Title
THE EFFECT OF YIELDING ON THE STRAIN ENERGY RELEASE RATE

Author
Key, P.L.

Publication Date
1968-06-01

University of California

Ernest O. Lawrence Radiation Laboratory

THE EFFECT OF YIELDING ON THE STRAIN ENERGY RELEASE RATE

P. L. Key

June 1968

Berkeley, California

DISCLAIMER

Submitted to The Journal
of Basic Engineering

UCRL-18155
Preprint

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract W-7405-eng-48

THE EFFECT OF YIELDING ON THE STRAIN ENERGY RELEASE RATE

P. L. Key

June 1968

THE EFFECT OF YIELDING ON THE STRAIN ENERGY RELEASE RATE

P. L. Key

Inorganic Materials Research Division, Lawrence Radiation Laboratory,
Department of Mineral Technology, College of Engineering,
University of California, Berkeley, California

ABSTRACT

The effect of local yielding on the strain energy release rate accompanying crack extension is evaluated by considering the Dugdale model of a yielded crack. An exact, analytical expression is obtained for the strain energy release rate of the Dugdale crack which shows an almost linear increase with plastic zone length from the value for the Griffith crack. When these results are applied to data on 2219-T87 aluminum alloy in the literature, the apparent decrease in toughness at high stress levels calculated by the ASTM procedure is removed.

INTRODUCTION

Fracture mechanics, although based on linear elasticity, recognizes the existence and importance of plastic deformation at the crack tip. Several approaches have been used to incorporate the effects of plasticity into the equations of linear elastic fracture mechanics. Irwin (1) made the a posteriori assumption that the effect of plastic deformation can be represented by an effective crack length equal to the actual crack length increased by the radius of the plastic zone at each end of the crack. The radius of the plastic zone is obtained by applying a yield criterion to the elastic stress distribution around the crack. This effective crack length is then used in the expressions for the stress intensity factor or strain energy release rate obtained from elasticity theory. This approach and other plasticity aspects of fracture mechanics have been reviewed by McClintock and Irwin (2) with emphasis on the results of elastic-plastic analyses of shear loaded cracks. The size of the plastic zone is calculated from these analyses and incorporated into the Irwin effective crack length to estimate the effect of yielding on the stress intensity factor.

In this paper, the effect of plastic flow at the crack tip on the strain energy release rate will be examined by considering the Dugdale model (3) for a yielded crack. In this model, yielding at the crack tip is simulated by applying a tensile pressure equal to the yield strength over short extensions at each end of the crack (Fig. 1). Although both the Dugdale and Irwin models of the effect of plastic flow at the crack tip involve crack extensions, the Dugdale model can be treated exactly by the theory of elasticity. Thus, the Dugdale model only involves assumptions in the description of the model whereas the Irwin approach has assumptions in both the model and its application.

The Dugdale model has been very successfully used to investigate yielding effects in fracture mechanics. The model was initially used by Dugdale to predict the length of plastic zones around the tips of cracks in mild steel (3) with excellent agreement being obtained with the measured values. The plastic energy absorption rate and crack tip displacement of the Dugdale model were calculated by Goodier and Field (4). Hahn and Rosenfield (5) obtained good agreement between the Dugdale model and measurements of the plastic zone size and crack tip displacements in various steels. In a recent paper, Forman (6) numerically calculated the strain energy release rate for a crack in a finite width plate using the Dugdale model to approximate the effects of plastic flow.

This latter application of the Dugdale model is re-examined below and an exact, analytical solution is obtained for the original Dugdale model of a crack in a plate of infinite width.

ANALYSIS

The Dugdale model combines two elastic stress distributions to simulate an elastic-plastic problem. Specifically, Dugdale (3) considerd a crack of length 2a in an infinite plate loaded by a uniaxial tensile stress normal to the crack. Yielding at the crack tip is simulated by extending each end of the crack by an amount, ρ, and applying a constant tensile pressure equal to Y, the yield stress of the material, over these extensions. The length ρ is determined by requiring that the stress be finite at the extended crack tip and thus mathematically removing the stress singularity normally observed in elastic crack tip stresses. This model thus consists of a crack in an elastic sheet loaded both by uniaxial stresses and by pressure on the crack faces , as shown in Fig. 1, and can be treated by the theory of elasticity.

For convenience, the strain energy release rate, G, will be formulated for a finite sheet of length L and cross-sectional area A_g containing a Dugdale crack. The explicit dependence on the size of the sheet does not appear in the final expression for G and thus the expression can be applied to the original Dugdale model (infinite sheet). The strain energy release rate associated with an extension of the crack and the corresponding extension of the plastic zones can be obtained from the work done by the stress, σ, and the elastic strain energy released:

$$G = A_g \sigma \frac{\partial \Delta}{\partial A_c} - \frac{\partial U_e}{\partial A_c} \qquad (1)$$

where σ = applied stress; Δ = axial displacement between the ends of the tensile member; A_c = area of crack in plane perpendicular to tensile axis; and U_e = elastic strain energy in the sheet.

The elastic strain energy can be calculated from the stresses, T_i, and displacements, u_i, on the boundary, S_e, of the elastic portion of the sheet:

$$U_e = \frac{1}{2} \int_{S_e} T_i \, u_i \, d\sigma$$

For the Dugdale crack:

$$U_e = \frac{\sigma}{2} A_g \Delta - 2Y \int_a^{a+\rho} w dx \tag{2}$$

where w is the axial component of displacement of the crack surface. The displacement, Δ, can also be expressed in terms of w by applying Greenspan's method modified for the effect of the tensile pressure loading (7):

$$\Delta = \frac{\sigma L}{E} + \frac{2}{A_g} \int_{A_c'} w dA \tag{3}$$

where A_c' is the area of the crack bounded by the elastic portion of the body. Thus, A_c' includes both the real crack and the crack extensions:

$$\Delta = \frac{\sigma L}{E} + \frac{4}{A_g} \int_c^{a+\rho} w dA \tag{4}$$

Introducing Eq. (2) and (4) into (1), and considering constant applied stress, the strain energy release rate becomes:

$$G = \sigma \frac{\partial}{\partial a} \int_c^{a+\rho} w dx + Y \frac{\partial}{\partial a} \int_a^{a+\rho} w dx \tag{5}$$

where the explicit dependence on the size of the sheet, L and Ag, no longer appears. As discussed by several authors (8), the assumption of constant stress does not restrict the generality of the result.

The displacement, w, has been given by Goodier and Field (4) for the case of plane stress:

$$w = \frac{(a+\rho)\,Y}{\pi E}\left\{\cos\theta\,\ln\left(\frac{\sin^2(\beta-\theta)}{\sin^2(\beta+\theta)}\right) + \cos\beta\,\ln\left(\frac{(\sin\beta+\sin\theta)^2}{(\sin\beta-\sin\theta)^2}\right)\right\} \quad (6)$$

where Y is the uniaxial yield strength of the material and

$$\theta = \cos^{-1}\,(x/a+\rho) \quad (7a)$$

$$\beta = \pi/2\,(\sigma/Y) \quad (7b)$$

$$\frac{\rho}{a} = \sec\beta - 1 \quad (7c)$$

After substituting Eq. (6) into Eq. (5), the indicated operations were performed with the aid of Goodier and Field's (4) Eqs. (A11) and (A12) (with a typographical error corrected by introducing a factor of $-\cos\theta_2$ into the third term of A12). In calculating the derivatives, the linear dependence of the plastic zone length, ρ, on the crack length, a, given by Eq. (7c) must be considered but the parameter β may be treated as a constant. Performing these operations yields the result:

$$\frac{G}{G_e} = 2\left\{\frac{\tan\beta}{\beta} - \frac{\ln\sec\beta}{\beta^2}\right\} \quad (8)$$

where $G_e = \frac{\pi\sigma^2 a}{E}$, the strain energy release rate for a Griffith crack. This result is plotted in Fig. 2 as a function of ρ/a and in Fig. 3 as a function of σ/Y using the relations between β, σ/Y and ρ/a given by Eq. (7). Figure 2 shows that the strain energy release rate for a Dugdale crack is almost a linear function of the plastic zone size. This fortuitous result enables the strain energy release rate for the Dugdale crack to be approximated closely by:

$$\frac{G}{G_e} = 1 + (\rho/a) \tag{9}$$

Alternately, the strain energy release rate may be expressed in terms of the displacement at the crack tip, $w(a) = w_c$ using the relation:

$$w_c = \frac{4Ya}{\pi E} \ln \sec \beta$$

Hence,

$$\frac{G}{2Yw_c} = \left\{\frac{\beta \tan\beta}{\ln \sec\beta} - 1\right\} \tag{10}$$

This expression is shown in Fig. 4 which demonstrates that G can be approximated by

$$G = 2Yw_c$$

for low stresses. A similar prediction has been made by Hahn and Rosenfield (5) and by Bilby et al. (8).

DISCUSSION

It is proposed that the strain energy release rate for the Dugdale crack may provide a useful general method of correcting elastic strain energy release rates for yielding effects. Accordingly, Eq. (8) has been put in the form of a correction factor to the elastic strain energy release rate and is compared to Irwin's correction factor in Fig. 3. Irwin's estimate of the effect of plastic deformation on the strain energy release rate in a plate of infinite width is given by (1):

$$\frac{G}{G_e} = \frac{1}{1 - \lambda/2(\sigma/Y)^2} \tag{11}$$

where the factor, λ, has been introduced for generality; Irwin's original estimate corresponded to the case of $\lambda = 1.0$.

As shown in Fig. 3, a value of $\lambda = 2.4$ gives good agreement between the Dugdale model and Irwin's estimate. It should be noted that ρ of the Dugdale model equals the total length of the plastic zone whereas r_y of the Irwin model is the radius of the zone (one-half the total zone length). This suggests that a value of $\lambda = 2.0$ should be used to make the two models consistent. Gerberich and Zackay (9) have recently shown that replacing the Irwin plastic zone radius, r_y, by the Dugdale zone size, ρ, results in better agreement between displacement gage results and those obtained using an explicit plastic zone correction. This corresponds approximately to using a value of the total plastic zone size ($2r_y$) rather than the plastic zone radius.

The analysis in this paper deals with plates of infinite width but could be applied to plates in which the crack length plus the plastic zone length is small compared to the plate width. The exact method to incorporate width effects would involve making a new elastic stress analysis for the finite width plate to obtain the correct crack surface displacements to use in Eq. (5). A simple approximation is to assume that the effects of width and plasticity may be considered separately. Thus, for a finite width plate, it is suggested that an equation of the form:

$$\left(\frac{G}{G_e}\right) = \left(\frac{G}{G_e}\right)_\infty f(\pi a/W)$$

can be used where $(G/G_e)_\infty$ is the value given by Eq. (8) and $f(\pi a/W)$ is a correction factor due to width. The recently suggested secant factor (10) is an example of such a function for a central crack of length 2a in a plate of width W:

$$f(\pi a/W) = \sec \pi a/W$$

Thus,

$$\frac{G}{G_e} = 2 \left\{ \frac{\tan\beta}{\beta} - \frac{\ln \sec\beta}{\beta^2} \right\} \sec \frac{\pi a}{W} \tag{12}$$

This approach assumes that the interaction of width and plasticity result in only small corrections to the respective factors and should be valid as long as the length of the plastic zone (Eq. 7c) is small relative to the plate width. Figure 5 shows the result of applying Eq. 12 to test data on 0.1 inch thick by 24 inches wide plates of 2219-T87 aluminum alloy having a yield strength of 58.8 ksi and ultimate strength of 69.5 ksi (11). These data have also been treated by Irwin and McClintock (2) and Gerberich and Zackay (9) and their results along with the ASTM analysis (12) are also shown in Fig. 5. All three of these latter analyses are based on the Irwin approach of increasing the crack length by the length of the plastic zone but the method of estimating the plastic zone length differs. Irwin and McClintock used a numerical procedure based on the elastic-plastic analyses of cracks in shear. Gerberich and Zackay used the Dugdale zone length modified by the Westergaard (13) width correction and the ASTM analysis derived a plastic zone length from the elastic stress distribution. In applying these analyses, the tensile strength rather than yield strength was used to compensate for work hardening since all the models were based on elastic-perfectly plastic solids; a similar approach was used by Irwin and McClintock and Gerberich and Zackay.

Figure 5 shows that a relatively constant fracture toughness is predicted by all three models which incorporate plasticity effects but that the ASTM analysis results in a decrease in K_c at smaller crack lengths. Because it is based on a model incorporating plasticity effects and because of its analytical simplicity, it is considered that Eq. (12) is worthy of further investigation as a method introducing plasticity corrections.

Goodier and Field (4) have shown that the plastic energy dissipation rate for the Dugdale crack is given by:

$$\frac{\partial W_\rho}{\partial A_c} = \frac{2\pi\sigma^2 a}{E} \left\{ \frac{\tan\beta}{\beta} - \frac{\ln\sec\beta}{\beta^2} \right\},$$

for the case of plane stress. Comparison with Eq. (8) shows that

$$G = \frac{\partial W_\rho}{\partial A_c} \tag{13}$$

for the Dugdale crack.

The change in energy accompanying crack extension can be written as

$$\Delta E = \left[-G + \frac{\partial W_p}{\partial A_c} + 2\gamma \right] \Delta A$$

which becomes, by virtue of Eq. (13)

$$\Delta E = 2\gamma \Delta A$$

independent of crack size and stress level. This result implies that an energy instability in the sense of Griffith cannot occur with the Dugdale crack. Thus a fracture criteria based on strain energy release rate would not be predicted for a Dugdale crack. Physically, this means that the energy supplied by the loading system and the release of strain energy is just balanced by the absorption of energy in inelastic deformation leaving no energy to provide for the new crack surfaces. A similar conclusion was reached by Goodier and Field (4) and attributed by them to be a consequence of the removal of the crack tip singularity in the Dugdale model. Recently, Rice (14) has shown that Eq. (13) applies for all load levels for the more general case of a crack in an elastic-perfectly plastic material.

As mentioned above, Forman (6) has also considered the strain energy release rate of the Dugdale crack. His analysis differs from the present one in three ways:

1) Forman considered finite width plates rather than the infinite width plates considered in this paper. Forman incorporated the width effect approximately by using the crack surface displacements for an infinite plate but modifying the applied stress, σ, with the stress distribution of the infinite plate. This greatly complicates the calculation since σ becomes a function of crack size. Because of these complications, Forman had to use numerical means to obtain values for the strain energy release rate. Although the results of this approach may be useful, it is a much more complicated approximation than treating the two effects separately as outlined above.

2) In calculating the strain energy of the plate, Forman considers only the applied stress, σ, thus neglecting the strain energy contribution of the stress, -Y, distributed along the elastic-plastic boundary. This contribution results from interaction of the two stress fields and neglecting it produces an underestimate of the strain energy release rate.

3) In calculating the axial rigidity of the plate, Forman integrated the crack displacements only over the true crack area. However, in order to obtain the rigidity of the elastic body, the crack displacements should be integrated over the boundary of the elastic region which is the extended crack $(a+\rho)$ and not the true crack. This also produces an underestimate of the strain energy release rate.

CONCLUSIONS

1. The Dugdale model of a crack with yielding provides a method of incorporating plastic flow effects into elastic fracture mechanics. The strain energy release rate for a Dugdale crack is shown to be almost a linear function of the plastic zone size.
2. The strain energy release rate for a Dugdale crack just equals the rate of energy absorption by plastic deformation. Hence, unstable, Griffith-type fracture cannot occur in this model.
3. The plastic zone correction proposed by Irwin can be made to more closely agree with the Dugdale model by increasing the effective half crack length by 2.4 r_y rather than r_y.
4. The strain energy release rate for a Dugdale crack is proportional to the crack tip displacement for low loads.

ACKNOWLEDGMENT

This work was done under the auspices of the United States Atomic Energy Commission. Discussions with E. R. Parker, W. W. Gerberich, and Y. Katz on this paper are gratefully acknowledged.

REFERENCES

1. G. R. Irwin, Plastic Zone Near a Crack and Fracture Toughness, Seventh Sagamore Ordnance Materials Research Conference (1960).

2. F. A. McClintock and G. R. Irwin, Plasticity Aspects of Fracture Mechanics, in Fracture Toughness Testing and Its Applications, ASTM STP No. 381, 84 (1965).

3. D. S. Dugdale, Yielding of Steel Sheets Containing Slits, J. Mechanics and Phys. of Solids 8, 8 (1960).

4. J. H. Goodier and F. A. Field, Plastic Energy Dissipation in Crack Propagation, Fracture of Solids (Interscience, New York, 1963), p. 103.

5. G. T. Hahn and A. R. Rosenfield, Local Yielding and Extension of a Crack Under Plane Stress, Acta Met. 13, 293 (1965).

6. R. G. Forman, Effect of Plastic Deformation on the Strain Energy Release Rate in a Centrally Notched Plate Subjected to Uniaxial Tension, J. of Basic Engineering 88, 82 (1966).

7. P. L. Key, On the Relationship Between Crack Surface Displacements and the Strain Energy Release Rate, University of California, Berkeley, UCRL-18065 (1968).

8. P. C. Paris and G. C. Sih, Stress Analysis of Cracks, in Fracture Toughness Testing and Its Applications (ASTM, Philadelphia, 1965), p. 30.

9. B. A. Bilby, A. H. Cottrell, and K. H. Swinden, The Spread of Plastic Yield from a Notch, Proc. Roy. Soc. (London) 272A, 304 (1963).

10. W. W. Gerberich and V. F. Zackay, On the Plane Stress Plastic Zone Correction in Fracture Mechanics, to be published in Welding Journal.

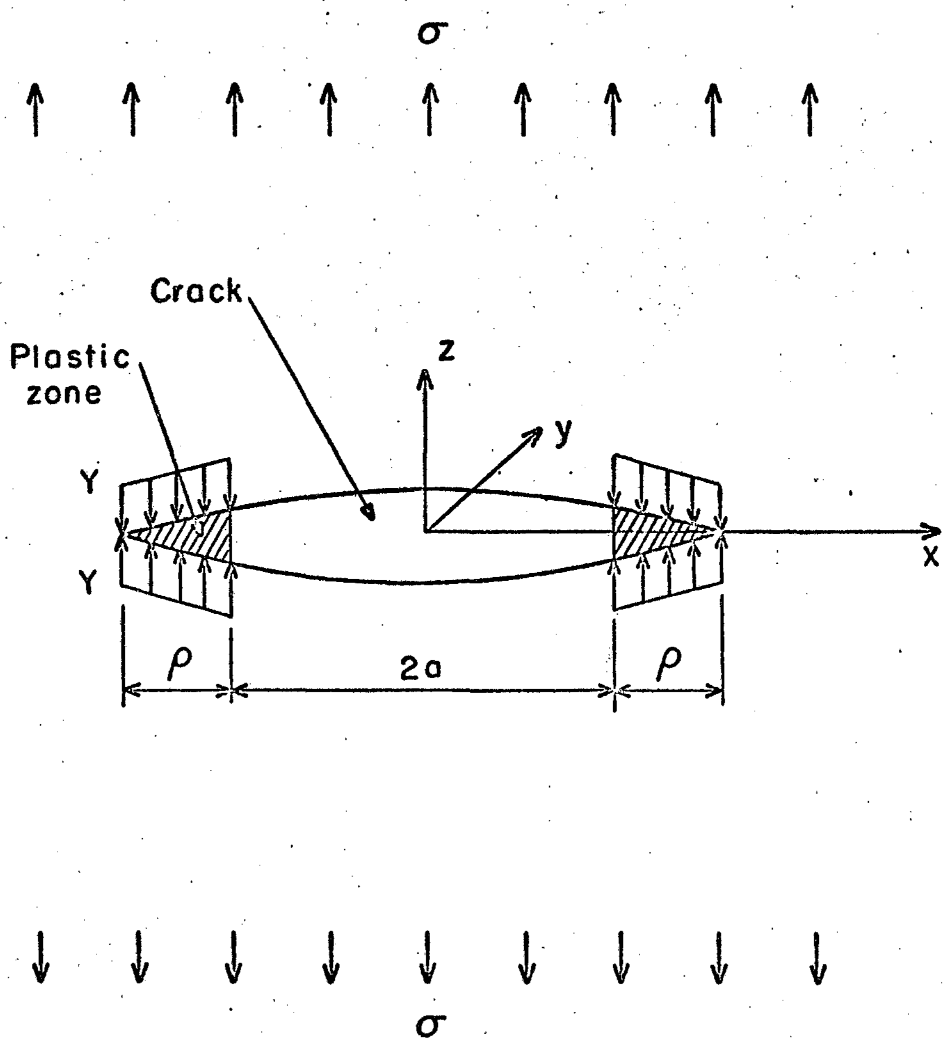
11. C. E. Feddersen, Discussion of Plane Strain Crack Toughness Testing of High Strength Metallic Materials, W. F. Brown and J. E. Srawley, ASTM, Philadelphia, 77 (1965).

12. Progress in Measuring Fracture Toughness and Using Fracture Mechanics, Materials Research and Standards 4, 107 (1964).

13. Fracture Testing of High-Strength Sheet Materials, ASTM Bulletin, 243, 29 (1960).

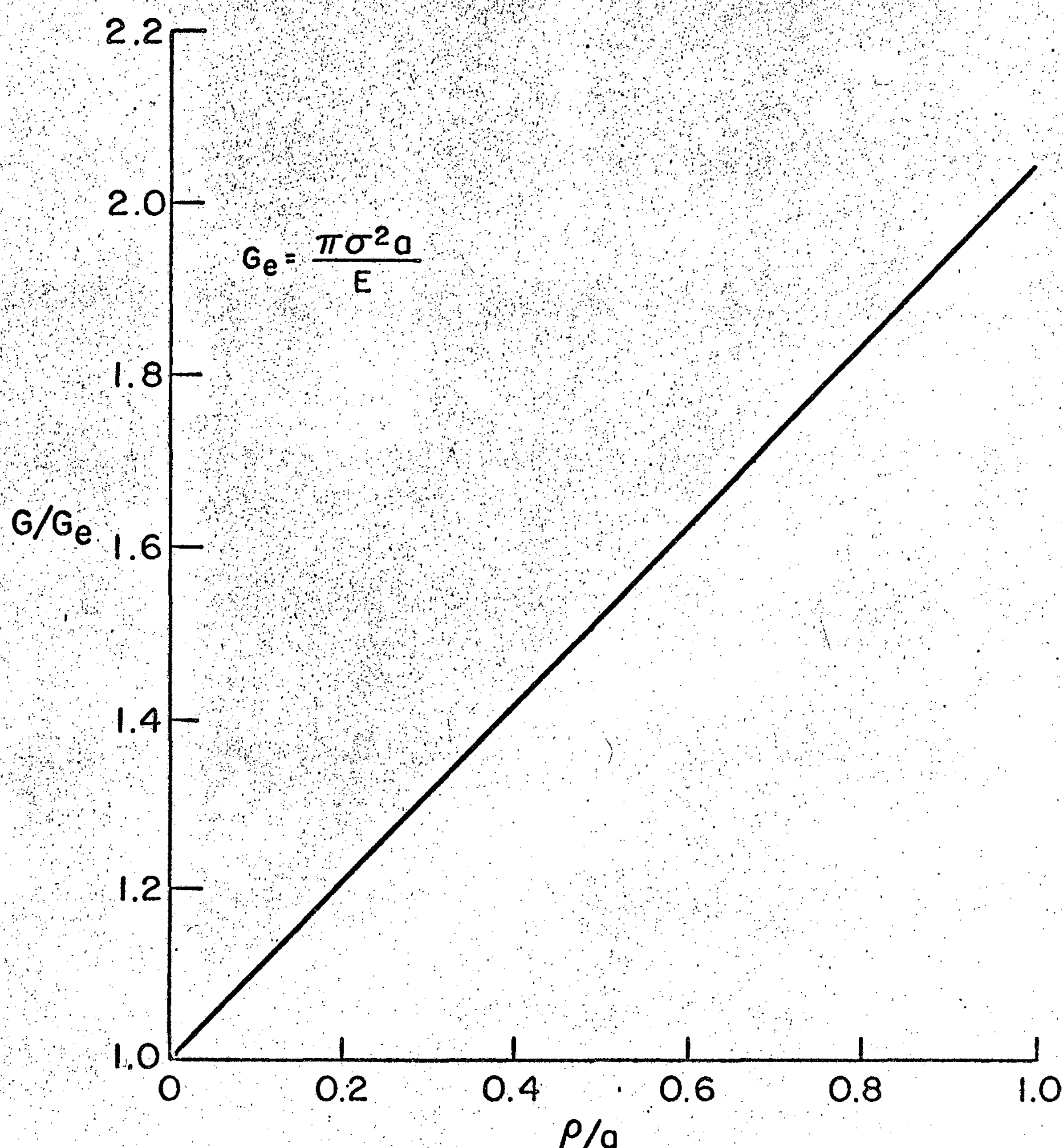
14. H. M. Westergaard, Bearing Pressures and Cracks, Trans. ASME 61, A49 (1939).

15. J. R. Rice, An Examination of the Fracture Mechanics Energy Balance from the Point of View of Continuum Mechanics, Proceedings of the First International Conference on Fracture, Sendai, Japan, 1965, 1, 309 (1965).



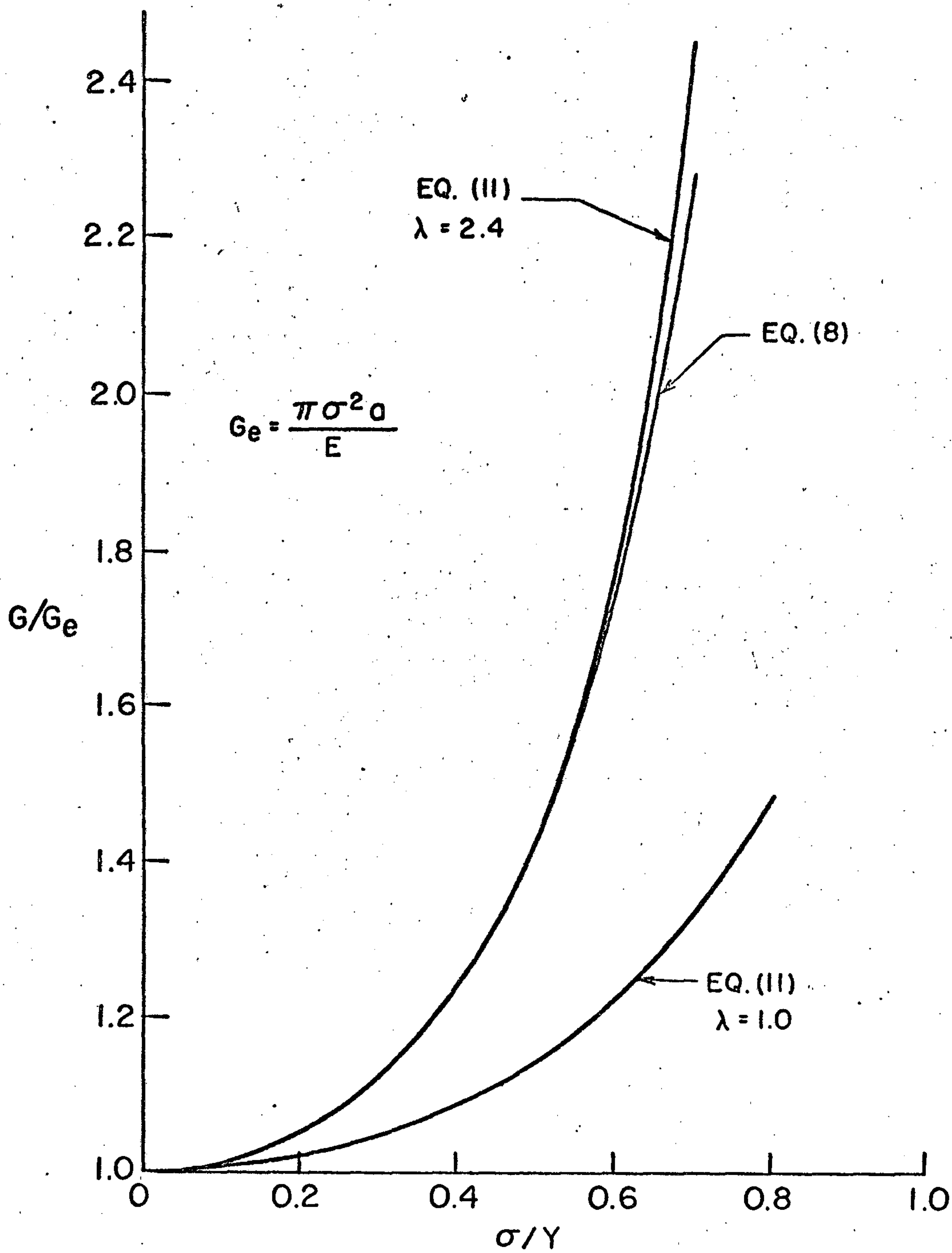
XBL 683-363

Fig. 1 The Dugdale model of a yielded crack



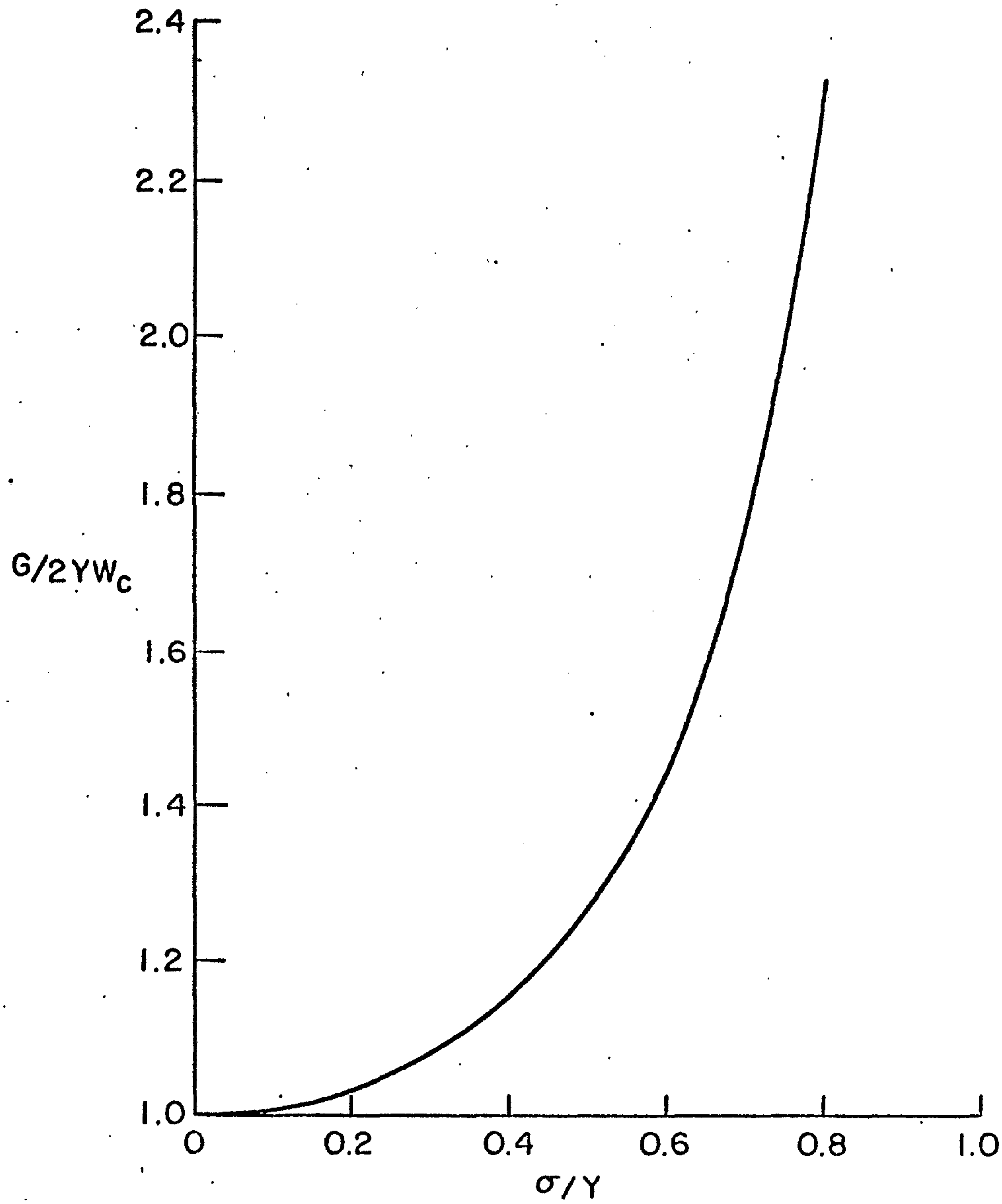
XBL 684-650

Fig. 2 Strain energy release rate of a yielded crack relative to a Griffith crack as a function of plastic zone length



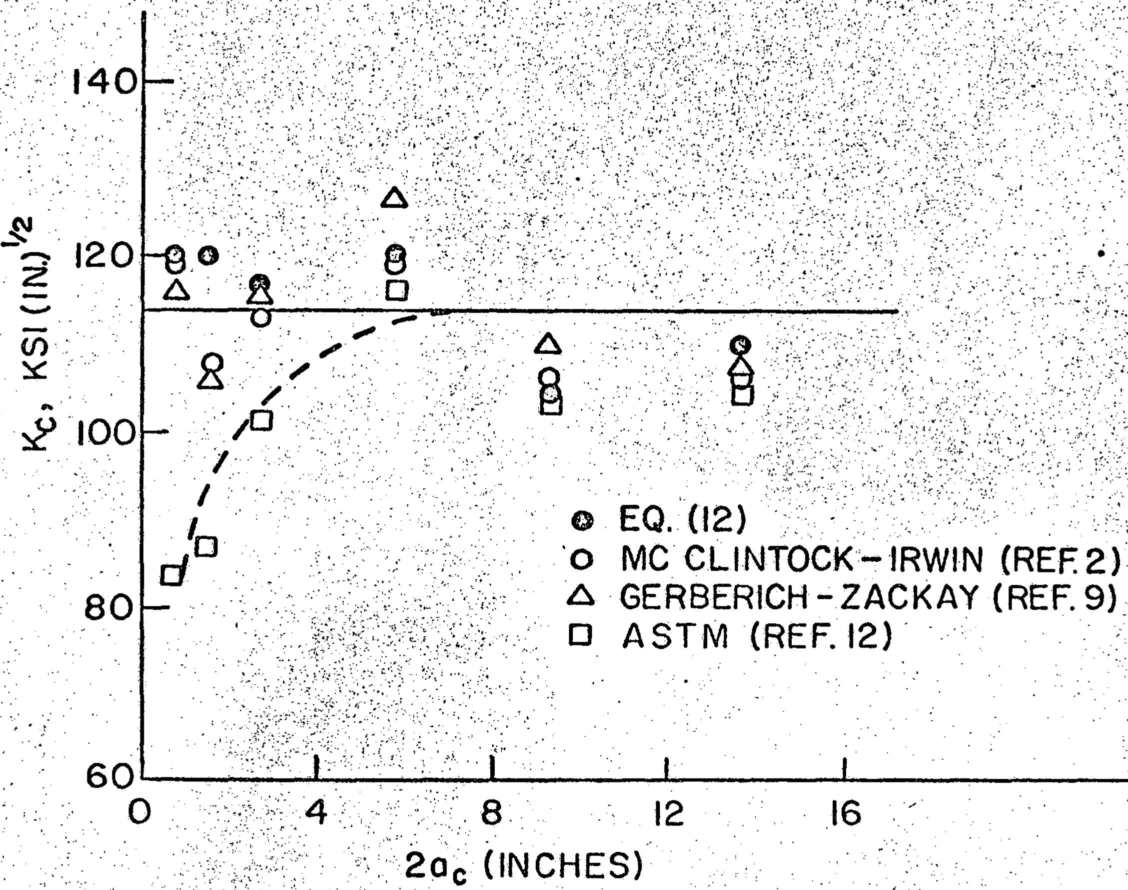
XBL 684-651

Fig. 3 Strain energy release rate of a yielded crack as a function of applied stress



XBL 684-652

Fig. 4 Strain energy release rate of a yielded crack relative to the crack opening displacement as a function of applied stress



XBL 685-749

Fig. 5 Fracture toughness of 2219-T87 aluminum alloy as a function of crack length. Plates were 0.1 thick by 24 inches wide with yield strength of 58.8 ksi and an ultimate strength of 69.5 ksi.