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THE ZERO-MASS-LIMIT APPROACH TO KINEMATICS AND VENEZIANO PARAMETERIZATIONS FOR PHOTON AMPLITUDES*

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ABSTRACT

The covariant photon amplitude is proved to be the smooth limit of the helicity amplitude for a massive vector boson. The kinematical consequences of this zero-mass limit and crossing are shown to include conservation of charge (or the equivalence principle in the case of the graviton), and the low energy theorems for photoproduction and Compton scattering. Consequently, the zero-mass limit is a practical alternative to the use of on-mass-shell gauge invariance. Moreover, the zero-mass limit provides a technique for the construction of Regge expansions and Veneziano parameterizations for photon amplitudes from the corresponding amplitudes for a massive vector boson. A model for the Pomeranchon which contribues to forward elastic Compton scattering and does not fall off at large q^2 is proposed. Also a Veneziano parameterization of the amplitude for Compton scattering of off-mass-shell charged currents off pions is presented which satisfies the current algebra sum rule of Fubini, Dashen, and Gell-Mann.

I. INTRODUCTION

Recently an approach to the kinematics of the photon has been developed that replaces the usual discussion of guage invariance with a zero-mass limit (ZML) on the helicity amplitudes for a massive vector boson.[1,2] Here (Sec. II) we give a rigorous derivation of the properties of this ZML based on Lorentz invariance and analyticity, and we related these properties to the standard assumptions of gauge invariance for tensor amplitudes and the conserved-vector-current (CVC) hypothesis.

Another recent development is the proposal of Veneziano-type parameterization for four-particle helicity amplitudes involving the photon[3] (photoproduction and Compton scattering). The main purpose of this paper is to show that the applications of the ZML to photon kinematics can be naturally extended to give a rederivation of the parameterization of Ref. 3. First (Sec. III or Ref. 1) by imposing crossing on the external line insertion (ELI) poles, one proves in the ZML the low energy theorems and charge conservation for the photon (or the equivalence principle for a spin-2 boson).[4] Then (Sec. IV) by applying the ZML to a Regge expansion, a modification of nonsense factors[2] is made which allows the ELI pole to appear on the exchanged trajectory.[4] It becomes apparent at this point (IV.B) that the peculiarities of a zero-mass vector particle favor, rather than conflict with, the duality picture. Therefore it is natural to seek a representation for photon amplitudes that combines crossing and Regge asymptotics. By applying the ZML to the Veneziano representation for a massive vector meson, one automatically obtains just such a representation in the zero-width approximation. Once the on-mass-shell amplitude has been constructed, one can in general continue off mass shell to obtain amplitudes for a conserved vector current (see Sec. II.B).

The two-current amplitude, which enters into the Fubini–Dashen–Gell-Mann sum rule, presents a more difficult problem. The ZML yields correct amplitudes only at $q^\mu \equiv 0$, and a fixed pole[5] must be inserted for $q^\mu \neq 0$. The important problem of fixed poles[3,6] in the two-current channel is discussed briefly as an introduction to the detailed consideration in the forthcoming papers on vector currents and current algebra in a dual, zero-width model.[7] These papers extend the parameterization of Ref. 3 to one- and two-current amplitudes with N spinless hadrons, and more importantly, they begin to impose the dynamical constraints of factorization and current algebra.

The ZML provides an alternative (but equivalent) approach to the <u>kinematics</u> of the zero-mass particle. However, the usefulness of this approach for converting a given hadron amplitude into an amplitude for a photon requires a close analogy between the photon and massive-vector-meson amplitudes. Roughly speaking, one is making a dynamical assumption that the photon amplitudes have the same Regge asymptotic and analyticity properties as hadron processes, <u>except</u> when this is in direct conflict with the requirements of the ZML, CVC, or some other basic principle. In Ref. 7, this point is developed more rigorously on the basis of unsubtracted dispersion relations in q^2, but the dynamical implications are much the same.

II. LORENTZ INVARIANCE AND ANALYTICITY IN MASS

In this section, we present that basic kinematical analysis for helicity amplitudes for zero-mass particles $(m_B = 0)$ and off-mass-shell (current) amplitudes. We begin in II.A by considering helicity amplitudes for entirely massive particles, and then study the zero-mass limit for one of the external bosons of mass m_B and spin J_B. If the amplitudes are bounded, the limit leads to Lorentz-invariant (hence gauge-invariant) amplitudes for massless particles.[8] Then the kinematical singularities in the "mass" of the photon (or equivalently the off-mass-shell invariant q^2) is determined by a fixed $J = 1$ projection of a diparticle state in a multibody amplitude. Lastly, the relationship between the helicity-amplitude and the tensor-amplitude approach is described. We have used the helicity amplitudes in this paper so that the kinematical singularities, crossing relations, and Reggeization procedure for photonic amplitudes can be more conveniently compared with, and made consistent with the standard results for the purely hadronic amplitudes. However, tensor amplitudes are convenient in our future extension to a study of current algebra in the zero-width model,[7] hence they are also introduced and the explicit relation to helicity amplitudes is given in Tables Ia,b.

A. Lorentz Invariance and the Zero-Mass Limit

In order to get gauge-invariant results from the direct application of the zero-mass limit, we discovered in Refs. 1 and 2 that only a mild condition on the amplitude H_0 for the zero-helicity massive photon (m_V) is required,

$$\lim_{m_V \to 0} m_V H_0(s,t,\cdots) = 0 \tag{2.1}$$

for some region in the space of invariants $(s,t,\cdots)$. One can immediately justify this condition on the basis of unitarity in the physical region, and go on to derive its consequences.[1] However, Weinberg[8] has demonstrated that Lorentz invariance for a photonic amplitude implies "on-mass-shell gauge invariance" and, therefore, its consequences - conservation of charge and low energy theorems. Therefore, it is more satisfactory to show that condition (2.1) is needed to derive Lorentz invariance for the massless photon from Lorentz invariance of the massive "photon" in the ZML. Since it is no more difficult, we shall consider the ZML for any boson of mass (m_B) and spin J_B.

The transformation law for a helicity amplitude, $H_{\mu_f;\lambda_i}(\bar{p}_f;\, p_i)$, with a massive boson of momentum q and helicity λ_B and N hadrons, is given by

$$H_{\mu_f;\lambda_i}(\bar{p}_f;p_i) = \prod_{f=K+1}^{N} \otimes \mathcal{D}^{*J_f}_{\mu_f'\,\mu_f}\left(R_W(\Lambda,\, \bar{p}_f)\right)$$

$$\prod_{i=0}^{K} \otimes \mathcal{D}^{J_i}_{\lambda_i'\lambda_i}\left(R_W(\Lambda,\, p_i)\right) H_{\mu_f';\lambda_i'}(\Lambda\bar{p}_f;\, \Lambda p_i) \quad , \tag{2.2}$$

where $\lambda_i(\mu_f)$ stand for the set of helicities of the incoming particles with momenta $p_i = q, p_1, \cdots, p_K$ (outgoing particles with momenta $\bar{p}_f = \bar{p}_{K+1}, \cdots, \bar{p}_N$) and $R_W(\Lambda, p_i)$ is the appropriate Wigner rotation. Since we are interested in the transformation properties of the massive boson (photon, graviton, etc.) we can abbreviate (2.2) to read

$$H_{\lambda_B}(q\cdots) = \mathcal{D}^{J_B}_{\lambda_B'\lambda_B}\left(R_W(\Lambda, q)\right) H_{\lambda_B'}(\Lambda q\cdots) \qquad (2.3)$$

by surpressing the hadrons' indices. However, it is important to deal with amplitudes which are directly related to a cross section, rather than with a single-particle state,[9] since our demonstration requires a boundedness condition (2.6) on the transformed amplitudes.

The problem is to take the zero-mass limit on the Wigner rotation $R_W(\Lambda, q) = L_{\Lambda q}^{-1}\, \Lambda\, L_q$.

$$L_q = R(\phi_q, \Theta_q, -\phi_q)\, B_z[\cosh \eta_q = (q^2 + m_B^2)^{\frac{1}{2}}/m_B] , \qquad (2.4)$$

and Λ is an arbitrary element of $SL(2, C)$. Choosing $q^\mu = [(\underset{\sim}{q}^2 + m_B^2)^{\frac{1}{2}}, 0, 0, |\underset{\sim}{q}|]$ we demonstrate in Appendix A that the limit of R_W is

$$R_W \approx e^{-i\Theta(\Lambda,\underset{\sim}{q})J_z}\, e^{-i\Theta J_y} , \qquad (2.5)$$

where $\sin\Theta \propto \frac{M_B}{|\underset{\sim}{q}|}$. The angle Θ is the z rotation of the

Wigner "rotation" of the photon, $\mathcal{R}_W(\Lambda, \underset{\sim}{q}) = \mathcal{L}_{\Lambda q}^{-1} \Lambda \mathcal{L}_q$, where $\mathcal{L}_q$ is a boost from the standard frame, $q = (1, 0, 0, 1)$. The infinite z boosts in L_q and $L^{-1}_{\Lambda q}$ are commuted through to give a finite R_W as $m_B \to 0$. Consequently, if the condition

$$\lim_{m_B \to 0} m_B^{J_B - |\lambda_B|} H_{\lambda_B} = 0 , \quad \lambda_B \neq \pm J_B \tag{2.6}$$

holds, Eq. (2.3) becomes the correct transformation law for an amplitude with an external zero-mass boson,

$$H_{\lambda_B = \pm 1}(q, \cdots) = e^{\mp i J_B \Theta(\Lambda, \underset{\sim}{q})} H_{\lambda_B = \pm J}(q, \cdots) . \tag{2.7}$$

The interesting group theoretical feature of this is that a Wigner rotation in the little group O(3) [or, more precisely SU(2)] for the massive particle can have a smooth limit into a subgroup of the light-like little group E_2. This is possible because the semisimple structure of E_2 and the requirement of a finite dimensional representative for the photon allows only the z rotation of $\mathcal{R}_W(\Lambda, q)$ to enter into the transformation law (2.7). Since the z rotation is in both little groups, the smooth limit can be achieved. The discontinuous group theoretical description often obscures the smoothness in m_B of the physical amplitudes.

The condition (2.1) $m_V H_0 \to 0$ is all that is required to construct gauge-invariant photon amplitudes and to prove charge conservation and the low energy theorems. The undesired amplitude H_0 can merely be ignored. However, for a physically reasonable off-mass-shell theory, one would expect H_0 to vanish as $q^2 \to 0$, since a finite H_0 would correspond to a spin-zero photon, in contradiction with experiment.

B. Off-Mass-Shell Continuation and Analyticity in q^2

The off-mass-shell amplitude $H_{\lambda_V}(q,\cdots)$ can be defined through the electron scattering amplitude $A_{\lambda_a\lambda_b}(p_a,p_b,\cdots)$, where $q = p_a - p_b$. This requires a complete knowledge of the off-mass-shell electron form factor and factorization at the $J = 1$ singularity in the electron-position channel. Clearly the weak coupling of the photon is essential to the physical meaningfulness of this definition. When $-q^2/m_p{}^2 > 1/e^2$ the two photon singularities may become important, so that the phenomenological meaning of this decomposition becomes obscure.

We have studied this $J = 1$ projection of the diparticle state (with spinless a b particle to simplify the kinematical factors at the "leptonic" vertex),

$$H_{\lambda_V}(q,p_1,\cdots)$$

$$= \int_{-1}^{1} d\cos\theta_r \int_0^{2\pi} d\phi_r\, e^{i\lambda_r\phi_r}\, d^1_{\lambda_V 0}(\theta_r)\, A(\theta_r,\phi_r,q,p_1,\cdots)\,, \tag{2.8}$$

where θ_r, ϕ_r are the polar angles for $r = p_a - p_b$ in the rest frame of q. We find[10] that a square root kinematical singularity in q^2 is introduced into the zero-helicity amplitude H_0. Therefore if A is to be analytic in q^2 <u>and</u> if H_0 is to be bounded at $q^2 = 0$, we must have

$$H_0(q,\cdots) = O[(q^2)^{\frac{1}{2}}] . \tag{2.9}$$

This gives a nontrivial constraint for continuing the photon amplitude off shell. In general, the $J = 1$, $\lambda_V = 0$ projection has an infinite singularity $1/(q^2)^{\frac{1}{2}}$ which can not be excluded. In the case of a photon this infinity is excluded because at $q^2 = 0$ there is a physical particle. The Adler-Weisberger relation for a dipion system also excludes this singularity, but one needs some dynamical explanation for this, as in the theory with a partially conserved axial current.

C. One- and Two-Current Amplitudes

It is completely adequate to use helicity amplitudes for the description of a conserved (pure $J = 1$) current, but it is traditional to use a tensor amplitude T^μ. (For convenience, Table I gives the explicit connection between the two sets of amplitudes.) We define the polarization vector for a <u>massive</u> vector particle $(q^2 = m_V^2)$ by

$$\epsilon^\mu(\lambda,q) = L_q{}^\mu{}_\nu\, \epsilon^\nu(\lambda,0) , \tag{2.10}$$

where $\epsilon^\nu(\lambda,0)$ is the rest frame value: $(0,0,0,1)$, $(0,-1,-i,0)/\sqrt{2}$, and $(0,1,-i,0)/\sqrt{2}$ for $\lambda = 0,1,-1$ respectively. This allows us to relate the four-vector T^μ to our helicity amplitude in the covariant equation

$$H_\lambda(q,\cdots) = \epsilon^\mu(\lambda,q)\, T_\mu(q,\cdots) . \tag{2.11}$$

The four-vector is usually identified as the matrix element[11] of a current operator

$$T_\mu(q,\cdots) = \langle \bar{p}_f | J_\mu^\dagger(0) | p_i \rangle \quad , \tag{2.12}$$

where $q = \sum \bar{p}_f - \sum p_i$.

From the condition $\epsilon^\mu(\lambda,q)q_\mu = 0$, we see that Eq. (2.11) defines only the conserved part of T_μ (denoted V_μ). We see this explicitly by inverting the equation to get

$$V^\mu = (g^{\mu\nu} - q^\mu q^\nu/q^2)\, T_\nu = -\sum_\lambda{}' \epsilon^{\mu *}(\lambda,q)\, H_\lambda(q\cdots) \quad . \tag{2.13}$$

Our condition[12] $H_0(q,\cdots) = O[(q^2)^{\frac{1}{2}}]$ and (2.11) can now be shown to imply

$$q_\mu\, T^\mu = O(q^2) \quad , \tag{2.14}$$

and this is sufficient to insure that the conserved part V^μ is nonsingular at $q^2 = 0$. The exclusion of this singularity is the real content of Lorentz invariance and analyticity for physical photons. This allows the tensor T^μ to be decomposed into an independent vector (V^μ) and a scalar $(S = q_\mu\, T^\mu)$ part even as $q^2 \to 0$,

$$T^\mu = V^\mu + \frac{q^\mu}{q^2} S \tag{2.15}$$

and the scalar part S can be set to zero in accordance with the conserved-vector-current (CVC) hypothesis. Setting S to zero has no effect on electroproduction, since the electron-positron state couples only to the conserved part. However, the mass difference between neutrino and electron allows S to be measured in weak interactions (charged current) and CVC has content there. We emphasize that the only rigorous result (2.14) applies to neutral currents, and if it were extended to charged currents instead of CVC, the experimental differences would be exceedingly small.[13]

For a two-current amplitude, $M_{ab}^{\mu\nu}$, analogous results are proved for the neutral currents

$$q_{1\mu} M_{33}^{\mu\nu} = O(q_1^2), \quad M_{33}^{\mu\nu} q_{2\nu} = O(q_2^2) \tag{2.16}$$

in order to have the correct correspondence with physical Compton scattering. If we assume CVC, we cannot prove the divergences are zero; indeed, for the charged currents, the divergences are proven to be nonzero in accordance with current algebra.[5,7] All that can be proven is that the divergences cannot have the unitarity cuts (normal threshold) in subenergies (s_{Vk}) that include (overlap) one photon momentum. The discontinuity around these cuts is related by unitarity to a single-photon amplitude,

$$\text{Disc}_{s_{Vk}}[M_{ab}^{\mu\nu}] = \sum_R \int d\,\Phi_k\, V_{a(k)}^{\mu}\, V_{b(k)}^{\nu *}\,, \tag{2.17}$$

and CVC is then applied to them [rhs of Eq. (2.17)]. In general, no simple relation exists between the entire two-current amplitude (covariant correlation tensor) and the matrix elements of the individual currents.

III. LOW ENERGY BEHAVIOR AND CROSSING

For an amplitude with a zero-mass particle (momentum q^μ) there is a soft pole term or external line insertion (ELI) that contributes at $q^\mu = 0$. For a photon, the residue of the ELI pole (at $q^\mu = 0$) is the charge, consequently conservation of charge is in a sense part of the low energy theorems for the photon. In addition, the kinematical singularities at threshold further specify the low energy behavior. For gravitons, the soft coupling is the inertial mass as demanded by the equivalence principle. We can demonstrate these points by considering the ZML in conjunction with crossing.

A. Conservation of Charge

Consider the four-particle process with the s,t, and u channels defined by

$$s : V\, S \to \bar{T}\, \bar{U} \,, \qquad t : V\, T \to \bar{S}\, \bar{U} \,, \qquad u : V\, U \to \bar{T}\, \bar{S} \,,$$

where we denote the particles in the initial state by the same letter as the channel. Let S, T, and U have masses m_S, m_T, and m_U and charges e_S, e_T, and e_U respectively.

For a massive photon (m_V), we consider the helicity amplitudes in the t channel,

$$H_1{}^t = -H_{-1}{}^t = \frac{(\phi)^{\frac{1}{2}}}{\mathcal{J}} \bar{H}_1{}^t \,, \qquad H_0{}^t = \frac{1}{\mathcal{J}} \bar{H}_0{}^t \,, \qquad (3.1)$$

where we have exhibited the kinematical singularities in terms of the Kibble function ϕ and the threshold and pseudothreshold factor

$\mathcal{J} = [(t - (m_T + m_V)^2)(t - (m_T - m_V)^2)]^{\frac{1}{2}}$. We can now use the Jacob and Wick crossing matrix to express the s-channel helicity amplitude $H_1{}^s = [(\phi)^{\frac{1}{2}}/\mathcal{S}]\, \bar{H}_1{}^s$ in terms of the t-channel amplitudes:

$$\bar{H}_1{}^s(s,t) = \frac{1}{\mathcal{J}^2} [P_V(s,t)\, \bar{H}_1{}^t + \sqrt{2}\, m_V\, \bar{H}_0{}^t] ,$$

where

$$P_V(s,t) = (s + m_V{}^2 - m_S{}^2)(t + m_V{}^2 - m_U{}^2) - 2\, m_V{}^2(m_V{}^2 + m_U{}^2 - m_T{}^2 - m_S{}^2) . \qquad (3.2)$$

The essential observation is that the direct-channel ELI pole at $s = m_S{}^2$ (or $t = m_T{}^2$) cannot contribute to the helicity-one amplitude $\bar{H}_1{}^s$ (or $\bar{H}_1{}^t$) because of angular momentum conservation. On the other hand, we can use the crossing relation to calculate the crossed-channel pole terms from their contribution to the zero-helicity amplitudes. For example, the residue of the pole in $\bar{H}_1{}^s$ at $t = m_T{}^2$ is given in terms of the residue of the pole in $H_0{}^t$ by use of Eq. (3.2). Therefore, we define the charge with the conventional factors through the zero-helicity amplitudes,

$$\lim_{X\to m_X{}^2} (X - m_X{}^2)\, H_0{}^x = - i(4m_X{}^2 - m_V{}^2)^{\frac{1}{2}}\, e_X\, g , \qquad (3.3)$$

where $X = S$, T, and U, and g is the coupling at the S T U vertex (g has units of mass). Then by use of the s-t (3.2), s-u, and t-u crossing relations, we determine the residue of the ELI poles in $\bar{H}_1{}^t$, and $\bar{H}_1{}^s$ and obtain the representations

$$\bar{H}_1^{\ t} = \frac{\sqrt{2}\ g\ e_S}{s - m_S^{\ 2}} - \frac{\sqrt{2}\ g\ e_U}{u - m_U^{\ 2}} + B_1^{\ t}(s,t) \ ,$$

$$\bar{H}_1^{\ s} = \frac{\sqrt{2}\ g\ e_U}{u - m_U^{\ 2}} - \frac{\sqrt{2}\ g\ e_T}{t - m_T^{\ 2}} + B_1^{\ s}(s,t) \ , \tag{3.4}$$

where $B_1^{\ t}$ and $B_1^{\ s}$ are analytic in a neighborhood of $t = m_T^{\ 2}$, $s = m_S^{\ 2}$, $(u = m_T^{\ 2} + m_V^{\ 2})$ for small m_V except for thresholds that are of higher order in e^2. With $m_V\ H_0^{\ t} \to 0$ as $m_V \to 0$, (2.1), the crossing relation becomes diagonal in the ZML and we arrive at the nontrivial consistency condition for the representations (3.3) of $\bar{H}_1^{\ t}$ and $\bar{H}_1^{\ s}$,

$$\sqrt{2}\ g(e_S + e_T + e_U) + (s - m_S^{\ 2})\ B_1^{\ t}(s,t) - (t - m_T^{\ 2})\ B_1^{\ s}(s,t) = 0. \tag{3.5}$$

This not only implies conservation of charge $e_S + e_T + e_U = 0$ for the nonzero hadron vertex $(g \neq 0)$, but also $B_1^{\ t} = (t - m_T^{\ 2})\ \bar{B}_1$ and $B_1^{\ s} = (s - m_S^{\ 2})\ \bar{B}_1$. Consequently we have the representation

$$H_1^{\ t} = \frac{(\phi)^{\frac{1}{2}}}{t - m_T^{\ 2}} \left[\frac{\sqrt{2}\ g\ e_S}{s - m_S^{\ 2}} - \frac{\sqrt{2}\ g\ e_U}{u - m_U^{\ 2}}\right] + (\phi)^{\frac{1}{2}}\ \bar{B}_1(s,t) \ , \tag{3.6}$$

and the residue of the pole at $t = m_T^{\ 2}$ is $i\sqrt{2}\ m_T g\ e_T$. Since the T pole enters into the amplitude with the standard factorized residue, and crossing symmetry puts it on the same footing as the S pole (e.g., consider the corresponding representation for H_1^s, which equals H_1^t by crossing), one is forced to consider it a dynamical pole.

In Ref. 1, we sketch the proof for hadrons of arbitrary spin (J_S, J_T, and J_U). The main features are the same. There is one nonsense amplitude $H^t_{\lambda_S\lambda_U;1-J_T}$ which does not have the pole at $t = m_T^2$ because of angular momentum conservation for $m_V \neq 0$, and in the ZML a kinematical factor $1/\mathcal{J}^2$ turns into a dynamical pole. In this proof, one has to use the crossing relation with much greater care, particularly the threshold and pseudothreshold relation at $t = (m_T \pm m_V)^2$. We note that, on the basis of Ref. 2, one can extend our proof of charge conservation to any N-hadron process by induction. If any two hadrons with arbitrary spins in the N-body amplitude have a pole, by factorization, we reduce the problem to an (N - 1)-body amplitude and a general three-body vertex presented in Ref. 1.

Finally, note that a more careful use of the crossing constraint and the condition $H_0^t \sim O(m_V)$ leads one to the approximate universality statement

$$e_S + e_T + e_U = O\left(\frac{m_V^2}{m^2}\right) , \tag{3.7}$$

where e_X is the $\rho X\bar{X}$-coupling constant, and m is a characteristic mass for the process involved. This may give some justification for an approximate rho universality.

B. Massless Particles of General Spin and Parity

With spinless hadrons, the massless particle must have natural spin-parity $\left(P = (-)^J\right)$ in order to have any ELI poles. For the graviton (2^+), one can essentially repeat the above discussion to show that the soft coupling f_X is proportional to the mass m_X in the rest frame of particle X. The soft coupling f_X is again defined through the zero-helicity amplitudes,

$$\lim_{X \to m_X^2} (X - m_X^2)\, H_0^{x} = -4\, m_X\, f_X\, g \quad . \tag{3.8}$$

Using the crossing relation for a massive graviton and the kinematical singularity-free amplitudes for t channel,

$$H_\lambda^{t} = \frac{[(\phi)^{\frac{1}{2}}]^{|\lambda|}}{\mathcal{J}^2}\, \overline{H}_\lambda^{t} \quad , \tag{3.9}$$

and similarly for s channel; we arrive at the representation

$$\begin{aligned} \overline{H}_2^{t} &= \frac{(6)^{\frac{1}{2}} f_S\, g}{m_S(s - m_S^2)} - \frac{(6)^{\frac{1}{2}} f_U\, g}{m_U(u - m_U^2)} + B_2^{t} \quad , \\ \overline{H}_2^{s} &= \frac{(6)^{\frac{1}{2}} f_T\, g}{m_T(t - m_T^2)} + \frac{(6)^{\frac{1}{2}} f_U\, g}{m_U(u - m_U^2)} + B_2^{s} \quad . \end{aligned} \tag{3.10}$$

If we now put $m_G = 0$, the crossing relation reduces to

$$(s - m_S^2)^2\, \overline{H}_2^{t} = (t - m_T^2)^2\, \overline{H}_2^{s} \quad . \tag{3.11}$$

Substituting Eq. (3.10) in Eq. (3.11), we find for the pole terms

$$\left(\frac{f_S}{m_S} - \frac{f_U}{m_U}\right)(s - m_S{}^2)^2 + \frac{f_S}{m_S}(s - m_S{}^2)(t - m_T{}^2)$$

$$= \left(\frac{f_T}{m_T} - \frac{f_U}{m_U}\right)(t - m_T{}^2)^2 + \frac{f_T}{m_T}(s - m_S{}^2)(t - m_T{}^2) \quad . \tag{3.12}$$

This equation is satisfied if

$$f_X = G_o\, m_X \;, \quad \text{for arbitrary } G_o \quad . \tag{3.13}$$

Since we have used center-of-mass helicity amplitudes, at $q^\mu = 0$ we are in the rest frame of particle X, and therefore this is the correct form of the equivalence principle.[8] Note that we have found the same sign for the coupling of the graviton to particles and antiparticles.

If the zero-mass particle has a spin higher than 2, and natural J parity, the kinematic singularities are given by (S, T, U spinless)

$$H_\lambda{}^t = \frac{[(\phi)^{\frac{1}{2}}]^{|\lambda|}}{\mathcal{J}^J}\; \bar{H}_\lambda{}^t \quad . \tag{3.14}$$

In the equation analogous to Eq. (3.12) there will be terms (2nd and 4th) proportional to $f_S(s - m_S{}^2)^{J-1}(t - m_T{}^2)$ and $f_T(t - m_T{}^2)^{J-1}(s - m_S{}^2)$ which cannot be matched. Therefore, the soft couplings of zero-mass particles with spin higher than 2 must vanish.

If the zero-mass particle has unnatural parity, then its coupling to spinless particles is zero, due to conservation of angular momentum and parity. We studied these couplings to particles with spin with the techniques introduced in Ref. 1. One can prove, for example, that all the soft couplings of an axial vector zero-mass particle must vanish. We can extend this result to all unnatural parity particles except the 0^-.[14]

C. Low Energy Theorems

1. Photoproduction

The representation (3.6) is the low energy theorem for photoproduction with spinless hadrons, analogous to the Kroll-Ruderman theorem for $\gamma N \to \pi N$. Usually these theorems are expressed in terms of the momenta $k(s)$ of the photon at fixed center-of-mass angle (Θ_s) in the s channel. Using $(\phi)^{\frac{1}{2}} = 2(s)^{\frac{1}{2}}\, k\, p(s) \sin \Theta_s$ and charge conservation, we can rewrite (3.6) as

$$H_1^{\,s} = p(s) \sin \Theta_s \left\{ \frac{\sqrt{2}\, g\, e_U}{u - m_U^{\,2}} - \frac{\sqrt{2}\, g\, e_T}{t - m_T^{\,2}} + 4\, k^2\, \overline{B}_1 \right\} . \qquad (3.15)$$

At fixed angle, we have the correct amplitude to order k from the Born term. In pion photoproduction, the physical threshold is at $s = (m_N + m_\pi)^2$, consequently the theorem is said to give orders $(m_\pi/m_N)^{-1}$ and $(m_\pi/m_N)^0$ (in the pion mass) relative to the background[15] of order (m_π/m_N). In either case, the low energy theorem is dependent only on identifying the correct ELI term (Born approximation) and the correct kinematical singularities.

2. Compton Scattering

Similar use of crossing relations and the ZML yield the low energy theorems for Compton scattering, $\gamma\pi^{\pm} \rightarrow \gamma\pi^{\pm}$. Since we shall need these results in Sec. IV, it is useful to present them here.

The independent t-channel helicity amplitudes $H^t_{\lambda_1\lambda_2}(\gamma\gamma \rightarrow \pi\pi)$ for the massive photon (m_V) are

$$H_{11}^{t} = \frac{1}{t - 4m_V^2} \bar{H}_{11}^{t} , \qquad H_{10}^{t} = \frac{(\phi)^{\frac{1}{2}}}{t - 4m_V^2} \bar{H}_{10}^{t} ,$$

$$H_{1-1}^{t} = \frac{\phi}{t(t - 4m_V^2)} \bar{H}_{1-1}^{t} , \quad H_{00}^{t} = \frac{1}{t - m_V^2} \bar{H}_{00}^{t} , \tag{3.16}$$

where parity, $H^t_{\lambda_1\lambda_2} = (-)^{\lambda_1-\lambda_2} H^t_{-\lambda_1-\lambda_2}$, and statistics, $H^t_{\lambda_1\lambda_2} = H^t_{\lambda_2\lambda_1}$, give the dependent ones. To prove the low energy theorem we also need the s-channel amplitudes

$$H_{1;1}^{s} = -\frac{\phi}{ts}\left[F^+ + \frac{1}{\mathcal{S}^2} F^-\right] , \quad H^s_{1;-1} = t\left[F^+ - \frac{1}{\mathcal{S}^2} F^-\right] , \tag{3.17}$$

where $\mathcal{S} = [(s - (m_\pi + m_V)^2)(s - (m_\pi - m_V)^2)]^{\frac{1}{2}}$ and $F^{\pm}$ are kinematical singularity-free amplitudes with parities $P = \pm(-)^J$ in leading order. There is a conspiracy relation at $s = 0$ of the form $F^+ = -\frac{1}{\mathcal{S}^2} F^-$.

Again we introduce the charge through the zero-helicity (sense) amplitude

$$\lim_{s\to m_\pi^2} (s - m_\pi^2)\, H_{00}^s = \lim_{u\to m_\pi^2} (u - m_\pi^2)\, H_{00}^u = -(4m_\pi^2 - m_V^2)e^2 . \tag{3.18}$$

With the use of the Jacob and Wick crossing matrix, the residue of poles can be calculated in $H_{1\pm1}^t$ and $H_{1;\pm1}^s$, and dropping terms of order m_V^2 in the residues of the pion poles, we get the representations

$$H_{11}^t = \frac{1}{t - 4m_V^2}\left[\frac{2e^2\, t\, m_\pi^2}{s - m_\pi^2} + \frac{2e^2\, t\, m_\pi^2}{u - m_\pi^2} + B_{11}^t\right] ,$$

$$H_{1-1}^t = \frac{\phi}{t(t - 4m_V^2)}\left[\frac{2e^2}{s - m_\pi^2} + \frac{2e^2}{u - m_\pi^2} + B_{1-1}^t\right] ,$$

$$H_{1;1}^s = -\frac{\phi}{t}\left[\frac{2e^2(s - m_\pi^2)}{\mathscr{S}^2(u - m_\pi^2)} + \frac{1}{s}\left(B_+^s + \frac{1}{\mathscr{S}^2} B_-^s\right)\right] , \tag{3.19}$$

$$H_{1;-1}^s = -t\left[\frac{2e^2\, m_\pi^2(s - m_\pi^2)}{\mathscr{S}^2\,(u - m_\pi^2)} - B_+^s + \frac{1}{\mathscr{S}^2} B_-^s\right] .$$

Note that only the u pole contributes to $H_{1;\pm1}^s$ because of angular momentum conservation, and the factor $1/\mathscr{S}^2$ introduces a first-order pole at $s = m_\pi^2$ in the ZML.

We take the limit $m_V \to 0$ and apply the resultant crossing relations $H_{11}{}^t = H^s_{1;-1}$, $H_{1-1}{}^t = H_{1;1}{}^s$. Since we have assumed charge conservation, the pole terms satisfy these relations by themselves, leaving constraints on the background terms. These constraints are solved with the new background terms $\bar{B}_{11}{}^t$ and $\bar{B}_{1-1}{}^t$, where

$$B_{11}{}^t = t^2 \bar{B}_{11}{}^t, \qquad B_{1-1}{}^t = t \bar{B}_{1-1}{}^t ,$$

$$B_+{}^s = \frac{1}{2} [\bar{B}_{11}{}^t + s \bar{B}_{1-1}{}^t], \qquad B_-{}^s = - \frac{(s - m_\pi{}^2)}{2}[\bar{B}_{11}{}^t - s \bar{B}_{1-1}{}^t]. \tag{3.20}$$

In addition to kinematical zeros, being forced into the background terms (B), there is the conspiracy relation $B_+{}^s = - m_\pi{}^2 B_-{}^s/2$ at $s = 0$.

The resultant expression for the Compton amplitude has the full content of the low energy theorem:

$$H^s_{1;-1} = H_{11}{}^t = \frac{-2e^2 m_\pi{}^2 t}{(s - m_\pi{}^2)(u - m_\pi{}^2)} + t \bar{B}_{11}{}^t ,$$

$$H_{1;1}{}^s = H_{1-1}{}^t = - \frac{\phi}{t} \frac{2e^2}{(s - m_\pi{}^2)(u - m_\pi{}^2)} + \frac{\phi}{t} \bar{B}_{1-1}{}^t . \tag{3.21}$$

For example, it is usually stated that the Born term gives the zeroth and first-order contributions in the momentum of the photon $k(s)$ for $H^s_{1;-1}$ at fixed angle, which comes from the factor $\phi/t = - 2k^2 s(1 + \cos \theta_s)$.

IV. REGGE ASYMPTOTICS AND CROSSING

The Veneziano representation[16] for the hadron amplitude, $\omega\pi \to \pi\pi$, gave a simultaneous solution to crossing and Regge asymptotics, provided the zero-width $(\mathrm{Im}\,\alpha = 0)$ and linear-trajectory approximations $\left(\alpha(t) = a + bt\right)$ are made. We demonstrate in this section that an equally simple solution to Regge asymptotics and crossing exists for photon amplitudes. The problem is to understand how the low energy constraints (i.e., gauge invariance, Sec. III) are incorporated in this representation. To this end, we first consider the ZML for a Regge asymptotic expansion for massive photons (Sec. IV.A), and then include crossing symmetry (Secs. IV.B and IV.C) in a zero-width model.

A. Nonsense Factors in the Regge Expansion

Even for large energy $(s \to \infty)$, the representation (3.6) poses the problem of an exchange of the particle T entering into the nonsense amplitude $H_1{}^t$. Let us consider the ZML for the standard Regge expansion for a massive photon (m_V) in the t channel $(t : V\, T \to \overline{S}\, \overline{U})$ helicity amplitudes:[4]

$$\tilde{H}_1{}^t = \frac{1}{(\phi)^{\frac{1}{2}}} H_1{}^t = \frac{1}{\mathcal{J}} \sum_k \frac{\alpha_k\, \gamma_1{}^k(t)}{\sin \pi\, \alpha_k} \left(1 \pm e^{-i\pi\alpha_k}\right) \left(\frac{s}{s_0}\right)^{\alpha_k - 1} , \tag{4.1}$$

$$H_0{}^t = \frac{1}{\mathcal{J}} \sum_k \frac{\gamma_0{}^k(t)}{\sin \pi\, \alpha_k} \left(1 \pm e^{-i\pi\alpha_k}\right) \left(\frac{s}{s_0}\right)^{\alpha_k} . \tag{4.2}$$

The nonsense factor $\alpha_k(t)$ excludes all zero-spin poles for helicity amplitude H_1^t as demanded by angular momentum conservation, and γ's are free of kinematical singularities.

If we take $m_V \to 0$, the factor $1/\mathcal{J}$ reinstates the T pole, but it apparently occurs multiplicatively on all terms in the expansion. This would mean that the dynamical T pole could not be considered to be exclusive on the T trajectory. (For example, the pion exchange in $\gamma p \to \pi^+ n$ would not be a Regge exchange!) The solution to this problem is the correct use of the threshold and pseudothreshold relations at $t_\pm = (m_T \pm m_V)^2$.

Using the crossing condition (3.2) on the expansion (4.1) and (4.2), one may derive the constraints

$$\pm\, \alpha_k(t_\pm)\ \gamma_1^{\,k}(t_\pm) \;=\; -\ \gamma_0^{\,k}(t_\pm)/(\sqrt{2}\ m_\pi\ s_o) \quad . \tag{4.3}$$

In addition, the residue for T has the normalization constraints from Eq. (3.3),

$$\gamma_0^{\,T}(m_T^{\,2}) \;=\; \frac{\pi}{2}\, \alpha_T'(m_T^{\,2})\ m_V(4m_T^{\,2} - m_V^{\,2})\ e\ g \quad . \tag{4.4}$$

If one expands $\gamma_1^{\,k}$ and $\gamma_0^{\,k}$ in a power series about $t = m_T^{\,2}$ and applies the condition (2.9), $\gamma_0^{\,k} = O(m_V)$, one discovers that in the ZML the pole at $t = m_T^{\,2}$ occurs only in the T Regge term and is properly normalized. More precisely, a comparison of the coefficient of $(m_V)^j$ in (4.3) and (4.4) yields

$$\bar{\gamma}_1^{\,T}(t) = \frac{\alpha_T(t)\ \gamma_1^{\,T}(t)}{t - m_T^{\,2}} = -\pi\ \alpha_T'\ (m_T^{\,2})\ \frac{e_T\ g}{\sqrt{2}\ s_o} + O(t - m_T^{\,2}) \tag{3.5}$$

and

$$\bar{\gamma}_1^{\,k}(t) = \frac{\gamma_1^{\,k}(t)}{t - m_T^{\,2}}, \quad k \neq T \ , \tag{3.6}$$

where $\bar{\gamma}_1$ are the new kinematic-singularity-free residue functions.

Hence, we arrive at the new Regge expansion for the kinematical-singularity-free amplitude $\tilde{H}_1^{\,t} = H_1^{\,t}/(\phi)^{\frac{1}{2}}$ at $m_V = 0$,

$$\tilde{H}_1^{\,t} = \frac{\gamma_1^{\,T}(t)}{\sin\pi\ \alpha_T}\,(1 + e^{-i\pi\alpha_T})\left(\frac{s}{s_o}\right)^{\alpha_T - 1}$$

$$+ \sum \frac{\alpha_k\ \bar{\gamma}_1^{\,k}}{\sin\pi\ \alpha_k}\,(1 \pm e^{-i\pi\lambda_k})\left(\frac{s}{s_o}\right)^{\alpha_k - 1} , \tag{4.7}$$

where the nonsense factor for the ELI pole (T) is absent and the remaining nonsense factors are in agreement with the rule "no zero-to-zero transitions." This expansion is in complete agreement with the "low energy" theorem (3.6) in the limit $t \to m_T^{\,2}$ and s large.

B. Duality and the Zero-Width Approximation

The Veneziano representation requires a particular interpolation (referred to as duality) between the low energy region and the asymptotic region. If we reinspect the Regge expansion (4.7) for massless particles, we can see that the peculiar constraints of gauge invariance are themselves suggestive of this duality interpolation. At threshold, the factor $(kp)^{\alpha-1}$ in the full residue requires that the Regge expansion in terms of $\nu = 2\ kp\ z_t$ is given exactly by the leading terms of $(kp)^{\alpha-1}\ P'_{\alpha}(z_t)$. For the zero-mass photons, the T exchange pole coincides with threshold $\left(k = (t - m_T^2)/2\right)$, so that the exact s dependence of the residue of the pole is determined by the Regge expansion. As $t \to m_T^2$ we obtain the result

$$(\tilde{H}_1{}^t)_{\text{Regge}} \to \frac{1}{t - m_T^2}\left[\frac{\sqrt{2}\ g\ e_S}{s - m_S^2} - \frac{\sqrt{2}\ g\ e_U}{u - m_U^2}\right] , \tag{4.8}$$

where we have used conservation of charge $e_T = -e_S - e_U$ and the kinematical relation $\nu = s - m_S^2 = -(u - m_U^2)$ for $t = m_T^2$. Consequently, we see that the full content of the low energy theorem is contained in the Regge expansion, by "summing up" the Regge expansion to get a direct channel pole. Far from being in conflict with duality, Eq. (4.8) may be considered an extreme example of duality for $\tilde{H}_1{}^t$.

There is apparently only one way to modify the Veneziano parameterization to accommodate the double ELI pole term. (We pick a channel with $e_U = 0$ for the present discussion.)

$$\tilde{H}_1{}^t = \frac{\sqrt{2}\ ge\ b^2}{\alpha_T(t)}\ B(1 - \alpha_T, - \alpha_S) = \sqrt{2}\ g\ e\ b^2\ \tilde{B}(-\alpha_T, - \alpha_S) , \tag{4.9}$$

where

$$\tilde{B}(-\alpha_T, -\alpha_S) = \frac{\Gamma(-\alpha_T)\,\Gamma(-\alpha_S)}{\Gamma(1-\alpha_T-\alpha_S)} \quad . \tag{4.10}$$

The reader may check that this "guess" simultaneously satisfies all the major conditions: (i) It is even under $\alpha_S \leftrightarrow \alpha_T$, as demanded by crossing $\tilde{H}_1{}^t = \tilde{H}_1{}^s$, (ii) it has the correct low energy behavior, (iii) it has Regge asymptotic behavior as $s \to \infty$ (or $t \to \infty$) with the correct helicity flip factors, (iv) the residues of the poles at $\alpha_T = J \geqslant 1$ are polynomials of order $J - 1$ in α_S (no ancestors).

To understand how these properties are obtained, we resort to the ZML once again. We take the residue of the $J = 1$ pole (mass $q^2 = m_V{}^2$, see Fig. 1) in the five-particle beta function,[17]

$$B\big(\alpha(q^2), \alpha_1, \sigma, \tau, \alpha_2\big) = \int_0^1 du_1 du_2\, u_1{}^{-1-\alpha(q^2)}(1-u_1)^{-1-\alpha_1}$$

$$\times\ (1-u_1u_2)^{\alpha_2-\alpha_1-\sigma}\,(1-u_2)^{-1-\sigma}\,u_2{}^{-1-\tau} \quad , \tag{4.11}$$

and observe explicitly how to deduce the above parameterization by the ZML. This deductive approach to Eq. (3.9) not only guarantees the satisfaction of properties (i) - (iv), but also can be extended to photoproduction of N hadrons with the N + 2-particle beta functions.[7]

It is convenient to calculate $H_\lambda{}^t$ (for $m_V \neq 0$) from the five-particle beta function through the tensor $B_{st}{}^\mu$ and the helicity $(J = 1)$ projection formula $H_\lambda{}^t = \epsilon_\mu(\lambda,q)\, B_{st}{}^\mu$. We discover that

$$B_{st}{}^{\mu} = \left(\tfrac{1}{2} q^{\mu} + p_S{}^{\mu}\right) B(1-\tau, -\sigma) - \left(\tfrac{1}{2} q^{\mu} + p_T{}^{\mu}\right) B(1-\sigma-\tau) \ . \qquad (4.12)$$

From this we calculate $H_1{}^t$ (see Table Ia),

$$H_1{}^t = \frac{(\phi)^{\frac{1}{2}}}{\mathcal{J}} \frac{1}{\sqrt{2}} B(1-\tau, -\sigma) \ . \qquad (4.13)$$

We are permitted to take the ZML, if $m_V H_o{}^t \to 0$ or equivalently[12] $q_{\mu} B_{st}{}^{\mu}/m_V \to 0$ as $m_V \to 0$. This condition holds only if S and T are the lowest members of the $\sigma(s)$ and $\tau(t)$ trajectories (i.e., $\sigma = \alpha_S$, where $\alpha_S(m_S{}^2) = 0$, and $\tau = \alpha_T$, where $\alpha_T(m_T{}^2) = 0$).

Moreover, an off-mass-shell continuation is possible because the condition $H_o{}^t = O[(q^2)^{\frac{1}{2}}]$ or $q_{\mu} B_{st}{}^{\mu} = O(q^2)$ also holds. We have chosen the "unphysical" q^{μ}-term so that

$$q_{\mu} B_{st}{}^{\mu} \equiv \alpha_S B(1-\alpha_T, -\alpha_S) - \alpha_T B(-\alpha_T, 1-\alpha_S) \equiv 0 \ , \qquad (4.14)$$

and we therefore can easily construct a conserved vector current amplitude V^{μ}. The following amplitude has the correct ELI poles at $q^{\mu} \equiv 0$ for $\gamma S \to \bar{T}\,\bar{U}$ and an appropriate vector-meson pole at $q^2 = m_{\rho}{}^2$:

$$V^{\mu} = F(q^2)[C_{st} B_{st}{}^{\mu} + C_{tu} B_{tu}{}^{\mu} + C_{us} B_{tu}{}^{\mu}] \ , \qquad (4.15)$$

where

$$F(q^2) = m_\rho^2/(m_\rho^2 - q^2) , \quad C_{st} - C_{tu} = \sqrt{2}\, g\, e_T\, b^2,$$

$$\text{and} \quad C_{tu} - C_{us} = \sqrt{2}\, g\, e^2\, e_U\, b^2 \quad .$$

The amplitude in brackets, $V^\mu/F(q^2)$, may be thought of as an off-mass-shell continuation of the vector meson amplitude, since s, t, and u are now constrained by

$$s + t + u = m_S^2 + m_T^2 + m_U^2 + q^2 \quad \text{for any} \quad q^2 \quad . \qquad (4.16)$$

Notice that there is one arbitrary constant in (4.15) that corresponds to a term $C(B_{st}^{\ \mu} + B_{us}^{\ \mu} + B_{tu}^{\ \mu})$ which, because of the condition $\alpha_S + \alpha_T + \alpha_U = b\, q^2$, does not affect the residue of the ELI poles.

C. Compton Scattering and The Pomeranchon

A similar discussion can be made for Compton scattering $(\gamma\pi \to \gamma\pi)$, but with some significant differences, which we will point out. We consider the projection of two $J = 1$ poles (mass m_V) from the six-particle beta function.[17] We must distinguish between projections which yield adjacent photons (Fig. 2, st and ut terms) and nonadjacent photons (Fig. 2, su term).

1. Nonadjacent Photons

For the nonadjacent photons, again the photon-hadron channel can contain the same hadron as the $J = 0$ particle on the sequence [i.e., $\sigma(s) = \alpha_\pi(s)$, $\mu(u) = \alpha_\pi(u)$] , so it is not surprising that the projection results in a doubly conserved nonsingular tensor (see Table Ib)

$$B_{su}^{\mu\nu} = \frac{g^{\mu\nu}}{2b} B\Big(1 - \alpha_\pi(s), 1 - \alpha_\pi(u)\Big) + P^\mu P^\nu B\Big(-\alpha_\pi(s), -\alpha_\pi(u)\Big)$$

$$+ q_2^{\mu} P^\nu [B\Big(1 - \alpha_\pi(s), -\alpha_\pi(u)\Big) - \frac{1}{2} B\Big(-\alpha_\pi(s), -\alpha_\pi(u)\Big)]$$

$$+ P^\mu q_1^{\nu} [B\Big(-\alpha_\pi(s), 1 - \alpha_\pi(u)\Big) - \frac{1}{2} B\Big(-\alpha_\pi(s), -\alpha_\pi(u)\Big)]$$

$$+ q_2^{\mu} q_1^{\nu} [B\Big(1 - \alpha_\pi(s), 1 - \alpha_\pi(u)\Big) - \frac{1}{4} B\Big(-\alpha_\pi(s), -\alpha_\pi(u)\Big)] . \tag{4.17}$$

From Table Ib one can deduce the helicity amplitudes and obtain

$$H_{1-1}^{\ t} = -\frac{\emptyset}{t} 2e^2 b^2 \tilde{B}\Big(-\alpha_\pi(s), -\alpha_\pi(u)\Big) ,$$

$$H_{11}^{\ t} = \frac{\emptyset}{t} 2e^2 b^2 \tilde{B}\Big(-\alpha_\pi(s), -\alpha_\pi(u)\Big) - 2e^2 B\Big(1 - \alpha_\pi(s), 1 - \alpha_\pi(u)\Big) . \tag{4.18}$$

In Ref. 3, we introduced the nonflip amplitudes

$$H_{11}^{\ t} = -2t\ m_\pi^{\ 2}\ e^2 b^2\ \tilde{B}(-\alpha_\pi(s), -\alpha_\pi(u)) , \tag{4.19}$$

and demonstrated that this corresponded to an $M = 1$ pion with a parity partner that chooses nonsense at $J = 0$. The solution found here with the ZML is an $M = 0$ pion[18] which indicates some of the flexibility in the parameterization.

Clearly the form factor $F(q_1^{\,2})\,F(q_2^{\,2})$ may be multiplied onto $B_{su}^{\;\mu\nu}$ to construct the off-mass-shell two-current amplitude $M^{\mu\nu}(q_1,q_2)$, just as in the single photon case.

2. Adjacent Photons

The projection of adjacent (massive) photon amplitudes from the six-point beta functions (st and ut terms in Fig. 2) in general does not lead to suitable amplitudes for the ZML. The difficulty arises because the two-photon trajectory does not correspond to the external masses ($q_1^{\,2}$ or $q_2^{\,2}$) and the resultant zero-helicity amplitudes diverge as $m_V \to 0$. The only exception is for a Pomeranchon in the t channel with intercept exactly at $J = 1$ at $t = 0$.

In this case for $m_V = 0$, we can add to $(-t/\emptyset)H_{1-1}^{\;\;t}$ arbitrary terms proportional to

$$S_{Pom} = \tilde{B}\big(1 - \alpha_p(t),\, -\alpha_\pi(s)\big) + \tilde{B}\big(1 - \alpha_p(t),\, -\alpha_\pi(u)\big) + \tilde{B}\big(-\alpha_\pi(s),\, -\alpha_\pi(u)\big) \quad , \qquad (4.20)$$

which gives a Pomeranchon that couples at $t = 0$, and because of the condition $\alpha_p(t) - 1 + \alpha_\pi(s) + \alpha_\pi(u) = 0$ does not affect the ELI poles. Such a Pomeranchuk is needed to give constant total cross section for photoproduction.

Even in this case the ZML does not lead to the necessary condition $H_{00}^{\;\;t} = O(q_1^{\,2}q_2^{\,2})$ [but instead $H_{00}^{\;\;t} = O(q_1^{\,2}) + O(q_2^{\,2})$] for continuing this adjacent-photon amplitude off-mass-shell. In fact, one is forced to introduce a fixed pole at $J = 0$ in the two-photon channel, in order to obtain an off-mass-shell continuation of (4.20), as discussed briefly in Sec. V.

V. FIXED J SINGULARITIES IN THE TWO-CURRENT CHANNEL

So far we have constructed pure Regge asymptotic amplitudes for physical (chargeless, massless) photons. It is a characteristic of these parameterizations, as well as all the Veneziano parameterizations for hadron processes, that there are nonsense wrong-signature fixed poles[19] due to the third term [e.g., $B\big(-\alpha(s), -\alpha(u)\big)$ has fixed poles at $\alpha(t) = -1, -3, \cdots$]. We shall discuss such a fixed pole in $H_{1-1}^{\ t}$ for $J = 1$, not because it is peculiar to weak amplitude, but because it has been postulated as a mechanism for canceling the nonsense factor $\alpha_P(t) - 1$ for the Pomeranchuk.[20]

In a unitary model for strong interaction, these nonsense wrong-signature fixed poles must by "covered" by a moving Mandelstam cut. Otherwise fixed poles would be excluded by the quadratic nature of unitarity for strong amplitudes. In weak processes unitarity becomes linear and fixed poles cannot generally be excluded. Indeed there is a right-signature $(J = 0)$ fixed pole associated with our Pomeranchuk solution off-mass-shell; and the right-signature $(J = 1)$ fixed pole[5] associated with charged currents. These right-signature fixed poles in nonsense amplitudes (and Kronecker delta term in sense amplitudes) are easily identified by the resultant fixed-power behavior of the amplitude. Although other fixed-power behavior is consistent with linear unitarity, it is interesting that so far there is evidence for fixed J singularities (at right signature point) only (a) in the two-current channel and (b) for either charged or massive currents (i.e., "unphysical photons").

A. The Pomeranchon and Fixed Poles

Abarbanel et al.[20] introduced, into the double helicity flip amplitude $H_{1-1}{}^{t}$ of Compton scattering, a Pomeranchuk with a singular residue to cancel the nonsense factor. On the basis of an N/D model for linear unitarity in the t channel $(\gamma\gamma \to \pi\pi)$, they found that the reduced partial-wave amplitude $b^{(+)}(J,t)$, where

$$a^{(+)}(J,t) = (pq)^{J-2} [(J+2)(J+1)J(J-1)^{\frac{1}{2}}] \, b^{(+)}(J,t)$$

for positive signature, has the form:

$$b^{(+)}(J,t) = 4e^2 \frac{1}{J - \alpha_P(t)} \frac{1}{t} + \frac{4e^2}{J-1} \frac{\tilde{R}(J,0)}{J - \alpha_P(t)} \qquad (5.1)$$

for $\alpha_P(0) = 1$. The first term is a Regge pole with a singular residue derived from the Born (pion pole) contribution to the left-hand cut, and the second term is a "multiplicative" fixed pole derived from more distant left-hand singularities. If one rewrites the "multiplicative" fixed pole via

$$\frac{1}{J-1} \frac{1}{J - \alpha_P(t)} = \frac{1}{\alpha_P(t) - 1} \frac{1}{J - \alpha_P(t)} - \frac{1}{\alpha_P(t) - 1} \frac{1}{J - 1} , \quad (5.2)$$

one can see that it is equivalent to a Regge singular pole minus a singular fixed pole, that does not contribute to the asymptotic behavior of the physical amplitude.

Our Pomeranchon has precisely this structure in the J plane, as we see from the amplitude for $\gamma\pi^{\pm} \to \gamma\pi^{\pm}$,

$$H_{1-1}{}^{t} = 4e^2 b^2 \frac{\phi}{-t} [C_o \tilde{B}\left(1 - \alpha_P, -\alpha_\pi(s)\right) + C_o \tilde{B}\left(1 - \alpha_P, -\alpha_\pi(u)\right)$$

$$+ (1 + C_o) \tilde{B}\left(-\alpha_\pi(s), -\alpha_\pi(u)\right)] . \qquad (5.3)$$

The first two terms result in a Pomeranchon with a singular residue, which gives the asymptotic contribution

$$H_{1-1}{}^{t} \sim 4e^2 b^2 \frac{\phi}{-t} C_o \Gamma\left(1 - \alpha_P(t)\right)(1 + e^{-i\pi\alpha_P})(bs)^{\alpha_P - 2} , \qquad (5.4)$$

and the third term has no asymptotic contribution, but it contains a fixed pole[19] in the J-plane at $J = 1$,

$$b^{(+)}(J,t) \propto \frac{1 + C_o}{J - 1} \frac{1}{bt} 2^{-bt} . \qquad (5.5)$$

This correspondence is more than an analogy. If we assume that there is no fixed pole $(C_o = -1)$, we arrive at the "predicted" cross section

$$\sigma_{TOT}(\infty) = 16\pi^2 \frac{e^2}{4\pi} \alpha_P'(0) \frac{1}{3} , \qquad (5.6)$$

which is identical to the result of Abarbanel et al. In our case, the Pomeranchuk slope $\alpha_P'(0)$ must be canonical $\left(\alpha_P'(0) = b \approx 1 \text{ GeV}^{-2}\right)$, and the value of this cross section is too large by a factor of 3. We see no reason in favor of this special value for C_o, especially since these additive fixed poles are a common feature of the zero-width model with no direct correlation with the Pomeranchuk coupling.

In Ref. 7, we have generalized our Pomeranchuk to all amplitudes for N hadrons and two currents at arbitrary q_1^2 and q_2^2. The resultant model yields an amplitude $M_1 = (2\mathcal{J}^2/\emptyset)\, H_{1-1}^{\ t}$ for the Pomeranchuk contribution with any intercept $\alpha_P(0)$,

$$M_1 = 8e^2 \left\{ \frac{2q_1 \cdot q_2 b}{\alpha_P(t) - 1} B\big(2 - \alpha_P(t), -\alpha_\pi(s)\big) - \frac{b(q_1^2 + q_2^2) + 1 - \alpha_P(0)}{\alpha_P(t) - 1} \times B\big(1, -\alpha_\pi(s)\big) + (s \leftrightarrow u) \right\} + 8e^2 b\, B\big(-\alpha_\pi(s), -\alpha_\pi(u)\big) . \tag{5.7}$$

An interesting feature of the right-signature $(J = 0)$ fixed pole, is that it is absent for the physical Compton scattering $(q_1^2 = q_2^2 = 0)$, only if $\alpha_P(0) = 1$. The divergence conditions $q_{1\mu} M^{\mu\nu} = O(q_1^2)$ and $M^{\mu\nu} q_{2\nu} = O(q_2^2)$ yield

$$q_1 \cdot p\, M_1 + q_1 \cdot q_2\, M_2 = O(q_1^2)$$

$$q_2 \cdot p\, M_1 + q_1 \cdot q_2\, M_3 = O(q_2^2) ,$$

which require that M_1 have a zero at $t = q_1^2 + q_2^2$ to order $q_1^2 q_2^2$. This is accomplished in (5.7) by the factor $q_1 \cdot q_2$ in the first term, symmetrization of the second term, and the zero at $\alpha(s) + \alpha_\pi(u) = -2bq_1 \cdot q_2 = 0$ in the third term.

The factor $1/\big(\alpha_P(t) - 1\big)$ compensates for the t dependence of $q_1 \cdot q_2$ as $t \to \infty$, and cancels the nonsense factor for fixed t and

$s \to \infty$. Then the fixed pole at $J = 0$ is introduced to cancel the singularity at $\alpha_P(t) - 1 = 0$ in lower orders of s.

We cannot be sure if the fixed pole will be necessary in more general construction with additional lower-order terms. However, if this $J = 0$ singularity persists, one cannot introduce form factors $F(q_1^2)\, F(q_2^2)$ multiplicatively on this amplitude because fixed poles at right-signature point must be avoided for hadronic amplitude. In this case, the Pomeranchuk contribution $(-C_o)$ cannot be established by a vector-dominance model, since it has no counterpart in hadronic or single-current processes. Also, the lack of strong damping as $q^2 \to \infty$ may be in agreement with the recent observation of the diffractive phenomenon in inelastic electron scattering.

B. Current-Algebra Fixed Poles

The current-algebra sum rule of Fubini—Dashen—Gell-Mann for $\mathrm{Im}\, M_1$ in the $I_t = 1$ (charged photon) channel requires that M_1 have a fixed right-signature fixed pole,

$$M_1 \sim \frac{F(t)}{2s}, \quad \text{as } s \to \infty \quad . \tag{5.8}$$

Independently of current algebra, it has been demonstrated by Bronzan et al.[5] that for conserved vector currents one has

$$\frac{1}{b} M_1 = \frac{F(t)}{\alpha_\pi(s)} - \frac{F(t)}{\alpha_\pi(u)} \tag{5.9}$$

at $q_1^2 = 0$, $q_2^2 = t$ (and $q_2^2 = 0$, $q_1^2 = t$), and therefore fixed-power behavior must exist.

In Ref. 3, we propose the parameterization

$$\frac{1}{b} M_1 = [F(t) - F(q_1^2)\, F(q_2^2)]\, B(2 - \alpha_\rho(t), -\alpha_\pi(s)) + F(t)/\alpha_\pi(s) - (s \leftrightarrow u) \quad , \tag{5.10}$$

which satisfies CVC (5.9), the sum rule (5.8), and for a single-pole form factor, $F(t) = m_\rho^2/(m_\rho^2 - t)$, has good analytic properties. The double projection of the six-point beta function leads to $B\big(2 - \alpha_\rho(t), -\alpha_\pi(s)\big) - (s \leftrightarrow u)$ for $\frac{1}{b} M_1$, which is only a slight help in our construction. But in the generalization[7] of (5.10) to all two-current amplitudes $M^{\mu\nu}$ for N spinless hadrons the projection of the amplitude $B_H^{\mu\nu}$ for $VV \to N$ hadrons from the $(N + 4)$-point beta function is an indispensable guide. This amplitude $B_H^{\mu\nu}$ allows one to construct the hadronic part of $M^{\mu\nu}$ $\big(F(q_1^2)\, F(q_2^2)\, B_H^{\mu\nu}\big)$ with the correct vector meson poles in q_1^2, the correct ELI poles at $q_i^\mu \to 0$, and the proper Regge powers. Then we add on by hand terms to satisfy CVC and to introduce the fixed-power behavior of current algebra. The resultant parameterization[7] factorizes on all poles except those on the nonleading trajectories in the channels overlapping the two-current channel.

Inspired by the integral representations for the N-point beta function,[17] several authors[6] have proposed integral representations (particularly for M_1) that give them an infinite number of vector-meson poles and fixed poles in the t channel. It is the opinion of this author that such an ambitious "guess" is premature, if one hopes to simultaneously get conserved currents, factorization, and rapidly falling form factors.

In our approach, arbitrary numbers of vector mesons from lower trajectories can be introduced, as follows:

$$\frac{1}{b} M_1 = [F(t) - F(q_1^2) F(q_2^2)] B(2 - \alpha_\rho(t), -\alpha_\pi(s)) + F(t)/\alpha_\pi(s)$$

$$+ \sum f_m [F_m(t) - F_m(q_1^2) F_m(q_2^2)][B(m + 1 - \alpha_\rho(t), -\alpha_\pi(s))$$

$$- B(2 - \alpha_\rho(t), -\alpha_\pi(s))] - (s \leftrightarrow u) \quad , \qquad (5.11)$$

where

$$F = \sum f_m F_m, \quad F_m(t) = \frac{m - \alpha_\rho(0)}{m - \alpha_\rho(t)} \quad .$$

This factorizes on the leading trajectory in the s channel and still has an infinite number of undetermined constants f_m. Consequently, tremendous freedom exists in these parameterizations until factorization (unitarity in the zero-width approximation) is applied to lower trajectories.

C. Concluding Remarks

Two distinct approaches can now be followed. One is the local adaptation of these parameterizations to the phenomenological analysis of a given reaction. Some work[7] is under way on the general features of these parameterizations for $\gamma\pi \to \gamma\pi$ and $\gamma K \to \gamma K$, but a detailed and quantitative comparison of various parameterizations and experiment for $\gamma N \to \gamma N$ (virtual Compton scattering and total electroproduction) would be vastly more instructive in determining the weaknesses and strengths of the present zero-width models.

The global or dynamical approach attempts to determine a large class of amplitudes through the imposition of factorization.[21] In Ref. 7, we have initiated the search for vector-current amplitudes consistent with a particularly simple factorized zero-width model of the hadron bootstrap. In the global approach, one first constructs single-current amplitudes for N hadrons that are conserved and whose poles at $q_1^2 = m_{\rho_n}^2$ are physical hadronic amplitudes $(\rho_n \to N$ hadrons$)$. Then one tries to construct two-current (covariant correlation) tensors that factor into a product of single-current amplitude on poles that overlap one current as well as yielding the current correspondence to the above single-current amplitudes for one $q_i^2 = m_{\rho_n}^2$. It is the existence of both linear and quadratic (unitarity) relations between the two-current amplitudes and the one-current amplitudes that makes this problem so highly constrained. The exciting question of whether currents can be constructed consistent with the hadron bootstrap and whether these currents will obey a particular current algebra may not be so forbidding within the approximation of a zero-width model.

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I thank F. Arbab and M. B. Halpern for their many contributions as co-authors of the papers that form the basis for this report, and also G. F. Chew, J. D. Jackson, and J. H. Weis for numerous and productive discussions. Particularly, I am indebted to G. F. Chew for generous advice on this report as part of my thesis and for guidance throughout my graduate studies.

APPENDIX

Zero-Mass Limit of the Wigner Rotation

In this appendix, we demonstrate that the Wigner rotation $R_W(\Lambda,q)$ for a particle of mass m_B ($q^2 = m_B^2$; actually R_W is only a function of Λ and $\underset{\sim}{q}/m_B$) becomes a pure z rotation $R_z(\Theta)$ to first order in the mass m_B, and this angle $\Theta(\Lambda,\underset{\sim}{q})$ is the z rotation part of the corresponding transformation $\mathcal{R}(\Lambda,\underset{\sim}{q})$ (defined in Sec. I.A) in the little group E(2) for a massless particle. Hence, for an arbitrary <u>fixed</u> Lorentz transformation Λ and <u>fixed</u> three-vector $\underset{\sim}{q}$, we must show that

$$R_W(\Lambda,q) \;\to\; R_z\big(\Theta(\Lambda,\underset{\sim}{q})\big) + O(m_B), \qquad \text{as } m_B \to 0. \tag{A.1}$$

The general Lorentz transformation may be parameterized

$$\Lambda = R_2(\alpha_2,\beta_2,\gamma_2)\, B_z(\eta)\, R_1(0,\beta_1,\gamma_1) \quad , \tag{A.2}$$

with the rotations

$$R(\alpha_1\beta_1\gamma) = e^{-i\alpha J_z}\, e^{-i\beta J_y}\, e^{-i\gamma J_z}$$

and the z boost

$$B_z(\eta) = e^{-i\eta K_z} \quad .$$

It follows from the definition of the Wigner rotation,

$$R_W(\Lambda,q) = L_{\Lambda q}^{-1}\,\Lambda\, Lq \quad , \tag{A.3}$$

that it can be written as a product of Wigner rotation,

$$R_W(\Lambda,q) = R_W(R_2,B_zR_1q)\, R_W(B_z,R_1q)\, R_W(R_1,q) \quad . \qquad (A.4)$$

Primarily by observing the convention that in

$L_p = R_p(\phi_p,\theta_p, -\phi_p)\, B_z(\eta_p)$ the first z rotation is the negative of the azimuthal angle ϕ_p for $\underset{\sim}{p}$, the reader may easily verify that the Wigner rotation of a rotation is a pure z rotation. Also, if the z axis is chosen to be along $\underset{\sim}{q}$ the vector $\underset{\sim}{R_1 q}$ is in the xz plane, and therefore $R_W(B_z,R_1q)$ is a pure y rotation $R_y(\theta)$. Hence, the product in Eq. (A.4) puts the Wigner rotation in the standard form

$$R_W(\Lambda,q) = R(\phi,\theta,\psi) = e^{-i\phi J_z}\, e^{-i\theta J_y}\, e^{-i\psi J_z} \quad . \qquad (A.5)$$

Without explicitly calculating ϕ and ψ from $R_W(R_2,B_zR_1q)$ and $R_W(R_1,q)$ respectively, one can demonstrate that ϕ and ψ are independent of the mass m_B. [In fact, if $R(\alpha,\beta,\gamma)\, R_p(\phi_p,\theta_p, -\phi_p) = R(\alpha',\beta',\gamma')$, then $R_W(R,p) = R_z(\alpha' + \gamma')$.] However, the mass dependence of θ is not so trivial. By representing the transformation by matrices in $O(3,1)$ acting on vectors $p^\mu = (p_x,p_y,p_z,p_t)$ [and therefore $q^\mu = (0,0,|q|, (\underset{\sim}{q}^2 + m_B{}^2)^{\frac{1}{2}}]$, and by performing the straightforward, but tedious, calculation of $R_W(B_z,R_1q)^\mu{}_\nu$ for $\nu = 1$, $\mu = 2$, one can verify that

$$\sin\theta = \frac{m_B \sinh\eta \sin\beta}{|\underset{\sim}{q}|} \quad . \qquad (A.6)$$

This demonstrates that for $m_B \to 0$ the Wigner rotation becomes a pure z rotation through some angle $\phi + \psi$.

We are not interested in the exact functional dependence of this angle $\phi + \psi$ on Λ and $\underset{\sim}{q}$, but we wish to show that it is the z rotation $\Theta(\Lambda,\underset{\sim}{q})$ in $\mathcal{R}(\Lambda,\underset{\sim}{q}) = \mathcal{L}_{\Lambda q}^{-1} \Lambda \mathcal{L}_q$. In the $O(\frac{1}{2},\frac{1}{2})$ representation (with rows and columns in the order 1230 or xyzt), we use the parameterization of $\mathcal{R}$ in terms of Θ, X_1, X_2 (see Weinberg, Ref. 8, Eq. (A.4))

$$\mathcal{R}^{\mu}{}_{\nu} = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 & 0 \\ -\sin\Theta & \cos\Theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -X_1 & X_1 \\ 0 & 1 & -X_2 & X_2 \\ X_1 & X_2 & 1-\frac{1}{2}X^2 & \frac{1}{2}X^2 \\ X_1 & X_1 & -\frac{1}{2}X^2 & 1+\frac{1}{2}X^2 \end{bmatrix}, \tag{A.7}$$

where

$$X^2 \equiv X_1{}^2 + X_2{}^2 .$$

To demonstrate that $\Theta = \phi + \psi$, we factor from L_q (and $L_{\Lambda q}$) the boost $B_z(\eta_o)$ which takes one from the rest frame with $q^{\mu} = (0,0,0,m_B)$ to the standard frame with $q^{\mu} = [0,0,1,(1+m_B{}^2)^{\frac{1}{2}}]$, so that $L_q\, B_z{}^{-1}(\eta_o)$ is the finite boost $\mathcal{L}_q$ in the limit $m_B \to 0$. Hence,

$$\mathcal{L}_{\Lambda q}^{-1}\,\Lambda\,\mathcal{L}_q = \lim_{m_B\to 0} B_z^{-1}(\eta_o)\,R_W(\Lambda,q)B_z(\eta_o)$$

$$= e^{-i\phi J_z}\lim_{m_B\to 0}\left\{B_z^{-1}(\eta_o)\,e^{-i\theta J_y}\,B_z(\eta_o)\right\} e^{-i\psi J_z} \quad . \tag{A.8}$$

In the $(\frac{1}{2},\frac{1}{2})$ representation, it is easy to explicitly calculate $\left\{B_z^{-1}(\eta_o)\,e^{-i\theta J_y}\,B_z(\eta_o)\right\}^\mu{}_\nu$ and take the limit. The result is

$$\lim_{m_B\to 0}\left\{B_z^{-1}(\eta_o)\,e^{-i\theta J_y}\,B_z(\eta_o)\right\}^\mu{}_\nu = \begin{bmatrix} 1 & 0 & -X & X \\ 0 & 1 & 0 & 0 \\ X & 0 & 1-\frac{1}{2}X^2 & \frac{1}{2}X^2 \\ X & 0 & -\frac{1}{2}X^2 & 1+\frac{1}{2}X^2 \end{bmatrix} , \tag{A.9}$$

where

$$X = -\lim_{m_B\to 0} \sin\theta/m_B \quad .$$

Notice that without a demonstration of $\sin\theta \propto m_B$, one could not be sure that the limit existed. Finally, conjugating Eq. (A.9) with $B_z(\psi)^\mu{}_\nu$, one arrive at an expression for $\mathcal{R}^\mu{}_\nu$ which is identical to (A.7) with the association $X_1 = X\cos\psi$, $X_2 = X\sin\psi$, and $\theta = \phi + \psi$.

FOOTNOTES AND REFERENCES

* This work was supported in part by the U.S. Atomic Energy Commission.

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3. R. C. Brower and M. B. Halpern, Phys. Rev. 182, (1969).

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5. J. B. Bronzan, I. S. Gerstein, B. W. Lee, and F. E. Low, Phys. Rev. 157, 1448 (1967), give a rigorous argument based on conservation of the vector current, but independent of current algebra, for a fixed pole $(J = 1, \ I = 1,$ right signature) in the double-helicity-flip amplitude for two currents $\rightarrow$ two pions.

6. M. Bander (Weizmann preprint 1969), H. Sugawara (Tokyo preprint), and M. Ademollo and E. DelGiudice (Massachusetts Institute of Technology preprint) give alternative approaches to the insertion of a fixed pole into the double-helicity-flip amplitude.

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We investigate space-space commutators through the Bjorken limit, electroproduction phenomenology, and electromagnetic mass differences.

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9. Factors of the mass m_B are essential in this argument, therefore for definiteness we choose our amplitudes to obey the unitarity condition

$$H_{\lambda_f;\lambda_i} - H^*_{\lambda_i;\lambda_f} = \frac{i}{2} \sum_{\lambda,\lambda'}{}' \int d\Phi_n \, H^*_{\lambda_i;\lambda'} \, H_{\lambda_f,\lambda'} ,$$

where

$$d\Phi_n = \prod_k \frac{d^4 p'_k}{(2\pi)^3} \, \delta^+(p_k{}^2 - m_k{}^2) \, \delta^4(\sum p'_k - \sum \bar{p}_f) .$$

10. The projection of high-spin particles from multibody amplitudes for spinless (and spin $\frac{1}{2}$) particles is a general technique for deriving all kinematical singularities, crossing relations, and threshold properties for amplitudes with high-spin particles. However, since high-spin particles are in general resonances, the direct use of multiparticle amplitudes is preferable, and perhaps more convenient, as indicated by the zero-width model based on the N point beta function or by the multiperipheral model.

11. Although the matrix element is easily defined, it is not obvious whether or not there actually exists a local operator $J^\mu(x)$ defined on the Hilbert space of asymptotic states.

12. By expanding T^μ in terms of the vectors available $(p_k{}^\mu)$, one can see that polarization vector

$$\epsilon^{\mu}(\lambda = 0,q) = \frac{1}{(q^2)^{\frac{1}{2}}} [|\underset{\sim}{q}|, \hat{\underset{\sim}{q}}\, q_o]$$

is "responsible" for the $(q^2)^{\frac{1}{2}}$ singularity in H_0. Also, from the expansion

$$\epsilon^{\mu}(\lambda = 0,q) = q^{\mu}/m_V + m_V/2|\underset{\sim}{q}|\ \delta_3^{\ \mu} + O(m_V) \quad ,$$

we see that $m_V H_0 \to 0$ is equivalent to on-mass-shell gauge invariance, $q_{\mu} T^{\mu}/m_V \to 0$ as $m_V \to 0$.

13. In field theory, one sometimes writes the exchange of an elementary $J = 1$ particle of mass (m_V) in the $(\frac{1}{2}, \frac{1}{2})$ representation, $\langle p_f|J^{\dagger\nu}|0\rangle[g_{\mu\nu} - q_{\mu}q_{\nu}/m_V^{\ 2}]\langle 0|J^{\nu}|p_i\rangle/(q^2 - m_V^{\ 2})$, and one must have a conserved current if the ZML $(m_V \to 0)$ is applied to this object. However, this off-mass-shell propagator has no precise meaning in an S-matrix theory, therefore one can just as well use the pure $J = 1$ propagator $[g_{\mu\nu} - q_{\mu}q_{\nu}/q^2]/(q^2 - m_V^{\ 2})$, which requires only condition (2.14) to remove the unwanted singularity at $q^2 = 0$.

14. The kinematical singularities in the helicity amplitudes makes the analysis for high-spin hadrons quite involved, but several interesting results are obtained, such as the restriction on the elastic form factor for arbitrary spin hadrons, $F_{\lambda_1\lambda_2}(q^2) = e\delta_{\lambda_1\lambda_2} + q^2\, a_{\lambda_1\lambda_2}$. As emphasized by Henry Stapp (Lawrence Radiation Laboratory, private communication) a more elegant formalism can probably be found in terms of the M function.

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Table Ia. Connection between invariant amplitudes in the tensor $T^{\mu} = p_T^{\ \mu}\, T_1 + p_S^{\ \mu}\, T_2$ and helicity amplitudes $H_{\lambda}^{\ t}$ for the process $\gamma(q_1,\mu) + T(p_T) \to S(p_S) + U(p_U)$.

$$H_1^{\ t} = \frac{(\phi)^{\frac{1}{2}}}{\mathcal{J}\sqrt{2}}\, T_1$$

$$H_0^{\ t} = -\frac{1}{4\, m_V\, \mathcal{J}} \left[\{(s - u)(t + m_V^{\ 2} - m_T^{\ 2}) - 2(m_S^{\ 2} - m_U^{\ 2})\left(t - 2(m_V^{\ 2} + m_T^{\ 2})\right)\}\, T_1 + \mathcal{J}^{\,2}(T_1 + 2\, T_2)\right]$$

$$H_{-\lambda}^{\ t} = \eta_S \eta_T \eta_U (-)^{\lambda}\, H_{\lambda}^{\ t} \qquad \text{by parity}$$

Table Ib. Connection between invariant amplitudes in the tensor $M^{\mu\nu} = g^{\mu\nu} M_0 + P^\mu P^\nu M_1 + q_2{}^\mu P^\mu M_2 + P^\mu q_1{}^\nu M_3 + q_2{}^\mu q_1{}^\nu M_4$ and helicity amplitudes $H^t_{\lambda_1\lambda_2}$ in the process $\gamma(q_1,\mu) + \gamma(q_2,\nu) \to \pi(p_1) + \pi(p_2)$, where $P = (p_1 - p_2)/2$.

$$H_{11}{}^t = - M_0 - \frac{\phi}{2\mathcal{J}^2} M_1$$

$$H_{1-1}{}^t = \frac{\phi}{2\mathcal{J}^2} M_1$$

$$H_{01}{}^t = \frac{(\phi)^{\frac{1}{2}}}{4\sqrt{2}\,\mathcal{J}^2 m_{V_1}} [(s - u)(t + m_{V_1}{}^2 - m_{V_2}{}^2) M_1 - 2\mathcal{J}^2 M_2]$$

$$H_{10}{}^t = \frac{(\phi)^{\frac{1}{2}}}{4\sqrt{2}\,\mathcal{J}^2 m_{V_2}} [(s - u)(t + m_{V_2}{}^2 - m_{V_1}{}^2) M_1 + 2\mathcal{J}^2 M_3]$$

$$H_{00}{}^t = \frac{1}{16\mathcal{J}^2 m_{V_1} m_{V_2}} [-8\mathcal{J}^2(t - m_{V_1}{}^2 - m_{V_2}{}^2)M_0$$

$$- (s - u)^2\left(t^2 - (m_{V_1}{}^2 - m_{V_2}{}^2)^2\right)M_1$$

$$+ 2\mathcal{J}^2(s - u)(t + m_{V_1}{}^2 - m_{V_2}{}^2)M_2$$

$$- 2\mathcal{J}^2(s - u)(t + m_{V_1}{}^2 - m_{V_2}{}^2)M_3 + 4\mathcal{J}^4 M_4]$$

Table Ib (Continued).

$H^t_{-\lambda_1,-\lambda_2}$	=	$(-)^{\lambda_1-\lambda_2} H^t_{\lambda_1\lambda_2}$	by parity.
$H^t_{\lambda_1\lambda_2}(s,u,t)$	=	$(-)^{I_t} H^t_{\lambda_2\lambda_1}(u,s,t)$ or	by Bose statistics for the photons.
$M^{\mu\nu}(q_1,q_2)$	=	$(-)^{I_t} M^{\nu\mu}(q_2,q_1)$	

FIGURE CAPTIONS

Fig. 1. Variables for the five-point beta function B_5 used in projecting out the term $B_{st}{}^{\mu}$.

Fig. 2. Adjacent photons in the st and ut terms and nonadjacent photons in the su term.

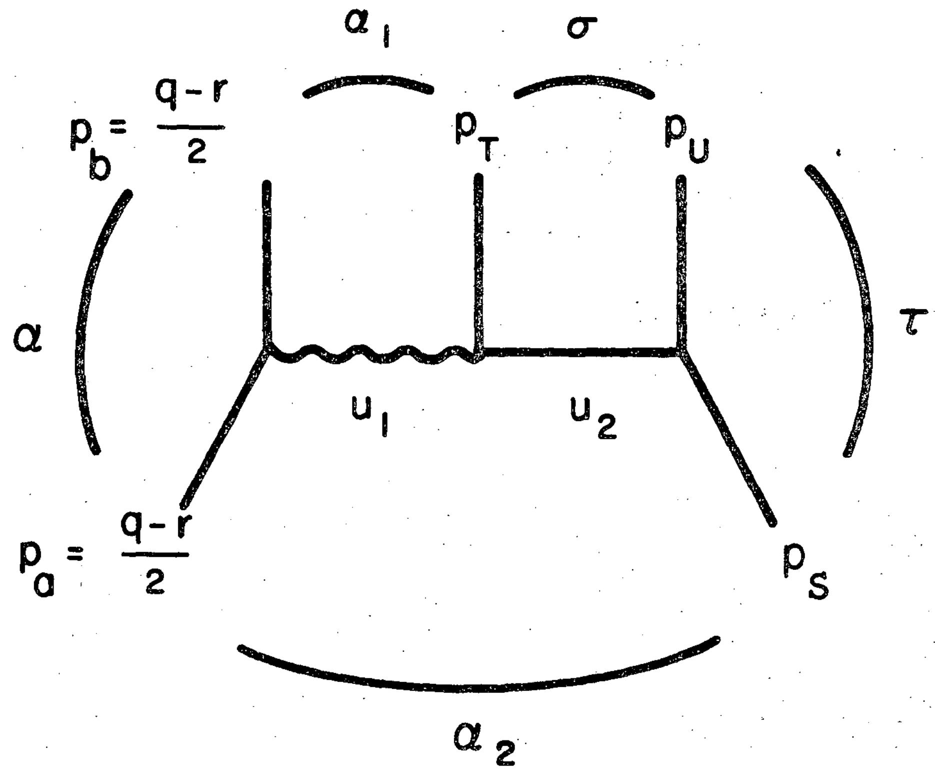

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Fig. 1

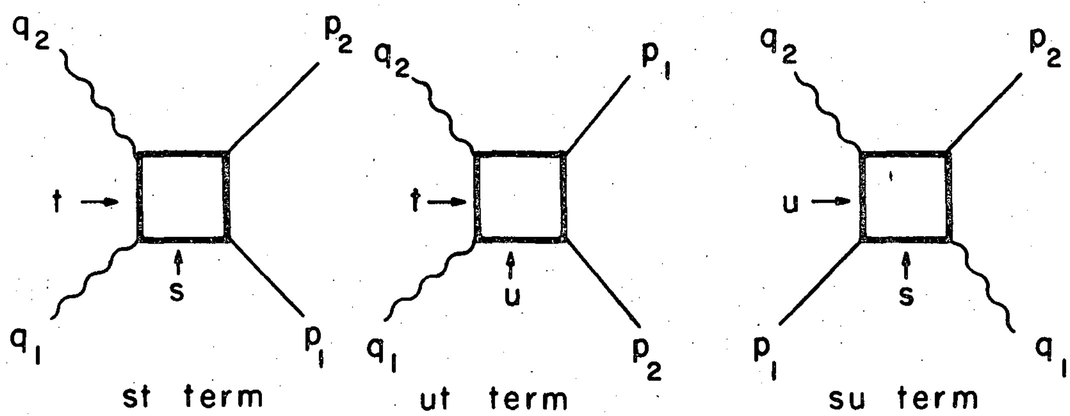

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Fig. 2

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