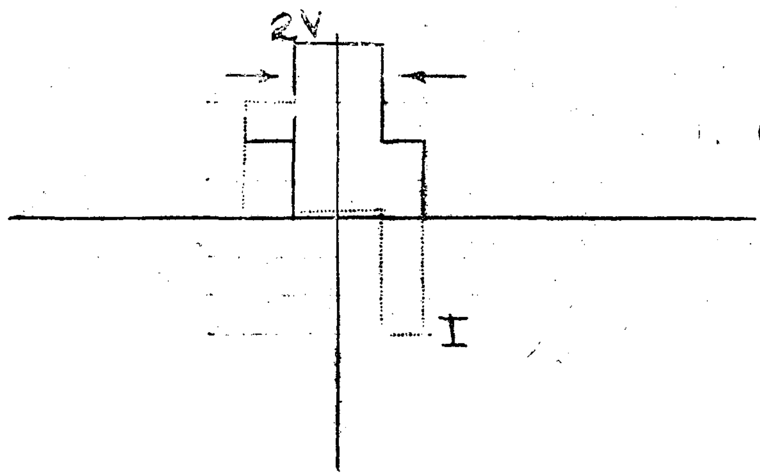
If both current and voltage are in phase for the pulse moving to the right, and the current at the terminus is zero, then the imaginary pulse will have to have a current of opposite phase, so the current of the two pulses will neutralize at the terminus. The voltages should be in the same phase, since the circuit is open.

Just as the pulse reaches the end of the line from the left, the imaginary pulse reaches the same point from the right



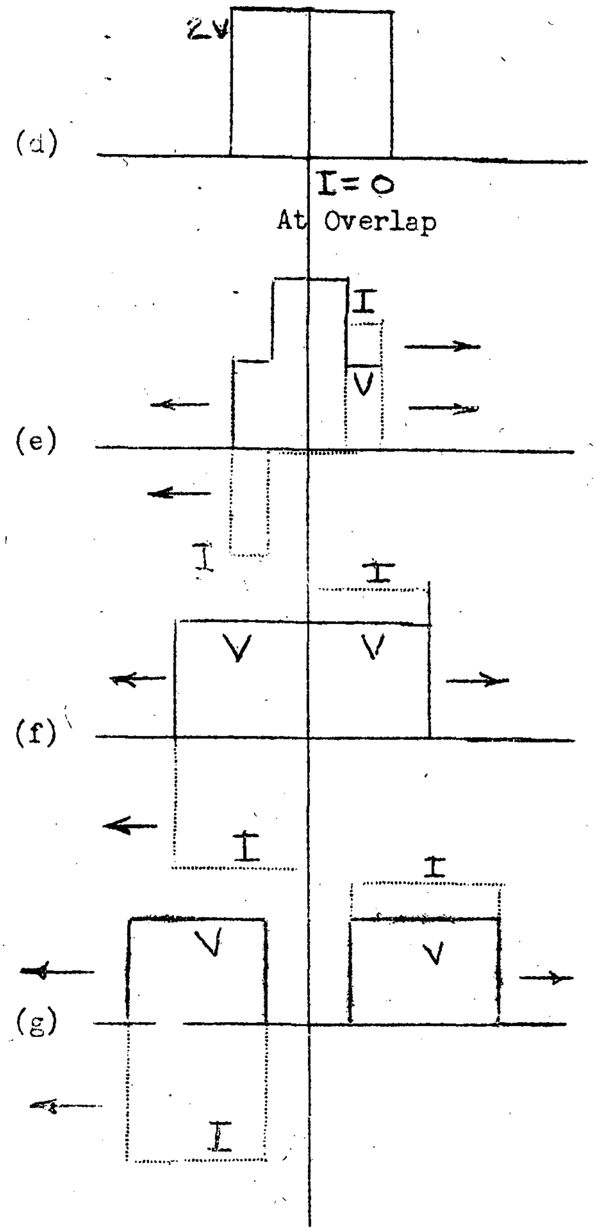
(b)

Now the voltages of the two pulses are in phase and add up where the two waves overlap, while the two currents are of opposite phase and nullify each other just at the end. Shortly after this the condition is as follows



(c)

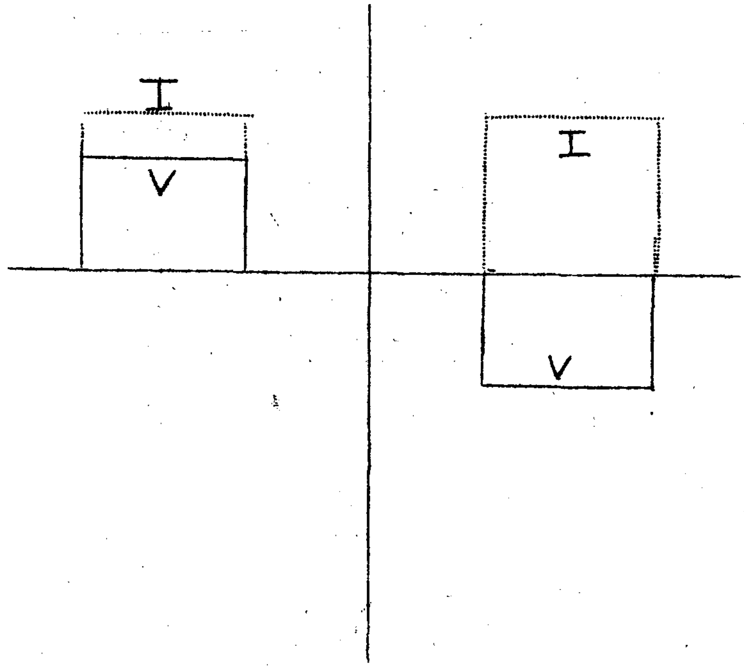
and then, in succession



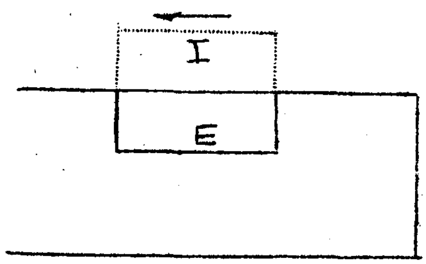
The voltages of the real and imaginary pulses add up while they are passing over the section of the line where they overlap, while the currents nullify each other over the sections of overlap. Finally when the elapsed time is emf for the width of the pulse to pass the end of the line, we have a pulse traveling to the left, but with current and voltage of opposite phase.

We may look at this as the result of two pulses of equal magnitude, but moving in opposite directions, meeting and passing at the juncture of the real line and its imaginary extension, and each pulse continuing in its original direction unchanged after they have passed each other, or we may consider that the pulse moving to the right has been reflected, but with a change in phase of voltage and current, and that the imaginary pulse moving to the left has also been reflected with change of phase.

Now consider what happens when the line is short circuited. In this case there can be a current at the end of the line, but no voltage.

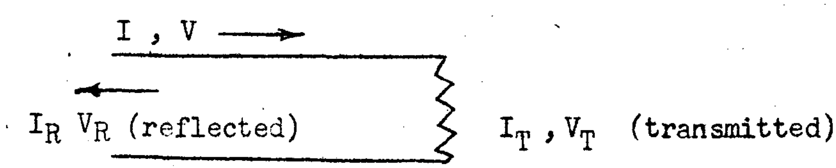


The imaginary pulse moving to the left must be considered as having the current of the same phase as that moving to the right but the voltage of opposite phase. The same process of reflection will occur except that the current will be reflected in the same phase as the original pulse with the voltage of opposite phase



Shorted Terminus

Now consider an intermediate case, where the two conductors are connected at the end by a resistance.



The square pulse coming from the left will have a voltage V in one conductor with respect to the other, and Current I. We may assume that there will be a reflected pulse having a current I_R and a voltage V_R ; also that there will be a current I_T thru the terminal resistance and a voltage V_T across it.

The current thru the terminal resistance will be, algebraically,

$$I_T = I + I_R \qquad (3\text{-}10)$$

and the voltage

$$V_T = V + V_R \qquad (3\text{-}11)$$

Now from (3-8)

$$\frac{V}{I} = + Z_c \qquad (3\text{-}12)$$

If the incoming wave has voltage and current in phase, the reflected wave will have voltage and current of opposite phases. Which of these is positive depends on the ratio of Z_T to Z_c, or Terminal Impedance to Characteristic Impedance of the line.

An open circuit is an extreme example of Z_T being larger than Z_c and a short circuit is an extreme example of Z_T being smaller than Z_c. Since the current and voltage phase relations are changed, the impedance has a negative sign for the reflected wave

so

$$\frac{V_R}{I_R} = - Z_c \qquad (3\text{-}13)$$

Also

$$\frac{V_T}{I_T} = R_T \qquad (3\text{-}14)$$

Combining these

$$I_T R_T = Z_c I - Z_c I_R \qquad (3\text{-}15)$$

$$R_T (I+I_R) = Z_c (I - I_a) \qquad (3\text{-}16)$$

whence

$$I_R = \frac{I (Z_c - R_T)}{(Z_c + R_T)} \qquad (3\text{-}17)$$

Let us check this for the two extreme cases of open and closed terminus.

For the open terminus, $R_T = \infty$

and

$$I_R = \frac{I (Z_c - \infty)}{(Z_c + \infty)} \qquad (3\text{-}18)$$

This is not convenient to draw conclusions from, so convert (3-17) by dividing numerator and denominator by R_T. This gives

$$I_R = \frac{I(\frac{Z_c}{R_T} - 1)}{(\frac{Z_c}{R_T} + 1)} \qquad (3\text{-}19)$$

When $R_T = \infty$, this becomes

$$I_R = - I \qquad (3\text{-}20)$$

This is what we got before for this case when $R_T = 0$

$$I_R = I$$

which is again what we got before for this case.

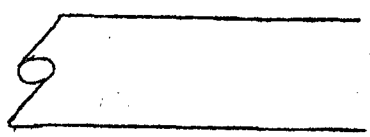
If $R_T = Z_c$,

$$I_R = 0 \qquad (3\text{-}22)$$

That is, no current is reflected, and all the power is consumed in the load. Since normally the transmission line is intended to put the power into the load, this indicates that this will be accomplished most efficiently by having a terminal resistance that is equal to the characteristic impedance of the transmission line.

SINUSOIDAL CURRENTS IN TRANSMISSION LINES

Now consider what happens if instead of a square wave, sinusoidal voltage and current is applied to one end of a transmission line having characteristics



R, L, G, and C per unit length.

In this case the voltage on the wave will change with x. We may write

$$\frac{dV}{dx} = - IR - L\frac{dI}{dt} \qquad (3\text{-}23)$$

For the current

$$\frac{dI}{dx} = - VG - C\frac{dV}{dt} \qquad (3\text{-}24)$$

which says that the rate of change of current along the line is proportional to the leakage and to the rate of energy storage in the capacity of the line.

In order to simplify the development, we will consider the case where R = 0 and G = 0, which gives a good approximation for most of our cases. After computing the current, however, provision must still be made for removing the heat lost thru the resistance, which may be considerable.

(3-24) in the simplified form will then be

$$\frac{dI}{dx} = - C\frac{dv}{dt} \qquad (3\text{-}25)$$

Differentiating again with respect to x

$$\frac{d^2I}{dx^2} = -C \frac{d}{dx} \frac{dV}{dt} \qquad (3\text{-}27)$$

Interchanging the order of differentiation

$$\frac{d^2I}{dx^2} = -C \frac{d}{dt} \frac{dV}{dx} \qquad (3\text{-}28)$$

But $$dV = -L \frac{dI}{dt} dx \qquad (3\text{-}29)$$

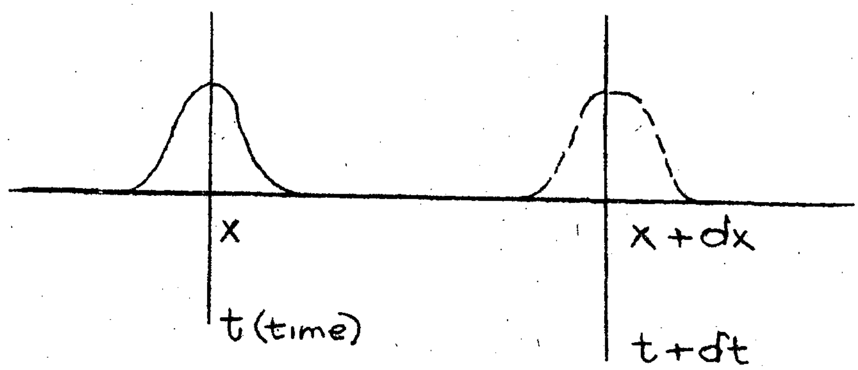
Whence $$\frac{d^2I}{dx^2} = LC \frac{d^2I}{dt^2}$$

This is generally known as the wave equation. Solutions of this equation are always some form of

$$I = f(x - vt) \qquad (3\text{-}31)$$

The function may be a cosine function or other shape.

What do we mean by a wave? It may be considered as hump passing down the line



For the wave to move with its shape unchanged, the value of I must be the same at a point δx further along which will be reached at a time δt later. Substituting these conditions in (3-31), and noting that of the wave is traveling at velocity v;

$$\delta x = v\,\delta t$$

$$I = f\{x + v\,\delta t - v(t + \delta t)\}$$
$$= f(x - vt) \qquad (3\text{-}32)$$

which is the same as (3-31)

If the wave is to travel in the negative direction, the velocity must be made negative, giving

$$I = f\{x - v\,\delta t + v(t + \delta t)\} \qquad (3\text{-}33)$$
$$= f(x + vt) \qquad (3\text{-}34)$$

We may check that (3-32) is a solution of the wave equation as follows

$$I = f(x - vt)$$

$$\frac{dI}{dx} = f'(x - vt)$$

$$\frac{d^2I}{dx^2} = f''(x - vt)$$

Also $\frac{d^4I}{dt} = f'(x - vt)(-v)$

$$\frac{d^2I}{dt^2} = f''(x - vt)\, v^2$$

Substituting in the wave equation (3-30)

$$f''(x - vt) = LCf''(x - vt)\, v^2$$

$$1 = v^2 LC$$

Or $$v = \frac{1}{\sqrt{LC}} \qquad (3\text{-}35)$$

This shows that the wave equation is satisfied for a wave moving with velocity $v = \frac{1}{\sqrt{LC}}$

A cosine or sine function will also satisfy the wave equation, as will exponential functions. This can be shown by using the exponential expression for the cosine function, as follows:

Let
$$I = I_o e^{j(\omega t - kx)} \qquad (3\text{-}36)$$

where I_o is the peak value of the current and I is value at any time t

If this is put in the form

$$I = I_o e^{-jk(-\frac{\omega t}{k} + x)}$$

$$= I_o e^{-jk(x - \frac{\omega t}{k})} \qquad (3\text{-}37)$$

it is obvious that this is of the form

$$I = f(x - vt)$$

providing $\frac{\omega}{k} = v$, or $k = \frac{\omega}{v}$

Now $= 2\pi f$

Where f is the frequency

So $$k = \frac{2\pi f}{v}$$

But $\frac{v}{f} = \lambda$, the wave length,

Therefore $$k = \frac{2\pi}{\lambda} \qquad (3\text{-}38)$$

Taking second derivatives of (3-37) with respect to x and t

$$\frac{d^2I}{dx^2} = -k^2 I_o e^{-jk(-\frac{\omega t}{k} + x)} \qquad (3-39)$$

$$\frac{d^2I}{dt^2} = -\omega^2 I_o e^{-jk(-\frac{\omega t}{k} + x)} \qquad (3-40)$$

Substituting in the wave equation

$$\frac{d^2I}{dx^2} = LC \frac{d^2I}{dt^2}$$

$$-k^2 I_o e^{-jk(x - \frac{\omega t}{k})} = -LC\,\omega^2 I_o e^{-jk(x - \frac{\omega t}{k})}$$

$$\frac{k^2}{\omega^2} = LC$$

But since $k = \frac{\omega}{v}$

$$\frac{1}{v^2} = LC$$

or $v = \frac{1}{\sqrt{LC}}$

Further consider the variation of voltage and current on the line to which a sinusoidal voltage is applied. The rate of change of voltage with respect to distance will be

$$\frac{dV}{dx} = -L\frac{dI}{dt}$$

If the voltage at any time is represented by

$$V = V_o e^{j(\omega t - kx)} \text{ and } I = I_o e^{j(\omega t - kx)}$$

where V_o is the peak voltage in the cycle

Then $\frac{dV}{dx} = -jk V_o e^{j(\omega t - kx)}$

$$= -j\omega L I_o e^{j(\omega t - kx)}$$

Whence $k V_o = \omega L I_o$

$$\frac{V_o}{I_o} = \frac{\omega L}{k} = \frac{V}{I}$$

$$= \frac{L}{\sqrt{LC}} = \frac{1}{\sqrt{L/C}} = Z_c \qquad (3-41)$$

 which is the characteristic Impedance of the line.

This represents a wave to the right.

For a wave to the left, the impedance is negative and

$$\frac{V_o}{I_o} = -Z_c = -\frac{1}{\sqrt{L/C}} = \frac{-\omega L}{k} \qquad (3\text{-}42)$$

More general expressions for I and V

$$I = A_o\, e^{j(\omega t - kx)} + B_o\, e^{j(\omega t - kx)} \qquad (3\text{-}43)$$

$$V - Z_c\, A_o\, e^{j(\omega t - kx)} - Z_c\, B_o\, e^{j(\omega t + kx)} \qquad (3\text{-}44)$$

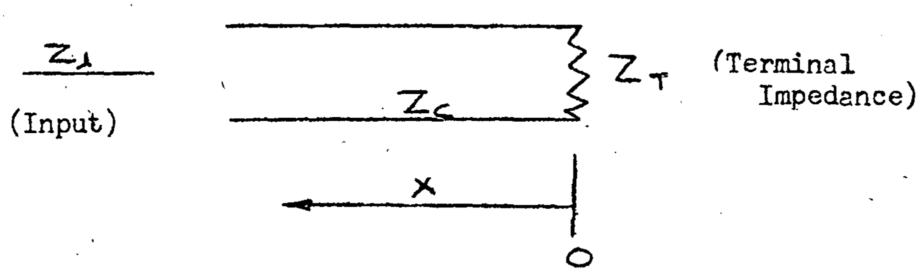
These represent a wave A from the left and a wave B from the right. In the expression for V, the impedance Z_c is taken negative for the B wave, and the sign of x also changes to indicate movement of this wave to the left.

If B = 0, this means no wave is coming back. This is, in general, what is desired, as it is usually desired to deliver power to the terminous, which occurs most efficiently when there is no energy reflected.

To determine what conditions favor this, we must find out what the terminal impedance does to the input wave.

Now usually the terminal conditions are fixed, and the problem is what must the input be. It is consequently conventional to measure the distance not from the input to the output, but from the output or terminus backward. This is done by changing the sign of x to negative.

We then have



This will change the general equation (3-43) to

$$I = A_o\, e^{j(\omega t + kx)} + B_o\, e^{j(\omega t - kx)} \qquad (3\text{-}45)$$

The terminal impedance will not always be a pure resistance, but at some point short of this, the effect of the line taken into account with the actual impedance of the terminal, will be such that at this point the section of line plus the terminal will appear as a pure resistance.

Equation (3-44) becomes

$$V = Z_c A_o e^{j(\omega t + kx)} - Z_c B_o e^{j(\omega t - kx)}$$

$$Z_i = \frac{V}{I} = Z_c \frac{A_o e^{jkx} - B_o e^{-jkx}}{A_o e^{jkx} + B e^{-jkx}} \qquad (3\text{-}46)$$

(Input Impedance)

(the term e^{jwt} cancels out of all the terms in numerator and denominator)

At $x = 0$, $Z_i = Z_t$, whence, since $e^o = 1$,

$$Z_i = Z_t = Z_c \frac{A_o - B_o}{A_o + B_o} \qquad (3\text{-}47)$$

From this the ratio $\frac{B_o}{A_o}$ may be obtained as follows

$$Z_t (A_o + B_o) = Z_c (A_o - B_o)$$

$$B_o (Z_t + Z_c) = A_o (Z_c - Z_t)$$

$$\frac{B_o}{A_o} = \frac{Z_c - Z_t}{Z + Z_c} \qquad (3\text{-}48)$$

Substituting this in (3-46)

$$Z_i = Z_c \frac{e^{jkx} - \frac{B_o}{A_o} e^{-jkx}}{e^{jkx} - \frac{B_o}{A_o} e^{-jkx}}$$

$$= Z_c \frac{e^{jkx} - \left(\frac{Z_c - Z}{Z_t + Z_c}\right) e^{-jkx}}{e^{jkx} + \left(\frac{Z_c - Z}{Z_t + Z_c}\right)}$$

$$= Z_c \frac{(Z_t + Z_c) e^{jkx} - (Z_c - Z_t) e^{-jkx}}{(Z_t + Z_c) e^{jkx} + (Z_c - Z_t) e^{-jkx}}$$

Rearranging

$$Z_i = Z_c \frac{Z_t (e^{jkx} + e^{-jkx}) + Z_c (e^{jkx} - e^{-jkx})}{Z_c (e^{jkx} + e^{-jkx}) + Z_t (e^{jkx} - e^{-jkx})}$$

$$= Z_c \frac{Z_t \cos kx + Z_e\, j \sin kx}{Z_c \cos kx + Z_t\, j \sin kx}$$

$$= Z_c \frac{(Z_t/Z_c) + j \tan kx}{1 + j\,(Z_t/Z_c) \tan kx}$$

whence

$$\frac{Z_i}{Z_c} = \frac{(Z_t/Z_c) + j \tan kx}{1 + j\,(Z_t/Z_c) \tan kx} \qquad (3\text{-}49)$$

Letting $a = \frac{Z_t}{Z_c}$, this becomes

$$\frac{Z_i}{Z_c} = \frac{a + j \tan kx}{1 + j\, a \tan kx} \qquad (3\text{-}50)$$

Z_i is obviously not a pure resistance even though Z_t may be resistive only, unless $kx = 0$ or $N \frac{\pi}{2}$ where N is an odd integer.

(The case for $kx = N \frac{\pi}{2}$ at first glance would appear to be indeterminate, of the form $\frac{\infty}{\infty}$. By dividing numerator and denominator by j tan kx, (3-50) becomes

$$\frac{\frac{a}{j \tan kx} + 1}{\frac{1}{j \tan kx} + a}$$

When $kx = \pi/2$ this becomes $\frac{0 + 1}{0 + a} = \frac{1}{a}$)

Multiply numerator and denominator of (3-50) by $(1 - j\, a \tan kx)$

$$\frac{Z_i}{Z_c} = \frac{a\,(1 + \tan^2 kx) + j\,(\tan kx - a^2 \tan kx)}{1 + a^2 \tan^2 kx} \qquad (3\text{-}51)$$

This is of the form $\frac{Z_i}{Z_c} = r_i + j\,X_i$

The resistive component is

$$r_i = \frac{a\,(1 + \tan^2 kx)}{1 + a^2 \tan^2 kx} \tag{3-52}$$

The reactive component is

$$X_i = \frac{(1 - a^2)\tan kx}{1 + a^2 \tan^2 kx} \tag{3-53}$$

This varies with x

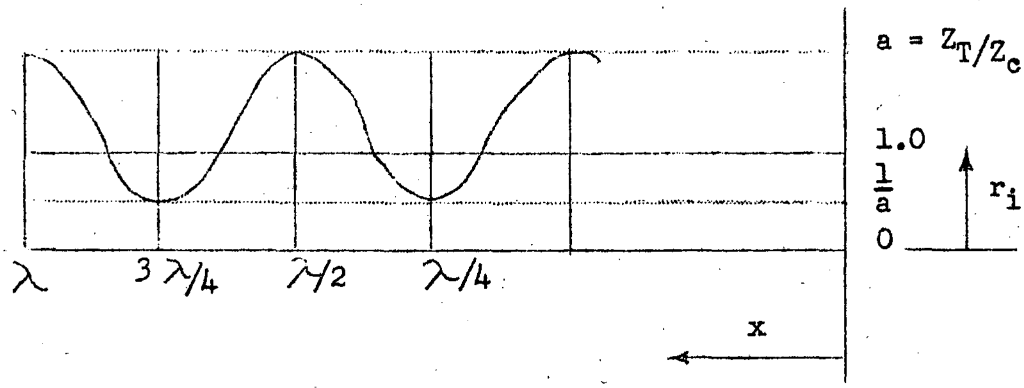
Let us look at some particular conditions

	Resistive Component	Reactive Component	Remarks
$kx = \frac{2\pi x}{\lambda}$	r_i	X_i	
$0, \pi, 2\pi$ ($\tan kx = 0$)	a	0	No reactive component when x = 0
$\pi/2, 3\pi/2, 5\pi/2$ ($\tan kx = \infty$)	1/a	0	
immaterial	a = 1	0	This is assumed for the case where $r_i = 1$ whence $Z_T/Z_c = 1$

For the case where $Z_T/Z_c = 1$ (or a = 1) the reactive component X_i (eq. 3-53) becomes zero and the resistive component r_i (eq. 3-52) equals a for all values of kx. This is to say that when the terminal impedance is equal to the characteristic impedance of the line, the input impedance will be purely resistive, and the length of the line is not critical, but may be any number of wave lengths.

It is of interest to see how the resistive component varies for the case where Z_T/Z_c is not unity. The two extremes are given in the two first cases in the table above. If the value of r_i is plotted as a function of the length of the transmission line measured in wave lengths we get something like this:

(See next page)



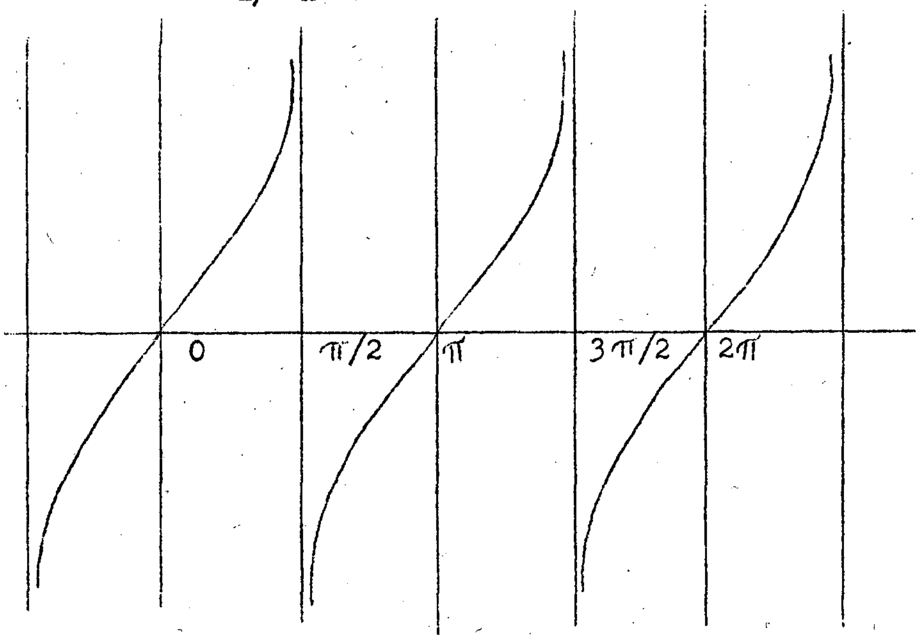
Now consider the reactive component when Z_T/Z_c is not unity. Eq. (3-53) may be put in the form

$$X_i = \frac{1 - a^2}{\tan kx \left(\frac{1}{\tan^2 kx} + a^2 \right)} \qquad (3\text{-}54)$$

When a = 0, corresponding to a short circuited terminal

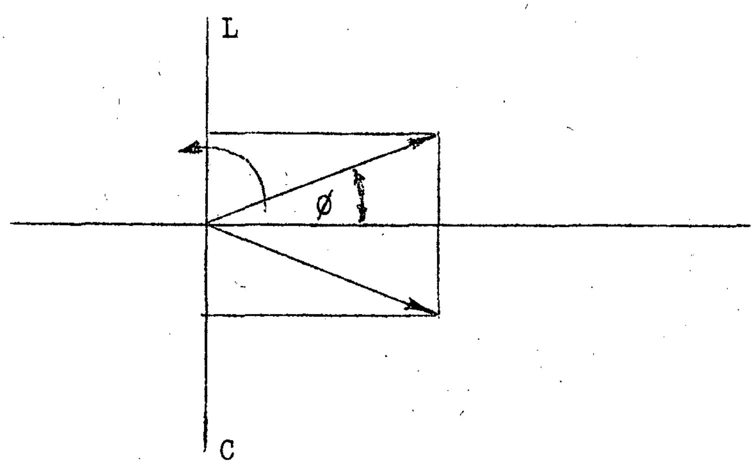
$$X_i = \frac{1}{1/\tan kx} = \tan kx \qquad (3\text{-}55)$$

This appears as shown



When kx lies between 0 and π/2, the reaction is inductive, as shown by the positive value. When kx lies between π/2 and π, the reaction is capacitive

This may be seen from the vector diagram.



Ø is the phase angle, indicating when the peak value of the current occurs relative to the peak value of the voltage. When the value of Θ is positive, the current lags behind the voltage, from the relative

$$I = I_o \cos(\omega t - \phi)$$

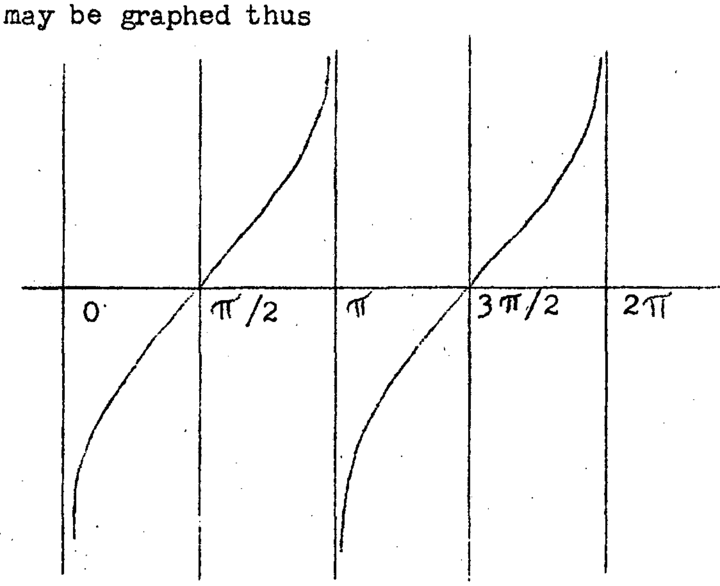
This is the effect of inductance.

When Ø is negative, the current leads the voltage, which is the effect that a capacitor has.

When the line is open at the end, a $= \infty$. Eq. (3-53) may be put in the form

$$X_i = \frac{\left(\frac{1}{a^2} - 1\right) \tan kx}{\frac{1}{a^2} + \tan^2 kx} = \frac{- \tan kx}{\tan^2 kx} = - \cot kx$$

This may be graphed thus



The reaction is capacitive for $0 < kx < \pi/2$ and inductive for $\pi/2 < k < \pi$

It follows from the diagram above showing the variation of r_i with kx, that a low resistance line can be transformed into a high one by adding (or subtracting) a quarter wave length. The range will be from 1/a to a, or the ratio of resistances from maximum to minimum will be a^2. If, for example $Z_T = 100$ ohms and $Z_c = 50$ ohms, $a = Z_T/Z_c = 100/50 = 2$.

A quarter wave length line would then have its input resistance

$$r_i = Z_c \cdot 1/a = 50/2 = 25 \text{ ohms}$$

Adding a quarter wave length

$$r_i = Z_c \cdot a = 50 \times 2 = 100 \text{ ohms}$$

A transmission line can thus act like a transformer.

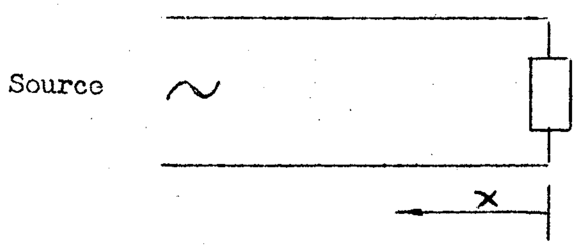
LECTURE 4 - DR. PANOFSKY

TRANSMISSION LINES

Transmission lines are of interest to us for three reasons:

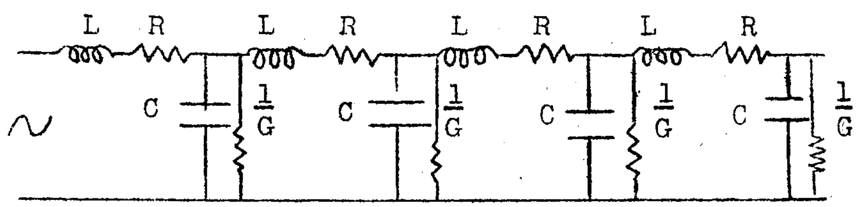
1. They may be used to transfer power to a cavity.
2. They differ from the usual lumped-constant systems in that the electric and magnetic quantities are distributed and mixed.
3. In spite of the last statement, under some conditions they can be used as lumped constants. At high frequencies there is no such thing as pure Inductance or Capacity.

This discussion will be from the point of view of a line with steady state A.C. current flow



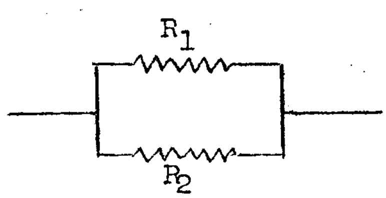
The distance x will be measured from the terminus back toward the source.

One way to consider a transmission line is to look at it as a series of small sections of unit length



L is the Inductance per unit length
C is the Capacity " "
R is the Resistance " "
$\frac{1}{G}$ is the Leakage Resistance per unit length

G is the reciprocal of the resistance. It is convenient to use reciprocals when resistances are in parallel, as is shown by this comparison



$$R_o = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{R_1 \cdot R_2}{R_1 + R_2}$$

$I = \frac{V}{R_o}$ may be applied after finding R_o

Ohms Law may be written

$I = VG$, where $G = \frac{1}{R}$ is the conductance.

For the diagram above

$$G_o = G_1 + G_2 \qquad (4\text{-}1)$$

and $$I = V(G_1 + G_2) \qquad (4\text{-}2)$$

The computations can be simplified by using reciprocals for quantities that are in parallel. The inductance and resistance of the line section above are in series, and the inductive impedance of this section of theline may be represented by

$$Z = j\omega L + R \qquad (4\text{-}3)$$

The capacity and the shunt leakage are in parallel and can be conveniently handled by adding their reciprocals. The reactance of the capacity alone

$$Z_C = \frac{-j}{\omega C}$$

The reciprocal of this is $\frac{\omega C}{-j} = j\omega C$

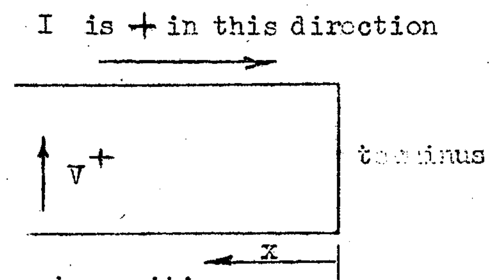
The resistance of the shunt leakage path per unit of length is $R_s = \frac{1}{G}$ and the reciprocal of this is G. The impedance of the capacity and leakage in parallel may then be written

$$Y = j\omega C + G \qquad (4\text{-}4)$$

The quantity Y is called the "Admittance"

Impedances in series are additive
Admittances in parallel are additive

Sign Conventions



distance is positive
in direction from terminus toward input.

The voltage change along a differential length of a transmission line is

$$dV = ZI\,dx \qquad (4\text{-}5)$$

The current change in differential length dx is equal to the voltage times the admittance, or

$$dI = VY\,dx \qquad (4\text{-}6)$$

Taking second derivatives

$$\frac{d^2V}{dx^2} = Z\frac{dI}{dx} = ZY\,V \qquad (4\text{-}7)$$

$$\frac{d^2I}{dx^2} = Y\frac{dV}{dx} = ZY\,I \qquad (4\text{-}8)$$

Solutions of these equations are exponentials or cosines, depending on whether the values of ZY are real or imaginary.

We now define the "Propogation Constant,"

$$K = \sqrt{ZY} \qquad (4\text{-}9)$$

and the solution to (4-7) may be shown to be

$$V = Ae^{kx} + Be^{-kx} \qquad (4\text{-}10)$$

We could solve for I in the same way, but since from (4-5),

$$I = \frac{}{Z}\left(Ae^{kx} - Be^{-kx}\right) \qquad (4\text{-}11)$$

In general

$$I = I_o e^{j\omega t} \cdot e^{kx} \qquad (4\text{-}12)$$

taking the real part. I_o is the peak voltage. If k is real, the line is attenuating, that is the amplitude of the current swing decreases exponentially with distance. If k is imaginary, we get standing waves.

Expanding (4-9)

$$k = j\,\omega\sqrt{\left(L - j\frac{R}{\omega}\right)\left(C - \frac{jG}{\omega}\right)} \qquad (4\text{-}13)$$

If R and G are zero (no resistance, no leakage) the expression is all imaginary and there is no attenuation, altho there will be a sinusoidal variation along the line.

If R and G are appreciable, there will be some attenuation.

In very high frequency circuits, the loss terms are small, and it simplifies computations to assume that R and G are zero; and compute the corresponding I and V. Having found I and V, and knowing the resistances, the actual value of the losses can be computed, as cooling may have to be provided for this amount of loss. Therefore figure on the basis that

$$k = j\,\omega\sqrt{LC} \qquad (4\text{-}14)$$

and compute I^2R as the loss.

FIM 1/8/51

In (4-12)

$$I = I_o e^{j\omega t} \cdot e^{kx} = I_o e^{j\omega t + kx} \quad (4\text{-}12)$$

$$= I_o e^{j\omega(t \pm \sqrt{LC} \cdot x)} \quad (4\text{-}15)$$

$$= I_o \cos\omega\,(t \pm \sqrt{LC} \cdot x) \quad (4\text{-}16)$$

This represents a wave.

At a given time t, I varies sinusoidally with x.

At a given x, I varies sinusoidally with t.

The wave moves with velocity v, that is to say

$$x \longrightarrow x + \Delta x$$

$$\text{when} \quad t \longrightarrow t + \frac{\Delta x}{v}$$

If the observer moves alongside the wave with velocity v, the wave form appears stationary.

If the line is homogeneous, that is, has the same cross-section or electrical properties all along its length

$$v = \frac{1}{\sqrt{LC}} = \frac{c}{\sqrt{k}} \quad (4\text{-}17)$$

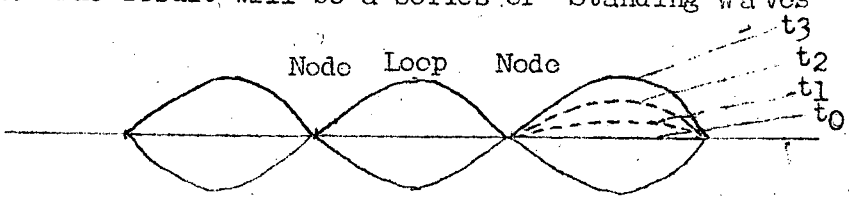
where c = velocity of light

(This is true for straight simple lines, but not always. In a loaded helix, the velocity is less)

If we know the capacity C, we can calculate L by the above relation.

If we have a wave in each direction the wave moving toward the terminus from the source has a positive sign, and is termed the incident wave, while the wave moving away toward the source has a negative sign and is termed the reflected wave.

If there are no losses, and the line is open or short circuited at the terminus, there will be places where the incident and reflected waves will be in phase, and add, and other places where they will be of opposite phase, and cancel. The result will be a series of "Standing Waves"

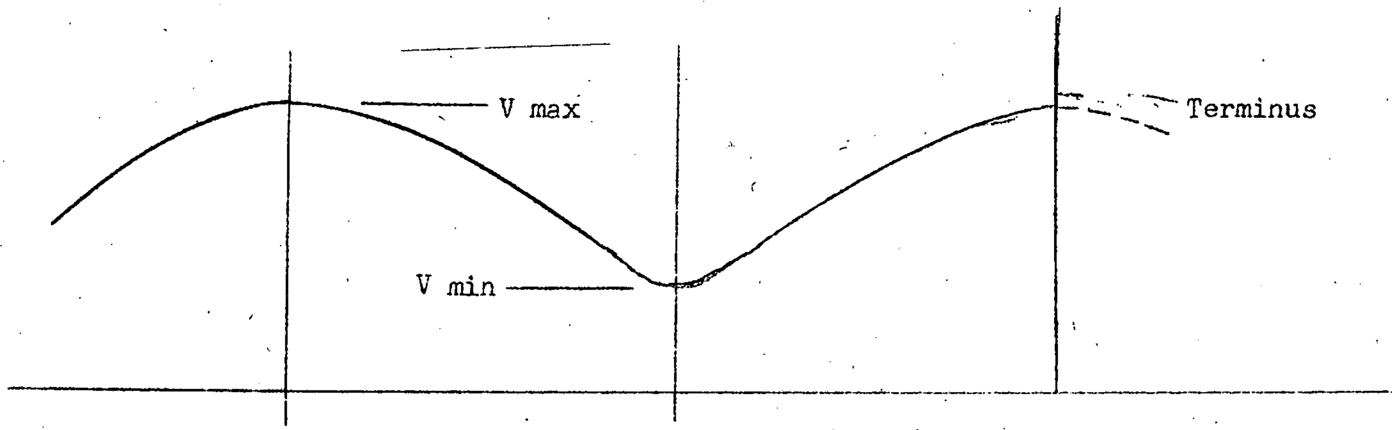


(This is for a short circuited terminus)

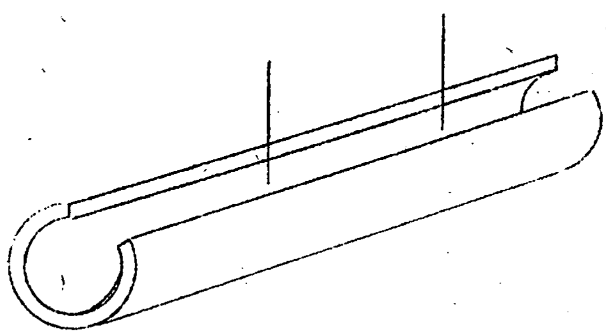
At the nodes there will be no voltage. In between the nodes, the voltage will vary sinusoidally with time.

For the above case, with total reflection and no losses, there will be no power transmitted down the line.

In general, where some power is transmitted and used at the terminus, the reflected wave will be of less amplitude than the incident wave. The voltages will therefore not cancel to form nodes, but the voltage variation will be somewhat as follows



The ratio of $\frac{V_{max}}{V_{min}}$ is calledthe "Voltage Standing Wave Ratio" (VSWR). This can be used as a means of measurement, since the incident and reflected waves are related by the terminal impedance Z_T. For very high frequencies, for example a slotted line may be used to find the standing wave by putting a probe into the slot and moving it along and measuring the voltage



Coming back to (4-11)

$$I = \frac{k}{Z} \left(Ae^{kx} - Be^{-kx}\right) \qquad (4\text{-}11)$$

where Z is the inductive impedance per unit length of line and remembering that

$$k = \sqrt{vY} = -\sqrt{Y/Z}$$

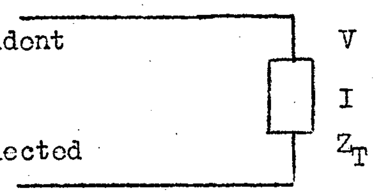
and that the characteristic impedance of the line is $Z_c = Z/Y$

$$Z_c\, I = (Ae^{kx} - Be^{-kx}) \qquad (4\text{-}18)$$

At the terminus we have

$$V_{incident} = Z_c\, I_{incident}$$

$$V_{reflected} = -Z_c\, I_{reflected}$$



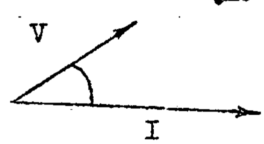
Given Z_T , what is the ratio B/A?

The easiest way to determine this is by a geometrical construction.

We have Z_T and given V, we can compute I

V and I are related by the phase angle of the impedance. This phase angle, from (4-16) is

$\pm\sqrt{LC}\;x$

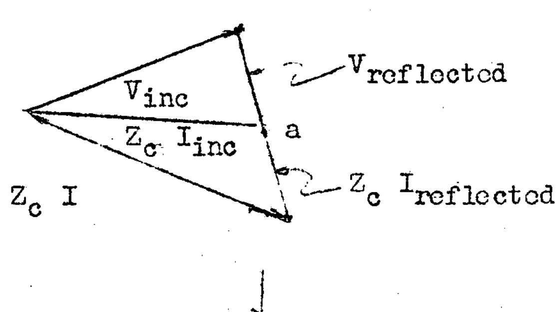


The current at the terminus will be

$$I = V/Z_T$$

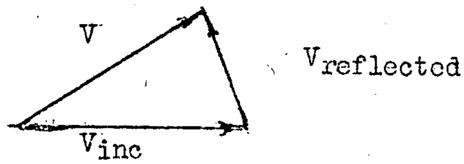
Multiply this by Z_c.

$$I\, Z_c = V\, Z_c/Z_T \qquad (4\text{-}19)$$

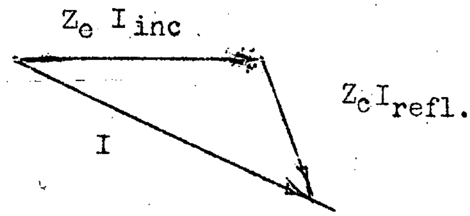


The construction is as follows:

Draw a vector equal to V and another equal to I at an angle equal to the phase angle. Connect the ends of these vectors and bisect the line. Then the two halves of the line represent V_{refl} and $Z_c\, I_{refl}$, which are equal and opposite. The upper triangle shows that the vector sum of V and V_{inc} = $V_{reflected}$



The lower triangle shows that the vector sum of Z_c I_{inc} and I is equal to Z_c $I_{reflected}$.



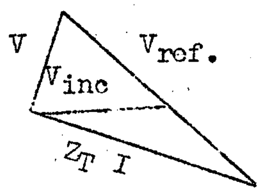
As an example, consider a case where

Z_T = $j X_T$, , where X_T is a pure reactance (C or L)

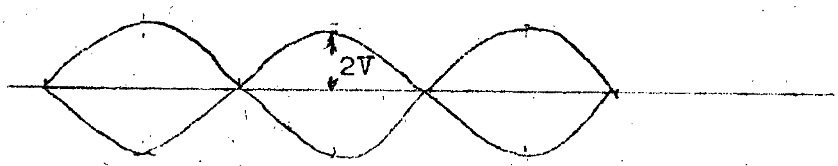
In this case the phase angle

ϕ = 90 Degrees

The diagram becomes a right angled triangle



Now from the geometric theorem that the three vertices of a right angled triangle lie on a circle whose center is the mid-point of the hypotenuse, it is evident that the incident and reflected waves are equal. They will therefore interfere and produce standing waves having a maximum voltage of 2 V and a minimum value of zero



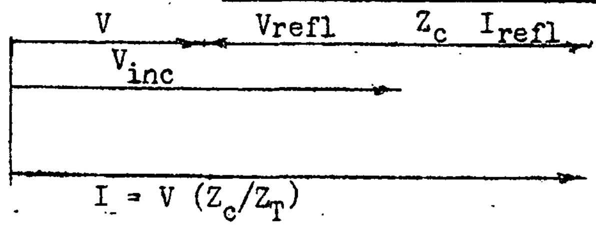
This cannot transmit power.

As another example, consider the case where

Z_T = R_T

The phase angle ϕ = 0 in this case, and the triangle flattens out to a line

(See next page)



From this relation

$$V\,(Z_c/Z_T) = V - V_{refl} + Z_c\, I_{refl}$$

and since $V_{refl} = -Z_c\, I_{refl}$

$$V\,(Z_c/Z_T) = V - 2\,V_{refl}$$

whence

$$V_{refl} = \frac{V}{2}\,(-Z_c/Z_T + 1) = \frac{V}{2}\,(1 - \frac{Z_c}{Z_T}) \qquad (4\text{-}19)$$

Two conclusions follow:

(1) V_{inc} is never equal to V_{refl}.

(2) If $Z_T = Z_c$, $\therefore$ $V_{refl} = 0$, that is,

there is no reflected wave.

Is this good or bad? This depends on the application.

In the case of an antenna, it is desired to transfer the power most efficiently to the antenna. This will occur if there is no reflected wave, because there are losses due to the reflected wave as well as to the incident wave. The effect of $Z_T = Z_c$ is in effect to connect the load direct from source to load.

If there is no reflected wave, there is no reflected effect on the driver.

These are good, for communication purposes.

If one is in the linear accelerator business this is not so good, because it is desirable to have the slight changes in the resonant frequency of the cavity reflect on the driver and affect its frequency so the driver remains in time with the cavity. For this purpose a "flat line" is not the best.

In a long line reflected waves absorb energy, and hence the transmission is less efficient than when there is no reflected wave. In the accelerator case, however, the line is so short that the losses are only a small fraction of the power transmitted, so that some increase in the losses can be tolerated if it is necessary in order to facilitate control. For this reason some reflection is desirable.

From the vector diagram above and 4-19) the general case can be written

$$V = V\left(\frac{Z_c}{Z_T}\right) + 2\,V_{refl}$$

whence $V_{refl} = \frac{V}{2}\left(1 - \frac{Z_c}{Z_T}\right)$ (4-20)

$$V = V_{inc} + V_{refl}$$

whence

$$V_{inc} = V - V_{refl} \qquad (4\text{-}21)$$

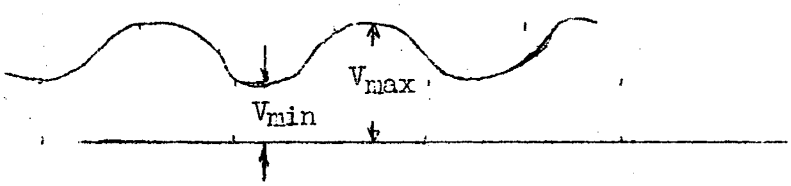
$$= V\left\{1 - \frac{1}{2}\left[1 - \frac{Z_c}{Z_T}\right]\right\}$$

$$= \frac{V}{2}\left\{1 - \frac{Z_c}{Z_T}\right\} \qquad (4\text{-}22)$$

From (4-20) and 4-21)

$$\frac{V_{refl}}{V_{inc}} = \frac{Z_T - Z_c}{Z_T + Z_c}$$

Z_T in general does not equal Z_c , and there will be some reflection, and the standing wave will not have a zero value in the valleys.



(One practical result of this is that insulators should be installed at the points of minimum voltage)

$$V_{max} = V_{refl} + V_{inc} = V \qquad (4\text{-}24)$$

$$V_{min} = V_{inc} - V_{refl} = \frac{Z_c}{Z_T}\,V \qquad (4\text{-}25)$$

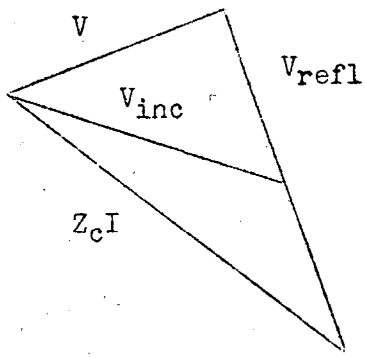
The ratio of the V_{max} to V_{min} is called the "Voltage Standing Wave Ratio" (abreviated VSWR) and from the above

$$VSWR = \frac{Z_T}{Z_c}$$

Phase of Voltage and Current

Consider conditions at a terminus as represented by the diagram

(see next page)



How do the conditions vary at different points on the line back from the terminus? First recall that the voltages given above in the diagram may be taken as the peak values of a sinusoidally varying alternating current.

The incident wave may be put in the form (See eq. between (3-45) and (3-46) in Lecture 3)

$$V = A_o e^{j\omega(t \pm \sqrt{LC} \cdot x)} \qquad (4\text{-}27)$$

This may also be written

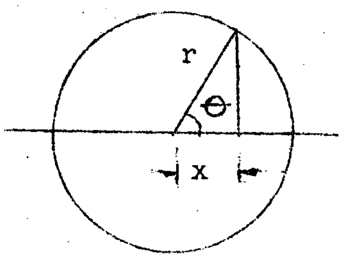
$$V = A_o \cos \omega(t \pm \sqrt{LC} \cdot x) \qquad (4\text{-}27)$$

This will have a maximum value when $(t \pm \sqrt{LC} \cdot x) = 0$, for which

$$\cos \omega (t \pm \sqrt{LC} \cdot x) = 1$$

If we consider the condition where $x = 0$ (the terminus) the incident voltage will vary sinusoidally with time. If we consider some point x distant from the terminus there is obviously some value of t that will make $(t \pm \sqrt{LC} \cdot x)$ = zero and at this point and time the same value of V will exist as will occur when the zero value of $(t \pm \sqrt{LC} \cdot x)$ occurs at the terminus. The voltage at any point will vary sinusoidally with time, or at any fixed time a sinusoidal voltage variation will exist down the line

Now a sinusoidal variation can be represented by the projection of a revolving radius of a circle on the diameter, because $x = r \cos \theta$

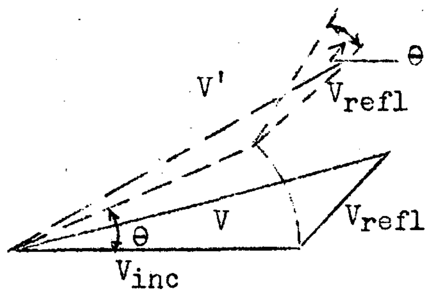


$$x = r \cos \theta$$

The variation of the incident voltage down the line can therefore be represented by the projection of a vector revolving in the positive direction (counterclockwise).

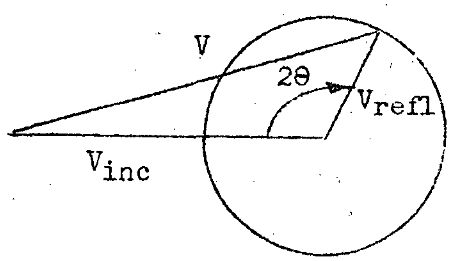
The reflected wave moves in the opposite direction, and so its variation may be represented by vector revolving in the negative (clockwise) direction.

The total voltage will be the vector sum of these two.



In the above diagram the full lines represent the relations of V_{inc}, V_{refl}, and V at time t_1 while the dotted lines indicate the condition at time t_2, after the incident voltage has advanced an angle θ.

If we are interested only in the magnitude of the voltage it can be noted that this can be obtained more simply by not revolving V_{inc}, but hold it stationary and revolve V_{refl} at twice the rate in the negative (clockwise) direction.



It will be noted that this will result in a maximum value

$$V_{max} = V_{inc} + V_{refl}$$

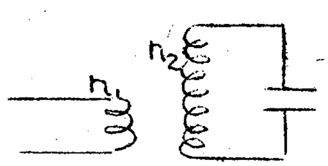
and a minimum value

$$V_{min} = V_{inc} - V_{refl}$$

The maximum and the minimum values will repeat at intervals of $\lambda/2$ where λ is the wave length. This shows that at every half-wave length down the line the input "sees" an impedance that is the same as the load impedance at the terminus.

Suppose our load is a resonant load. How do we match the line to it? This may be represented thus

Resonant Load



We may use a transformer. The secondary impedance is $Z_2 = Q\sqrt{L/C}$ (4-29)

If the turns ratio is $\frac{n_1}{n_2}$, the apparent impedance on the primary side is

$$Z_{prim} = \left(\frac{n_1}{n_2}\right)^2 Q\sqrt{L/C} \tag{4-30}$$

It is common to make the coupling into cavity resonators such that

$$Z_{prim} = \left(\frac{n_1}{n_2}\right)^2 Q\sqrt{L/C} \sim 5\,Z_c \tag{4-31}$$

This is satisfactory, and is not critical.

The next question is, what is the input impedance?

This can be gotten from the diagram. It may also be obtained analytically as follows:

$$V = A\,e^{kx} + B\,e^{-kx} \tag{4-32}$$

$$Z_c\,I = A\,e^{kx} - B\,e^{-kx} \tag{4-33}$$

At the load, where x = 0

$$V = A + B$$

$$Z_c\,I = A - B$$

whence

$$\frac{A+B}{A-B} = \frac{V}{Z_c I} \tag{4-34}$$

but

$$I = \frac{V}{Z_T}$$

so

$$\frac{A+B}{A-B} = \frac{V}{Z_c}\frac{Z_T}{V} = \frac{Z_T}{Z_c} \tag{4-35}$$

$$\frac{1+B/A}{1-B/A} = \frac{Z_T}{Z_c}$$

$$1 + B/A = \frac{Z_T}{Z_c}(1 - B/A)$$

$$B/A\left(1 + \frac{Z_T}{Z_c}\right) = \frac{Z_T}{Z_c} - 1$$

$$B/A = \frac{(Z_T/Z_c) - 1}{(Z_T/Z_c) + 1} = \frac{Z_T - Z_c}{Z_T + Z_c} \tag{4-36}$$

Substituting this in (4-28)

$$V = A\left(e^{kx} + \frac{Z_T - Z_c}{Z_T + Z_c}\, e^{-kx}\right) \tag{4-37}$$

$$Z_c\, I = A\left(e^{kx} - \frac{Z_T - Z_c}{Z_T + Z_c}\, e^{-kx}\right) \tag{4-38}$$

The Effective Impedance is $\frac{V}{I}$

So $$\frac{Z_{input}}{Z_c} = \frac{(Z_T + Z_c)e^{kx} + (Z_T - Z_c)e^{-kx}}{(Z_T + Z_c)e^{kx} - (Z_T - Z_c)e^{-kx}} \tag{4-39}$$

(This is general)

For a lossless line

$$kx = j\,\omega\sqrt{LC}\; x \tag{4-40}$$

$$\omega = 2\pi f \tag{4-41}$$

$$\sqrt{LC} = \frac{1}{v} \quad \text{and } v = f\lambda \tag{4-42}$$

whence $$kx = j\left(\frac{2\pi x}{\lambda}\right) \tag{4-43}$$

Substituting this in 4-34) and rearranging

$$\frac{Z_{input}}{Z_c} = \frac{Z_T\left(e^{\frac{j2\pi x}{\lambda}} + e^{\frac{-j2\pi x}{\lambda}}\right) + Z_c\left(e^{\frac{j2\pi x}{\lambda}} - e^{\frac{-j2\pi x}{\lambda}}\right)}{Z_c\left(e^{\frac{j2\pi x}{\lambda}} + e^{\frac{-j2\pi x}{\lambda}}\right) + Z_T\left(e^{\frac{j2\pi x}{\lambda}} \quad e^{\frac{-j2\pi x}{\lambda}}\right)} \tag{4-44}$$

$$= \frac{Z_T \cos\frac{2\pi x}{\lambda} + j\, Z_c \sin\frac{2\pi x}{\lambda}}{Z_c \cos\frac{2\pi x}{\lambda} + j\, Z_T \sin\frac{2\pi x}{\lambda}} \tag{4-45}$$

If $Z_T = 0$ (Short circuited Terminus)

$$Z_i = j\, Z_c \frac{\sin\frac{2\pi x}{\lambda}}{\cos\frac{2\pi x}{\lambda}} = j\, Z_c \tan\frac{2\pi x}{\lambda} \tag{4-46}$$

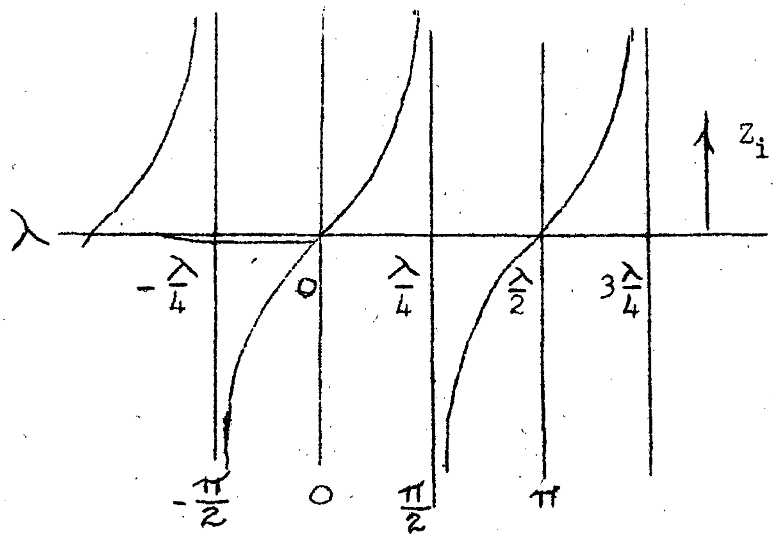
If $x < \frac{\lambda}{4}$ the line looks inductive

If $\frac{\lambda}{4} < x < \frac{\lambda}{2}$ the line looks capacitive

(Note that when $x = \frac{\lambda}{4}$, $\frac{2\pi\lambda}{4\lambda} = \frac{\pi}{2}$)

This may be shown by plotting Z_i as a function of λ

Shorted Terminus



Angle

Positive values of Z_i correspond to inductive reactance (lagging current). Negative values to capacitive reactance (leading current).

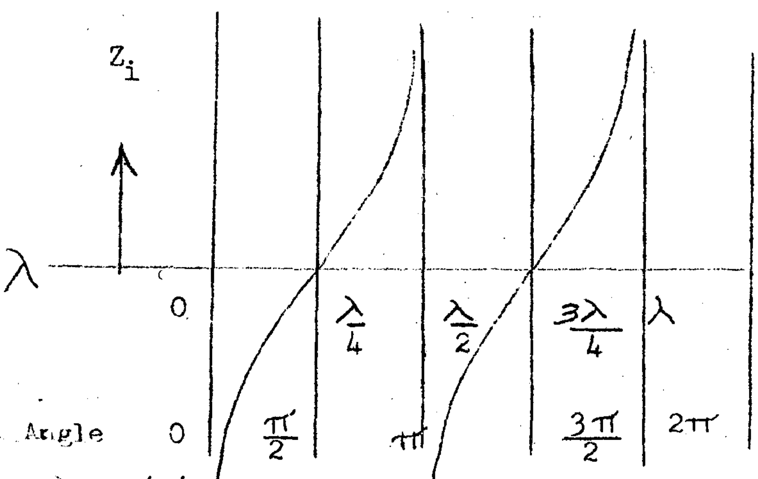
When $x = \frac{\lambda}{4}$, the input looks as though the line is an open circuit. It is therefore equivalent to a choke. Radio frequency will not get thru, and a D. C. connection can be made here.

When $Z_T = \infty$ (Open Circuited Terminus)

(4-45) reduces to $Z_i = Z_c \dfrac{1}{j \tan \frac{2\pi x}{\lambda}} = Z \frac{1}{j} \cot \frac{2\pi x}{\lambda}$ (4-47)

This may be plotted as follows

Open Circuited Terminus



When $0 < x < \frac{\lambda}{4}$, the line looks capacitive

When $\frac{\lambda}{4} < x < \frac{\lambda}{2}$, the line looks inductive

The line is an R.F. short for $\frac{\lambda}{4}$

An R.F. connection can be made thru by pass condensers, which will block D.C. current.

For $x = \frac{\lambda}{4}$,

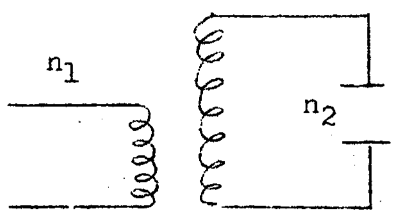
(4-45) reduces to

$$\frac{Z_i}{Z_c} = \frac{Z_c}{Z_T} \qquad (4\text{-}48)$$

whence

$$Z_i = \frac{Z_c^2}{Z_T} \qquad (4\text{-}49)$$

This makes a line of this length equivalent to a transformer, with turns ratio $\frac{n_1}{n_2} = M$



for which $Z_i = \frac{\omega M^2}{Z_T}$

The input impedance is, for $x = \frac{\lambda}{4}$, the reciprocal of the terminal impedance.

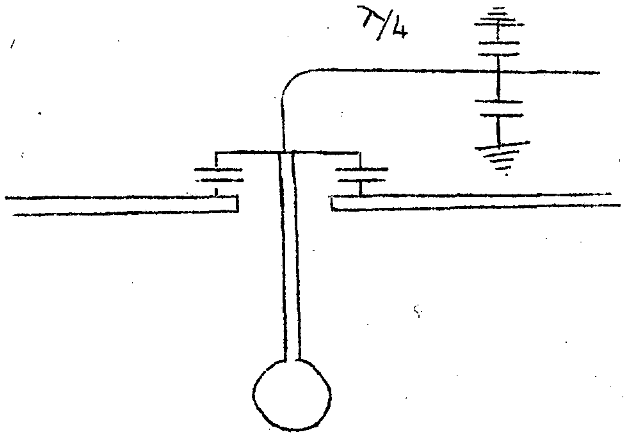
From the analytical expression of (4-45) it can be noted that if $Z_T = 0$,

$Z/Z_c = \infty$

for $x = \frac{\lambda}{4}$

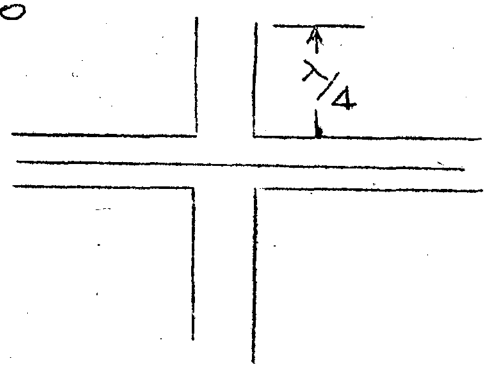
This has the properties of a pure inductance and can be used for the choke of a radio-frequency filter.

(See next page)

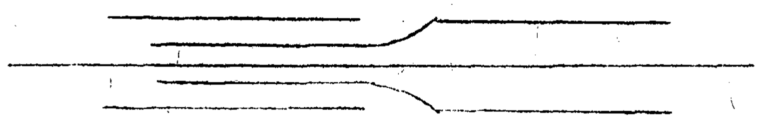


For example, a drift tube in a linear accelerator with a connection for putting on a bias voltage may have a quarter wave length section with by-pass condensers at each end which will prevent radio-frequency energy from the inside passing out along the bias connection. This can often be arranged by using things already lying around, such as a portion of the bias connection. For an open circuited terminus

for $x < \frac{\lambda}{4}$, the line appears as capacitive and at $\frac{\lambda}{4}$ the effective capacity is ∞



A transmission line that has a pair of skirt flanges of length $\lambda/4$ will have an infinite capacity for the R.F. current and act as a by-pass but will be insulated so that a direct current connection can be made to one side that will be insulated from the other



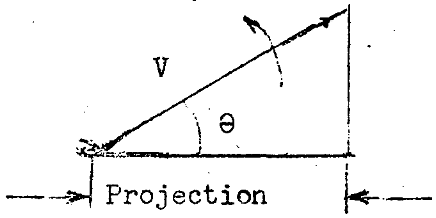
This can also be arranged thus.

These involve some losses to pay for the advantage gained.

LECTURE 5 - DR. PANOFSKY

ELECTROMAGNETIC FIELDS

The discussions of the previous lectures have been concerned with circuits. In connection with cavities and wave guides it is convenient to deal with FIELDS. These are best handled by the use of the mathematics of VECTORS. It should be noted that the term "Vector" is used in two senses. We used it in the preceding lecture to denote a rotating vector, the projection of which would then give the magnitude at any time of a quantity, such as voltage.



The projection of a vector of length V on the horizontal axis is

$$x = V \cos \theta$$

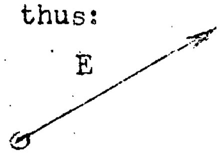
If V is of constant value and θ varies uniformly, this represents a sinusoidal variation of constant maximum amplitude.

In the other sense, as is used in VECTOR ANALYSIS, a vector is a quantity that has magnitude and direction.

The direction of a vector is a direction in space that is defined by its relation to axes of reference. The magnitude of the vector is defined as the number of times it will contain a vector of the same direction but having a unit value, that is the number of UNIT VECTORS it consists of.

Consider, for example, an electromagnetic vector of electric potential, E. At some point there may be a potential gradient of E volts in a particular direction. If the unit of potential gradient is 1 volt per meter, the symbol $\underline{E}$ is a vector consisting of E unit vectors in the given direction. (Vectors are variously represented. In printed matter it is customary to use heavy faced type. In manuscript the fact that a quantity is a vector may be indicated by drawing a line over it, thus $\bar{E}$, or putting an arrow over it, thus $\vec{E}$, or an accent, thus, $\hat{E}$, or underlining it, thus: $\underline{E}$. In these notes, which are typewritten, it is most convenient to use the underline method, $\underline{E}$, as this can be done without necessity of reversing the rotation of the platten that is required to form an overline. Use of the underline, however, has the disadvantage of requiring care to keep the underline distinguished from the line indicating division. The arrow may occasionally be useful to distinguish equal and opposite vectors.)

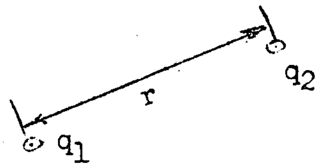
The voltage vector $\underline{E}$ therefore represents a voltage gradient of magnitude E in a particular direction, thus:



If there is a charge of q coulombs subject to a voltage gradient of E volts/meter the charge will be subjected to a force in the same direction as the voltage gradient, which will also be a vector, thus:

$$\underline{F} = q\,\underline{E}$$

If there are two charges q_1 and q_2



at a distance r apart, the force between the charges may be shown to be

$$\underline{F} = \frac{q_1 \, q_2}{4 \pi k_o \, r^2} \, \underline{r_1} \qquad (5\text{-}1)$$

where
- q is in coulombs
- r is in meters
- $\underline{r_1}$ is a vector of unit length with direction from q_1 to q_2
- $\underline{F}$ is in Newtons (= 10^5 dynes) with the same direction as $\underline{r}$
- k_o is a constant called the "permittivity of free space" with the dimensions (coulombs)^2 per joule or (coulombs)^2 per Newton meter or farads/meter

if q is in coulombs
$\underline{r}$ in meters
F in Newtons

$$k_o = 8.85 \times 10^{-12} \, \frac{(\text{coulombs})^2}{\text{joule meter}} \qquad (5\text{-}2)$$

$$= 8.85 \times 10^{-12} \, \frac{\text{farads}}{\text{meter}}$$

Along with this unit is

μ_o , the permeability of free space

$\mu_o = 1.257 \times 10^{-6}$ henries/meter

However it is not necessary to remember these values if the two following relations are remembered, which are more useful

$$\sqrt{\mu_o / k_o} = 376 \text{ ohms} \qquad (5\text{-}3)$$

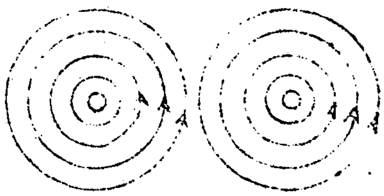
$$\frac{1}{\sqrt{\mu_o \, k_o}} = \frac{1}{c} = \frac{1}{3 \times 10^8 \text{ meters/sec.}} \qquad (5\text{-}4)$$

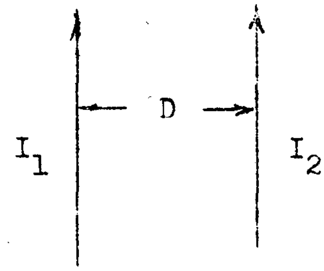
There are a number of systems of units for electromagnetic quantities. The most common is the MKS system, where

Distances are in Meters
Masses in Kilograms
Time in Seconds
Force in Newtons (=10^5 dynes)($\approx$1/5 lb. force)

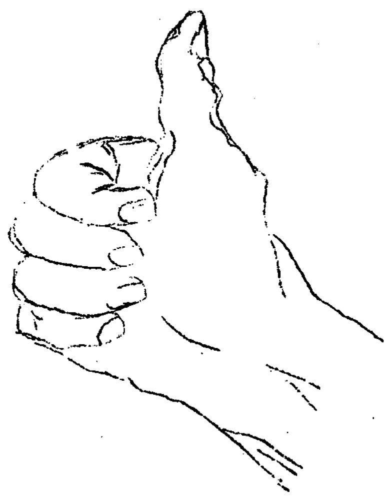
Quality of charge in Coulombs
Current in Amperes
Potential Difference in Volts
Electric Field in volts/meter
k_o = $(\text{coulombs})^2/\text{joule meter}$

If we have currents flowing in two parallel conductors, each conductor will be surrounded by an electromagnetic field





The direction of the magnetic field is in accordance with the right hand rule.



If the thumb of the right hand is extended in the direction of the current flow, the closed fingers will be in the direction of the lines of magnetic force.

When a current flows in a magnetic field, a force acts upon the conductor tending to move it in a direction at right angles to the field. Since each of the two parallel conductors above is surrounded by a magnetic field thru which the other conductor passes, the two conductors tend to move toward each other, if the flows in both conductors are in the same direction, or away from each other if the flows are in opposite direction.

The force per unit length may be shown to be

$$F/L = \frac{\mu_o \; I_1 \; I_2}{2 \pi D} \qquad (5\text{-}5)$$

where F is in Newtons
L is in Meters
I is in Amperes
μ_o is permeability of free space
$= 1.257 \times 10^{-6}$ henries/meter (5-6)

Where there is a number of separate currents or charges or magnetic fields it may be possible to add up the effects of each one on each other, but it is convenient to consider instead the effect on one charge or current of the <u>field</u> due to the others. This has the effect of breaking up the separate computations and puts the computations on a basis parallel to the mechanism by which experimental measurements may be made. In cavities, for example, measurements of the fields may be made by inserting probes at various points and measuring the intensity of the magnetic field, or other quantities.

The force acting on an element may, for example, be expressed as

$$\underline{F} = q_1 \times \text{field of second element}$$

In an electrostatic field the force on a charge is

$$\underline{F} = q_1 \times \underline{E}$$

The potential gradient is

$$\underline{E} = \frac{1}{4\pi k_o} \times \frac{q}{r^2} \cdot \underline{r_1}$$

The force per unit length on a conductor is

$$\underline{F}/L = I\,\underline{B}$$

where $\underline{B}$ is the flux density

$$= \frac{\mu_o I}{2\pi D}$$

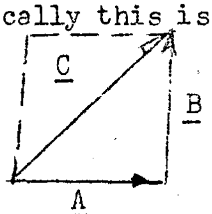
Manipulation of Vectors Algebraically

Because of the quality of <u>direction</u> involved in a vector, there are some differences that appear in their algebraic manipulation.

Two vectors may be added, which is represented thus

$$\underline{A} + \underline{B} = \underline{C} \qquad (5\text{-}7)$$

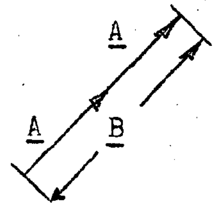
Geometrically this is represented thus



This is similar to the familiar method of adding forces in mechanics by the parallelogram rule (which is not surprising, since forces are vectors)

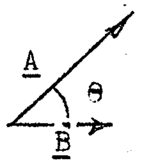
Vectors can be multiplied by numerical factors. This merely produces a vector of greater magnitude, with the same direction, thus

$$2\underline{A} = \underline{B}$$



There are two ways to multiply vectors. The first is called the SCALAR or DOT product (from the method of representing the operation)

$$\underline{A} \cdot \underline{B} = \underline{C} \qquad (5\text{-}8)$$



The SCALAR product is equal to the Length of $\underline{A}$ times the length of $\underline{B}$ times the cosine of the angle θ between the vectors.

Thus, in mechanics the power involved in a force $\underline{F}$ acting on a body moving with a velocity $\underline{V}$ at angle θ to the force is

$$P = \underline{F} \cdot \underline{V} = |F| \cdot |V| \cos\theta \qquad (5\text{-}9)$$

where $|F|$ and $|V|$ are the numerical values of the force and velocity. This also illustrates that the DOT product of 2 Vectors produces a scalar quantity, since power has no direction.

The other method of multiplying vectors is the CROSS PRODUCT

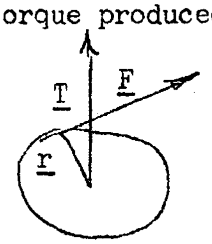
$$\underline{A} \times \underline{B} = \underline{C} \qquad (5\text{-}10)$$

$\underline{C}$ is a Vector at right angles to the plane of $\underline{A}$ and $\underline{B}$, and numerically equal to the product of the numerical values of the vectors times the sine of the angle between them.

$$\underline{A} \times \underline{B} = |A|\,|B| \sin\theta \qquad (5\text{-}11)$$

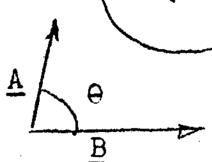
A convention for signs is needed here. This is taken by the right hand rule, that is, if the direction of the vector is in the direction of the extended thumb of the right hand, the positive direction of the angle is in the direction of the closed fingers. Another way to say this is that the positive direction of the vector is in the direction of the motion of a nut on a right hand screw thread when the nut is turned in the clockwise direction.

The Torque produced by a force $\underline{F}$ with lever arm $\underline{r}$ is



$$\underline{T} = \underline{F} \times \underline{r} \qquad (5\text{-}11\text{-}a)$$

The torque is represented by a vector ⊥ to the plane of force and lever arm.



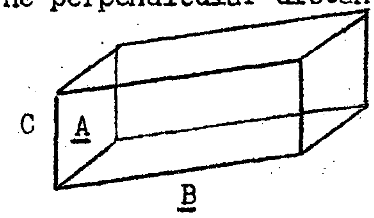
The area of a parallelogram is equal to

$$\underline{A} = \underline{A} \times \underline{B} = |A|\,|B| \sin\theta \qquad (5\text{-}12)$$

The volume of a parallelepiped is

$$(\underline{A} \times \underline{B}) \cdot C \qquad (5\text{-}13)$$

Where C is the perpendicular distance between the faces



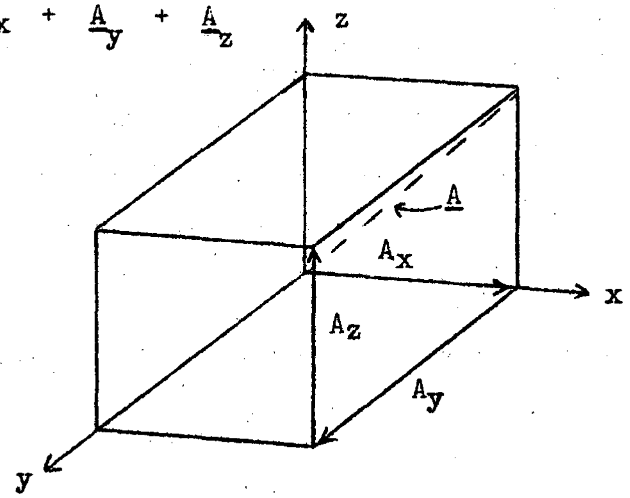
If there are several factors, with mixed x and • products, the factors can be commuted. Thus:

$$(\underline{A} \times \underline{B}) \cdot C = \underline{A} \times (\underline{B} \cdot \underline{C}) \qquad (5\text{-}14)$$

Vectors in 3 Dimensions

A vector $\underline{A}$ may be considered to be the sum of three vectors $\underline{A}_x$, $\underline{A}_y$, $\underline{A}_z$ parallel to three axes of coordinates, thus

$$\underline{A} = \underline{A}_x + \underline{A}_y + \underline{A}_z \qquad (5\text{-}15)$$



The dot product of vectors $\underline{A}$ and $\underline{B}$ may be represented by the sum of the products of the components

$$\underline{A} \cdot \underline{B} = \underline{A}_x \underline{B}_x + \underline{A}_y \underline{B}_y + \underline{A}_z \underline{B}_z \qquad (5\text{-}16)$$

The components of $\underline{A}$ parallel to the x, y, and z axis may be represented by taking the magnitudes of the three components of $\underline{A}$, multiplied by the unit vectors $\underline{i}$, $\underline{j}$, and $\underline{k}$ parallel respectively to the x, y, and z axes.

Thus

$$\underline{A} = i A_x + j A_y + k A_z \tag{5-17}$$

(note that this is a different usage of i and j from that in complex expressions where i or j represents $\sqrt{-1}$).

From the definition of the scalar or dot product

$$\underline{A} \cdot \underline{B} = |A| \; |B| \; \cos\theta \tag{5-18}$$

it follows that

$$\underline{i} \cdot \underline{i} = \underline{j} \cdot \underline{j} = \underline{k} \cdot \underline{k} = 1.0 \tag{5-19}$$

and

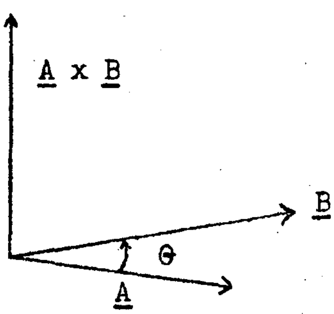
$$\underline{i} \cdot \underline{j} = \underline{j} \cdot \underline{k} = \underline{k} \cdot \underline{i} = 0 \tag{5-20}$$

It follows that

$$\begin{aligned} \underline{A} \cdot \underline{B} &= (\underline{i} A_x + \underline{j} A_y + \underline{k} A_z)(\underline{i} B_x + \underline{j} B_y + \underline{k} B_z) \\ &= \underline{i} \cdot \underline{i} \; A_x B_x + \underline{j} \cdot \underline{j} \; A_y B_y + \underline{k} \cdot \underline{k} \; A_z B_z \\ &= A_x B_x + A_y B_y + A_z B_z \end{aligned} \tag{5-21}$$

(All of the coefficient products involving $\underline{i} \cdot j$, $\underline{j} \cdot k$ etc. equal zero and the terms containing them drop out).

The cross product becomes more complicated. The cross or vector product is defined to be a vector perpendicular to the two given vectors with a direction in the sense of the direction of a right hand screw rotated from the first to the second of the given vectors through the smaller of the two angles between them.

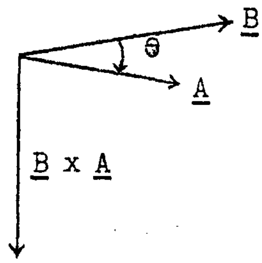


The magnitude of the vector product is

$$\underline{A} \times \underline{B} = \mathcal{E} \; |A| \; |B| \; \sin\theta \tag{5-22}$$

where $\mathcal{E}$ is a unit vector in the direction of a normal to the plane of $\underline{A}$ and $\underline{B}$.

Note however that $\underline{B} \times \underline{A}$ by this rule becomes



In this type of multiplicatien

$$\underline{A} \times \underline{B} = - \underline{B} \times \underline{A}$$

For Vector Products, the order of multiplication makes a difference

Since

$$\underline{A} \times \underline{B} = \mathcal{E} |A| \ |B| \ \sin \theta$$

it is obvious that the vector product is zero when the vectors are parallel.

By the definition of vector products it follows that

$$\underline{i} \times \underline{j} = \underline{k} , \ \underline{j} \times \underline{k} = \underline{i} , \ \underline{k} \times \underline{i} = \underline{j} \qquad (5\text{-}23)$$

also that

$$\underline{i} \times \underline{i} = 0 , \ \underline{j} \times \underline{j} = 0 , \ \underline{k} \times \underline{k} = 0 \qquad (5\text{-}24)$$

The vector product

$$\underline{A} \times \underline{B} = (\underline{i} A_x + \underline{j} A_y + \underline{k} A_z) \times (\underline{i} B_x + \underline{j} B_y + \underline{k} B_z)$$

$$= \underline{i} (A_y B_z - A_z B_y) + \underline{j} (A_z B_x - A_x B_z) + \underline{k} (A_x B_y - A_y B_x) \qquad (5\text{-}25)$$

This may be written more compactly in the form of a determinant, thus

$$\underline{A} \times \underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad (5\text{-}26)$$

Application of Calculus to Vectors

A derivative of a vector may be taken with respect to a scalar quantity, say, for example, time. Thus

$$\frac{d\underline{A}}{dt} = \lim_{\Delta t \to 0} \frac{\underline{A}(t + \Delta t) - \underline{A}(t)}{\Delta t} \qquad (5\text{-}27)$$

This is still a vector

so also

$$\frac{d}{dt} (\underline{A} + \underline{B}) = \frac{d\underline{A}}{dt} + \frac{d\underline{B}}{dt} \qquad (5\text{-}28)$$

The derivative of the product of a scalar times a vector is shown by the following example

$$\frac{d}{dt}(u\,\underline{a}) = \underline{a}\,\frac{du}{dt} + u\,\frac{d\underline{a}}{dt} \qquad (5\text{-}29)$$

Also

$$\frac{d}{dt}(\underline{A}\cdot\underline{B}) = \underline{B}\cdot\frac{d\underline{A}}{dt} + A\cdot\frac{d\underline{B}}{dt} \qquad (5\text{-}30)$$

and

$$\frac{d}{dt}(\underline{A}\times\underline{B}) = \frac{d\underline{A}}{dt}\times\underline{B} + \underline{A}\times\frac{d\underline{B}}{dt} \qquad (5\text{-}31)$$

Now consider the symbol

$$\underline{\nabla} = \underline{i}\,\frac{\partial}{\partial x} + \underline{j}\,\frac{\partial}{\partial y} + \underline{k}\,\frac{\partial}{\partial z} \qquad (5\text{-}32)$$

This is called "Del" and is an "operator." It can operate on a scalar or on a vector. If ϕ is a scalar quantity

$$\nabla\phi = \underline{i}\,\frac{\partial\phi}{\partial x} + \underline{j}\,\frac{\partial\phi}{\partial y} + \underline{k}\,\frac{\partial\phi}{\partial z} \qquad (5\text{-}33)$$

This is the gradient operator. The meaning of this may be clarified by considering a scalar ϕ which is a function of position in space, that is

$$\phi\,(x, y, z)$$

Then

$$\phi\,(x, y, z) = C_1 \text{ (constant)} \qquad (5\text{-}34)$$

represents an equipotential surface, or, in the case of heat flow, a surface of constant temperature or isothermal surface. For any other value of C, there will be another equipotential surface adjacent to, but not necessarily parallel to, the first equipotential surface. If we consider the distance from a point on the first surface to the second surface, having a difference of potential $d\phi$, with length $d\underline{r}$, then

$$\frac{d\phi}{d\underline{r}} \text{ is the rate of change of } \phi$$

with respect to the distance in the direction $d\underline{r}$. This will be a maximum, for a fixed value of $d\phi$, when $d\underline{r}$ is as short as possible, which will be when $\underline{r}$ is normal to the equipotential surfaces. This maximum value of the rate of change of ϕ with respect to the distance is called the GRADIENT.

In Electrostatics,

$$\underline{E} = -\nabla\phi \text{ is the gradient of } \phi$$

(The minus sign because the force is from higher to lower voltage)

$$E_x = - \frac{\partial \phi}{\partial x} \tag{5-35}$$

$$E_y = - \frac{\partial \phi}{\partial y} \tag{5-36}$$

$$E_z = - \frac{\partial \phi}{\partial z} \tag{5-37}$$

So

$$\underline{E} = \underline{E}_x + \underline{E}_y + \underline{E}_z = - \underline{i} \frac{\partial \phi}{\partial x} - \underline{j} \frac{\partial \phi}{\partial y} - \underline{k} \frac{\partial \phi}{\partial z} \tag{5-38}$$

$$= - \underline{\nabla} \phi \tag{5-39}$$

$$= - \text{GRAD } \phi \tag{5-40}$$

The operator $\underline{\nabla}$ has many properties similar to ordinary differentiation.

Thus: $$\underline{\nabla}(F + G) = \underline{\nabla} F + \underline{\nabla} G \tag{5-41}$$

and $$\underline{\nabla} FG = F \underline{\nabla} G + G \underline{\nabla} F \tag{5-42}$$

also $$\underline{\nabla}\left(\frac{F}{G}\right) = \frac{G \underline{\nabla} F - F \underline{\nabla} G}{G^2} \tag{5-43}$$

In the above, the operator $\underline{\nabla}$ has been applied to scalar quantities. It may also be applied to Vectors, but in two ways, by analogy to the scalar or dot product, or by analogy to the Vector or cross product.

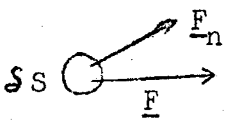
DIVERGENCE

The operation represented by $\underline{\nabla} \cdot \underline{A}$ is a scalar that is called the DIVERGENCE of $\underline{A}$ or DIV $\underline{A}$.

The meaning of this can be clarified by considering a related quantity called the FLUX.

Flux of a Vector.

If we have a vector $\underline{F}$ at a certain point in space and an element of area δS at this point, then $\underline{F}_n \delta S$, which is the projection of $\underline{F}$ on the normal to the surface times the area δS is the Flux of the Vector $\underline{F}$ across δS .



Some physical examples will illustrate the meaning of the term.

If we have a moving fluid, of which the velocity at any point is represented by the Vector $\underline{F}$, then the Flux of $\underline{F}$ or $\underline{F}_n \delta S$ would represent the total flow thru the cross section δS.

If the vector $\underline{F} = \rho \underline{V}$

where ρ = density

and $\underline{V}$ = velocity

Then the flux of $\underline{F} = \rho \underline{V}_n \delta S$ would represent the mass per unit time flowing across area δS.

If $\underline{F}$ is an Electric Field then the Flux of $\underline{F}$ is $F_n \delta S$ or the total lines in a given area δS.

If $\underline{q}$ is the intensity of heat flow normal to a small surface δS, then $q \delta S$ is the flux of heat flow across the surface. This may be expressed, for example, in Btu/hr.

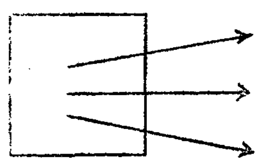
If the surface is large enuf so that the value of $\underline{F}_n$ is different at different places, then the Flux across the entire surface would have to be obtained by integrating over the entire surface the values of the vector $\underline{F}_n$ at every point. This may be written

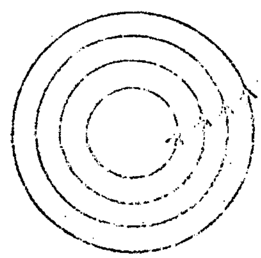
$$\text{Flux of } \underline{F} = \iint_S \underline{F}_n \, d\underline{S} \tag{5-44}$$

In a magnetic field, the total flux is

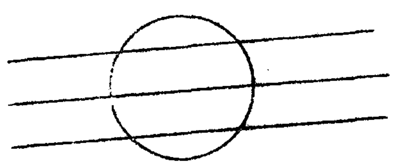
$$N = \int_S \underline{B} \cdot d\underline{S} \tag{5-45}$$

A FLUX must start somewhere. It can be considered to be made somewhere or to be a circulation closing on itself





If the net flux from an area or volume is positive, the area is a SOURCE, and if negative, it is a SINK. These terms originate in hydrodynamics.



If the flux entering an area is equal to that leaving, it is neither a source nor a sink.

If we consider the net flux from a volume V we can represent the total flux of $\underline{A}$ as

$$\int_S \underline{A} \cdot dS \tag{5-46}$$

and the flux per unit volume as

$$\frac{\int_S \underline{A} \cdot dS}{V} \tag{5-47}$$

The limit as $V \longrightarrow 0$ of the Flux per unit Volume is called the DIVERGENCE.

If we consider a volume as made up of a number of small volumes the flux per unit volume will vary from one small volume to another. The total Flux from all the small elements of volume will be

$$\iiint_V \text{div}\ \underline{F} \cdot dV \qquad (5\text{-}48)$$

This total flux can also be expressed in terms of the total external surface of the volume. (The net flux across any surface in the interior of the volume will be zero, since what leaves one small interior volume thru its surface (positive) enters the adjacent small volume (negative). These fluxes in the interior accordingly cancel, and only the flux thru the exterior surface of the total volume can be counted. The summation of the fluxes on the differential volumes is evidently equal to the total flux across the exterior surface, which may be expressed

$$\iiint_V \text{div}\ \underline{F} \propto dV = \iiint_V \underline{\nabla}\ \underline{F}\ dV = \iint_S \underline{F} \cdot dS \qquad (5\text{-}49)$$

The choice of the term DIVERGENCE may be explained by considering $\underline{F}$ as equal to $\rho \underline{v}$ where ρ is the density and $\underline{v}$ is the velocity. Then the integrals of eq (5-49) indicate the mass of fluid flowing per unit time from the volume. When the volume is infinitesimal, it represents the mass of fluid flowing or diverging per unit time from a point.

If we consider a small volume dx dy dz , the change in mass per unit time per unit volume will be

$$\frac{\partial \rho}{\partial t}.$$

This will be, from the principle of conservation of mass, equal and opposite to the total flux per unit volume, or

$$\frac{\partial \rho}{\partial t} = -\ \underline{\nabla} M = -\underline{\nabla}(\rho \underline{v}) \qquad (5\text{-}50)$$

This is called the Equation of Continuity.

When the fluid is incompressible, ρ = constant and $\frac{\partial \rho}{\partial t} = 0$, so $\underline{\nabla} M = 0$.

When the divergence of a function vanishes in a region, the function is said to be Solenoidal in that region.

The other means of applying the operator $\underline{\nabla}$ to vector is

$$\underline{\nabla} \times \underline{A} = \text{Curl } A \tag{5-51}$$

$$\text{Curl } \underline{A} = \underline{\nabla} A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \tag{5-52}$$

$$= \underline{i} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \underline{j} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \underline{k} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \tag{5-53}$$

The term Curl is derived from hydrodynamics. It may be shown that the motion of a small particle of fluid in time dt may be considered as made up of a translation, a deformation and a rotation about an instantaneous axis. The curl of the vector v (the velocity) may be shown to have the direction of this axis and a magnitude equal to twice the instantaneous angular velocity.

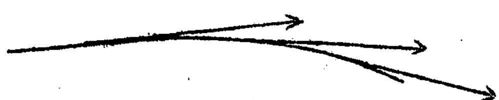
We may get a picture of what is meant by

$$\underline{\nabla} \times \underline{A} = \text{curl } A \tag{5-54}$$

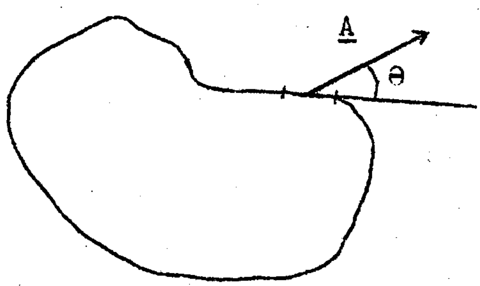
as follows.

There are two ways to make a field. One of these is to stir it up, to form lines of force that make closed figures.

What is a line of force? It has direction and magnitude.



The direction at any point is that of the tangent to the curve at any point



Suppose we consider a short section dL of a path and a vector $\underline{A}$ making an angle θ with the tangent. Then $\underline{A}$ cos θ is the component of $\underline{A}$ parallel to dL. Suppose, for example, that $\underline{A}$ represents the force acting on a particle moving along the path. Then $|A|$ cos θ dL is the work done on the particle during its motion thru distance dL but $|A| \cos\theta \, dL = \underline{A} \cdot dL$

The total work done by the force $\underline{A}$ in moving around the path is

$$W = \int \underline{A} \cdot dL \qquad (5\text{-}55)$$

This is a line integral. If this is carried out for the entire closed path we have

$$\oint \underline{A} \cdot dL \qquad (5\text{-}56)$$

The circle on the integral sign indicates taking the line integral clear around to the point of beginning.

If $\underline{A}$ is the gradient $\underline{\nabla} \phi$ of a scalar function of position, then

$$\int_A^B \underline{A} \cdot dL = \int_A^B (\underline{\nabla} \phi) \cdot dL = \int_A^B \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right) \qquad (5\text{-}57)$$

$$= \int_A^B d\phi = \phi_B - \phi_A \qquad (5\text{-}58)$$

It follows from this that the line integral of the gradient of any scalar function of position ϕ around a closed curve vanishes, because $\phi_B \equiv \phi_A$.

If the value of the line integral around a closed curve is not zero, the positive value is termed the CIRCULATION.

If, for example, v is the velocity of water around a closed path and

$$\oint \underline{v} \cdot d\underline{L} \neq 0 \qquad (5\text{-}59)$$

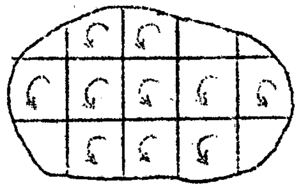
its value is the quantity of water/sec/unit volume that is being carried around. If the integration in the direction A to B gives the positive value, the integration from B to A gives a negative value. The direction giving the positive value is the direction of circulation. The circulation of the electric intensity round a given curve represents the total voltage that would be impressed on a conducting wire coinciding with the curve. The circulation of any force represents the work done on a particle moving around the closed curve. If the circulation vanishes for every closed curve the field of force is said to be conservative, that is the field does no work on a particle returning to its original position.

The circulation around a small loop in a field depends on its size and orientation. As the loop contracts to a point, the circulation per unit area may approach a finite limit. This limit is called the component of CURL of F in that direction, normal to the plane in which the circulation appears to be clockwise

$$\lim_{A \to 0} \frac{\oint \underline{A} \cdot d\underline{L}}{\text{Area}} = \text{component of } \underline{\nabla} \times \underline{A} \qquad (5\text{-}60)$$

that is perpendicular to the area.

We may consider a larger area, it may be considered to be made up of a large number of small areas



The line integrals in the interior of the area cancel each other, and only the portions on the outer periphery remain to form portions of the line integral around the boundary of the area. The components of the $\underline{\nabla} \times \underline{A}$ perpendicular to the area may be summed up, so

$$\oint \underline{A} \cdot d\underline{L} = \iint_S (\underline{\nabla} \cdot \underline{A}) \cdot dS \qquad (5\text{-}61)$$

This is known as Stokes' Theorem.

Summarizing:

Del and scalar

$$\underline{\nabla}\, \phi = \text{grad}\, \phi = \text{gradient of scalar}\ \phi \qquad (5\text{-}62)$$

(This is a vector)

Dot Product of
Del and Vector

$$\underline{\nabla} \cdot \underline{A} = \text{div}\, \underline{A} = \text{divergence of}\ \underline{A} \qquad (5\text{-}63)$$

(This is a scalar)

Vector Product of
Del and a Vector

$$\underline{\nabla} \times \underline{A} = \text{Curl}\, \underline{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \qquad (5\text{-}64)$$

With this background we can now proceed to write Maxwell's Equations. This will be taken up in the next lecture.

LECTURE 6 - DR. PANOFSKY
MAXWELL'S EQUATIONS

Maxwell's Equations in Electromagnetic Theory are

$$\nabla \cdot \underline{D} = \rho \qquad (6-1)$$

$$\nabla \cdot \underline{B} = 0 \qquad (6-2)$$

$$\nabla \times \underline{E} = -\frac{\partial B}{\partial t} \qquad (6-3)$$

$$\nabla \times \underline{H} = \underline{J} + \frac{\partial D}{\partial t} \qquad (6-4)$$

When the MKS (Meter, Kilogram, Second) system of units is used, the current I is in amperes and V is in volts. The other terms are

$\underline{D}$ = Electric Displacement (coulombs/meter^2)

$\underline{B}$ = Magnetic Induction (weber/meter^2)

$\underline{E}$ = Electric Intensity (volts/meter)

$\underline{H}$ = Magnetic Intensity (amperes/meter)

$\underline{J}$ = Current Density (amperes/meter^2)

σ = Electric Conductivity (1/ohm-meter)

$K = K_r K_o$ = Electric inductive capacity of the medium

$\mu = \mu_r \mu_o$ = Magnetic inductive capacity of the medium

K_r = Dialectric constant

μ_r = Permeability

$K_o = 8.854 \times 10^{-12}$ (Farad/meter)

$\mu_o = 4\pi \times 10^{-7} = 1.257 \times 10^{-6}$ (henry/meter)

ρ = Charge density (coulombs/meter^3)

$c = \dfrac{1}{\sqrt{K_o \mu_o}} = 2.998 \times 10^8$ meters/sec
(velocity of light (close enuf to $3. \times 10^8$ M/S for ordinary computations)

FLM 1/29/51

In a vacuum

$$\underline{D} = \frac{\underline{E}}{K_o} \qquad (6\text{-}5)$$

$$\underline{H} = \frac{\underline{B}}{\mu_o} \text{ or } \mu_o \underline{H} = \underline{B} \qquad (6\text{-}6)$$

Also $\mu_o K_o = \frac{1}{c^2}$ (6-7)

and $\sqrt{\frac{\mu_o}{K_o}} = 376\ \Omega$ (6-8)

Equation (6-1) is

$$\underline{\nabla} \cdot \underline{D} = \rho \qquad (6\text{-}1)$$

This says that the divergence of the Electric Displacement, or number of lines of electric force per unit area is equal to the charge per unit volume.

ρ = Charge per unit volume

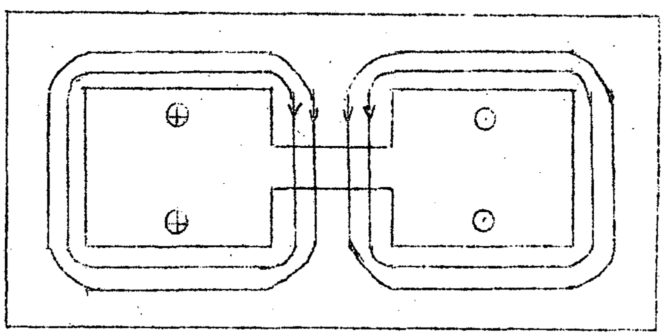
The number of lines originating in a unit volume is equal to the charge present per unit volume. The electric force lines start and end on charges.

Equation (6-2) is

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad (6\text{-}2)$$

This says that the divergence of $\underline{B}$, the magnetic induction, is zero. This means that magnetic lines cannot originate in a source or disappear into a sink.

There is no such thing as a single magnetic pole, but the lines of magnetic force constitute a circulation.



Equation (6-3) is

$$\underline{\nabla} \times \underline{E} = -\frac{\partial B}{\partial t} \qquad (6\text{-}3)$$

This says that the curl of $\underline{E}$, the electric intensity, is equal to minus the time rate of change of the magnetic induction $\underline{B}$. This says that an electric field can be set in circulation

The surface Integral — The Line Integral

$$\int (\nabla \times \underline{E}) \cdot dS = \int \underline{E} \cdot dL = -\frac{\partial}{\partial t} \text{(Total magnetic force)} \qquad (6\text{-}9)$$

In other words, the total electric force around any closed loop is equal to minus the rate of change of magnetic flux. (The minus sign enters because the counter E.M.F. developed by a rise in magnetic flux is in the sense opposite to that which produces the magnetic field rise; also the E.M.F. produced when the magnetic flux decreases is in the sense to keep the current flowing.

Equation (6-4) is

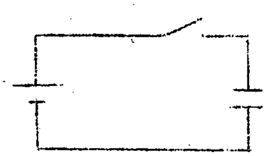
$$\underline{\nabla} \times \underline{H} = \underline{J} + \frac{\partial D}{\partial t} \qquad (6\text{-}4)$$

This says that the curl of the magnetic intensity is equal to the current density plus the rate of change of the Electric Displacement.

Circulation Density — Magnetic Motive Force — Total Current

$$\int (\underline{\nabla} \times \underline{H})\, ds = \oint \underline{H}\, dL = \int \underline{J} \cdot ds \qquad (6\text{-}10)$$

In a magnet, the integral $\int \underline{H} \cdot d\underline{L}$ = total current being made. The total current includes the displacement current, which also produces a magnetomotive force. If we have a circuit like this,



there will be current started when the switch is closed. The magnetic field accompanying it is

$$2\pi r H = I$$

OR

$$\underline{H} = \frac{I}{2\pi r} \qquad (6\text{-}11)$$

This current continues only until the condenser is charged, since there is no current across the gap. This is a displacement current. If the charge on the condenser is q ,

$$I = \frac{\partial q}{\partial t} \tag{6-12}$$

If the charge is known, the electric field can be computed

$$\int \underline{D} \cdot dS = \int \rho \, dV = q \tag{6-13}$$

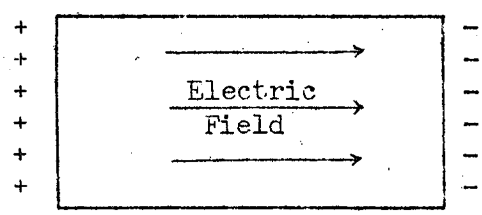
This says the total flux out of a given volume is equal to the charge q. The Electric Displacement is in terms of coulombs/meter^2 so the charge

$$q = \underline{D} \times \text{Area} \tag{6-14}$$

The Current per unit area is equal to the time rate of change of the electric displacement current

$$\frac{I}{A} = \frac{d\underline{D}}{dt} \tag{6-15}$$

Consider the conditions in a cavity



The current flowing down the sides produces a magnetic field circulating around the axis and an electric field between the two ends.

In a vacuum neither electric nor magnetic fields have sources. A change in magnetic field changes the electric field and vice versa.

Why do we have two similar quantities

$\underline{D}$ and $\underline{E}$ for Electric

quantities and two similar quantities

$\underline{B}$ and $\underline{H}$ for Magnetic

quantities?

In a vacuum these are related by simple constants

$$\mu_0 \underline{H} = \underline{B}$$

$$\text{and} \quad K_0 \; \underline{D} = \underline{E}$$

In other materials

$$\underline{B} = \mu \mu_o \underline{H} \tag{6-17}$$

where μ is the permeability of the material

and $\mu_o = 4\pi \times 10^{-7} = 1.257 \times 10^{-8}$ henry/meter

Also

$$\underline{D} = K K_o \underline{E} \tag{6-18}$$

where K is the dialectric constant and

$$K_o = 8.85 \times 10^{-12} \text{ (Farad/meter)}$$

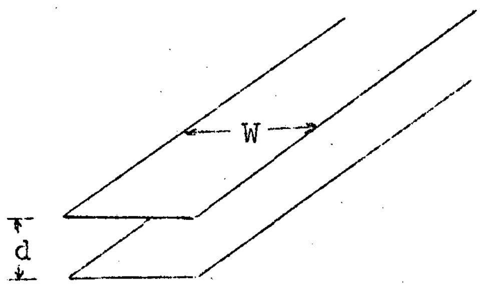
Eq. (6-1) can be expressed in terms of components

$$\text{Div } \underline{D} = \underline{\nabla} \cdot \underline{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \tag{6-19}$$

and (6-3) as a determinant

$$\underline{\nabla} \times \underline{E} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} \tag{6-20}$$

Consider the application of these equations to a transmission line consisting of two parallel plates



Since it was shown (3-7) that

$$c = \frac{1}{\sqrt{LC}} \tag{3-7}$$

it follows that either L or C can be computed if the other is known. The capacity can be computed as follows.

The capacity of a parallel plate condenser is

$$C_1 = \frac{K_o A}{d} \tag{6-21}$$

The capacity per unit length of the transmission line above is

$$C = \frac{K_o w}{d} \tag{6-22}$$

From Maxwell's equations

$$q = C V = C (\underline{E}\, d) = C \frac{D \cdot d}{K_o} \tag{6-23}$$

Field x Gap

The total flux of $\underline{D}$ = total charge enclosed

$$\underline{D} = D \cdot w$$

$$q = \underline{D}w = \frac{C}{K_o} (\underline{D} \cdot d) \tag{6-24}$$

$$\text{Whence} \quad C = \frac{wK_o}{d} \tag{6-22}$$

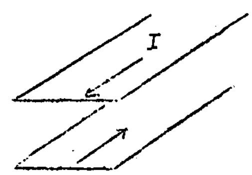
Having this, the inductance L can be computed.

It can also be computed from first principles thus:

L is defined thus

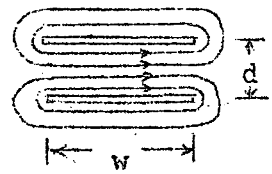
$$\phi = L I \tag{6-25}$$

Flux = Inductance x Current



In the transmission line above current is flowing in opposite directions in the two plates.

Looking endways at the plates, each plate will be surrounded by a magnetic field



The current flow in this plate is up from the paper
The current flow in this plate is down

The lines from each current that encircle the other plate cancel, being of opposite sense, leaving only a weak stray field. Those in between reinforce each other.

The Integral

$$\oint \underline{H} \, dl = \underline{H}w \qquad (6\text{-}26)$$

$$I = w\underline{H} = \frac{w\underline{B}}{\mu_0} = \frac{w\underline{B}d}{\mu_0 d} \qquad (6\text{-}27)$$

This is equal to the total flux per unit length. This may be written

$$\frac{\mu_0 d}{w} = \frac{\underline{B}d}{I} = L \qquad (6\text{-}28)$$

which is the inductance

We now have

$$C = \frac{k_o w}{d} \qquad (6\text{-}22)$$

and

$$L = \frac{\mu_o d}{w} \qquad (6\text{-}28)$$

whence $LC = K_o \mu_o$

and

$$\frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{K_o \mu_o}} = C = 3 \times 10^8 \text{ m/sec} \qquad (5\text{-}4)$$

Previously it was shown that the characteristic impedance of free space is

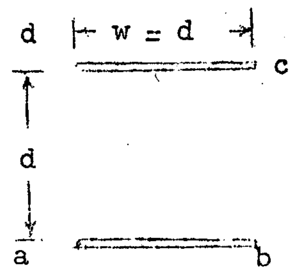
$$Z_c = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu_o}{K_o}} \cdot \frac{d}{w} = 376 \frac{d}{w} \, \Omega \qquad (6\text{-}29)$$

or 376 Ohms per square.

376 Ohms per square what?

This says that the characteristic impedance of a transmission line consisting of two parallel plates may be determined by dividing the space between in square areas of any size and then taking the impedance of each square as 376 ohms and combining them in series and parallel, get the impedance of the transmission line.

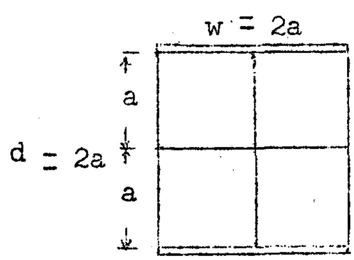
For example, take the case where d = w



$$Z_c = 376 \frac{d}{w} = 376 \frac{w}{w} = 376\ \Omega$$

Consider section of cross-section ab cd and one unit in length perpendicular to the paper.

We may look at this as made up of 4 squares thus

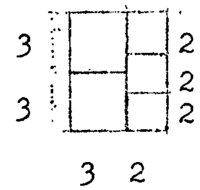


Then from the top plate to the bottom plate we have 2 squares in series vertically and 2 columns in parallel horizontally. The resistance of any one square is 376 ohms, so the resistance of one column 2 squares high will be

2 x 376 ohms

There are two such columns, and the resistance of these two columns in parallel will be half of that of each column, so the total resistance of the 2 Columns is $\frac{2 \times 376}{2} = 376$ ohms.

The squares do not have to be of the same size. For example, take this case



This has $w = 5$ and $d = 6$, and according to Eq. (6-29)

$$Z_c = 376 \frac{d}{w} = 376 \times \frac{6}{5}$$

Looking at the subdivision into squares, we have one column 2 squares high in parallel with a second column 3 squares high. The two columns will have impedances of 2 x 376 ohms and 3 x 376 ohms respectively. We may get the equivalent impedance by adding the admittances and taking the reciprocal of this

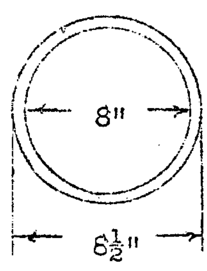
Thus $$Z_c = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{1}{\frac{R_1 + R_2}{R_1 R_2}} = \frac{R_1 R_2}{R_1 + R_2}$$

In this case this becomes

$$Z_c = 376 \frac{1}{\frac{1}{3} + \frac{1}{2}} = 376 \times \frac{2 \times 3}{2 + 3} = 376 \times \frac{6}{5}$$

which is the same as found above.

The usefulness of this concept is that it permits making graphical approximations of characteristic impedances of shapes that might be difficult or impossible to compute analytically. For example, consider a transmission line composed of two concentric cylinders, with an annulus 1/4"



wide. Considering this as equivalent to a pair of flat plates (6-29)

$$Z_c = 376 \frac{d}{w} = 376 \cdot \frac{0.25}{8.25\,\pi} = 3.66\ \Omega$$

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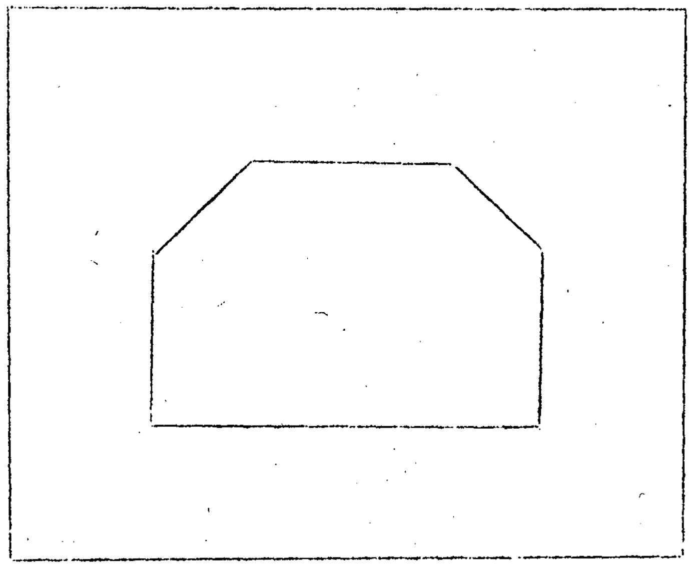
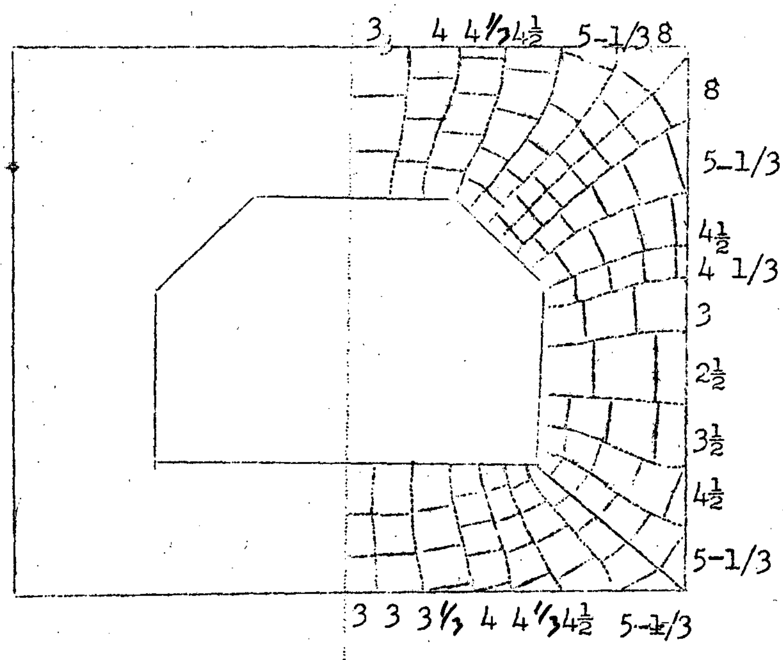
Apply the idea of squares to a more complicated transmission line shape, such as this





Apply to this the method of curvilinear squares.

First we can note that if there is an axis of symmetry we can consider only one half and get the total impedance by dividing the impedance found for one half in two, since the two halves are in parallel.

Next we can conclude for the two corners that the conditions will be more or less symmetrical about a 45 degree line extending to the corners.

Then, noting that lines starting on the inner boundary must spread out as they approach the outer boundary, lines of potential are sketched in by eye, and divided in squares. The spacings are adjusted by eye to permit division into areas as nearly as possible square, keeping in mind that the lines must be at right angles to the inner and outer boundaries. The number of squares in each column is then noted. This gives the number in series in each column. These are then combined in parallel by the rule

$$Z_c = 376 \frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \ldots}$$

In the case sketched this can be handled as follows, starting from top center and going to the right

No. of Squares Per Column	Reciprocal	No. of Squares Per Column	Reciprocal
3	.333	Bro't For'd - - -	2.842
4	.250	3-1/2	.285
4-1/3	.230	4-1/2	.222
4-1/2	.222	5-1/3	.186
5-1/3	.186	5-1/3	.186
8	.125	4-1/2	.222
8	.125	4-1/3	.230
5-1/3	.186	4	.250
4-1/2	.222	3-1/3	.300
4-1/3	.230	3	.333
3	.333	3	.333
2-1/2	.400		5.389

$$\frac{1}{5.389} = 0.186$$

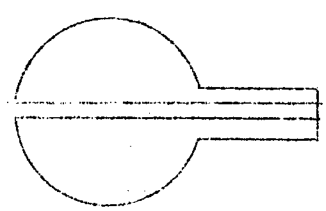
Divide by 2 to take account of other half

$$\frac{0.186}{2} = 0.093$$

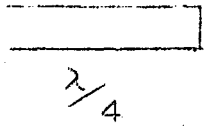
Then $Z_c = 376 \times 0.093 = 35\,\Omega$

It doesn't make any difference what the sizes of the squares are, as long as they are properly added in series, and the series sums combined in parallel.

The process can be applied to a more complicated structure, such as a pair of Dees for a cyclotron



The connections may be made into quarterwave transmission lines with shorted termini

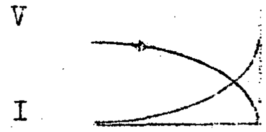


For this case

$$V = V_c \cos \frac{2\pi x}{L}$$

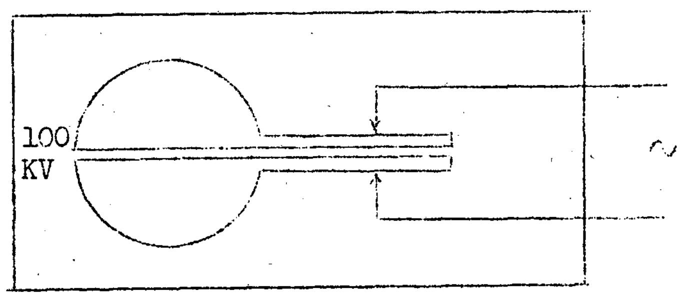
$$I = \frac{V}{Z_c} \sin \frac{2\pi x}{L}$$

The voltage and current appear thus



The two dees of a cyclotron are computed separately and the result multiplied by two

Excitation



Suppose it is desired to excite the Cyclotron Dee's to 100 KV and the available source is limited to 10 KV.

An estimate is made as to where the 10 KV points are in the stems, and the power is fed in there by connecting to the grid and plate of the R-F power source, making the connecting lines an integral number of wave lengths long.

Adjustment will have to be made by trial and error.

APPENDIX TO LECTURE 6

THE FOLLOWING IS EXTRACTED FROM "MATHEMATICS OF MODERN ENGINEERING, VOLUME I, BY DOHERTY AND KELLER

Chapter III, Part II, Pages 206-220, "Derivation of the Partial Differential Equations of Mathematical Physics or Vector Fields"

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The following is extracted from "Mathematics of Modern Engineering, I" by Doherty and Keller, John Wiley & Sons--1936:

DERIVATION OF THE PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS OR VECTOR FIELDS

The derivation of the partial differential equations of mathematical physics is little more than expressing vector relations which hold within a vector field or between vector fields. The basis relations themselves are, in general, physical relations in vector form accompanied by certain mathematical transformations resulting in the partial differential equations of Section 79.

77. SOME VECTOR FIELDS. There are many kinds of vector fields. In the study of heat conduction, it is known that the flow of heat is in the direction of the greatest decrease of temperature and has a magnitude per unit area proportional to the rate of change of temperature. This statement is expressed simply by the equation $\underline{q} = - k\nabla V$, where $\underline{q}$ is the heat flowing through a cross-section of unit area per unit time, the direction being that to give maximum q; V is the temperature, a scalar function; and k is the thermal conductivity of the body. Near every mass there is a field of force called the gravitational attraction. This force of attraction at any point may be obtained by taking the gradient of a scalar point function called the gravitational potential (see Section 82.). Likewise, near an electrically charged body, there is the electrostatic field.

At points exterior to the charge, there exists a scalar point function, the electrostatic potential, whose gradient taken at the point P(x,y,z) gives the negative of the electric intensity at that point. Near a magnetized body there is a magnetic field. The negative of the magnetic intensity of this field is given by the gradient of the scalar magnetic potential. Within a body of flowing fluid there is a vector field or velocity field. If the curl of this field is zero then there exists a function ϕ, called the velocity potential, such that the gradient of ϕ at any point gives the negative of the velocity of the fluid at that point.

78. PRELIMINARY THEOREMS. Before deriving the partial differential equations of mathematical physics, it is necessary to understand the very important theorems, in vector analysis, of Gauss, Stokes, and Green. In Section 70, line and surface integrals involving vectors were defined and illustrated. The concept of the volume integral, $\int \underline{F}dv$, of a vector function is also needed. The integral $\int_{Vol} \underline{F}dv$ is defined by the equation

$$\int_{Vol} \underline{F}dv = \underline{i} \int_{Vol} F_x dv + \underline{j} \int_{Vol} F_y dv + \underline{k} \int_{Vol} F_z dv. \qquad (249)$$

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The three theorems of this section are the machinery by which transformations are made between line, surface, and volume integrals. Eqs. (250-252) state in symbols, respectively, Gauss's, Stokes's, and Green's theorems as follows:

$$\iiint_{Vol} \underline{\nabla} \cdot \underline{F} dv = \iint_S \underline{F} \cdot d\underline{S}, \tag{250}$$

$$\iint_S \underline{\nabla} \times \underline{F} \cdot d\underline{S} = \int_c \underline{F} \cdot d\underline{r}, \tag{251}$$

$$\iiint_{Vol} (U \underline{\nabla} \cdot \underline{\nabla} V - V \underline{\nabla} \cdot \underline{\nabla} U) dv = \iint_S (U \underline{\nabla} V - V \underline{\nabla} U) \cdot \underline{n} dS \tag{252}$$

Gauss's theorem stated in words is: The volume integral of the divergence of a vector function of position in space taken over a volume is equal to the surface integral of the vector function taken over a closed surface bounding the volume. To illustrate Gauss's theorem qualitatively, consider a mass of metal within which heat is generated, say by electric current. Gauss's theorem states that the total heat flowing, in the steady state, out through the surface is equal to the volume integral of the divergence of the heat-flow vector, which can be shown to be equal to the amount of heat generated in the solid.

Stokes's theorem is: The surface integral of the curl of a vector function of position in space taken over a surface S is equal to the line integral of the vector function taken around the periphery of the surface. A physical illustration of Stokes's theorem may be had in the magnetic field about a wire carrying a current. According to the circuital theorem the work done in carrying a unit pole around a closed path is 4π times the current enclosed by the path, or if the path lies in air ($\mu = 1$), in symbols

$$\int_c \underline{B} \cdot d\underline{r} = 4\pi \underline{I}.$$

But $\underline{I}$ is equal to the surface integral of the current density $\underline{j}$ over any surface bounded by the closed path c,

$$\int_c \underline{B} \cdot d\underline{r} = 4\pi \iint \underline{j} \cdot d\underline{S}.$$

DERIVATION OF THE PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS OR VECTOR FIELDS

The circuital theorem may also be written $\underline{\nabla} \times \underline{B} = 4\pi \underline{j}$. To see that this is true it is only necessary to refer to the third definition of curl in Section 75. If in Eq. (248), $\underline{F}$ is replaced by $\underline{B}$ and $\int_c \underline{B} \cdot d\underline{r}$ by $4\pi \underline{I}$ we obtain

$$(\underline{\nabla} \times \underline{B})_j = \lim_{a \to 0} \frac{\int_c \underline{B} \cdot d\underline{r}}{a} = \lim_{a \to 0} \frac{4\pi \underline{I}}{a} = 4\pi j.$$

Replacing in the double integral above, $4\pi \underline{j}$ by $\nabla \times \underline{B}$ we obtain

$$\int_c \underline{B} \cdot d\underline{r} = \int_s\int \underline{\nabla} \times \underline{B} \cdot d\underline{S},$$

which is Stokes's theorem. The material of this article is a statement and illustration of these theorems by means of physical examples. But these theorems depend in no way upon physical experiment. They are mathematical identities.

79. THE PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS. The chief partial differential equations of Mathematical Physics are the following:

a. Laplace's equation $\underline{\nabla} \cdot \underline{\nabla} V^* = 0$, which is satisfied by the functions:

1. Gravitational potential in regions unoccupied by attracting matter.
2. Electrostatic potential at points where no charge is present.
3. Magnetic potential in regions free from magnetic charges.
4. Temperature in steady state.
5. Velocity potential at points of a homogeneous non-viscous fluid moving irrotationally.

* The operator

$$\underline{\nabla} \cdot \underline{\nabla} V = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) V$$

and the equation

$$\underline{\nabla} \cdot \underline{\nabla} V = 0$$

is called the LAPLACIAN EQUATION.

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DERIVATION OF THE PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS OR VECTOR FIELDS

6. Electric potential in homogeneous conductors in which a current is flowing.

b. Poisson's equation $\nabla \cdot \nabla V = - e$.
c. Equation of heat conduction without sources, $a^2 \nabla \cdot \nabla \theta = \theta_t$.
d. Equation of heat conduction with sources, $a^2(\nabla \cdot \nabla \theta + e) = \theta_t$.
e. Wave equation, $a^2 \nabla \cdot \nabla \psi = \psi_{tt}$, and
f. $a^2(\nabla \cdot \nabla \psi + e) = \psi_{tt}$.
g. Equations of elasticity.
h. Telegraphists' equation, $a\phi_{tt} + b\phi_t + c \nabla \cdot \nabla \phi = - ce$.
i. Maxwell's field equations.
j. Euler's equation for the motion of a fluid.

The single subscript t indicates one partial differentiation with respect to time; two subscripts, partial differentiation twice. We now derive, in vector notation, some of the above equations.

80. EQUATION OF HEAT CONDUCTION WITHOUT SOURCES, $a^2 \nabla \cdot \nabla \theta = \theta_t$. Consider the following problem: A mass of iron has been heated to a certain temperature and left to cool. What is the temperature at any point of the mass at any time t? The differential equation giving this temperature may be found from the following physical facts:

(a) The flow of heat will be in the direction of the greatest decrease of temperature and will have a magnitude per unit area proportional to this rate of change of temperature.

(b) The rate at which heat is lost lost by a given region of the body is the heat flux passing through the surface bounding the region.

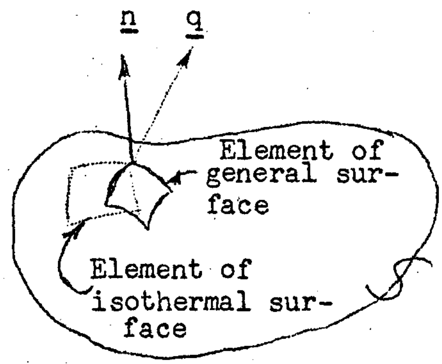

Fig. 47.

The rate of heat loss from an element of volume dv in terms of temperature θ and specific heat c is $- c\rho \frac{\partial \theta}{\partial t}$ dv where ρ is the density.

Thus the rate of heat loss from a general region of volume V (See Fig. 47) bounded by surface S is

$$-\frac{\partial Q}{\partial t} = -\iiint_V c\rho \frac{\partial \theta}{\partial t}\, dv.$$

DERIVATION OF THE PARTIAL DIFFERENTIAL EQUATIONS OF
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In general S is not an isothermal surface. We may also express the rate of heat loss in terms of the heat current density $\underline{q}$ (heat flow per unit of time per unit area normal to the flow) as

$$-\frac{\partial Q}{\partial t} = \int_S\int \underline{n}\cdot\underline{q}dS.$$

Equating these two expressions by relation (b)

$$-\int\int\int \rho c\frac{\partial\theta}{\partial t}dv = \int_S\int \underline{n}\cdot\underline{q}dS. \qquad (253)$$

By means of Gauss's theorem, the last equation becomes

$$-\int\int_V\int \rho c\frac{\partial\theta}{\partial t}dv = \int\int_V\int \underline{\nabla}\cdot\underline{q}dS.$$

Since these integrals are equal for every volume, the integrands must be equal. Hence

$$-\rho c\frac{\partial\theta}{\partial t} = \underline{\nabla}\cdot\underline{q}.$$

But by relation (a), $\underline{q} = -k\underline{\nabla}\theta$, where k is the thermal conductivity. The last equation then becomes

$$a^2\underline{\nabla}\cdot\underline{\nabla}\ \theta = \theta_t\ ,$$

where $a^2 = \frac{k}{\rho c}$.

81. EQUATION OF HEAT CONDUCTION WITH SOURCES. In this case, physical relations (a) and (b), Section 80, still obtain, and also one additional one. Each element of the mass within the volume V may have heat generated in it by some means, for example, by an electric current. The density of strength of source e of heat is defined by the equation

$$e = \lim_{V\to 0}\frac{\text{Total heat created within V per unit time}}{V}.$$

The additional physical relation is: the rate at which heat is emitted from the element of volume dv may be considered as consisting of two parts: first, that which is the rate of cooling the element if no source were present, namely, $-\rho c\frac{\partial\theta}{\partial t}dv$; and secondly, that due to the source edv. Returning to Eq. (253) of the preceding paragraph, we write

$$\int\int_{Vol}\int\left(-\rho c\frac{\partial\theta}{\partial t} + e\right)dv = \int_S\int \underline{n}\cdot\underline{q}dS.$$

Since this equation holds for every volume, it follows that

$$a^2(\underline{\nabla} \cdot \underline{\nabla}\, \theta + e) = \theta_t \ .$$

82. CONCEPT OF POTENTIAL AND THEOREMS OF GENERAL VECTOR FIELDS. It has been noted in Section 77 that the gradient of a scalar point function (called various kinds of potential) gives a vector field. This leads to the definition of a potential. A potential is a scalar point function whose gradient is a vector field. In such a case, the vector field is said to possess a potential. It is by no means true that all vector fields possess a potential. The simple criterion for the existence of a potential is given by the theorem:

I. A necessary and sufficient condition that a field $\underline{F}$ possess a potential is that $\underline{\nabla} \times \underline{F} = 0$. (See Section 27 for the meaning of necessary and sufficient.)

To determine whether the curl of a field is zero, it is necessary to know physical facts about the field and then to apply Eq. (248),

which is

$$(\text{Curl } \underline{F})_s = \lim_{a \to 0} \frac{\int_c \underline{F} \cdot d\underline{r}}{a}$$

where a = area

For instance, in the magnetostatic case, if the line integral $\int_c \underline{B} \cdot d\underline{r}$ is calculated around a closed path which encloses no currents, by the circuital theorem, $\int \underline{B} \cdot d\underline{r} = 0$, and consequently, by (248), curl $\underline{B} = 0$ in such regions. Similarly, the line integrals of the force of attraction and electric intensity, taken around closed paths, are zero for gravitational and electrostatic fields.

The concept of potential function is one of the most important in mathematical physics because, once the potential (if it exists) of the field is known, the field is determined. This raises the question, why not find the field due to the distribution of charge, current, or mass at once, and dispense with the intermediate potential? The answer is that the potential satisfies certain partial differential equations which can be integrated and hence the potential may be found with less difficulty than the field. The following table displays some of the most important potentials and their definitions.

DERIVATION OF THE PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS OR VECTOR FIELDS

	Definition by line integral	Definition by volume integral	Definition by partial differential equation. Solution, subject to boundary conditions of:
Newtonian potential	Negative Work $= \int_{\infty}^{r} \underline{F} \cdot (+\, d\underline{r})$ Per unit mass	$V = \int \frac{\mu\, dv}{r}$	$\nabla \cdot \nabla V = 0$, or $\nabla \cdot \nabla V = -4\pi\mu$.
Electrostatic potential	Work $= \int_{\infty}^{r} \underline{E} \cdot (-\, d\underline{r})$ Per unit charge	$V = \int \frac{\rho\, dv}{\underline{r}}$	$\nabla \cdot \nabla V = 0$, or $\nabla \cdot \nabla V = -4\pi\rho$.
Magnetic potential	Work $= \int_{\infty}^{r} \underline{H} \cdot (-\, d\underline{r})$ Per unit pole	$\Omega = \int \frac{\sigma\, dv}{r}$	$\nabla \cdot \nabla \Omega = 0$, or $\nabla \cdot \nabla \Omega = -4\pi\sigma$
Magnetic vector potential	....	$\underline{A} = \int \frac{\underline{j}\, dv}{r}$	$\nabla \cdot \nabla \underline{A} = 0$, or $\nabla \cdot \nabla \underline{A} = -4\pi \underline{j}$
Velocity potential	....	....	$\nabla \cdot \nabla \phi = 0.$
Velocity vector potential	....	$\underline{\Phi} = \frac{1}{2\pi} \int_{vol} \frac{\underline{\omega}}{r} dV$	$\nabla \cdot \nabla \underline{\Phi} = 0$, or $\nabla \cdot \nabla \underline{\Phi} = -2\underline{\omega}$.

In the above table:

μ = mass per unit volume,
ρ = density of charge per unit volume,
σ = pole strength per unit volume,
$\underline{\Phi}$ = velocity vector potential,
$\underline{\omega}$ = angular velocity of fluid = $\frac{1}{2}$ curl of linear velocity,
$\underline{F}$ = gravitational force,
$\underline{H}$ = magnetic intensity (force per unit pole) = B/μ,
$\underline{\phi}$ = velocity potential,
$\underline{E}$ = electric intensity.

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DERIVATION OF THE PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS OR VECTOR FIELDS

In the case of vector potentials the fields desired are obtained not by taking the gradient but by taking the curl of the vector potential.

From theorem I, it is evident that vector fields possessing potentials are not the most general fields, since the curls of such fields have the special value zero. What then is the nature of a general vector field, and what must be known about a general field to determine it? The answer to these two questions are theorems II and III.

II. Let $\underline{F}$ be a single-valued vector function which, along with its derivatives, is finite and continuous and vanishes at infinity. Then $\underline{F}$ can be written

$$\underline{F} = \underline{\nabla} \phi + \underline{\nabla} \times \underline{H},$$

$$\text{or} \quad \underline{F} = \text{grad } \phi + \text{curl } \underline{H}$$

where ϕ and $\underline{H}$ are respectively a scalar and a vector point function. This is the Helmholtz theorem in vector analysis.

III. A vector field is uniquely determined if the divergence and curl be specified, and if the normal component of the field be known over a closed surface, or if the vector vanish as $1/r^2$ at infinity. If neither of the last two conditions is satisfied, the field is determined except for an additive constant vector.

We now resume the derivation of equations.

83. PARTIAL DIFFERENTIAL EQUATIONS OF GRAVITATIONAL, ELECTROSTATIC, AND MAGNETOSTATIC FIELDS. These derivations are based upon Gauss's law and in the case of the magnetostatic field, the circuital theorem of Ampere.

1. Gauss's law. In electrostatics the force between two charges q_1 and q_2 is given by the inverse square law

$$f = \frac{q_1 q_2}{r^2}.$$

The field vector $\underline{E}$ is defined as the force per unit charge. Gauss's law relates the surface integral of $\underline{E}$ over a closed surface S to the charge Q within S. For a region containing no polarized dielectric it is

$$\iint_S \underline{E} \cdot d\underline{S} = 4\pi Q.$$

This can readily be proved from the inverse square law; in fact, it is a mathematical equivalent which is based on no further experimental evidence. Thus if a phenomenon is characterized by Coulomb's inverse square law, as the magnetostatic and gravitational fields are, Gauss's law also holds. For the magnetostatic field $\underline{E}$ in Gauss's law is replaced by $\underline{H}$, the force per unit pole, and Q, by the number of unit poles enclosed. For the gravitational field, $\underline{E}$ is replaced by $\underline{F}$, the force per unit mass, and Q by — M, the negative of the total mass enclosed. The negative sign occurs because the force between masses is attraction whereas that between like charges or like poles is repulsion.

2. The circuital theorem. The line integral $\int_c \underline{H} \cdot d\underline{r}$ of the magnetic intensity $\underline{H}$, due to a current, taken around any closed path c encircling a conductor is equal to $4\pi I$, where I is the total current flowing in the conductor.

By means of 1 and 2 above, most of the fundamental laws governing gravitational, electrostatic, and magnetostatic fields are quickly obtained.

(a) Gravitation. In gravitational fields Gauss's law is

$$\int_S\int \underline{F} \cdot d\underline{S} = -4\pi M,$$

where M is the total mass enclosed. The last equation may be written

$$\int_S\int \underline{F} \cdot d\underline{S} = -4\pi\int_{Vol} \mu \, dv,$$

where μ is the mass density. Applying Gauss's theorem (250), we have

$$\int_{Vol} \underline{\nabla} \cdot \underline{F} dv = -4\pi \int_{Vol} \mu \, dv.$$

Since the last equation holds for every volume, it follows that the integrands are equal, that is,

$$\underline{\nabla} \cdot \underline{F} = -4\pi\mu. \qquad (254)$$

By applying the definition of curl (248) to a gravitational field, which obeys the inverse square law, it can be shown that $\underline{\nabla} \times \underline{F} = 0$ everywhere. By theorem I, Section 82, a potential V exists such that $\underline{\nabla} V = \underline{F}$. Hence (254) can be written

$$\underline{\nabla} \cdot \underline{\nabla} V = -4\pi\mu. \qquad (255)$$

This is Poisson's equation. It holds at all points occupied by matter. At points free from attracting matter $\mu = 0$, and Poisson's equation becomes Laplace's equation

$$\underline{\nabla} \cdot \underline{\nabla} V = 0. \qquad (256)$$

Eqs. (255) and (256) are the important partial differential equations of gravitational theory.

DERIVATION OF THE PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS OR VECTOR FIELDS

(b) Magnetostatics. Replacing $\underline{E}$ by $\underline{H}$ and Q by $\int_{Vol} \sigma\, dv$ in Gauss's law, and repeating the reasoning immediately preceding (254), we have

$$\nabla \cdot \underline{H} = 4\pi\sigma. \tag{257}$$

The quantity σ is the pole strength per unit volume. By the circuital theorem and the definition of curl (248), it follows that in non-current-carrying regions

$$\nabla \times \underline{H} = 0 .$$

Hence by theorem I, Section 82, potential function Ω exists in non-current-carrying regions such that $\nabla \Omega = -\underline{H}$. Hence (257) becomes

$$\nabla \cdot \nabla \Omega = -4\pi\sigma. \tag{258}$$

At points devoid of magnetic poles the last equation becomes

$$\nabla \cdot \nabla \Omega = 0. \tag{259}$$

In current-carrying regions, by the circuital theorem, $\nabla \times \underline{H} \neq 0$, and consequently no scalar potential Ω exists.

(c) Electrostatics. By retracing the steps employed in (a) of this article, it follows that

$$\left.\begin{aligned} \nabla \cdot \underline{E} &= 4\pi\rho, \\ \nabla \times \underline{E} &= 0, \\ \nabla V &= -\underline{E}, \\ \nabla \cdot \nabla V &= -4\pi\rho . \end{aligned}\right\} \tag{260}$$

The quantities ρ and V are defined in Section 82. So far, the electrostatic charges considered in the application of Gauss's law have been free charges. Gauss's law as stated above holds only if there is no dielectric medium within the closed surface. Suppose now, in addition to free charges, there is within S a dielectric containing bound charges which are influenced by an electric field. The field causes the positive atom cores and negative electrons of an atom to be displaced from their equilibrium (normal) position. The result is that the atom forms a DIPOLE. The product of either charge of a dipole by the separating distance is called the magnitude of the electric moment of the dipole. If the direction is taken from the negative charge to the positive charge, the product of this unit vector by the magnitude of the moment is called the ELECTRIC MOMENT, a vector quantity. The polarization $\underline{P}$ of a dielectric is defined to be the total electric moment per unit volume. It can be shown that the POLARIZATION of the atoms of dielectric is equivalent to a mean charge per unit volume of $-\underline{\nabla} \cdot \underline{P}$. Hence Gauss's theorem becomes

$$\int_S \underline{E} \cdot d\underline{S} = 4\pi(Q - \int_V \underline{\nabla} \cdot \underline{P}dv)$$

$$= 4\pi(Q - \int_S \underline{P} \cdot dS),$$

$$\int_S (\underline{E} + 4\pi\underline{P}) \cdot dS = 4\pi Q. \qquad (261)$$

The quantity $\underline{E} + 4\pi\underline{P}$ is called the ELECTRIC DISPLACEMENT and is denoted by $\underline{D}$. Hence Gauss's law for all charges within S is

$$\int_S \underline{D} \cdot d\underline{S} = 4\pi Q.$$

Again proceeding as in (a), we have

$$\underline{\nabla} \cdot \underline{D} = 4\pi\rho \qquad (262)$$

instead of

$$\underline{\nabla} \cdot \underline{E} = 4\pi\rho$$

We are now in a position to derive Maxwell's field equations.

84. MAXWELL'S EQUATIONS. For the derivation of these equations, in general form, there are needed: (a) certain results of Section 83, (b) the experimental results of Faraday and Ampere, (c) vector relations, and (d) Maxwell's generalization.

(a) Results of Section 83. In Eqs. (257-262) electrostatic and electromagnetic units are employed. The most important of these equations, if written in Heaviside-Lorentz rational units (to eliminate the 4π's), are

$$\left.\begin{aligned} \underline{\nabla} \cdot \underline{D} &= \rho, \\ \underline{\nabla} \cdot \underline{B} &= 0, \\ \underline{\nabla} \cdot \underline{E} &= \rho, \\ \int_C \underline{H} \cdot \underline{dr} &= \underline{j}. \end{aligned}\right\} \qquad (263)$$

Eqs. (263) hold for steady currents and stationary electrostatic charges and stationary circuits. It is natural to expect the existence of a set of simultaneous partial differential equations describing the more general electromagnetic configurations, that is, those configurations or systems in which there are moving circuits and charges not at rest. These equations are the well-known FIELD EQUATIONS.

(b) Experiments of Faraday and Ampere. In 1831 Faraday discovered the fact that, whenever the magnetic flux through a closed single-turn circuit varies, there is induced in the circuit an electromotive force whose magnitude is equal to the time rate of decrease of flux. The direction of the electromotive force is related to the direction of flux through the circuit as shown in Fig. 48. If the electromotive force is induced in a conductor a current flows.

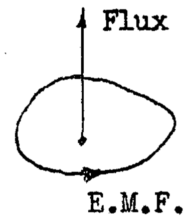


Fig. 48.

Ampere first obtained experimentally the results upon which the theorem stated in Section 83 is based.

(c) Mathematical expression of Faraday's and Ampere's laws. The electromotive force e around the closed curve C formed by a circuit is defined by the line integral

$$e = \int_c \underline{E} \cdot d\underline{r},$$

taken around the curve. By Stokes's theorem,

$$\int_c \underline{E} \cdot d\underline{r} = \int_S\int \underline{\nabla} \times \underline{E} \cdot d\underline{S}, \tag{264}$$

where S is a cap (surface) whose periphery is the circuit or curve C. Faraday's experimental result, expressed in vector form, is

$$\int_S\int (\underline{\nabla} \times \underline{E}) \cdot d\underline{S} = -\frac{1}{c}\frac{d}{dt}\int_S\int \underline{B} \cdot d\underline{S} = -\frac{1}{c}\int\int \frac{\partial \underline{B}}{\partial t} \cdot d\underline{S},$$

or

$$\int\int (\underline{\nabla} \times \underline{E}) \cdot d\underline{S} = -\frac{1}{c}\int\int \dot{\underline{B}} \cdot d\underline{S},$$

where the dot over a quantity indicates partial time differention, and c is a constant of proportionality, equal to the velocity of light, necessary in this system of units. Since the last equation is true for every surface S, it follows that

$$\underline{\nabla} \times \underline{E} = -\frac{1}{c}\dot{\underline{B}}. \tag{265}$$

Eq. (265) is Faraday's law in differential form. Ampere's circuital theorem, in vector notation, is

$$\int_c \underline{H} \cdot d\underline{r} = \int_S\int \underline{j} \cdot d\underline{S},$$

where S is a cap whose periphery is C. By Stokes's theorem, we also have

$$\int_c \underline{H} \cdot d\underline{r} = \int_S\int \underline{\nabla} \times \underline{H} \cdot d\underline{S}.$$

Consequently,

$$\int_S\int \underline{\nabla} \times \underline{H} \cdot d\underline{S} = \int_S\int \underline{j} \cdot d\underline{S},$$

or by the reasoning preceding Eq. (265)

$$\underline{\nabla} \times \underline{H} = \underline{j}. \tag{266}$$

DERIVATION OF THE PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS OR VECTOR FIELDS

Eq. (266) is Ampere's law in differential form. If it is assumed that Gauss's law is valid for variable fields as well as for electrostatic and magnetostatic fields, we then have the four equations:

$$\left.\begin{aligned} \underline{\nabla} \cdot \underline{D} &= \rho, \\ \underline{\nabla} \cdot \underline{B} &= 0, \\ \underline{\nabla} \times \underline{E} &= -\frac{1}{c}\dot{\underline{B}} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t}, \\ \underline{\nabla} \times \underline{H} &= \underline{j} = \frac{\rho \underline{v}}{c}, \end{aligned}\right\} \qquad (267)$$

where $\underline{v}$ is the drift velocity of charge of density ρ.

(d) Maxwell's generalization. Maxwell noted that Eqs. (267) are inconsistent with the equation of continuity of charge. The equation of continuity of mass, $\frac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot \underline{M}$, was derived in Section 74. If ρ denotes charge per unit volume and $\underline{v}$ its velocity, the equation of continuity, in electromagentic theory, becomes

$$\frac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot (\rho \underline{v}). \qquad (268)$$

This equation merely states that the time rate of increase of charge in any region is equal to the excess of charge flowing in, per unit time, over that flowing out. All experimental evidence indicates that the law of continuity holds, that electricity is neither created nor destroyed.

The contradiction between Eq. (268) and the first and last of (267) is seen as follows. Taking the divergence of $\underline{\nabla} \times \underline{H} = \frac{\rho \underline{v}}{c}$, we have

$$\underline{\nabla} \cdot \left(\frac{\rho \underline{v}}{c}\right) = \underline{\nabla} \cdot \underline{\nabla} \times \underline{H} = 0, \qquad (269)$$

or

$$\underline{\nabla} \cdot \left(\frac{\rho \underline{v}}{c}\right) = 0.$$

But (268) gives

$$\underline{\nabla} \cdot (\rho \underline{v}) = -\frac{\partial \rho}{\partial t}.$$

Moreover, if the first of (267) is differentiated with respect to time, there is

$$\frac{\partial \rho}{\partial t} = \underline{\nabla} \cdot \dot{\underline{D}},$$

and from the value of $\frac{\partial \rho}{\partial t}$ in the equation of continuity

$$\underline{\nabla} \cdot (\rho \underline{v}) = -\underline{\nabla} \cdot \dot{\underline{D}}. \qquad (270)$$

DERIVATION OF THE PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS OR VECTOR FIELDS

Equations (269) and (270) do not agree. Accordingly, Maxwell revised Ampere's law as follows. Let the total current consist of a convection current $\frac{\rho \underline{v}}{c}$ and a displacement current $\frac{\dot{\underline{D}}}{c}$. The $\underline{j}$ in equation (266) is then replaced by $\frac{1}{c}(\rho \underline{v} + \dot{\underline{D}})$, and Ampere's equation as revised by Maxwell becomes

$$\underline{\nabla} \times \underline{H} = \frac{1}{c}(\rho \underline{v} + \dot{\underline{D}}). \qquad (271)$$

If the divergence of (271) is taken, Eq. (270) is obtained. But (270) is a consequence of (268) and the first of (267). Thus the equation of continuity is satisfied if system (267) be replaced by the equations

$$\left.\begin{aligned} \underline{\nabla} \cdot \underline{D} &= \rho, \\ \underline{\nabla} \cdot \underline{B} &= 0, \\ \underline{\nabla} \times \underline{E} &= -\frac{1}{c}\dot{\underline{B}}, \\ \underline{\nabla} \times \underline{H} &= \frac{1}{c}(\rho \underline{v} + \dot{\underline{D}}). \end{aligned}\right\} \qquad (272)$$

These are the field equations of Maxwell.

If the currents are steady the $\dot{\underline{D}} = 0$ and (272), in this special case, reduce to (267).

In regions devoid of charge and current equations (272), since $\underline{B} = \mu \underline{H}$ and $\underline{D} = k\underline{E}$, become

$$\left.\begin{aligned} \underline{\nabla} \cdot \underline{E} &= 0, \\ \underline{\nabla} \cdot \underline{H} &= 0, \\ \underline{\nabla} \times \underline{E} &= -\frac{\mu \dot{\underline{H}}}{c}, \\ \underline{\nabla} \times \underline{H} &= \frac{k}{c}\dot{\underline{E}}. \end{aligned}\right\} \qquad (273)$$

The constants μ and k are respectively the permeability and dielectric constant of the space for which (273) are valid.

The nature of the solution of these equations is discussed in Section 86, and the equations are solved in Vol. II, Chap. III, for configurations of charge and current of great industrial importance.

LECTURE 7

By Dr. Andrew Longacre
2/12/51

WAVE GUIDES

Maxwell computed the velocity of electromagnetic waves as

$$V = \frac{1}{\sqrt{\mu_o K_o}} = 3 \times 10^{10} \text{ cm/sec} \qquad (7\text{-}1)$$

where $\mu_o = 4\pi \times 10^{-7} = 1.257 \times 10^{-6}$ (henry/meter)

$K_o = 8.854 \times 10^{-12}$ (farad/meter)

From the known measured valves of these constants giving a velocity of the electromagnetic wave equal to c, the velocity of light, Maxwell concluded immediately that light was also an electromagnetic wave.

It was some time before Hertz succeeded in generating and detecting waves generated by purely electromagnetic means, which eventually developed into symphony concerts, wooden dummies, political misinformation, and singing commercials now available to all on radio and television sets.

The form of the equations indicated that electromagnetic waves vibrate transverse to the direction of propagation. This is quite different from sound waves, in which the vibration is longitudinal, consisting of alternate zones of compression and rarefaction. (A stretched rope may transmit longitudinal waves, but can also transmit transverse waves, such as may be initiated by a sudden transverse displacement of one end of the rope, which forms a disturbance that travels down the line.)

That light consists of transverse vibrations may be demonstrated by the following experiment.

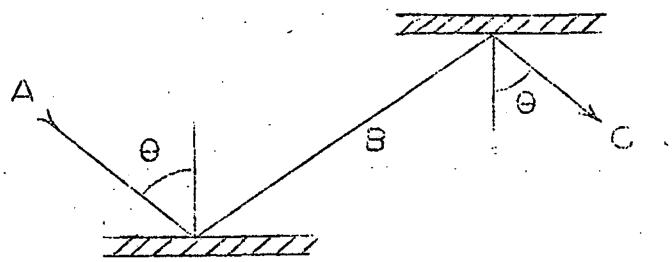


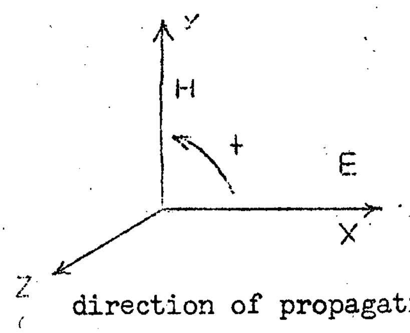
Fig. (7-1)

If light is reflected from 2 polished pieces of black glass (black to absorb the component of the rays that penetrate the surface, so that this portion will not be reflected by the opposite surface, and wreck the experiment) there is a certain critical angle at which light will be seen after the double reflection. At other angles all light is absorbed.

This can be explained as follows. The incident beam A is composed of numerous waves vibrating transversely, but in planes of random orientation relative to the plane of the glass. Most of the light penetrates the glass and is absorbed, but those components that are perpendicular to the plane of the glass will be reflected. The reflected ray B is "plane polarized", that is, it contains only elements vibrating in a single plane. This plane polarized light will also be reflected from the second glass if this is at the critical angle. This phenomenon can only be explained if the vibration is transverse.

A few years ago, after means were developed for generating waves of very short wave lengths, it was found that it was possible to transmit electromagnetic waves down hollow pipes of conducting material. This phenomenon is not the same as transmission by a coaxial cable, in which the current is carried on the surface of the inner and outer conductors, in accordance with the principles of transmission lines that have already been discussed, but the transmission is by means of multiple reflections back and forth between the opposing surfaces.

A transverse wave in space has an E vector (electric) and an H vector (magnetic). These two are in phase, but vibrate in planes at 90° to each other.

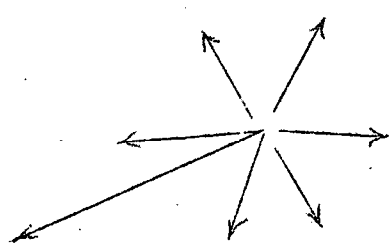


direction of propagation

Fig (7-2)

This is a right hand system. The direction of propagation Z is that of a nut moving along a right hand screw. This is toward the observer if the rotation appears to be counterclockwise.

Light from ordinary sources does not have the directions of E and H fixed, but sucessions of waves vibrate transversely at random angles, as below.



Here the E vector only is shown. The H vector accompanying each E vector is perpendicular to it.

Fig (7-3)

If the direction of the E vector (and the H vector at right angles to it) are pinned down to be in a single plane, the wave is polarized.

Now consider a wave with direction of propagation making an angle of incidence θ with a conducting metal wall.

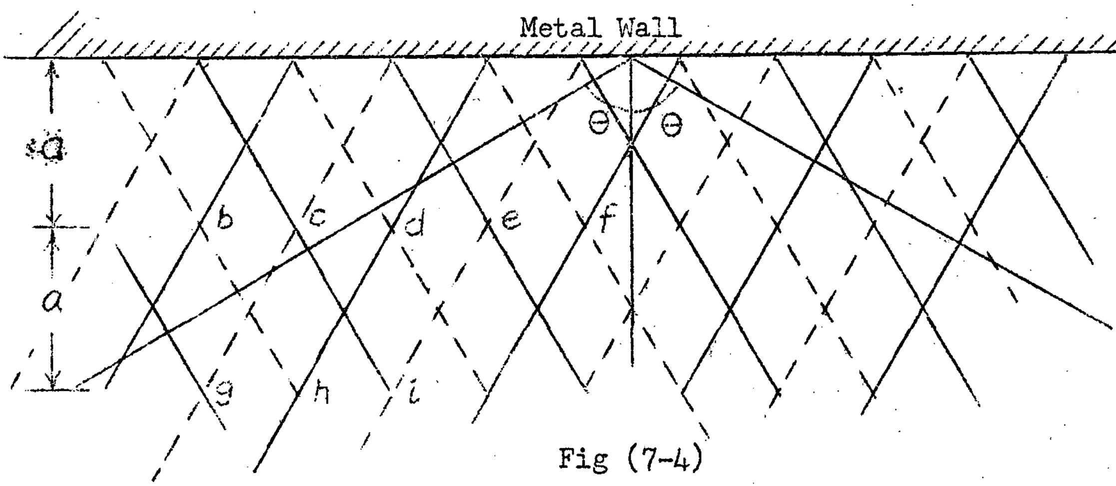
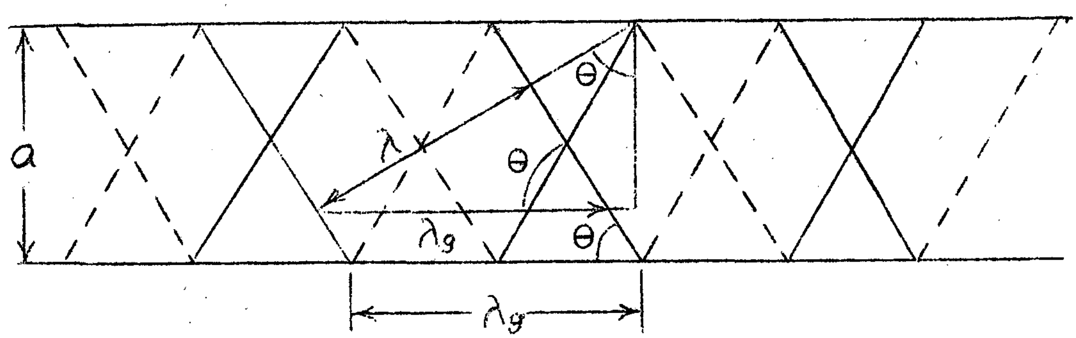


Fig (7-4)

The wave front is perpendicular to the direction of propagation. Consider the full lines represent E vectors coming up from the paper, and the dotted lines represent vectors going down. The E vectors that intersect the surface cause currents that cancel the incident E vector and are of such magnitude and direction that they send out another E vector of opposite phase that constitute the reflected wave front. Now it will be noted that at points b,c,d,e, etc. there are intersections of equal vectors of opposite phase, which will therefore cancel. A similar cancellation will occur at the second group of intersections g,h,i etc. The planes containing these intersections are parallel to the upper metal wall, and are spaced multiples of distance a apart. At any such plane of intersection another metal wall can be placed, which will permit a second series of reflections, but will leave the pattern of waves between the plates undisturbed. The waves will then travel down between the two guiding planes, being reflected back and forth between them, but without causing currents in the metal, such as exist in transmission lines. Two additional planes can then be put in parallel to the plane of the paper, making a rectangular wave guide.

Fig (7-5)



The distance between two successive wave fronts of the same phase, measured perpendicular to the wave fronts, is equal to the wave length λ. The distance between the wave fronts parallel to the wave guide is λ_g. The angle of incidence of the wave against the wall is θ. It follows that

$$\lambda = \lambda_g \sin\theta \quad \text{or} \quad \frac{\lambda}{\lambda_g} = \sin\theta$$

The wave fronts make an angle θ with the wall, which is the same as the angle of incidence of the direction of propagation with the normal. The perpendicular distance "a" bisects the angle between incident and reflected wave fronts, so it follows that

$$\frac{a}{\lambda_g/2} = \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad (7\text{-}2)$$

or

$$\frac{2a}{\lambda_g} = \frac{\lambda/\lambda_g}{\sqrt{1-\left(\frac{\lambda}{\lambda_g}\right)^2}} \qquad (7\text{-}3)$$

This may be solved for λ_g as follows:
Divide by λ/λ_g

$$\frac{2a}{\lambda} = \frac{1}{\sqrt{1-(\lambda/\lambda_g)^2}}$$

$$\frac{\lambda^2}{4a^2} = 1 - \left(\frac{\lambda}{\lambda_g}\right)^2$$

$$1 - \frac{\lambda^2}{4a^2} = \frac{\lambda^2}{\lambda_g^2}$$

$$\lambda_g = \frac{\lambda}{\sqrt{1-\left(\frac{\lambda}{2a}\right)^2}} \qquad (7\text{-}4)$$

This gives the relation λ_g in the quide to λ and a .

If $\lambda = 2a$, $\lambda_g = \infty$

This is the CUT-OFF wave length = 2a. The wave guide then acts as a high-pass filter, and will not pass waves with greater wave length than 2a (that is, it will only pass waves having higher frequency than corresponds to $\lambda = 2a$.
(That this passes waves only with wave lengths shorter than $\lambda = 2a$ can be seen from the fact that for $\lambda > 2a$, the term in the denominator under the radical will be negative, which would make λ_g imaginary.)

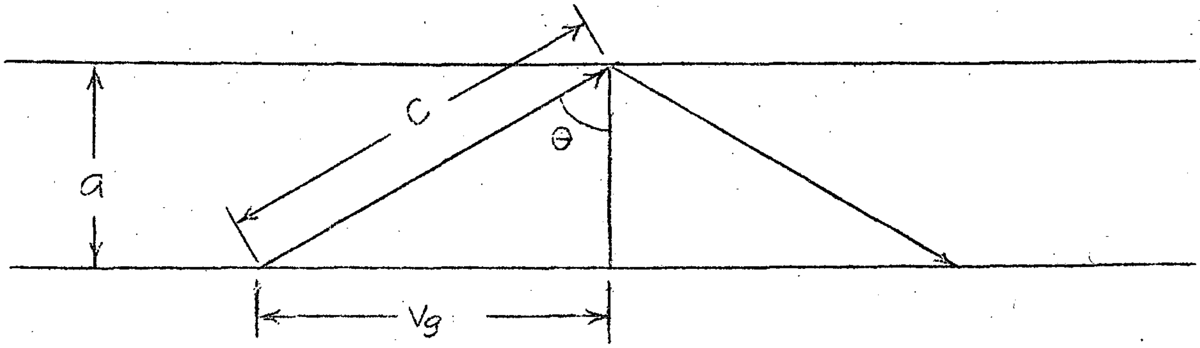


Fig (7-6)

Since the direction of propagation of the waves goes back and forth between the two walls, it will take the energy longer to pass down the wave guide than if it were traveling straight in free space at the velocity of light, C. The velocity of energy transmission is called the GROUP VELOCITY and, as can be seen from the diagram above

$$V_g = C \sin \theta \qquad (7\text{-}5)$$

From Figure (7-5) it can be seen that

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{a}{\lambda_g/2} = \frac{2a}{\lambda_g} = \frac{2a}{\lambda/\sin \theta}$$

$$\text{so} \quad \cos \theta = \frac{\lambda}{2a} \qquad (7\text{-}6)$$

$$\text{and} \quad \sin \theta = \sqrt{1 - \left(\frac{\lambda}{2a}\right)^2} \qquad (7\text{-}7)$$

$$\text{so} \quad V_g = C \sqrt{1 - \left(\frac{\lambda}{2a}\right)^2}$$

In Figure (7-5) the distance λ_g is greater than λ, the relation being

$$\lambda_g = \frac{\lambda}{\sin \theta} \qquad (7\text{-}9)$$

The apparent velocity of the waves in the guide is <u>greater</u> than the velocity of light. This is called the PHASE VELOCITY and

$$V_p = \frac{\lambda_g}{\lambda} C = \frac{c}{\sin \Theta} \qquad (7\text{-}10)$$

Combining (7-5) and (7-10)

$$V_g V_p = C \sin \Theta \cdot \frac{c}{\sin \Theta} = C^2 \qquad (7\text{-}11)$$

Consider a wave guide with plates a distance a apart (This view is looking into the end of the guide)

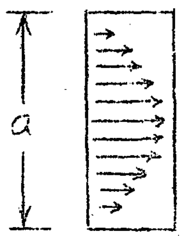


Fig (7-7

The E vector is now in the plane of the paper, and is a cosine function, with zero value at the top and bottom and a maximum in the center.

If the guide width (shown vertically) is doubled to 2a, there will be a second portion of the cosine wave of opposite sign. (fig. 7-8)

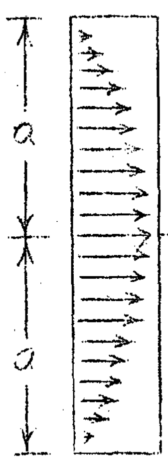


Fig (7-9)

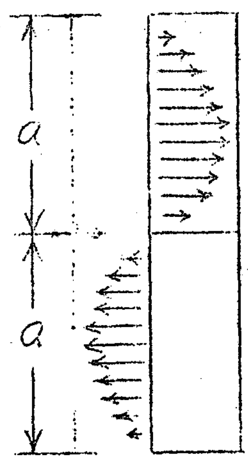
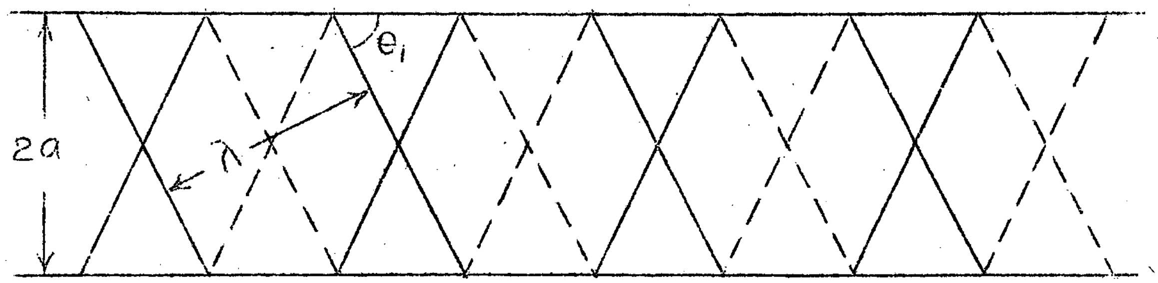


Fig (7-8)

This guide of width 2a can also have passing thru it simultaneously another wave of the same frequency. (fig. 7-9)

Looking at this in longitudinal section the last case would be represented by this arrangement of wave fronts.

Fig. (7-10)



And the first one by this (Fig 7-8):

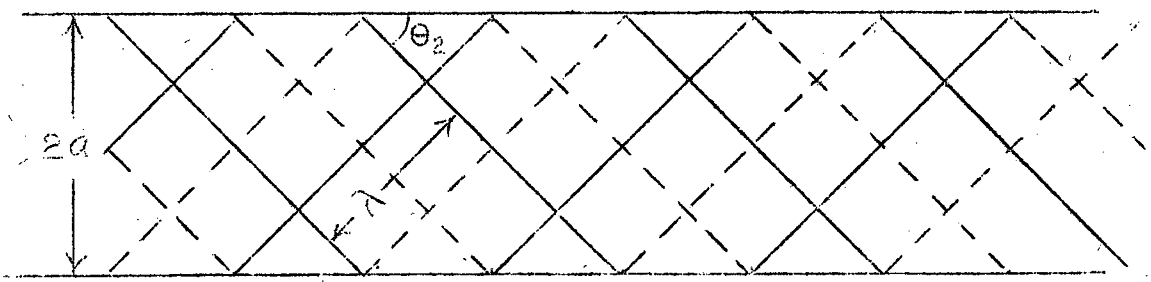


Fig. (7-11)

This shows there may be at least two waves of the same frequency passing down the wave guide simultaneously, but being reflected differently.

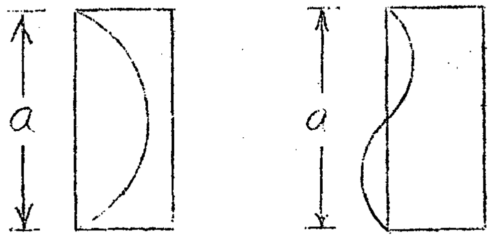
In Figure (7-11), for a guide of width a ,

$$\theta = \cos^{-1} \frac{\lambda}{2a} \tag{7-12}$$

and when $\lambda = 2a$, $\theta = 0$. This means that the wave is bouncing back and forth normal to the side walls, and is not progressing down the guide. Under these conditions no power is transmitted. This is the cut - off frequency.

For a given a , there are an infinite number of wave lengths that can move down the line, all shorter than $\lambda = 2a$. The attenuation is a minimum for the simplest mode; and increases as the frequency increases.

If there are two loops



in width a , the cut off wave length will be

$$\frac{2\lambda}{2a} = \frac{\lambda}{a} = 1$$

and in general, the condition for transmission of energy is that

$$\frac{n\lambda}{2a} < 1$$

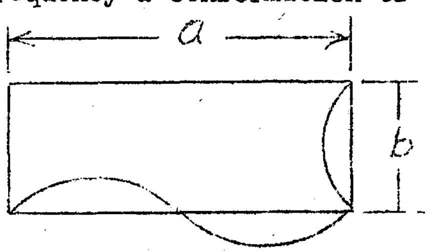
For a given ratio of λ to a , there is a definite angle of incidence and reflection θ. To show what this looks like take a rectangular guide of width a and the particular case where $\lambda = 0.8a$ and n has the value 1 or 2. from (7-12) for n = 1

$$\theta = \cos^{-1}\frac{\lambda}{2a} = \cos^{-1}\frac{0.8a}{2a} = \cos^{-1}0.4 = 63.5^{\circ}$$

For n = 2

$$\theta = \cos^{-1}\frac{2\lambda}{2a} = \cos^{-1}0.8 = 37^{\circ}$$

In a rectangular guide there may be waves reflected back and forth between each pair of side walls. If the guide is square, there are many wave lengths that may be moving down both ways simultaneously. If the wave guide is made with one dimension twice the other, there may be for one given frequency a conformation of two loops in the a direction and one in the



b direction, and this is the cross section generally used.

Now let us consider how the currents act in a wave guide.

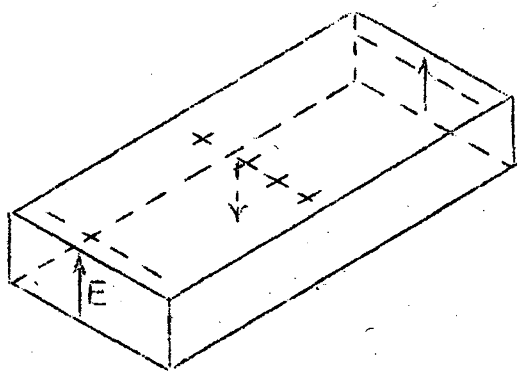


Fig. (7-12)

The electric vectors move up and down, causing charges alternately negative and positive in the top and bottom surfaces. It might be that at first that there would be current flows along the top and bottom guide plates, but this is not the case. As the waves progress, the charges pass from the top to the bottom plates by going down the sides, and there are strong currents

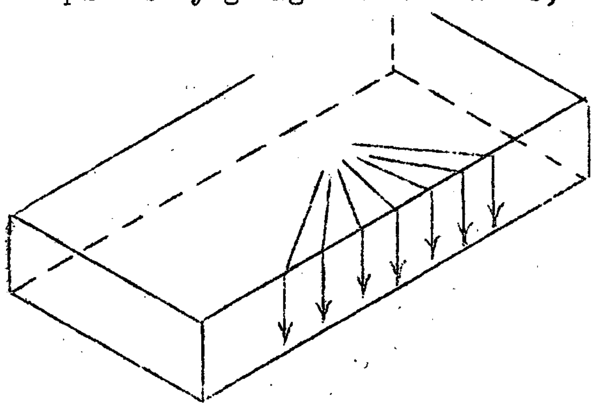


Fig. (7-13)

in the sides of the wave guide.

A slot put in the side walls may be a source of heavy radiation.

Looking down on the wave guide in plan

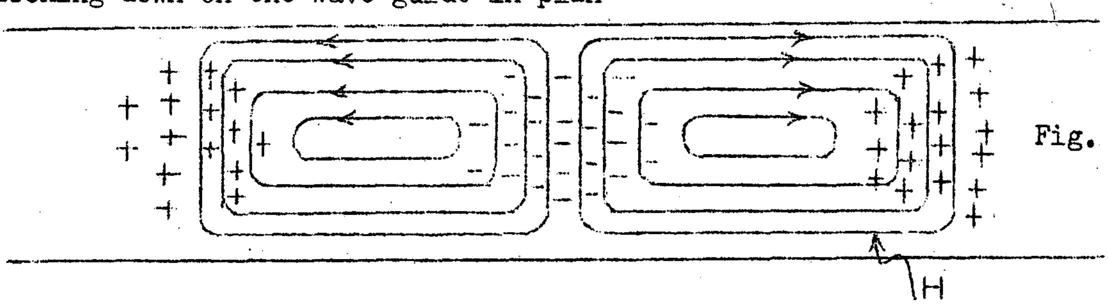
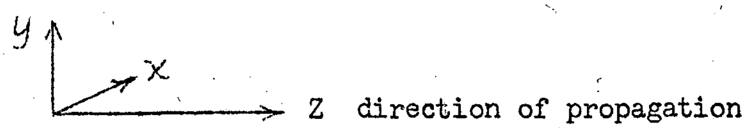


Fig. (7-14)

There are circulating lines of magnetic vectors. These move along the guide as the wave progresses.

Either the E or H vectors may have components along the direction of propagation. If the direction of



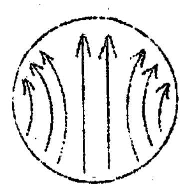
propagation is Z , and the electric vector is in the y direction we have

$E_x = 0$ and $H_x \neq 0$
$E_z = 0$ $H_z \neq 0$
$E_y = E$ $H_y = 0$

The wave mode just described is a transverse electric wave, denoted TE. Transverse magnetic waves are denoted TM. Subscripts are added to indicate the number of bumps or half cycles in the longer and narrower (a and b) directions of the guide. Thus $TE_{1,0}$ represents a transverse electric mode with one hump in the a direction, and none in the b direction.

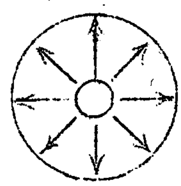
(See Terman "Radio Engineering p 116 et seq for further discussion)

A circular wave guide can be used. This is convenient, for example, when a swivel connection is desired, to permit rotating a guide, but it is otherwise avoided. The pattern of electric vectors for the dominant mode is thus



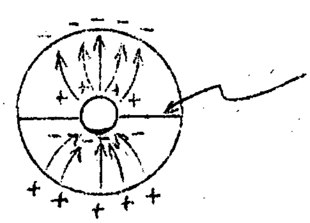
$TE_{1,0}$ Fig. (7-15)

In a coaxial circular guide it is possible to have a pattern thus



$TE_{0,1}$ Fig. (7-16)

By putting in a metal partition the following mode can be obtained.



$TE_{2,0}$ Fig. (7-17)

One objection to a circular guide is that there is nothing to maintain the plane of polarization, and the power may be hard to pickup, as the plane may rotate, due to bends or roughnesses in the line.

For the circular guide for the $TE_{1,0}$ mode of Fig (7-15) the cut-off wave length is

$$\lambda = 3.41\ a$$

where a is the radius in this case

The TM mode may be shown

thus

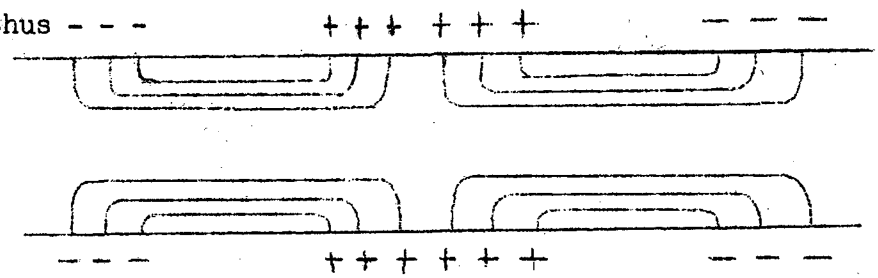


Fig.(7-18)

In this case the lines are E vectors terminating in charges on the walls, and these rings of charges move down the guide walls. The cut-off wave length here is

$$\lambda c = 2.61\ r$$

A circular guide that will transmit this mode will also transmit the $TE_{1,0}$ mode with $\lambda = 3.41\ r$. It would be desirable to prevent both modes occuring.

If diaphragms are put across the cylinder, there will be reflections.

If the diaphragms are spaced a limited number of half guide wave lengths apart, the reflections will superimpose and form a standing wave pattern.

If there are longitudinal E vectors they must end normal to a surface. A resonating cavity can be made thus

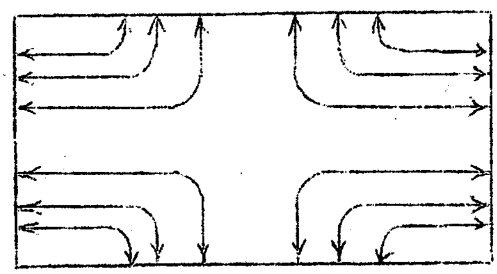


Fig. (7-19)

or by joining a series of such that has an even number of half guide wave lengths.

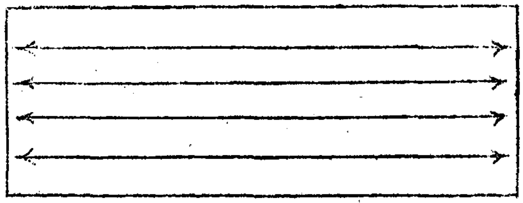
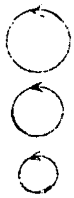


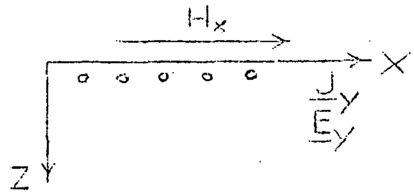
Fig. (7-20)

There can be longitudinal electric waves terminating in the ends. This can happen when $\lambda = 2.61 r$, and the length is immaterial. Since this can also transmit $\lambda = 3.41\ a$, it is necessary to avoid exciting the latter wave length.



As an anolog, consider loops as shown. A varying magnetic field applied to the top loop will produce a current in it. This current has an accompanying magnetic field of which part passes through the next loop and induces a current in the next loop below.

The conditions may be set up in terms of Maxwell's Equations as follows



current is flowing normal to the plane of the paper (upwards)

Take axes as shown, with y normal to the plane of the paper. The plane x y is then the surface of the metal and z is normal to the surface in the direction of the depth. In other words the view above is a transverse section normal to the surface x y

The magnetic vectors have a component $\underline{H}_x$ parallel to x.

There are currents $\underline{J}_y$ normal to the plane of the paper and normal to $\underline{H}_x$.

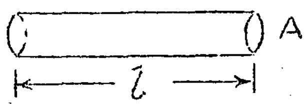
The electric field $\underline{E}_y$ is also normal to the plane of the paper, and this is the only component of this field.

Now $\underline{J} = \sigma \underline{E}$ (8-2)

where σ is the conductivity.

This says the current is proportional to the field and the conductivity, and is only a form of Ohm's Law applied to a field.

This is shown below



Ohms law
$V = IR$

The voltage V is equal to the field E times the length, or

$V = E \cdot l$

The Current I is equal to the current per unit area, J, times the area, or

$I = JA$ (8-3)

The resistance $R = \frac{l}{A} \times \frac{1}{\sigma}$

(σ = conductivity)

so that $\underline{E}l = \underline{J} A \left(\frac{l}{A} \cdot \frac{1}{\sigma} \right) = \underline{J}$

and so $V = I R$

becomes $\underline{E}l = \underline{J} A \frac{l}{A\sigma}$

or $\sigma \underline{E} = J$ (8-4)

Since a current driven by a field must be parallel to the field, $\underline{J}$ and $\underline{E}$ can only have components in the y direction.

In copper, $\frac{\partial D}{\partial t}$ is always $\ll \underline{J}$

so the term $\frac{\partial D}{\partial t}$ can be neglected.

The fourth Maxwell Equation is

$$\underline{\nabla} \times \underline{H} = \underline{J} + \frac{\partial D}{\partial t} \qquad (8\text{-}5)$$

which may be written

$$\underline{\nabla} \cdot \underline{H} = \underline{J} + i\omega K_o \underline{E} \qquad (8\text{-}6)$$

In copper, for, 10 cm waves the last term is only $\frac{1}{10000}$ of $\underline{J}$
For very high frequencies, e.g. light, this isn't so. For technical wave lengths however, the term $\frac{\partial D}{\partial t}$ may be neglected

$$\text{so} \quad \underline{\nabla} \times \underline{H} = \underline{J} \qquad (8\text{-}7)$$

This is the curl of $\underline{H}$, which may be expressed by the determinant

$$\underline{\nabla} \times \underline{H} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = J \qquad (8\text{-}8)$$

Since the $\underline{H}$ vector has only an x component, $\underline{H}_y = \underline{H}_z = 0$
and by the rules for evaluating determinants and the coordinate system assumed

this reduces to

$$\frac{\partial}{\partial z} (\underline{H}_x) = J_y \qquad (8\text{-}9)$$

This says the rate of change of magnetic field with respect to depth is proportional to the current flowing in the y direction.

The third Maxwell equation is

$$\underline{\nabla} \times \underline{E} = -\frac{\partial B}{\partial t} \qquad (8\text{-}10)$$

$\underline{E}$ has only a component in the y direction

$$\underline{\nabla} \times \underline{E} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} \qquad (8\text{-}11)$$

with $\underline{E}_x$ and $\underline{E}_z = 0$, this reduces to

$$-\frac{\partial}{\partial z} \underline{E}_y = -\frac{\partial}{\partial t} \mu_o H_x \quad \text{and since } \underline{H}_x = e^{i\omega t}, \text{ to } -i\omega\mu_o \underline{H}_x \qquad (8\text{-}12)$$

(The i here is $\sqrt{-1}$, not the $\underline{i}$ that is the unit vector in the x direction in the determinant)

Now $E_y = \frac{J}{\sigma}$ from (8-4)

and $J = \frac{\partial H_x}{\partial z}$ from (8-9)

so $E_y = \frac{\partial}{\partial z} \frac{H_x}{\sigma}$ (8-14)

So from (8-12) and (8-14) we get

$$\frac{\partial^2 H_x}{\partial z^2} + (i\omega\mu_0\sigma) H_x = 0 \quad (8\text{-}15)$$

This is of the form

$$\frac{\partial^2 x}{\partial z^2} + A_x = 0$$

which has the solution

$$x = Ae^{i\sqrt{a}\,z} \quad (8\text{-}16)$$

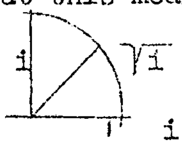
whence

$$H_x = C_1 e^{i\sqrt{i\omega\mu\sigma}\,z}$$

$$= C_1 e^{\pm\sqrt{i}\sqrt{\omega\mu\sigma}\,z} \quad (8\text{-}17)$$

which gives H_x as a function of z

Note that $\sqrt{i}$ appears in the exponent of e.
What this means may be arrived at by looking at the following diagram.



$$\sqrt{i} = \frac{1+i}{\sqrt{2}}$$

Since $i = e^{i\pi/2}$

$$\sqrt{i} = e^{i\pi/4}$$

This can be separated into real and imaginary parts. When we use $e^{i\omega t}$ for representing an alternating sinusoidal current, only the real part is used for the voltage and the imaginary part gives the phase angle

So $$H_x = A\,e^{-\sqrt{\frac{\omega\mu_0\sigma}{2}}\,z}$$

$$H_x = A\,e^{i\left[\omega t - \sqrt{\frac{\omega\mu_0\sigma}{2}}\,z\right]} \quad (8\text{-}18)$$

This shows that the magnetic field decreases exponentially with the depth and that there is a phase change. The radical has both plus and minus values mathematically but the plus value has no physical meaning.

It would indicate that the field increased indefinitely with depth, which is not reasonable. This merely shows that the mathematics doesn't know that there isn't a field on both sides of the surface, and that there is no source of energy beneath the surface.

If we set $\sqrt{\frac{2}{\omega\mu_0\sigma}} = \delta$ (8-19)

(8-18) becomes

$$H_x = (Ae^{\frac{-z}{\delta}} = Ae^{i(\omega t - \frac{z}{\delta})} \qquad (8\text{-}20)$$

δ is called the skin depth, and represents the equivalent depth of the skin that would have the same resistance if the current were being carried uniformly across the cross section. Another way to look at this is to consider δ as the depth at which the current has been reduced in value to 1/e of its value at the surface.

From (8-19) it will be noted that if the conductivity is reduced from 100% to 90% or   is reduced 10%, the power loss is increased 5%, since the skin depth increases 5% while the conductivity is decreased 10%.

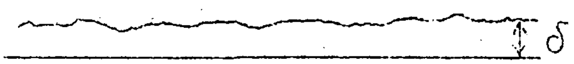
Note also that

$$\delta \text{ varies as } \frac{1}{\sqrt{a}}$$

At 60 cycles, $\delta \doteq 1$ inch
and at 12 megacycles $\delta \doteq 0.00075$ inch

On very large conductors, even for 60 cycles it is desirable to subdivide the conductors to avoid having material that is not acting at good efficiency.

For very high frequencies, only a small thickness carries the current. It is consequently necessary to keep the surface smooth in the direction of current flow



Even very fine emery paper produces grooves relatively large compared to δ at 12 megacycles. This increases the length of the current path.

If the conductivity test is made at too high a frequency, it will indicate higher relative losses than will occur at lower frequencies.

In the distance δ, the current has dropped to 1/e of its surface values and the power has dropped to 10% of the total. With cavities of very high Q, the effect of the losses due to too thin skins cannot be ignored until the power loss is less than 10^{-5} of the total. The higher the Q, the greater number of effective skin depths are necessary to get the Q.

For copper and 12 megacycles

$$\omega = 2 \quad 12 \times 10^6$$
$$\mu_0 = 4 \quad \times 10^{-7}$$
$$\sigma = \frac{1}{1.7 \times 10^4} \text{ mhos/meter}$$

These values substituted in (8-19) give the skin depth in meters.

This reduces to the following to give the skin depth in inches

$$\delta = \frac{0.00260}{\sqrt{f}}$$

where f is the frequency in megacycles (8-21)

Given the magnetic field at the edge of a conducting sheet, how do we determine the loss?

We had

$$\underline{J} = \sigma \underline{E} \qquad (8\text{-}2)$$

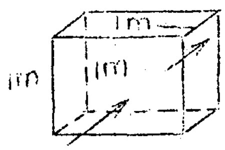
current = conductivity x Field.

from which

$$\underline{P} = \frac{J^2}{\sigma} \qquad (8\text{-}22)$$

Power per unit volume

This follows from the definition of the conductivity and Ohms Law



The resistance of a one meter cube is $\frac{1}{\sigma}$ and Ohm's Law

$P = J^2R$ becomes

$$P = \frac{J^2}{\sigma}$$

For alternating currents, the current value to be taken is the R.M.S. value. In fields, it is usual to talk about peak values instead of R.M.S., and since for sine waves the peak value = $\sqrt{2}$ times the R.M.S. value, (8-22) must take the form

$$P = \frac{J^2_{peak}}{2\sigma} \qquad (8\text{-}23)$$

Now the current will vary with depth below the surface in the same manner as the field varies as given by (8-20), so we can write

$$J_{\text{at any depth}} = J_{\text{at surface}}\, e^{-\frac{z}{\delta}}\; e^{-\frac{iz}{\delta}} \cdot e^{i\omega t}$$

whence

$$J^2\text{ peak}_{\text{at any depth}} = J^2\text{ peak}_{\text{at surface}}\; e^{-\frac{2z}{\delta}} \qquad (8\text{-}24)$$

$$P = \frac{J^2\text{ surface}}{2\sigma} \int e^{\frac{-2z}{\delta}}\, dz \qquad (8\text{-}25)$$

This is the power per unit area of surface

$$P = \frac{\delta}{2\sigma} J^2 \text{ surface}$$

This permits computing the losses if the surface current density is known. Frequently the surface current is not known, but the integral of the current at various depths is known.

$$I = J \text{ surface} \int e^{-\frac{z}{\delta}} e^{\frac{iz}{\delta}} dz \qquad (8\text{-}26)$$

$$= J \text{ surface } \frac{\delta}{1 + i} \qquad (8\text{-}27)$$

I is Amperes per meter width

J surface is Amperes /meter^2

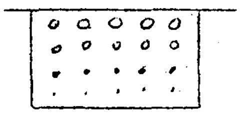
In magnitude

$$|I| = \left|J \text{ surface}\right| \frac{\delta}{\sqrt{2}}$$

$$P = \frac{I^2}{2\sigma\delta}$$

If we have so many amperes per square meter, how many watts loss?

Total Current I $\propto$ total field H



$\int \underline{H} \cdot d\underline{L}$ = I for 1 meter length

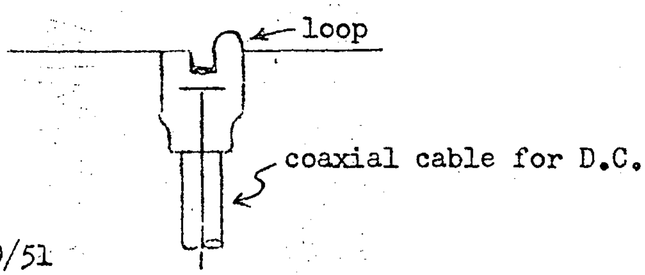
$\underline{H}$ = Total current enclosed

If we know the current we can compute $\underline{H}$ or vice versa ($\underline{H}$ may be measured with a loop)

$$\underline{V} = (A\mu_0\omega)\ \underline{H}$$
$$= (A\mu_0\omega)\ \underline{I} \qquad (8\text{-}28)$$

If I = 100 Amperes/inch, = 4000 Amperes / meter

(8-28) gives the voltage in a loop of Area A. Consider the loop and its connection to the instrument as a transmission line or put the voltage measuring instrument close to the loop. This can be done in effect by putting a diode rectifier at the loop, thus



With this, the crest voltage can be measured.

Other means are to terminate the transmission line at its characteristic impedance or use a vacuum tube volt meter.