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Authors
Ghoul, Tej-Eddine
Ibrahim, Slim
Lin, Quyuan
et al.

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ON THE EFFECT OF ROTATION ON THE LIFE-SPAN OF ANALYTIC SOLUTIONS TO THE $3D$ INVISCID PRIMITIVE EQUATIONS

TEJ EDDINE GHOUL, SLIM IBRAHIM, QUYUAN LIN, AND EDRISS S. TITI

ABSTRACT. In this paper, we study the effect of the rotation on the life-span of solutions to the $3D$ inviscid Primitive equations (PEs). The space of analytic functions appears to be the natural space to study the initial value problem for the inviscid PEs with general initial data, as they have been recently shown to exhibit Kelvin-Helmholtz type instability. First, for a short interval of time, that is independent of the rate of rotation $|\Omega|$, we establish the local well-posedness of the inviscid PEs in the space of analytic functions. In addition, thanks to a fine analysis of the barotropic and baroclinic modes decomposition, we establish two results about the long time existence of solutions. (i) Independently of $|\Omega|$, we show that the life-span of the solution tends to infinity as the analytic norm of the initial baroclinic mode goes to zero. Moreover, we show in this case that the solution of the $3D$ inviscid PEs converges to the solution of the limit system, which is governed by the $2D$ Euler equations. (ii) We show that the life-span of the solution goes toward infinity, with $|\Omega| \to \infty$, which is the main result of this paper. This is established for "well-prepared" initial data, namely, when only the Sobolev norm (but not the analytic norm) of the baroclinic mode is small enough, depending on $|\Omega|$. Furthermore, for large $|\Omega|$ and "well-prepared" initial data, we show that the solution to the $3D$ inviscid PEs is approximated by the solution to a simple limit resonant system with the same initial data.

MSC Subject Classifications: 35Q35, 35Q86, 86A10, 76E07.

Keywords: inviscid primitive equations; fast rotation; limit resonant system

1. Introduction

For large-scale oceanic and atmospheric dynamics the vertical scale (a few kilometers for the ocean, 10-20 kilometers for the atmosphere) is much smaller than the horizontal scales (several thousands of kilometers). The following $3D$ viscous primitive equations (PEs) has been a standard framework for studying geostrophic adjustment of frontal anomalies in a rotating continuously stratified fluid of strictly rectilinear fronts and jets (see, e.g., [10, 29, 30, 34, 36, 46, 53, 55] and references therein):

$$\partial_t v + v \cdot \nabla v + w \partial_z v - \nu_h \Delta v - \nu_z \partial_{zz} v + \Omega v^{\perp} + \nabla p = 0, \tag{1.1}$$

$$\partial_z p + T = 0, \tag{1.2}$$

$$\partial_t T + v \cdot \nabla T + w \partial_z T - \kappa_h \Delta T - \kappa_z \partial_{zz} T = 0, \tag{1.3}$$

$$\nabla \cdot v + \partial_z w = 0 \tag{1.4}$$

in the horizontal channel $\{(x_1, x_2, z) : 0 \leq z \leq H, (x_1, x_2) \in \mathbb{T}^2\}$, subject to the following initial and boundary conditions:

$$(v, T)|_{t=0} = (v_0, T_0), \tag{1.5}$$

$$(v_z, w, T_z)|_{z=0,H} = 0, \tag{1.6}$$

$$v, w, T \text{ are periodic in } (x_1, x_2) \text{ with period } 1. \tag{1.7}$$

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Here the horizontal velocity field $v = (v_1, v_2)$, the vertical velocity w, the temperature T, and the pressure p are the unknown quantities which are functions of the independent variables $(\boldsymbol{x}', z, t) = (x_1, x_2, z, t)$. The $2D$ horizontal gradient and Laplacian are denoted by $\nabla = (\partial_{x_1}, \partial_{x_2})$ and $\Delta = \partial_{x_1 x_1} + \partial_{x_2 x_2}$, respectively. The nonnegative constants ν_h, ν_z, κ_h and κ_z are the horizontal viscosity, the vertical viscosity, the horizontal diffusivity and the vertical diffusivity coefficients, respectively. The parameter $\Omega \in \mathbb{R}$ stands for the speed of rotation in the Coriolis force, and $v^\perp = (-v_2, v_1)$. The $3D$ viscous PEs is derived by performing a formal asymptotic limit of the small aspect ratio (the ratio of the depth or the height to the horizontal length scale) from the Rayleigh-Bénard (Boussinesq) system, and this limit is justified rigorously first by Azérad and Guillén [2] in a weak sense then later by Li and Titi [49] in a strong sense with error estimates.

The global existence of strong solutions to the $3D$ PEs with full viscosity and full diffusion was first established by Cao and Titi in [19], and later by Kobelkov in [39], see also the subsequent articles of Kukavica and Ziane [44, 45] for different boundary conditions, as well as Hieber and Kashiwabara [35] for some progress towards relaxing the smoothness on the initial data by using the semigroup method. This result has been improved later by Cao, Li and Titi [15, 16, 17], where the authors proved global well-posedness for $3D$ PEs with only horizontal viscosity, i.e., with $\nu_h > 0$ and $\nu_z = 0$. On the other hand, with only vertical viscosity, i.e., $\nu_h = 0$ and $\nu_z > 0$, Cao, Lin and Titi established recently [18] the local well-posedness of the PEs in Sobolev spaces by considering an additional weak dissipation, which is the linear (Rayleigh-like friction) damping. This linear damping helps the system overcome the ill-posedness in Sobolev spaces established in [54]. See also [20] for a similar idea on the effect of this linear damping.

When $\nu_h = \nu_z = 0$, the inviscid PEs without coupling with the temperature is also called the hydrostatic Euler equations. In the absence of rotation ($\Omega = 0$), the linear ill-posedness of the inviscid PEs, near certain shear-flows, has been established by Renardy in [54]. Later on, the nonlinear ill-posedness of the inviscid PEs without rotation was established by Han-Kwan and Nguyen in [33], where they built an abstract framework to show that the inviscid PEs are ill-posed in any Sobolev space. Moreover, it was proven that smooth solutions to the inviscid PEs, in the absence of rotation, can develop singularities in finite time (cf. Cao, Ibrahim, Nakanishi and Titi [14], and Wong [57]). It is shown in [37] that these results on the finite-time blowup and the ill-posedness can be extended to the $3D$ inviscid PEs with rotation, i.e., $\Omega \neq 0$. By virtue of the finite-time blowup results, one can conclude that there is no hope to show the global well-posedness of the $3D$ inviscid PEs, even with fast rotation. The optimal result one can expect is that fast rotation prolongs the life-span of solutions to the $3D$ inviscid PEs.

The linear ill-posedness results mentioned above show that the linearized $2D$ inviscid PEs (as well as the $3D$ case [37]), around a special steady state background flow, has unstable solutions of the form $u(t, x, z) = e^{2\pi ikx} e^{\sigma_k t} u_k(z)$, where $\Re\sigma_k = \lambda k$ for some $\lambda \in \mathbb{R}$ and $\lambda \neq 0$. Such Kelvin-Helmholtz type instability, which is similar to the one appears in the context of vortex sheets (see, e.g., [13], and the survey paper [8] and reference therein), precludes the construction of solutions in Sobolev spaces for general initial data. To overcome this strong instability, one should consider initial data u_0 that are strongly localized in Fourier, typically for which $|\hat{u}_0(k, z)| \lesssim e^{-\delta |k|^{1/s}}$ with $\delta > 0$ and $s \geq 1$. Such localization condition corresponds to Gevrey class of order s in the x variable. Kelvin-Helmholtz type instability forces us to choose $s = 1$ for the well-posedness result, which is the space of analytic functions. This is consistent with positive results reported in [43] and in this paper. Notably, for the Prandtl equations, which have some similarities in its structure with the PEs, is shown in [28] that its linearization around a special background flow has unstable solutions of similar form, but with $\Re\sigma_k \sim \lambda\sqrt{k}$ for $k \gg 1$ arbitrarily large and some positive $\lambda \in \mathbb{R}_+$. This implies that the optimal Gevrey class order s for Prandtl equation is $s = 2$, which is consistent with the positive results reported in [23, 48]. This shows that the linear instability of the inviscid PEs is "worse" than that of the Prandtl equations.

Due to the ill-posedness discussed above, in order to show the well-posedness of the inviscid PEs, one needs to assume either some special structures (local Rayleigh condition) on the initial data or real analyticity for general initial data [11, 12, 31, 42, 43, 51]. In particular, the authors in [43] established the local well-posedness of the $3D$ inviscid PEs in the space of analytic functions, but the time of existence they obtained shrinks to zero as the rate of rotation $|\Omega|$ increases toward infinity. This is contrary to the cases of the $3D$ fast rotating Euler, Navier–Stokes and Boussinesq equations, where the limit of fast rotation leads to either strong "dispersion" or averaging mechanism that weakens the nonlinear effects and hence allows for establishing the global regularity result in the Navier-Stokes case, and prolongs the life-span of the solutions in the Euler case, by Babin, Mahalov and Nicolaenko [4, 5, 6, 7] (see also [21, 24, 25, 38, 40] and references therein). In addition, we refer to [3, 32, 41, 50] for simple examples demonstrating the above mechanism. This suggests that one should be able to show that the fast rotation prolongs the life-span of the solution the $3D$ inviscid PEs.

For mathematical simplicity, we consider system (1.1)–(1.4) with $T_0 = 0$, which implies $T \equiv 0$ for smooth solutions. By considering the inviscid case, i.e., $\nu_h = \nu_z = 0$, in this paper, we are interested in the effect of rotation on the $3D$ inviscid PEs

$$\partial_t v + v \cdot \nabla v + w\partial_z v + \Omega v^{\perp} + \nabla p = 0, \tag{1.8}$$

$$\partial_z p = 0, \tag{1.9}$$

$$\nabla \cdot v + \partial_z w = 0, \tag{1.10}$$

in three-dimensional unit torus $\mathbb{T}^3$, subject to the following initial and boundary conditions:

$$v|_{t=0} = v_0, \tag{1.11}$$

$$v, w \text{ are periodic in } (\boldsymbol{x}', z) \text{ with period } 1, \tag{1.12}$$

$$v \text{ is even in } z \text{ and } w \text{ is odd in } z. \tag{1.13}$$

Observe that the space of periodic functions with respect to z with the symmetry condition (1.13) is invariant under the dynamics of system (1.8)–(1.10). If $H = \frac{1}{2}$, the solution to system (1.8)–(1.10) in $\mathbb{T}^3$ subject to (1.11)–(1.13) restricted on the horizontal channel $\left\{(\boldsymbol{x}', z) : 0 \leq z \leq \frac{1}{2}, \boldsymbol{x}' \in \mathbb{T}^2\right\}$ is the solution to system (1.8)–(1.10) subject to the physical boundary conditions, i.e., $w|_{z=0,\frac{1}{2}} = 0$ and v, w are periodic in $\boldsymbol{x}'$ with period 1, and initial condition v_0 being even extendable in z variable. Working in $\mathbb{T}^3$ allows us to use Fourier analysis, and makes the mathematical presentation simpler and more beautiful.

The paper is organized as follows. In section 2, we introduce the main notation and collect some preliminary results. In section 3, we establish the local well-posedness of the $3D$ inviscid PEs (1.8)–(1.10) subject to (1.11)–(1.13) in the space of analytic functions. The short time of existence of the solution we obtain is independent of Ω. This improves the result in [43], in which the short time of existence of the solution shrinks to zero as $|\Omega|$ increases toward infinity. This Ω dependence of the short time of existence in [43] arises from the pressure estimates that the authors obtain in terms of the Coriolis parameter. The key idea to avoid pressure estimates is to decompose the velocity field into the barotropic and baroclinic modes, and to apply the Leray projection to the barotropic mode evolution to eliminate the pressure. In section 4, independently of $|\Omega|$, we show that the life-span of the solution tends to infinity as the analytic norm of the initial baroclinic mode goes to zero. Moreover, we show in this case that the solution of the $3D$ inviscid PEs converges to the solution of the limit system, which is governed by the $2D$ Euler equations. The intuition comes from the observation that the $3D$ inviscid PEs is reduced to the $2D$ Euler equations when the baroclinic mode is zero initially. In section 5, we explore further the structure of the inviscid PEs with rotation and derive its formal limit resonant system when $|\Omega| \to \infty$. Let us emphasize that this limit resonant system is not solely the $2D$ Euler equations when the initial baroclinic mode is not zero. Moreover, we investigate this limit resonant system and establish its global regularity in both Sobolev and the analytic functions spaces. In section 6, we establish the main result of this paper, namely, the life-span of the solution to the $3D$ inviscid PEs goes toward infinity, with $|\Omega| \to \infty$. This is established for well-prepared initial data, namely, when only the Sobolev norm (but not the analytic norm) of the baroclinic mode is small enough, depending on $|\Omega|$. Furthermore, for large $|\Omega|$ and "well-prepared" initial data, we show that the solution to the $3D$ inviscid PEs is indeed approximated by the solution to the limit resonant system that is the main feature of section 5. In section 6, we also provide some rationals and discussions on why we need the smallness condition in the well-prepared initial data. The last section is an appendix, which is devoted to stating and proving technical lemmas concerning key nonlinear estimates.

2. Preliminaries

In this section, we introduce the notation and collect some preliminary results that will be used in this paper. The universal constant C appears in this paper may change from step to step. When we use subscript for C, e.g., C_r, it means that the constant depends only on r.

2.1. **Functional Settings.** We use the notation $\boldsymbol{x} := (\boldsymbol{x}', z) = (x_1, x_2, z) \in \mathbb{T}^3$, where $\boldsymbol{x}'$ and z represent the horizontal and vertical variables, respectively. $\mathbb{T}^3$ is the three dimensional torus with unit length. Denote by $L^2(\mathbb{T}^3)$, the Lebesgue space of real valued periodic functions $f(\boldsymbol{x})$ satisfying $\int_{\mathbb{T}^3} |f(\boldsymbol{x})|^2 d\boldsymbol{x} < \infty$, endowed with the norm

$$\|f\| := \|f\|_{L^2(\mathbb{T}^3)} = (\int_{\mathbb{T}^3} |f(\boldsymbol{x})|^2 d\boldsymbol{x})^{\frac{1}{2}}, \tag{2.1}$$

coming from the inner product

$$\langle f, g\rangle = \int_{\mathbb{T}^3} f(\boldsymbol{x}) g(\boldsymbol{x})\, d\boldsymbol{x} \tag{2.2}$$

for $f, g \in L^2(\mathbb{T}^3)$. Given time $\mathcal{T} > 0$, denote by $L^p(0, \mathcal{T}; X)$ the space of functions $f : [0, T] \to X$ satisfying $\int_0^{\mathcal{T}} \|f(t)\|_X^p dt < \infty$, where X is a Banach space and $\|\cdot\|_X$ represents its norm. Similarly, denote by $C([0, \mathcal{T}]; X)$ the space of continuous functions $f : [0, T] \to X$. For function $f \in L^2(\mathbb{T}^3)$, we use $\hat{f}_{\boldsymbol{k}}$ to denote its Fourier coefficient, so that

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^3} \hat{f}_{\boldsymbol{k}} e^{2\pi i \boldsymbol{k}\cdot\boldsymbol{x}}, \qquad \hat{f}_{\boldsymbol{k}} = \int_{\mathbb{T}^3} e^{-2\pi i \boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}) d\boldsymbol{x}. \tag{2.3}$$

For $r \geq 0$, define the following H^r norm

$$\|f\|_{H^r} := \Big(\sum_{\boldsymbol{k}\in\mathbb{Z}^3} (1 + |\boldsymbol{k}|^{2r}) |\hat{f}_{\boldsymbol{k}}|^2\Big)^{1/2}. \tag{2.4}$$

The Sobolev space $H^r(\mathbb{T}^3)$ is the set of all $L^2(\mathbb{T}^3)$ functions for which (2.4) is finite. We also denote the corresponding H^r semi-norm by

$$\|f\|_{\dot{H}^r} := \Big(\sum_{\boldsymbol{k}\in\mathbb{Z}^3} |\boldsymbol{k}|^{2r} |\hat{f}_{\boldsymbol{k}}|^2\Big)^{1/2}. \tag{2.5}$$

For more details about Sobolev spaces, see [1]. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ denote mult-indices. The notation

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_z^{\alpha_3}, \qquad |\alpha| = \sum_{j=1}^3 \alpha_j, \qquad \alpha! = \prod_{j=1}^3 \alpha_j \tag{2.6}$$

will be used throughout. For $s > 0$, a function $f \in C^\infty(\mathbb{T}^3)$ is said to be in Gevrey class of order s, denoted by $f \in G^s(\mathbb{T}^3)$, if there exist constants $\rho > 0$ and $M > 0$ such that for every $x \in \mathbb{T}^3$ and $\alpha \in \mathbb{N}^3$, one has

$$|\partial^\alpha f(\boldsymbol{x})| \leq M\Big(\frac{\alpha!}{\rho^{|\alpha|}}\Big)^s. \tag{2.7}$$

Denote by $A = \sqrt{-(\Delta + \partial_{zz})}$, subject to periodic boundary condition. For each $s > 0$ and $r \geq 0$, we define a family, parameterized by $\tau \geq 0$, of normed spaces

$$\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3)) := \{f \in H^r(\mathbb{T}^3) : \|e^{\tau A^{1/s}} f\|_{H^r} < \infty\}, \tag{2.8}$$

where the norm is defined by

$$\|e^{\tau A^{1/s}} f\|_{H^r} := \Big(\sum_{\boldsymbol{k} \in \mathbb{Z}^3} (1 + |\boldsymbol{k}|^{2r} e^{2\tau |\boldsymbol{k}|^{1/s}}) |\hat{f}_{\boldsymbol{k}}|^2 \Big)^{1/2}. \tag{2.9}$$

Let us denote the semi-norm by

$$\|A^r e^{\tau A^{1/s}} f\| := \Big(\sum_{\boldsymbol{k} \in \mathbb{Z}^3} |\boldsymbol{k}|^{2r} e^{2\tau |\boldsymbol{k}|^{1/s}} |\hat{f}_{\boldsymbol{k}}|^2 \Big)^{1/2}, \tag{2.10}$$

then it is easy to see that

$$\|e^{\tau A^{1/s}} f\|_{H^r}^2 = \|A^r e^{\tau A^{1/s}} f\|^2 + \|f\|^2. \tag{2.11}$$

For more details about Gevrey class, we refer the readers to [26, 27, 47]. The next lemma comes from [47], addressing the relation between Gevrey class $G^s(\mathbb{T}^3)$ and $\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3))$.

Lemma 2.1. *For any $s > 0$ and $r \geq 0$, we have*

$$G^s(\mathbb{T}^3) = \bigcup_{\tau > 0} \mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3)). \tag{2.12}$$

Althought our definition of the norm $\|e^{\tau A^{1/s}} f\|_{H^r}$ is slightly different from [47], the proof of this lemma is almost the same, and we refer the readers to [47]. The next lemma comes from [47] (see also [26]), addressing an important property of the space $\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3))$.

Lemma 2.2. *If $s \geq 1$, $\tau \geq 0$, and $r > 3/2$, then $\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3))$ is a Banach algebra, and for any $f, g \in \mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3))$, we have*

$$\|e^{\tau A^{1/s}} (fg)\|_{H^r} \leq C_{r,s} \|e^{\tau A^{1/s}} f\|_{H^r} \|e^{\tau A^{1/s}} g\|_{H^r}. \tag{2.13}$$

For the semi-norm, we also have similar estimate

$$\|A^r e^{\tau A^{1/s}} (fg)\| \leq C_{r,s} \Big(|\hat{f}_0| + \|A^r e^{\tau A^{1/s}} f\| \Big) \Big(|\hat{g}_0| + \|A^r e^{\tau A^{1/s}} g\| \Big). \tag{2.14}$$

For the proof, we refer the readers to [26] for the case when $s = 1$, and to [52] for the case when $s > 1$.

Remark 1. Since the inviscid PEs is linearly ill-posed in Sobolev spaces and Gevrey class of order $s > 1$ [37, 54], we focus on Gevrey class of order $s = 1$, which is equivlent to the space of analytic function.

2.2. **Projections and reformulation of the problem.** In this paper, we assume that $\int_{\mathbb{T}^3} v_0(\boldsymbol{x}) d\boldsymbol{x} = 0$. This assumption is made to simplify the mathematical presentation. See Remark 3 for detailed explanation. Integrating (1.8) in $\mathbb{T}^3$, by integration by parts, thanks to (1.10) and (1.12), we obtain

$$\partial_t \int_{\mathbb{T}^3} v d\boldsymbol{x} + \Omega \int_{\mathbb{T}^3} v^\perp d\boldsymbol{x} = 0. \tag{2.15}$$

Therefore, for any time $t \geq 0$, v has zero mean in $\mathbb{T}^3$:

$$\int_{\mathbb{T}^3} v d\boldsymbol{x} = \hat{v}_0 = 0. \tag{2.16}$$

Denote by

$$\dot{L}^2 := \Big\{\varphi \in L^2(\mathbb{T}^3, \mathbb{R}^2) : \int_{\mathbb{T}^3} \varphi(\boldsymbol{x}) d\boldsymbol{x} = 0\Big\}. \tag{2.17}$$

Denote the barotropic mode $\overline{v}$ and baroclinic mode $\widetilde{v}$ by

$$\overline{v}(\boldsymbol{x}') := \int_0^1 v(\boldsymbol{x}', z) dz = \sum_{\boldsymbol{k}\in\mathbb{Z}^3, k_3=0} \hat{v}_{\boldsymbol{k}} e^{2\pi i \boldsymbol{k}\cdot\boldsymbol{x}}, \qquad \widetilde{v}(\boldsymbol{x}) := v - \overline{v} = \sum_{\boldsymbol{k}\in\mathbb{Z}^3, k_3\neq 0} \hat{v}_{\boldsymbol{k}} e^{2\pi i \boldsymbol{k}\cdot\boldsymbol{x}}. \tag{2.18}$$

From boundary condition (1.13) and the incompressible condition (1.10), we know that

$$\nabla \cdot \overline{v} = \int_0^1 \nabla \cdot v(\boldsymbol{x}', z) dz = -\int_0^1 \partial_z w(\boldsymbol{x}', z) dz = 0. \tag{2.19}$$

Since $\nabla \cdot \overline{v} = 0$, and $\overline{v}$ has zero mean over $\mathbb{T}^2$ due to (2.16), we know there exists a stream function $\psi(\boldsymbol{x}')$ such that $\overline{v} = \nabla^{\perp}\psi = (-\partial_{x_2}\psi, \partial_{x_1}\psi)$. That is, $v \in \mathcal{S}$, where

$$\mathcal{S} := \Big\{\varphi \in \dot{L}^2 : \nabla \cdot \overline{\varphi} = 0\Big\} = \Big\{\varphi \in \dot{L}^2 : \varphi = \nabla^{\perp}\psi(\boldsymbol{x}') + \widetilde{\varphi}(\boldsymbol{x})\Big\}. \tag{2.20}$$

For $\varphi \in \dot{L}^2$, the rotating opeartor is

$$\mathcal{J}\varphi := \varphi^{\perp} = (-\varphi_2, \varphi_1). \tag{2.21}$$

Denote the $2D$ Leray projection by

$$\mathbb{P}_h \overline{\varphi} := \overline{\varphi} - \nabla\Delta^{-1}\nabla \cdot \overline{\varphi}. \tag{2.22}$$

Here, we denote by $\overline{\phi} = \Delta^{-1}\overline{\varphi}$ when $\Delta\overline{\phi} = \overline{\varphi}$ and $\int_{\mathbb{T}^2} \overline{\phi}(\boldsymbol{x}')d\boldsymbol{x}' = \int_{\mathbb{T}^2} \overline{\varphi}(\boldsymbol{x}')d\boldsymbol{x}' = 0$. Inspired by the $2D$ Leray projection, we define the projection $P_S : \dot{L}^2 \to \mathcal{S}$ as

$$P_S\varphi := \widetilde{\varphi} + \mathbb{P}_h\overline{\varphi}. \tag{2.23}$$

Then, we can define an operator $P : \mathcal{S} \to \mathcal{S}$ as

$$P\varphi := P_S(\mathcal{J}\varphi). \tag{2.24}$$

A direct computation using $\nabla \cdot \overline{\varphi} = 0$, we obtain

$$P\varphi = \widetilde{\varphi}^{\perp}. \tag{2.25}$$

It is easy to see that the kernel of P is

$$\ker P = \Big\{\varphi \in \mathcal{S} : \widetilde{\varphi}^{\perp} = 0\Big\} = \Big\{\varphi \in \mathcal{S} : \varphi = \overline{\varphi}\Big\}. \tag{2.26}$$

Therefore, we define the projection $P_0 : \mathcal{S} \to \ker P$ as

$$P_0\varphi := \overline{\varphi} = \int_0^1 \varphi(\boldsymbol{x}', z) dz, \tag{2.27}$$

which actually projects any vector $\varphi \in \mathcal{S}$ to its barotropic mode. Now applying P_S to equation (1.8), thanks to (1.9), and since $v \in \mathcal{S}$, we get

$$\partial_t v + P_S(v \cdot \nabla v + w\partial_z v) + \Omega\widetilde{v}^{\perp} = 0. \tag{2.28}$$

Next, applying P_0 and $I - P_0$ to equation (2.28), by integration by parts, thanks to (1.13) and (2.19), we derive the evolution equations for the barotropic mode $\overline{v}$ and the baroclinic mode $\widetilde{v}$:

$$\partial_t\overline{v} + \mathbb{P}_h\Big(\overline{v} \cdot \nabla\overline{v}\Big) + \mathbb{P}_h P_0\Big((\nabla \cdot \widetilde{v})\widetilde{v} + \widetilde{v} \cdot \nabla\widetilde{v}\Big) = 0, \tag{2.29}$$

$$\partial_t\widetilde{v} + \widetilde{v} \cdot \nabla\widetilde{v} + \widetilde{v} \cdot \nabla\overline{v} + \overline{v} \cdot \nabla\widetilde{v} - P_0\Big(\widetilde{v} \cdot \nabla\widetilde{v} + (\nabla \cdot \widetilde{v})\widetilde{v}\Big) - \Big(\int_0^z \nabla \cdot \widetilde{v}(\boldsymbol{x}', s) ds\Big)\partial_z\widetilde{v} + \Omega\widetilde{v}^{\perp} = 0. \tag{2.30}$$

In summary, we have the following lemma.

Lemma 2.3. *For $v \in \mathcal{S}$, system (1.8)–(1.10) is equivalent to system (2.29)–(2.30).*

Notice that if we consider $v_0 \in \ker P$, i.e., consider $\widetilde{v}_0 = 0$, then from (2.30) we can see $\widetilde{v}$ remains zero. Therefore, system (2.29)–(2.30) reduces to the $2D$ Euler equations, which is globally well-posed. Based on this observation, we establish the first long time existence result in section 4 by assuming the analytic norm of $\widetilde{v}_0$ is small. In order to investigate the effect of rotation, we further study the evolution of the baroclinic mode. This can be done by further decomposing the baroclinic mode in order to identify the resonant and non-resonant parts due to the rotation.

Since the rotation matrix $\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ corresponding to $\mathcal{J}\widetilde{v} = \widetilde{v}^\perp$ has eigenvalues $\pm i$, with corresponding eigenvectors $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp i \end{pmatrix}$, we can define

$$P_+\varphi := \Big\langle (I-P_0)\varphi, \overline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}} \Big\rangle_E \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}\Big\langle \widetilde{\varphi}, \overline{\begin{pmatrix} 1 \\ i \end{pmatrix}} \Big\rangle_E \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}(\widetilde{\varphi} + i\widetilde{\varphi}^\perp), \tag{2.31}$$

and

$$P_-\varphi := \Big\langle (I-P_0)\varphi, \overline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}} \Big\rangle_E \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}\Big\langle \widetilde{\varphi}, \overline{\begin{pmatrix} 1 \\ -i \end{pmatrix}} \Big\rangle_E \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}(\widetilde{\varphi} - i\widetilde{\varphi}^\perp). \tag{2.32}$$

Here the inner product $\langle \cdot, \cdot \rangle_E$ is the usual Euclidean inner product. Similar ideas and projections for $3D$ rotating Euler equations can be found in [24, 40]. In fact, the operator P has three eigenvalues, 0 and $\pm i$. These three projections P_0 and $P_\pm$ project v into the eigenspaces corresponding to 0 and $\mp i$. To be more specific, the following lemma addresses that we can use P_0 and $P_\pm$ to decompose any vector filed φ into three parts that are orthogonal to each other.

Lemma 2.4. *For any $\varphi \in L^2(\mathbb{T}^3)$, we have the following decomposition:*

$$\varphi = P_0\varphi + P_+\varphi + P_-\varphi. \tag{2.33}$$

Moreover, we have the following properties:

$$P_\pm P_\pm \varphi = P_\pm\varphi, \qquad P_0P_0\varphi = P_0\varphi, \qquad P_\pm P_\mp\varphi = P_0P_\pm\varphi = P_\pm P_0\varphi = 0. \tag{2.34}$$

Proof. The proof is straightforward from the definition of P_0 and $P_\pm$, and the fact that $\overline{\widetilde{\varphi}} = \widetilde{\overline{\varphi}} = 0$. □

For projections $P_0, P_\pm$, we have the following properties.

Lemma 2.5. *For $f, g \in L^2(\mathbb{T}^3)$, we have*

$$\langle P_0 f, g\rangle = \langle f, P_0 g\rangle = \langle P_0 f, P_0 g\rangle, \tag{2.35}$$

and

$$\langle P_\pm f, g\rangle = \langle f, P_\mp g\rangle. \tag{2.36}$$

Here the L^2 inner product is defined as (2.2). If $f \in H^r(\mathbb{T}^3)$ with $r \geq 0$, then for $|\alpha| \leq r$, we have

$$\partial^\alpha P_0 f = P_0\partial^\alpha f \quad \text{and} \quad \partial^\alpha P_\pm f = P_\pm \partial^\alpha f. \tag{2.37}$$

Moreover, if $f \in \mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3))$ with $s > 0$ and $r \geq 0$, we have

$$A^r e^{\tau A^{1/s}} P_0 f = P_0 A^r e^{\tau A^{1/s}} f. \tag{2.38}$$

Proof. For (2.35), we compute

$$\begin{aligned}\langle P_0 f, g\rangle &= \int_{\mathbb{T}^3}\Big(\int_0^1 f(\boldsymbol{x}', z)dz\Big)g(\boldsymbol{x}', z)d\boldsymbol{x}'dz\\ &= \int_{\mathbb{T}^2}\Big(\int_0^1 f(\boldsymbol{x}', z)dz\Big)\Big(\int_0^1 g(\boldsymbol{x}', z)dz\Big)d\boldsymbol{x}' = \langle P_0 f, P_0 g\rangle\\ &= \int_{\mathbb{T}^3} f(\boldsymbol{x}', z)\Big(\int_0^1 g(\boldsymbol{x}', z)dz\Big)d\boldsymbol{x}'dz = \langle f, P_0 g\rangle. \end{aligned}\tag{2.39}$$

For (2.36), one has

$$\begin{aligned}\langle P_\pm f, g\rangle &= \frac{1}{2}\int_{\mathbb{T}^3}(\widetilde{f} \pm i\widetilde{f}^\perp)(\boldsymbol{x})g(\boldsymbol{x})d\boldsymbol{x} = \frac{1}{2}\int_{\mathbb{T}^3}\Big((fg - \overline{f}g) \pm i(f^\perp g - \overline{f}^\perp g)\Big)(\boldsymbol{x})d\boldsymbol{x}\\ &= \frac{1}{2}\int_{\mathbb{T}^3}\Big((fg - f\overline{g}) \mp i(fg^\perp - f\overline{g}^\perp)\Big)(\boldsymbol{x})d\boldsymbol{x} = \frac{1}{2}\int_{\mathbb{T}^3} f(\boldsymbol{x})(\widetilde{g} \mp i\widetilde{g}^\perp)(\boldsymbol{x})d\boldsymbol{x} = \langle f, P_\mp g\rangle. \end{aligned}\tag{2.40}$$

For (2.37), if $\alpha_3 = 0$, we have

$$\begin{aligned}\partial^\alpha P_0 f &= \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\int_0^1 f(\boldsymbol{x}', z)dz = \int_0^1 \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} f(\boldsymbol{x}', z)dz = P_0\partial^\alpha f,\\ \partial^\alpha P_\pm f &= \frac{1}{2}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\Big[(f - P_0 f) \pm i(f - P_0 f)^\perp\Big]\\ &= \frac{1}{2}\Big[(\partial^\alpha f - P_0\partial^\alpha f) \pm i(\partial^\alpha f - P_0\partial^\alpha f)^\perp\Big] = P_\pm\partial^\alpha f. \end{aligned}\tag{2.41}$$

If $\alpha_3 > 0$, thanks to periodic boundary condition, we have

$$\begin{aligned}\partial^\alpha P_0 f &= \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_z^{\alpha_3}\int_0^1 f(\boldsymbol{x}', z)dz = 0 = \int_0^1 \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_z^{\alpha_3} f(\boldsymbol{x}', z)dz = P_0\partial^\alpha f,\\ \partial^\alpha P_\pm f &= \frac{1}{2}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_z^{\alpha_3}\Big[(f - P_0 f) \pm i(f - P_0 f)^\perp\Big] = \frac{1}{2}(\partial^\alpha f \pm i\partial^\alpha f^\perp)\\ &= \frac{1}{2}\Big[(\partial^\alpha f - P_0\partial^\alpha f) \pm i(\partial^\alpha f - P_0\partial^\alpha f)^\perp\Big] = P_\pm\partial^\alpha f. \end{aligned}\tag{2.42}$$

Therefore, for any $|\alpha| \le r$, (2.37) holds. The proof of (2.38) is straightforward, so we omit it. □

For Leray projection $\mathbb{P}_h$, we have the following properties.

Lemma 2.6. *For $f, g \in L^2(\mathbb{T}^3)$, we have*

$$\langle \mathbb{P}_h f, g\rangle = \langle f, \mathbb{P}_h g\rangle, \tag{2.43}$$

and

$$\mathbb{P}_h P_0 f = P_0\mathbb{P}_h f. \tag{2.44}$$

If $f \in H^r(\mathbb{T}^3)$ with $r \ge 0$, then for $|\alpha| \le r$, we have

$$\partial^\alpha\mathbb{P}_h f = \mathbb{P}_h\partial^\alpha f. \tag{2.45}$$

Moreover, if $f \in \mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3))$ with $s > 0$ and $r \ge 0$, we have

$$A^r e^{\tau A^{1/s}}\mathbb{P}_h f = \mathbb{P}_h A^r e^{\tau A^{1/s}} f. \tag{2.46}$$

Proof. For the proof of (2.43) and (2.45), see [22]. For (2.44), we compute

$$\mathbb{P}_h P_0 f = P_0 f - \nabla\Delta^{-1}\nabla\cdot(P_0 f) = P_0 f - P_0(\nabla\Delta^{-1}\nabla\cdot f) = P_0\mathbb{P}_h f. \tag{2.47}$$

For (2.46), one has

$$\begin{aligned} A^r e^{\tau A^{1/s}}\mathbb{P}_h f &= A^r e^{\tau A^{1/s}}\Big[\sum_{\boldsymbol{k}\neq\boldsymbol{0}}(\hat{f}_{\boldsymbol{k}} - \frac{\boldsymbol{k}\cdot\hat{f}_{\boldsymbol{k}}}{|\boldsymbol{k}|^2}\boldsymbol{k})e^{2\pi i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{f}_{\boldsymbol{0}}\Big] \\ &= \sum_{\boldsymbol{k}\neq\boldsymbol{0}}|\boldsymbol{k}|^r e^{\tau|\boldsymbol{k}|^{1/s}}(\hat{f}_{\boldsymbol{k}} - \frac{\boldsymbol{k}\cdot\hat{f}_{\boldsymbol{k}}}{|\boldsymbol{k}|^2}\boldsymbol{k})e^{2\pi i\boldsymbol{k}\cdot\boldsymbol{x}} = \mathbb{P}_h A^r e^{\tau A^{1/s}} f. \end{aligned} \tag{2.48}$$

□

For the relation between the norm of v and the norms of $\overline{v}, \widetilde{v}$ in $L^2(\mathbb{T}^3)$ and $\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3))$, we have the following Lemma.

Lemma 2.7. *Let $v = P_0 v + (I - P_0)v = \overline{v} + \widetilde{v}$. Suppose that $r \geq 0$, $s > 0$, and $\tau \geq 0$, we have*

$$\|v\|^2 = \|\overline{v}\|^2 + \|\widetilde{v}\|^2, \tag{2.49}$$

and

$$\|e^{\tau A^{1/s}}v\|_{H^r}^2 = \|e^{\tau A^{1/s}}\overline{v}\|_{H^r}^2 + \|e^{\tau A^{1/s}}\widetilde{v}\|_{H^r}^2. \tag{2.50}$$

Proof. Using Fourier representation of $v, \overline{v}, \widetilde{v}$, one has

$$v = \sum_{\boldsymbol{k}\in\mathbb{Z}^3}\hat{v}_{\boldsymbol{k}}e^{2\pi i\boldsymbol{k}\cdot\boldsymbol{x}}, \quad \overline{v} = \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3=0}}\hat{v}_{\boldsymbol{k}}e^{2\pi i\boldsymbol{k}\cdot\boldsymbol{x}}, \quad \widetilde{v} = \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq0}}\hat{v}_{\boldsymbol{k}}e^{2\pi i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{2.51}$$

Then we have

$$\|v\|^2 = \sum_{\boldsymbol{k}\in\mathbb{Z}^3}|\hat{v}_{\boldsymbol{k}}|^2 = \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3=0}}|\hat{v}_{\boldsymbol{k}}|^2 + \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq0}}|\hat{v}_{\boldsymbol{k}}|^2 = \|\overline{v}\|^2 + \|\widetilde{v}\|^2, \tag{2.52}$$

and

$$\begin{aligned} &\|e^{\tau A^{1/s}}v\|_{H^r}^2 = \sum_{\boldsymbol{k}\in\mathbb{Z}^3}(1 + |\boldsymbol{k}|^{2r}e^{2\tau|\boldsymbol{k}|^{1/s}})|\hat{v}_{\boldsymbol{k}}|^2 \\ &= \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3=0}}(1 + |\boldsymbol{k}|^{2r}e^{2\tau|\boldsymbol{k}|^{1/s}})|\hat{v}_{\boldsymbol{k}}|^2 + \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq0}}(1 + |\boldsymbol{k}|^{2r}e^{2\tau|\boldsymbol{k}|^{1/s}})|\hat{v}_{\boldsymbol{k}}|^2 = \|e^{\tau A^{1/s}}\overline{v}\|_{H^r}^2 + \|e^{\tau A^{1/s}}\widetilde{v}\|_{H^r}^2. \end{aligned} \tag{2.53}$$

□

Observe that we can write $\widetilde{v}^\perp$ in equation (2.30) as $\widetilde{v}^\perp = -i(P_+ v - P_- v)$. Hence applying $P_\pm$ to (2.30), we have

$$\begin{aligned} \partial_t P_\pm v + P_\pm\Big(&\widetilde{v}\cdot\nabla\widetilde{v} + \widetilde{v}\cdot\nabla\overline{v} + \overline{v}\cdot\nabla\widetilde{v} - P_0(\widetilde{v}\cdot\nabla\widetilde{v} + (\nabla\cdot\widetilde{v})\widetilde{v}) \\ &- (\int_0^z \nabla\cdot\widetilde{v}(\boldsymbol{x}', s)ds)\partial_z\widetilde{v}\Big) \mp i\Omega P_\pm v = 0. \end{aligned} \tag{2.54}$$

By setting $u_+ = e^{-i\Omega t}P_+ v,\ u_- = e^{i\Omega t}P_- v$, multiplying $e^{-i\Omega t}$ to the equation for $P_+ v$ and $e^{i\Omega t}$ to the equation for $P_- v$, we can rewrite (2.54) as

$$\begin{aligned}\partial_t u_\pm + e^{\mp i\Omega t}P_\pm\Big(\widetilde{v}\cdot\nabla\widetilde{v} + \widetilde{v}\cdot\nabla\overline{v} + \overline{v}\cdot\nabla\widetilde{v} - P_0(\widetilde{v}\cdot\nabla\widetilde{v} + (\nabla\cdot\widetilde{v})\widetilde{v})\\ - (\int_0^z \nabla\cdot\widetilde{v}(\boldsymbol{x}', s)ds)\partial_z\widetilde{v}\Big) = 0.\end{aligned} \tag{2.55}$$

For the u_+ part, thanks to Lemma 2.4 and (2.31), we have

$$\begin{aligned}P_+(\widetilde{v}\cdot\nabla\widetilde{v}) &= \frac{1}{2}(\widetilde{v}\cdot\nabla\widetilde{v} + i\widetilde{v}\cdot\nabla\widetilde{v}^\perp) - \frac{1}{2}P_0\Big(\widetilde{v}\cdot\nabla\widetilde{v} + i\widetilde{v}\cdot\nabla\widetilde{v}^\perp\Big)\\ &= \frac{1}{2}\widetilde{v}\cdot\nabla(\widetilde{v} + i\widetilde{v}^\perp) - \frac{1}{2}P_0\Big(\widetilde{v}\cdot\nabla(\widetilde{v} + i\widetilde{v}^\perp)\Big) = e^{i\Omega t}\Big(\widetilde{v}\cdot\nabla u_+ - P_0(\widetilde{v}\cdot\nabla u_+)\Big),\end{aligned} \tag{2.56}$$

$$P_+(\widetilde{v}\cdot\nabla\overline{v}) = \frac{1}{2}(\widetilde{v}\cdot\nabla\overline{v} + i\widetilde{v}\cdot\nabla\overline{v}^\perp) = \frac{1}{2}\widetilde{v}\cdot\nabla(\overline{v} + i\overline{v}^\perp), \tag{2.57}$$

$$P_+(\overline{v}\cdot\nabla\widetilde{v}) = \frac{1}{2}(\overline{v}\cdot\nabla\widetilde{v} + i\overline{v}\cdot\nabla\widetilde{v}^\perp) = e^{i\Omega t}(\overline{v}\cdot\nabla u_+), \tag{2.58}$$

$$P_+P_0\Big(\widetilde{v}\cdot\nabla\widetilde{v} + (\nabla\cdot\widetilde{v})\widetilde{v}\Big) = 0. \tag{2.59}$$

Observe that by integration by parts one has

$$\begin{aligned}P_+\Big((\int_0^z \nabla\cdot\widetilde{v}(\boldsymbol{x}', s)ds)\partial_z\widetilde{v}\Big) =& \frac{1}{2}\Big((\int_0^z \nabla\cdot\widetilde{v}(\boldsymbol{x}', s)ds)\partial_z\widetilde{v} + i(\int_0^z \nabla\cdot\widetilde{v}(\boldsymbol{x}', s)ds)\partial_z\widetilde{v}^\perp\Big)\\ &- \frac{1}{2}P_0\Big((\int_0^z \nabla\cdot\widetilde{v}(\boldsymbol{x}', s)ds)\partial_z\widetilde{v} + i(\int_0^z \nabla\cdot\widetilde{v}(\boldsymbol{x}', s)ds)\partial_z\widetilde{v}^\perp\Big)\\ =& e^{i\Omega t}(\int_0^z \nabla\cdot\widetilde{v}(\boldsymbol{x}', s)ds)\partial_z u_+ + e^{i\Omega t}P_0\Big((\nabla\cdot\widetilde{v})u_+\Big).\end{aligned} \tag{2.60}$$

Therefore, u_+ part in (2.55) becomes

$$\begin{aligned}\partial_t u_+ = &-\Big(\widetilde{v}\cdot\nabla u_+ + \overline{v}\cdot\nabla u_+ - P_0(\widetilde{v}\cdot\nabla u_+ + (\nabla\cdot\widetilde{v})u_+) - (\int_0^z \nabla\cdot\widetilde{v}(\boldsymbol{x}', s)ds)\partial_z u_+\Big)\\ &-\frac{1}{2}e^{-i\Omega t}(\widetilde{v}\cdot\nabla)(\overline{v} + i\overline{v}^\perp).\end{aligned} \tag{2.61}$$

Using $\widetilde{v} = u_+ e^{i\Omega t} + u_- e^{-i\Omega t}$, we can furthermore rewrite (2.61) as

$$\begin{aligned}\partial_t u_+ = &-e^{i\Omega t}\Big(u_+\cdot\nabla u_+ - P_0(u_+\cdot\nabla u_+ + (\nabla\cdot u_+)u_+) - (\int_0^z \nabla\cdot u_+(\boldsymbol{x}', s)ds)\partial_z u_+\Big)\\ &-\Big(\overline{v}\cdot\nabla u_+ + \frac{1}{2}(u_+\cdot\nabla)(\overline{v} + i\overline{v}^\perp)\Big) - e^{-2i\Omega t}\frac{1}{2}(u_-\cdot\nabla)(\overline{v} + i\overline{v}^\perp)\\ &-e^{-i\Omega t}\Big(u_-\cdot\nabla u_+ - P_0(u_-\cdot\nabla u_+ + (\nabla\cdot u_-)u_+) - (\int_0^z \nabla\cdot u_-(\boldsymbol{x}', s)ds)\partial_z u_+\Big).\end{aligned} \tag{2.62}$$

From (2.62), one can identify the resonant and non-resonant parts due to the rotation. Notice that u_- is the complex conjugate of u_+, therefore, by taking complex conjugate of (2.62), we obtain the evolution equation for u_- as:

$$\begin{aligned}\partial_t u_- = &-e^{-i\Omega t}\Big(u_-\cdot\nabla u_- - P_0(u_-\cdot\nabla u_- + (\nabla\cdot u_-)u_-) - (\int_0^z \nabla\cdot u_-(\boldsymbol{x}', s)ds)\partial_z u_-\Big)\\ &-\Big(\overline{v}\cdot\nabla u_- + \frac{1}{2}(u_-\cdot\nabla)(\overline{v} - i\overline{v}^\perp)\Big) - e^{2i\Omega t}\frac{1}{2}(u_+\cdot\nabla)(\overline{v} - i\overline{v}^\perp)\\ &-e^{i\Omega t}\Big(u_+\cdot\nabla u_- - P_0(u_+\cdot\nabla u_- + (\nabla\cdot u_+)u_-) - (\int_0^z \nabla\cdot u_+(\boldsymbol{x}', s)ds)\partial_z u_-\Big).\end{aligned} \tag{2.63}$$

Using $\widetilde{v} = u_+ e^{i\Omega t} + u_- e^{-i\Omega t}$, we can rewrite (2.29) as:

$$\partial_t \overline{v} + \mathbb{P}_h(\overline{v}\cdot\nabla\overline{v}) + e^{2i\Omega t}\mathbb{P}_h P_0\Big(u_+\cdot\nabla u_+ + (\nabla\cdot u_+)u_+\Big) + e^{-2i\Omega t}\mathbb{P}_h P_0\Big(u_-\cdot\nabla u_- + (\nabla\cdot u_-)u_-\Big)$$
$$+\mathbb{P}_h P_0\Big(u_+\cdot\nabla u_- + u_-\cdot\nabla u_+ + (\nabla\cdot u_+)u_- + (\nabla\cdot u_-)u_+\Big) = 0.$$

Since $u_\pm = e^{\mp i\Omega t}P_\pm v = \frac{1}{2}e^{\mp i\Omega t}(\widetilde{v} \pm i\widetilde{v}^\perp)$, and $\mathbb{P}_h$ commutes with P_0, the last term becomes

$$\begin{aligned}
&\mathbb{P}_h P_0\Big(u_+\cdot\nabla u_- + u_-\cdot\nabla u_+ + (\nabla\cdot u_+)u_- + (\nabla\cdot u_-)u_+\Big)\\
&= P_0\mathbb{P}_h\Big(u_+\cdot\nabla u_- + u_-\cdot\nabla u_+ + (\nabla\cdot u_+)u_- + (\nabla\cdot u_-)u_+\Big)\\
&= \frac{1}{2}P_0\mathbb{P}_h\Big(\widetilde{v}\cdot\nabla\widetilde{v} + \widetilde{v}^\perp\cdot\nabla\widetilde{v}^\perp + (\nabla\cdot\widetilde{v})\widetilde{v} + (\nabla\cdot\widetilde{v}^\perp)\widetilde{v}^\perp\Big) = \frac{1}{2}P_0\mathbb{P}_h(\nabla|\widetilde{v}|^2) = 0.
\end{aligned}$$

Therefore, one obtains

$$\begin{aligned}
\partial_t \overline{v} + \mathbb{P}_h(\overline{v}\cdot\nabla\overline{v}) + e^{2i\Omega t}\mathbb{P}_h P_0\Big(u_+\cdot\nabla u_+ + (\nabla\cdot u_+)u_+\Big)&\\
+e^{-2i\Omega t}\mathbb{P}_h P_0\Big(u_-\cdot\nabla u_- + (\nabla\cdot u_-)u_-\Big) &= 0.
\end{aligned} \tag{2.64}$$

In summary, we have the following lemma.

Lemma 2.8. *For $v\in\mathcal{S}$, system (1.8)–(1.10) is equivalent to system (2.62)–(2.64).*

For the relation between the norm of $\widetilde{v}$ and the norms of $u_\pm$ in $L^2(\mathbb{T}^3)$ and $\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3))$, we have the following Lemma.

Lemma 2.9. *Let $u_\pm = \frac{1}{2}e^{\mp i\Omega t}(\widetilde{v}\pm i\widetilde{v}^\perp)$. Suppose that $r\geq 0$, $s>0$, and $\tau\geq 0$, we have*

$$\|u_+\|^2 = \|u_-\|^2 = \frac{1}{2}\|\widetilde{v}\|^2, \tag{2.65}$$

and

$$\|e^{\tau A^{1/s}}u_+\|_{H^r}^2 = \|e^{\tau A^{1/s}}u_-\|_{H^r}^2 = \frac{1}{2}\|e^{\tau A^{1/s}}\widetilde{v}\|_{H^r}^2. \tag{2.66}$$

Proof. For (2.65), we have

$$\|u_+\|^2 = \|u_-\|^2 = \langle u_+, u_-\rangle = \frac{1}{4}\langle\widetilde{v}+i\widetilde{v}^\perp, \widetilde{v}-i\widetilde{v}^\perp\rangle = \frac{1}{2}\|\widetilde{v}\|^2. \tag{2.67}$$

For (2.66), notice that

$$\begin{aligned}
\|A^r e^{\tau A^{1/s}}u_+\|^2 &= \|A^r e^{\tau A^{1/s}}u_-\|^2 = \langle A^r e^{\tau A^{1/s}}u_+, A^r e^{\tau A^{1/s}}u_-\rangle\\
&= \frac{1}{4}\langle A^r e^{\tau A^{1/s}}(\widetilde{v}+i\widetilde{v}^\perp), A^r e^{\tau A^{1/s}}(\widetilde{v}-i\widetilde{v}^\perp)\rangle = \frac{1}{2}\|A^r e^{\tau A^{1/s}}\widetilde{v}\|^2.
\end{aligned} \tag{2.68}$$

Thanks to (2.11), we know (2.66) holds. □

In sections 3 and 4, we focus on system (2.29)–(2.30) since the results are independent of the rate of rotation. In section 5 and 6, our focus is on the effect of rotation, and we focus on system (2.62)–(2.64).

We also need the following Aubin-Lions theorem.

Lemma 2.10. *(Aubin-Lions Lemma, cf. Simon [56] Corollary 4) Assume that X, B and Y are three Banach spaces, with $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$. Then it holds that*

(i) If $\mathcal{F}$ is a bounded subset of $L^p(0,T;X)$, where $1 \leq p < \infty$, and $\mathcal{F}_t := \{\frac{\partial f}{\partial t} | f \in \mathcal{F}\}$ is bounded in $L^1(0,T;Y)$, then $\mathcal{F}$ is relative compact in $L^p(0,T;B)$.

(ii) If $\mathcal{F}$ is a bounded subset of $L^\infty(0,T;X)$ and $\mathcal{F}_t$ is bounded in $L^q(0,T;Y)$, where $q > 1$, then $\mathcal{F}$ is relative compact in $C([0,T];B)$.

3. Local Well-posedness

The local well-posedness of system (1.8)–(1.10) in the space of analytic functions is established in [43], The time of existence of solutions obtained in [43] shrinks to zero as $|\Omega|$ increases toward infinite in the $3D$ case. The reason behind this is that the pressure term was computed explicitly, which contains Ω. This makes the estimates dependent on Ω, and thus forces the time of existence of solutions shrink to zero as $|\Omega|$ increases toward infinite. In this section, we show that the time of existence of solutions is independent of Ω. The idea is that instead of system (1.8)–(1.10), we consider (2.29)–(2.30), where the pressure disappears due to Leray projection. To be precise, in this section, we study the local well-posedness in the space of analytic functions of the following system:

$$\partial_t \overline{v} + \mathbb{P}_h\Big(\overline{v} \cdot \nabla \overline{v}\Big) + \mathbb{P}_h P_0\Big((\nabla \cdot \widetilde{v})\widetilde{v} + \widetilde{v} \cdot \nabla \widetilde{v}\Big) = 0, \tag{3.1}$$

$$\partial_t \widetilde{v} + \widetilde{v} \cdot \nabla \widetilde{v} + \widetilde{v} \cdot \nabla \overline{v} + \overline{v} \cdot \nabla \widetilde{v} - P_0\Big(\widetilde{v} \cdot \nabla \widetilde{v} + (\nabla \cdot \widetilde{v})\widetilde{v}\Big) - \Big(\int_0^z \nabla \cdot \widetilde{v}(\boldsymbol{x}', s) ds\Big)\partial_z \widetilde{v} + \Omega \widetilde{v}^\perp = 0 \tag{3.2}$$

in $\mathbb{T}^3$, subject to the following symmetry boundary conditions and initial conditions:

$$\overline{v}, \widetilde{v} \text{ are periodic in } \mathbb{T}^3 \text{ and are even in } z; \tag{3.3}$$

$$\overline{v}|_{t=0} = \overline{v}_0 = P_0 v_0, \quad \widetilde{v}|_{t=0} = \widetilde{v}_0 = (I - P_0)v_0, \quad \nabla \cdot \overline{v}_0 = 0. \tag{3.4}$$

Observe that whenever $v \in \mathcal{S}$ then $\overline{v}, \widetilde{v} \in \mathcal{S}$.

We have the following theorem concerning the local well-posedness of system (3.1)–(3.4).

Theorem 3.1. *Assume $\overline{v}_0, \widetilde{v}_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau_0 A} : H^r(\mathbb{T}^3))$ with $r > 5/2$ and $\tau_0 > 0$. Let $\Omega \in \mathbb{R}$ be arbitrary and fixed. Then there exist a time*

$$\mathcal{T} = \frac{\tau_0}{1 + 2C_r(1 + \|e^{\tau_0 A}\overline{v}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{v}_0\|_{H^r}^2)} > 0, \tag{3.5}$$

and a function

$$\tau(t) = \tau_0 - 2tC_r(1 + \|e^{\tau_0 A}\overline{v}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{v}_0\|_{H^r}^2), \tag{3.6}$$

both independent of Ω, such that there exists a unique solution

$$(\overline{v}, \widetilde{v}) \in \mathcal{S} \cap L^\infty\big(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r(\mathbb{T}^3))\big) \cap L^2\big(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^{r+1/2}(\mathbb{T}^3))\big) \tag{3.7}$$

to system (3.1)–(3.4) on $[0, \mathcal{T}]$. Moreover, the unique solution $(\overline{v}, \widetilde{v})$ depends continuously on the initial data, in the sense of (3.71).

Thanks to Lemma 2.3 and 2.7, we have the following corollary for the original system (1.8)–(1.13).

Corollary 3.2. *Assume $v_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau_0 A} : H^r(\mathbb{T}^3))$ with $r > 5/2$ and $\tau_0 > 0$. Let $\Omega \in \mathbb{R}$ be arbitrary and fixed. Then there exist a time $\mathcal{T}$ defined in (3.5) and a function $\tau(t)$ defined in (3.6), both independent of Ω, such that there exists a unique solution*

$$v \in \mathcal{S} \cap L^\infty\big(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r(\mathbb{T}^3))\big) \cap L^2\big(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^{r+1/2}(\mathbb{T}^3))\big) \tag{3.8}$$

to system (1.8)–(1.13) on $[0,T]$. Moreover, the unique solution v depends continuously on the initial data.

For the proof of Theorem 3.1, we first work on Galerkin approximating system of (3.1)–(3.4), and provide energy estimates. Then, using Aubin-Lions compactness theorem, one can show the existence of solutions to system (3.1)–(3.4). Finally, we establish the uniqueness of solutions and its continuous dependence on the initial data.

3.1. **Galerkin approximating system.** In this section, we employ the standard Galerkin approximation procedure. For $\boldsymbol{k}\in\mathbb{Z}^3$, let

$$\phi_{\boldsymbol{k}} = \phi_{k_1,k_2,k_3} := \begin{cases} \sqrt{2}e^{2\pi i(k_1x_1+k_2x_2)}\cos(2\pi k_3 z) & \text{if } k_3\neq 0\\ e^{2\pi i(k_1x_1+k_2x_2)} & \text{if } k_3 = 0,\end{cases} \tag{3.9}$$

and

$$\mathcal{E} := \{\phi\in L^2(\mathbb{T}^3)\mid \phi = \sum_{\boldsymbol{k}\in\mathbb{Z}^3} a_{\boldsymbol{k}}\phi_{\boldsymbol{k}},\ a_{-k_1,-k_2,k_3} = a^*_{k_1,k_2,k_3},\ \sum_{\boldsymbol{k}\in\mathbb{Z}^3}|a_{\boldsymbol{k}}|^2<\infty\}. \tag{3.10}$$

Notice that $\mathcal{E}$ is a closed subspace of $L^2(\mathbb{T}^3)$, and consists of real valued functions which are even in z variable. For any $m\in\mathbb{N}$, denote by

$$\mathcal{E}_m := \{\phi\in L^2(\mathbb{T}^3)\mid \phi = \sum_{|\boldsymbol{k}|\leq m} a_{\boldsymbol{k}}\phi_{\boldsymbol{k}},\ a_{-k_1,-k_2,k_3} = a^*_{k_1,k_2,k_3}\}, \tag{3.11}$$

the finite-dimensional subspaces of $\mathcal{E}$. For any function $f\in L^2(\mathbb{T}^3)$, we write $\Pi_m f := \sum_{|\boldsymbol{k}|\leq m}\hat{f}_{\boldsymbol{k}}\phi_{\boldsymbol{k}}$. Then Π_m are orthogonal projections from $L^2(\mathbb{T}^3)$ to $\mathcal{E}_m$. Now let

$$\overline{v}_m = \sum_{0\neq|\boldsymbol{k}|\leq m, k_3=0} a_{\boldsymbol{k}}(t)\phi_{\boldsymbol{k}}(\boldsymbol{x}'), \qquad \widetilde{v}_m = \sum_{|\boldsymbol{k}|\leq m, k_3\neq 0} b_{\boldsymbol{k}}(t)\phi_{\boldsymbol{k}}(\boldsymbol{x}',z). \tag{3.12}$$

From this definition, we know that $\overline{v}_m = P_0(\overline{v}_m+\widetilde{v}_m)$ and $\widetilde{v}_m = (I-P_0)(\overline{v}_m+\widetilde{v}_m)$. Consider the following Galerkin approximation system for our model (3.1)–(3.2):

$$\partial_t\overline{v}_m + \Pi_m\mathbb{P}_h\Big(\overline{v}_m\cdot\nabla\overline{v}_m\Big) + \Pi_m\mathbb{P}_h P_0\Big((\nabla\cdot\widetilde{v}_m)\widetilde{v}_m + \widetilde{v}_m\cdot\nabla\widetilde{v}_m\Big) = 0, \tag{3.13}$$

$$\begin{aligned}\partial_t\widetilde{v}_m + \Pi_m\Big[&\widetilde{v}_m\cdot\nabla\widetilde{v}_m + \widetilde{v}_m\cdot\nabla\overline{v}_m + \overline{v}_m\cdot\nabla\widetilde{v}_m - P_0\Big(\widetilde{v}_m\cdot\nabla\widetilde{v}_m + (\nabla\cdot\widetilde{v}_m)\widetilde{v}_m\Big)\\ &- \Big(\int_0^z\nabla\cdot\widetilde{v}_m(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{v}_m\Big] + \Omega\widetilde{v}_m^{\perp} = 0,\end{aligned} \tag{3.14}$$

subject to the following initial conditions:

$$\overline{v}_m|_{t=0} = \Pi_m\overline{v}_0, \qquad \widetilde{v}_m|_{t=0} = \Pi_m\widetilde{v}_0. \tag{3.15}$$

For each $m\in\mathbb{N}$, the Galerkin approximation, system (3.13)–(3.15), corresponds to a first order system of ordinary differential equations, in the coefficients $a_{\boldsymbol{k}}$ and $b_{\boldsymbol{k}}$, for $0\leq|\boldsymbol{k}|\leq m$, with quadratic nonlinearity. Therefore, by the theory of ordinary differential equations, there exists some $t_m>0$ such that system (3.13)–(3.15) admits a unique solution $(\overline{v}_m,\widetilde{v}_m)$ on the interval $[0,t_m]$. Observe that from (3.15), we have $a_{\boldsymbol{k}}(0)$ and $b_{\boldsymbol{k}}(0)\in\mathbb{C}$ satisfying $a_{-k_1,-k_2,k_3}(0) = a^*_{k_1,k_2,k_3}(0)$ and $b_{-k_1,-k_2,k_3}(0) = b^*_{k_1,k_2,k_3}(0)$. Thanks to the uniqueness of the solutions of the ODE system, we conclude that $a_{-k_1,-k_2,k_3}(t) = a^*_{k_1,k_2,k_3}(t)$ and $b_{-k_1,-k_2,k_3}(t) = b^*_{k_1,k_2,k_3}(t)$, for $t\in[0,t_m]$. Therefore, $\overline{v}_m,\widetilde{v}_m\in\mathcal{E}_m$. Thanks to (3.15), we know that $\nabla\cdot\overline{v}_m(t=0)=0$. Applying $2D$ divergence on (3.13), we have $\partial_t(\nabla\cdot\overline{v}_m)=0$. Therefore, we know $\nabla\cdot\overline{v}_m = 0$. In next section, we provide the energy estimates for the Galerkin approximation system.

3.2. **Energy Estimates.** In this section, we establish the energy estimates for the Galerkin approximation system (3.13)–(3.15). By virtue of Lemma 2.5 and Lemma 2.6, and since $\nabla \cdot \overline{v}_m = 0$, we have

$$\frac{1}{2}\frac{d}{dt}(\|\overline{v}_m\|^2 + \|\widetilde{v}_m\|^2) = 0. \tag{3.16}$$

Integrating in time yields

$$\|\overline{v}_m(t)\|^2 + \|\widetilde{v}_m(t)\|^2 = \|\overline{v}_m(0)\|^2 + \|\widetilde{v}_m(0)\|^2 \le \|\overline{v}_0\|^2 + \|\widetilde{v}_0\|^2. \tag{3.17}$$

Therefore, (3.17) implies that the solution $\overline{v}_m$ and $\widetilde{v}_m$ exist global in time. Next, employing Lemma 2.5 and Lemma 2.6, we derive the estimate for the analytic norm, that is,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A^r e^{\tau A}\overline{v}_m\|^2 &= \dot{\tau}\|A^{r+1/2}e^{\tau A}\overline{v}_m\|^2 - \Big\langle A^r e^{\tau A}\Big(\overline{v}_m \cdot \nabla \overline{v}_m\Big), A^r e^{\tau A}\overline{v}_m\Big\rangle \\ &\quad -\Big\langle A^r e^{\tau A}\Big((\nabla\cdot\widetilde{v}_m)\widetilde{v}_m\Big), A^r e^{\tau A}\overline{v}_m\Big\rangle - \Big\langle A^r e^{\tau A}\Big(\widetilde{v}_m\cdot\nabla\widetilde{v}_m\Big), A^r e^{\tau A}\overline{v}_m\Big\rangle,\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A^r e^{\tau A}\widetilde{v}_m\|^2 &= \dot{\tau}\|A^{r+1/2}e^{\tau A}\widetilde{v}_m\|^2 - \Big\langle A^r e^{\tau A}\Big(\widetilde{v}_m \cdot \nabla \widetilde{v}_m\Big), A^r e^{\tau A}\widetilde{v}_m\Big\rangle \\ &\quad -\Big\langle A^r e^{\tau A}\Big(\widetilde{v}_m\cdot\nabla\overline{v}_m\Big), A^r e^{\tau A}\widetilde{v}_m\Big\rangle - \Big\langle A^r e^{\tau A}\Big(\overline{v}_m\cdot\nabla\widetilde{v}_m\Big), A^r e^{\tau A}\widetilde{v}_m\Big\rangle \\ &\quad +\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla\cdot\widetilde{v}_m(\boldsymbol{x}', s)ds)\partial_z\widetilde{v}_m\Big), A^r e^{\tau A}\widetilde{v}_m\Big\rangle.\end{aligned} \tag{3.19}$$

Add estimates (3.18)–(3.19) together, and add $\|A^{r+1/2}e^{\tau A}\overline{v}_m\|^2 + \|A^{r+1/2}e^{\tau A}\widetilde{v}_m\|^2$ to both sides. By Lemma A.1–A.3, since $\overline{v}_m$ and $\widetilde{v}_m$ have zero mean, thanks to Young's inequality we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|A^r e^{\tau A}\overline{v}_m\|^2 + \|A^r e^{\tau A}\widetilde{v}_m\|^2\Big) + \Big(\|A^{r+1/2}e^{\tau A}\overline{v}_m\|^2 + \|A^{r+1/2}e^{\tau A}\widetilde{v}_m\|^2\Big) \\ &\quad\le \Big(\dot{\tau} + C_r(\|A^r e^{\tau A}\overline{v}_m\| + \|A^r e^{\tau A}\widetilde{v}_m\|) + 1\Big)\Big(\|A^{r+1/2}e^{\tau A}\overline{v}_m\|^2 + \|A^{r+1/2}e^{\tau A}\widetilde{v}_m\|^2\Big) \\ &\quad\le \Big(\dot{\tau} + C_r(1 + \|e^{\tau A}\overline{v}_m\|^2_{H^r} + \|e^{\tau A}\widetilde{v}_m\|^2_{H^r})\Big)\Big(\|A^{r+1/2}e^{\tau A}\overline{v}_m\|^2 + \|A^{r+1/2}e^{\tau A}\widetilde{v}_m\|^2\Big).\end{aligned} \tag{3.20}$$

Remark 2. Here we add the term $\|A^{r+1/2}e^{\tau A}\overline{v}_m\|^2 + \|A^{r+1/2}e^{\tau A}\widetilde{v}_m\|^2$ to both sides so that one can obtain the regularity in $L^2\big(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^{r+1/2}(\mathbb{T}^3))\big)$.

Let τ satisfy

$$\dot{\tau} + 2C_r(1 + \|e^{\tau_0 A}\overline{v}_0\|^2_{H^r} + \|e^{\tau_0 A}\widetilde{v}_0\|^2_{H^r}) = 0, \tag{3.21}$$

for which we can solve out that

$$\tau(t) = \tau_0 - 2tC_r(1 + \|e^{\tau_0 A}\overline{v}_0\|^2_{H^r} + \|e^{\tau_0 A}\widetilde{v}_0\|^2_{H^r}). \tag{3.22}$$

Denote by

$$\mathcal{T} = \frac{\tau_0}{1 + 2C_r(1 + \|e^{\tau_0 A}\overline{v}_0\|^2_{H^r} + \|e^{\tau_0 A}\widetilde{v}_0\|^2_{H^r})} > 0, \tag{3.23}$$

we know that

$$\tau(t) \ge \tau(\mathcal{T}) = \frac{\tau_0}{1 + 2C_r(1 + \|e^{\tau_0 A}\overline{v}_0\|^2_{H^r} + \|e^{\tau_0 A}\widetilde{v}_0\|^2_{H^r})} > 0 \tag{3.24}$$

on $t \in [0, \mathcal{T}]$. Here we require C_r to be large enough such that

$$C_r \ge 2(\widetilde{C}_r + C_{r-\frac{1}{2}}), \tag{3.25}$$

where $\widetilde{C}_r$ appears in (3.67) and $C_{r-\frac{1}{2}}$ appears in (3.68). By the continuity of τ, $\|e^{\tau A}\overline{v}_m\|_{H^r}^2$, and $\|e^{\tau A}\widetilde{v}_m\|_{H^r}^2$, there exists a maximal time $\mathcal{T}_1 \in (0,\mathcal{T}]$ such that

$$\|e^{\tau(t)A}\overline{v}_m(t)\|_{H^r}^2 + \|e^{\tau(t)A}\widetilde{v}_m(t)\|_{H^r}^2 \leq 2(\|e^{\tau_0 A}\overline{v}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{v}_0\|_{H^r}^2) + 1 \tag{3.26}$$

on $t\in[0,\mathcal{T}_1]$. We claim that $\mathcal{T}_1 = \mathcal{T}$. On $[0,\mathcal{T}_1]$, from (3.26), we know

$$\dot{\tau} + C_r(1 + \|e^{\tau A}\overline{v}_m\|_{H^r}^2 + \|e^{\tau A}\widetilde{v}_m\|_{H^r}^2) \leq \dot{\tau} + 2C_r(1 + \|e^{\tau_0 A}\overline{v}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{v}_0\|_{H^r}^2) = 0. \tag{3.27}$$

From (3.20), on $t\in[0,\mathcal{T}_1]$, we have

$$\frac{1}{2}\frac{d}{dt}\Big(\|A^r e^{\tau A}\overline{v}_m\|^2 + \|A^r e^{\tau A}\widetilde{v}_m\|^2\Big) + \Big(\|A^{r+1/2}e^{\tau A}\overline{v}_m\|^2 + \|A^{r+1/2}e^{\tau A}\widetilde{v}_m\|^2\Big) \leq 0, \tag{3.28}$$

and thus

$$\|A^r e^{\tau(\mathcal{T}_1)A}\overline{v}_m(\mathcal{T}_1)\|^2 + \|A^r e^{\tau(\mathcal{T}_1)A}\widetilde{v}_m(\mathcal{T}_1)\|^2 \leq \|A^r e^{\tau_0 A}\overline{v}_0\|^2 + \|A^r e^{\tau_0 A}\widetilde{v}_0\|^2. \tag{3.29}$$

This together with (3.17) give us

$$\|e^{\tau(\mathcal{T}_1)A}\overline{v}_m(\mathcal{T}_1)\|_{H^r}^2 + \|e^{\tau(\mathcal{T}_1)A}\widetilde{v}_m(\mathcal{T}_1)\|_{H^r}^2 \leq \|e^{\tau_0 A}\overline{v}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{v}_0\|_{H^r}^2. \tag{3.30}$$

Therefore, if $\mathcal{T}_1 < \mathcal{T}$, then by continuity, there exists some $\mathcal{T}_2$ such that $\mathcal{T}_1 < \mathcal{T}_2 < \mathcal{T}$ and

$$\begin{aligned}\|e^{\tau(\mathcal{T}_2)A}\overline{v}_m(\mathcal{T}_2)\|_{H^r}^2 + \|e^{\tau(\mathcal{T}_2)A}\widetilde{v}_m(\mathcal{T}_2)\|_{H^r}^2 &\leq \|e^{\tau(\mathcal{T}_1)A}\overline{v}_m(\mathcal{T}_1)\|_{H^r}^2 + \|e^{\tau(\mathcal{T}_1)A}\widetilde{v}_m(\mathcal{T}_1)\|_{H^r}^2 + 1\\ &\leq 2(\|e^{\tau_0 A}\overline{v}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{v}_0\|_{H^r}^2) + 1,\end{aligned} \tag{3.31}$$

which contradicts to the maximum assumption on $\mathcal{T}_1$. Therefore, $\mathcal{T}_1 = \mathcal{T}$. Thus, (3.26)–(3.28) are satisfied on $[0,\mathcal{T}]$. Therefore, (3.27) holds on $[0,\mathcal{T}]$, and we obtain

$$\|e^{\tau(t)A}\overline{v}_m(t)\|_{H^r}^2 + \|e^{\tau(t)A}\widetilde{v}_m(t)\|_{H^r}^2 \leq \|e^{\tau_0 A}\overline{v}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{v}_0\|_{H^r}^2. \tag{3.32}$$

For arbitrary fixed $\mathcal{T}^* \in [0,\mathcal{T}]$, from (3.22), we know that $\min\limits_{t\in[0,\mathcal{T}^*]}\tau(t) = \tau(\mathcal{T}^*)$. For $t\in[0,\mathcal{T}^*]$, integrating (3.20) from 0 to t in time, we obtain

$$\begin{aligned}&\|A^r e^{\tau(\mathcal{T}^*)A}\overline{v}_m(t)\|^2 + \|A^r e^{\tau(\mathcal{T}^*)A}\widetilde{v}_m(t)\|^2\\ &\quad + 2\int_0^t \Big(\|A^{r+1/2}e^{\tau(\mathcal{T}^*)A}\overline{v}_m(s)\|^2 + \|A^{r+1/2}e^{\tau(\mathcal{T}^*)A}\widetilde{v}_m(s)\|^2\Big)ds\\ &\leq \|A^r e^{\tau(t)A}\overline{v}_m(t)\|^2 + \|A^r e^{\tau(t)A}\widetilde{v}_m(t)\|^2\\ &\quad + 2\int_0^t \Big(\|A^{r+1/2}e^{\tau(s)A}\overline{v}_m(s)\|^2 + \|A^{r+1/2}e^{\tau(s)A}\widetilde{v}_m(s)\|^2\Big)ds\\ &\leq \|A^r e^{\tau_0 A}\overline{v}_m(0)\|^2 + \|A^r e^{\tau_0 A}\widetilde{v}_m(0)\|^2 \leq \|e^{\tau_0 A}\overline{v}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{v}_0\|_{H^r}^2.\end{aligned} \tag{3.33}$$

The estimates (3.17) and (3.33) together imply that

$$\overline{v}_m, \widetilde{v}_m \text{ are uniformly bounded in } L^\infty\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(t)A}:H^r)\big)\cap L^2\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(t)A}:H^{r+1/2})\big), \tag{3.34}$$

and

$$\overline{v}_m, \widetilde{v}_m \text{ are uniformly bounded in } L^\infty\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(\mathcal{T}^*)A}:H^r)\big)\cap L^2\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(\mathcal{T}^*)A}:H^{r+1/2})\big). \tag{3.35}$$

By Banach–Alaoglu theorem, there exist a subsequence, denoted also by $\overline{v}_m, \widetilde{v}_m$, and corresponding limits, $\overline{v}, \widetilde{v}$, respectively, such that

$$\begin{aligned}&\overline{v}_m \to \overline{v},\ \ \widetilde{v}_m \to \widetilde{v} \text{ weakly* in } L^\infty\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(\mathcal{T}^*)A}:H^r)\big)\\ &\text{and weakly in } L^2\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(\mathcal{T}^*)A}:H^{r+1/2})\big).\end{aligned} \tag{3.36}$$

Moreover, $\overline{v}$ and $\widetilde{v}$ also satisfy the bound in (3.33). By virtue of $P_0\overline{v}_m = \overline{v}_m$ and $P_0\widetilde{v}_m = 0$ for any $m\in\mathbb{N}$, thanks to the convergence in (3.36), one has $P_0\overline{v} = \overline{v}$ and $P_0\widetilde{v} = 0$, which clarifies the notation.

In order to apply Aubin-Lions compactness theorem, we need some estimates on $\partial_t \overline{v}_m$ and $\partial_t \widetilde{v}_m$. By taking L^2 inner product of (3.13) and (3.14) with arbitrary $\phi \in L^2(\mathbb{T}^3)$, thanks to Lemma 2.5 and 2.6, we have

$$\Big\langle \partial_t \overline{v}_m, \phi \Big\rangle + \Big\langle \overline{v}_m \cdot \nabla \overline{v}_m, \mathbb{P}_h \Pi_m \phi \Big\rangle + \Big\langle (\nabla \cdot \widetilde{v}_m)\widetilde{v}_m + \widetilde{v}_m \cdot \nabla \widetilde{v}_m, P_0 \mathbb{P}_h \Pi_m \phi \Big\rangle = 0, \tag{3.37}$$

$$\begin{aligned} &\Big\langle \partial_t \widetilde{v}_m, \phi \Big\rangle + \Big\langle \widetilde{v}_m \cdot \nabla \widetilde{v}_m + \widetilde{v}_m \cdot \nabla \overline{v}_m + \overline{v}_m \cdot \nabla \widetilde{v}_m - P_0\Big(\widetilde{v}_m \cdot \nabla \widetilde{v}_m + (\nabla \cdot \widetilde{v}_m)\widetilde{v}_m\Big) \\ &\qquad - \Big(\int_0^z \nabla \cdot \widetilde{v}_m(\boldsymbol{x}', s)ds\Big)\partial_z \widetilde{v}_m + \Omega \widetilde{v}_m^{\perp}, \Pi_m \phi \Big\rangle = 0. \end{aligned} \tag{3.38}$$

By Hölder inequality and Sobolev inequality, thanks to $\|\Pi_m \phi\| \leq \|\phi\|$, $\|\mathbb{P}_h \phi\| \leq \|\phi\|$, and $\|P_0 \phi\| \leq \|\phi\|$ for any $\phi \in L^2(\mathbb{T}^3)$, since $r > 5/2$, we have

$$\Big|\langle \partial_t \overline{v}_m, \phi \rangle\Big| \leq C_r(\|\overline{v}_m\|_{H^r}^2 + \|\widetilde{v}_m\|_{H^r}^2)\|\phi\|, \tag{3.39}$$

$$\Big|\langle \partial_t \widetilde{v}_m, \phi \rangle\Big| \leq C_r(\|\overline{v}_m\|_{H^r}^2 + \|\widetilde{v}_m\|_{H^r}^2 + |\Omega|\|\widetilde{v}_m\|)\|\phi\|. \tag{3.40}$$

Next, applying $A^{r-1/2}e^{\tau(\mathcal{T}^*)A}$ to (3.13) and (3.14), and taking L^2 inner product of (3.13) and (3.14) with arbitrary $\phi \in L^2(\mathbb{T}^3)$, thanks to Lemma 2.5 and 2.6, we have

$$\begin{aligned} &\Big\langle A^{r-1/2}e^{\tau(\mathcal{T}^*)A}\partial_t \overline{v}_m, \phi \Big\rangle + \Big\langle A^{r-1/2}e^{\tau(\mathcal{T}^*)A}(\overline{v}_m \cdot \nabla \overline{v}_m), \mathbb{P}_h \Pi_m \phi \Big\rangle \\ &\qquad + \Big\langle A^{r-1/2}e^{\tau(\mathcal{T}^*)A}\Big((\nabla \cdot \widetilde{v}_m)\widetilde{v}_m + \widetilde{v}_m \cdot \nabla \widetilde{v}_m\Big), P_0 \mathbb{P}_h \Pi_m \phi \Big\rangle = 0, \end{aligned} \tag{3.41}$$

$$\begin{aligned} &\Big\langle A^{r-1/2}e^{\tau(\mathcal{T}^*)A}\partial_t \widetilde{v}_m, \phi \Big\rangle + \Big\langle A^{r-1/2}e^{\tau(\mathcal{T}^*)A}\Big[\widetilde{v}_m \cdot \nabla \widetilde{v}_m + \widetilde{v}_m \cdot \nabla \overline{v}_m + \overline{v}_m \cdot \nabla \widetilde{v}_m \\ &\quad - P_0\Big(\widetilde{v}_m \cdot \nabla \widetilde{v}_m + (\nabla \cdot \widetilde{v}_m)\widetilde{v}_m\Big) - \Big(\int_0^z \nabla \cdot \widetilde{v}_m(\boldsymbol{x}', s)ds\Big)\partial_z \widetilde{v}_m + \Omega \widetilde{v}_m^{\perp}\Big], \Pi_m \phi \Big\rangle = 0. \end{aligned} \tag{3.42}$$

By Cauchy–Schwarz inequality and Lemma 2.2, since $r > 5/2$, we have

$$\begin{aligned} &\Big|\Big\langle A^{r-1/2}e^{\tau(\mathcal{T}^*)A}\partial_t \overline{v}_m, \phi \Big\rangle\Big| \\ &\quad \leq C_r\Big(\|e^{\tau(\mathcal{T}^*)A}\overline{v}_m\|_{H^r}\|e^{\tau(\mathcal{T}^*)A}\overline{v}_m\|_{H^{r+1/2}} + \|e^{\tau(\mathcal{T}^*)A}\widetilde{v}_m\|_{H^r}\|e^{\tau(\mathcal{T}^*)A}\widetilde{v}_m\|_{H^{r+1/2}}\Big)\|\phi\|, \end{aligned} \tag{3.43}$$

$$\begin{aligned} &\Big|\Big\langle A^{r-1/2}e^{\tau(\mathcal{T}^*)A}\partial_t \widetilde{v}_m, \phi \Big\rangle\Big| \\ &\quad \leq C_r\Big(\|e^{\tau(\mathcal{T}^*)A}\overline{v}_m\|_{H^{r+1/2}}^2 + \|e^{\tau(\mathcal{T}^*)A}\widetilde{v}_m\|_{H^{r+1/2}}^2 + |\Omega|\|A^r e^{\tau(\mathcal{T}^*)A}\widetilde{v}_m\|\Big)\|\phi\|. \end{aligned} \tag{3.44}$$

By virtue of the bound (3.35), from (3.39)–(3.40) and (3.43)–(3.44), we have

$$\begin{aligned} &\partial_t \overline{v}_m \text{ are uniformly bounded in } L^2\big(0, \mathcal{T}^*; \mathcal{D}(e^{\tau(\mathcal{T}^*)A} : H^{r-1/2})\big) \cap L^\infty\big(0, \mathcal{T}^*; L^2\big), \\ &\partial_t \widetilde{v}_m \text{ are uniformly bounded in } L^1\big(0, \mathcal{T}^*; \mathcal{D}(e^{\tau(\mathcal{T}^*)A} : H^{r-1/2})\big) \cap L^\infty\big(0, \mathcal{T}^*; L^2\big). \end{aligned} \tag{3.45}$$

By Banach–Alaoglu theorem, we have

$$\begin{aligned} &\partial_t \overline{v}_m \to \partial_t \overline{v} \text{ weakly in } L^2\big(0, \mathcal{T}^*; \mathcal{D}(e^{\tau(\mathcal{T}^*)A} : H^{r-1/2})\big), \text{ weakly* in } L^\infty\big(0, \mathcal{T}^*; L^2\big), \\ &\partial_t \widetilde{v}_m \to \partial_t \widetilde{v} \text{ weakly* in } L^1\big(0, \mathcal{T}^*; \mathcal{D}(e^{\tau(\mathcal{T}^*)A} : H^{r-1/2})\big) \cap L^\infty\big(0, \mathcal{T}^*; L^2\big). \end{aligned} \tag{3.46}$$

From (3.35) and (3.45), since $\mathcal{D}(e^{\tau A} : H^{r_1}) \hookrightarrow\hookrightarrow \mathcal{D}(e^{\tau A} : H^{r_2})$ when $r_1 > r_2$, by proposition 2.10, for a subsequence and $0 < \epsilon < 1/2$, the following strong convergence holds:

$$\overline{v}_m \to \overline{v}, \;\; \widetilde{v}_m \to \widetilde{v} \text{ strongly in } C\big(0, \mathcal{T}^*; \mathcal{D}(e^{\tau(\mathcal{T}^*)A} : H^{r-\epsilon})\big) \cap L^2\big(0, \mathcal{T}^*; \mathcal{D}(e^{\tau(\mathcal{T}^*)A} : H^{r+1/2-\epsilon})\big). \tag{3.47}$$

3.3. **Existence of Solutions.** In this section, we establish the local in time existence of solutions to system (3.1)–(3.4). More specifically, we show the limit functions $\overline{v}$ and $\widetilde{v}$ we get from previous section satisfy (3.1)–(3.2) and (3.7). First, since $\nabla\cdot\overline{v}_m=0$ for any $m\in\mathbb{N}$ and thanks to the convergence (3.47), one has $\nabla\cdot\overline{v}=0$. By taking inner product of equation (3.13) and (3.14) with test function $\phi\in L^2(0,\mathcal{T}^*;L^2)$ in $L^2((0,\mathcal{T}^*)\times\mathbb{T}^3)$, we have

$$\Big\langle\partial_t\overline{v}_m+\Pi_m\mathbb{P}_h\Big(\overline{v}_m\cdot\nabla\overline{v}_m\Big)+\Pi_m\mathbb{P}_hP_0\Big((\nabla\cdot\widetilde{v}_m)\widetilde{v}_m+\widetilde{v}_m\cdot\nabla\widetilde{v}_m\Big),\phi\Big\rangle=0, \tag{3.48}$$

$$\begin{aligned}&\Big\langle\partial_t\widetilde{v}_m+\Pi_m\Big[\widetilde{v}_m\cdot\nabla\widetilde{v}_m+\widetilde{v}_m\cdot\nabla\overline{v}_m+\overline{v}_m\cdot\nabla\widetilde{v}_m-P_0\Big(\widetilde{v}_m\cdot\nabla\widetilde{v}_m+(\nabla\cdot\widetilde{v}_m)\widetilde{v}_m\Big)\\&\qquad-\Big(\int_0^z\nabla\cdot\widetilde{v}_m(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{v}_m\Big]+\Omega\widetilde{v}_m^{\perp},\phi\Big\rangle=0.\end{aligned}\tag{3.49}$$

From (3.36) and (3.46), we know that $\langle\Omega\widetilde{v}_m^{\perp},\phi\rangle\to\langle\Omega\widetilde{v}^{\perp},\phi\rangle$, $\langle\partial_t\overline{v}_m,\phi\rangle\to\langle\partial_t\overline{v},\phi\rangle$ and $\langle\partial_t\widetilde{v}_m,\phi\rangle\to\langle\partial_t\widetilde{v},\phi\rangle$.

For nonlinear terms, we consider, for example,

$$\begin{aligned}&\Big|\Big\langle\Pi_m\mathbb{P}_h\Big(\overline{v}_m\cdot\nabla\overline{v}_m\Big),\phi\Big\rangle-\Big\langle\mathbb{P}_h\Big(\overline{v}\cdot\nabla\overline{v}\Big),\phi\Big\rangle\Big|=\Big|\Big\langle\overline{v}_m\cdot\nabla\overline{v}_m,\Pi_m\mathbb{P}_h\phi\Big\rangle-\Big\langle\overline{v}\cdot\nabla\overline{v},\mathbb{P}_h\phi\Big\rangle\Big|\\&\le\Big|\Big\langle\overline{v}_m\cdot\nabla(\overline{v}_m-\overline{v}),\Pi_m\mathbb{P}_h\phi\Big\rangle\Big|+\Big|\Big\langle(\overline{v}_m-\overline{v})\cdot\nabla\overline{v},\Pi_m\mathbb{P}_h\phi\Big\rangle\Big|+\Big|\Big\langle\overline{v}\cdot\nabla\overline{v},(\Pi_m\mathbb{P}_h\phi-\mathbb{P}_h\phi)\Big\rangle\Big|\\&\le C_r\Big(\|\overline{v}_m\|_{L^\infty(0,\mathcal{T}^*;H^r)}+\|\overline{v}\|_{L^\infty(0,\mathcal{T}^*;H^r)}\Big)\|\overline{v}_m-\overline{v}\|_{L^2(0,\mathcal{T}^*;H^r)}\|\phi\|_{L^2(0,\mathcal{T}^*;L^2)}\\&\quad+C_r\|\overline{v}\|^2_{L^4(0,\mathcal{T}^*;H^r)}\|\Pi_m\phi-\phi\|_{L^2(0,\mathcal{T}^*;L^2)},\end{aligned}\tag{3.50}$$

and

$$\begin{aligned}&\Big|\Big\langle\Pi_m\Big(\int_0^z\nabla\cdot\widetilde{v}_m(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{v}_m,\phi\Big\rangle-\Big\langle\Big(\int_0^z\nabla\cdot\widetilde{v}(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{v},\phi\Big\rangle\Big|\\&\le\Big|\Big\langle\Big(\int_0^z\nabla\cdot\widetilde{v}_m(\boldsymbol{x}',s)ds\Big)\partial_z(\widetilde{v}_m-\widetilde{v}),\Pi_m\phi\Big\rangle\Big|+\Big|\Big\langle\Big(\int_0^z\nabla\cdot(\widetilde{v}_m-\widetilde{v})(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{v},\Pi_m\phi\Big\rangle\Big|\\&\quad+\Big|\Big\langle\Big(\int_0^z\nabla\cdot\widetilde{v}(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{v},(\Pi_m\phi-\phi)\Big\rangle\Big|\\&\le C_r\Big(\|\widetilde{v}_m\|_{L^\infty(0,\mathcal{T}^*;H^r)}+\|\widetilde{v}\|_{L^\infty(0,\mathcal{T}^*;H^r)}\Big)\|\widetilde{v}_m-\widetilde{v}\|_{L^2(0,\mathcal{T}^*;H^r)}\|\phi\|_{L^2(0,\mathcal{T}^*;L^2)}\\&\quad+C_r\|\widetilde{v}\|^2_{L^4(0,\mathcal{T}^*;H^r)}\|\Pi_m\phi-\phi\|_{L^2(0,\mathcal{T}^*;L^2)},\end{aligned}\tag{3.51}$$

where we have used Hölder inequality, Sobolev inequality, and $r>5/2$. By virtue of of (3.35), (3.36) and (3.47), the right hand side of (3.50) and (3.51) go to zero as $m\to\infty$.

One can show similarly that all other nonlinear terms converge to corresponding limit terms. Therefore, for arbitrary $\phi\in L^2(0,\mathcal{T}^*;L^2)$, we have

$$\Big\langle\partial_t\overline{v}+\mathbb{P}_h\Big(\overline{v}\cdot\nabla\overline{v}\Big)+\mathbb{P}_hP_0\Big((\nabla\cdot\widetilde{v})\widetilde{v}+\widetilde{v}\cdot\nabla\widetilde{v}\Big),\phi\Big\rangle=0, \tag{3.52}$$

$$\begin{aligned}&\Big\langle\partial_t\widetilde{v}+\widetilde{v}\cdot\nabla\widetilde{v}+\widetilde{v}\cdot\nabla\overline{v}+\overline{v}\cdot\nabla\widetilde{v}-P_0\Big(\widetilde{v}\cdot\nabla\widetilde{v}_m+(\nabla\cdot\widetilde{v})\widetilde{v}\Big)\\&\qquad-\Big(\int_0^z\nabla\cdot\widetilde{v}(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{v}+\Omega\widetilde{v}^{\perp},\phi\Big\rangle=0,\end{aligned}\tag{3.53}$$

and this implies that (3.1) and (3.2) hold in $L^2(0,\mathcal{T}^*;L^2)$. Moreover, from (3.46), we conclude that (3.1) holds in $L^2\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(\mathcal{T}^*)A}:H^{r-1/2})\big)\cap L^\infty\big(0,\mathcal{T}^*;L^2\big)$ and (3.2) holds in $L^1\big(0,\mathcal{T}^*;\mathcal{D}(e^{\tau(\mathcal{T}^*)A}:H^{r-1/2})\big)\cap L^\infty\big(0,\mathcal{T}^*;L^2\big)$. Next, due to (3.47), one has, for every $t\in[0,\mathcal{T}^*]$, $\overline{v}_m(t)\to\overline{v}(t)$, $\widetilde{v}_m(t)\to\widetilde{v}(t)$ in L^2. In particular, $\overline{v}_m(0)\to\overline{v}(0)$, $\widetilde{v}_m(0)\to\widetilde{v}(0)$ in L^2. On the other hand, by (3.15), we have $\overline{v}_m(0)\to\overline{v}_0$, $\widetilde{v}_m(0)\to\widetilde{v}_0$ in L^2. As a result, $\overline{v},\widetilde{v}$ satisfy the desired initial condition: $\overline{v}(0)=\overline{v}_0$ and $\widetilde{v}(0)=\widetilde{v}_0$. Recall that the choice of $\mathcal{T}^* \in [0,\mathcal{T}]$ is arbitrary, and in particular we can choose $\mathcal{T}^* = \mathcal{T}$ so that all of the results above hold with $\mathcal{T}^*$ replaced by $\mathcal{T}$.

Finally, we need to show (3.7). First, it is obvious that $(\overline{v},\widetilde{v}) \in \mathcal{S}$. Recall that for arbitrary $\mathcal{T}^* \in [0,\mathcal{T}]$, by (3.33) and the convergence (3.36), we have

$$\|A^r e^{\tau(\mathcal{T}^*)A}\overline{v}(t)\|^2 + \|A^r e^{\tau(\mathcal{T}^*)A}\widetilde{v}(t)\|^2 \leq \|A^r e^{\tau_0 A}\overline{v}_0\|^2 + \|A^r e^{\tau_0 A}\widetilde{v}_0\|^2, \tag{3.54}$$

for $t \in [0,\mathcal{T}^*]$, and especailly for $t = \mathcal{T}^*$. Therefore,

$$\|A^r e^{\tau(\mathcal{T}^*)A}\overline{v}(\mathcal{T}^*)\|^2 + \|A^r e^{\tau(\mathcal{T}^*)A}\widetilde{v}(\mathcal{T}^*)\|^2 \leq \|A^r e^{\tau_0 A}\overline{v}_0\|^2 + \|A^r e^{\tau_0 A}\widetilde{v}_0\|^2, \tag{3.55}$$

for any $\mathcal{T}^* \in [0,\mathcal{T}]$. Since the L^2 energy is conversed, we have

$$\|e^{\tau(\mathcal{T}^*)A}\overline{v}(\mathcal{T}^*)\|_{H^r}^2 + \|e^{\tau(\mathcal{T}^*)A}\widetilde{v}(\mathcal{T}^*)\|_{H^r}^2 \leq \|e^{\tau_0 A}\overline{v}_0\|_{H^r}^2 + \|e^{\tau_0 A}\widetilde{v}_0\|_{H^r}^2. \tag{3.56}$$

Therefore, the solution $(\overline{v},\widetilde{v}) \in L^\infty\big(0,\mathcal{T};\mathcal{D}(e^{\tau(t)A} : H^r)\big)$ for $\tau = \tau(t)$ defined in (3.22).

In order to show $(\overline{v},\widetilde{v}) \in L^2\big(0,\mathcal{T};\mathcal{D}(e^{\tau(t)A} : H^{r+1/2})\big)$, define the inner product:

$$\langle f,g\rangle_H := \sum_{\boldsymbol{k}\in\mathbb{Z}^3}\int_0^{\mathcal{T}} (1+|\boldsymbol{k}|^{2r+1}e^{2\tau(t)|\boldsymbol{k}|})(\hat{f}_{\boldsymbol{k}}\cdot\hat{g}_{\boldsymbol{k}})dt. \tag{3.57}$$

$L^2\big(0,\mathcal{T};\mathcal{D}(e^{\tau(t)A} : H^{r+1/2})\big)$ with inner product defined by (3.57) is a Hilbert space. By setting $\mathcal{T}^* = \mathcal{T}$ in (3.34), we know $\{\overline{v}_m\}$ and $\{\widetilde{v}_m\}$ are bounded sequence in this Hilbert space, and therefore there exist weak limit $\overline{v}^*$ and $\widetilde{v}^*$ such that

$$\int_0^{\mathcal{T}} \|e^{\tau(t)A}\overline{v}^*(t)\|_{H^{r+1/2}}^2 dt \leq \liminf_{m\to\infty}\int_0^{\mathcal{T}} \|e^{\tau(t)A}\overline{v}_m(t)\|_{H^{r+1/2}}^2 dt < \infty, \tag{3.58}$$

$$\int_0^{\mathcal{T}} \|e^{\tau(t)A}\widetilde{v}^*(t)\|_{H^{r+1/2}}^2 dt \leq \liminf_{m\to\infty}\int_0^{\mathcal{T}} \|e^{\tau(t)A}\widetilde{v}_m(t)\|_{H^{r+1/2}}^2 dt < \infty. \tag{3.59}$$

Thus, $(\overline{v}^*,\widetilde{v}^*) \in L^2\big(0,\mathcal{T};\mathcal{D}(e^{\tau(t)A} : H^{r+1/2})\big)$. By uniqueness of weak limit, we know $\overline{v} = \overline{v}^*, \widetilde{v} = \widetilde{v}^*$, thus, $(\overline{v},\widetilde{v}) \in L^2\big(0,\mathcal{T};\mathcal{D}(e^{\tau(t)A} : H^{r+1/2})\big)$. Therefore, (3.7) holds. The existence of solutions to system (3.1)–(3.4) is proved.

3.4. **Uniqueness of Solutions and Continuous Dependence on the Initial Data.** In this section, we show the uniqueness of solutions and the continuous dependence on the initial data. Let $(\overline{v}_1,\widetilde{v}_1)$ and $(\overline{v}_2,\widetilde{v}_2)$ be two strong solutions to system (3.1)–(3.4) with initial data $((\overline{v}_0)_1,(\widetilde{v}_0)_1)$ and $((\overline{v}_0)_2,(\widetilde{v}_0)_2)$, respectively. Assume the radius of analyticity for initial data $((\overline{v}_0)_1,(\widetilde{v}_0)_1)$ is τ_{10}, and for $((\overline{v}_0)_2,(\widetilde{v}_0)_2)$ is τ_{20}. Let $\tau_0 = \min\{\tau_{10},\tau_{20}\}$, and

$$M = \max\Big\{\|e^{\tau_{10}A}(\overline{v}_0)_1\|_{H^r}^2 + \|e^{\tau_{10}A}(\widetilde{v}_0)_1\|_{H^r}^2, \|e^{\tau_{20}A}(\overline{v}_0)_2\|_{H^r}^2 + \|e^{\tau_{20}A}(\widetilde{v}_0)_2\|_{H^r}^2\Big\}. \tag{3.60}$$

Denote by $\overline{v} = \overline{v}_1 - \overline{v}_2$ and $\widetilde{v} = \widetilde{v}_1 - \widetilde{v}_2$. By virtue of (3.22) and (3.23), we define

$$\widetilde{\tau}(t) = \tau_0 - 2tC_r(1+M), \quad \widetilde{\mathcal{T}} = \frac{\tau_0}{1+2C_r(1+M)}. \tag{3.61}$$

Here C_r satisfies (3.25).

From previous sections, and by the definition of τ_0 and M, we know

$$(\overline{v}_i,\widetilde{v}_i),(\overline{v},\widetilde{v}) \in L^\infty\big(0,\widetilde{\mathcal{T}};\mathcal{D}(e^{\widetilde{\tau}(t)A} : H^r)\big) \cap L^2\big(0,\widetilde{\mathcal{T}};\mathcal{D}(e^{\widetilde{\tau}(t)A} : H^{r+1/2})\big), \tag{3.62}$$

and

$$\|e^{\widetilde{\tau}A}\overline{v}_i\|_{H^r}^2 + \|e^{\widetilde{\tau}A}\widetilde{v}_i\|_{H^r}^2 \leq M \tag{3.63}$$

for $i=1,2$. From (3.1)–(3.2), it is clear that

$$\partial_t\overline{v}+\mathbb{P}_h\Big(\overline{v}\cdot\nabla\overline{v}_1+\overline{v}_2\cdot\nabla\overline{v}\Big)+\mathbb{P}_hP_0\Big((\nabla\cdot\widetilde{v})\widetilde{v}_1+(\nabla\cdot\widetilde{v}_2)\widetilde{v}+\widetilde{v}\cdot\nabla\widetilde{v}_1+\widetilde{v}_2\cdot\nabla\widetilde{v}\Big)=0, \tag{3.64}$$

$$\begin{aligned}&\partial_t\widetilde{v}+\widetilde{v}\cdot\nabla\widetilde{v}_1+\widetilde{v}_2\cdot\nabla\widetilde{v}+\widetilde{v}\cdot\nabla\overline{v}_1+\widetilde{v}_2\cdot\nabla\overline{v}+\overline{v}\cdot\nabla\widetilde{v}_1+\overline{v}_2\cdot\nabla\widetilde{v}-P_0\Big((\nabla\cdot\widetilde{v})\widetilde{v}_1+(\nabla\cdot\widetilde{v}_2)\widetilde{v}\\&\qquad+\widetilde{v}\cdot\nabla\widetilde{v}_1+\widetilde{v}_2\cdot\nabla\widetilde{v}\Big)-\Big(\int_0^z\nabla\cdot\widetilde{v}(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{v}_1-\Big(\int_0^z\nabla\cdot\widetilde{v}_2(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{v}+\Omega\widetilde{v}^{\perp}=0.\end{aligned} \tag{3.65}$$

Taking L^2 inner product of (3.64) with $\overline{v}$ and (3.65) with $\widetilde{v}$, applying $A^{r-1/2}e^{\widetilde{\tau}A}$ to (3.64) and (3.65) and taking L^2 inner product with $A^{r-1/2}e^{\widetilde{\tau}A}\overline{v}$ and $A^{r-1/2}e^{\widetilde{\tau}A}\widetilde{v}$, correspondingly, thanks to Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|e^{\widetilde{\tau}(t)A}\overline{v}(t)\|^2_{H^{r-1/2}}+\|e^{\widetilde{\tau}(t)A}\widetilde{v}(t)\|^2_{H^{r-1/2}}\Big)-\dot{\widetilde{\tau}}\Big(\|A^re^{\widetilde{\tau}A}\overline{v}\|^2+\|A^re^{\widetilde{\tau}A}\widetilde{v}\|^2\Big)\\&+\Big\langle\overline{v}\cdot\nabla\overline{v}_1+\overline{v}_2\cdot\nabla\overline{v}+(\nabla\cdot\widetilde{v})\widetilde{v}_1+(\nabla\cdot\widetilde{v}_2)\widetilde{v}+\widetilde{v}\cdot\nabla\widetilde{v}_1+\widetilde{v}_2\cdot\nabla\widetilde{v},\overline{v}\Big\rangle\\&+\Big\langle A^{r-1/2}e^{\widetilde{\tau}A}\Big(\overline{v}\cdot\nabla\overline{v}_1+\overline{v}_2\cdot\nabla\overline{v}+(\nabla\cdot\widetilde{v})\widetilde{v}_1+(\nabla\cdot\widetilde{v}_2)\widetilde{v}+\widetilde{v}\cdot\nabla\widetilde{v}_1+\widetilde{v}_2\cdot\nabla\widetilde{v}\Big),A^{r-1/2}e^{\widetilde{\tau}A}\overline{v}\Big\rangle\\&+\Big\langle\widetilde{v}\cdot\nabla\widetilde{v}_1+\widetilde{v}_2\cdot\nabla\widetilde{v}+\widetilde{v}\cdot\nabla\overline{v}_1+\widetilde{v}_2\cdot\nabla\overline{v}+\overline{v}\cdot\nabla\widetilde{v}_1+\overline{v}_2\cdot\nabla\widetilde{v}\\&\qquad-\Big(\int_0^z\nabla\cdot\widetilde{v}(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{v}_1-\Big(\int_0^z\nabla\cdot\widetilde{v}_2(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{v},\widetilde{v}\Big\rangle\\&+\Big\langle A^{r-1/2}e^{\widetilde{\tau}A}\Big[\widetilde{v}\cdot\nabla\widetilde{v}_1+\widetilde{v}_2\cdot\nabla\widetilde{v}+\widetilde{v}\cdot\nabla\overline{v}_1+\widetilde{v}_2\cdot\nabla\overline{v}+\overline{v}\cdot\nabla\widetilde{v}_1+\overline{v}_2\cdot\nabla\widetilde{v}\\&\qquad-\Big(\int_0^z\nabla\cdot\widetilde{v}(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{v}_1-\Big(\int_0^z\nabla\cdot\widetilde{v}_2(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{v}\Big],A^{r-1/2}e^{\widetilde{\tau}A}\widetilde{v}\Big\rangle=0.\end{aligned} \tag{3.66}$$

Thanks to Hölder inequality, Young's inequality and Sobolev inequality, since $r>5/2$, and noticing that $\overline{v}$ and $\widetilde{v}$ have zero mean over $\mathbb{T}^3$, we can apply Poincaré inequality to have

$$\begin{aligned}&\Big|\Big\langle\overline{v}\cdot\nabla\overline{v}_1+\overline{v}_2\cdot\nabla\overline{v}+(\nabla\cdot\widetilde{v})\widetilde{v}_1+(\nabla\cdot\widetilde{v}_2)\widetilde{v}+\widetilde{v}\cdot\nabla\widetilde{v}_1+\widetilde{v}_2\cdot\nabla\widetilde{v},\overline{v}\Big\rangle\\&\quad+\Big\langle\widetilde{v}\cdot\nabla\widetilde{v}_1+\widetilde{v}_2\cdot\nabla\widetilde{v}+\widetilde{v}\cdot\nabla\overline{v}_1+\widetilde{v}_2\cdot\nabla\overline{v}+\overline{v}\cdot\nabla\widetilde{v}_1+\overline{v}_2\cdot\nabla\widetilde{v}\\&\qquad-\Big(\int_0^z\nabla\cdot\widetilde{v}(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{v}_1-\Big(\int_0^z\nabla\cdot\widetilde{v}_2(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{v},\widetilde{v}\Big\rangle\Big|\\&\le\widetilde{C}_r\Big(\|\overline{v}_1\|_{H^r}+\|\overline{v}_2\|_{H^r}+\|\widetilde{v}_1\|_{H^r}+\|\widetilde{v}_2\|_{H^r}\Big)\Big(\|\overline{v}\|^2_{H^{r-1/2}}+\|\widetilde{v}\|^2_{H^{r-1/2}}\Big)\\&\le\widetilde{C}_r\Big(\|\overline{v}_1\|_{H^r}+\|\overline{v}_2\|_{H^r}+\|\widetilde{v}_1\|_{H^r}+\|\widetilde{v}_2\|_{H^r}\Big)\Big(\|A^re^{\widetilde{\tau}A}\overline{v}\|^2+\|A^re^{\widetilde{\tau}A}\widetilde{v}\|^2\Big),\end{aligned} \tag{3.67}$$

where the last step we apply Poincaré inequality. For higher order part, thanks to Lemma A.1–A.3, by Young's inequality, we have

$$\begin{aligned}&\Big|\Big\langle A^{r-1/2}e^{\widetilde{\tau}A}\Big(\overline{v}\cdot\nabla\overline{v}_1+\overline{v}_2\cdot\nabla\overline{v}+(\nabla\cdot\widetilde{v})\widetilde{v}_1+(\nabla\cdot\widetilde{v}_2)\widetilde{v}+\widetilde{v}\cdot\nabla\widetilde{v}_1+\widetilde{v}_2\cdot\nabla\widetilde{v}\Big),A^{r-1/2}e^{\widetilde{\tau}A}\overline{v}\Big\rangle\\&+\Big\langle A^{r-1/2}e^{\widetilde{\tau}A}\Big[\widetilde{v}\cdot\nabla\widetilde{v}_1+\widetilde{v}_2\cdot\nabla\widetilde{v}+\widetilde{v}\cdot\nabla\overline{v}_1+\widetilde{v}_2\cdot\nabla\overline{v}+\overline{v}\cdot\nabla\widetilde{v}_1+\overline{v}_2\cdot\nabla\widetilde{v}\\&\qquad-\Big(\int_0^z\nabla\cdot\widetilde{v}(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{v}_1-\Big(\int_0^z\nabla\cdot\widetilde{v}_2(\boldsymbol{x}',s)ds\Big)\partial_z\widetilde{v}\Big],A^{r-1/2}e^{\widetilde{\tau}A}\widetilde{v}\Big\rangle\Big|\\&\le C_{r-\frac{1}{2}}\Big(\|e^{\widetilde{\tau}A}\overline{v}_1\|_{H^r}+\|e^{\widetilde{\tau}A}\widetilde{v}_1\|_{H^r}+\|e^{\widetilde{\tau}A}\overline{v}_2\|_{H^r}+\|e^{\widetilde{\tau}A}\widetilde{v}_2\|_{H^r}\Big)\Big(\|A^re^{\widetilde{\tau}A}\overline{v}\|^2+\|A^re^{\widetilde{\tau}A}\widetilde{v}\|^2\Big).\end{aligned} \tag{3.68}$$

Combining (3.66)–(3.68), thanks to (3.25), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|e^{\widetilde{\tau}(t)A}\overline{v}(t)\|^2_{H^{r-1/2}}+\|e^{\widetilde{\tau}(t)A}\widetilde{v}(t)\|^2_{H^{r-1/2}}\Big)\\ &\leq\Big[\dot{\widetilde{\tau}}+\frac{1}{2}C_r\Big(\|e^{\widetilde{\tau}A}\overline{v}_1\|_{H^r}+\|e^{\widetilde{\tau}A}\widetilde{v}_1\|_{H^r}+\|e^{\widetilde{\tau}A}\overline{v}_2\|_{H^r}+\|e^{\widetilde{\tau}A}\widetilde{v}_2\|_{H^r}\Big)\Big]\\ &\quad\times\Big(\|A^r e^{\widetilde{\tau}A}\overline{v}\|^2+\|A^r e^{\widetilde{\tau}A}\widetilde{v}\|^2\Big).\end{aligned}\tag{3.69}$$

Since $\|e^{\widetilde{\tau}A}\overline{v}_i\|^2_{H^r}+\|e^{\widetilde{\tau}A}\widetilde{v}_i\|^2_{H^r}\leq M$ for $i=1,2$, by Cauchy–Schwarz inequality, we know that

$$\begin{aligned}&\dot{\widetilde{\tau}}+\frac{1}{2}C_r\Big(\|e^{\widetilde{\tau}A}\overline{v}_1\|_{H^r}+\|e^{\widetilde{\tau}A}\widetilde{v}_1\|_{H^r}+\|e^{\widetilde{\tau}A}\overline{v}_2\|_{H^r}+\|e^{\widetilde{\tau}A}\widetilde{v}_2\|_{H^r}\Big)\\ &\leq -2C_r(1+M)+\sqrt{2}C_r\sqrt{M}\leq(\frac{\sqrt{2}}{2}-2)C_r(1+M)<0,\end{aligned}\tag{3.70}$$

for $t\in[0,\widetilde{\mathcal{T}}]$. Therefore, for $t\in[0,\widetilde{\mathcal{T}}]$, we have

$$\|e^{\widetilde{\tau}(t)A}\overline{v}(t)\|^2_{H^{r-1/2}}+\|e^{\widetilde{\tau}(t)A}\widetilde{v}(t)\|^2_{H^{r-1/2}}\leq\|e^{\widetilde{\tau}_0A}\overline{v}_0\|^2_{H^{r-1/2}}+\|e^{\widetilde{\tau}_0A}\widetilde{v}_0\|^2_{H^{r-1/2}}.\tag{3.71}$$

The above inequality proves the continuous dependence of the solutions on the initial data, and in particular, when $\overline{v}_0=\widetilde{v}_0=0$ and $\tau_{10}=\tau_{20}$, we have $\overline{v}=\widetilde{v}=0$ for all $t\in[0,\widetilde{\mathcal{T}}]$. Moreover, from (3.23), (3.61), and the definition of M in (3.60), we know $\widetilde{\mathcal{T}}=\mathcal{T}$. Therefore, the solution is unique, and we complete the proof of Theorem 3.1.

Remark 3. In case that $\int_{\mathbb{T}^3}v(\boldsymbol{x})d\boldsymbol{x}=\int_{\mathbb{T}^2}\overline{v}(\boldsymbol{x}')d\boldsymbol{x}'\neq 0$, the only change in system (3.1)–(3.4) is in (3.1) which will become

$$\partial_t\overline{v}+\mathbb{P}_h\Big(\overline{v}\cdot\nabla\overline{v}\Big)+\mathbb{P}_hP_0\Big((\nabla\cdot\widetilde{v})\widetilde{v}+\widetilde{v}\cdot\nabla\widetilde{v}\Big)+\Omega\int_{\mathbb{T}^2}\overline{v}^{\perp}(\boldsymbol{x}')d\boldsymbol{x}'=0.\tag{3.72}$$

The additional term $\Omega\int_{\mathbb{T}^2}\overline{v}^{\perp}(\boldsymbol{x}')d\boldsymbol{x}'$ appearing in (3.72) does not change the energy estimates. Since

$$\int_{\mathbb{T}^2}\overline{v}^{\perp}(\boldsymbol{x}')d\boldsymbol{x}'\cdot\int_{\mathbb{T}^2}\overline{v}(\boldsymbol{x}')d\boldsymbol{x}'=0,\tag{3.73}$$

the conservation of L^2 norm does not change. Since $\Omega\int_{\mathbb{T}^2}\overline{v}^{\perp}(\boldsymbol{x}')d\boldsymbol{x}'$ is a constant vector in spatial variables, when we apply the operator $A^re^{\tau A}$ to it, it will disappear. Therefore, this additional term does not affect the higher order energy estimates. Thus, when $\int_{\mathbb{T}^2}\overline{v}(\boldsymbol{x}')d\boldsymbol{x}'\neq 0$, we still have the same results.

4. Long time existence of solutions

In this section, we establish the long time existence of solutions to system (3.1)–(3.4) provided that the analytic norm of $\widetilde{v}_0$ is small. Notice that we do not assume any smallness in $\overline{v}_0$, and therefore, we do not have smallness in v_0. The motivation is that, when $\widetilde{v}=0$, system (3.1)–(3.4) reduces to $2D$ Euler equations, for which we have global solution in the space of analytic functions (see [47]). Therefore, if $\widetilde{v}_0$ is small in the analytic norm, one can expect that the solution to system (3.1)–(3.4) exists for a long time. In section 6, however, we will demonstrate that system (3.1)–(3.4) exhibits a different behaviour for large value of $|\Omega|$ when we assume $\widetilde{v}_0$ is small only in Sobolev norm, but not in the analytic norm.

4.1. $2D$ **Euler equations.** Consider the following $2D$ Euler equations in $\mathbb{T}^3$:

$$\partial_t \overline{V} + \overline{V} \cdot \nabla \overline{V} + \nabla P = 0, \tag{4.1}$$

$$\nabla \cdot \overline{V} = 0, \tag{4.2}$$

$$\overline{V}(0) = \overline{V}_0. \tag{4.3}$$

Here $\overline{V}$ depends only on two horizontal variables $\boldsymbol{x}'$. The global existence of solutions to system (4.1)–(4.3) in Sobolev spaces H^r with $r \geq 3$ is a classical result, see, e.g., [9]. Moreover, from equation (3.84) in [9], for $r \geq 3$, we have

$$\frac{d}{dt}\|\overline{V}\|_{H^r} \leq C_r \|\overline{V}\|_{H^r}(1 + \ln^+ \|\overline{V}\|_{H^r}). \tag{4.4}$$

Let $\|\overline{V}_0\|_{H^r} \leq M$ for some $M \geq 0$. Since $\ln^+ x + 1 \leq 2\ln(x+e)$, by setting $W(t) = \|\overline{V}(t)\|_{H^r} + e$, from (4.4), we have

$$\frac{d}{dt} W \leq C_r W \ln W. \tag{4.5}$$

Therefore, we get the following bound:

$$\|\overline{V}(t)\|_{H^r} \leq W(t) \leq W(0)^{e^{C_r t}} = (\|\overline{V}_0\|_{H^r} + e)^{e^{C_r t}} \leq (M+e)^{e^{C_r t}} =: \theta_{M,r}(t). \tag{4.6}$$

We need the following lemma for the purpose of this section. For its proof, we refer the reader to [47].

Lemma 4.1. *For $f, g \in \mathcal{D}(e^{\tau A} : H^{r+1/2})$ where $r > 5/2$ and $\tau \geq 0$, one has*

$$\begin{aligned} \left| \langle A^r e^{\tau A}(f \cdot \nabla g), A^r e^{\tau A} g \rangle \right| \leq\ & C_r (\|A^r f\| \|A^r g\|^2 + \|\nabla \cdot f\|_{L^\infty} \|A^r e^{\tau A} g\|^2) \\ & + C_r \tau \|A^{r+1/2} e^{\tau A} f\| \|A^{r+1/2} e^{\tau A} g\|^2. \end{aligned} \tag{4.7}$$

Moreover, if $r > 3$, then $\|A^{r+1/2} e^{\tau A} f\|$ can be replaced by $\|A^r e^{\tau A} f\|$.

Based on Lemma 4.1, the authors in [47] proved the global existence of solutions to system (4.1)–(4.3) for initial data in the space of analytic functions. For completion, we state it here, with slight difference compared with the original statement in [47].

Proposition 4.2. *Assume $\overline{V}_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau_0 A} : H^r(\mathbb{T}^3))$ with $r > 3$ and $\tau_0 > 0$, and suppose that $\|e^{\tau_0 A} \overline{V}_0\|_{H^r} \leq M$ for some $M \geq 0$. There exists a non-increasing function*

$$\tau(t) = \tau_0 \exp\Big(- C_r \int_0^t h(s) ds \Big), \tag{4.8}$$

where

$$h^2(t) := \|e^{\tau_0 A} \overline{V}_0\|_{H^r}^2 + C_r \int_0^t \theta_{M,r}^3(s) ds, \tag{4.9}$$

and $\theta_{M,r}(t)$ defined in (4.6), such that for any given time $\mathcal{T} > 0$, there exists a unique solution

$$\overline{V} \in L^\infty(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r(\mathbb{T}^3))) \tag{4.10}$$

to system (4.1)–(4.3). Moreover, there exist constants $C_M > 1$ and $C_r > 1$ such that

$$\|e^{\tau(t)A} \overline{V}(t)\|_{H^r}^2 \leq h^2(t) \leq C_M^{\exp(C_r t)}. \tag{4.11}$$

4.2. **Long time existence of the $3D$ inviscid PEs.** The following is the main theorem of this section, which concerns the long time exsitence of solutions to system (3.1)–(3.4) in the case when the analytic norm of $\widetilde{v}_0$ is small.

Theorem 4.3. *Assume $\overline{v}_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau_0 A} : H^{r+1}(\mathbb{T}^3))$, $\widetilde{v}_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau_0 A} : H^r(\mathbb{T}^3))$ with $r > 5/2$ and $\tau_0 > 0$. Let $\Omega \in \mathbb{R}$ be arbitrary and fixed. Let $M \geq 0$ and $\epsilon \geq 0$, and suppose that $\|e^{\tau_0 A}\overline{v}_0\|_{H^{r+1}} \leq M$ and $\|e^{\tau_0 A}\widetilde{v}_0\|_{H^r} \leq \epsilon$. Then there are constants $C_M > 1$ and $C_r > 1$, and a function $K(t) = C_M^{\exp(C_r t)}$, such that if $\mathcal{T} = \mathcal{T}(\tau_0, \epsilon, M, r)$ satisfies*

$$\int_0^{\mathcal{T}} e^{K(s)} ds = \frac{\tau_0}{2\epsilon}, \tag{4.12}$$

then the unique solution obtained in Theorem 3.1 satisfies $(\overline{v}, \widetilde{v}) \in \mathcal{S} \cap L^\infty(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r(\mathbb{T}^3)))$, with

$$\tau(t) = e^{-\int_0^t K(s)ds}\left(\tau_0 - \epsilon \int_0^t e^{K(s)} ds\right). \tag{4.13}$$

In particular, from (4.12), $\mathcal{T} \gtrsim \ln(\ln(\ln(\frac{1}{\epsilon}))) \to \infty$, as $\epsilon \to 0^+$.

Thanks to Lemma 2.7, we immediately have the following corollary.

Corollary 4.4. *Assume $v_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau_0 A} : H^{r+1}(\mathbb{T}^3))$, and the conditions of Theorem 4.3 hold. Then the unique solution obtained in Corollary 3.2 satisfies $v \in \mathcal{S} \cap L^\infty(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r(\mathbb{T}^3)))$, with $\mathcal{T}$ defined in (4.12) and τ defined in (4.13).*

Remark 4. For the proof of Theorem 4.3, we only establish formal energy estimates. However, these formal estimates can be justified rigorously by establishing them first for the Galerkin approximation system and then passing to the limit using the Aubin-Lions compactness theorem, as we did in the previous section.

Remark 5. The constants C_M and C_r in $K(t)$ may change from step to step in the proof, and always larger than 1. When necessary, we use $K_1(t), K_2(t), ...$ to emphasize the changes in C_M and C_r. Two useful inequalities are

$$\int_0^t K(s)ds \leq K_1(t), \quad \int_0^t e^{K(s)} ds \leq e^{K_1(t)} \tag{4.14}$$

for some new $K_1(t)$. At the end, we choose some suitable and large enough C_M and C_r for the $K(t)$ in Theorem 4.3. Similar abuse of notation will also be used in the rest of sections.

Proof. (proof of Theorem 4.3.) Let $\overline{V}$ be the unique global solution to the $2D$ Euler equations (4.1)–(4.3) in the space $\mathcal{D}(e^{\tau_1(t)A} : H^{r+1}(\mathbb{T}^3))$, with initial condition $\overline{V}_0 = \overline{v}_0$ and $\tau_1(t)$ satisfying (4.8). Let $\overline{\phi} = \overline{v} - \overline{V}$. Applying $\mathbb{P}_h$ to (4.1), taking the difference between (3.1) and (4.1), and writing (3.2) in terms of $\overline{V}$ and $\overline{\phi}$, we have

$$\partial_t \overline{\phi} + \mathbb{P}_h\Big(\overline{\phi} \cdot \nabla\overline{\phi} + \overline{\phi} \cdot \nabla\overline{V} + \overline{V} \cdot \nabla\overline{\phi}\Big) + \mathbb{P}_h P_0\Big((\nabla \cdot \widetilde{v})\widetilde{v} + \widetilde{v} \cdot \nabla\widetilde{v}\Big) = 0, \tag{4.15}$$

$$\begin{aligned}\partial_t \widetilde{v} + \widetilde{v} \cdot \nabla\widetilde{v} + \overline{\phi} \cdot \nabla\widetilde{v} + \overline{V} \cdot \nabla\widetilde{v} + \widetilde{v} \cdot \nabla\overline{\phi} + \widetilde{v} \cdot \nabla\overline{V} - P_0\Big((\nabla \cdot \widetilde{v})\widetilde{v} + \widetilde{v} \cdot \nabla\widetilde{v}\Big) \\ -\Big(\int_0^z \nabla \cdot \widetilde{v}(s)ds\Big)\partial_z\widetilde{v} + \Omega\widetilde{v}^\perp = 0,\end{aligned} \tag{4.16}$$

with initial condition

$$\overline{\phi}(0) = \overline{v}_0 - \overline{V}_0 = 0, \qquad \widetilde{v}(0) = \widetilde{v}_0. \tag{4.17}$$

First, by virtue of (3.17), and since the L^2 energy is conserved for $\overline{V}$, thanks to triangle inequality, we have

$$\|\overline{\phi}\|^2 + \|\widetilde{v}\|^2 \leq 2(\|\overline{v}\|^2 + \|\overline{V}\|^2 + \|\widetilde{v}\|^2) = 4\|\overline{v}_0\|^2 + 2\|\widetilde{v}_0\|^2. \tag{4.18}$$

Next, applying $A^r e^{\tau A}$ to equation (4.15) and (4.16), and taking L^2 inner product with $A^r e^{\tau A}\overline{\phi}$ and $A^r e^{\tau A}\widetilde{v}$, respectively, thanks to Lemma 2.5 and Lemma 2.6, since $P_0\widetilde{v} = 0$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A^r e^{\tau A}\overline{\phi}\|^2 = {} & \dot{\tau}\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 - \Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla\overline{\phi}), A^r e^{\tau A}\overline{\phi}\Big\rangle \\ & -\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla\overline{V}), A^r e^{\tau A}\overline{\phi}\Big\rangle - \Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla\overline{\phi}), A^r e^{\tau A}\overline{\phi}\Big\rangle \\ & -\Big\langle A^r e^{\tau A}(\widetilde{v}\cdot\nabla\widetilde{v}), A^r e^{\tau A}\overline{\phi}\Big\rangle - \Big\langle A^r e^{\tau A}((\nabla\cdot\widetilde{v})\widetilde{v}), A^r e^{\tau A}\overline{\phi}\Big\rangle,\end{aligned} \tag{4.19}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A^r e^{\tau A}\widetilde{v}\|^2 = {} & \dot{\tau}\|A^{r+1/2}e^{\tau A}\widetilde{v}\|^2 - \Big\langle A^r e^{\tau A}(\widetilde{v}\cdot\nabla\widetilde{v}), A^r e^{\tau A}\widetilde{v}\Big\rangle \\ & -\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla\widetilde{v}), A^r e^{\tau A}\widetilde{v}\Big\rangle - \Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla\widetilde{v}), A^r e^{\tau A}\widetilde{v}\Big\rangle \\ & -\Big\langle A^r e^{\tau A}(\widetilde{v}\cdot\nabla\overline{\phi}), A^r e^{\tau A}\widetilde{v}\Big\rangle - \Big\langle A^r e^{\tau A}(\widetilde{v}\cdot\nabla\overline{V}), A^r e^{\tau A}\widetilde{v}\Big\rangle \\ & +\Big\langle A^r e^{\tau A}\big((\int_0^z \nabla\cdot\widetilde{v}(\boldsymbol{x}', s)ds)\partial_z\widetilde{v}\big), A^r e^{\tau A}\widetilde{v}\Big\rangle.\end{aligned} \tag{4.20}$$

Thanks to Lemma A.1–A.3, we have

$$\begin{aligned}& \Big|\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla\overline{\phi}), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| + \Big|\Big\langle A^r e^{\tau A}(\widetilde{v}\cdot\nabla\widetilde{v}), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| \\ & +\Big|\Big\langle A^r e^{\tau A}((\nabla\cdot\widetilde{v})\widetilde{v}), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| + \Big|\Big\langle A^r e^{\tau A}(\widetilde{v}\cdot\nabla\widetilde{v}), A^r e^{\tau A}\widetilde{v}\Big\rangle\Big| \\ & +\Big|\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla\widetilde{v}), A^r e^{\tau A}\widetilde{v}\Big\rangle\Big| + \Big|\Big\langle A^r e^{\tau A}(\widetilde{v}\cdot\nabla\overline{\phi}), A^r e^{\tau A}\widetilde{v}\Big\rangle\Big| \\ & +\Big|\Big\langle A^r e^{\tau A}\big((\int_0^z \nabla\cdot\widetilde{v}(\boldsymbol{x}', s)ds)\partial_z\widetilde{v}\big), A^r e^{\tau A}\widetilde{v}\Big\rangle\Big| \\ \leq {} & C_r(\|A^r e^{\tau A}\overline{\phi}\| + \|A^r e^{\tau A}\widetilde{v}\|)(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2}e^{\tau A}\widetilde{v}\|^2).\end{aligned} \tag{4.21}$$

Here we use the fact that $\widetilde{v}$ and $\overline{\phi}$ have zero mean value. For $\widetilde{v}$, since $\overline{\widetilde{v}} = 0$, so its mean is zero. For $\overline{\phi}$, since $\overline{V}$ and $\overline{v}$ both have zero mean value, therefore, $\overline{\phi}$ also has zero mean value.

By virtue of Lemma 4.1, since $\nabla\cdot\overline{V} = 0$, one obtains

$$\begin{aligned}& \Big|\Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla\overline{\phi}), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| + \Big|\Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla\widetilde{v}), A^r e^{\tau A}\widetilde{v}\Big\rangle\Big| \\ \leq {} & C_r\|A^r\overline{V}\|(\|A^r\overline{\phi}\|^2 + \|A^r\widetilde{v}\|^2) + C_r\tau\|A^{r+1/2}e^{\tau A}\overline{V}\|(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2}e^{\tau A}\widetilde{v}\|^2).\end{aligned} \tag{4.22}$$

From Lemma 2.2, thanks to Cauchy–Schwarz inequality, and since $\widetilde{v}$ and $\overline{\phi}$ have zero mean, we have

$$\begin{aligned}& \Big|\Big\langle A^r e^{\tau A}(\widetilde{v}\cdot\nabla\overline{V}), A^r e^{\tau A}\widetilde{v}\Big\rangle\Big| + \Big|\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla\overline{V}), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| \\ \leq {} & C_r\|e^{\tau A}\overline{V}\|_{H^{r+1}}(\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\widetilde{v}\|^2).\end{aligned} \tag{4.23}$$

Combining all of the estimates above, we have

$$\begin{aligned}& \frac{1}{2}\frac{d}{dt}(\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\widetilde{v}\|^2) \\ \leq {} & \Big(\dot{\tau} + C_r(\|A^r e^{\tau A}\overline{\phi}\| + \|A^r e^{\tau A}\widetilde{v}\|) + C_r\tau\|e^{\tau A}\overline{V}\|_{H^{r+1}}\Big)\Big(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2}e^{\tau A}\widetilde{v}\|^2\Big)\end{aligned}$$

$$+C_r\|e^{\tau A}\overline{V}\|_{H^{r+1}}\Big(\|A^r e^{\tau A}\overline{\phi}\|^2+\|A^r e^{\tau A}\widetilde{v}\|^2\Big). \tag{4.24}$$

As indicated in Remark 5, we will use $K_0, K_1, K_2, \ldots$ to indicate the change in $K(t)$ from step to step, and all of them are increasing double exponentially in t. Recall that τ_1 is defined by (4.8). Indeed, there exists a function $K_0(t)$ such that $\tau_1(t)\geq \tau_0 e^{-\int_0^t K_0(s)ds}$. Let $\tau\leq\tau_1$. Recall from (4.11), we have

$$\|e^{\tau(t)A}\overline{V}(t)\|_{H^{r+1}}\leq \|e^{\tau_1(t)A}\overline{V}(t)\|_{H^{r+1}}\leq C_M^{\exp(\widetilde{C_r}t)}=:K_1(t). \tag{4.25}$$

Denote by

$$F=\|A^r e^{\tau A}\overline{\phi}\|^2+\|A^r e^{\tau A}\widetilde{v}\|^2, \quad G=\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2+\|A^{r+1/2}e^{\tau A}\widetilde{v}\|^2. \tag{4.26}$$

We can rewrite (4.24) as

$$\frac{d}{dt}F\leq 2(\dot{\tau}+C_rF^{1/2}+\tau K_2)G+K_2F. \tag{4.27}$$

Notice that when τ satisfies

$$\dot{\tau}+C_rF^{1/2}+\tau K_2\leq 0, \tag{4.28}$$

we have

$$F(t)\leq F(0)e^{\int_0^t K_2(s)ds}\leq F(0)e^{K_3(t)}, \tag{4.29}$$

and therefore

$$C_rF(t)^{1/2}\leq F(0)^{1/2}e^{K_4(t)}. \tag{4.30}$$

Notice that $F(0)=\|A^r e^{\tau_0 A}\widetilde{v}_0\|^2\leq\|e^{\tau_0 A}\widetilde{v}_0\|_{H^r}^2\leq\epsilon^2$. From (4.28), we require that

$$\frac{d}{dt}(\tau e^{\int_0^t K_2(s)ds})+\epsilon e^{\int_0^t K_2(s)ds}e^{K_4(t)}\leq 0. \tag{4.31}$$

Thanks to (4.14), we have

$$e^{\int_0^t K_2(s)ds}e^{K_4(t)}\leq e^{K_5(t)} \tag{4.32}$$

for some new $K_5(t)$. Therefore, instead of (4.31), we require that

$$\frac{d}{dt}(\tau e^{\int_0^t K_2(s)ds})+\epsilon e^{K_5(t)}\leq 0. \tag{4.33}$$

Integrating (4.31) from 0 to t in time, we have

$$\tau(t)e^{\int_0^t K_2(s)ds}\leq\tau_0-\epsilon\int_0^t e^{K_5(s)}ds. \tag{4.34}$$

Recall that we also need $\tau(t)\leq\tau_1(t)$ and we know that $\tau_1(t)\geq\tau_0 e^{-\int_0^t K_0(s)ds}$. Therefore, for a new and suitable function $K(t)$, we can set

$$\tau(t)=e^{-\int_0^t K(s)ds}(\tau_0-\epsilon\int_0^t e^{K(s)}ds) \tag{4.35}$$

such that $\tau(t)$ satisfies the condition in (4.28) and also $\tau(t)\leq\tau_1(t)$. One can see $\tau(t)>0$ on $t\in[0,\mathcal{T}]$ when $\mathcal{T}$ satisfies

$$\int_0^{\mathcal{T}} e^{K(s)}ds=\frac{\tau_0}{2\epsilon}. \tag{4.36}$$

Since $K(t)$ is double exponential in time, and $\int_0^{\mathcal{T}} e^{K(s)}ds\leq\mathcal{T}e^{K(\mathcal{T})}\leq e^{2K(\mathcal{T})}$, we have $\mathcal{T}\gtrsim\ln(\ln(\ln(\frac{1}{\epsilon})))\to\infty$ as $\epsilon\to 0^+$.

From (4.29), since $\overline{\phi}$ and $\widetilde{v}$ have zero mean, we can apply Poincaré inequality to obtain

$$\|e^{\tau(t)A}\overline{\phi}(t)\|_{H^r}^2 + \|e^{\tau(t)A}\widetilde{v}(t)\|_{H^r}^2 \leq \epsilon^2 e^{K(t)} \tag{4.37}$$

when $K(t)$ is chosen suitably, on $t \in [0, \mathcal{T}]$, with $\tau(t)$ defined by (4.35). From (4.11), and since $\tau \leq \tau_1$, we know $\|e^{\tau(t)A}\overline{V}(t)\|_{H^r}$ is also bounded on $t \in [0, \mathcal{T}]$. By triangle inequality, we have

$$\|e^{\tau(t)A}\overline{v}(t)\|_{H^r} + \|e^{\tau(t)A}\widetilde{v}(t)\|_{H^r} \leq \|e^{\tau(t)A}\overline{\phi}(t)\|_{H^r} + \|e^{\tau(t)A}\overline{V}(t)\|_{H^r} + \|e^{\tau(t)A}\widetilde{v}(t)\|_{H^r} < \infty \tag{4.38}$$

on $t \in [0, \mathcal{T}]$. Therefore, the time of existence of the solution to system (3.1)–(3.4) satisfies (4.12). □

4.3. **Convergence to the $2D$ Euler equations.** Based on Theorem 4.3, we have the following result concerning the convergence of solutions of the $3D$ inviscid PEs (3.1)–(3.2) to solutions of the $2D$ Euler equations (4.1)–(4.3) in the space of analytic functions.

Theorem 4.5. *Assume a sequence of initial data $\{\overline{v}_0^n = \overline{v}_0\}_{n\in\mathbb{N}} \subset \mathcal{S}\cap\mathcal{D}(e^{\tau_0 A} : H^{r+1}(\mathbb{T}^3))$ and $\{\widetilde{v}_0^n\}_{n\in\mathbb{N}} \subset \mathcal{S}\cap\mathcal{D}(e^{\tau_0 A} : H^r(\mathbb{T}^3))$ with $r > 5/2$ and $\tau_0 > 0$. Let $\Omega \in \mathbb{R}$ be arbitrary and fixed. Suppose $\|e^{\tau_0 A}\overline{v}_0\|_{H^{r+1}} \leq M$ for some $M \geq 0$, and $\|e^{\tau_0 A}\widetilde{v}_0^n\|_{H^r} \leq \epsilon_n$ with $\epsilon_n \to 0$, as $n \to \infty$. Then there are constants $C_M > 1$, $C_r > 1$, and a function $K(t) = C_M^{\exp(C_r t)}$, such that for each $n \in \mathbb{N}$, if the function $\tau^n(t)$ and the time $\mathcal{T}_n$ satisfy*

$$\tau^n(t) = e^{-\int_0^t K(s)ds}(\tau_0 - \epsilon_n \int_0^t e^{K(s)}ds), \qquad \int_0^{\mathcal{T}_n} e^{K(s)}ds = \frac{\tau_0}{2\epsilon_n}, \tag{4.39}$$

the solution to system (3.1)–(3.4) with initial data $(\overline{v}_0^n, \widetilde{v}_0^n)$ satisfies $(\overline{v}^n, \widetilde{v}^n) \in \mathcal{S} \cap L^\infty(0, \mathcal{T}_n; \mathcal{D}(e^{\tau^n A} : H^r(\mathbb{T}^3)))$. Let $\overline{V} \in \mathcal{S} \cap L^\infty(0, \infty; \mathcal{D}(e^{\tau^0(t)A} : H^r(\mathbb{T}^3)))$ be the unique global solution to the $2D$ Euler equations (4.1)–(4.3) with initial data $\overline{V}(0) = \overline{v}_0$. Then, $(\overline{v}^n, \widetilde{v}^n)$ converges to $\overline{V}$ for $t \in [0, \mathcal{T}_0]$, as $n \to \infty$, in the following sense:

$$\|e^{\tau^0(t)A}(\overline{v}^n + \widetilde{v}^n - \overline{V})(t)\|_{H^r} \leq \epsilon_n e^{K(t)} \to 0, \quad as \ n \to \infty. \tag{4.40}$$

Proof. Denote by $\overline{\phi}^n = \overline{v}^n - \overline{V}$. By virtue of the proof of Theorem 4.3, we just need to prove the estimate (4.40). Since $\tau^0(t) \leq \tau^n(t)$ for any $n \in \mathbb{N}$, from (4.37), one has

$$\|e^{\tau^0(t)A}\widetilde{v}^n(t)\|_{H^r} + \|e^{\tau^0(t)A}\overline{\phi}^n(t)\|_{H^r} \leq \|e^{\tau^n(t)A}\widetilde{v}^n(t)\|_{H^r} + \|e^{\tau^n(t)A}\overline{\phi}^n(t)\|_{H^r} \leq \epsilon_n e^{K(t)} \tag{4.41}$$

when the function $K(t)$ is chosen suitably. Therefore, we have

$$\|e^{\tau^0(t)A}(\overline{v}^n + \widetilde{v}^n - \overline{V})(t)\|_{H^r} \leq \|e^{\tau^0(t)A}\widetilde{v}^n(t)\|_{H^r} + \|e^{\tau^0(t)A}\overline{\phi}^n(t)\|_{H^r} \leq \epsilon_n e^{K(t)}. \tag{4.42}$$

As $n \to \infty$, we have $\epsilon_n \to 0$, and therefore, $\epsilon_n e^{K(t)} \to 0$. This completes the proof. □

5. Limit resonant system

In this section, we derive the formal resonant limit resonant system of the original system (3.1)–(3.2) as $|\Omega| \to \infty$, and establish some properties of the limit resonant system. Recall from (2.62), we have

$$\begin{aligned}
\partial_t u_+ = &-e^{i\Omega t}\Big(u_+ \cdot \nabla u_+ - P_0(u_+ \cdot \nabla u_+ + (\nabla \cdot u_+)u_+) - (\int_0^z \nabla \cdot u_+(\boldsymbol{x}', s)ds)\partial_z u_+\Big) \\
&-\Big(\overline{v} \cdot \nabla u_+ + \frac{1}{2}(u_+ \cdot \nabla)(\overline{v} + i\overline{v}^\perp)\Big) \\
&-e^{-i\Omega t}\Big(u_- \cdot \nabla u_+ - P_0(u_- \cdot \nabla u_+ + (\nabla \cdot u_-)u_+) - (\int_0^z \nabla \cdot u_-(\boldsymbol{x}', s)ds)\partial_z u_+\Big) \\
&-e^{-2i\Omega t}\frac{1}{2}(u_- \cdot \nabla)(\overline{v} + i\overline{v}^\perp) = -e^{i\Omega t}I_1 - I_0 - e^{-i\Omega t}I_{-1} - e^{-2i\Omega t}I_{-2},
\end{aligned} \tag{5.1}$$

where

$$
\begin{aligned}
I_1 &:= u_+ \cdot \nabla u_+ - P_0\Big(u_+ \cdot \nabla u_+ + (\nabla \cdot u_+)u_+\Big) - \Big(\int_0^z \nabla \cdot u_+(\boldsymbol{x}', s)ds\Big)\partial_z u_+,\\
I_0 &:= \overline{v} \cdot \nabla u_+ + \frac{1}{2}(u_+ \cdot \nabla)(\overline{v} + i\overline{v}^\perp),\\
I_{-1} &:= u_- \cdot \nabla u_+ - P_0\Big(u_- \cdot \nabla u_+ + (\nabla \cdot u_-)u_+\Big) - \Big(\int_0^z \nabla \cdot u_-(\boldsymbol{x}', s)ds\Big)\partial_z u_+,\\
I_{-2} &:= \frac{1}{2}(u_- \cdot \nabla)(\overline{v} + i\overline{v}^\perp).
\end{aligned}
\tag{5.2}
$$

Observe that I_0 is a typical resonant term. Unlike the case of the $3D$ Euler equations where there are frequency selection resonances, in this resonance term, I_0, all frequencies resonate.

We can rewrite (5.1) as

$$
\partial_t\Big[u_+ - \frac{i}{\Omega}\Big(e^{i\Omega t}I_1 - e^{-i\Omega t}I_{-1} - \frac{1}{2}e^{-2i\Omega t}I_{-2}\Big)\Big] = -\frac{i}{\Omega}\Big(e^{i\Omega t}\partial_t I_1 - e^{-i\Omega t}\partial_t I_{-1} - \frac{1}{2}e^{-2i\Omega t}\partial_t I_{-2}\Big) - I_0. \tag{5.3}
$$

Denote by the formal limits of $u_+, u_-, \overline{v}$ to be $U_+, U_-, \overline{V}$. By taking limit $\Omega \to \infty$ formally, we obtain the limit resonant equation for u_+ is

$$
\partial_t U_+ = -(\overline{V} \cdot \nabla)U_+ - \frac{1}{2}(U_+ \cdot \nabla)(\overline{V} + i\overline{V}^\perp). \tag{5.4}
$$

By taking the complex conjugate, we obtain the limit resonant equation for u_- is

$$
\partial_t U_- = -(\overline{V} \cdot \nabla)U_- - \frac{1}{2}(U_- \cdot \nabla)(\overline{V} - i\overline{V}^\perp). \tag{5.5}
$$

For the limit equation for $\overline{v}$, recall from (2.64) that

$$
\begin{aligned}
\partial_t\overline{v} + \mathbb{P}_h(\overline{v} \cdot \nabla\overline{v}) + e^{2i\Omega t}\mathbb{P}_h P_0\Big(u_+ \cdot \nabla u_+ + (\nabla \cdot u_+)u_+\Big)&\\
+e^{-2i\Omega t}\mathbb{P}_h P_0\Big(u_- \cdot \nabla u_- + (\nabla \cdot u_-)u_-\Big) &= 0.
\end{aligned}
$$

Observe that $\mathbb{P}_h(\overline{v} \cdot \nabla\overline{v})$ is a typical resonant term. Using the similar method in the derivation of U_+, we can derive the limit resonant equation for $\overline{v}$ as

$$
\partial_t\overline{V} + \mathbb{P}_h(\overline{V} \cdot \nabla\overline{V}) = 0. \tag{5.6}
$$

Observe that (5.6) is the $2D$ Euler system. Consider the initial conditions

$$
(\overline{V}_0, (U_+)_0, (U_-)_0) = (\overline{v}_0, \frac{1}{2}(\widetilde{v}_0 + i\widetilde{v}_0^\perp), \frac{1}{2}(\widetilde{v}_0 - i\widetilde{v}_0^\perp)) \tag{5.7}
$$

for system (5.4)–(5.6). Since $v_0 \in \mathcal{S}$, we have $\nabla \cdot \overline{V} = 0$, $P_0\overline{V} = \overline{V}$, and $P_0 U_\pm = 0$.

Besides the equations for $U_\pm$ and $\overline{V}$, we also want a baroclinic mode $\widetilde{V}$ similar as in the original system. Since initially $U_\pm(0) = \frac{1}{2}(\widetilde{v}_0 \pm i\widetilde{v}_0^\perp)$, we define $\widetilde{V} := U_+ + U_-$ so that $U_\pm = \frac{1}{2}(\widetilde{V} \pm i\widetilde{V}^\perp)$. From (5.4)–(5.5), we have

$$
\partial_t\widetilde{V} + (\overline{V} \cdot \nabla)\widetilde{V} + \frac{1}{2}(\widetilde{V} \cdot \nabla\overline{V} - \widetilde{V}^\perp \cdot \nabla\overline{V}^\perp) = 0. \tag{5.8}
$$

Since $\nabla \cdot \overline{V} = 0$, (5.8) is equivelent to

$$
\partial_t\widetilde{V} + \overline{V} \cdot \nabla\widetilde{V} + \frac{1}{2}\widetilde{V}^\perp(\nabla^\perp \cdot \overline{V}) = 0. \tag{5.9}
$$

Since $P_0 U_\pm = 0$, we see $P_0\widetilde{V} = 0$.

Therefore, we consider the following limit resonant system

$$\partial_t\overline{V} + \mathbb{P}_h(\overline{V}\cdot\nabla\overline{V}) = 0, \tag{5.10}$$

$$\partial_t\widetilde{V} + \overline{V}\cdot\nabla\widetilde{V} + \frac{1}{2}\widetilde{V}^\perp(\nabla^\perp\cdot\overline{V}) = 0, \tag{5.11}$$

$$\overline{V}(0) = \overline{V}_0, \;\; \widetilde{V}(0) = \widetilde{V}_0, \tag{5.12}$$

with $P_0\overline{V} = \overline{V}$ and $P_0\widetilde{V} = 0$. Now notice that (5.10) is the $2D$ Euler system, and (5.11) is a linear transport equation with an additional stretching term.

Next, we establish the global well-posedness of limit resonant system (5.10)–(5.12) in both Sobolev spaces and the space of analytic functions. Notice that the global well-posedness of (5.10) has been established in Proposition 4.2.

Proposition 5.1. *Assume $\overline{V}_0\in\mathcal{S}\cap H^{r+1}(\mathbb{T}^3)$ and $\widetilde{V}_0\in\mathcal{S}\cap H^r(\mathbb{T}^3)$ with $r>5/2$. Let $M\geq 0$, and suppose that $\|\overline{V}_0\|_{H^{r+1}}\leq M$. Then there exist constants $C_M>1$ and $C_r>1$, and a function $K(t):=C_M^{\exp(C_r t)}$, such that for any give time $\mathcal{T}>0$, there exists a unique solution $\overline{V}\in\mathcal{S}\cap L^\infty(0,\mathcal{T};H^{r+1}(\mathbb{T}^3))$ and $\widetilde{V}\in\mathcal{S}\cap L^\infty(0,\mathcal{T};H^r(\mathbb{T}^3))$ of system (5.10)–(5.12) on $[0,\mathcal{T}]$, and satisfies*

$$\|\overline{V}(t)\|_{H^{r+1}}\leq K(t), \quad \|\widetilde{V}(t)\|_{H^r}\leq\|\widetilde{V}_0\|_{H^r}e^{K(t)}. \tag{5.13}$$

Moreover, assume $\overline{V}_0\in\mathcal{D}(e^{\tau_0 A}:H^{r+1}(\mathbb{T}^3))$ and $\widetilde{V}_0\in\mathcal{D}(e^{\tau_0 A}:H^r(\mathbb{T}^3))$ with $r>5/2$ and $\tau_0>0$, and suppose that $\|e^{\tau_0 A}\overline{V}_0\|_{H^{r+1}}\leq M$. Then there exists a function

$$\tau(t) = \tau_0\exp(-\int_0^t K(s)ds), \tag{5.14}$$

such that for any given time $\mathcal{T}>0$, there exists a unique solution $\overline{V}\in\mathcal{S}\cap L^\infty(0,\mathcal{T};\mathcal{D}(e^{\tau(t)A}:H^{r+1}(\mathbb{T}^3)))$ and $\widetilde{V}\in\mathcal{S}\cap L^\infty(0,\mathcal{T};\mathcal{D}(e^{\tau(t)A}:H^r(\mathbb{T}^3)))$ of system (5.10)–(5.12) on $[0,\mathcal{T}]$ such that

$$\|e^{\tau(t)A}\overline{V}(t)\|_{H^{r+1}}\leq K(t), \quad \|e^{\tau(t)A}\widetilde{V}(t)\|_{H^r}\leq\|e^{\tau_0 A}\widetilde{V}_0\|_{H^r}e^{K(t)}. \tag{5.15}$$

Proof. We will use the notation $K_1, K_2, ...$ as indicated in Remark 5. The global well-posedness of the $2D$ Euler equations in Sobolev spaces and corresponding growth estimate is classical, see [9]. From (4.6), we obtain that $\|\overline{V}\|_{H^{r+1}}\leq K_1(t)$ for some function $K_1(t)$.

For the growth of $\|\widetilde{V}\|_{H^r}$, by standard energy estimate, since $\nabla\cdot\overline{V}=0$ and $r>\frac{5}{2}$, we have

$$\frac{d}{dt}\|\widetilde{V}\|_{H^r}^2\leq C_r\|\overline{V}\|_{H^{r+1}}\|\widetilde{V}\|_{H^r}^2. \tag{5.16}$$

By Gronwall inequality, and by virture of the growth of $\|\overline{V}\|_{H^{r+1}}$, we obtain

$$\|\widetilde{V}(t)\|_{H^r}\leq\|\widetilde{V}_0\|_{H^r}\exp(\frac{1}{2}C_r\int_0^t K_1(s)ds)\leq\|\widetilde{V}_0\|_{H^r}e^{K(t)} \tag{5.17}$$

for some suitable function $K(t)$, such that $\|\overline{V}(t)\|_{H^{r+1}}\leq K(t)$ also holds. By virtue of these formal energy estimates, the global well-posedness of system (5.10)–(5.12) in Sobolev spaces follows.

The global well-posedness of the $2D$ Euler equations in the space of analytic functions and the corresponding growth estimate are established in Proposition 4.2. From Proposition 4.2, we can first choose some suitable functions $K_1(t)$ and $K_2(t)$ such that $\tau(t)\leq\tau_0\exp(-\int_0^t K_1(s)ds)$ and $\|e^{\tau(t)A}\overline{V}(t)\|_{H^{r+1}}\leq K_2(t)$.

For the baroclinic mode $\widetilde{V}$, first, it is easy to see the L^2 energy is conserved. Next, using Lemma 2.2 and Lemma 4.1, since $r>5/2$ and $\int_{\mathbb{T}^3}\widetilde{V}(\boldsymbol{x})d\boldsymbol{x}=0$, we have

$$\frac{1}{2}\frac{d}{dt}\|A^r e^{\tau A}\widetilde{V}\|^2$$

$$= \dot{\tau}\|A^{r+1/2}e^{\tau A}\widetilde{V}\|^2 - \Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla\widetilde{V}), A^r e^{\tau A}\widetilde{V}\Big\rangle - \frac{1}{2}\Big\langle A^r e^{\tau A}(\nabla^\perp\cdot\overline{V})\widetilde{V}^\perp, A^r e^{\tau A}\widetilde{V}\Big\rangle$$
$$\leq (\dot{\tau} + C_r\tau\|A^{r+1}e^{\tau A}\overline{V}\|)\|A^{r+1/2}e^{\tau A}\widetilde{V}\|^2 + C_r\|e^{\tau A}\overline{V}\|_{H^{r+1}}\|A^r e^{\tau A}\widetilde{V}\|^2. \tag{5.18}$$

For suitable $K_1(t)$ and $K_2(t)$, we have

$$\dot{\tau} + C_r\tau\|A^{r+1}e^{\tau A}\overline{V}\| \leq \tau(-K_1 + C_r K_2) \leq 0. \tag{5.19}$$

Therefore, by Gronwall inequality, thanks to (4.14), for some suitable function $K(t)$, we have

$$\|A^r e^{\tau(t)A}\widetilde{V}(t)\|^2 \leq \|A^r e^{\tau_0 A}\widetilde{V}_0\|^2 \exp(\int_0^t C_r\|e^{\tau(s)A}\overline{V}(s)\|_{H^{r+1}}ds)$$
$$\leq \|e^{\tau_0 A}\widetilde{V}_0\|^2_{H^r}e^{K(t)}. \tag{5.20}$$

Since L^2 energy is conserved, we have

$$\|e^{\tau(t)A}\widetilde{V}(t)\|_{H^r} \leq \|e^{\tau_0 A}\widetilde{V}_0\|_{H^r}e^{K(t)}. \tag{5.21}$$

We can choose $K(t)$ large enough such that $\tau(t) = \tau_0\exp(-\int_0^t K(s)ds)$ and $\|e^{\tau(t)A}\overline{V}\|_{H^{r+1}} \leq K(t)$. Notice that $\tau(\mathcal{T}) > 0$ for any finite time $\mathcal{T} < \infty$. Therefore, the solution $(\overline{V}, \widetilde{V})$ exists in the space of analytic functions globally in time. □

Remark 6. The use of $K(t)$ above still follows Remark 5. The conclusion is that the growth of $\|\overline{V}(t)\|_{H^{r+1}}$ and $\|e^{\tau(t)A}\overline{V}(t)\|_{H^{r+1}}$ are double exponential in time, while the growth of $\|\widetilde{V}(t)\|_{H^r}$ and $\|e^{\tau(t)A}\widetilde{V}(t)\|_{H^r}$ are triple exponential in time.

Remark 7. Since $U_\pm = \frac{1}{2}(\widetilde{V} + i\widetilde{V}^\perp)$, similar as Lemma 2.9, for $r \geq 0$ and $\tau \geq 0$, we have

$$\|U_+\|^2 = \|U_-\|^2 = \frac{1}{2}\|\widetilde{V}\|^2, \tag{5.22}$$

and

$$\|e^{\tau A}U_+\|^2_{H^r} = \|e^{\tau A}U_-\|^2_{H^r} = \frac{1}{2}\|e^{\tau A}\widetilde{V}\|^2_{H^r}. \tag{5.23}$$

Therefore, the growing bounds of $\|\widetilde{V}\|_{H^r}$ and $\|e^{\tau(t)A}\widetilde{V}(t)\|_{H^r}$ also apply to $\|U_\pm(t)\|_{H^r}$ and $\|e^{\tau(t)A}U_\pm(t)\|_{H^r}$.

6. EFFECT OF ROTATION

In section 4, we see that by requiring $\|e^{\tau_0 A}\widetilde{v}_0\|_{H^r} \leq \epsilon$, the life-span of the solution to system (3.1)–(3.4) has a lower bound $\mathcal{T} \gtrsim \ln(\ln(\ln(\frac{1}{\epsilon})))$, as $\epsilon \to 0^+$, and this result is uniform in $\Omega \in \mathbb{R}$. In this section, we establish the effect of the rate of rotation $|\Omega|$ on the life-span $\mathcal{T}$. With the help of fast rotation, i.e., when $|\Omega|$ is large, we show that the time of existence of the solution in the space of analytic functions can be prolonged as long as the Sobolev norm $\|\widetilde{v}_0\|_{H^r}$ is small depending on Ω, while the analytic norm $\|e^{\tau_0 A}\widetilde{v}_0\|_{H^r}$ can be large (of order 1). We call such initial data as "well-prepared" initial data. The following theorem is the main result of this paper.

Theorem 6.1. *Assume $\overline{v}_0 \in \mathcal{S}\cap\mathcal{D}(e^{\tau_0 A} : H^{r+3}(\mathbb{T}^3))$, $\widetilde{v}_0 \in \mathcal{S}\cap\mathcal{D}(e^{\tau_0 A} : H^{r+2}(\mathbb{T}^3))$ with $r > 5/2$ and $\tau_0 > 0$. Let $M \geq 0$ and $\delta > 0$, then there exist constants $C_{\tau_0} > 1$, $C_{M,\tau_0} > 1$, $C_r > 1$, $\widetilde{C}_{M,\tau_0} > 1$, $\widetilde{C}_r > 1$, and functions $\widetilde{K}(t) := e^{C_{M,\tau_0}^{\exp(C_r t)}}$, $\widetilde{K}_0(t) := e^{\widetilde{C}_{M,\tau_0}^{\exp(\widetilde{C}_r t)}}$, with $\widetilde{K}(t) > \widetilde{K}_0(t)$. Suppose that $|\Omega_0| \geq C_{\tau_0}e^{\widetilde{K}(1)}$, and that $\|e^{\tau_0 A}\overline{v}_0\|_{H^{r+3}} + \|e^{\tau_0 A}\widetilde{v}_0\|_{H^{r+2}} \leq M$ with $\|\widetilde{v}_0\|_{H^{3+\delta}} \leq \frac{1}{|\Omega_0|}$. Then there exists a time $\mathcal{T} = \mathcal{T}(\tau_0, |\Omega_0|, M, r) \geq 1$ satisfying*

$$C_{\tau_0}e^{\widetilde{K}(\mathcal{T})} = |\Omega_0|, \tag{6.1}$$

such that when $|\Omega| \geq |\Omega_0|$, the unique solution $(\overline{v}, \widetilde{v})$ to system (3.1)–(3.4) obtained in Theorem 3.1 satisfies $(\overline{v}, \widetilde{v}) \in \mathcal{S} \cap L^\infty(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r(\mathbb{T}^3)))$, with

$$\tau(t) = \Big(\tau_0 - \int_0^t \frac{e^{\widetilde{K}_0(s)}}{\sqrt{|\Omega_0| - e^{\widetilde{K}_0(s)}}} ds - \int_0^t \frac{e^{\widetilde{K}_0(s)}}{|\Omega_0|} ds\Big) e^{-\int_0^t \widetilde{K}_0(s) ds} > 0. \tag{6.2}$$

In particular, from (6.1), $\mathcal{T} \gtrsim \ln(\ln(\ln(\ln|\Omega_0|))) \to \infty$, as $|\Omega_0| \to \infty$.

Thanks to Lemma 2.3 and Lemma 2.7, we immediately have the following corollary.

Corollary 6.2. *Suppose $v_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau_0 A} : H^{r+3}(\mathbb{T}^3))$, and the conditions of Theorem 6.1 hold. Then the unique solution v obtained in Corollary 3.2 satisfies $v \in \mathcal{S} \cap L^\infty(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r(\mathbb{T}^3)))$, when $|\Omega| \geq |\Omega_0|$, with $\mathcal{T}$ defined in (6.1) and τ defined in (6.2).*

In this section, we focus on system (2.62)–(2.64), which is equivalent to system (3.1)–(3.2) due to Lemma 2.3 and Lemma 2.8. To prove Theorem 6.1, in section 6.2, we consider the difference between the original system (2.62)–(2.64) and the limit resonant system (5.4)–(5.6). We call such difference system as perturbed system. In section 6.3, by the formal energy estimate, we show that the solution to the perturbed system exists for a long time. This together with the global existence of the solution to system (5.4)–(5.6) give us the long time existence of the solution to system (2.62)–(2.64), and therefore the long time existence of the solution to system (3.1)–(3.2).

In section 6.1, we first give a rational behind the smallness of the initial baroclinic mode.

6.1. **A rational behind the smallness of the initial baroclinic mode.** The result of Theorem 6.1 is for "well-prepared" initial data, namely, for a given fixed $\delta > 0$, $\|\widetilde{v}_0\|_{H^{3+\delta}} \leq \frac{1}{|\Omega_0|}$. Before we go into the proof of Theorem 6.1, we briefly rationalize, below, the reason behind this smallness condition on the baroclinic mode.

Consider the linear inviscid PEs:

$$\partial_t \mathcal{V} + \Omega \mathcal{V}^\perp + \nabla p = 0, \tag{6.3}$$

$$\partial_z p = 0, \tag{6.4}$$

$$\nabla \cdot \mathcal{V} + \partial_z w = 0, \tag{6.5}$$

whose explicit solution is

$$\mathcal{V}(\boldsymbol{x}, t) = \overline{\mathcal{V}}_0(\boldsymbol{x}') + \mathcal{R}(t) \widetilde{\mathcal{V}}_0(\boldsymbol{x}), \tag{6.6}$$

where

$$\mathcal{R}(t) := \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{pmatrix}. \tag{6.7}$$

We see there is no "decay" due to rotation in the linear level. This is different from the linearized $3D$ Euler equations with rotation, for which one can obtain certain decay due to dispersion/averaging mechanism, see, e.g., [24, 40].

Now let us look back to our nonlinear inviscid PEs (3.1)–(3.2). The first equation (3.1) is the evolution of the barotropic mode, which is the $2D$ Euler with source terms coming from the baroclinic mode. The second equation (3.2) is the evolution of the baroclinic mode, which is the Burger's equations with rotation and other nonlinear coupling terms. For the Burger's equations with rotation, it is shown in [3, 50] that when the rotation rate $|\Omega|$ is large enough depending on the initial data, the solution exists globally in time because of the absence of resonance between the rotation and nonlinearity, which allows a very strong averaging mechanism that weakens the nonlinearity. In our case, however, the additional coupling nonlinear terms in (3.2) resonate with the rotation term, which does not allow for this simple scenario to take place. However, thanks to the smallness assumption on the initial baroclinic mode, the additional coupling nonlinear terms are initially small, which allows us to push this argument further.

Another reason behind this smallness assumption is indicated in [37], where a finite-time blowup of solutions to the inviscid PEs with rotation is established. Indeed, for the initial data

$$v_0(\boldsymbol{x}) = v_0(x,z) = \Big(\lambda(-z^2+\frac{1}{3})\sin x, -\Omega\sin x\Big) \tag{6.8}$$

with $\lambda > 0$, it is shown that $\frac{9}{2\lambda}$ is an upper bound for the blowup time. Notably here $\overline{v}_0 = (0, -\Omega\sin x)$ and $\widetilde{v}_0 = (\lambda(-z^2+\frac{1}{3})\sin x, 0)$. Therefore, when $|\Omega| \gg 1$, we have:

- when $\lambda = |\Omega|$, the baroclinic mode satisfies $\widetilde{v}_0 \sim |\Omega|$, and the whole initial data satisfies $v_0 \sim |\Omega|$. An upper bound of blowup time in this case satisfies $\mathcal{T} \sim \frac{1}{|\Omega|}$;
- when $\lambda = 1$, the baroclinic mode satisfies $\widetilde{v}_0 \sim 1$, while the whole initial data satisfies $v_0 \sim |\Omega|$. An upper bound of blowup time in this case satisfies $\mathcal{T} \sim 1$;
- when $\lambda = \frac{1}{|\Omega|}$, this implies a smallness condition on the baroclinic $\widetilde{v}_0 \sim \frac{1}{|\Omega|}$, while the whole initial data satisfies $v_0 \sim |\Omega|$. An upper bound of blowup time in this case satisfies $\mathcal{T} \sim |\Omega|$.

The above, in particular, the last item suggest that the smallness condition on the baroclinic mode is required to guarantee the long time existence of solutions to the $3D$ inviscid PEs with fast rotation.

Further reasoning for the smallness condition on the initial baroclinic mode will be provided in Remark 9 and Remark 10, below.

6.2. **The perturbed system around $|\Omega| = \infty$.** In section 5, we see that the limit resonant system (5.4)–(5.6) is globally well-posed. Therefore, the idea to show long time existence of the solution is to consider the difference between the original system (2.62)–(2.64) and the limit resonant system (5.4)–(5.6).

Denote by

$$\overline{\phi} = \overline{v} - \overline{V}, \quad \phi_\pm = u_\pm - U_\pm. \tag{6.9}$$

Taking the difference between (2.64) and (5.6), (2.62) and (5.4), (2.63) and (5.5), we obtain

$$\begin{aligned}\partial_t\overline{\phi} + \mathbb{P}_h\Big[&\overline{\phi}\cdot\nabla\overline{V} + \overline{\phi}\cdot\nabla\overline{\phi} + \overline{V}\cdot\nabla\overline{\phi} + e^{2i\Omega t}P_0\Big(Q_{1,+,+} + Q_{2,+,+}\Big)\\ &+e^{-2i\Omega t}P_0\Big(Q_{1,-,-} + Q_{2,-,-}\Big)\Big] = 0,\end{aligned} \tag{6.10}$$

$$\begin{aligned}\partial_t\phi_+ &+ \overline{\phi}\cdot\nabla U_+ + \overline{\phi}\cdot\nabla\phi_+ + \overline{V}\cdot\nabla\phi_+ + \frac{1}{2}(\phi_+\cdot\nabla)(\overline{V} + i\overline{V}^\perp) + \frac{1}{2}(\phi_+\cdot\nabla)(\overline{\phi} + i\overline{\phi}^\perp)\\ &+\frac{1}{2}(U_+\cdot\nabla)(\overline{\phi} + i\overline{\phi}^\perp) + e^{i\Omega t}\Big(Q_{1,+,+} - P_0Q_{1,+,+} - P_0Q_{2,+,+} - Q_{3,+,+}\Big)\\ &+e^{-i\Omega t}\Big(Q_{1,-,+} - P_0Q_{1,-,+} - P_0Q_{2,-,+} - Q_{3,-,+}\Big) + e^{-2i\Omega t}Q_{4,-,+} = 0,\end{aligned} \tag{6.11}$$

$$\begin{aligned}\partial_t\phi_- &+ \overline{\phi}\cdot\nabla U_- + \overline{\phi}\cdot\nabla\phi_- + \overline{V}\cdot\nabla\phi_- + \frac{1}{2}(\phi_-\cdot\nabla)(\overline{V} - i\overline{V}^\perp) + \frac{1}{2}(\phi_-\cdot\nabla)(\overline{\phi} - i\overline{\phi}^\perp)\\ &+\frac{1}{2}(U_-\cdot\nabla)(\overline{\phi} - i\overline{\phi}^\perp) + e^{-i\Omega t}\Big(Q_{1,-,-} - P_0Q_{1,-,-} - P_0Q_{2,-,-} - Q_{3,-,-}\Big)\\ &+e^{i\Omega t}\Big(Q_{1,+,-} - P_0Q_{1,+,-} - P_0Q_{2,+,-} - Q_{3,+,-}\Big) + e^{2i\Omega t}Q_{4,+,-} = 0,\end{aligned} \tag{6.12}$$

where

$$Q_{1,\pm,\mp} = \phi_\pm\cdot\nabla U_\mp + \phi_\pm\cdot\nabla\phi_\mp + U_\pm\cdot\nabla\phi_\mp + U_\pm\cdot\nabla U_\mp, \tag{6.13}$$

$$Q_{2,\pm,\mp} = (\nabla\cdot\phi_\pm)U_\mp + (\nabla\cdot\phi_\pm)\phi_\mp + (\nabla\cdot U_\pm)\phi_\mp + (\nabla\cdot U_\pm)U_\mp, \tag{6.14}$$

$$
\begin{aligned}
Q_{3,\pm,\mp} &= \Big(\int_0^z \nabla\cdot\phi_\pm(\boldsymbol{x}',s)ds\Big)\partial_z U_\mp + \Big(\int_0^z \nabla\cdot\phi_\pm(\boldsymbol{x}',s)ds\Big)\partial_z\phi_\mp \\
&\quad + \Big(\int_0^z \nabla\cdot U_\pm(\boldsymbol{x}',s)ds\Big)\partial_z\phi_\mp + \Big(\int_0^z \nabla\cdot U_\pm(\boldsymbol{x}',s)ds\Big)\partial_z U_\mp, \qquad (6.15)\\
Q_{4,\pm,\mp} &= \frac{1}{2}\Big[(\phi_\pm\cdot\nabla)(\overline{V}\mp i\overline{V}^\perp) + (\phi_\pm\cdot\nabla)(\overline{\phi}\mp i\overline{\phi}^\perp) \\
&\quad + (U_\pm\cdot\nabla)(\overline{\phi}\mp i\overline{\phi}^\perp) + (U_\pm\cdot\nabla)(\overline{V}\mp i\overline{V}^\perp)\Big]. \qquad (6.16)
\end{aligned}
$$

We supplement the initial conditions for the limit resonant system (5.4)–(5.6) as

$$
\overline{V}_0 = \overline{v}_0, \quad (U_\pm)_0 = (u_\pm)_0 = \frac{1}{2}(\widetilde{v}_0 \pm i\widetilde{v}_0^\perp). \qquad (6.17)
$$

Therefore, the initial conditions for the perturbed system is

$$
\overline{\phi}_0 = 0, \quad (\phi_\pm)_0 = 0. \qquad (6.18)
$$

6.3. **Proof of Theorem 6.1.** In this subsection, we prove Theorem 6.1. From Proposition 5.1, let $\overline{V}$ and $U_\pm$ be the global solution in $\mathcal{S}\cap\mathcal{D}(e^{\tau(t)A}: H^{r+3}(\mathbb{T}^3))$ and $\mathcal{S}\cap\mathcal{D}(e^{\tau(t)A}: H^{r+2}(\mathbb{T}^3))$, respectively, to system (5.4)-(5.6), with initial data (6.17) and $\tau(t)$ defined by (5.14). Next, we provide the energy estimate in the space of analytic functions for system (6.10)–(6.12). Applying $A^re^{\tau A}$ to (6.10)–(6.12), and taking L^2 inner product of (6.10) with $A^re^{\tau A}\overline{\phi}$, (6.11) with $2A^re^{\tau A}\phi_-$, and (6.12) with $2A^re^{\tau A}\phi_+$, thanks to Lemma 2.5 and Lemma 2.6, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|A^re^{\tau A}\overline{\phi}\|^2 &= \dot{\tau}\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 - \Big\langle A^re^{\tau A}(\overline{\phi}\cdot\nabla\overline{V}), A^re^{\tau A}\overline{\phi}\Big\rangle - \Big\langle A^re^{\tau A}(\overline{\phi}\cdot\nabla\overline{\phi}), A^re^{\tau A}\overline{\phi}\Big\rangle \\
&\quad - \Big\langle A^re^{\tau A}(\overline{V}\cdot\nabla\overline{\phi}), A^re^{\tau A}\overline{\phi}\Big\rangle - e^{2i\Omega t}\Big\langle A^re^{\tau A}(Q_{1,+,+}+Q_{2,+,+}), A^re^{\tau A}\overline{\phi}\Big\rangle \\
&\quad - e^{-2i\Omega t}\Big\langle A^re^{\tau A}(Q_{1,-,-}+Q_{2,-,-}), A^re^{\tau A}\overline{\phi}\Big\rangle, \qquad (6.19)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}(\|A^re^{\tau A}\phi_+\|^2 + \|A^re^{\tau A}\phi_-\|^2) = 2\dot{\tau}(\|A^{r+1/2}e^{\tau A}\phi_+\|^2 + \|A^{r+1/2}e^{\tau A}\phi_-\|^2) \\
&\quad -2\Big\langle A^re^{\tau A}(\overline{\phi}\cdot\nabla U_+), A^re^{\tau A}\phi_-\Big\rangle - 2\Big\langle A^re^{\tau A}(\overline{\phi}\cdot\nabla U_-), A^re^{\tau A}\phi_+\Big\rangle \\
&\quad -2\Big\langle A^re^{\tau A}(\overline{\phi}\cdot\nabla \phi_+), A^re^{\tau A}\phi_-\Big\rangle - 2\Big\langle A^re^{\tau A}(\overline{\phi}\cdot\nabla \phi_-), A^re^{\tau A}\phi_+\Big\rangle \\
&\quad -2\Big\langle A^re^{\tau A}(\overline{V}\cdot\nabla \phi_+), A^re^{\tau A}\phi_-\Big\rangle - 2\Big\langle A^re^{\tau A}(\overline{V}\cdot\nabla \phi_-), A^re^{\tau A}\phi_+\Big\rangle \\
&\quad -\Big\langle A^re^{\tau A}(\phi_+\cdot\nabla(\overline{V}+i\overline{V}^\perp)), A^re^{\tau A}\phi_-\Big\rangle - \Big\langle A^re^{\tau A}(\phi_-\cdot\nabla(\overline{V}-i\overline{V}^\perp)), A^re^{\tau A}\phi_+\Big\rangle \\
&\quad -\Big\langle A^re^{\tau A}(\phi_+\cdot\nabla(\overline{\phi}+i\overline{\phi}^\perp)), A^re^{\tau A}\phi_-\Big\rangle - \Big\langle A^re^{\tau A}(\phi_-\cdot\nabla(\overline{\phi}-i\overline{\phi}^\perp)), A^re^{\tau A}\phi_+\Big\rangle \\
&\quad -\Big\langle A^re^{\tau A}(U_+\cdot\nabla(\overline{\phi}+i\overline{\phi}^\perp)), A^re^{\tau A}\phi_-\Big\rangle - \Big\langle A^re^{\tau A}(U_-\cdot\nabla(\overline{\phi}-i\overline{\phi}^\perp)), A^re^{\tau A}\phi_+\Big\rangle \\
&\quad -2e^{i\Omega t}\Big(\Big\langle A^re^{\tau A}(Q_{1,+,+}-Q_{3,+,+}), A^re^{\tau A}\phi_-\Big\rangle + \Big\langle A^re^{\tau A}(Q_{1,+,-}-Q_{3,+,-}), A^re^{\tau A}\phi_+\Big\rangle\Big) \\
&\quad -2e^{-i\Omega t}\Big(\Big\langle A^re^{\tau A}(Q_{1,-,+}-Q_{3,-,+}), A^re^{\tau A}\phi_-\Big\rangle + \Big\langle A^re^{\tau A}(Q_{1,-,-}-Q_{3,-,-}), A^re^{\tau A}\phi_+\Big\rangle\Big) \\
&\quad -2e^{2i\Omega t}\Big\langle A^re^{\tau A}Q_{4,+,-}, A^re^{\tau A}\phi_+\Big\rangle - 2e^{-2i\Omega t}\Big\langle A^re^{\tau A}Q_{4,-,+}, A^re^{\tau A}\phi_-\Big\rangle. \qquad (6.20)
\end{aligned}
$$

There are totally 71 different nonlinear terms in (6.19) and (6.20). We separate them into the following four different types. We use V to denote the velocity field of the limit resonant system, i.e., $\overline{V}$ and $U_\pm$, and use ϕ to denote the velocity filed of the perturbed system, i.e., $\overline{\phi}$ and $\phi_\pm$.

- Type 1: terms that are trilinear in ϕ, e.g., $\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla\overline{\phi}), A^r e^{\tau A}\overline{\phi}\Big\rangle$.
- Type 2: terms that are bilinear in ϕ with no derivative of ϕ, e.g., $\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla\overline{V}), A^r e^{\tau A}\overline{\phi}\Big\rangle$.
- Type 3: terms that are linear in ϕ, e.g., $e^{2i\Omega t}\Big\langle A^r e^{\tau A}(U_+\cdot\nabla U_+), A^r e^{\tau A}\overline{\phi}\Big\rangle$.
- Type 4: terms that are bilinear in ϕ and a derivative of ϕ, e.g., $\Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla\overline{\phi}), A^r e^{\tau A}\overline{\phi}\Big\rangle$.

For type 1 nonlinear terms (19 terms), using Lemma A.1–A.3, and for type 2 nonlinear terms (15 terms), using Lemma 2.2, since $\overline{\phi}$, $\phi_\pm$, $\overline{V}$ and $U_\pm$ all have zero mean value in $\mathbb{T}^3$, we have

$$\begin{aligned}
&\Big|\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla\overline{V}), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| + \Big|\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla\overline{\phi}), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| \\
&+\Big|e^{2i\Omega t}\Big\langle A^r e^{\tau A}\Big(\phi_+\cdot\nabla U_+ + \phi_+\cdot\nabla\phi_+ + (\nabla\cdot U_+)\phi_+ + (\nabla\cdot\phi_+)\phi_+\Big), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| \\
&+\Big|e^{-2i\Omega t}\Big\langle A^r e^{\tau A}\Big(\phi_-\cdot\nabla U_- + \phi_-\cdot\nabla\phi_- + (\nabla\cdot U_-)\phi_- + (\nabla\cdot\phi_-)\phi_-\Big), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| \\
&+2\Big|\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla U_+), A^r e^{\tau A}\phi_-\Big\rangle\Big| + 2\Big|\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla U_-), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&+2\Big|\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla \phi_+), A^r e^{\tau A}\phi_-\Big\rangle\Big| + 2\Big|\Big\langle A^r e^{\tau A}(\overline{\phi}\cdot\nabla \phi_-), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&+\Big|\Big\langle A^r e^{\tau A}\Big(\phi_+\cdot\nabla(\overline{V}+i\overline{V}^\perp)\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big| + \Big|\Big\langle A^r e^{\tau A}\Big(\phi_-\cdot\nabla(\overline{V}-i\overline{V}^\perp)\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&+\Big|\Big\langle A^r e^{\tau A}\Big(\phi_+\cdot\nabla(\overline{\phi}+i\overline{\phi}^\perp)\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big| + \Big|\Big\langle A^r e^{\tau A}\Big(\phi_-\cdot\nabla(\overline{\phi}-i\overline{\phi}^\perp)\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&+2\Big|e^{i\Omega t}\Big\langle A^r e^{\tau A}\Big(\phi_+\cdot\nabla U_+ + \phi_+\cdot\nabla\phi_+ - (\int_0^z \nabla\cdot\phi_+(\boldsymbol{x}',s)ds)\partial_z\phi_+\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&+2\Big|e^{i\Omega t}\Big\langle A^r e^{\tau A}\Big(\phi_+\cdot\nabla U_- + \phi_+\cdot\nabla\phi_- - (\int_0^z \nabla\cdot\phi_+(\boldsymbol{x}',s)ds)\partial_z\phi_-\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&+2\Big|e^{-i\Omega t}\Big\langle A^r e^{\tau A}\Big(\phi_-\cdot\nabla U_+ + \phi_-\cdot\nabla\phi_+ - (\int_0^z \nabla\cdot\phi_-(\boldsymbol{x}',s)ds)\partial_z\phi_+\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&+2\Big|e^{-i\Omega t}\Big\langle A^r e^{\tau A}\Big(\phi_-\cdot\nabla U_- + \phi_-\cdot\nabla\phi_- - (\int_0^z \nabla\cdot\phi_-(\boldsymbol{x}',s)ds)\partial_z\phi_-\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&+\Big|e^{2i\Omega t}\Big\langle A^r e^{\tau A}\Big(\phi_+\cdot\nabla(\overline{V}-i\overline{V}^\perp) + \phi_+\cdot\nabla(\overline{\phi}-i\overline{\phi}^\perp)\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&+\Big|e^{-2i\Omega t}\Big\langle A^r e^{\tau A}\Big(\phi_-\cdot\nabla(\overline{V}+i\overline{V}^\perp) + \phi_-\cdot\nabla(\overline{\phi}+i\overline{\phi}^\perp)\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
\le\; & C_r\Big(\|A^{r+1}e^{\tau A}\overline{V}\| + \|A^{r+1}e^{\tau A}U_+\| + \|A^{r+1}e^{\tau A}U_-\|\Big)\Big(\frac{1}{2}\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_+\|^2 + \|A^r e^{\tau A}\phi_-\|^2\Big) \\
&+C_r\Big(\|A^r e^{\tau A}\overline{\phi}\| + \|A^r e^{\tau A}\phi_+\| + \|A^r e^{\tau A}\phi_-\|\Big)\Big(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2}e^{\tau A}\phi_+\|^2 + \|A^{r+1/2}e^{\tau A}\phi_-\|^2\Big).
\end{aligned} \tag{6.21}$$

For type 3 nonlinear terms (14 terms), when $\Omega \neq 0$, we first explain the idea on the sample term $e^{2i\Omega t}\Big\langle A^r e^{\tau A}(U_+\cdot\nabla U_+), A^r e^{\tau A}\overline{\phi}\Big\rangle$. Indeed, by differentiation by parts, we have

$$e^{2i\Omega t}\Big\langle A^r e^{\tau A}(U_+\cdot\nabla U_+), A^r e^{\tau A}\overline{\phi}\Big\rangle$$

$$= \frac{1}{2i\Omega}\partial_t\Big(e^{2i\Omega t}\Big\langle A^r e^{\tau A}(U_+\cdot\nabla U_+), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big) - \frac{1}{2i\Omega}e^{2i\Omega t}\partial_t\Big(\Big\langle A^r e^{\tau A}(U_+\cdot\nabla U_+), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big). \tag{6.22}$$

We leave the first term until integrating in time. For the second term, we have

$$\begin{aligned}
&-\frac{1}{2i\Omega}e^{2i\Omega t}\partial_t\Big(\Big\langle A^r e^{\tau A}(U_+\cdot\nabla U_+), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big)\\
&\le \frac{1}{|\Omega|}|\dot\tau|\Big|\Big\langle A^{r+1}e^{\tau A}(U_+\cdot\nabla U_+), A^r e^{\tau A}\overline{\phi} >\Big| + \frac{1}{2|\Omega|}\Big|\Big\langle A^r e^{\tau A}\partial_t(U_+\cdot\nabla U_+), A^r e^{\tau A}\overline{\phi} >\Big|\\
&+\frac{1}{2|\Omega|}\Big|\Big\langle A^r e^{\tau A}(U_+\cdot\nabla U_+), A^r e^{\tau A}\partial_t\overline{\phi} >\Big| := I_1 + I_2 + I_3.
\end{aligned} \tag{6.23}$$

Thanks to Cauchy–Schwarz inequality, Lemma 2.2, and Lemma 2.6, since $\overline{\phi}$, $\phi_\pm$, $\overline{V}$ and $U_\pm$ all have zero mean value in $\mathbb{T}^3$, and since $r > 5/2$, from (5.4) and (6.10), we have

$$\begin{aligned}
I_1 &\le \frac{C_r}{|\Omega|}|\dot\tau|\|A^{r+1}e^{\tau A}U_+\|\|A^{r+2}e^{\tau A}U_+\|\|A^r e^{\tau A}\overline{\phi}\|\\
&\le \frac{C_r}{|\Omega|^2}|\dot\tau|^2 + C_r\|A^{r+2}e^{\tau A}U_+\|^4\|A^r e^{\tau A}\overline{\phi}\|^2,
\end{aligned} \tag{6.24}$$

$$\begin{aligned}
I_2 &\le \frac{C}{|\Omega|}\Big(\Big|\Big\langle A^r e^{\tau A}\Big\{\Big(\overline{V}\cdot\nabla U_+ + \frac{1}{2}(U_+\cdot\nabla)(\overline{V}+i\overline{V}^\perp)\Big)\cdot\nabla U_+\Big\}, A^r e^{\tau A}\overline{\phi}\Big\rangle\Big|\\
&\qquad +\Big|\Big\langle A^r e^{\tau A}\Big\{U_+\cdot\nabla\Big(\overline{V}\cdot\nabla U_+ + \frac{1}{2}(U_+\cdot\nabla)(\overline{V}+i\overline{V}^\perp)\Big)\Big\}, A^r e^{\tau A}\overline{\phi}\Big\rangle\Big|\Big)\\
&\le \frac{C_r}{|\Omega|}\|A^{r+2}e^{\tau A}U_+\|^2\|A^{r+2}e^{\tau A}\overline{V}\|\|A^r e^{\tau A}\overline{\phi}\|\\
&\le C_r\|A^{r+2}e^{\tau A}U_+\|^2\|A^{r+2}e^{\tau A}\overline{V}\|^2\|A^r e^{\tau A}\overline{\phi}\|^2 + \frac{C_r}{|\Omega|^2}\|A^{r+2}e^{\tau A}U_+\|^2,
\end{aligned} \tag{6.25}$$

$$\begin{aligned}
I_3 &\le \frac{C}{|\Omega|}\Big|\Big\langle A^r e^{\tau A}\mathbb{P}_h(U_+\cdot\nabla U_+), A^r e^{\tau A}\Big\{\overline{\phi}\cdot\nabla\overline{V} + \overline{\phi}\cdot\nabla\overline{\phi} + \overline{V}\cdot\nabla\overline{\phi}\\
&\qquad +e^{2i\Omega t}P_0\Big(Q_{1,+,+}+Q_{2,+,+}\Big) + e^{-2i\Omega t}P_0\Big(Q_{1,-,-}+Q_{2,-,-}\Big)\Big\}\Big\rangle\Big|\\
&\le \frac{C}{|\Omega|}\Big|\Big\langle A^{r+1}e^{\tau A}\mathbb{P}_h(U_+\cdot\nabla U_+), A^{r-1}e^{\tau A}\Big\{\overline{\phi}\cdot\nabla\overline{V} + \overline{\phi}\cdot\nabla\overline{\phi} + \overline{V}\cdot\nabla\overline{\phi}\\
&\qquad +e^{2i\Omega t}P_0\Big(Q_{1,+,+}+Q_{2,+,+}\Big) + e^{-2i\Omega t}P_0\Big(Q_{1,-,-}+Q_{2,-,-}\Big)\Big\}\Big\rangle\Big|\\
&\le \frac{C_r}{|\Omega|}\|A^{r+2}e^{\tau A}U_+\|^2\Big[\|A^r e^{\tau A}\overline{V}\|^2 + \|A^r e^{\tau A}U_+\|^2 + \|A^r e^{\tau A}U_-\|^2\\
&\qquad +\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_+\|^2 + \|A^r e^{\tau A}\phi_-\|^2\Big].
\end{aligned} \tag{6.26}$$

Applying differentiation by parts to all the type 3 nonlinear terms (14 terms), one obtains

$$\begin{aligned}
&-e^{2i\Omega t}\Big[\Big\langle A^r e^{\tau A}(U_+\cdot\nabla U_+), A^r e^{\tau A}\overline{\phi}\Big\rangle + \Big\langle A^r e^{\tau A}\Big((\nabla\cdot U_+)U_+\Big), A^r e^{\tau A}\overline{\phi}\Big\rangle\\
&\qquad +\Big\langle A^r e^{\tau A}\Big((U_+\cdot\nabla)(\overline{V}-i\overline{V}^\perp)\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big]\\
&-e^{-2i\Omega t}\Big[\Big\langle A^r e^{\tau A}(U_-\cdot\nabla U_-), A^r e^{\tau A}\overline{\phi}\Big\rangle + \Big\langle A^r e^{\tau A}\Big((\nabla\cdot U_-)U_-\Big), A^r e^{\tau A}\overline{\phi}\Big\rangle\\
&\qquad +\Big\langle A^r e^{\tau A}\Big((U_-\cdot\nabla)(\overline{V}+i\overline{V}^\perp)\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big]
\end{aligned}$$

$$
\begin{aligned}
&-2e^{i\Omega t}\Big[\Big\langle A^r e^{\tau A}(U_+\cdot\nabla U_+), A^r e^{\tau A}\phi_-\Big\rangle - \Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla\cdot U_+(\boldsymbol{x}',s)ds)\partial_z U_+\Big), A^r e^{\tau A}\phi_-\Big\rangle\\
&\qquad+\Big\langle A^r e^{\tau A}(U_+\cdot\nabla U_-), A^r e^{\tau A}\phi_+\Big\rangle - \Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla\cdot U_+(\boldsymbol{x}',s)ds)\partial_z U_-\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big]\\
&-2e^{-i\Omega t}\Big[\Big\langle A^r e^{\tau A}(U_-\cdot\nabla U_+), A^r e^{\tau A}\phi_-\Big\rangle - \Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla\cdot U_-(\boldsymbol{x}',s)ds)\partial_z U_+\Big), A^r e^{\tau A}\phi_-\Big\rangle\\
&\qquad+\Big\langle A^r e^{\tau A}(U_-\cdot\nabla U_-), A^r e^{\tau A}\phi_+\Big\rangle - \Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla\cdot U_-(\boldsymbol{x}',s)ds)\partial_z U_-\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big]\\
&= -\frac{1}{2i\Omega}\partial_t\Big\{e^{2i\Omega t}\Big[\Big\langle A^r e^{\tau A}(U_+\cdot\nabla U_+), A^r e^{\tau A}\overline{\phi}\Big\rangle + \Big\langle A^r e^{\tau A}\Big((\nabla\cdot U_+)U_+\Big), A^r e^{\tau A}\overline{\phi}\Big\rangle\\
&\qquad+\Big\langle A^r e^{\tau A}\Big((U_+\cdot\nabla)(\overline{V}-i\overline{V}^{\perp})\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big]\Big\}\\
&\quad+\frac{1}{2i\Omega}\partial_t\Big\{e^{-2i\Omega t}\Big[\Big\langle A^r e^{\tau A}(U_-\cdot\nabla U_-), A^r e^{\tau A}\overline{\phi}\Big\rangle + \Big\langle A^r e^{\tau A}\Big((\nabla\cdot U_-)U_-\Big), A^r e^{\tau A}\overline{\phi}\Big\rangle\\
&\qquad+\Big\langle A^r e^{\tau A}\Big((U_-\cdot\nabla)(\overline{V}+i\overline{V}^{\perp})\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big]\Big\}\\
&\quad-\frac{2}{i\Omega}\partial_t\Big\{e^{i\Omega t}\Big[\Big\langle A^r e^{\tau A}(U_+\cdot\nabla U_+), A^r e^{\tau A}\phi_-\Big\rangle - \Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla\cdot U_+(\boldsymbol{x}',s)ds)\partial_z U_+\Big), A^r e^{\tau A}\phi_-\Big\rangle\\
&\qquad+\Big\langle A^r e^{\tau A}(U_+\cdot\nabla U_-), A^r e^{\tau A}\phi_+\Big\rangle - \Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla\cdot U_+(\boldsymbol{x}',s)ds)\partial_z U_-\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big]\Big\}\\
&\quad+\frac{2}{i\Omega}\partial_t\Big\{e^{-i\Omega t}\Big[\Big\langle A^r e^{\tau A}(U_-\cdot\nabla U_+), A^r e^{\tau A}\phi_-\Big\rangle - \Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla\cdot U_-(\boldsymbol{x}',s)ds)\partial_z U_+\Big), A^r e^{\tau A}\phi_-\Big\rangle\\
&\qquad+\Big\langle A^r e^{\tau A}(U_-\cdot\nabla U_-), A^r e^{\tau A}\phi_+\Big\rangle - \Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla\cdot U_-(\boldsymbol{x}',s)ds)\partial_z U_-\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big]\Big\}\\
&\quad+R =: \partial_t N + R,
\end{aligned}
\tag{6.27}
$$

where R corresponds the remaining terms.

Using the similar estimates for (6.23), thanks to Young's inequality, when $|\Omega|>1$, we have

$$
\begin{aligned}
|R| \le\, & C_r\Big(\|A^{r+2}e^{\tau A}\overline{V}\|^4 + \|A^{r+2}e^{\tau A}U_+\|^4 + \|A^{r+2}e^{\tau A}U_-\|^4 + 1\Big)\\
&\times\Big(\frac{1}{2}\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi\|^2 + \|A^r e^{\tau A}\phi\|^2\Big)\\
&+\frac{C_r}{|\Omega|}\Big(|\dot{\tau}|^2 + \|A^{r+2}e^{\tau A}\overline{V}\|^4 + \|A^{r+2}e^{\tau A}U_+\|^4 + \|A^{r+2}e^{\tau A}U_-\|^4 + 1\Big).
\end{aligned}
\tag{6.28}
$$

For $\partial_t N$, since $\overline{\phi}(0)=\phi_+(0)=\phi_-(0)=0$, using Lemma 2.2, since $\overline{V}$ and $U_\pm$ have zero mean value in $\mathbb{T}^3$, by Young's inequality, we have

$$
\begin{aligned}
\Big|\int_0^t \partial_s N(s)ds\Big| = |N(t)| \le\, & \frac{C_r}{|\Omega|}\Big(\|A^{r+1}e^{\tau A}\overline{V}\|^2 + \|A^{r+1}e^{\tau A}U_+\|^2 + \|A^{r+1}e^{\tau A}U_-\|^2\Big)\\
&\times\Big(\|A^r e^{\tau A}\overline{\phi}\| + \|A^r e^{\tau A}\phi_+\| + \|A^r e^{\tau A}\phi_+\|\Big).
\end{aligned}
\tag{6.29}
$$

The difficulties are on the estimate of type 4 nonlinear terms (23 terms). Thanks to Lemma 4.1, since $\nabla\cdot\overline{V}=0$, we have

$$
\Big|\Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla\overline{\phi}), A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| \le C_r\|A^r e^{\tau A}\overline{V}\|\|A^r e^{\tau A}\overline{\phi}\|^2 + C_r\tau\|A^{r+1/2}e^{\tau A}\overline{V}\|\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2. \tag{6.30}
$$

Thanks to Lemma A.4, by integration by parts, we have

$$\begin{aligned}
&\Big|\Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla\phi_+), A^r e^{\tau A}\phi_-\Big\rangle + \Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla\phi_-), A^r e^{\tau A}\phi_+ > \Big| \\
&\le \Big|\Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla\phi_+), A^r e^{\tau A}\phi_-\Big\rangle - \Big\langle \overline{V}\cdot\nabla A^r e^{\tau A}\phi_+, A^r e^{\tau A}\phi_- > \Big| \\
&\quad + \Big|\Big\langle A^r e^{\tau A}(\overline{V}\cdot\nabla\phi_-), A^r e^{\tau A}\phi_+\Big\rangle - \Big\langle \overline{V}\cdot\nabla A^r e^{\tau A}\phi_-, A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&\quad + \Big|\Big\langle \overline{V}\cdot\nabla A^r e^{\tau A}\phi_+, A^r e^{\tau A}\phi_-\Big\rangle + \Big\langle \overline{V}\cdot\nabla A^r e^{\tau A}\phi_-, A^r e^{\tau A}\phi_+ > \Big| \\
&\le C_r\|A^r e^{\tau A}\overline{V}\|(\|A^r e^{\tau A}\phi_+\|^2 + \|A^r e^{\tau A}\phi_-\|^2) \\
&\quad + C_r\tau\|A^{r+1/2} e^{\tau A}\overline{V}\|(\|A^{r+1/2} e^{\tau A}\phi_+\|^2 + \|A^{r+1/2} e^{\tau A}\phi_-\|^2),
\end{aligned} \tag{6.31}$$

where

$$\Big|\Big\langle \overline{V}\cdot\nabla A^r e^{\tau A}\phi_+, A^r e^{\tau A}\phi_-\Big\rangle + \Big\langle \overline{V}\cdot\nabla A^r e^{\tau A}\phi_-, A^r e^{\tau A}\phi_+\Big\rangle\Big| = 0 \tag{6.32}$$

by integration by parts and $\nabla\cdot\overline{V} = 0$.

Thanks to Lemma A.4 and Lemma A.6, since $r > 5/2$, by integration by parts and by Sobolev inequality, we have

$$\begin{aligned}
&\Big|e^{i\Omega t}\Big\langle A^r e^{\tau A}(U_+\cdot\nabla\phi_+), A^r e^{\tau A}\phi_-\Big\rangle + e^{i\Omega t}\Big\langle A^r e^{\tau A}(U_+\cdot\nabla\phi_-), A^r e^{\tau A}\phi_+\Big\rangle \\
&\quad - e^{i\Omega t}\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla\cdot U_+(\boldsymbol{x}', s)ds)\partial_z\phi_+\Big), A^r e^{\tau A}\phi_-\Big\rangle \\
&\quad - e^{i\Omega t}\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla\cdot U_+(\boldsymbol{x}', s)ds)\partial_z\phi_-\Big), A^r e^{\tau A}\phi_+ > \Big| \\
&\le \Big|\Big\langle A^r e^{\tau A}(U_+\cdot\nabla\phi_+), A^r e^{\tau A}\phi_-\Big\rangle - \Big\langle U_+\cdot\nabla A^r e^{\tau A}\phi_+, A^r e^{\tau A}\phi_- > \Big| \\
&\quad + \Big|\Big\langle A^r e^{\tau A}(U_+\cdot\nabla\phi_-), A^r e^{\tau A}\phi_+\Big\rangle - \Big\langle U_+\cdot\nabla A^r e^{\tau A}\phi_-, A^r e^{\tau A}\phi_+ > \Big| \\
&\quad + \Big|\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla\cdot U_+(\boldsymbol{x}', s)ds)\partial_z\phi_+\Big), A^r e^{\tau A}\phi_-\Big\rangle \\
&\qquad - \Big\langle(\int_0^z \nabla\cdot U_+(\boldsymbol{x}', s)ds) A^r e^{\tau A}\partial_z\phi_+, A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&\quad + \Big|\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla\cdot U_+(\boldsymbol{x}', s)ds)\partial_z\phi_-\Big), A^r e^{\tau A}\phi_+\Big\rangle \\
&\qquad - \Big\langle(\int_0^z \nabla\cdot U_+(\boldsymbol{x}', s)ds) A^r e^{\tau A}\partial_z\phi_-, A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&\quad + \Big|\Big\langle U_+\cdot\nabla A^r e^{\tau A}\phi_+, A^r e^{\tau A}\phi_-\Big\rangle + \Big\langle U_+\cdot\nabla A^r e^{\tau A}\phi_-, A^r e^{\tau A}\phi_+\Big\rangle \\
&\qquad - \Big\langle(\int_0^z \nabla\cdot U_+(\boldsymbol{x}', s)ds) A^r e^{\tau A}\partial_z\phi_+, A^r e^{\tau A}\phi_-\Big\rangle \\
&\qquad - \Big\langle(\int_0^z \nabla\cdot U_+(\boldsymbol{x}', s)ds) A^r e^{\tau A}\partial_z\phi_-, A^r e^{\tau A}\phi_+ > \Big| \\
&\le C_r\|A^{r+1} e^{\tau A}U_+\|(\|A^r e^{\tau A}\phi_+\|^2 + \|A^r e^{\tau A}\phi_-\|^2) \\
&\quad + C_r\tau\|A^{r+3/2} e^{\tau A}U_+\|(\|A^{r+1/2} e^{\tau A}\phi_+\|^2 + \|A^{r+1/2} e^{\tau A}\phi_-\|^2),
\end{aligned} \tag{6.33}$$

where

$$\Big|\Big\langle U_+\cdot\nabla A^r e^{\tau A}\phi_+, A^r e^{\tau A}\phi_-\Big\rangle + \Big\langle U_+\cdot\nabla A^r e^{\tau A}\phi_-, A^r e^{\tau A}\phi_+\Big\rangle$$

$$-\Big\langle (\int_0^z \nabla\cdot U_+(\boldsymbol{x}',s)ds)A^re^{\tau A}\partial_z\phi_+, A^re^{\tau A}\phi_-\Big\rangle$$
$$-\Big\langle (\int_0^z \nabla\cdot U_+(\boldsymbol{x}',s)ds)A^re^{\tau A}\partial_z\phi_-, A^re^{\tau A}\phi_+ > \Big| = 0 \tag{6.34}$$

by integration by parts. Similarly, we have

$$\Big|e^{-i\Omega t}\Big\langle A^re^{\tau A}(U_-\cdot\nabla\phi_+), A^re^{\tau A}\phi_-\Big\rangle + e^{-i\Omega t}\Big\langle A^re^{\tau A}(U_-\cdot\nabla\phi_-), A^re^{\tau A}\phi_+\Big\rangle$$
$$-e^{-i\Omega t}\Big\langle A^re^{\tau A}\Big((\int_0^z \nabla\cdot U_-(\boldsymbol{x}',s)ds)\partial_z\phi_+\Big), A^re^{\tau A}\phi_-\Big\rangle$$
$$-e^{-i\Omega t}\Big\langle A^re^{\tau A}\Big((\int_0^z \nabla\cdot U_-(\boldsymbol{x}',s)ds)\partial_z\phi_-\Big), A^re^{\tau A}\phi_+\Big\rangle\Big|$$
$$\le C_r\|A^{r+1}e^{\tau A}U_-\|(\|A^re^{\tau A}\phi_+\|^2+\|A^re^{\tau A}\phi_-\|^2)$$
$$+C_r\tau\|A^{r+3/2}e^{\tau A}U_-\|(\|A^{r+1/2}e^{\tau A}\phi_+\|^2+\|A^{r+1/2}e^{\tau A}\phi_-\|^2). \tag{6.35}$$

Next, since $-iU_+ = U_+^\perp$, we have

$$\Big|\Big\langle U_+\cdot\nabla A^re^{\tau A}\phi_+, A^re^{\tau A}\overline{\phi}\Big\rangle + \Big\langle (\nabla\cdot A^re^{\tau A}\phi_+)U_+, A^re^{\tau A}\overline{\phi}\Big\rangle$$
$$+\Big\langle U_+\cdot\nabla A^re^{\tau A}(\overline{\phi}-i\overline{\phi}^\perp), A^re^{\tau A}\phi_+\Big\rangle\Big|$$
$$\le \Big|\Big\langle U_+\cdot\nabla A^re^{\tau A}\phi_+, A^re^{\tau A}\overline{\phi}\Big\rangle + \Big\langle U_+\cdot\nabla A^re^{\tau A}\overline{\phi}, A^re^{\tau A}\phi_+\Big\rangle\Big|$$
$$+\Big|\Big\langle (\nabla\cdot A^re^{\tau A}\phi_+)U_+, A^re^{\tau A}\overline{\phi}\Big\rangle + \Big\langle U_+^\perp\cdot\nabla A^re^{\tau A}\overline{\phi}^\perp, A^re^{\tau A}\phi_+ > \Big|$$
$$\le \Big|\Big\langle (\nabla\cdot U_+)A^re^{\tau A}\phi_+, A^re^{\tau A}\overline{\phi}\Big\rangle\Big| + \Big|\Big\langle A^re^{\tau A}\phi_+\cdot\nabla U_+, A^re^{\tau A}\overline{\phi}\Big\rangle\Big|$$
$$+\Big|\Big\langle U_+^\perp\cdot\nabla A^re^{\tau A}\overline{\phi}^\perp, A^re^{\tau A}\phi_+\Big\rangle - \Big\langle A^re^{\tau A}\phi_+\cdot\nabla A^re^{\tau A}\overline{\phi}, U_+\Big\rangle\Big|. \tag{6.36}$$

Notice that

$$\Big|\Big\langle U_+^\perp\cdot\nabla A^re^{\tau A}\overline{\phi}^\perp, A^re^{\tau A}\phi_+\Big\rangle - \Big\langle A^re^{\tau A}\phi_+\cdot\nabla A^re^{\tau A}\overline{\phi}, U_+\Big\rangle\Big|$$
$$= \Big|\Big\langle (\nabla\cdot A^re^{\tau A}\overline{\phi})U_+, A^re^{\tau A}\phi_+\Big\rangle\Big| = 0, \tag{6.37}$$

therefore, by Sobolev inequality and Hölder inequality, and since $r > 5/2$, we have

$$\Big|\Big\langle U_+\cdot\nabla A^re^{\tau A}\phi_+, A^re^{\tau A}\overline{\phi}\Big\rangle + \Big\langle (\nabla\cdot A^re^{\tau A}\phi_+)U_+, A^re^{\tau A}\overline{\phi}\Big\rangle$$
$$+\Big\langle U_+\cdot\nabla A^re^{\tau A}(\overline{\phi}-i\overline{\phi}^\perp), A^re^{\tau A}\phi_+\Big\rangle\Big|$$
$$\le C_r\|\nabla U_+\|_{L^\infty}(\frac{1}{2}\|A^re^{\tau A}\overline{\phi}\|^2+\|A^re^{\tau A}\phi_+\|^2)$$
$$\le C_r\|A^re^{\tau A}U_+\|(\frac{1}{2}\|A^re^{\tau A}\overline{\phi}\|^2+\|A^re^{\tau A}\phi_+\|^2). \tag{6.38}$$

Based on this, thanks to Lemma A.4 and Lemma A.5, we have

$$\Big|e^{2i\Omega t}\Big\langle A^re^{\tau A}\Big(U_+\cdot\nabla\phi_+ + (\nabla\cdot\phi_+)U_+\Big), A^re^{\tau A}\overline{\phi}\Big\rangle$$
$$+e^{2i\Omega t}\Big\langle A^re^{\tau A}\Big(U_+\cdot\nabla(\overline{\phi}-i\overline{\phi}^\perp)\Big), A^re^{\tau A}\phi_+\Big\rangle\Big|$$
$$\le \Big|\Big\langle A^re^{\tau A}\Big(U_+\cdot\nabla\phi_+\Big), A^re^{\tau A}\overline{\phi}\Big\rangle - \Big\langle U_+\cdot\nabla A^re^{\tau A}\phi_+, A^re^{\tau A}\overline{\phi}\Big\rangle\Big|$$

$$
\begin{aligned}
&+\Big|\Big\langle A^r e^{\tau A}\Big((\nabla\cdot\phi_+)U_+\Big), A^r e^{\tau A}\overline{\phi}\Big\rangle - \Big\langle(\nabla\cdot A^r e^{\tau A}\phi_+)U_+, A^r e^{\tau A}\overline{\phi}\Big\rangle\Big| \\
&+\Big|\Big\langle A^r e^{\tau A}\Big(U_+\cdot\nabla\overline{\phi}\Big), A^r e^{\tau A}\phi_+\Big\rangle - \Big\langle U_+\cdot\nabla A^r e^{\tau A}\overline{\phi}, A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&+\Big|\Big\langle A^r e^{\tau A}\Big(U_+\cdot\nabla\overline{\phi}^\perp\Big), A^r e^{\tau A}\phi_+\Big\rangle - \Big\langle U_+\cdot\nabla A^r e^{\tau A}\overline{\phi}^\perp\Big\rangle, A^r e^{\tau A}\phi_+\Big| \\
&+\Big|\Big\langle U_+\cdot\nabla A^r e^{\tau A}\phi_+, A^r e^{\tau A}\overline{\phi}\Big\rangle + \Big\langle(\nabla\cdot A^r e^{\tau A}\phi_+)U_+, A^r e^{\tau A}\overline{\phi}\Big\rangle \\
&\quad+\Big\langle U_+\cdot\nabla A^r e^{\tau A}(\overline{\phi}-i\overline{\phi}^\perp), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&\le C_r\|A^r e^{\tau A}U_+\|(\frac{1}{2}\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_+\|^2) \\
&\quad+C_r\tau\|A^{r+1/2}e^{\tau A}U_+\|(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2}e^{\tau A}\phi_+\|^2). \qquad (6.39)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\Big|e^{-2i\Omega t}\Big\langle A^r e^{\tau A}\Big(U_-\cdot\nabla\phi_- + (\nabla\cdot\phi_-)U_-\Big), A^r e^{\tau A}\overline{\phi}\Big\rangle \\
&\quad+e^{-2i\Omega t}\Big\langle A^r e^{\tau A}\Big(U_-\cdot\nabla(\overline{\phi}+i\overline{\phi}^\perp)\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&\le C_r\|A^r e^{\tau A}U_-\|(\frac{1}{2}\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_-\|^2) \\
&\quad+C_r\tau\|A^{r+1/2}e^{\tau A}U_-\|(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2}e^{\tau A}\phi_-\|^2). \qquad (6.40)
\end{aligned}
$$

For the rest parts in type 4, there is no cancellation as above. First, by Hölder inequality, we have

$$
\begin{aligned}
&\Big|\Big\langle U_+\cdot\nabla A^r e^{\tau A}(\overline{\phi}+i\overline{\phi}^\perp)), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&\le \Big|\Big\langle A^{1/2}U_+\cdot\nabla A^{r-1/2}e^{\tau A}(\overline{\phi}+i\overline{\phi}^\perp)), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&\quad+\Big|\Big\langle U_+\cdot\nabla A^{r-1/2}e^{\tau A}(\overline{\phi}+i\overline{\phi}^\perp)), A^{r+1/2}e^{\tau A}\phi_-\Big\rangle\Big| \\
&\le C_r(\|U_+\|_{L^\infty} + \|A^{1/2}U_+\|_{L^\infty})(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2}e^{\tau A}\phi_-\|^2). \qquad (6.41)
\end{aligned}
$$

Based on this, using Lemma A.4 to A.7, we have

$$
\begin{aligned}
&\Big|\Big\langle A^r e^{\tau A}(U_+\cdot\nabla(\overline{\phi}+i\overline{\phi}^\perp)), A^r e^{\tau A}\phi_- >\Big| \\
&\le \Big|\Big\langle A^r e^{\tau A}(U_+\cdot\nabla(\overline{\phi}+i\overline{\phi}^\perp)), A^r e^{\tau A}\phi_-\Big\rangle - \Big\langle U_+\cdot\nabla A^r e^{\tau A}(\overline{\phi}+i\overline{\phi}^\perp)), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&\quad+\Big|\Big\langle U_+\cdot\nabla A^r e^{\tau A}(\overline{\phi}+i\overline{\phi}^\perp)), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&\le C_r\|A^r e^{\tau A}U_+\|(\frac{1}{2}\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_-\|^2) \\
&\quad+C_r(\tau\|A^{r+1/2}e^{\tau A}U_+\| + \|U_+\|_{L^\infty} + \|A^{1/2}U_+\|_{L^\infty})(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2}e^{\tau A}\phi_-\|^2). \qquad (6.42)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\Big|\Big\langle A^r e^{\tau A}(U_-\cdot\nabla(\overline{\phi}-i\overline{\phi}^\perp)), A^r e^{\tau A}\phi_+ >\Big| \\
&\le C_r\|A^r e^{\tau A}U_-\|((\|A^r e^{\tau A}\overline{\phi}\|^2 + \|A^r e^{\tau A}\phi_+\|^2) \\
&\quad+C_r(\tau\|A^{r+1/2}e^{\tau A}U_-\| + \|U_-\|_{L^\infty} + \|A^{1/2}U_-\|_{L^\infty})(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + \|A^{r+1/2}e^{\tau A}\phi_+\|^2). \qquad (6.43)
\end{aligned}
$$

Next, by Hölder inequality, we have

$$\begin{aligned}
&\Big|\Big\langle(\partial_z U_+)A^r e^{\tau A}(\int_0^z \nabla\cdot\phi_+(\boldsymbol{x'},s)ds), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&\le \Big|\Big\langle(A^{1/2}\partial_z U_+)A^{r-1/2} e^{\tau A}(\int_0^z \nabla\cdot\phi_+(\boldsymbol{x'},s)ds), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&\quad +\Big|\Big\langle(\partial_z U_+)A^{r-1/2} e^{\tau A}(\int_0^z \nabla\cdot\phi_+(\boldsymbol{x'},s)ds), A^{r+1/2} e^{\tau A}\phi_-\Big\rangle\Big| \\
&\le C_r(\|\partial_z U_+\|_{L^\infty}+\|A^{1/2}\partial_z U_+\|_{L^\infty})(\|A^{r+1/2}e^{\tau A}\phi_+\|^2+\|A^{r+1/2}e^{\tau A}\phi_-\|^2).
\end{aligned} \tag{6.44}$$

Based on this, thanks to Lemma A.4 to A.7, we obtain

$$\begin{aligned}
&\Big|\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla\cdot\phi_+(\boldsymbol{x'},s)ds)\partial_z U_+\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&\le \Big|\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla\cdot\phi_+(\boldsymbol{x'},s)ds)\partial_z U_+\Big), A^r e^{\tau A}\phi_-\Big\rangle \\
&\qquad -\Big\langle(\partial_z U_+)A^r e^{\tau A}(\int_0^z \nabla\cdot\phi_+(\boldsymbol{x'},s)ds), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&\qquad +\Big|\Big\langle(\partial_z U_+)A^r e^{\tau A}(\int_0^z \nabla\cdot\phi_+(\boldsymbol{x'},s)ds), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&\le C_r\|A^{r+1}e^{\tau A}U_+\|(\|A^r e^{\tau A}\phi_+\|^2+\|A^r e^{\tau A}\phi_-\|^2) \\
&\quad +C_r(\tau\|A^{r+3/2}e^{\tau A}U_+\|+\|\partial_z U_+\|_{L^\infty}+\|A^{1/2}\partial_z U_+\|_{L^\infty}) \\
&\qquad \times(\|A^{r+1/2}e^{\tau A}\phi_+\|^2+\|A^{r+1/2}e^{\tau A}\phi_-\|^2).
\end{aligned} \tag{6.45}$$

Similarly, we have

$$\begin{aligned}
&\Big|\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla\cdot\phi_+(\boldsymbol{x'},s)ds)\partial_z U_-\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&\quad +\Big|\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla\cdot\phi_-(\boldsymbol{x'},s)ds)\partial_z U_+\Big), A^r e^{\tau A}\phi_-\Big\rangle\Big| \\
&\quad +\Big|\Big\langle A^r e^{\tau A}\Big((\int_0^z \nabla\cdot\phi_-(\boldsymbol{x'},s)ds)\partial_z U_-\Big), A^r e^{\tau A}\phi_+\Big\rangle\Big| \\
&\le C_r(\|A^{r+1}e^{\tau A}U_+\|+\|A^{r+1}e^{\tau A}U_-\|)(\|A^r e^{\tau A}\phi_+\|^2+\|A^r e^{\tau A}\phi_-\|^2) \\
&\quad +C_r\Big(\tau\|A^{r+3/2}e^{\tau A}U_+\|+\tau\|A^{r+3/2}e^{\tau A}U_-\|+\|\partial_z U_+\|_{L^\infty}+\|\partial_z U_-\|_{L^\infty} \\
&\qquad +\|A^{1/2}\partial_z U_+\|_{L^\infty}+\|A^{1/2}\partial_z U_-\|_{L^\infty}\Big)\Big(\|A^{r+1/2}e^{\tau A}\phi_+\|^2+\|A^{r+1/2}e^{\tau A}\phi_-\|^2\Big).
\end{aligned} \tag{6.46}$$

Finally, taking summation of (6.19) and (6.20), and using estimates (6.21)–(6.46) for all the nonlinear terms (71 terms), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{2}\|A^r e^{\tau A}\overline{\phi}\|^2+\|A^r e^{\tau A}\phi_+\|^2+\|A^r e^{\tau A}\phi_-\|^2\Big) \\
&\le \Big[\dot{\tau}+C_r\big(\|A^r e^{\tau A}\overline{\phi}\|+\|A^r e^{\tau A}\phi_+\|+\|A^r e^{\tau A}\phi_+\|\big) \\
&\qquad +C_r\tau\big(\|A^{r+1/2}e^{\tau A}\overline{V}\|+\|A^{r+3/2}e^{\tau A}U_+\|+\|A^{r+3/2}e^{\tau A}U_-\|\big) \\
&\qquad +C_r\big(\|U_+\|_{L^\infty}+\|U_-\|_{L^\infty}+\|\partial_z U_+\|_{L^\infty}+\|\partial_z U_-\|_{L^\infty} \\
&\qquad\quad +\|A^{1/2}U_+\|_{L^\infty}+\|A^{1/2}U_-\|_{L^\infty}+\|A^{1/2}\partial_z U_+\|_{L^\infty}+\|A^{1/2}\partial_z U_-\|_{L^\infty}\big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&\times\Big(\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + 2\|A^{r+1/2}e^{\tau A}\phi_+\|^2 + 2\|A^{r+1/2}e^{\tau A}\phi_-\|^2\Big)\\
&+C_r\Big(\|A^{r+2}e^{\tau A}\overline{V}\|^4 + \|A^{r+2}e^{\tau A}U_+\|^4 + \|A^{r+2}e^{\tau A}U_-\|^4 + 1\Big)\\
&\qquad\times\Big(\frac{1}{2}\|A^{r}e^{\tau A}\overline{\phi}\|^2 + \|A^{r}e^{\tau A}\phi_+\|^2 + \|A^{r}e^{\tau A}\phi_-\|^2\Big)\\
&+\frac{C_r}{|\Omega|}\Big(|\dot{\tau}|^2 + \|A^{r+2}e^{\tau A}\overline{V}\|^4 + \|A^{r+2}e^{\tau A}U_+\|^4 + \|A^{r+2}e^{\tau A}U_-\|^4 + 1\Big) + \partial_t N. \qquad (6.47)
\end{aligned}
$$

Notice that eventually we will set

$$
\begin{aligned}
&\dot{\tau} + C_r\big(\|A^{r}e^{\tau A}\overline{\phi}\| + \|A^{r}e^{\tau A}\phi_+\| + \|A^{r}e^{\tau A}\phi_+\|\big)\\
&\quad+C_r\tau\big(\|A^{r+1/2}e^{\tau A}\overline{V}\| + \|A^{r+3/2}e^{\tau A}U_+\| + \|A^{r+3/2}e^{\tau A}U_-\|\big)\\
&\quad+C_r\big(\|U_+\|_{L^\infty} + \|U_-\|_{L^\infty} + \|\partial_z U_+\|_{L^\infty} + \|\partial_z U_-\|_{L^\infty}\\
&\qquad+\|A^{1/2}U_+\|_{L^\infty} + \|A^{1/2}U_-\|_{L^\infty} + \|A^{1/2}\partial_z U_+\|_{L^\infty} + \|A^{1/2}\partial_z U_-\|_{L^\infty}\big) = 0. \qquad (6.48)
\end{aligned}
$$

Therefore, by Sobolev inequality, Poincare' inequality, and Young's inequality, since $r > 5/2$, $\tau \le \tau_0$, and $U_\pm$ have zero mean value, we have

$$
\begin{aligned}
|\dot{\tau}|^2 &\le C_r\big(\|A^{r}e^{\tau A}\overline{\phi}\|^2 + \|A^{r}e^{\tau A}\phi_+\|^2 + \|A^{r}e^{\tau A}\phi_+\|^2\big)\\
&\quad+C_r(\tau_0^2+1)\big(\|A^{r+1/2}e^{\tau A}\overline{V}\|^2 + \|A^{r+3/2}e^{\tau A}U_+\|^2 + \|A^{r+3/2}e^{\tau A}U_-\|^2\big). \qquad (6.49)
\end{aligned}
$$

By Young's inequality, the term $\frac{|\dot{\tau}|^2}{|\Omega|}$ can be combined with other terms, and we can rewrite (6.47) as

$$
\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{2}\|A^{r}e^{\tau A}\overline{\phi}\|^2 + \|A^{r}e^{\tau A}\phi_+\|^2 + \|A^{r}e^{\tau A}\phi_-\|^2\Big)\\
&\le\Big[\dot{\tau} + C_r\big(\|A^{r}e^{\tau A}\overline{\phi}\| + \|A^{r}e^{\tau A}\phi_+\| + \|A^{r}e^{\tau A}\phi_+\|\big)\\
&\quad+C_r\tau\big(\|A^{r+1/2}e^{\tau A}\overline{V}\| + \|A^{r+3/2}e^{\tau A}U_+\| + \|A^{r+3/2}e^{\tau A}U_-\|\big)\\
&\quad+C_r\big(\|U_+\|_{L^\infty} + \|U_-\|_{L^\infty} + \|\partial_z U_+\|_{L^\infty} + \|\partial_z U_-\|_{L^\infty}\\
&\qquad+\|A^{1/2}U_+\|_{L^\infty} + \|A^{1/2}U_-\|_{L^\infty} + \|A^{1/2}\partial_z U_+\|_{L^\infty} + \|A^{1/2}\partial_z U_-\|_{L^\infty}\big)\Big]\\
&\quad\times\Big[\|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + 2\|A^{r+1/2}e^{\tau A}\phi_+\|^2 + 2\|A^{r+1/2}e^{\tau A}\phi_-\|^2\Big]\\
&\quad+C_r\Big(\|A^{r+2}e^{\tau A}\overline{V}\|^4 + \|A^{r+2}e^{\tau A}U_+\|^4 + \|A^{r+2}e^{\tau A}U_-\|^4 + 1\Big)\\
&\qquad\times\Big(\frac{1}{2}\|A^{r}e^{\tau A}\overline{\phi}\|^2 + \|A^{r}e^{\tau A}\phi_+\|^2 + \|A^{r}e^{\tau A}\phi_-\|^2\Big)\\
&\quad+\frac{C_{r,\tau_0}}{|\Omega|}\Big(\|A^{r+2}e^{\tau A}\overline{V}\|^4 + \|A^{r+2}e^{\tau A}U_+\|^4 + \|A^{r+2}e^{\tau A}U_-\|^4 + 1\Big) + \partial_t N. \qquad (6.50)
\end{aligned}
$$

We set

$$
F := \frac{1}{2}\|A^{r}e^{\tau A}\overline{\phi}\|^2 + \|A^{r}e^{\tau A}\phi_+\|^2 + \|A^{r}e^{\tau A}\phi_-\|^2, \qquad (6.51)
$$

$$
G := \|A^{r+1/2}e^{\tau A}\overline{\phi}\|^2 + 2\|A^{r+1/2}e^{\tau A}\phi_+\|^2 + 2\|A^{r+1/2}e^{\tau A}\phi_-\|^2, \qquad (6.52)
$$

and denote by

$$
K(t) := C_{M,\tau_0}^{\exp(C_r t)}, \qquad \widetilde{K}(t) := e^{K(t)}, \qquad (6.53)
$$

which are double exponential and triple exponential in time. We will follow the rule on the use of notation as indicated in Remark 5. From Proposition 5.1 and thanks to Lemma 2.9, when $\|e^{\tau_0 A}\overline{V}_0\|_{H^{r+3}} +$

$\|e^{\tau_0 A}\widetilde{V}_0\|_{H^{r+2}} \leq M$, we have

$$\|e^{\tau(t)A}\overline{V}(t)\|_{H^{r+3}} \leq K(t) \leq \widetilde{K}_1(t), \qquad \|e^{\tau(t)A}U_\pm(t)\|_{H^{r+2}} \leq \widetilde{K}_1(t), \tag{6.54}$$

provided that $\tau(t)$ satisfies (5.14). Observe that in (6.50), $\|U_\pm\|_{L^\infty}$, $\|A^{1/2}U_\pm\|_{L^\infty}$, $\|\partial_z U_\pm\|_{L^\infty}$, and $\|A^{1/2}\partial_z U_\pm\|_{L^\infty}$ are the terms force the smallness assumption on Sobolev norm of the baroclinic mode. For $\delta > 0$, by Proposition 5.1 and Lemma 2.9, thanks to Sobolev inequality, when $\|\widetilde{V}_0\|_{H^{3+\delta}} = \|\widetilde{v}_0\|_{H^{3+\delta}} \leq \frac{1}{|\Omega_0|}$, we have

$$\|U_\pm\|_{L^\infty} + \|A^{1/2}U_\pm\|_{L^\infty} + \|\partial_z U_\pm\|_{L^\infty} + \|A^{1/2}\partial_z U_\pm\|_{L^\infty} \leq C\|\widetilde{V}\|_{H^{3+\delta}} \leq \frac{C\widetilde{K}_1(t)}{|\Omega_0|}. \tag{6.55}$$

Since $|\Omega| \geq |\Omega_0|$, we can rewrite (6.50) as

$$\frac{dF}{dt} \leq (\dot{\tau} + C_r F^{1/2} + \tau\widetilde{K}_2 + \frac{\widetilde{K}_2}{|\Omega_0|})G + \widetilde{K}_2 F + \frac{\widetilde{K}_2}{|\Omega_0|} + \partial_t N. \tag{6.56}$$

By setting $\dot{\tau} + C_r F^{1/2} + \tau\widetilde{K}_2 + \frac{\widetilde{K}_2}{|\Omega_0|} = 0$, we have

$$\frac{dF}{dt} \leq \widetilde{K}_2 F + \frac{\widetilde{K}_2}{|\Omega_0|} + \partial_t N. \tag{6.57}$$

By Grönwall inequality, we have

$$\frac{d}{dt}(Fe^{-\int_0^t \widetilde{K}_2(s)ds}) \leq \frac{\widetilde{K}_2}{|\Omega_0|} + (\partial_t N)e^{-\int_0^t \widetilde{K}_2(s)ds}. \tag{6.58}$$

Integrating from 0 to t, noticing that $F(0) = 0$, we have

$$F(t)e^{-\int_0^t \widetilde{K}_2(s)ds} \leq \frac{1}{|\Omega_0|}\int_0^t \widetilde{K}_2(s)ds + \int_0^t (\partial_s N(s))e^{-\int_0^s \widetilde{K}_2(\xi)d\xi}ds. \tag{6.59}$$

From (6.29), we know $|N(t)| \leq \frac{1}{|\Omega_0|}\widetilde{K}_3(t)F^{1/2}$. Moreover, $\frac{1}{|\Omega_0|}\widetilde{K}_3(t)F^{1/2}$ is increasing in time. By integration by parts in time, thanks to Cauchy–Schwarz inequality, since $N(0) = 0$, we have

$$\begin{aligned}&\int_0^t (\partial_s N(s))e^{-\int_0^s \widetilde{K}_2(\xi)d\xi}ds \leq |N(t)| + \int_0^t |N(s)||\partial_s e^{-\int_0^s \widetilde{K}_2(\xi)d\xi}|ds\\ &\leq \frac{1}{|\Omega_0|}\widetilde{K}_3 F^{1/2} + \frac{t}{|\Omega_0|}\widetilde{K}_3 F^{1/2}\widetilde{K}_2 \leq \frac{1}{|\Omega_0|}\widetilde{K}_4 + \frac{1}{|\Omega_0|}F.\end{aligned} \tag{6.60}$$

Thus, we have

$$F(t) \leq \frac{1}{|\Omega_0|}e^{\widetilde{K}_5(t)} + \frac{1}{|\Omega_0|}e^{\widetilde{K}_5(t)}F(t), \tag{6.61}$$

which is equivalent to

$$F(t) \leq \frac{e^{\widetilde{K}_5(t)}}{|\Omega_0| - e^{\widetilde{K}_5(t)}}. \tag{6.62}$$

Plugging this back to $\dot{\tau} + C_r F^{1/2} + \tau\widetilde{K}_2 + \frac{\widetilde{K}_2}{|\Omega_0|} = 0$, we can require that

$$\dot{\tau} + \frac{e^{\widetilde{K}_6(t)}}{\sqrt{|\Omega_0| - e^{\widetilde{K}_6(t)}}} + \tau\widetilde{K}_6 + \frac{1}{|\Omega_0|}\widetilde{K}_6 \leq 0. \tag{6.63}$$

By Gronwall inequality, we can require

$$\frac{d}{dt}(\tau e^{\int_0^t \widetilde{K}_6(s)ds}) \leq \frac{-e^{\widetilde{K}_7(t)}}{\sqrt{|\Omega_0| - e^{\widetilde{K}_6(t)}}} - \frac{e^{\widetilde{K}_7(t)}}{|\Omega_0|}. \tag{6.64}$$

Integrating from 0 to t, for some suitable function $\widetilde{K}_0(t)$, we can require

$$\tau(t) = (\tau_0 - \int_0^t \frac{e^{\widetilde{K}_0(s)}}{\sqrt{|\Omega_0| - e^{\widetilde{K}_0(s)}}} ds - \int_0^t \frac{e^{\widetilde{K}_0(s)}}{|\Omega_0|} ds) e^{-\int_0^t \widetilde{K}_0(s)ds}. \tag{6.65}$$

Notice that τ in (6.65) also satisfies (5.14) when $\widetilde{K}_0(t)$ is chosen suitably. In order to have $\tau(t) > 0$, we just need to require that

$$\tau_0 \geq \frac{3e^{\widetilde{K}_8(t)}}{\sqrt{|\Omega_0| - e^{\widetilde{K}_8(t)}}} \text{ and } \tau_0 \geq \frac{3e^{\widetilde{K}_8(t)}}{|\Omega_0|} \tag{6.66}$$

for some suitable function $\widetilde{K}_8(t) > \widetilde{K}_0(t)$.

For some new $\widetilde{K}(t) > \widetilde{K}_8(t)$ and the given Ω_0, let $\mathcal{T}$ satisfy

$$C_{\tau_0} e^{\widetilde{K}(\mathcal{T})} = |\Omega_0|, \tag{6.67}$$

then the two conditions in (6.66) are satisfied on $t \in [0, \mathcal{T}]$. Thus, $\tau(t) > 0$ on $t \in [0, \mathcal{T}]$. From (6.67), we know that $e^{\widetilde{K}(\mathcal{T})} \geq \frac{|\Omega_0|}{2C_{\tau_0}}$, and thus the time $\mathcal{T}$ satisfies

$$\mathcal{T} \gtrsim \ln(\ln(\ln(\ln|\Omega_0|))) \to \infty, \tag{6.68}$$

as $|\Omega_0| \to \infty$.

When $\widetilde{K}(t)$ is chosen suitably, from (6.62), we know

$$\|A^r e^{\tau(t)A}\overline{\phi}(t)\|^2 + \|A^r e^{\tau(t)A}\phi_+(t)\|^2 + \|A^r e^{\tau(t)A}\phi_-(t)\|^2 \leq \frac{e^{\widetilde{K}(t)}}{|\Omega_0| - e^{\widetilde{K}(t)}} < \infty \tag{6.69}$$

on $t \in [0, \mathcal{T}]$. Since $\overline{\phi}$ and $\phi_\pm$ have zero mean value in $\mathbb{T}^3$, by Poincaré inequality, the L^2 norm can be bounded by the higher order norm. Therefore, we have

$$\|e^{\tau(t)A}\overline{\phi}(t)\|_{H^r}^2 + \|e^{\tau(t)A}\phi_+(t)\|_{H^r}^2 + \|e^{\tau(t)A}\phi_-(t)\|_{H^r}^2 \leq \frac{2e^{\widetilde{K}(t)}}{|\Omega_0| - e^{\widetilde{K}(t)}} < \infty \tag{6.70}$$

on $t \in [0, \mathcal{T}]$. Since $\tau(t)$ satisfies (5.14), we know that

$$\|e^{\tau(t)A}\overline{V}(t)\|_{H^r}^2 + \|e^{\tau(t)A}U_+(t)\|_{H^r}^2 + \|e^{\tau(t)A}U_-(t)\|_{H^r}^2 < \infty \tag{6.71}$$

on $t \in [0, \mathcal{T}]$. Since $\overline{v} = \overline{\phi} + \overline{V}$ and $\widetilde{u}_\pm = \widetilde{\phi}_\pm + \widetilde{U}_\pm$, by triangle inequality, thanks to Lemma 2.9, we have

$$\|e^{\tau(t)A}\overline{v}(t)\|_{H^r}^2 + \|e^{\tau(t)A}\widetilde{v}(t)\|_{H^r}^2 < \infty \tag{6.72}$$

on $t \in [0, \mathcal{T}]$. Therefore, we obtain

$$(\overline{v}, \widetilde{v}) \in L^\infty(0, \mathcal{T}; \mathcal{D}(e^{\tau(t)A} : H^r(\mathbb{T}^3))). \tag{6.73}$$

This completes the proof of Theorem 6.1.

6.4. **Approximation by the limit resonant system.** As a consequence of the proof of Theorem 6.1, the following theorem describes the approximation of the solution to the original system (3.1)–(3.4) by the solution to the limit resonant system (5.10)–(5.12) in the space of analytic functions, for large rotation rate $|\Omega|$ and small initial baroclinic mode in Sobolev norm.

Theorem 6.3. *Suppose the conditions in Theorem 6.1 hold, and let* $(\overline{V}, \widetilde{V})$ *be the solution to system (5.10)–(5.12) with initial data* $(\overline{v}_0, \widetilde{v}_0)$*. Denote by* $\overline{\phi} = \overline{v} - \overline{V}$ *and* $\widetilde{\phi} = \widetilde{v} - \widetilde{V}$*, then, for* $|\Omega| \geq |\Omega_0|$*, one has*

$$\|e^{\tau(t)A}\overline{\phi}(t)\|_{H^r} + \|e^{\tau(t)A}\widetilde{\phi}(t)\|_{H^r} \lesssim \frac{e^{\widetilde{K}(t)}}{|\Omega_0| - e^{\widetilde{K}(t)}}, \tag{6.74}$$

for $t \in [0, \mathcal{T}]$ *with* $\mathcal{T}$ *given by (6.1) and* $\tau(t)$ *given by* (6.2)*.*

Proof. The proof is an immediate consequence of (6.70). □

6.5. **Remarks and discussions.**

Remark 8. To emphasize the difference between smallness in analytic norm and in Sobolev norm, for $|\Omega| \gg 1$, consider

$$\widetilde{v}_0 = c_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \quad k_3 \neq 0, \tag{6.75}$$

with $|\boldsymbol{k}| = \lceil \tau_0^{-1} \ln|\Omega| \rceil$ and $|c_{\boldsymbol{k}}| = (\ln|\Omega|)^{-r-2}|\Omega|^{-1}$. When $0 < \delta < 1$, since $r > 5/2$, we have $\|\widetilde{v}_0\|_{H^{3+\delta}} \leq \|\widetilde{v}_0\|_{H^{r+2}} \sim |\Omega|^{-1}$, $\|e^{\tau_0 A}\widetilde{v}_0\|_{H^{r+2}} \sim 1$. Therefore, one can construct a sequence of initial data

$$\{(\widetilde{v}_0)_\Omega\} = c_{\boldsymbol{k}(\Omega)} e^{i\boldsymbol{k}(\Omega)\cdot\boldsymbol{x}}, \tag{6.76}$$

where $|\boldsymbol{k}(\Omega)| = \lceil \tau_0^{-1} \ln|\Omega| \rceil$ and $|c_{\boldsymbol{k}(\Omega)}| = (\ln|\Omega|)^{-r-2}|\Omega|^{-1}$. Then as $|\Omega| \to \infty$, the existence time of solutions $\mathcal{T} \to \infty$, with initial condition $\|e^{\tau_0 A}(\widetilde{v}_0)_\Omega\|_{H^{r+2}} \sim 1$. This result needs fast rotation, and is very different from Theorem 4.3.

Remark 9. In estimate (6.42) we have the resonance term

$$(U_+ \cdot \nabla)(\overline{\phi} + i\overline{\phi}^\perp) = (U_+ \cdot \nabla\overline{\phi} - U_+^\perp \cdot \nabla\overline{\phi}^\perp) = U_+^\perp(\nabla^\perp \cdot \overline{\phi}), \tag{6.77}$$

which involves the vorticity $\nabla^\perp \cdot \overline{\phi}$. Notice that in the limit resonant system (5.10)–(5.11), the evolution of the barotropic mode $\overline{V}$ is independent of the baroclinic mode $\widetilde{V}$, and therefore we can control the vorticity $\nabla^\perp \cdot \overline{V}$. However, for the original system (3.1)–(3.2) (or the perturbed system (6.10)–(6.12)), the evolution of the barotropic mode $\overline{v}$ (or $\overline{\phi}$) depends also on the baroclinic mode $\widetilde{v}$ (or $\phi_\pm$). Therefore, we are unable to control (6.77) without the smallness condition on the initial baroclinic mode.

Remark 10. In estimate (6.45), we have the term

$$e^{i\Omega t}\Big(\int_0^z \nabla \cdot \phi_+(\boldsymbol{x}', s)ds\Big)\partial_z U_+. \tag{6.78}$$

Despite the oscillation, we are unable apply similar methods as in type 3 due to the loss of derivative on the baroclinic mode. For this term, we do not have cancellation as other terms in type 4. Therefore, we are forced to require the smallness condition on the initial baroclinic mode.

Appendix A. Estimates of nonlinear terms

In this appendix, we list the estimates of nonlinear terms in the space of analytic functions. Lemma A.1–A.3 will be used to prove the local well-posedness.

First, we estimate nonlinear terms of the form $f \cdot \nabla g$.

Lemma A.1. *For $f, g, h \in \mathcal{D}(e^{\tau A} : H^{r+1/2})$, where $r > 2$ and $\tau \geq 0$, one has*

$$\left|\left\langle A^r e^{\tau A}(f \cdot \nabla g), A^r e^{\tau A} h\right\rangle\right| \leq C_r \Big[(\|A^r e^{\tau A} f\| + |\hat{f}_0|)\|A^{r+1/2} e^{\tau A} g\|\|A^{r+1/2} e^{\tau A} h\| + \|A^{r+1/2} e^{\tau A} f\|\|A^r e^{\tau A} g\|\|A^r e^{\tau A} h\|\Big]. \tag{A.1}$$

Proof. First, notice that $\left|\left\langle A^r e^{\tau A}(f \cdot \nabla g), A^r e^{\tau A} h\right\rangle\right| = \left|\left\langle f \cdot \nabla g, A^r e^{\tau A} H\right\rangle\right|$, where $H = A^r e^{\tau A} h$. We use Fourier representation of f, g and H, in which we can write

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{j} \in \mathbb{Z}^3} \hat{f}_{\boldsymbol{j}} e^{2\pi i \boldsymbol{j} \cdot \boldsymbol{x}}, \tag{A.2}$$

$$g(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^3} \hat{g}_{\boldsymbol{k}} e^{2\pi i \boldsymbol{k} \cdot \boldsymbol{x}}, \tag{A.3}$$

$$A^r e^{\tau A} H(\boldsymbol{x}) = \sum_{\boldsymbol{l} \in \mathbb{Z}^3} |\boldsymbol{l}|^r e^{\tau |\boldsymbol{l}|} \hat{H}_{\boldsymbol{l}} e^{2\pi i \boldsymbol{l} \cdot \boldsymbol{x}}. \tag{A.4}$$

Therefore,

$$\left|\left\langle f \cdot \nabla g, A^r e^{\tau A} H\right\rangle\right| \leq \sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0} |\hat{f}_{\boldsymbol{j}}||\boldsymbol{k}||\hat{g}_{\boldsymbol{k}}||\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|}|\hat{H}_{\boldsymbol{l}}|. \tag{A.5}$$

From $|\boldsymbol{l}| = |\boldsymbol{j} + \boldsymbol{k}| \leq |\boldsymbol{j}| + |\boldsymbol{k}|$ we have the following inequalities:

$$|\boldsymbol{l}|^r \leq (|\boldsymbol{j}| + |\boldsymbol{k}|)^r \leq C_r(|\boldsymbol{j}|^r + |\boldsymbol{k}|^r), \quad e^{\tau|\boldsymbol{l}|} \leq e^{\tau|\boldsymbol{j}|} e^{\tau|\boldsymbol{k}|}. \tag{A.6}$$

Applying these inequalities, we have

$$\left|\left\langle f \cdot \nabla g, A^r e^{\tau A} H\right\rangle\right| \leq \sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0} C_r |\hat{f}_{\boldsymbol{j}}||\boldsymbol{k}||\hat{g}_{\boldsymbol{k}}|(|\boldsymbol{j}|^r + |\boldsymbol{k}|^r) e^{\tau|\boldsymbol{j}|} e^{\tau|\boldsymbol{k}|} |\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|} |\hat{h}_{\boldsymbol{l}}|. \tag{A.7}$$

Since $|\boldsymbol{k}|, |\boldsymbol{j}|, |\boldsymbol{l}|$ are all nonnegative, we have $|\boldsymbol{k}|^{1/2} \leq (|\boldsymbol{j}| + |\boldsymbol{l}|)^{1/2} \leq |\boldsymbol{j}|^{1/2} + |\boldsymbol{l}|^{1/2}$, therefore,

$$\begin{aligned}
\left|\left\langle f \cdot \nabla g, A^r e^{\tau A} H\right\rangle\right| &\leq \sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0} C_r |\hat{f}_{\boldsymbol{j}}||\boldsymbol{k}|^{1/2}(|\boldsymbol{j}|^{1/2} + |\boldsymbol{l}|^{1/2})|\hat{g}_{\boldsymbol{k}}|(|\boldsymbol{j}|^r + |\boldsymbol{k}|^r) e^{\tau|\boldsymbol{j}|} e^{\tau|\boldsymbol{k}|} |\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|} |\hat{h}_{\boldsymbol{l}}| \\
\leq \sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0} C_r \Big(&|\boldsymbol{k}|^{1/2}|\boldsymbol{j}|^{r+1/2}|\boldsymbol{l}|^r + |\boldsymbol{k}|^{r+1/2}|\boldsymbol{j}|^{1/2}|\boldsymbol{l}|^r + |\boldsymbol{k}|^{1/2}|\boldsymbol{j}|^r|\boldsymbol{l}|^{r+1/2} + |\boldsymbol{k}|^{r+1/2}|\boldsymbol{l}|^{r+1/2}\Big) \\
&\times e^{\tau|\boldsymbol{j}|} e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{l}|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}| =: A_1 + A_2 + A_3 + A_4.
\end{aligned} \tag{A.8}$$

Thanks to Cauchy–Schwarz inequality, since $r > 2$, we have

$$\begin{aligned}
A_1 &= \sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0} C_r |\boldsymbol{k}|^{1/2}|\boldsymbol{j}|^{r+1/2}|\boldsymbol{l}|^r e^{\tau|\boldsymbol{j}|} e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{l}|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}| \\
&= C_r \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}^3 \\ \boldsymbol{k} \neq 0}} |\boldsymbol{k}|^{1/2}|\hat{g}_{\boldsymbol{k}}| e^{\tau|\boldsymbol{k}|} \sum_{\substack{\boldsymbol{j} \in \mathbb{Z}^3 \\ \boldsymbol{j} \neq 0, -\boldsymbol{k}}} |\boldsymbol{j}|^{r+1/2} e^{\tau|\boldsymbol{j}|} |\hat{f}_{\boldsymbol{j}}||\boldsymbol{j} + \boldsymbol{k}|^r e^{\tau|\boldsymbol{j}+\boldsymbol{k}|} |\hat{h}_{-\boldsymbol{j}-\boldsymbol{k}}| \\
&\leq C_r \Big(\sum_{\substack{\boldsymbol{k} \in \mathbb{Z}^3 \\ \boldsymbol{k} \neq 0}} |\boldsymbol{k}|^{1-2r}\Big)^{1/2} \Big(\sum_{\substack{\boldsymbol{k} \in \mathbb{Z}^3 \\ \boldsymbol{k} \neq 0}} |\boldsymbol{k}|^{2r} e^{2\tau|\boldsymbol{k}|} |\hat{g}_{\boldsymbol{k}}|^2\Big)^{1/2} \\
&\quad \times \sup_{\boldsymbol{k} \in \mathbb{Z}^3} \Big(\sum_{\substack{\boldsymbol{j} \in \mathbb{Z}^3 \\ \boldsymbol{j} \neq 0, -\boldsymbol{k}}} |\boldsymbol{j}|^{2r+1} e^{2\tau|\boldsymbol{j}|} |\hat{f}_{\boldsymbol{j}}|^2\Big)^{1/2} \Big(\sum_{\substack{\boldsymbol{j} \in \mathbb{Z}^3 \\ \boldsymbol{j} \neq 0, -\boldsymbol{k}}} |\boldsymbol{j} + \boldsymbol{k}|^{2r} e^{2\tau|\boldsymbol{j}+\boldsymbol{k}|} |\hat{h}_{-\boldsymbol{j}-\boldsymbol{k}}|^2\Big)^{1/2}
\end{aligned}$$

$$\leq C_r\|A^{r+1/2}e^{\tau A}f\|\|A^r e^{\tau A}g\|\|A^r e^{\tau A}h\|, \tag{A.9}$$

Similarly, we have

$$\begin{aligned} A_2 = &\sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0} C_r|\boldsymbol{k}|^{r+1/2}|\boldsymbol{j}|^{1/2}|\boldsymbol{l}|^r e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{l}|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}| \\ &\leq C_r\|A^r e^{\tau A}f\|\|A^{r+1/2}e^{\tau A}g\|\|A^r e^{\tau A}h\|, \end{aligned} \tag{A.10}$$

and

$$\begin{aligned} A_3 = &\sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0} C_r|\boldsymbol{k}|^{1/2}|\boldsymbol{j}|^{r}|\boldsymbol{l}|^{r+1/2} e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{l}|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}| \\ &\leq C_r\|A^r e^{\tau A}f\|\|A^r e^{\tau A}g\|\|A^{r+1/2}e^{\tau A}h\|. \end{aligned} \tag{A.11}$$

For A_4, thanks to Cauchy–Schwarz inequality, since $r > 2$, we have

$$\begin{aligned} A_4 &= \sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0} C_r|\boldsymbol{k}|^{r+1/2}|\boldsymbol{l}|^{r+1/2}e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{l}|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}| \\ &= C_r\sum_{\boldsymbol{j}\in\mathbb{Z}^3} e^{\tau|\boldsymbol{j}|}|\hat{f}_{\boldsymbol{j}}| \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ \boldsymbol{k}\neq 0,-\boldsymbol{j}}} |\boldsymbol{k}|^{r+1/2}|\hat{g}_{\boldsymbol{k}}|e^{\tau|\boldsymbol{k}|}|\boldsymbol{j}+\boldsymbol{k}|^{r+1/2}e^{\tau|\boldsymbol{j}+\boldsymbol{k}|}|\hat{h}_{-\boldsymbol{j}-\boldsymbol{k}}| \\ &\leq C_r\Big\{|\hat{f}_0| + \Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ \boldsymbol{j}\neq 0}}|\boldsymbol{j}|^{-2r}\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ \boldsymbol{j}\neq 0}}|\boldsymbol{j}|^{2r}e^{2\tau|\boldsymbol{j}|}|\hat{f}_{\boldsymbol{j}}|^2\Big)^{1/2}\Big\} \\ &\quad\times\sup_{\boldsymbol{j}\in\mathbb{Z}^3}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ \boldsymbol{k}\neq 0,-\boldsymbol{j}}}|\boldsymbol{k}|^{2r+1}e^{2\tau|\boldsymbol{k}|}|\hat{g}_{\boldsymbol{k}}|^2\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ \boldsymbol{k}\neq 0,-\boldsymbol{j}}}|\boldsymbol{j}+\boldsymbol{k}|^{2r+1}e^{2\tau|\boldsymbol{j}+\boldsymbol{k}|}|\hat{h}_{-\boldsymbol{j}-\boldsymbol{k}}|^2\Big)^{1/2} \\ &\leq C_r(\|A^r e^{\tau A}f\| + |\hat{f}_0|)\|A^{r+1/2}e^{\tau A}g\|\|A^{r+1/2}e^{\tau A}h\|. \end{aligned} \tag{A.12}$$

Combine the estimates for A_1 to A_4, and since $\|A^r e^{\tau A}g\| \leq \|A^{r+1/2}e^{\tau A}g\|$, $\|A^r e^{\tau A}h\| \leq \|A^{r+1/2}e^{\tau A}h\|$, we achieve the desired inequality. □

Similarly, we estimate $(\nabla\cdot f)g$ in the following:

Lemma A.2. *For $f, g, h \in \mathcal{D}(e^{\tau A} : H^{r+1/2})$, where $r > 2$ and $\tau \geq 0$, one has*

$$\begin{aligned} \Big|\Big\langle A^r e^{\tau A}\big((\nabla\cdot f)g\big), A^r e^{\tau A}h\Big\rangle\Big| \leq &C_r\Big[(\|A^r e^{\tau A}g\| + |\hat{g}_0|)\|A^{r+1/2}e^{\tau A}f\|\|A^{r+1/2}e^{\tau A}h\| \\ &+ \|A^{r+1/2}e^{\tau A}g\|\|A^r e^{\tau A}f\|\|A^r e^{\tau A}h\|\Big]. \end{aligned} \tag{A.13}$$

The proof of Lemma A.2 is almost the same as Lemma A.1, so we omit it.

Finally, we provide an estimate for $(\int_0^z \nabla\cdot f(\boldsymbol{x}', s)ds)\partial_z g$ in the following:

Lemma A.3. *For $f, g, h \in \mathcal{D}(e^{\tau A} : H^{r+1/2})$, where $r > 2$, $\tau \geq 0$, and $\overline{f} = 0$, one has*

$$\begin{aligned} \Big|\Big\langle A^r e^{\tau A}\big((\int_0^z \nabla\cdot f(\boldsymbol{x}', s)ds)\partial_z g\big), A^r e^{\tau A}h\Big\rangle\Big| \leq C_r\Big(&\|A^r e^{\tau A}f\|\|A^{r+1/2}e^{\tau A}g\|\|A^{r+1/2}e^{\tau A}h\| \\ +\|A^r e^{\tau A}g\|\|A^{r+1/2}e^{\tau A}f\|\|A^{r+1/2}e^{\tau A}h\| &+ \|A^r e^{\tau A}h\|\|A^{r+1/2}e^{\tau A}f\|\|A^{r+1/2}e^{\tau A}g\|\Big). \end{aligned} \tag{A.14}$$

Proof. First, $\left|\left\langle A^r e^{\tau A}\big((\int_0^z \nabla\cdot f(\boldsymbol{x}',s)ds)\partial_z g\big), A^r e^{\tau A} h\right\rangle\right| = \left|\left\langle (\int_0^z \nabla\cdot f(\boldsymbol{x}',s)ds)\partial_z g, A^r e^{\tau A} H\right\rangle\right|$. Using Fourier representation of f, and noticing that $\overline{f}=0$, we have

$$f(\boldsymbol{x}) = \sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3\neq 0}} \hat{f}_{\boldsymbol{j}} e^{2\pi(i\boldsymbol{j}'\cdot\boldsymbol{x}'+ij_3z)}, \tag{A.15}$$

where $\boldsymbol{j}'=(j_1,j_2)$. Then we have

$$\int_0^z \nabla\cdot f(\boldsymbol{x}',s)ds = \sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3\neq 0,\boldsymbol{j}'\neq 0}} \frac{j_1+j_2}{j_3}\hat{f}_{\boldsymbol{j}} e^{2\pi(i\boldsymbol{j}'\cdot\boldsymbol{x}'+ij_3z)} - \sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3\neq 0,\boldsymbol{j}'\neq 0}} \frac{j_1+j_2}{j_3}\hat{f}_{\boldsymbol{j}} e^{2\pi i\boldsymbol{j}'\cdot\boldsymbol{x}'}. \tag{A.16}$$

Therefore, we have

$$\begin{aligned}\left|\left\langle (\int_0^z \nabla\cdot f(s)ds)\partial_z g, A^r e^{\tau A} H\right\rangle\right| &\le \left|\left\langle (\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3\neq 0,\boldsymbol{j}'\neq 0}} \frac{j_1+j_2}{j_3}\hat{f}_{\boldsymbol{j}} e^{2\pi(i\boldsymbol{j}'\cdot\boldsymbol{x}'+ij_3z)})\partial_z g, A^r e^{\tau A} H\right\rangle\right| \\ &\quad + \left|\left\langle (\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3\neq 0,\boldsymbol{j}'\neq 0}} \frac{j_1+j_2}{j_3}\hat{f}_{\boldsymbol{j}} e^{i\boldsymbol{j}'\cdot\boldsymbol{x}'})\partial_z g, A^r e^{\tau A} H\right\rangle\right| =: I_1 + I_2.\end{aligned} \tag{A.17}$$

Let us estimate I_2 first. For $\boldsymbol{l}=(\boldsymbol{l}',l_3)=(-\boldsymbol{j}'-\boldsymbol{k}',-k_3)$, by using the inequalities

$$|\boldsymbol{j}'|^{1/2} \le C(|\boldsymbol{k}|^{1/2}+|\boldsymbol{l}|^{1/2}), \quad |\boldsymbol{k}|^{1/2} \le C(|\boldsymbol{j}'|^{1/2}+|\boldsymbol{l}|^{1/2}), \quad |\boldsymbol{l}|^r \le C_r(|\boldsymbol{j}'|^r+|\boldsymbol{k}|^r), \tag{A.18}$$

we have

$$\begin{aligned} I_2 &\le \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r \frac{1}{|j_3|}|\boldsymbol{j}'||k_3||\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}|(|\boldsymbol{j}'|^r+|\boldsymbol{k}|^r)e^{\tau|\boldsymbol{j}'|}e^{\tau|\boldsymbol{k}|}|\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|}|\hat{h}_{\boldsymbol{l}}| \\ &\le \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}|(|\boldsymbol{j}'|^{r+1}|\boldsymbol{k}|+|\boldsymbol{j}'||\boldsymbol{k}|^{r+1})e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}|\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|}|\hat{h}_{\boldsymbol{l}}| \\ &\le \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r \frac{1}{|j_3|}\Big(|\boldsymbol{k}|^{3/2}|\boldsymbol{j}'|^{r+1/2}|\boldsymbol{l}|^r + |\boldsymbol{k}||\boldsymbol{j}'|^{r+1/2}|\boldsymbol{l}|^{r+1/2} + |\boldsymbol{j}'|^{3/2}|\boldsymbol{k}|^{r+1/2}|\boldsymbol{l}|^r \\ &\qquad + |\boldsymbol{j}'||\boldsymbol{k}|^{r+1/2}|\boldsymbol{l}|^{r+1/2}\Big)e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{l}|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}| =: B_1+B_2+B_3+B_4. \end{aligned} \tag{A.19}$$

When $k_3\neq 0$ and $r>2$, we know that $|\boldsymbol{k}|^{1-r} \le |(\boldsymbol{k}',\pm 1)|^{1-r}$ and $\sum_{\boldsymbol{k}'\in\mathbb{Z}^2} |(\boldsymbol{k}',\pm1)|^{2-2r} \le C_r$ is finite. Thanks to Cauchy–Schwarz inequality, we have

$$\begin{aligned} B_1 &= \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r \frac{1}{|j_3|}|\boldsymbol{k}|^{3/2}|\boldsymbol{j}'|^{r+1/2}|\boldsymbol{l}|^r e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{l}|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}| \\ &= C_r \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} |\boldsymbol{k}|^{3/2}|\hat{g}_{\boldsymbol{k}}|e^{\tau|\boldsymbol{k}|} \sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\boldsymbol{j}'|^{r+1/2}e^{\tau|\boldsymbol{j}|}|\hat{f}_{\boldsymbol{j}}||(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^r e^{\tau|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}| \end{aligned}$$

$$
\begin{aligned}
&\le C_r \sum_{\boldsymbol{k}'\in\mathbb{Z}^2} |(\boldsymbol{k}',\pm 1)|^{1-r} \sum_{k_3\neq 0} |\boldsymbol{k}|^{r+1/2}|\hat{g}_{\boldsymbol{k}}|e^{\tau|\boldsymbol{k}|}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ \boldsymbol{j}\neq 0}} |\boldsymbol{j}|^{2r+1}e^{2\tau|\boldsymbol{j}|}|\hat{f}_{\boldsymbol{j}}|^2\Big)^{1/2}\\
&\quad\times\Big(\sum_{j_3\neq 0}\frac{1}{|j_3|^2}\sum_{\boldsymbol{j}'\in\mathbb{Z}^2}|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^{2r}e^{2\tau|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|^2\Big)^{1/2}\\
&\le C_r\|A^{r+1/2}e^{\tau A}f\|\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm 1)|^{1-r}\Big(\sum_{k_3\neq 0}|\boldsymbol{k}|^{2r+1}|\hat{g}_{\boldsymbol{k}}|^2e^{2\tau|\boldsymbol{k}|}\Big)^{1/2}\\
&\quad\times\Big(\sum_{k_3\neq 0}\sum_{\boldsymbol{j}'\in\mathbb{Z}^2}|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^{2r}e^{2\tau|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|^2\Big)^{1/2}\\
&\le C_r\|A^{r+1/2}e^{\tau A}f\|\|A^re^{\tau A}h\|\Big(\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm 1)|^{2-2r}\Big)^{1/2}\Big(\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}\sum_{k_3\neq 0}|\boldsymbol{k}|^{2r+1}|\hat{g}_{\boldsymbol{k}}|^2e^{2\tau|\boldsymbol{k}|}\Big)^{1/2}\\
&\le C_r\|A^{r+1/2}e^{\tau A}f\|\|A^{r+1/2}e^{\tau A}g\|\|A^re^{\tau A}h\|. \qquad\qquad (A.20)
\end{aligned}
$$

The estimate for B_2 is similar as B_1, and we can get $B_2 \le C_r\|A^{r+1/2}e^{\tau A}f\|\|A^re^{\tau A}g\|\|A^{r+1/2}e^{\tau A}h\|$.

For B_3, thanks to Cauchy–Schwarz inequality, since $r>2$, we have

$$
\begin{aligned}
B_3 &= \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r\frac{1}{|j_3|}|\boldsymbol{j}'|^{3/2}|\boldsymbol{k}|^{r+1/2}|\boldsymbol{l}|^re^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{l}|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}|\\
&= C_r\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|}|\boldsymbol{j}'|^{3/2}|\hat{f}_{\boldsymbol{j}}|e^{\tau|\boldsymbol{j}|}\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|\boldsymbol{k}|^{r+1/2}e^{\tau|\boldsymbol{k}|}|\hat{g}_{\boldsymbol{k}}||(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^re^{\tau|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|\\
&\le C_r\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|^2}|\boldsymbol{j}'|^{2-2r}\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}|\boldsymbol{j}|^{2r+1}|\hat{f}_{\boldsymbol{j}}|^2e^{2\tau|\boldsymbol{j}|}\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|\boldsymbol{k}|^{2r+1}e^{2\tau|\boldsymbol{k}|}|\hat{g}_{\boldsymbol{k}}|^2\Big)^{1/2}\\
&\quad\times\sup_{\boldsymbol{j}\in\mathbb{Z}^3}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^{2r}e^{2\tau|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|^2\Big)^{1/2}\\
&\le C_r\|A^{r+1/2}e^{\tau A}f\|\|A^{r+1/2}e^{\tau A}g\|\|A^re^{\tau A}h\|. \qquad\qquad (A.21)
\end{aligned}
$$

The estimate for B_4 is similar as B_3, and we can get $B_4 \le C_r\|A^re^{\tau A}f\|\|A^{r+1/2}e^{\tau A}g\|\|A^{r+1/2}e^{\tau A}h\|$. The estimates of B_1 to B_4 indicate that I_2 satisfies the desired inequality.

Now let us estimate on I_1. For $\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0$, by using the inequalities

$$
|\boldsymbol{j}|^{1/2}\le C(|\boldsymbol{k}|^{1/2}+|\boldsymbol{l}|^{1/2}),\quad |\boldsymbol{k}|^{1/2}\le C(|\boldsymbol{j}|^{1/2}+|\boldsymbol{l}|^{1/2}),\quad |\boldsymbol{l}|^r\le C_r(|\boldsymbol{j}|^r+|\boldsymbol{k}|^r), \qquad (A.22)
$$

we have

$$
\begin{aligned}
I_1 &\le \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r\frac{1}{|j_3|}|\boldsymbol{j}'||k_3||\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}|(|\boldsymbol{j}|^r+|\boldsymbol{k}|^r)e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}|\boldsymbol{l}|^re^{\tau|\boldsymbol{l}|}|\hat{h}_{\boldsymbol{l}}|\\
&\le \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r\frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}|(|\boldsymbol{j}|^{r+1}|\boldsymbol{k}|+|\boldsymbol{j}||\boldsymbol{k}|^{r+1})e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}|\boldsymbol{l}|^re^{\tau|\boldsymbol{l}|}|\hat{h}_{\boldsymbol{l}}|
\end{aligned}
$$

$$\leq \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r\frac{1}{|j_3|}\Big(|\boldsymbol{k}|^{3/2}|\boldsymbol{j}|^{r+1/2}|\boldsymbol{l}|^r+|\boldsymbol{k}||\boldsymbol{j}|^{r+1/2}|\boldsymbol{l}|^{r+1/2}+|\boldsymbol{j}|^{3/2}|\boldsymbol{k}|^{r+1/2}|\boldsymbol{l}|^r$$
$$+|\boldsymbol{j}||\boldsymbol{k}|^{r+1/2}|\boldsymbol{l}|^{r+1/2}\Big)e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{l}|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}| =: \widetilde{B}_1+\widetilde{B}_2+\widetilde{B}_3+\widetilde{B}_4. \tag{A.23}$$

Thanks to Cauchy–Schwarz inequality, since $r>2$, we have

$$\begin{aligned}
\widetilde{B}_1 &= \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r\frac{1}{|j_3|}|\boldsymbol{k}|^{3/2}|\boldsymbol{j}|^{r+1/2}|\boldsymbol{l}|^r e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{l}|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}|\\
&= C_r\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|\boldsymbol{k}|^{3/2}|\hat{g}_{\boldsymbol{k}}|e^{\tau|\boldsymbol{k}|}\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|}|\boldsymbol{j}|^{r+1/2}e^{\tau|\boldsymbol{j}|}|\hat{f}_{\boldsymbol{j}}||\boldsymbol{j}+\boldsymbol{k}|^r e^{\tau|\boldsymbol{j}+\boldsymbol{k}|}|\hat{h}_{-\boldsymbol{j}-\boldsymbol{k}}|\\
&\leq C_r\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm 1)|^{1-r}\sum_{k_3\neq 0}|\boldsymbol{k}|^{r+1/2}|\hat{g}_{\boldsymbol{k}}|e^{\tau|\boldsymbol{k}|}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ \boldsymbol{j}\neq 0}}|\boldsymbol{j}|^{2r+1}e^{2\tau|\boldsymbol{j}|}|\hat{f}_{\boldsymbol{j}}|^2\Big)^{1/2}\\
&\quad\times\Big(\sum_{j_3\neq 0}\frac{1}{|j_3|^2}\sum_{\boldsymbol{j}'\in\mathbb{Z}^2}|(\boldsymbol{j}'+\boldsymbol{k}',j_3+k_3)|^{2r}e^{2\tau|(\boldsymbol{j}'+\boldsymbol{k}',j_3+k_3)|}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',j_3+k_3)}|^2\Big)^{1/2}\\
&\leq C_r\|A^{r+1/2}e^{\tau A}f\|\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm 1)|^{1-r}\Big(\sum_{k_3\neq 0}|\boldsymbol{k}|^{2r+1}|\hat{g}_{\boldsymbol{k}}|^2e^{2\tau|\boldsymbol{k}|}\Big)^{1/2}\\
&\quad\times\Big(\sum_{j_3\neq 0}\frac{1}{|j_3|^2}\sum_{k_3\neq 0}\sum_{\boldsymbol{j}'\in\mathbb{Z}^2}|(\boldsymbol{j}'+\boldsymbol{k}',j_3+k_3)|^{2r}e^{2\tau|(\boldsymbol{j}'+\boldsymbol{k}',j_3+k_3)|}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',j_3+k_3)}|^2\Big)^{1/2}\\
&\leq C_r\|A^{r+1/2}e^{\tau A}f\|\|A^re^{\tau A}h\|\Big(\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm 1)|^{2-2r}\Big)^{1/2}\Big(\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}\sum_{k_3\neq 0}|\boldsymbol{k}|^{2r+1}|\hat{g}_{\boldsymbol{k}}|^2e^{2\tau|\boldsymbol{k}|}\Big)^{1/2}\\
&\leq C_r\|A^{r+1/2}e^{\tau A}f\|\|A^{r+1/2}e^{\tau A}g\|\|A^re^{\tau A}h\|,
\end{aligned}\tag{A.24}$$

where in the second inequality, we use Fubini theorem to exchange the order of $\sum\limits_{j_3\neq 0}$ and $\sum\limits_{k_3\neq 0}$. The estimate for $\widetilde{B}_2$ is similar to $\widetilde{B}_1$, and we can get $\widetilde{B}_2\leq C_r\|A^{r+1/2}e^{\tau A}f\|\|A^re^{\tau A}g\|\|A^{r+1/2}e^{\tau A}h\|$.

For $\widetilde{B}_3$, thanks to Cauchy–Schwarz inequality, since $r>2$, we have

$$\begin{aligned}
\widetilde{B}_3 &= \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} C_r\frac{1}{|j_3|}|\boldsymbol{j}|^{3/2}|\boldsymbol{k}|^{r+1/2}|\boldsymbol{l}|^r e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{l}|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}|\\
&= C_r\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|}|\boldsymbol{j}|^{3/2}e^{\tau|\boldsymbol{j}|}|\hat{f}_{\boldsymbol{j}}|\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|\boldsymbol{k}|^{r+1/2}|\hat{g}_{\boldsymbol{k}}|e^{\tau|\boldsymbol{k}|}|\boldsymbol{j}+\boldsymbol{k}|^r e^{\tau|\boldsymbol{j}+\boldsymbol{k}|}|\hat{h}_{-\boldsymbol{j}-\boldsymbol{k}}|\\
&\leq C_r\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|^2}|\boldsymbol{j}'|^{2-2r}\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}|\boldsymbol{j}|^{2r+1}|\hat{f}_{\boldsymbol{j}}|^2e^{2\tau|\boldsymbol{j}|}\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|\boldsymbol{k}|^{2r+1}e^{2\tau|\boldsymbol{k}|}|\hat{g}_{\boldsymbol{k}}|^2\Big)^{1/2}\\
&\quad\times\sup_{\boldsymbol{j}\in\mathbb{Z}^3}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|\boldsymbol{j}+\boldsymbol{k}|^{2r}e^{2\tau|\boldsymbol{j}+\boldsymbol{k}|}|\hat{h}_{-\boldsymbol{j}-\boldsymbol{k}}|^2\Big)^{1/2}\\
&\leq C_r\|A^{r+1/2}e^{\tau A}f\|\|A^{r+1/2}e^{\tau A}g\|\|A^re^{\tau A}h\|,
\end{aligned}\tag{A.25}$$

where in the first inequality we use $|\boldsymbol{j}|^{2-2r} \le |\boldsymbol{j}'|^{2-2r}$ due to $r > 2$. The estimate for $\widetilde{B}_4$ is similar as $\widetilde{B}_3$, and we can get $\widetilde{B}_4 \le C_r \|A^r e^{\tau A} f\| \|A^{r+1/2} e^{\tau A} g\| \|A^{r+1/2} e^{\tau A} h\|$. The estimates of $\widetilde{B}_1$ to $\widetilde{B}_4$ indicate that I_1 satisfies the desired inequality. □

Lemma A.4–A.7 play an essential role in the proof of Theorem 6.1. First, let us state the following:

Lemma A.4. *For $f, g, h \in \mathcal{D}(e^{\tau A} : H^{r+1/2})$, where $r > 5/2$ and $\tau \ge 0$, one has*

$$\begin{aligned}
&\Big|\Big\langle A^r e^{\tau A}(f\cdot\nabla g), A^r e^{\tau A} h\Big\rangle - \Big\langle f\cdot\nabla A^r e^{\tau A} g, A^r e^{\tau A} h\Big\rangle\Big| \\
&\le C_r \|A^r f\| \|A^r g\| \|A^r h\| + C_r \tau \|A^{r+1/2} e^{\tau A} f\| \|A^{r+1/2} e^{\tau A} g\| \|A^{r+1/2} e^{\tau A} h\|.
\end{aligned} \tag{A.26}$$

Next, we have

Lemma A.5. *For $f, g, h \in \mathcal{D}(e^{\tau A} : H^{r+1/2})$, where $r > 5/2$ and $\tau \ge 0$, one has*

$$\begin{aligned}
&\Big|\Big\langle A^r e^{\tau A}\big((\nabla\cdot f) g\big), A^r e^{\tau A} h\Big\rangle - \Big\langle (\nabla\cdot A^r e^{\tau A} f) g, A^r e^{\tau A} h\Big\rangle\Big| \\
&\le C_r \|A^r f\| \|A^r g\| \|A^r h\| + C_r \tau \|A^{r+1/2} e^{\tau A} f\| \|A^{r+1/2} e^{\tau A} g\| \|A^{r+1/2} e^{\tau A} h\|.
\end{aligned} \tag{A.27}$$

The proof of Lemma A.4 is similarly to that of Lemma 8 in [47] since it involves nonlinear term similar to that appearing in the Euler equations. The proof of Lemma A.5 is similarly to that of Lemma A.4. Therefore, they are omitted.

The next two lemmas provide the estimates for nonlinear terms which are specific to the structure of the PEs.

Lemma A.6. *For $f \in \mathcal{D}(e^{\tau A} : H^{r+3/2})$ and $g, h \in \mathcal{D}(e^{\tau A} : H^{r+1/2})$, where $r > 5/2$, $\tau \ge 0$, and $\overline{f} = 0$, one has*

$$\begin{aligned}
&\Big|\Big\langle A^r e^{\tau A}\Big(\big(\int_0^z \nabla\cdot f(\boldsymbol{x}', s) ds\big)\partial_z g\Big), A^r e^{\tau A} h\Big\rangle - \Big\langle \big(\int_0^z \nabla\cdot f(\boldsymbol{x}', s) ds\big) A^r e^{\tau A}\partial_z g, A^r e^{\tau A} h\Big\rangle\Big| \\
&\le C_r \|A^{r+1} f\| \|A^r g\| \|A^r h\| + C_r \tau \|A^{r+3/2} e^{\tau A} f\| \|A^{r+1/2} e^{\tau A} g\| \|A^{r+1/2} e^{\tau A} h\|.
\end{aligned}$$

Lemma A.7. *For $g \in \mathcal{D}(e^{\tau A} : H^{r+3/2})$ and $f, h \in \mathcal{D}(e^{\tau A} : H^{r+1/2})$, where $r > 5/2$, $\tau \ge 0$, and $\overline{f} = 0$, one has*

$$\begin{aligned}
&\Big|\Big\langle A^r e^{\tau A}\Big(\big(\int_0^z \nabla\cdot f(\boldsymbol{x}', s) ds\big)\partial_z g\Big), A^r e^{\tau A} h\Big\rangle - \Big\langle \partial_z g A^r e^{\tau A}\big(\int_0^z \nabla\cdot f(\boldsymbol{x}', s) ds\big), A^r e^{\tau A} h\Big\rangle\Big| \\
&\le C_r \|A^{r+1} g\| \|A^r f\| \|A^r h\| + C_r \tau \|A^{r+3/2} e^{\tau A} g\| \|A^{r+1/2} e^{\tau A} f\| \|A^{r+1/2} e^{\tau A} h\|.
\end{aligned}$$

Since the proof of Lemma A.6 is similar to that of Lemma A.7, we first focus below on the proof of Lemma A.7, and later we sketch the proof of Lemma A.6 with emphasis on the main differences.

Proof. (proof of Lemma A.7) First, denote by $H = A^r e^{\tau A} h$, and let

$$\begin{aligned}
I &:= \Big|\Big\langle A^r e^{\tau A}\Big(\big(\int_0^z \nabla\cdot f(\boldsymbol{x}', s) ds\big)\partial_z g\Big), A^r e^{\tau A} h\Big\rangle - \Big\langle \partial_z g A^r e^{\tau A}\big(\int_0^z \nabla\cdot f(\boldsymbol{x}', s) ds\big), A^r e^{\tau A} h\Big\rangle\Big| \\
&= \Big|\Big\langle \big(\int_0^z \nabla\cdot f(\boldsymbol{x}', s) ds\big)\partial_z g, A^r e^{\tau A} H\Big\rangle - \Big\langle \partial_z g A^r e^{\tau A}\big(\int_0^z \nabla\cdot f(\boldsymbol{x}', s) ds\big), H\Big\rangle\Big|.
\end{aligned} \tag{A.28}$$

Similar as in the proof of Lemma A.3, using Fourier representation of f, since $\overline{f} = 0$, we have

$$\int_0^z \nabla\cdot f(\boldsymbol{x}', s) ds = \sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3 \\ j_3\neq 0}} \frac{j_1 + j_2}{j_3} \hat{f}_{\boldsymbol{j}} e^{2\pi(i\boldsymbol{j}'\cdot\boldsymbol{x}' + i j_3 z)} - \sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3 \\ j_3\neq 0}} \frac{j_1 + j_2}{j_3} \hat{f}_{\boldsymbol{j}} e^{2\pi i\boldsymbol{j}'\cdot\boldsymbol{x}'}, \tag{A.29}$$

where $\boldsymbol{j}' = (j_1, j_2)$. Using Fourier representation of g and H, we have

$$
\begin{aligned}
I \leq C & \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0 \\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}| \Big| |\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|} - |\boldsymbol{j}|^r e^{\tau|\boldsymbol{j}|} \Big| \\
& + C \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0 \\ k_3+l_3=0 \\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}| \Big| |\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|} - |(\boldsymbol{j}',0)|^r e^{\tau|(\boldsymbol{j}',0)|} \Big| := I_1 + I_2.
\end{aligned}
\tag{A.30}
$$

We estimate I_2 first. By virtue of the following observation [47]:

For $r \geq 1$ and $\tau \geq 0$, and for all positive $\xi, \eta \in \mathbb{R}$, we have

$$
|\xi^r e^{\tau\xi} - \eta^r e^{\tau\eta}| \leq C_r |\xi - \eta| \Big(|\xi-\eta|^{r-1} + \eta^{r-1} + \tau(|\xi-\eta|^r + \eta^r) e^{\tau|\xi-\eta|} e^{\tau\eta} \Big); \tag{A.31}
$$

with $\xi = |\boldsymbol{l}|$, $\eta = |(\boldsymbol{j}',0)| = |\boldsymbol{j}'|$, and $|\xi - \eta| = \Big| |\boldsymbol{l}| - |(\boldsymbol{j}',0)| \Big| \leq \Big| -\boldsymbol{l} - (\boldsymbol{j}',0) \Big| = |\boldsymbol{k}|$, inequality (A.31) implies

$$
I_2 \leq C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0 \\ k_3+l_3=0 \\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|^2 \Big(|\boldsymbol{k}|^{r-1} + |\boldsymbol{j}'|^{r-1} + \tau(|\boldsymbol{k}|^r + |\boldsymbol{j}'|^r) e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|} \Big). \tag{A.32}
$$

By the definition of H, and since $e^x \leq 1 + xe^x$ for any $x \geq 0$, we have

$$
|\hat{H}_{\boldsymbol{l}}| = |\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|} |\hat{h}_{\boldsymbol{l}}| \leq |\boldsymbol{l}|^r (1 + \tau|\boldsymbol{l}| e^{\tau|\boldsymbol{l}|}) |\hat{h}_{\boldsymbol{l}}| \leq |\boldsymbol{l}|^r |\hat{h}_{\boldsymbol{l}}| + \tau(|\boldsymbol{j}'| + |\boldsymbol{k}|) |\hat{H}_{\boldsymbol{l}}|. \tag{A.33}
$$

Therefore, one obtains that

$$
\begin{aligned}
& |\hat{H}_{\boldsymbol{l}}| \Big(|\boldsymbol{k}|^{r-1} + |\boldsymbol{j}'|^{r-1} + \tau(|\boldsymbol{k}|^r + |\boldsymbol{j}'|^r) e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|} \Big) \\
\leq & \Big(|\boldsymbol{l}|^r |\hat{h}_{\boldsymbol{l}}| + \tau(|\boldsymbol{j}'| + |\boldsymbol{k}|) |\hat{H}_{\boldsymbol{l}}| \Big) \Big(|\boldsymbol{k}|^{r-1} + |\boldsymbol{j}'|^{r-1} \Big) + |\hat{H}_{\boldsymbol{l}}| \Big(\tau(|\boldsymbol{k}|^r + |\boldsymbol{j}'|^r) e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|} \Big) \\
\leq & |\hat{h}_{\boldsymbol{l}}||\boldsymbol{l}|^r (|\boldsymbol{k}|^{r-1} + |\boldsymbol{j}'|^{r-1}) + \tau C_r |\hat{H}_{\boldsymbol{l}}| (|\boldsymbol{k}|^r + |\boldsymbol{j}'|^r) e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|}.
\end{aligned}
\tag{A.34}
$$

Based on this, one has

$$
\begin{aligned}
I_2 \leq C_r & \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0 \\ k_3+l_3=0 \\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|^2 |\boldsymbol{l}|^r (|\boldsymbol{k}|^{r-1} + |\boldsymbol{j}'|^{r-1}) \\
& + \tau C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0 \\ k_3+l_3=0 \\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|^2 (|\boldsymbol{k}|^r + |\boldsymbol{j}'|^r) e^{\tau|\boldsymbol{k}|} e^{\tau|\boldsymbol{j}|} := I_{21} + I_{22}.
\end{aligned}
\tag{A.35}
$$

Here

$$
I_{21} = C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0 \\ k_3+l_3=0 \\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|^{r+1} |\boldsymbol{l}|^r + \frac{1}{|j_3|} |\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}'|^r |\boldsymbol{k}|^2 |\boldsymbol{l}|^r := I_{211} + I_{212}. \tag{A.36}
$$

Thanks to Cauchy–Schwarz inequality, since $r > 5/2$, we have

$$
I_{211} = C_r \sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3 \\ j_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|} |\boldsymbol{j}'||\hat{f}_{\boldsymbol{j}}| \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3 \\ k_3\neq 0}} |\boldsymbol{k}|^{r+1} |(\boldsymbol{j}'+\boldsymbol{k}', k_3)|^r |\hat{g}_{\boldsymbol{k}}||\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|
$$

$$
\begin{aligned}
&\le C_r\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|^2}|\boldsymbol{j}'|^{2-2r}\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}|\boldsymbol{j}|^{2r}|\hat{f}_{\boldsymbol{j}}|^2\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|\boldsymbol{k}|^{2r+2}|\hat{g}_{\boldsymbol{k}}|^2\Big)^{1/2}\\
&\quad\times\sup_{\boldsymbol{j}\in\mathbb{Z}^3}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^{2r}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|^2\Big)^{1/2}\\
&\le C_r\|A^r f\|\|A^{r+1}g\|\|A^r h\|, \qquad\qquad (A.37)
\end{aligned}
$$

and

$$
\begin{aligned}
I_{212}&=C_r\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|\boldsymbol{k}|^2|\hat{g}_{\boldsymbol{k}}|\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|}|\boldsymbol{j}'|^r|\hat{f}_{\boldsymbol{j}}||(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^r|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|\\
&\le C_r\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm1)|^{1-r}\sum_{k_3\neq 0}|\boldsymbol{k}|^{r+1}|\hat{g}_{\boldsymbol{k}}|\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ \boldsymbol{j}\neq 0}}|\boldsymbol{j}|^{2r}|\hat{f}_{\boldsymbol{j}}|^2\Big)^{1/2}\\
&\qquad\times\Big(\sum_{j_3\neq 0}\frac{1}{|j_3|^2}\sum_{\boldsymbol{j}'\in\mathbb{Z}^2}|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^{2r}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|^2\Big)^{1/2}\\
&\le C_r\|A^r f\|\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm1)|^{1-r}\Big(\sum_{k_3\neq 0}|\boldsymbol{k}|^{2r+2}|\hat{g}_{\boldsymbol{k}}|^2\Big)^{1/2}\\
&\qquad\times\Big(\sum_{k_3\neq 0}\sum_{\boldsymbol{j}'\in\mathbb{Z}^2}|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^{2r}|\hat{h}_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|^2\Big)^{1/2}\\
&\le C_r\|A^r f\|\|A^r h\|\Big(\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm1)|^{2-2r}\Big)^{1/2}\Big(\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}\sum_{k_3\neq 0}|\boldsymbol{k}|^{2r+2}|\hat{g}_{\boldsymbol{k}}|^2\Big)^{1/2}\\
&\le C_r\|A^r f\|\|A^{r+1}g\|\|A^r h\|. \qquad\qquad (A.38)
\end{aligned}
$$

Next, for I_{22}, we have

$$
\begin{aligned}
I_{22}&=\tau C_r\sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|^{r+2}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}\\
&\quad+\tau C_r\sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'|^{r+1}|\boldsymbol{k}|^2e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}:=I_{221}+I_{222}. \qquad (A.39)
\end{aligned}
$$

Noticing that $|\boldsymbol{k}|^{1/2}\le C(|\boldsymbol{j}'|^{1/2}+|\boldsymbol{l}|^{1/2})$ and $|\boldsymbol{j}'|^{1/2}\le C(|\boldsymbol{k}|^{1/2}+|\boldsymbol{l}|^{1/2})$, thanks to Cauchy–Schwarz inequality, since $r>5/2$, we have

$$
\begin{aligned}
I_{221}&=\tau C_r\sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|^{r+2}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}\\
&\le\tau C_r\sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{l}|^r|\boldsymbol{k}|^{r+3/2}(|\boldsymbol{j}'|^{1/2}+|\boldsymbol{l}|^{1/2})e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{l}|}
\end{aligned}
$$

$$\le \tau C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat f_{\boldsymbol{j}}||\hat g_{\boldsymbol{k}}||\hat h_{\boldsymbol{l}}||\boldsymbol{j}'|^{3/2}|\boldsymbol{l}|^{r+1/2}|\boldsymbol{k}|^{r+3/2}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{l}|}$$

$$\le \tau C_r\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|^2}|\boldsymbol{j}'|^{2-2r}\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}}|\boldsymbol{j}|^{2r+1}e^{2\tau|\boldsymbol{j}|}|\hat f_{\boldsymbol{j}}|^2\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|\boldsymbol{k}|^{2r+3}|e^{2\tau|\boldsymbol{k}|}\hat g_{\boldsymbol{k}}|^2\Big)^{1/2}$$

$$\times\sup_{\boldsymbol{j}\in\mathbb{Z}^3}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}}|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^{2r+1}e^{2\tau|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|}|\hat h_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|^2\Big)^{1/2}$$

$$\le \tau C_r\|A^{r+1/2}e^{\tau A}f\|\|A^{r+3/2}e^{\tau A}g\|\|A^{r+1/2}e^{\tau A}h\|, \tag{A.40}$$

and

$$I_{222} = \tau C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat f_{\boldsymbol{j}}||\hat g_{\boldsymbol{k}}||\hat H_{\boldsymbol{l}}||\boldsymbol{j}'|^{r+1}|\boldsymbol{k}|^2e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}$$

$$\le \tau C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat f_{\boldsymbol{j}}||\hat g_{\boldsymbol{k}}||\hat h_{\boldsymbol{l}}||\boldsymbol{j}'|^{r+1/2}|\boldsymbol{k}|^2|\boldsymbol{l}|^r(|\boldsymbol{k}|^{1/2}+|\boldsymbol{l}|^{1/2})e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{l}|}$$

$$\le \tau C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat f_{\boldsymbol{j}}||\hat g_{\boldsymbol{k}}||\hat h_{\boldsymbol{l}}||\boldsymbol{j}'|^{r+1/2}|\boldsymbol{k}|^{5/2}|\boldsymbol{l}|^{r+1/2}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{l}|}$$

$$\le \tau C_r\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm1)|^{1-r}\sum_{k_3\neq 0}|\boldsymbol{k}|^{r+3/2}e^{\tau|\boldsymbol{k}|}|\hat g_{\boldsymbol{k}}|\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ \boldsymbol{j}\neq 0}}|\boldsymbol{j}|^{2r+1}e^{2\tau|\boldsymbol{j}|}|\hat f_{\boldsymbol{j}}|^2\Big)^{1/2}$$

$$\times\Big(\sum_{j_3\neq 0}\frac{1}{|j_3|^2}\sum_{\boldsymbol{j}'\in\mathbb{Z}^2}|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^{2r+1}e^{2\tau|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|}|\hat h_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|^2\Big)^{1/2}$$

$$\le \tau C_r\|A^{r+1/2}e^{\tau A}f\|\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm1)|^{1-r}\Big(\sum_{k_3\neq 0}|\boldsymbol{k}|^{2r+3}e^{2\tau|\boldsymbol{k}|}|\hat g_{\boldsymbol{k}}|^2\Big)^{1/2}$$

$$\times\Big(\sum_{k_3\neq 0}\sum_{\boldsymbol{j}'\in\mathbb{Z}^2}|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|^{2r+1}e^{2\tau|(\boldsymbol{j}'+\boldsymbol{k}',k_3)|}|\hat h_{-(\boldsymbol{j}'+\boldsymbol{k}',k_3)}|^2\Big)^{1/2}$$

$$\le \tau C_r\|A^{r+1/2}e^{\tau A}f\|\|A^{r+1/2}e^{\tau A}h\|\Big(\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}|(\boldsymbol{k}',\pm1)|^{2-2r}\Big)^{1/2}\Big(\sum_{\boldsymbol{k}'\in\mathbb{Z}^2}\sum_{k_3\neq 0}|\boldsymbol{k}|^{2r+3}e^{2\tau|\boldsymbol{k}|}|\hat g_{\boldsymbol{k}}|^2\Big)^{1/2}$$

$$\le \tau C_r\|A^{r+1/2}e^{\tau A}f\|\|A^{r+3/2}e^{\tau A}g\|\|A^{r+1/2}e^{\tau A}h\|. \tag{A.41}$$

Therefore, I_2 satisfies the desired estimates.

To estimate I_1, we use (A.31) with $\xi = |\boldsymbol{l}|$, $\eta = |\boldsymbol{j}|$, and with $|\xi-\eta| = \Big||\boldsymbol{l}|-|\boldsymbol{j}|\Big| \le |-\boldsymbol{l}-\boldsymbol{j}| = |\boldsymbol{k}|$, to obtain

$$I_1 \le C_r\sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}}\frac{1}{|j_3|}|\hat f_{\boldsymbol{j}}||\hat g_{\boldsymbol{k}}||\hat H_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|^2\Big(|\boldsymbol{k}|^{r-1}+|\boldsymbol{j}|^{r-1}+\tau(|\boldsymbol{k}|^r+|\boldsymbol{j}|^r)e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}\Big). \tag{A.42}$$

Thanks to (A.34), one obtains that

$$
\begin{aligned}
I_1 \le C_r & \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat f_{\boldsymbol{j}}||\hat g_{\boldsymbol{k}}||\hat h_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|^2|\boldsymbol{l}|^r(|\boldsymbol{k}|^{r-1}+|\boldsymbol{j}|^{r-1})\\
&+\tau C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat f_{\boldsymbol{j}}||\hat g_{\boldsymbol{k}}||\hat H_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|^2(|\boldsymbol{k}|^r+|\boldsymbol{j}|^r)e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|} := I_{11}+I_{12}.
\end{aligned} \tag{A.43}
$$

Here

$$
I_{11} \le C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat f_{\boldsymbol{j}}||\hat g_{\boldsymbol{k}}||\hat h_{\boldsymbol{l}}||\boldsymbol{j}||\boldsymbol{k}|^{r+1}|\boldsymbol{l}|^r + \frac{1}{|j_3|}|\hat f_{\boldsymbol{j}}||\hat g_{\boldsymbol{k}}||\hat h_{\boldsymbol{l}}||\boldsymbol{j}|^r|\boldsymbol{k}|^2|\boldsymbol{l}|^r := I_{111}+I_{112}. \tag{A.44}
$$

Thanks to Cauchy–Schwarz inequality, since $r > 5/2$, we have

$$
\begin{aligned}
I_{111} &= C_r \sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\boldsymbol{j}||\hat f_{\boldsymbol{j}}| \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} |\boldsymbol{k}|^{r+1}|\boldsymbol{j}+\boldsymbol{k}|^r|\hat g_{\boldsymbol{k}}||\hat h_{-(\boldsymbol{j}+\boldsymbol{k})}|\\
&\le C_r\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}} |\boldsymbol{j}|^{2-2r}\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}} |\boldsymbol{j}|^{2r}|\hat f_{\boldsymbol{j}}|^2\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} |\boldsymbol{k}|^{2r+2}|\hat g_{\boldsymbol{k}}|^2\Big)^{1/2}\\
&\quad\times \sup_{\boldsymbol{j}\in\mathbb{Z}^3}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} |\boldsymbol{j}+\boldsymbol{k}|^{2r}|\hat h_{-(\boldsymbol{j}+\boldsymbol{k})}|^2\Big)^{1/2}\\
&\le C_r\|A^r f\|\|A^{r+1}g\|\|A^r h\|,
\end{aligned} \tag{A.45}
$$

and

$$
\begin{aligned}
I_{112} &= C_r \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} |\boldsymbol{k}|^2|\hat g_{\boldsymbol{k}}| \sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\boldsymbol{j}|^r|\hat f_{\boldsymbol{j}}||\boldsymbol{j}+\boldsymbol{k}|^r|\hat h_{-(\boldsymbol{j}+\boldsymbol{k})}|\\
&\le C_r\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ \boldsymbol{k}\neq 0}} |\boldsymbol{k}|^{2-2r}\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ \boldsymbol{k}\neq 0}} |\boldsymbol{k}|^{2r+2}|\hat g_{\boldsymbol{k}}|^2\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ \boldsymbol{j}\neq 0}} |\boldsymbol{j}|^{2r}|\hat f_{\boldsymbol{j}}|^2\Big)^{1/2}\\
&\quad\times \sup_{\boldsymbol{k}\in\mathbb{Z}^3}\Big(\sum_{\boldsymbol{j}\in\mathbb{Z}^3} |\boldsymbol{j}+\boldsymbol{k}|^{2r}|\hat h_{-(\boldsymbol{j}+\boldsymbol{k})}|^2\Big)^{1/2}\\
&\le C_r\|A^r f\|\|A^{r+1}g\|\|A^r h\|.
\end{aligned} \tag{A.46}
$$

Next, for I_{12}, we have

$$
\begin{aligned}
I_{12} \le \tau C_r & \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat f_{\boldsymbol{j}}||\hat g_{\boldsymbol{k}}||\hat H_{\boldsymbol{l}}||\boldsymbol{j}||\boldsymbol{k}|^{r+2}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}\\
&+\tau C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat f_{\boldsymbol{j}}||\hat g_{\boldsymbol{k}}||\hat H_{\boldsymbol{l}}||\boldsymbol{j}|^{r+1}|\boldsymbol{k}|^2e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|} := I_{121}+I_{122}.
\end{aligned} \tag{A.47}
$$

Since $|\boldsymbol{k}|^{1/2} \leq C(|\boldsymbol{j}|^{1/2}+|\boldsymbol{l}|^{1/2})$ and $|\boldsymbol{j}|^{1/2} \leq C(|\boldsymbol{k}|^{1/2}+|\boldsymbol{l}|^{1/2})$, thanks to Cauchy–Schwarz inequality, since $r>5/2$, we have

$$\begin{aligned}
I_{121} &= \tau C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}||\boldsymbol{k}|^{r+2}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}\\
&\leq \tau C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}',\boldsymbol{l}\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}||\boldsymbol{l}|^{r}|\boldsymbol{k}|^{r+3/2}(|\boldsymbol{j}|^{1/2}+|\boldsymbol{l}|^{1/2})e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{l}|}\\
&\leq \tau C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}',\boldsymbol{l}\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}|^{3/2}|\boldsymbol{l}|^{r+1/2}|\boldsymbol{k}|^{r+3/2}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{l}|}\\
&\leq \tau C_r \Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}} |\boldsymbol{j}|^{2-2r}\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ j_3,\boldsymbol{j}'\neq 0}} |\boldsymbol{j}|^{2r+1}e^{2\tau|\boldsymbol{j}|}|\hat{f}_{\boldsymbol{j}}|^2\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} |\boldsymbol{k}|^{2r+3}|e^{2\tau|\boldsymbol{k}|}\hat{g}_{\boldsymbol{k}}|^2\Big)^{1/2}\\
&\qquad\times \sup_{\boldsymbol{j}\in\mathbb{Z}^3}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ k_3\neq 0}} |\boldsymbol{j}+\boldsymbol{k}|^{2r+1}e^{2\tau|\boldsymbol{j}+\boldsymbol{k}|}|\hat{h}_{-(\boldsymbol{j}+\boldsymbol{k})}|^2\Big)^{1/2}\\
&\leq \tau C_r \|A^{r+1/2}e^{\tau A}f\|\|A^{r+3/2}e^{\tau A}g\|\|A^{r+1/2}e^{\tau A}h\|,
\end{aligned}\tag{A.48}$$

and

$$\begin{aligned}
I_{122} &= \tau C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}|^{r+1}|\boldsymbol{k}|^{2}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}\\
&\leq \tau C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}',\boldsymbol{l}\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}|^{r+1/2}|\boldsymbol{k}|^{2}|\boldsymbol{l}|^{r}(|\boldsymbol{k}|^{1/2}+|\boldsymbol{l}|^{1/2})e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{l}|}\\
&\leq \tau C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}',\boldsymbol{l}\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{h}_{\boldsymbol{l}}||\boldsymbol{j}|^{r+1/2}|\boldsymbol{k}|^{5/2}|\boldsymbol{l}|^{r+1/2}e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}e^{\tau|\boldsymbol{l}|}\\
&\leq \tau C_r \Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ \boldsymbol{k}\neq 0}} |\boldsymbol{k}|^{2-2r}\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^3\\ \boldsymbol{k}\neq 0}} |\boldsymbol{k}|^{2r+3}e^{2\tau|\boldsymbol{k}|}|\hat{g}_{\boldsymbol{k}}|^2\Big)^{1/2}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ \boldsymbol{j}\neq 0}} |\boldsymbol{j}|^{2r+1}e^{2\tau|\boldsymbol{j}|}|\hat{f}_{\boldsymbol{j}}|^2\Big)^{1/2}\\
&\qquad\times \sup_{\boldsymbol{k}\in\mathbb{Z}^3}\Big(\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}^3\\ \boldsymbol{j}\neq 0}} |\boldsymbol{j}+\boldsymbol{k}|^{2r+1}e^{2\tau|\boldsymbol{j}+\boldsymbol{k}|}|\hat{h}_{-(\boldsymbol{j}+\boldsymbol{k})}|^2\Big)^{1/2}\\
&\leq \tau C_r \|A^{r+1/2}e^{\tau A}f\|\|A^{r+3/2}e^{\tau A}g\|\|A^{r+1/2}e^{\tau A}h\|.
\end{aligned}\tag{A.49}$$

Therefore, I_1 satisfies the desired estimates. The proof is completed. □

Finally, we sketch the proof of Lemma A.6.

Proof. (proof of Lemma A.6) Similar as the proof of Lemma A.7, we have

$$I := \Big|\Big\langle A^r e^{\tau A}\Big(\big(\int_0^z \nabla\cdot f(\boldsymbol{x}',s)ds\big)\partial_z g\Big), A^r e^{\tau A}h\Big\rangle - \Big\langle \big(\int_0^z \nabla\cdot f(\boldsymbol{x}',s)ds\big)A^r e^{\tau A}\partial_z g, A^r e^{\tau A}h\Big\rangle\Big|$$

$$= \Big|\Big\langle (\int_0^z \nabla\cdot f(\boldsymbol{x}',s)ds)\partial_z g, A^r e^{\tau A} H\Big\rangle - \Big\langle (\int_0^z \nabla\cdot f(\boldsymbol{x}',s)ds) A^r e^{\tau A}\partial_z g, H\Big\rangle\Big|.$$

$$\le C \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|\Big||\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|} - |\boldsymbol{k}|^r e^{\tau|\boldsymbol{k}|}\Big|$$

$$+C \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'||\boldsymbol{k}|\Big||\boldsymbol{l}|^r e^{\tau|\boldsymbol{l}|} - |\boldsymbol{k}|^r e^{\tau|\boldsymbol{k}|}\Big| := I_1 + I_2. \tag{A.50}$$

For I_1, since $\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0$, we use (A.31) with $\xi = |\boldsymbol{l}|$, $\eta = |\boldsymbol{k}|$ and $|\xi-\eta| = \Big||\boldsymbol{l}|-|\boldsymbol{k}|\Big| \le \Big|-\boldsymbol{l}-\boldsymbol{k}\Big| = |\boldsymbol{j}|$, to conclude

$$I_1 \le C_r \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'|^2|\boldsymbol{k}|\Big(|\boldsymbol{k}|^{r-1} + |\boldsymbol{j}|^{r-1} + \tau(|\boldsymbol{k}|^r + |\boldsymbol{j}|^r)e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}\Big). \tag{A.51}$$

For I_2, since $(\boldsymbol{j}',0)+\boldsymbol{k}+\boldsymbol{l}=0$, we use (A.31) with $\xi = |\boldsymbol{l}|$, $\eta = |\boldsymbol{k}|$ and $|\xi-\eta| = \Big||\boldsymbol{l}|-|\boldsymbol{k}|\Big| \le \Big|-\boldsymbol{l}-\boldsymbol{k}\Big| = |\boldsymbol{j}'|$, to obtain

$$I_2 \le C_r \sum_{\substack{\boldsymbol{j}'+\boldsymbol{k}'+\boldsymbol{l}'=0\\ k_3+l_3=0\\ j_3,k_3,\boldsymbol{j}'\neq 0}} \frac{1}{|j_3|}|\hat{f}_{\boldsymbol{j}}||\hat{g}_{\boldsymbol{k}}||\hat{H}_{\boldsymbol{l}}||\boldsymbol{j}'|^2|\boldsymbol{k}|\Big(|\boldsymbol{k}|^{r-1} + |\boldsymbol{j}'|^{r-1} + \tau(|\boldsymbol{k}|^r + |\boldsymbol{j}'|^r)e^{\tau|\boldsymbol{k}|}e^{\tau|\boldsymbol{j}|}\Big). \tag{A.52}$$

Observe that the difference between the sums in the right-hand sides of (A.51) and (A.42) (and between (A.52) and (A.32)), is manifested in the factors $|\boldsymbol{j}'|^2|\boldsymbol{k}|$ and $|\boldsymbol{j}'||\boldsymbol{k}|^2$, respectively. Therefore, one can follow the estimates of I_1 in (A.42) and I_2 in (A.32), and obtain that I_1 in (A.51) and I_2 in (A.52) satisfy the desired bound in Lemma A.6.

□

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(T. E. Ghoul) Department of Mathematics, New York University in Abu Dhabi, Saadyat Island, Abu Dhabi, UAE.

Email address: teg6@nyu.edu

(S. Ibrahim) Department of Mathematics and Statistics, University of Victoria, 3800 Finnerty Road, Victoria, B.C., Canada V8P 5C2.

Email address: ibrahims@uvic.ca

(Q. Lin) Department of Mathematics, Texas A&M University, College Station, Texas, TX 77840, USA.

Email address: abellyn@hotmail.com

(E.S. Titi) Department of Mathematics, Texas A&M University, College Station, Texas, TX 77840, USA. Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK. Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel.

Email address: titi@math.tamu.edu

Email address: Edriss.Titi@damtp.cam.ac.uk

Email address: edriss.titi@weizmann.ac.il