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THE A-FIBERED BURNSIDE RING AS A-FIBERED BISET FUNCTOR IN CHARACTERISTIC ZERO

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Abstract

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Let A be an abelian group and let $\mathbb{K}$ be a field of characteristic zero containing roots of unity of all orders equal to finite element orders in A. In this thesis we prove foundational properties of the A-fibered Burnside ring functor $B_{\mathbb{K}}^A$ as an A-fibered biset functor over $\mathbb{K}$. This includes the determination of the lattice of subfunctors of $B_{\mathbb{K}}^A$ and the determination of the composition factors of $B_{\mathbb{K}}^A$. The results of the paper extend results of Coşkun and the author for the A-fibered Burnside ring functor restricted to p-groups and results of Bouc in the case that A is trivial, i.e., the case of the Burnside ring functor over fields of characteristic zero.

Devran'a

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Chapter 1

Introduction

The study of group actions on sets plays an important role in representation theory. As the simplest case, the action of a finite group on a finite set leads to the theory of Burnside rings. Considering the common features shared by Burnside rings and other representation rings, Dress [D73] and Green [Gn71] introduced Mackey functors to give a unified treatment of these objects. The structure of Mackey functors was later studied extensively by Thévenaz and Webb [TW95]. The theory of Mackey functors has several important applications such as the canonical induction formualae introduced by Boltje [Bo98].

Let G and H be finite groups. A (G, H)-biset X is a finite set X equipped with a left G-action and a right H-action that commute with each other. Equivalently, a (G, H)-biset is a left $G \times H$-set. The translation between these two points of view is given by $(g, h)x = gxh^{-1}$, for $g \in G$, $h \in H$, and $x \in X$. The biset notation is more useful when the composition of bisets is considered whereas the other point of view is used when the description of the transitive $G \times H$-sets is given. The study of bisets leads us to the theory of biset functors introduced by Bouc in [Bc96]. One of the most important applications of biset functors, among many others, is the final determination of the structure of the Dade group by Bouc [Bc06].

Let A be an abelian group. Let X be an $A \times G$-set with finitely many A-orbits and the property that the A-action on X is free. Since the A-action is free, such an action of $A \times G$ on X can be considered as G acting on the A-fibers and in this case, the set X is called an A-fibered G-set. These objects were introduced by Dress in [D71] and studied by Boltje [Bo] and Barker [Ba04].

Representation rings carry more structure by also considering multiplication with one-dimensional representations as structural maps compared to the biset functor structure. Motivated from this fact, Boltje and Coşkun generalized the notions of bisets and fibered G-sets, and introduced fibered bisets and fibered biset functors [BC18].

Let A be an abelian group and let k be a commutative ring. An A-fibered biset functor F over k is, informally speaking, a functor that assigns to each finite group G a k-module $F(G)$ together with maps $\mathrm{res}^G_H \colon F(G) \to F(H)$ and $\mathrm{ind}^G_H \colon F(H) \to F(G)$, whenever $H \leqslant G$, called restriction and induction, maps $\mathrm{inf}^G_{G/N} \colon F(G/N) \to F(G)$ and $\mathrm{def}^G_{G/N} \colon F(G) \to F(G/N)$, whenever N is a normal subgroup of G, called inflation and deflation, and maps $\mathrm{iso}_f \colon F(G) \to F(H)$, whenever $f \colon G \to H$ is an isomorphism. Moreover, the abelian group $G^* := \mathrm{Hom}(G, A)$ acts k-linearly on $F(G)$ for every finite group G. These operations satisfy natural relations. Standard examples are various representation rings of KG-modules, for a field K and $A = K^\times$. In this case, G^* is the group of one-dimensional KG-modules acting by multiplication on these representation rings. In [BC18] the simple A-fibered biset functors were parametrized. If A is the trivial group then one obtains the well-established theory of biset functors, see [Bc10] as special case. A-fibered biset functors over k can also be interpreted as the modules over the Green biset functor B_k^A, where $B_k^A(G)$ is the A-fibered Burnside ring of G over k (also called the K-monomial Burnside ring of G over k, when $A = K^\times$ for a field K). Another natural example of A-fibered biset functors (without deflation) is the unit group functor $G \mapsto B^A(G)^\times$. This structure was established in a recent paper by Bouc and Mutlu, see [BM19] and generalizes the biset functor structure on the unit group $B(G)^\times$ of the Burnside ring.

Representation rings carry more structure when viewed as A-fibered biset functors compared to the biset functor structure. One of the goals is to understand their composition factors as such functors. By various induction theorems they can be viewed as quotient functors of the functor B_k^A for various A. Thus, it is natural to first investigate the lattice of subfunctors of B_k^A and its composition factors. This is the objective of this thesis, where k is a field of characteristic zero containing sufficiently many roots of unity. Coşkun and the author achieved this already in [CY19] for the same functor category restricted to finite p-groups for fixed p.

The thesis is arranged as follows. In Chapter 2 we recall the notions of G-sets, A-fibered G-sets, A-fibered (G, H)-bisets, and their Grothendieck groups $B_k^A(G)$ and $B_k^A(G, H)$ over a commutative base ring k. The tensor product of A-fibered bisets leads to the A-fibered biset category $\mathcal{C}_k^A$ and its functor category $\mathcal{F}_k^A$ of k-linear functors from

$\mathcal{C}_k^A$ to ${}_k\mathsf{Mod}$, the category of A-fibered biset functors over k. The relevant definitions and constructions are recalled in Section 2.5. In Section 2.6 we recall the parametrization of simple A-fibered biset functors from [BC18]. In Chapter 3 we parametrize the set of primitive idempotents of $B_{\mathbb{K}}^A(G)$ over a field $\mathbb{K}$ of characteristic 0 which contains enough roots of unity in relation to the finite element orders of A. We also derive an explicit formula for these idempotents, using results from [BRV19] on the $-_+$ construction. We take advantage of the fact that the Green biset functor $B_{\mathbb{K}}^A$ arises as the $-_+$ construction of the Green biset functor $G \mapsto \mathbb{K}G^*$. Interestingly, the idempotent formula we derive in Theorem 3.0.2 is different from the one given by Barker in [Ba04], which was proved for more restrictive cases of A. It is used as a crucial tool in the following sections. In Chapter 4 we provide formulas for the action of inductions, restrictions, inflations, deflations, isomorphisms and twists by $\phi \in G^*$ on these idempotents. Crucial among those is the action of $\mathrm{def}_{G/N}^G$, which maps a primitive idempotent of $B_{\mathbb{K}}^A(G)$ to a scalar multiple of a primitive idempotent of $B_{\mathbb{K}}^A(G/N)$. After establishing three technical lemmas in Chapter 5, this mysterious scalar is studied in more depth in Chapter 6. The vanishing of this scalar is a condition that leads to the notion of a B^A-pair (G, Φ), where G is a finite group and $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$, in Chapter 7. There, we also study particular subfunctors $E_{(G,\Phi)}$ of $B_{\mathbb{K}}^A$. In Chapter 8, we show that every subfunctor of $B_{\mathbb{K}}^A$ is a sum of the functors $E_{(G,\Phi)}$ and that the subfunctors of $B_{\mathbb{K}}^A$ are in bijection with the set of subsets of isomorphism classes of B^A-pairs that are closed from above with respect to a natural partial order $\preccurlyeq$, see Theorem 8.0.7. In Chapter 9 we determine the composition factors of $B_{\mathbb{K}}^A$ in terms of the parametrization of simple functors from

[BC18], see Theorem 9.0.2. Finally, in Section 10, we consider the special case that A is a subgroup of $\mathbb{K}^\times$. In this case, a natural isomorphism $G/O(G) \xrightarrow{\sim} \mathrm{Hom}(G^*, \mathbb{K}^\times)$ for a normal subgroup $O(G)$ of G depending on A, allows to reinterpret the set of B^A-pairs and makes our results compatible with the language and setup in [Ba04] and [CY19].

The approach in this thesis follows closely the blueprint in [Bc10, Section 5] for the case $A = \{1\}$. However, the presence of the fiber group A requires additional ideas and techniques to achieve the analogous results. The main technical problem is that a transitive A-fibered biset with stabilizer pair (U, ϕ), does not in general factor through the group $q(U) \cong p_i(U)/k_i(U)$, $i = 1, 2$, since ϕ is in general non-trivial when restricted to $k_1(U) \times k_2(U)$.

1.1 Notation

For a finite group G we denote by $\exp(G)$ the exponent of G. If X is a left G-set, we write $x =_G y$ if two elements x and y of X belong to the same G-orbit. For $x \in X$, we denote by G_x or $\mathrm{stab}_G(x)$ the stabilizer of x in G. By X^G we denote the set of G-fixed points in X and by $[G\backslash X]$ a set of representatives of the G-orbits of X. For subgroups H and K of G, we denote by $[H\backslash G/K]$ a set of representatives of the (H, K)-double cosets of G.

For an abelian group A, we denote by $\mathrm{tor}(A)$ its subgroup of elements of finite order, and, for a ring R, we denote by $R^\times$ its group of units.

Chapter 2

Preliminaries

2.1 G-sets and Burnside rings

Throughout this section, G denotes a finite group. We recall the basic notions and results regarding G-sets and the Burnside ring of G. We refer the reader to [Bc10, Chapter 2] for more details.

Recall that a left G-set X is a set X equipped with a map

$$G \times X \to X\,, \quad (g,x) \mapsto g \cdot x = gx\,, \tag{2.1}$$

such that the following conditions holds

(i) For $g, h \in G$ and $x \in X$ we have $g \cdot (h \cdot x) = (gh) \cdot x$.

(ii) For any $x \in X$, we have $1_G \cdot x = x$ where 1_G is the identity element of G.

Throughout the thesis we use G-set X to mean a finite G-set. A morphism between two G-sets is a G-equivariant map. The G-sets and their morphisms form a category which we denote by ${}_G\mathsf{set}$. Similarly one defines the category set_G of right G-sets.

Let X be a G-set and $x \in X$. For a subgroup H of G, the set

$$H \cdot x = \{hx \mid h \in H\}$$

is called the *H-orbit of x*. We denote by $[G\backslash X]$ a set of representatives of G-orbits of X. The G-set X is called *transitive* if it has a single G-orbit. The set

$$X^H = \{x \in X \mid \forall h \in H, hx = x\}$$

is called *the set of H-fixed points of X*. The set

$$G_x = \{g \in G \mid gx = x\}$$

is called *the stabilizer of x in G*.

Let X be a transitive G-set. Then X is isomorphic, as G-sets, to the G-set G/G_x of left cosets of G_x in G for any $x \in X$. Here the G-action on G/G_x is defined by

$$g \cdot hG_x = ghG_x \tag{2.2}$$

for any $g, h \in G$. If H and K are subgroups of G, then the G-sets G/H and G/K are isomorphic if and only if H and K are conjugate in G.

There is an obvious notion of disjoint union of G-sets which is also a categorical coproduct. The Burnside group $B(G)$ of G is defined as the Grothendieck group of the category ${}_G\mathsf{set}$ with respect to disjoint union. In other words, it is defined as the quotient of the free abelian group on the set of isomorphism classes of finite G-sets, by the subgroup generated by the elements of the form

$$[X \sqcup Y] - [X] - [Y]$$

where X and Y are finite G-sets, $\sqcup$ denotes the disjoint union, and $[X]$ denotes the isomorphism class of X. Note that the classes of transitive G-sets form a $\mathbb{Z}$-basis of $B(G)$. For a commutative ring k, we set $B_k(G) := k \otimes_{\mathbb{Z}} B(G)$.

Note that if X and Y are G-sets, then the cartesian product $X \times Y$ of X and Y is again a G-set with the diagonal G-action. This induces a ring structure on the Burnside group $B(G)$. Note that $B(G)$ is a commutative ring and the class of a G-set with cardinality one is the identity element. Dress showed in [D69] that the $\mathbb{Q}$-algebra $B_{\mathbb{Q}}(G)$ is split semisimple and the primitive idempotents of $B_{\mathbb{Q}}(G)$ are indexed by the set of conjugacy classes of subgroups of G. We have the following explicit formula by Gluck [G81] and Yoshida [Y83].

Theorem 2.1.1. *Let G be a finite group. There is a bijective correspondence between primitive idempotents e_H^G of $B_{\mathbb{Q}}(G)$ and conjugacy classes of subgroups H of G. Moreover we have*

$$e_H^G = \frac{1}{N_G(H)} \sum_{K \leqslant H} |K| \mu(K, H) [G/K] \,, \tag{2.3}$$

where μ is the Möbius function on the poset of the subgroups of G.

2.2 Bisets

Throughout this section G, H and K denote finite groups and k denotes a commutative ring. We recall the definition and the basic properties of bisets. We refer the reader to [Bc10, Chapters 2,3] for more details.

A (G,H)-biset X is a finite set X equipped with a left G-action and a right H-action that commute with each other. Equivalently, a (G,H)-biset is a left $G\times H$-set. The translation between these two points of view is given by $(g,h)x = gxh^{-1}$, for $g \in G$, $h \in H$, and $x \in X$. We call the latter notation the *biset notation.* A morphism between two (G,H)-bisets is a $G \times H$-equivariant map. The (G,H)-bisets and their morphisms form a category which we denote by ${}_G\mathsf{set}_H$. The Grothendieck group of the category ${}_G\mathsf{set}_H$ is denoted by $B(G,H)$. The class of a (G,H)-biset X is again denoted $[X]$.

Let X be a (G,H)-biset and Y be an (H,K)-biset. Then the cartesian product $X \times Y$ is an H-set via the action $h \cdot (x,y) = (xh^{-1}, hy)$. The set of H-orbits of $X \times Y$ under this action is denoted by $X \otimes Y$ and it is called *the composition of X and Y*. The orbit of an element $(x,y) \in X \times Y$ is denoted by $x \otimes y$. Note that $X \otimes Y$ has a (G,K)-biset structure where the actions are given by $g \cdot (x \otimes y) \cdot k = gx \otimes yk$. Note that the composition of bisets induces a bilinear map

$$- \underset{H}{\cdot} -\colon B_k(G,H) \times B_k(H,K) \to B_k(G,K)\,. \tag{2.4}$$

If K is the trivial group, then via the identifications ${}_G\mathsf{set}_K \cong {}_G\mathsf{set}$ and ${}_H\mathsf{set}_K \cong {}_H\mathsf{set}$, the map in 2.4 induces the map

$$- \underset{H}{\cdot} -\colon B_k(G,H) \times B_k(H) \to B_k(G)\,.$$

The map in 2.4 also allows us to define the *biset category.*

Definition 2.2.1. *The biset category $\mathcal{C}_k$ of finite groups over a commutative ring k is the category defined as follows:*

- *The objects of* $\mathcal{C}_k$ *are finite groups.*
- *If* G *and* H *are finite groups, then* $\mathrm{Hom}_{\mathcal{C}_k}(G, H) = B_k(H, G)$.
- *If* G, H *and* K *are finite groups, then the composition* $v \circ u$ *of the morphism* $u \in \mathrm{Hom}_{\mathcal{C}_k}(G, H)$ *and the morphism* $v \in \mathrm{Hom}_{\mathcal{C}_k}(H, K)$ *is equal to* $v \underset{H}{\cdot} u$.
- *For any finite group* G*, the identity morphism of* G *in* $\mathcal{C}_k$ *is equal to* $[G]$ *the class of the* (G, G)*-biset* G *where the actions are given by multiplications from both sides.*

Since the sets of morphisms in $\mathcal{C}_k$ are k-modules, and the composition in $\mathcal{C}_k$ is k-bilinear, it follows that the category $\mathcal{C}_k$ is a k-linear category.

Definition 2.2.2. *A biset functor over* k *is a* k*-linear functor from* $\mathcal{C}_k$ *to* ${}_k\mathsf{Mod}$.

Together with natural transformations, biset functors over k form an abelian category which we denote by $\mathcal{F}_k$.

Example 2.2.3 (Burnside functor)**.** *The map*

- $G \mapsto B_k(G)$
- $x \in B_k(H, G) \mapsto \left(x \underset{H}{\cdot} - : B_k(G) \to B_k(H) \right)$

is a biset functor over k.

2.3 A-fibered G-sets and A-fibered Burnside rings

Throughout the thesis we fix an abelian group A. For any finite group G we set

$$G^* := \mathrm{Hom}(G, A)$$

and view G^* again as abelian group with point-wise multiplication.

Throughout this section G denotes a finite group. We introduce A-fibered G-sets and the associated monomial Burnside ring $B^A(G)$ (or A-fibered Burnside rings). These structures were first considered by Dress in [D71]. We recall the most important definitions and results and refer the reader to [D71] for more details and proofs.

We denote by $\mathcal{M}(G) = \mathcal{M}^A(G)$ the set of all pairs (H, ϕ) such that $H \leqslant G$ and $\phi \in H^*$. The set $\mathcal{M}(G)$ has a partial order: $(K, \psi) \leqslant (H, \phi)$ if and only if $K \leqslant H$ and $\psi = \phi|_K$. Moreover, G acts by conjugation on $\mathcal{M}(G)$: For $(H, \phi) \in \mathcal{M}(G)$ and $g \in G$ one sets ${}^g(H, \phi) := ({}^gH, {}^g\phi)$, with ${}^g\phi(x) := \phi(g^{-1}xg)$, for $x \in {}^gH$. This way, $\mathcal{M}(G)$ becomes a G-poset, i.e., G acts by poset automorphisms. We denote the G-orbit of $(H, \phi) \in \mathcal{M}(G)$ by $[H, \phi]_G$.

Recall that a (left) *A-fibered G-set* X is a left $(G \times A)$-set with only finitely many A-orbits and the property that every A-orbit is regular, i.e., $ax = bx$ for $a, b \in A$ and $x \in X$ implies $a = b$. Here and in the sequel we consider A and G as subgroups of $G \times A$ via $a \mapsto (1, a)$ and $g \mapsto (g, 1)$ so that expressions like gax, for $g \in G$, $a \in A$, and $x \in X$, are defined. Together with $(G \times A)$-equivariant maps, one obtains a category ${}_G\mathsf{set}^A$. Similarly, we define the category set_G^A of right A-fibered G-sets. We sometimes view a left A-fibered G-set X also as a right A-fibered G-set. In this case the right A-action is given by $xa := ax$ and the right G-action by $xg := g^{-1}x$ for $a \in A$, $g \in G$ and $x \in X$. We view the A-action in analogy to the action of the base ring k when considering a module for a k-algebra. The A-orbits of X are called the *A-fibers* of X. Since the actions of G and A on X commute, we can consider the set of A-fibers of X as a G-set.

There is an obvious notion of *disjoint union* in ${}_G\mathsf{set}^A$ which is also a categorical coproduct. Moreover, there is a *tensor product* $X \otimes Y = X \otimes_A Y \in {}_G\mathsf{set}^A$ defined as the set of A-orbits of the direct product $X \times Y$ under the A-action $a(x,y) = (xa^{-1}, ay)$. The orbit of (x,y) is denoted by $x \otimes y$. One has $a(x \otimes y) = ax \otimes y = x \otimes ay$ and $g(x \otimes y) = gx \otimes gy$ for all $a \in A$, $g \in G$, $x \in X$ and $y \in Y$.

Every element x of an A-fibered G-set X has an associated *stabilizing pair*, $(H_x, \phi_x) \in \mathcal{M}(G)$, where H_x denotes the stabilizer in G of the A-orbit of x, and $\phi_x \in H_x^*$ is defined by the relation $hx = \phi_x(h)x$, for $h \in H_x$. The stabilizing pair of $(g,a)x$ is equal to ${}^g(H_x, \phi_x)$. If X is transitive, the associated stabilizing pairs form a G-orbit in $\mathcal{M}(G)$ which determines X up to isomorphism. If $(H, \phi) \in \mathcal{M}(G)$, then the set $(G \times A)/(\{(h, \phi(h^{-1})|h \in H\})$ is a transitive A-fibered G-set with a stabilizing pair (H, ϕ).

The set of isomorphism classes of ${}_G\mathsf{set}^A$ is a commutative semiring with respect to disjoint union as addition and $\otimes$ as multiplication. The associated Grothendieck ring is denoted by $B^A(G)$. The ring $B^A(G)$ is called the *monomial Burnside ring of G*, or sometimes the *A-fibered Burnside ring of G*. The class of $X \in {}_G\mathsf{set}^A$ in $B^A(G)$ is denoted by $[X]$. The classes of the transitive A-fibered G-sets form a $\mathbb{Z}$-basis of $B^A(G)$. For computational purposes we will often view $B^A(G)$ as the free abelian group with basis elements $[H,\phi]_G$, where $(H,\phi)\in[G\backslash\mathcal{M}(G)]$, and refer to those elements as the *standard basis* of $B^A(G)$. The multiplication in $B_k^A(G)$ is given by

$$[H,\phi]_G\cdot[V,\psi]_G=\sum_{g\in[U\backslash G/V]}[U\cap{}^gV,\phi|_{U\cap{}^gV}\cdot({}^g\psi)|_{U\cap{}^gV}]_G\,, \tag{2.5}$$

for $(H,\phi),(V,\psi)\in\mathcal{M}(G)$. The identity element of $B_k^A(G)$ is $[G,1]_G$.

In the case that $A=\{1\}$, the trivial group, we may identify $G\times A$ with G and obtain $B^A(G)=B(G)$, the Burnside ring of G.

2.4 A-fibered bisets

Throughout this section, G, H, and K denote finite groups and k denotes a commutative ring. We recall the notions and basic results regarding A-fibered bisets, the A-fibered biset category, and A-fibered biset functors from [BC18, Sections 1–3].

The category of *A-fibered (G,H)-bisets* is formally defined as the category ${}_{G\times H}\mathsf{set}^A$ and is also denoted by ${}_G\mathsf{set}_H^A$. We often consider an A-fibered (G,H)-biset X as equipped with a left G-action, a right H-action and a two-sided A-action, all four actions commuting with each other, via $gaxhb:=((g,h^{-1}),ab)x$ for $g\in G$, $h\in H$, $a,b\in A$, and $x\in X$ (*fibered biset notation*). Note that b is not inverted consistent with the convention in 2.3.

If H is the trivial group, we can identify ${}_G\mathsf{set}_H$ with ${}_G\mathsf{set}$ and ${}_G\mathsf{set}_H^A$ with ${}_G\mathsf{set}^A$ in the obvious way. Similar identifications apply if G is the trivial group.

There exists a functor

$$- \otimes_{AH} -\colon {}_G\mathsf{set}_H^A \times {}_H\mathsf{set}_K^A \to {}_G\mathsf{set}_K^A\,, \tag{2.6}$$

the *tensor product* of A-fibered bisets, given on objects $X \in {}_G\mathsf{set}_H^A$ and $Y \in {}_H\mathsf{set}_K^A$ as the set of those $(H \times A)$-orbits of $X \times Y$ under the action $(h,a)(x,y) := (x(ha)^{-1}, hay)$ (in fibered biset notation) which are A-free under the induced A-action on these orbits. The orbit of (x,y) is denoted by $x \otimes y$. Thus, in fibered biset notation, $xah \otimes y = x \otimes ahy$. The set $X \otimes_{AH} Y$ is an A-fibered (G,K)-biset via $(g,k,a)(x \otimes y) = (g,a)x \otimes ky$ in formal notation and $ga(x \otimes y)k = gax \otimes yk$ in fibered biset notation. This construction is associative, i.e., the map $(x \otimes y) \otimes z \to x \otimes (y \otimes z)$ is a well-defined isomorphism in ${}_G\mathsf{set}_L^A$ between $(X \otimes_{AH} Y) \otimes_{AK} Z$ and $X \otimes_{AH} (Y \otimes_{AK} Z)$, whenever L is a finite group and $Z \in {}_K\mathsf{set}_L^A$. It is functorial in X, Y and Z.

If K is the trivial group, then via the identifications ${}_H\mathsf{set}_K^A \cong {}_H\mathsf{set}^A$ and ${}_G\mathsf{set}_K^A \cong {}_G\mathsf{set}^A$, the tensor product functor in (2.6) induces a functor

$$- \otimes_{AH} -\colon {}_G\mathsf{set}_H^A \times {}_H\mathsf{set}^A \to {}_G\mathsf{set}^A\,. \tag{2.7}$$

Similarly, choosing G as the trivial group, we obtain a functor

$$- \otimes_{AH} -\colon \mathsf{set}_H^A \times {}_H\mathsf{set}_K^A \to \mathsf{set}_K^A\,. \tag{2.8}$$

There is an obvious notion of disjoint union in ${}_G\mathsf{set}_H^A$ which is also a categorical coproduct. We denote the Grothendieck group of ${}_G\mathsf{set}_H^A$ with respect to disjoint union by $B^A(G,H)$. For any commutative ring k we set $B_k^A(G,H) := k \otimes B^A(G,H)$. The class of an A-fibered (G,H)-biset X in $B^A(G,H)$ is denoted by $[X]$. We also write $[X]$ for

$1 \otimes [X] \in B_k^A(G,H)$. If $X \in {}_G\mathsf{set}_H^A$ is transitive with stabilizing pair $(U,\phi) \in \mathcal{M}(G \times H)$ for some $x \in X$, then we often write $\left[\frac{G\times H}{U,\phi}\right]$ for $[X]$. Thus, if (U,ϕ) runs through a set of representatives of the $(G \times H)$-orbits of $\mathcal{M}(G \times H)$ then the elements $\left[\frac{G\times H}{U,\phi}\right]$ form a k-basis of $B_k^A(G,H)$.

The tensor product constructions above induce the following k-bilinear maps on the Grothendieck group levels:

$$- \underset{H}{\cdot} - \colon B_k^A(G,H) \times B_k^A(H,K) \to B_k^A(G,K)\,, \tag{2.9}$$

$$- \underset{H}{\cdot} - \colon B_k^A(G,H) \times B_k^A(H) \to B_k^A(G)\,, \tag{2.10}$$

$$- \underset{H}{\cdot} - \colon B_k^A(H) \times B_k^A(H,K) \to B_k^A(K)\,. \tag{2.11}$$

The first bilinear map allows us to define the *fibered biset category* $\mathcal{C}_k^A$.

Definition 2.4.1. *The fibered biset category $\mathcal{C}_k^A$ of finite groups over a commutative ring k is the category defined as follows:*

- *The objects of $\mathcal{C}_k^A$ are finite groups.*
- *If G and H are finite groups, then $\mathrm{Hom}_{\mathcal{C}_k^A}(G,H) = B_k^A(H,G)$.*
- *If G, H and K are finite groups, then the composition $v \circ u$ of the morphism $u \in \mathrm{Hom}_{\mathcal{C}_k^A}(G,H)$ and the morphism $v \in \mathrm{Hom}_{\mathcal{C}_k^A}(H,K)$ is equal to $v \underset{H}{\cdot} u$.*
- *For any finite group G, the identity morphism of G in $\mathcal{C}_k^A$ is the element $\left[\frac{G\times G}{\Delta(G),1}\right] \in B_k(G,G)$, which is also the class of the A-fibered (G,G)-biset $G \times A$ with the multiplication actions from both sides. Here, $\Delta(G) := \{(g,g) \mid g \in G\}$.*

In order to state a formula for the tensor product of two transitive A-fibered bisets we need the following notation.

Let U be a subgroup of $G\times H$. Then $U \leqslant p_1(U)\times p_2(U)$, where $p_1\colon G\times H \to G$ and $p_2\colon G \times H \to H$ denote the projection maps. Moreover, setting

$$k_1(U) := \{g \in G \mid (g,1) \in U\} \quad \text{and} \quad k_2(U) := \{h \in H \mid (1,h) \in U\}\,,$$

we obtain $k_i(U) \trianglelefteq p_i(U)$ for $i = 1,2$ and $k_1(U) \times k_2(U) \trianglelefteq U$. The projections p_i induce isomorphisms $q(U) := U/(k_1(U) \times k_2(U)) \xrightarrow{\sim} p_i(U)/k_i(U)$ for $i = 1,2$, so that one obtains an isomorphism

$$\eta_U\colon p_2(U)/k_2(U) \xrightarrow{\sim} p_1(U)/k_1(U) \tag{2.12}$$

given by $\eta_U(hk_2(U)) = gk_1(U)$ if and only if $(g,h) \in U$. We call $q(U)$ the *quotient of* U. Note that it is isomorphic to a subquotient of G and a subquotient of H. For $\phi \in U^* = \mathrm{Hom}(U,A)$, we can write $\phi|_{k_1(U)\times k_2(U)} = \phi_1 \times \phi_2^{-1}$, with uniquely determined $\phi_1 \in k_1(U)^*$ and $\phi_2 \in k_2(U)^*$. We also associate to the pair (U,ϕ) its *left invariants* and *right invariants*

$$l(U,\phi) := (p_1(U), k_1(U), \phi_1) \quad \text{and} \quad r(U,\phi) := (p_2(U), k_2(U), \phi_2)\,.$$

If additionally V is a subgroup of $H \times K$, one sets

$$U * V := \{(g,k) \in G \times K \mid \exists h \in H : (g,h) \in U, (h,k) \in V\}$$

a subgroup of $G \times K$. Moreover, if $\phi \in U^*$ and $\psi \in V^*$ with the property that $\phi_2|_{k_2(U)\cap k_1(V)} = \psi_1|_{k_2(U)\cap k_1(V)}$, then one obtains a homomorphism $\phi*\psi \in \mathrm{Hom}(U*V, A)$

defined by

$$(\phi * \psi)(g,k) := \phi(g,h)\psi(h,k)\,,$$

where $h \in H$ is chosen such that $(g,h) \in U$ and $(h,k) \in V$.

When K is the trivial group we identify $H \times K$ with H, and obtain

$$U * V := \{g \in G \mid \exists h \in V : (g,h) \in U\} \leqslant G\,,$$

for $V \leqslant H$. Moreover, for $(U,\phi) \in \mathcal{M}(G \times H)$ and $(V,\psi) \in \mathcal{M}(H)$ with $\phi_2|_{k_2(U)\cap V} = \psi|_{k_2(U)\cap V}$, we define $\phi*\psi \in \mathrm{Hom}(U*V, A)$ by $(\phi*\psi)(g) := \phi(g,h)\psi(h)$ for any $g \in U*V$, where $h \in V$ is chosen such that $(g,h) \in U$.

Similarly, choosing G to be the trivial group, we define $U * V$ for $U \leqslant H$ and $V \leqslant H \times K$, and a product $\phi * \psi$ for $(U,\phi) \in \mathcal{M}(H)$ and $(V,\psi) \in \mathcal{M}(H \times K)$ if ϕ^{-1} and ψ_1 coincide on $U \cap k_1(V)$.

The following theorem gives explicit formulas for the tensor products (2.9)–(2.11) of standard basis elements. We will refer to it as the *Mackey formula*.

Theorem 2.4.2. *([BC18, Corollary 2.5])* (a) *For $(U,\phi) \in \mathcal{M}(G \times H)$ and $(V,\psi) \in \mathcal{M}(H \times K)$ one has*

$$\left[\frac{G \times H}{U,\phi}\right] \mathop{\cdot}_{H} \left[\frac{H \times K}{V,\psi}\right] = \sum_{\substack{t \in [p_2(U)\backslash H/p_1(V)] \\ \phi_2|_{H_t} = {}^{t}\psi_1|_{H_t}}} \left[\frac{G \times K}{U * {}^{(t,1)}V, \phi * {}^{(t,1)}\psi}\right]\,,$$

where $H_t := k_2(U) \cap {}^{t}k_1(V)$.

(b) *For $(U,\phi) \in \mathcal{M}(G \times H)$ and $(V,\psi) \in \mathcal{M}(H)$ one has*

$$\left[\frac{G \times H}{U,\phi}\right] \mathop{\cdot}_{H} [V,\psi]_H = \sum_{\substack{t \in [p_2(U)\backslash H/V] \\ \phi_2|_{k_2(U)\cap {}^{t}V} = {}^{t}\psi|_{k_2(U)\cap {}^{t}V}}} [U * {}^{t}V, \phi * {}^{t}\psi]_G\,.$$

(c) *For* $(U,\phi)\in\mathcal{M}(H)$ *and* $(V,\psi)\in\mathcal{M}(H\times K)$ *one has*

$$[U,\phi]\underset{H}{\cdot}\left[\frac{H\times K}{V,\psi}\right]=\sum_{\substack{t\in[U\backslash H/p_1(V)]\\ \phi^{-1}|_{U\cap{}^{t}k_1(V)}={}^{t}\psi_1|_{U\cap{}^{t}k_1(V)}}}[U*{}^{(t,1)}V,\phi*{}^{(t,1)}\psi]_K\,.$$

Proof. (a) See [BC18, Corollary 2.5].

(b) This follows immediately from Part (a) by choosing K to be the trivial group:

$$\begin{aligned}\left[\frac{G\times H}{U,\phi}\right]\underset{H}{\cdot}[V,\psi]_H&=\left[\frac{G\times H}{U,\phi}\right]\underset{H}{\cdot}\left[\frac{H\times 1}{V\times 1,\psi\times 1}\right]\\&=\sum_{\substack{t\in[p_2(U)\backslash H/V]\\ \phi_2|_{k_2(U)\cap{}^{t}V}={}^{t}\psi|_{k_2(U)\cap{}^{t}V}}}\left[\frac{G\times 1}{U*{}^{(t,1)}(V\times 1),\phi*{}^{(t,1)}(\psi\times 1)}\right]\\&=\sum_{\substack{t\in[p_2(U)\backslash H/V]\\ \phi_2|_{k_2(U)\cap{}^{t}V}={}^{t}\psi|_{k_2(U)\cap{}^{t}V}}}[U*{}^{t}V,\phi*{}^{t}\psi]_G\,.\end{aligned}$$

(c) This is proved similarly after choosing G to be trivial group in Part (a).

□

If A is the trivial group, we can identify ${}_G\mathsf{set}_H^A$ with ${}_G\mathsf{set}_H$, $B_k^A(G,H)$ with $B_k(G,H)$, and $\mathcal{C}_k^A$ with $\mathcal{C}_k$.

For general A we have an embedding $B_k(G,H)\to B_k^A(G,H)$, induced by the functor $X\mapsto X\times A$ where the A-action is defined as the multiplication on the second component. This functor maps a standard basis element $[(G\times H)/U]$ to the standard basis element $\left[\frac{G\times H}{U,1}\right]$. This induces an embedding of categories $\mathcal{C}_k\subseteq\mathcal{C}_k^A$ and we may view the *elementary bisets* of *induction*, *restriction*, *inflation*, *deflation*, and *isomorphism* (see [Bc10, 2.3.9]) also as morphisms in $\mathcal{C}_k^A$. These morphisms are defined as follows. For $H \leqslant G$ one sets

$$\mathrm{res}_H^G := \left[\frac{H \times G}{\Delta(H), 1}\right] = [G] \in B(H,G) \quad \text{and} \quad \mathrm{ind}_H^G := \left[\frac{G \times H}{\Delta(H), 1}\right] = [G] \in B(G,H)\,,$$

with G viewed as (H,G)-biset and as (G,H)-biset via left and right multiplication. For $N \trianglelefteq G$, one sets

$$\mathrm{inf}_{G/N}^G := \left[\frac{G \times G/N}{\{(g, gN) \mid g \in G\}, 1}\right] = [G/N] \in B(G, G/N)$$

and

$$\mathrm{def}_{G/N}^G := \left[\frac{G/N \times G}{\{(gN, g) \mid g \in G\}, 1}\right] = [G/N] \in B(G/N, G)\,,$$

with G/N viewed as $(G, G/N)$-biset and as $(G/N, G)$-biset via multiplication and the natural epimorphism $G \to G/N$. Finally, if $f\colon H \to G$ is an isomorphism, one sets

$$\mathrm{iso}_f := \left[\frac{G \times H}{\{(f(h), h) \mid h \in H\}, 1}\right] = [G] \in B(G,H)\,,$$

where G is viewed as (G,H)-biset via $gxh := gxf(h)$ for $g \in G$ and $h \in H$. In the special case that $H \leqslant G$, $x \in G$ and $f = c_x\colon H \to {}^xH$ is conjugation by x, we also set $c_x := c_x^H := \mathrm{iso}_{c_x} \in B({}^xH, H)$.

For $\phi \in G^*$ we define $\Delta(\phi) \in \Delta(G)^*$ by $\Delta(\phi)(g,g) := \phi(g)$ and define the map

$$\Delta\colon B_k^A(G) \to B_k^A(G,G)\,, \quad [H,\phi]_G \mapsto \left[\frac{G \times G}{\Delta(H), \Delta(\phi)}\right]\,,$$

for $(H,\phi) \in \mathcal{M}(G)$. Using the explicit multiplication rule from 2.5 and the Mackey formula in Theorem 2.4.2(a) it is straightforward to see that this map is a k-algebra homomorphism. Moreover, it is injective. In fact, for $(H,\phi), (K,\psi) \in \mathcal{M}(G)$, one has

$(\Delta(H), \Delta(\phi)) =_{G\times G} (\Delta(K), \Delta(\psi))$ if and only if $(H,\phi) =_G (K,\psi)$. The explicit Mackey formulas in Theorem 2.4.2(b) and (c) imply that

$$x \underset{G}{\cdot} \Delta(y) = x \cdot y = \Delta(x) \underset{G}{\cdot} y \tag{2.13}$$

for all $x, y \in B_k^A(G)$.

For $\phi \in G^*$ we set

$$\mathrm{tw}_\phi := \Delta([G,\phi]_G) \in B^A(G,G)\,,$$

the *twist* with ϕ. The Mackey formula shows that, for $(U,\phi) \in \mathcal{M}(G \times H)$, $\kappa \in G^*$, and $\lambda \in H^*$, one has

$$\mathrm{tw}_\kappa \underset{G}{\cdot} \left[\frac{G\times H}{U,\phi}\right] = \left[\frac{G\times H}{U,(\kappa\times 1)|_U\cdot\phi}\right] \quad \text{and} \quad \left[\frac{G\times H}{U,\phi}\right] \underset{H}{\cdot} \mathrm{tw}_\lambda = \left[\frac{G\times H}{U,\phi\cdot(1\times\lambda)|_U}\right]. \tag{2.14}$$

We will call the elementary bisets in (b) together with the twists, the *elementary A-fibered bisets.*

Remark 2.4.3. *Recall from [BC18, 1.6, 1.7] that the construction of the* opposite *A-fibered biset induces a k-linear map* $-^\circ\colon B_k^A(G,H) \to B_k^A(H,G)$, $\left[\frac{G\times H}{U,\phi}\right] \mapsto \left[\frac{H\times G}{U^\circ,\phi^\circ}\right]$, *with* $U^\circ := \{(h,g) \in H\times G \mid (g,h) \in U$ *and* $\phi^\circ(h,g) := \phi^{-1}(g,h)$ *for* $(h,g) \in U^\circ$. *Note that* $(U^\circ)^\circ = U$, $p_1(U^\circ) = p_2(U)$, $k_1(U^\circ) = k_2(U)$, $(\phi^\circ)^\circ = \phi$, *and* $(\phi^\circ)_1 = \phi_2$. *The Mackey formula in Theorem 2.4.2(a) implies that, for* $x \in B^A(G,H)$ *and* $y \in B^A(H,K)$, *one has*

$$(x \cdot_H y)^\circ = y^\circ \cdot_H x^\circ \in B^A(K,G)\,. \tag{2.15}$$

Moreover, one has a ring isomorphism $-^\circ\colon B^A(G) \to B^A(G)$ *given by* $[U,\phi]_G \mapsto [U,\phi^{-1}]_G$ *for* $(U,\phi) \in \mathcal{M}(G)$*. The Mackey formulas in Theorem 2.4.2(b) and (c) imply that, for* $x \in B^A(G)$*,* $y \in B^A(G,H)$ *and* $z \in B^A(H)$*, one has*

$$\Delta(x^\circ) = \Delta(x)^\circ\,, \quad (x \underset{G}{\cdot} y)^\circ = y^\circ \underset{G}{\cdot} x^\circ \in B^A(H)\,, \quad \textit{and} \quad (y \underset{H}{\cdot} z)^\circ = z^\circ \underset{H}{\cdot} y^\circ \in B^A(G)\,. \tag{2.16}$$

Using the above notation, we obtain a canonical decomposition of a standard basis element of $B^A(G,H)$ into elementary bisets and a standard basis element for smaller groups.

Theorem 2.4.4. *([BC18, Proposition 2.8]) Let* $(U,\phi) \in \mathcal{M}(G\times H)$ *and set* $P := p_1(U)$*,* $Q := p_2(U)$*,* $K := \ker(\phi_1)$*, and* $L := \ker(\phi_2)$*. Then* $K \trianglelefteq P$*,* $L \trianglelefteq Q$*,* $K \times L \trianglelefteq U$*, and*

$$\left[\frac{G\times H}{U,\phi}\right] = \mathrm{ind}_P^G \underset{P}{\cdot} \mathrm{inf}_{P/K}^P \underset{P/K}{\cdot} \left[\frac{P/K \times Q/L}{U/(K\times L), \bar{\phi}}\right] \underset{Q/L}{\cdot} \mathrm{def}_{Q/L}^Q \underset{Q}{\cdot} \mathrm{res}_L^H\,,$$

where $\bar{\phi} \in (U/(K\times L))^*$ *is induced by* ϕ *and* $U/(K\times L)$ *is viewed as subgroup of* $P/K \times Q/L$ *via the canonical isomorphism* $(P\times Q)/(K\times L) \cong P/K \times Q/L$*.*

2.5 A-fibered biset functors

Throughout this section, G and H denote finite groups and k denotes a commutative ring.

Definition 2.5.1. *([Bc10, Definition 3.2.2], [BC18, Definition 3.1]) An* A-fibered biset functor over k *is a* k*-linear functor* $F\colon \mathcal{C}_k^A \to {}_k\mathsf{Mod}$*.*

Together with natural transformations, the A-fibered biset functors over k form an abelian category that we denote by $\mathcal{F}_k^A$. This allows us to define subfunctors, quotient functors, simple functors etc. Clearly, via restriction along the embedding $\mathcal{C}_k \subseteq \mathcal{C}_k^A$, every A-fibered biset functor can also be viewed as a biset functor. Thus, if $F \in \mathcal{F}_k^A$ or $F \in \mathcal{F}_k$, we can apply F to the elementary fibered bisets, see Section 2.4. We denote the resulting homomorphisms with the same symbols: $\mathrm{res}_H^G = F(\mathrm{res}_H^G)\colon F(G) \to F(H)$ for $H \leqslant G$, etc. Moreover, for $H \leqslant G$, $g \in G$, and $m \in F(H)$, we denote by ${}^g m \in F({}^g H)$ the element $c_g(m)$.

Example 2.5.2 (The character ring). *[BC18, Section 11B] Let $A = \mathbb{C}^\times$ be the unit group of the complex numbers and let $R_\mathbb{C}(G)$ denote the character ring of $\mathbb{C}G$-modules. Using the properties of the linearization map*

$$\mathrm{lin}_G\colon B^{\mathbb{C}^\times}(G) \to R_\mathbb{C}(G)\,, \quad [H,\phi]_G \mapsto \mathrm{ind}_H^G(\phi)\,,$$

one can show that the functor that assigns to each finite group G the character ring $R_\mathbb{C}(G)$ defines an $\mathbb{C}^\times$-fibered biset functor $R_{\mathbb{C}^\times}$. Clearly we can extend the coefficients from $\mathbb{Z}$ to any commutative ring k. If k is a field, then the $\mathbb{C}^\times$-fibered biset functor $kR_\mathbb{C}$ is simple, see [BC18, Theorem 11.3]. It is worth noting that the character ring as a biset functor, i.e., when the fiber group A is trivial, is semisimple being a direct sum of infinitely many simple biset functors [Bc10, Chapter 7].

Example 2.5.3 (The fibered Burnside functor). *Mapping a finite group G to $B_k^A(G)$ and an element $x \in B_k^A(G,H)$ to the k-linear map $x \cdot_H -\colon B_k^A(H) \to B_k^A(G)$ from (2.10) defines an A-fibered biset functor B_k^A over k, which we can also view as a biset functor* *over k.*

In this thesis, our main goal is to describe the subfunctor lattice and the composition factors of B_k^A.

By the formula in Theorem 2.4.2(b), we have the following explicit elementary biset operations on $B_k^A(G)$: For $H \leqslant G$, $(U,\phi) \in \mathcal{M}(G)$ and $(V,\psi) \in \mathcal{M}(H)$, one has

$$\mathrm{res}_H^G([U,\phi]_G) = \sum_{g\in[H\backslash G/U]} [H \cap {}^gU, ({}^g\phi)|_{H\cap {}^gU}]_H \quad \text{and} \quad \mathrm{ind}_H^G([V,\psi]_H) = [V,\psi]_G\,.$$

For $N \trianglelefteq G$ and $(U/N,\phi) \in \mathcal{M}(G/N)$ one has

$$\mathrm{inf}_{G/N}^G([U/N,\phi]_{G/N}) = [U,\phi\circ\nu]_G\,,$$

where $\nu\colon U \to U/N$ is the natural epimorphism, and

$$\mathrm{def}_{G/N}^G([V,\psi]_G) = \begin{cases} [VN/N,\tilde{\psi}]_{G/N}, & \text{if } V\cap N \leqslant \ker(\psi), \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{\psi}(vN) := \psi(v)$ for $v \in V$. For an isomorphism $f\colon H \to G$ and $(V,\psi) \in \mathcal{M}(H)$, we have

$$\mathrm{iso}_f([V,\psi]_H) = [f(V),\psi\circ f^{-1}|_{f(V)}]_G\,.$$

In particular, for $g \in G$ and $H \leqslant G$, and $(K,\psi) \in \mathcal{M}(H)$, we have

$${}^g[K,\psi]_H = [{}^gK, {}^g\psi]_{{}^gH}\,.$$

Finally, for $\lambda \in G^*$ and $(H,\phi) \in \mathcal{M}(G)$ one has

$$\mathrm{tw}_\lambda([H,\phi]_G) = [H,\lambda|_H\cdot\phi]_G\,.$$

It is easily verified that res^G_H, $\mathrm{inf}^G_{G/N}$ and iso_f are ring homomorphisms, and that $B_k^A(G)$ is a kG^*-algebra via tw.

The ring structure of $B_k^A(G)$ defined in Section 2.3 provides the biset functor B_k^A even with the structure of a Green biset functor over k. The category of A-fibered biset functors over k is isomorphic to the category of B_k^A-modules, see [Bc10, Sections 8.6, 8.7] and also [R12]. Again, the necessary axioms can be verified immediately with the formulas in Theorem 2.4.2.

Let $\mathcal{D}_k$ be the subcategory of $\mathcal{C}_k$ with the same objects as $\mathcal{C}_k$, but with morphism sets generated by all elementary bisets, excluding inductions. In other words, $\mathrm{Hom}_{\mathcal{D}_k}(G,H) \subseteq B_k(G,H)$ is the free k-module generated by all standard basis elements $[(G\times H)/U]$ with $p_1(U) = G$. Mapping G to the group algebra kG^* defines a Green biset functor F on $\mathcal{D}_k$ over k in the sense of [Bc10, Definition 8.5.1] with restriction, inflation and isomorphisms defined as usual, viewing $\mathrm{Hom}(-,A)$ as contravariant functor, and deflation defined by

$$\mathrm{def}^G_{G/N}(\phi) := \begin{cases} \bar{\phi}, & \text{if } \phi|_N = 1, \\ 0, & \text{otherwise,} \end{cases}$$

whenever N is a normal subgroup of G and $\phi \in G^*$. Here, $\bar{\phi} \in (G/N)^*$ is induced by ϕ. In fact, it is straightforward to check that all the relations in [Bc10, 1.1.3] that do not involve inductions are satisfied. Thus, we are in the situation of [BRV19, Theorem 7.3(a),(b)] and obtain via the $-_+$-construction a Green biset functor F_+ on $\mathcal{D}_+ = \mathcal{C}$. This Green biset functor is isomorphic to the Green biset functor B_k^A by

[BRV19, Theorem 4.7(c)] and the explicit formulas for the elementary biset operations in [BRV19, Remark 4.8]. We will use this point of view in Section 3 in order to determine the primitive idempotents of $B_k^A(G)$ for special cases of k and A.

Note that the construction of $B_k^A(G)$ is also functorial in A. If $f\colon A' \to A$ is a homomorphism between abelian groups, then we obtain an induced homomorphim $\mathrm{Hom}(G, A') \to \mathrm{Hom}(G, A)$ for all finite groups G and an induced G-equivariant map $\mathcal{M}^{A'}(G) \to \mathcal{M}^A(G)$. We also obtain a group homomorphism $B_k^{A'}(G) \to B_k^A(G)$, $[H, \phi]_G \mapsto [H, f \circ \phi]_G$. This group homomorphism is even a k-algebra homomorphism. Moreover, it commutes with the tensor product maps in (2.9) and (2.10). In this way we obtain a morphism $B_k^{A'} \to B_k^A$ of Green biset functors and a functor $\mathcal{C}_k^{A'} \to \mathcal{C}_k^A$. If $A' \leqslant A$ then $B_k^{A'}(G)$ is a k-subalgebra of $B_k^A(G)$ and $\mathcal{C}_k^{A'}$ becomes a subcategory of $\mathcal{C}_k^A$.

2.6 Simple A-fibered biset functors

Let A be an abelian group and k a commutative ring. In this section we recall the parametrization of simple A-fibered biset functors over k. We refer the reader to [BC18] for more details. We first recall several constructions from [Bc96].

For a finite group G, let $E_G = E_k^A(G)$ denote the endomorphism algebra of G in $\mathcal{C}_k^A$, i.e.,

$$E_G = \mathrm{Hom}_{\mathcal{C}_k^A}(G, G) = B_k^A(G, G).$$

Note that for any A-fibered biset functor F, the k-module $F(G)$ is a left E_G-module. Let $I_G = I_k^A(G)$ denote the ideal of E_G generated by the morphisms that factor through groups of smaller order, i.e.,

$$I_G = \sum_{|H|<|G|} B_k^A(G,H) \underset{H}{\cdot} B_k^A(H,G) \subseteq E_G.$$

The factor k-algebra $\bar{E}_G = E_G/I_G$ is called the *the essential algebra of* G. Given a finite group G and an irreducible module V of E_G, we define two functors in $\mathcal{F}_k^A$. The functor $L_{G,V}$ is defined on the objects as

$$L_{G,V}(H) = \mathrm{Hom}_{\mathcal{C}_k^A}(G,H) \otimes_{E_G} V = B_k^A(H,G) \otimes_{E_G} V\,,$$

for any finite group H. Here $\mathrm{Hom}_{\mathcal{C}_k^A}(G,H)$ is considered as a right E_G-module via composition of morphisms. The functor $J_{G,V}$ is defined on the objects as

$$J_{G,V}(H) = \left\{ \sum_i x_i \otimes v_i \in B_k^A(H,G) \otimes_{E_G} V \,|\forall y \in B_k^A(G,H) : \sum_i (y \underset{H}{\cdot} x_i)(v_i) = 0 \right\}\,,$$

for any finite group H. By the general theory in [Bc96] the functor $J_{G,V}$ is the unique maximal subfunctor of $L_{G,V}$. Moreover, the simple quotient $S_{G,V} = L_{G,V}/J_{G,V}$ satisfies $S_{G,V}(G) \cong V$.

Definition 2.6.1. *Let $F \in \mathcal{F}_k^A$ be an A-fibered biset functor. A finite group G is called a* minimal group *for F if G is a group of minimal order with $F(G) \neq \{0\}$.*

Let $S \in \mathcal{F}_k^A$ be a simple A-fibered biset functor and let G be a minimal group of S. Then $V := S(G)$ is a simple E_G-module and we have $S \cong S_{G,V}$ in $\mathcal{F}_k^A$. Moreover the simple E_G-module V is annihilated by I_G. This implies, in particular, that every simple A-fibered biset functor is isomorphic to $S_{G,V}$ for some finite group G and some simple E_G-module V which is annihilated by I_G. Conversely, if G is a finite group and V

is a simple E_G-module V which is annihilated by I_G, then G is a minimal group of $S_{G,V}$. Therefore, the isomorphism classes of simple A-fibered biset functors are parametrized by the isomorphism classes of pairs (G, V) where G is a finite group and V is a simple $\bar{E}_G$-module.

In [BC18], Boltje and Coşkun provide a further parametrization by showing that the essential algebra is isomorphic to a direct product of matrix algebras over some group algebras. To review this parametrization briefly, we start with some notations and definitions. We refer the reader to [BC18] for more details.

Recall that the group G acts on the set $\mathcal{M}(G)$ via conjugation. Let $\mathcal{M}(G)^G$ denote the set of G-fixed points of $\mathcal{M}(G)$. A pair $(K, \kappa) \in \mathcal{M}(G)$ is G-fixed if and only K is a normal subgroup of G and κ is a G-stable homomorphism. Let $(K, \kappa) \in \mathcal{M}(G)^G$ be a pair. The set

$$\Delta_K(G) = \{(g_1, g_2) \in G \times G \,|\, g_1K = g_2K\} = (K \times \{1\})\Delta(G) = (\{1\} \times K)\Delta(G)$$

is a subgroup of $G \times G$ since K is a normal subgroup of G. Moreover the map $\phi_\kappa(g_1, g_2) = \kappa(g_1 g_2^{-1}) = \kappa(g_2^{-1} g_1)$ is a homomorphism on $\Delta_K(G)$ since κ is G-stable. We set $e_{(K,\kappa)}$ to be the class of the A-fibered (G, G)-biset

$$\left(\frac{G \times G}{\Delta_K(G), \phi_\kappa}\right)$$

in $E_G = B_k^A(G, G)$. We now set

$$f_{(K,\kappa)} = \sum_{(K,\kappa) \leqslant (L,\lambda) \in \mathcal{M}(G)^G} \mu^{\triangleleft}(K, L) e_{(L,\lambda)}$$

where $\mu^{\triangleleft}$ is the Möbius function on the poset of normal subgroups of G. The elements $f_{(K,\kappa)}$, $(K,\kappa) \in \mathcal{M}(G)^G$, are mutually orthogonal idempotents [BC18, Proposition 4.4].

Let H be another finite group. Let also $(K,\kappa) \in \mathcal{M}(G)^G$ and $(L,\lambda) \in \mathcal{M}(H)^H$. We say that the pairs (G,K,κ) and (H,L,λ) are *linked*, if there exists $(U,\phi) \in \mathcal{M}(G\times H)$ such that $l(U,\phi) = (G,K,\kappa)$ and $r(U,\phi) = (H,L,\lambda)$.

Let $(K,\kappa) \in \mathcal{M}(G)^G$. Then the set of standard basis elements $\left[\frac{G\times G}{U,\phi}\right] \in E_G$ with $l(U,\phi) = (G,K,\kappa) = r(U,\phi)$ form a finite group $\Gamma_{(G,K,\kappa)}$ under multiplication, with identity element $e_{(K,\kappa)}$ and inverses induced by taking opposite of bisets.

Suppose $(K,\kappa) \in \mathcal{M}(G)^G$ and $(L,\lambda) \in \mathcal{M}(H)^H$ are linked. Then any pair $(U,\phi) \in \mathcal{M}(G\times H)$ with the property that $l(U,\phi) = (G,K,\kappa)$ and that $r(U,\phi) = (H,L,\lambda)$ induces an isomorphism $\Gamma_{(H,L.\lambda)} \to \Gamma_{(G,K,\kappa)}$ and as a result one obtains a canonical bijection

$$\mathrm{Irr}(k\Gamma_{(H,L.\lambda)}) \to \mathrm{Irr}(k\Gamma_{(G,K,\kappa)}). \tag{2.17}$$

See [BC18, Section 6.1] for more details.

We now set $\mathcal{R}_G = \mathcal{R}_k^A(G) = \{(K,\kappa) \in \mathcal{M}(G)^G \,|\, e_{(K,\kappa)} \notin I_G\}$. We call the pair $(K,\kappa) \in \mathcal{M}(G)$ a *reduced pair* if $(K,\kappa) \in \mathcal{R}_G$. Note that if $(K,\kappa),(K',\kappa') \in \mathcal{M}(G)^G$ are G-linked, then (K,κ) is reduced if and only if (K',κ') is reduced.

Let $\mathcal{S}_G = \mathcal{S}_k^A(G) = \{((K,\kappa),[V]) \,|\, (K,\kappa) \in \mathcal{R}_G, [V] \in \mathrm{Irr}(k\Gamma_{(G,K,\kappa)})\}$. We call two elements $((K,\kappa),[V]),((K',\kappa'),[V']) \in \mathcal{S}_G$ *equivalent*, if (K,κ) and (K',κ') are G-linked and $[V]$ corresponds to $[V']$ via the canonical bijection in 2.17. Let $\tilde{\mathcal{R}}_G$ denote a set of representatives of the linkage classes of $\mathcal{R}_G$. Then

$$\tilde{\mathcal{S}}_G = \{((K,\kappa),[V])\,|(K,\kappa)\in\tilde{\mathcal{R}}_G, [V]\in \mathrm{Irr}(k\Gamma_{(G,K,\kappa)})\}$$

is a set of representatives of the equivalence classes of $\mathcal{S}_G$.

Theorem 2.6.2 (Proposition 8.4(c), Corollary 8.5, [BC18])**.** (a) *For any reduced pair $(K,\kappa)\in\mathcal{R}_G$, the map*

$$k\Gamma_{(G,K,\kappa)} \to \bar{f}_{(K,\kappa)}\bar{E}_G\bar{f}_{(K,\kappa)} \quad a\mapsto \bar{f}_{(K,\kappa)}\bar{a}\bar{f}_{(K,\kappa)}$$

is a k-algebra isomorphism.

(b) *The map*

$$((K,\kappa),[V])\mapsto \tilde{V} := \bar{E}_G\bar{f}_{(K,\kappa)}\otimes_{k\Gamma_{(G,k,\kappa)}} V$$

induces a bijection between a set of equivalence classes of $\mathcal{S}_G$ and $\mathrm{Irr}(\bar{E}_G)$.

Note that Part (b) of the theorem above implies that for any $((K,\kappa),[V])\in\mathcal{S}_G$ we obtain a simple A-fibered biset functor $S_{(G,K,\kappa,V)} := S_{G,\tilde{V}}\in\mathcal{F}_k^A$. Set

$$\mathcal{S} = \{(G,K,\kappa,[V])\,|G\in \mathrm{Ob}(\mathcal{C}_k^A), (K,\kappa)\in\mathcal{R}_G, [V]\in\mathrm{Irr}(k\Gamma_{(G,K,\kappa)})\}.$$

Two quadruples $(G,K,\kappa,[V])$, $(H,L,\lambda,[W])$ in $\mathcal{S}$ are called *linked*, if $(G,K,\kappa)\sim(H,L,\lambda)$ and the module $[V]$ corresponds to the module $[W]$ via the canonical bijection $\mathrm{Irr}(k\Gamma_{(H,L,\lambda)})\to\mathrm{Irr}(k\Gamma_{(G,K,\kappa)})$. Note that the linkage defines an equivalence relation on $\mathcal{S}$. Let $\bar{\mathcal{S}}$ denote the set of linkage classes $[(G,K,\kappa,[V])]$ of $\mathcal{S}$.

Theorem 2.6.3 (Theorem 9.2, [BC18]). *The map*

$$\bar{\mathcal{S}} \to \mathrm{Irr}(\mathcal{F}_k^A), \quad [(G, K, \kappa, [V])] \mapsto [S_{(G,K,\kappa,V)}]$$

is a bijection between the set of the linkage classes of $\mathcal{S}$ and the set of isomorphism classes of the simple A-fibered biset functors.

Chapter 3

Primitive idempotents of $B_{\mathbb{K}}^A(G)$

Throughout this section we assume that G is a finite group such that $H^* = \mathrm{Hom}(H, A)$ is a finite abelian group for every $H \leqslant G$. This is equivalent to $\mathrm{tor}_{\exp(G)}(A)$ being finite. Moreover, we assume that $\mathbb{K}$ is a splitting field of characteristic zero for all H^*, $H \leqslant G$. Note that this holds if and only if $\mathbb{K}$ has a root of unity of order $\exp(\mathrm{tor}_{\exp(G)}(A))$. Also note that in this case S^* is finite and $\mathbb{K}$ is a splitting field for S^*, for each subquotient S of G.

We define $\mathcal{X}(G)$ as the set of all pairs (H, Φ) with $H \leqslant G$ and $\Phi \in \mathrm{Hom}(H^*, \mathbb{K}^\times)$ and note that G acts on $\mathcal{X}(G)$ by conjugation: ${}^g(H, \Phi) := ({}^gH, {}^g\Phi)$, with ${}^g\Phi(\phi) := \Phi({}^{g^{-1}}\phi)$, for $g \in G$, $(H, \Phi) \in \mathcal{X}(G)$, and $\phi \in H^*$. The assumptions on $\mathbb{K}$ imply that, for any $H \leqslant G$,

$$\mathbb{K}H^* \to \prod_{\Phi \in \mathrm{Hom}(H^*, \mathbb{K}^\times)} \mathbb{K}\,, \quad a \mapsto (s_\Phi^H(a))_\Phi\,, \tag{3.1}$$

is an isomorphism of $\mathbb{K}$-algebras. Here, we $\mathbb{K}$-linearly extended Φ to a $\mathbb{K}$-algebra homomorphism

$$s_\Phi^H\colon \mathbb{K}H^* \to \mathbb{K}\,.$$

The first orthogonality relation implies that, for $\Psi \in \mathrm{Hom}(H^*, \mathbb{K}^\times)$, the element

$$e_\Psi^H := \frac{1}{|H^*|} \sum_{\phi \in H^*} \Psi(\phi^{-1})\phi \in \mathbb{K}H^* \tag{3.2}$$

is the primitive idempotent of $\mathbb{K}H^*$ which is mapped under the isomorphism in (3.1) to the primitive idempotent $\epsilon_\Psi^H \in \prod_\Phi \mathbb{K}$ whose Φ-component is $\delta_{\Phi,\Psi}$.

For any $H \leqslant G$ we consider the map

$$\pi_H\colon B_\mathbb{K}^A(H) \to \mathbb{K}H^*\,, \quad [U,\phi]_H \mapsto \begin{cases} \phi & \text{if } U = H, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen by the multiplication formula in 2.5 that π_H is a $\mathbb{K}$-algebra homomorphism and we obtain for every $(H,\Phi) \in \mathcal{X}(G)$, a $\mathbb{K}$-algebra homomorphism

$$s_{(H,\Phi)}^G := s_\Phi^H \circ \pi_H \circ \mathrm{res}_H^G\colon B_\mathbb{K}^A(G) \to B_\mathbb{K}^A(H) \to \mathbb{K}H^* \to \mathbb{K}\,.$$

Theorem 3.0.1. *The map*

$$B_\mathbb{K}^A(G) \to \left(\prod_{(H,\Phi)\in\mathcal{X}(G)} \mathbb{K}\right)^G\,, \quad x \mapsto \left(s_{(H,\Phi)}^G(x)\right)_{(H,\Phi)}\,, \tag{3.3}$$

is a $\mathbb{K}$-algebra isomorphism. Here, G acts on $\prod_{(H,\Phi)\in\mathcal{X}(G)} \mathbb{K}$ by permuting the coordinates according to the G-action on $\mathcal{X}(G)$. In particular, every $\mathbb{K}$-algebra homomorphism $B_\mathbb{K}^A(G) \to \mathbb{K}$ is of the form $s_{(H,\Phi)}^G$ for some $(H,\Phi) \in \mathcal{X}(G)$. For $(H,\Phi),(K,\Psi) \in \mathcal{X}(G)$ one has $s_{(H,\Phi)}^G = s_{(K,\Psi)}^G$ if and only if $(H,\Phi) =_G (K,\Psi)$.

Proof. By Theorem [BRV19, Theorems 6.1 and 7.3(c)] and using the point of view from Section 2.5, the *mark morphism*

$$m_G\colon B_{\mathbb{K}}^A(G) \to \left(\prod_{H\leqslant G} \mathbb{K}H^*\right)^G, \quad x \mapsto \big(\pi_H(\mathrm{res}_H^G(x))\big)_{H\leqslant G}, \tag{3.4}$$

is a homomorphism of $\mathbb{K}$-algebras and by Theorem [BRV19, Corollary 6.4] it is an isomorphism, since $|G|$ is invertible in $\mathbb{K}$. Here, G acts on $\prod_{H\leqslant G} \mathbb{K}H^*$ by ${}^g((a_H)_{H\leqslant G}) := ({}^g a_{g^{-1}Hg})_{H\leqslant G}$. Using the K-algebra isomorphisms from (3.1), we obtain a G-equivariant $\mathbb{K}$-algebra isomorphism

$$\prod_{H\leqslant G} \mathbb{K}H^* \to \prod_{(H,\Phi)\in\mathcal{X}(G)} \mathbb{K}\,.$$

Taking G-fixed points of this isomorphism and composing it with the isomorphism in (3.4), we obtain the isomorphism in (3.3). The remaining assertions follow immediately.

□

Clearly, for each $(H,\Phi)\in\mathcal{X}(G)$, we obtain a primitive idempotent $\epsilon_{(H,\Phi)}^G$ of the right hand side of the isomorphism (3.3). More precisely, $\epsilon_{(H,\Phi)}^G$ has entries equal to 1 at indices labelled by the G-conjugates of (H,Φ) and entries equal to 0 everywhere else. We denote the idempotent of $B_{\mathbb{K}}^A(G)$ corresponding to $\epsilon_{(H,\Phi)}^G$ by $e_{(H,\Phi)}^G \in B_{\mathbb{K}}^A(G)$. If (H,Φ) runs through a set of representatives of the G-orbits of $\mathcal{X}(G)$ then $e_{(H,\Phi)}^G$ runs through the set of primitive idempotents of $B_{\mathbb{K}}^A(G)$, without repetition. Thus, we have

$$s_{(H,\Phi)}^G(e_{(K,\Psi)}^G) = \begin{cases} 1, & \text{if } (H,\Phi) =_G (K,\Psi), \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad x\cdot e_{(H,\Phi)}^G = s_{(H,\Phi)}^G(x)e_{(H,\Phi)}^G\,, \tag{3.5}$$

for any $(H,\Phi),(K,\Psi)\in\mathcal{X}(G)$ and any $x\in B_{\mathbb{K}}^A(G)$.

The following theorem gives an explicit formula for $e^G_{(H,\Phi)}$. A different formula for particular choices of A was given by Barker in [Ba04, Theorem 5.2]. For any $H \leqslant G$ and $a = \sum_{\phi\in H^*} a_\phi \phi \in \mathbb{K}H^*$ we will use the notation $[H,a]_G := \sum_{\phi\in H^*} a_\phi [H,\phi]_G \in B^A_{\mathbb{K}}(G)$. Moreover, $N_G(H,\Phi)$ denotes the stabilizer of (H,Φ) under G-conjugation.

Theorem 3.0.2. *For $(H,\Phi) \in \mathcal{X}(G)$ one has*

$$e^G_{(H,\Phi)} = \frac{1}{|N_G(H,\Phi)|} \sum_{K\leqslant H} |K|\mu(K,H)[K, \mathrm{res}^H_K(e^H_\Phi)]_G \tag{3.6}$$

$$= \frac{1}{|N_G(H,\Phi)|} \sum_{\substack{K\leqslant H\\ \Phi|_{K^\perp}=1}} |K|\mu(K,H)[K, \mathrm{res}^H_K(e^H_\Phi)]_G \tag{3.7}$$

$$= \frac{1}{|N_G(H,\Phi)|\cdot|H^*|} \sum_{\substack{K\leqslant H\\ \Phi|_{K^\perp}=1}} \sum_{\phi\in H^*} |K|\mu(K,H)\Phi(\phi^{-1})[K,\phi|_K]_G \in B^A_{\mathbb{K}}(G)\,, \tag{3.8}$$

where $K^\perp := \{\phi \in H^* \mid \phi|_K = 1\} \leqslant H^*$ and μ is the Möbius function on the poset of all subgroups of G.

Proof. We use the inversion formula of the $\mathbb{K}$-algebra isomorphism (3.4) from [BRV19, Proposition 6.3] and obtain

$$e^G_{(H,\Phi)} = \frac{1}{|G|} \sum_{L\leqslant K\leqslant G} |L|\mu(L,K)[L, \mathrm{res}^K_L(a_K)]_G\,, \tag{3.9}$$

with $a_K \in \mathbb{K}K^*$, $K \leqslant G$, given by $a_H = \sum_{x\in[N_G(H)/N_G(H,\Phi)]} {}^x e^H_\Phi$, $a_{{}^gH} = {}^g a_H$, for any $g \in G$, and $a_K = 0$ for all K not G-conjugate to H. Thus, in the above sum, it suffices to sum only over G-conjugate subgroups of H in place of K, and we obtain

$$e^G_{(H,\Phi)} = \frac{1}{|G|} \sum_{x\in[G/N_G(H)]} \sum_{L\leqslant {}^xH} |L|\mu(L, {}^xH)[L, \mathrm{res}^{{}^xH}_L({}^x a_H)]_G\,. \tag{3.10}$$

Replacing L with ${}^{x}K$, for $K \leqslant H$, we see that the sum over L is independent of x and we obtain

$$e^G_{(H,\Phi)} = \frac{[G : N_G(H)]}{|G|} \sum_{K \leqslant H} |K|\mu(K, H)[K, \mathrm{res}^H_K(a_H)]_G \,. \tag{3.11}$$

Substituting $a_H = \sum_{g \in [N_G(H)/N_G(H,\Phi)]} {}^{g}e^H_\Phi$ and using the same argument as above, we obtain the formula in (3.6). In order to prove the formula in (3.7) it suffices to show that $\mathrm{res}^H_K(e^H_\Phi) = 0$ if $\Phi|_{K\perp} \neq 1$. Substituting the formula (3.2) for e^H_Φ, we obtain

$$\mathrm{res}^H_K(e^H_\Phi) = \frac{1}{|H^*|} \sum_{\phi \in H^*} \Phi(\phi^{-1})\phi|_K \,. \tag{3.12}$$

Note that $K^\perp = \ker(\mathrm{res}^H_K \colon H^* \to K^*)$ and choose for every $\psi \in \mathrm{im}(\mathrm{res}^H_K) \leqslant K^*$ an element $\tilde{\psi} \in H^*$ with $\tilde{\psi}|_K = \psi$. Then the right hand side in (3.12) is equal to

$$\frac{1}{|H^*|} \sum_{\psi \in \mathrm{im}(\mathrm{res}^H_K)} \sum_{\lambda \in K^\perp} \Phi(\tilde{\psi}^{-1}\lambda^{-1})\psi = \frac{1}{|H^*|} \sum_{\psi \in \mathrm{im}(\mathrm{res}^H_K)} \Phi(\tilde{\psi}^{-1})\big(\sum_{\lambda \in K^\perp} \Phi(\lambda^{-1})\big)\psi \,,$$

and it suffices to show that $\sum_{\lambda \in K^\perp} \Phi|_{K^\perp}(\lambda^{-1}) = 0$. But

$$\sum_{\lambda \in K^\perp} \Phi|_{K^\perp}(\lambda^{-1}) = [K^\perp : K^\perp \cap \ker(\Phi)] \sum_{\bar{\lambda} \in K^\perp/(K^\perp \cap \ker(\Phi))} \overline{\Phi|_{K^\perp}}(\bar{\lambda}^{-1}) \,, \tag{3.13}$$

with an injective homomorphism from $\overline{\Phi|_{K^\perp}} \colon K^\perp/(K^\perp \cap \ker(\Phi)) \to \mathbb{K}^\times$. It follows that $K^\perp/(K^\perp \cap \ker(\Phi))$ is cyclic, say of order n. Our assumption on $\mathbb{K}$ implies that $\mathbb{K}$ has a primitive n-th root of unity. Moreover, since $\Phi|_{K^\perp} \neq 1$, we have $n > 1$. Thus the sum on the right hand side of (3.13) is equal to the sum of all n-th roots of unity in $\mathbb{K}$, which is 0. This proves Equation (3.7). Formula (3.8) is now immediate after substituting the formula for e^H_Φ. □

Remark 3.0.3. *If A' is the trivial subgroup of A we have $B_{\mathbb{K}}^{A'}(G) = B_{\mathbb{K}}(G)$, the Burnside algebra over $\mathbb{K}$. Using the functoriality properties in Section 2.5, we obtain a commutative diagram*

$$\begin{array}{ccccc} B_{\mathbb{K}}(G) & \xrightarrow{m_G} & (\prod_{H\leqslant G} \mathbb{K})^G & \xrightarrow{\mathrm{id}} & (\prod_{H\leqslant G} \mathbb{K})^G \\ \downarrow & & \downarrow & & \downarrow \\ B_{\mathbb{K}}^A(G) & \xrightarrow[m_G]{} & (\prod_{H\leqslant G} \mathbb{K}H^*)^G & \longrightarrow & (\prod_{(H,\Phi)\in\mathcal{X}(G)} \mathbb{K})^G \end{array} \tag{3.14}$$

of $\mathbb{K}$-algebra homomorphisms, where the left horizontal maps are the mark isomorphisms m_G from (3.4) given by $(\pi_H \circ \mathrm{res}_H^G)$, the right top horizontal map is the identity, the right bottom horizontal map is the product of the isomorphisms from (3.1), the middle vertical map is the product of the unique $\mathbb{K}$-algebra homomorphisms $\mathbb{K} \to \mathbb{K}H^*$, and the right vertical map is induced by the G-equivariant map $\mathcal{X}(G) \to \{H \leqslant G\}$, $(H, \Phi) \mapsto H$, between the indexing sets. We denote the primitive idempotents of $B_{\mathbb{K}}(G)$ by e_H^G, for any $H \leqslant G$. Thus, by the above commutative diagram,

$$e_G^G = \sum_{\Phi\in\mathrm{Hom}(G^*,\mathbb{K}^\times)} e_{(G,\Phi)}^G \,. \tag{3.15}$$

Lemma 3.0.4. *For any $x \in B_{\mathbb{K}}^A(G)$ one has $e_G^G \cdot x = 0$ if and only if $\pi_G(x) = 0$.*

Proof. Since $m_G \colon B_{\mathbb{K}}^A(G) \to \prod_{H\leqslant G}$ is injective and multiplicative, one has $e_G^G \cdot x = 0$ if and only if $m_G(e_G^G) \cdot m_G(x) = 0$. But $m_G(e_G^G)$ has entry 1 in the G-component and entry 0 everywhere else. Thus, $e_G^G \cdot x = 0$ if and only if the entry of $m(x)$ in the G-component is equal to 0. But this entry equals $\pi_G(x)$. $\square$

Chapter 4

Elementary operations on primitive idempotents

Throughout this section we assume as in Section 3 that G is a finite group such that $S^* = \mathrm{Hom}(S, A)$ is finite for all subquotients S of G, and that $\mathbb{K}$ is a field of characteristic 0 which is a splitting field of S^* for all subqotients S of G.

In this section we will establish formulas for elementary fibered biset operations on the primitive idempotents of $B_{\mathbb{K}}^A(S)$ for subquotients S of G. These formulas will be used in later sections.

Proposition 4.0.1. *Let $H \leqslant G$.*

(a) *For $(L, \Psi) \in \mathcal{X}(H)$ one has $s_{(L,\Psi)}^H \circ \mathrm{res}_H^G = s_{(L,\Psi)}^G \colon B_{\mathbb{K}}^A(G) \to \mathbb{K}$.*

(b) *For $(K, \Phi) \in \mathcal{X}(G)$ one has*

$$\mathrm{res}_H^G(e_{(K,\Phi)}^G) = \sum_{\substack{(L,\Psi)\in[H\backslash\mathcal{X}(H)] \\ (L,\Psi)=_G(K,\Phi)}} e_{(L,\Psi)}^H \,.$$

(c) *For* $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$ *and* $H < G$ *one has* $\mathrm{res}^G_H(e^G_{(G,\Phi)}) = 0$.

Proof. (a) We use the point of view from Section 2.5. By [BRV19, Equation (13) and Theorem 6.1] the left square in

$$\begin{array}{ccccc} B^A_{\mathbb{K}}(G) & \xrightarrow{m_G} & \prod\limits_{K\leqslant G} \mathbb{K}K^* & \longrightarrow & \prod\limits_{(K,\Phi)\in\mathcal{X}(G)} \mathbb{K} \\ \Big\downarrow{\scriptstyle \mathrm{res}^G_H} & & \Big\downarrow & & \Big\downarrow \\ B^A_{\mathbb{K}}(H) & \xrightarrow[m_H]{} & \prod\limits_{L\leqslant H} \mathbb{K}L^* & \longrightarrow & \prod\limits_{(L,\Psi)\in\mathcal{X}(H)} \mathbb{K} \end{array} \tag{4.1}$$

is commutative, where the left horizontal maps are the mark homomorphisms from (3.4), the right horizontal maps are given by the isomorphisms in (3.1), and the middle and right vertical maps are the canonical projections. Since the right hand square commutes as well, following up with the projection onto the (L,Ψ)-component of $\prod_{(L,\Psi)\in\mathcal{X}(H)} \mathbb{K}$, yields the result.

(b) Since $\mathrm{res}^G_H \colon B^A_{\mathbb{K}}(G) \to B^A_{\mathbb{K}}(H)$ is a $\mathbb{K}$-algebra homomorphism, $\mathrm{res}^G_H(e_{(K,\Phi)})$ is the sum of certain primitive idempotents $e^H_{(L,\Psi)}$, $(L,\Psi) \in [H\backslash\mathcal{X}(H)]$. Moreover, $e^H_{(L,\Psi)}$ occurs in this sum if and only if $s^H_{(L,\Psi)}(\mathrm{res}^G_H(e^G_{(K,\Phi)})) \neq 0$. The result follows now immediately from (a).

(c) This follows immediately from Part (b). □

Proposition 4.0.2. *Let* $N \trianglelefteq G$.

(a) *For* $(H,\Phi) \in \mathcal{X}(G)$ *one has* $s^G_{(H,\Phi)} \circ \mathrm{inf}^G_{G/N} = s^{G/N}_{(HN/N,\Phi_N)}$, *where* $\Phi_N := \Phi \circ \nu^* \circ \alpha^* \in \mathrm{Hom}((HN/N)^*, \mathbb{K}^\times)$ *with* $\alpha\colon H/(H\cap N) \xrightarrow{\sim} HN/N$ *denoting the canonical isomorphism and* $\nu\colon H \to H/(H\cap N)$ *denoting the canonical epimorphism.*

(b) *For* $(U/N, \Psi) \in \mathcal{X}(G/N)$ *with* $N \leqslant U \leqslant G$*, one has*

$$\inf_{G/N}^{G}(e_{(U/N,\Psi)}^{G/N}) = \sum_{\substack{(H,\Phi)\in[G\backslash\mathcal{X}(G)] \\ (HN/N,\Phi_N)=_{G/N}(U/N,\Psi)}} e_{(H,\Phi)}^{G}\,.$$

Proof. (a) We use again the point of view from Section 2.5. By [BRV19, Equation (12)] applied to $D := \{(g, gN) \mid g \in G\} \leqslant G \times G/N$ and [BRV19, Theorem 6.1] the left square in

$$\begin{array}{ccccc} B_{\mathbb{K}}^{A}(G/N) & \xrightarrow{m_{G/N}} & \prod\limits_{N\leqslant U\leqslant G} \mathbb{K}(U/N)^{*} & \longrightarrow & \prod\limits_{(U/N,\Psi)\in\mathcal{X}(G/N)} \mathbb{K} \\ \Big\downarrow \inf_{G/N}^{G} & & \Big\downarrow & & \Big\downarrow \\ B_{\mathbb{K}}^{A}(G) & \xrightarrow[m_G]{} & \prod\limits_{H\leqslant G} \mathbb{K}H^{*} & \longrightarrow & \prod\limits_{(H,\Phi)\in\mathcal{X}(G)} \mathbb{K} \end{array} \tag{4.2}$$

is commutative, where the left horizontal maps are the mark homomorphisms from (3.4), the right horizontal maps are given by the isomorphisms in (3.1), the middle vertical homomorphism maps the family $(a_{U/N})_{N\leqslant U\leqslant G}$ to $(\inf_{H/(H\cap N)}^{H}(\alpha^{*}(a_{HN/N})))_{H\leqslant G}$ with $\alpha\colon H/(H\cap N) \xrightarrow{\sim} HN/N$ denoting the canonical isomorphism, and the right vertical homomorphism maps the family $(a_{(U/N,\Psi)})_{(U/N,\Psi)\in\mathcal{X}(G/N)}$ to $(a_{(HN/N,\Phi_N)})_{(H,\Phi)\in\mathcal{X}(G)}$. Since the right hand square commutes as well (note that $\inf_{H/(H\cap N)}^{H}\colon \mathbb{K}(H/(H\cap N))^{*} \to \mathbb{K}H^{*}$ is the $\mathbb{K}$-linear extension of ν^{*} from (a)), following up with the projection onto the (H,Φ)-component of $\prod_{(H,\Phi)\in\mathcal{X}(G)} \mathbb{K}$, yields the result.

(b) Since $\inf_{G/N}^{G}\colon B_{\mathbb{K}}^{A}(G/N) \to B_{\mathbb{K}}^{A}(G)$ is a $\mathbb{K}$-algebra homomorphism, the element $\inf_{G/N}^{G}(e_{(U/N,\Psi)}^{G/N})$ is the sum of certain primitive idempotents $e_{(H,\Phi)}^{G}$, $(H,\Phi) \in [G\backslash\mathcal{X}(G)]$. Moreover, $e_{(H,\Phi)}^{G}$ occurs in this sum if and only if $s_{(H,\Phi)}^{G}(\inf_{G/N}^{G}(e_{(U/N,\Psi)}^{G/N})) \neq 0$. Part (a) now implies the result. $\square$

Proposition 4.0.3. *Let $N \trianglelefteq G$.*

(a) *For all $(H,\Phi) \in \mathcal{X}(G)$, there exists $m^{G,N}_{(H,\Phi)} \in \mathbb{K}$ such that*

$$\mathrm{def}^G_{G/N}(e^G_{(H,\Phi)}) = m^{G,N}_{(H,\Phi)} \cdot e^G_{(HN/N,\Phi_N)} \tag{4.3}$$

with Φ_N defined as in Proposition 4.0.2(a).

(b) *For all $\Phi \in G^*$ one has*

$$m^N_{(G,\Phi)} := m^{G,N}_{(G,\Phi)} = \frac{|(G/N)^*|}{|G| \cdot |G^*|} \sum_{\substack{K \leqslant G \\ KN=G \\ \Phi|_{K^\perp}=1}} |K| \cdot |K^\perp| \cdot \mu(K,G) \in \mathbb{Q}\,. \tag{4.4}$$

Proof. (a) For any $x \in B^A_{\mathbb{K}}(G/N)$ we have

$$\begin{aligned} x \cdot \mathrm{def}^G_{G/N}(e^G_{(H,\Phi)}) &= \mathrm{def}^G_{G/N}(\mathrm{inf}^G_{G/N}(x) \cdot e^G_{(H,\Phi)}) = \mathrm{def}^G_{G/N}(s^G_{(H,\Phi)}(\mathrm{inf}^G_{G/N}(x)) \cdot e^G_{(H,\Phi)}) \\ &= \mathrm{def}^G_{G/N}(s^{G/N}_{(HN/N,\Phi_N)}(x) \cdot e^G_{(H,\Phi)}) = s^{G/N}_{(HN/N,\Phi_N)}(x) \cdot \mathrm{def}^G_{G/N}(e^G_{(H,\Phi)})\,. \end{aligned}$$

In fact, the first equation follows from the Green biset functor axioms (see [BRV19, Definition 7.2(a)] and [R11, Definición 3.2.7, Lema 4.2.3]), the second from (3.5), and the third from Proposition 4.0.2(a). Choosing $x = e^{G/N}_{(HN/N,\Phi_N)}$, and reading the above equations backward, we obtain

$$\mathrm{def}^G_{G/N}(e^G_{(H,\Phi)}) = \mathrm{def}^G_{G/N}(e^G_{(H,\Phi)}) \cdot e^G_{(HN/N,\Phi_N)}\,.$$

Now, (3.5) implies the result with $m^G_{(H,\Phi)} = s^{G/N}_{(HN/N,\Phi_N)}(\mathrm{def}^G_{G/N}(e^G_{(H,\Phi)}))$.

(b) Substituting the explicit idempotent formula (3.8) for $e^G_{(G,\Phi)}$ and using the explicit formula for $\mathrm{def}^G_{G/N} \colon B^G_{\mathbb{K}}(G) \to B^A_{\mathbb{K}}(G/N)$ from Section 2.5, the left hand side of

(4.3) is equal to

$$\frac{1}{|G|\cdot|G^*|}\sum_{\substack{K\leqslant G\\ \Phi|_{K^\perp}=1}}\ \sum_{\substack{\phi\in G^*\\ \phi|_{K\cap N}=1}} |K|\,\mu(K,G)\,\Phi(\phi^{-1})\,[KN/K,\widetilde{\phi|_K}]_{G/N}\,,$$

where $\widetilde{\phi|_K}(kN) := \phi(k)$ for $k\in K$. Moreover, using the explicit formula (3.8) for $e^{G/N}_{(G/N,\Phi_N)}$, the right hand side of (4.3) is equal to

$$\frac{m_N^{(G,\Phi)}}{|G/N|\cdot|(G/N)^*|}\sum_{\substack{U/N\leqslant G/N\\ \Phi_N|_{(U/N)^\perp}=1}}\ \sum_{\psi\in(G/N)^*} |U/N|\,\mu(U/N,G/N)\,\Phi_N(\psi^{-1})\,[U/N,\psi|_{U/N}]_{G/N}\,.$$

Next we compare the coefficients at the standard basis element $[G/N,1]_{G/N}$ of $B^A_{\mathbb{K}}(G/N)$ on both sides. On the left hand side, we only have to sum over those $K\leqslant G$ with $KN=G$ and those $\phi\in G^*$ with $\widetilde{\phi|_K}=1$. By the definition of $\widetilde{\phi|_K}$, this is equivalent to $\phi\in K^\perp$. But then $\Phi|_{K^\perp}=1$ implies that $\Phi(\phi^{-1})=1$ for all such ϕ. Thus, the coefficient of $[G/N,1]_{G/N}$ on the left hand side of (4.3) is equal to

$$\frac{1}{|G|\cdot|G^*|}\sum_{\substack{K\leqslant G\\ \Phi|_{K^\perp}=1\\ KN=G}} |K|\,|K^\perp|\,\mu(K,G)\,.$$

On the right hand side of (4.3) only the summands with $U=G$ and $\psi=1$ contribute to the coefficient of $[G/N,1]_{G/N}$ and this coefficient evaluates to $m^N_{(G,\Phi)}/|(G/N)^*|$. The result follows. □

Proposition 4.0.4. *Let H be a subgroup of G and let $(K,\Psi)\in\mathcal{X}(H)$. Then*

$$\mathrm{ind}_H^G(e^H_{(K,\Psi)}) = \frac{|N_G(K,\Psi)|}{|N_H(K,\Psi)|}\cdot e^G_{(K,\Psi)}\,.$$

Proof. This is an immediate consequence of the explicit formula in (3.8), since for any $(L,\phi)\in\mathcal{M}(H)$ we have $\mathrm{ind}_H^G([L,\phi]_H)=[L,\phi]_G$. □

Proposition 4.0.5. (a) *For every isomorphism* $f\colon G \xrightarrow{\sim} G'$ *and* $(H,\Phi) \in \mathcal{X}(G)$ *one has* $\mathrm{iso}_f(e^G_{(H,\Phi)}) = e^{G'}_{(f(H),\Phi\circ(f|_H)^*)}$.

(b) *For every* $g \in G$, $H \leqslant G$, *and* $(K,\Psi) \in \mathcal{X}(H)$ *one has* ${}^g e^H_{(K,\Psi)} = e^{{}^gH}_{({}^gK,\,{}^g\Psi)}$.

(c) *For every* $(H,\Phi) \in \mathcal{X}(G)$ *and* $\alpha \in G^*$ *one has* $\mathrm{tw}_\alpha(e^G_{(H,\Phi)}) = \Phi(\alpha|_H)\, e^G_{(H,\Phi)}$ *and* $\Delta(e^G_{(H,\Phi)}) \cdot_G \mathrm{tw}_\alpha = \Phi(\alpha|_H)\, \Delta(e^G_{(H,\Phi)})$.

Proof. All three parts follow immediately from the explicit formulas for the three operations in Section 2.5, the explicit idempotent formula (3.8), and the Mackey formula. □

Chapter 5

Three lemmata

Throughout this section, G and H denote finite groups and $\mathbb{K}$ a field of characteristic 0 such that S^* is finite and $\mathbb{K}$ is a splitting field of S^* for all subquotients S of G and H.

Lemma 5.0.1. *Let G and H be finite groups and let k be a commutative ring. For $(U, \phi) \in \mathcal{M}(G \times H)$ with $p_1(U) = G$ and $p_2(U) = H$ the following are equivalent:*

(i) *There exists $\alpha \in G^*$ such that $\alpha|_{k_1(U)} = \phi_1$.*

(ii) *There exists $\beta \in H^*$ such that $\beta|_{k_2(U)} = \phi_2$.*

(iii) *There exists $\psi \in (G \times H)^*$ such that $\psi|_U = \phi$.*

(iv) *In the category $\mathcal{C}^A$, the morphism $\left[\frac{G \times H}{U, \phi}\right]$ factors through $q(U)$.*

(v) *In the category $\mathcal{C}_k^A$, the morphism $\left[\frac{G \times H}{U, \phi}\right]$ factors through $q(U)$.*

Proof. Clearly, (iii) implies (i) and (ii).

We next show that (i) implies (iii). Let $\alpha \in G^*$ be an extension of ϕ_1. Since $\phi \in U^*$ and $\alpha \times 1 \in (G \times \{1\})^*$ coincide on $U \cap (G \times \{1\}) = k_1(U) \times \{1\}$ and since $G \times H = (G \times \{1\})U$ with $G \times \{1\}$ normal in $G \times H$, the function $\psi \colon G \times H \to A$, $(g,1)u \mapsto \alpha(g)\phi(u)$ is well-defined and extends ϕ. It is also a homomorphism, since $G \times \{1\}$ is normal in $G \times H$ and $\alpha \times 1$ is U-stable.

Similarly one proves that (ii) implies (iii).

Next we show that (iii) and implies (iv). Let $\psi = \alpha \times \beta \in (G \times H)^*$ extend $\phi \in U^*$. By (2.14), we have

$$\left[\frac{G \times H}{U, \phi}\right] = \mathrm{tw}_\alpha \underset{G}{\cdot} \left[\frac{G \times H}{U, 1}\right] \underset{H}{\cdot} \mathrm{tw}_\beta$$

and the morphism $\left[\frac{G \times H}{U,1}\right]$ factors through $G/k_1(U)$ by Theorem 2.4.4.

Clearly, (iv) implies (v).

Finally, we show that (v) implies (iii). Assume that $\left[\frac{G\times H}{U,\phi}\right]$ factors in $\mathcal{C}_k^A$ through $K := q(U)$ with $K \cong G/k_1(U)$. Then there exist $(V, \Psi) \in \mathcal{M}(G \times K)$ and $(W, \rho) \in \mathcal{M}(K \times H)$ such that $\left[\frac{G\times H}{U,\phi}\right]$ occurs with nonzero coefficient in $\left[\frac{G\times K}{V,\psi}\right] \cdot_K \left[\frac{K\times H}{W,\rho}\right]$. By the Mackey formula, this implies that there exists $t \in K$ such that $\psi_2|_{K_t} = \rho_1|_{K_t}$ with $K_t := k_2(V) \cap {}^t k_1(W)$ and $(U, \phi) = {}^{(g,h)}(V * {}^{(t,1)}W, \psi * {}^{(t,1)}\rho)$. Replacing (V, ψ) with ${}^{(g,1)}(V, \psi)$ and (W, ρ) with ${}^{(t,h)}(W, \rho)$, we may assume that there exist $(V, \psi) \in \mathcal{M}(G \times K)$ and $(W, \rho) \in \mathcal{M}(K \times H)$ with $\psi_2|_{k_2(V)\cap k_1(W)} = \rho_1|_{k_2(V)\cap k_1(W)}$ and $(V * W, \psi * \rho) = (U, \phi)$. By [Bc10, 2.3.22.2] one has

$$k_1(V) \leqslant k_1(V * W) = k_1(U) \leqslant p_1(U) \leqslant p_1(V)\,.$$

Since $p_1(U) = G$, this implies that $p_1(V) = G$. Moreover, since $p_1(U)/k_1(U)$ is isomorphic to a subquotient of $p_1(V)/k_1(V) \cong p_2(V)/k_2(V)$, which in turn is a subquotient of $K \cong p_1(U)/k_1(U)$, we otain that $k_1(V) = k_1(U)$, $p_2(V) = K$ and $k_2(V) = \{1\}$. Since $\psi_2 = 1$, it extends trivially to $K = p_2(V)$. By the first part of the proof (note that $p_1(V) = G$ and $p_2(V) = K$) we also obtain that ψ extends to $\alpha \times 1 \in (G \times K)^*$ for some $\alpha \in G^*$. Similarly one shows that ρ extends to $1 \times \beta \in (K \times H)^*$ for some $\beta \in H^*$. But then $\phi = \psi * \rho$ is the restriction of $(\alpha \times 1) * (1 \times \beta) = \alpha \times \beta$ and (iii) holds. □

Lemma 5.0.2. *Let $(U, \phi) \in \mathcal{M}(G \times H)$.*

(a) *If $\Phi \in \mathrm{Hom}(H^*, \mathbb{K}^\times)$ satisfies $\left[\frac{G\times H}{U,\phi}\right] \cdot_H e^H_{(H,\Phi)} \neq 0$ then $p_2(U) = H$ and ϕ_2 extends to H^*.*

(b) *If $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$ satisfies $e^G_{(G,\Phi)} \cdot_G \left[\frac{G\times H}{U,\phi}\right] \neq 0$ then $p_1(U) = G$ and ϕ_1 extends to G^*.*

Proof. Using the explicit formula (3.8) it is easy to check that $(e^H_{(H,\Phi)})^\circ = e^H_{(H,\Phi^{-1})}$. Thus, by 2.4.3 and Equation (2.16), it suffices to show Part (a).

Since $\left[\frac{G\times H}{U,\phi}\right] = \left[\frac{G\times p_2(U)}{U,\phi}\right] \cdot_{p_2(U)} \mathrm{res}^H_{p_2(U)}$, Proposition 4.0.1(c) implies that $p_2(U) = H$.

We will show that ϕ_2 extends to H by induction on $|G|$. If $|G| = 1$ then ϕ_1 is trivial, thus extends to G, and Lemma 5.0.1 implies that ϕ_2 extends to H. From now on assume that $|G| > 1$. We distinguish two cases.

Case 1: $\pi_G\left(\left[\frac{G\times H}{U,\phi}\right] \cdot_H e^H_{(H,\Phi)}\right) = 0$. By Lemma 3.0.4, this implies that $e^G_G \cdot \left(\left[\frac{G\times H}{U,\phi}\right] \cdot_H e^H_{(H,\Phi)}\right) = 0$ and therefore, using the primitive idempotents e^G_K of the

Burnside ring (see Remark 3.0.3) and (2.13), we have

$$
\begin{aligned}
0 \neq \left[\frac{G\times H}{U,\phi}\right] \underset{H}{\cdot} e^H_{(H,\Phi)} &= (1-e^G_G)\cdot\left(\left[\frac{G\times H}{U,\phi}\right] \underset{H}{\cdot} e^H_{(H,\Phi)}\right) \\
&= \sum_{\substack{K\in[G\backslash\mathcal{S}(G)]\\ K<G}} e^G_K\cdot\left(\left[\frac{G\times H}{U,\phi}\right] \underset{H}{\cdot} e^H_{(H,\Phi)}\right) = \sum_{\substack{K\in[G\backslash\mathcal{S}(G)]\\ K<G}} \Delta(e^G_K)\underset{G}{\cdot}\left(\left[\frac{G\times H}{U,\phi}\right] \underset{H}{\cdot} e^H_{(H,\Phi)}\right).
\end{aligned}
$$

Moreover, using the explicit formula for e^G_K (in the special case that A is trivial), we have

$$
\begin{aligned}
0 \neq \left[\frac{G\times H}{U,\phi}\right] \underset{H}{\cdot} e^H_{(H,\Phi)} &\in \sum_{K<G} \mathbb{K}\cdot\left[\frac{G\times G}{\Delta(K),1}\right]\underset{G}{\cdot}\left(\left[\frac{G\times H}{U,\phi}\right] \underset{H}{\cdot} e^H_{(H,\Phi)}\right) \\
&= \sum_{K<G} \mathbb{K}\cdot\left(\left[\frac{G\times G}{\Delta(K),1}\right]\underset{G}{\cdot}\left[\frac{G\times H}{U,\phi}\right]\right) \underset{H}{\cdot} e^H_{(H,\Phi)} = \sum_{K<G}\mathbb{K}\cdot\left[\frac{G\times H}{\Delta(K)*U,1*\phi}\right]\underset{H}{\cdot} e^H_{(H,\Phi)}.
\end{aligned}
$$

Therefore, there exists $K<G$ such that $\left[\frac{G\times H}{\Delta(K)*U,1*\phi}\right]\cdot_H e^H_{(H,\Phi)} \neq 0$. Note that $p_1(\Delta(K)*U)\leqslant K$, so that we can decompose

$$
\left[\frac{G\times H}{\Delta(K)*U,1*\phi}\right] = \mathrm{ind}^G_K \underset{K}{\cdot}\left[\frac{K\times H}{\Delta(K)*U,1*\phi}\right].
$$

Thus,

$$
\left[\frac{K\times H}{\Delta(K)*U,1*\phi}\right]\underset{H}{\cdot} e^H_{(H,\Phi)} \neq 0\,. \tag{5.1}
$$

By the first part of the proof this implies that $p_2(\Delta(K)*U)=H$. Moreover, it is straightforward to verify that $k_2(\Delta(K)*U)=k_2(U)$ and that $(1*\phi)_2=\phi_2\in k_2(U)^*$. Since $K<G$, the inductive hypothesis applied to (5.1) yields that ϕ_2 extends to H.

Case 2: $\pi_G\Big(\left[\frac{G\times H}{U,\phi}\right]\cdot_H e^H_{(H,\Phi)}\Big)\neq 0$. The explicit formula for $e^H_{(H,\Phi)}$ in (3.8) implies that

$$
\sum_{\substack{K\leqslant H\\ \Phi|_{K^\perp}=1}}\ \sum_{\psi\in H^*} |K|\,\mu(K,H)\,\Phi(\psi^{-1})\,\pi_G\left(\left[\frac{G\times H}{U,\phi}\right]\underset{H}{\cdot}[K,\psi|_K]_H\right)\neq 0\,.
$$

Thus, there exists $K \leqslant H$ and $\psi \in H^*$ such that $\pi_G\left(\left[\frac{G\times H}{U,\phi}\right] \cdot_H [K, \psi|_K]_H\right) \neq 0$. This implies that $U * K = G$ and $\phi_2|_{k_2(U)\cap K} = \psi|_{k_2(U)\cap K}$. Since ϕ_2 is stable under $p_2(U) = H$ and ϕ_2 and ψ coincide on $k_2(U) \cap K$, there exists an extension $\beta \in (k_2(U)K)^*$ of ϕ_2 and ψ. Moreover, $U * K = G$ implies $k_2(U)K = H$. In fact, if $h \in H = p_2(U)$ then there exists $g \in G$ with $(g, h) \in U$. Since $g \in G = U * K$, there exists $k \in K$ such that $(g, k) \in U$. But then $hk^{-1} \in k_2(U)$ and $h \in k_2(U)K$. This completes the lemma. □

Lemma 5.0.3. *Let $(U, \phi) \in \mathcal{M}(G \times H)$, $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$ and $\Psi \in \mathrm{Hom}(H^*, \mathbb{K}^\times)$. Then*

$$e^G_{(G,\Phi)} \cdot \left(\left[\frac{G \times H}{U, \phi}\right] \underset{H}{\cdot} e^H_{(H,\Psi)}\right) = 0$$

unless $p_1(U) = G$, $p_2(U) = H$, ϕ has an extension $\alpha \times \beta \in (G \times H)^*$, $\Phi_{k_1(U)} = \Psi_{k_2(U)} \circ \eta_U^*$, and $m^{k_2(U)}_{(H,\Phi)} \neq 0$, in which case one has

$$e^G_{(G,\Phi)} \cdot \left(\left[\frac{G \times H}{U, \phi}\right] \underset{H}{\cdot} e^H_{(H,\Psi)}\right) = \Phi(\alpha)\, \Psi(\beta)\, m^{k_2(U)}_{(H,\Psi)} \cdot e^G_{(G,\Phi)}\,.$$

Proof. Assume that

$$e^G_{(G,\Phi)} \cdot \left(\left[\frac{G \times H}{U, \phi}\right] \underset{H}{\cdot} e^H_{(H,\Psi)}\right) \neq 0\,. \tag{5.2}$$

By Lemma 5.0.2, $p_2(U) = H$ and ϕ_2 extends to H. Assume $p_1(U) < G$. Then $\left[\frac{G\times H}{U,\phi}\right] \cdot e^H_{(H,\Psi)}$ is in the $\mathbb{K}$-span of elements of the form $[U * L, \psi]_G$, with $L \leqslant H$ and $\psi \in (U * L)^*$. Since $U * L \leqslant p_1(U) < G$ we obtain $m_G(e_{(G,\Phi)} \cdot [U * L, \psi]_G) = m_G(e_{(G,\Phi)} \cdot m_G([U * L, \psi]_G) = 0$, because the first factor has vanishes in the components indexed by proper subgroups of G and the second factor vanishes in the G-component. By the injectivity of m_G this implies that the element in (5.2) vanishes, a contradiction. Thus, $p_1(U) = G$.

By Lemma 5.0.1, ϕ has an extension $\alpha \times \beta \in (G \times H)^*$. With (2.14) we obtain

$$e^G_{(G,\Phi)} \cdot \left(\left[\frac{G \times H}{U, \phi} \right] \underset{H}{\cdot} e^H_{(H,\Psi)} \right) = e^G_{(G,\Phi)} \cdot \left(\mathrm{tw}_\alpha \underset{G}{\cdot} \left[\frac{G \times H}{U, 1} \right] \underset{H}{\cdot} \mathrm{tw}_\beta \underset{H}{\cdot} e^H_{(H,\Psi)} \right)$$
$$= \Delta(e^G_{(G,\Phi)}) \underset{G}{\cdot} \mathrm{tw}_\alpha \underset{G}{\cdot} \left[\frac{G \times H}{U, 1} \right] \underset{H}{\cdot} \mathrm{tw}_\beta \underset{H}{\cdot} e^H_{(H,\Psi)}$$

by (2.13). By Proposition 4.0.5(c) we have $\mathrm{tw}_\beta \cdot_H e^H_{(H,\Psi)} = \mathrm{tw}_\beta(e^H_{(H,\Psi)}) = \Psi(\beta) \cdot e^H_{(H,\Psi)}$ and $\Delta(e^G_{(G,\Phi)}) \cdot_G \mathrm{tw}_\alpha = \Phi(\alpha) \cdot \Delta(e^G_{(G,\Phi)})$. Thus,

$$e^G_{(G,\Phi)} \cdot \left(\left[\frac{G \times H}{U, \phi} \right] \underset{H}{\cdot} e^H_{(H,\Psi)} \right) = \Phi(\alpha)\Psi(\beta) \cdot \Delta(e^G_{(G,\Phi)}) \underset{G}{\cdot} \left[\frac{G \times H}{U, 1} \right] \underset{H}{\cdot} e^H_{(H,\Psi)}$$
$$= \Phi(\alpha)\, \Psi(\beta)\, e_{(G,\Phi)} \cdot \left(\mathrm{inf}^G_{G/k_1(U)} \underset{G/k_1(U)}{\cdot} \left[\frac{G \times H}{\bar{U}, 1} \right] \underset{H/k_2(U)}{\cdot} \mathrm{def}^H_{H/k_2(U)} \underset{H}{\cdot} e^H_{(H,\Psi)} \right)$$
$$= \Phi(\alpha)\, \Psi(\beta)\, e_{(G,\Phi)} \cdot \left(\mathrm{inf}^G_{G/k_1(U)} \underset{G/k_1(U)}{\cdot} \mathrm{iso}_{\eta_U} \underset{H/k_2(U)}{\cdot} \mathrm{def}^H_{H/k_2(U)} \underset{H}{\cdot} e^H_{(H,\Psi)} \right)$$

Using Propositions 4.0.3(a), 4.0.5(a) and 4.0.2(b), we obtain

$$e^G_{(G,\Phi)} \cdot \left(\left[\frac{G \times H}{U, \phi} \right] \underset{H}{\cdot} e^H_{(H,\Psi)} \right) = \Phi(\alpha)\, \Psi(\beta)\, m^{k_2(U)}_{(H,\Psi)} \sum_{(K,\Theta)} e^G_{(G,\Phi)} \cdot e^G_{(K,\Theta)}\,, \tag{5.3}$$

where the sum runs over those $(K, \Theta) \in [G\backslash\mathcal{X}(G)]$ satisfying

$$(Kk_1(U)/k_1(U), \Theta_{k_1(U)}) =_{G/k_1(U)} (G/k_1(U), \Psi_{k_2(U)} \circ \eta_U^*)\,.$$

Since the term in (5.3) is nonzero, one of these pairs (K, Θ) must be G-conjugate, and then also equal, to (G, Φ). This implies the result. $\square$

Chapter 6

The constant $m^N_{(G,\Phi)}$

Throughout this section, G and H denote finite groups and $\mathbb{K}$ denotes a field of characteristic 0 such that for any subquotients S of G and T of H the groups S^* and T^* are finite and $\mathbb{K}$ is a splitting field for S^* and T^*.

In this section we prove the crucial Propsition 6.0.4 stating that $m^M_{(G,\Phi)} = m^N_{(G,\Phi)}$ if $(G/M, \Phi_M) \cong (G/N, \Phi_N)$ (see Definition 7.0.3(a)).

Proposition 6.0.1. *Let N and M be normal subgroups of G with $N \leqslant M$ and let $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$. Then*

$$m^M_{(G,\Phi)} = m^N_{(G,\Phi)} \cdot m^{M/N}_{(G/N,\Phi_N)} .$$

Proof. This follows immediately from Proposition 4.0.3(a) and applying

$$\mathrm{iso}_f \circ \mathrm{def}^{G/N}_{(G/N)/(M/N)} \circ \mathrm{def}^G_{G/N} = \mathrm{def}^G_{G/M}$$

to $e^G_{(G,\Phi)}$, where $f\colon (G/N)/(M/N) \to G/M$ is the canonical isomorphism. □

Lemma 6.0.2. *Let $f_1, f_2 \colon G \to H$ be group homomorphisms, let $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$, and let $K \leqslant G$ be such that $\Phi|_{K^\perp} = 1$ and $f_1|_K = f_2|_K$. Then $\Phi \circ f_1^* = \Phi \circ f_2^* \in \mathrm{Hom}(H^*, \mathbb{K}^\times)$.*

Proof. Let $\lambda \in H^*$. Then $(\Phi \circ f_1^*)(\lambda) = (\Phi \circ f_2^*)(\lambda)$ if and only if $\Phi(\lambda \circ f_1) = \Phi(\lambda \circ f_2)$ which in turn is equivalent to $\Phi((\lambda \circ f_1) \cdot (\lambda \circ f_2)^{-1}) = 1$. But, since $f_1|_K = f_2|_K$, we have $(\lambda \circ f_1) \cdot (\lambda \circ f_2)^{-1} \in K^\perp$ and since $\Phi|_{K^\perp} = 1$ the result follows. $\square$

Proposition 6.0.3. *For $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$ and normal subgroups M and N of G one has*

$$m_{(G,\Phi)}^{M} = \frac{1}{|G| \cdot |G^*|} \sum_{\substack{K \leqslant G \\ KN = KM = G \\ \Phi|_{K^\perp} = 1}} |K| \, \mu(K, G) \, |\Sigma_{M,N}^{K}| \, m_{(G/N, \Phi_N)}^{(K \cap M)N/N} \,,$$

where $\Sigma_{M,N}^{K}$ is the set of elements $\phi \in G^*$ such that $\phi|_{K \cap M \cap N} = 1$ and such that there exists $\psi \in ((G/M) \times (G/N))^*$ with $\psi(kM, kN) = \phi(k)$ for all $k \in K$.

Proof. Consider the element

$$v := e_{(G/M, \Phi_M)}^{G/M} \cdot \left(\mathrm{def}_{G/M}^{G} \underset{G}{\cdot} \Delta(e_{(G,\Phi)}^{G}) \underset{G}{\cdot} \mathrm{inf}_{G/N}^{G} \underset{G/N}{\cdot} e_{(G/N, \Phi_N)}^{G/N} \right) \in B_{\mathbb{K}}^{A}(G/M) \,.$$

Then, on the one hand,

$$v = m_{(G,\Phi)}^{M} e_{(G/M, \Phi_M)}^{G/M} \in B_{\mathbb{K}}^{A}(G/M) \,. \tag{6.1}$$

In fact, by (2.13) and Proposition 4.0.2(b),

$$\Delta(e_{(G,\Phi)}^{G}) \underset{G}{\cdot} \mathrm{inf}_{G/N}^{G} \underset{G/N}{\cdot} e_{(G/N, \Phi_N)}^{G/N} = e_{(G,\Phi)}^{G} \cdot (\mathrm{inf}_{G/N}^{G} \underset{G/N}{\cdot} e_{(G/N, \Phi_N)}^{G/N}) = e_{(G,\Phi)}^{G} \,,$$

and then Proposition 4.0.3(a) implies (6.1). On the other hand, using the formula in (3.8) for $e^G_{(G,\Phi)}$ we obtain

$$v = \frac{1}{|G|\cdot|G^*|} \sum_{\substack{K\leqslant G\\ \Phi|_{K^\perp}=1}} \sum_{\phi\in G^*} |K|\,\mu(K,G)\,\Phi(\phi^{-1})\,e^{G/M}_{(G/M,\Phi_M)}\cdot\left(x_{K,\phi}\underset{G/N}{\cdot} e^{G/N}_{(G/N,\Phi_N)}\right) \quad (6.2)$$

with

$$x_{K,\phi} := \mathrm{def}^G_{G/M}\underset{G}{\cdot}\Delta([K,\phi|_K]_G)\underset{G}{\cdot}\mathrm{inf}^G_{G/N} = \begin{cases} \left[\frac{G/M\times G/N}{\Delta^K_{M,N},\bar\phi}\right], & \text{if } \phi|_{K\cap M\cap N}=1,\\ 0, & \text{otherwise,}\end{cases}$$

for $K\leqslant G$ with $\Phi|_{K^\perp}=1$, by the Mackey formula in Theorem 2.4.2(a), where, in the first case, $\Delta^K_{M,N} := \{(kM,kN)\mid k\in K\}$ and $\bar\phi((kM,kN) := \phi(k)$ for $k\in K$. Note that $A_K := k_1(\Delta^K_{M,N}) = (K\cap N)M/M$, $B_K := k_2(\Delta^K_{M,N}) = (K\cap M)N/N$, $p_1(\Delta^K_{M,N}) = KM/M$, and $p_2(\Delta^K_{M,N}) = KN/N$. Lemma 5.0.3 implies that, if the term $e^{G/M}_{(G/M,\Phi_M)}\cdot(x_{K,\phi}\underset{G/N}{\cdot} e^{G/N}_{(G/N,\Phi_N)})$ in (6.2) is nonzero then $KM=G=KN$, $\phi|_{K\cap M\cap N}=1$, and $\bar\phi$ extends to some $\psi=\alpha\times\beta\in((G/M)\times(G/N))^*$, and the formula in Lemma 5.0.3 yields

$$v = \frac{1}{|G|\cdot|G^*|} \sum_{\substack{K\leqslant G\\ KM=G=KN\\ \Phi|_{K^\perp}=1}} \sum_{\phi\in\Sigma^K_{M,N}} |K|\,\mu(K,G)\,\Phi(\phi^{-1})\,\Phi_M(\alpha_{K,\phi})\,\Phi_N(\beta_{K,\phi})\,m^{B_K}_{(G/N,\Phi_N)}\cdot$$
$$\cdot\, e^{G/M}_{(G/M,\Phi_M)}\,, \quad (6.3)$$

where, for K and ϕ as above, $\alpha_{K,\phi}\in(G/M)^*$ and $\beta_{K,\psi}\in(G/N)^*$ are chosen such that $\alpha_{K,\phi}\times\beta_{K,\phi}$ is an extension of $\bar\phi$. Denoting the inflations of $\alpha_{K,\psi}$ and $\beta_{K,\psi}$ to G by $\tilde\alpha_{K,\psi}\in$ and $\tilde\beta_{K,\psi}$, we have

$$\Phi(\phi^{-1})\Phi_M(\alpha_{K,\phi})\Phi_N(\beta_{K,\phi}) = \Phi(\phi^{-1}\tilde\alpha_{K,\phi}\tilde\beta_{K,\phi}) = 1$$

since $\tilde{\alpha}_{K,\phi}(k)\tilde{\beta}_{K,\phi}(k) = \alpha_{K,\phi}(kM)\beta_{K,\phi}(kN) = \bar{\phi}(kM, kN) = \phi(k)$ for all $k \in K$ and $\Phi|_{K^\perp} = 1$. Thus,

$$v = \frac{1}{|G| \cdot |G^*|} \sum_{\substack{K \leqslant G \\ KM=G=KN \\ \Phi|_{K^\perp}=1}} |\Sigma_{M,N}^K| \, |K| \, \mu(K, G) \, m_{(G/N,\Phi_N)}^{B_K} \, e_{(G/M,\Phi_M)}^{G/M} \, . \tag{6.4}$$

Comparing Equations (6.1) and (6.4), the formula for $m_{(G,\Phi)}^M$ follows. □

Proposition 6.0.4. *Let M and N be normal subgroups of G such that there exists an isomorphism $f\colon G/N \xrightarrow{\sim} G/M$ and let $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$ be such that $\Phi_N \circ f^* = \Phi_M \in \mathrm{Hom}((G/M)^*, \mathbb{K}^\times)$. Then one has $m_{(G,\Phi)}^M = m_{(G,\Phi)}^N$.*

Proof. We proceed by induction on $|G|$. If $|G| = 1$ the result is clearly true. So assume that $|G| > 1$. If $M = N$ is the trivial subgroup of G then the result holds for trivial reasons. So assume that M and N are not trivial. By Proposition 6.0.3 one has

$$m_{(G,\Phi)}^M = \frac{1}{|G| \cdot |G^*|} \sum_{\substack{K \leqslant G \\ KN=KM=G \\ \Phi|_{K^\perp}=1}} |K| \, \mu(K, G) \, |\Sigma_{M,N}^K| \, m_{(G/N,\Phi_N)}^{(K\cap M)N/N}$$

and

$$m_{(G,\Phi)}^N = \frac{1}{|G| \cdot |G^*|} \sum_{\substack{K \leqslant G \\ KN=KM=G \\ \Phi|_{K^\perp}=1}} |K| \, \mu(K, G) \, |\Sigma_{N,M}^K| \, m_{(G/M,\Phi_M)}^{(K\cap N)M/M} \, .$$

We will show that these sums coincide by comparing them summand by summand. Since $\Sigma_{M,N}^K = \Sigma_{N,M}^K$, it suffices to show that $m_{(G/N,\Phi_N)}^{(K\cap M)N/N} = m_{(G/M,\Phi_M)}^{(K\cap N)M/M}$. By Proposition 4.0.3(b), for any $X \trianglelefteq G/N$ we have

$$m_{(G/N,\Phi_N)}^X = \frac{|((G/N)/X)^*|}{|G/N| \cdot |(G/N)^*|} \sum_{\substack{L \leqslant G/N \\ LX=G/N \\ (\Phi_N)|_{L^\perp}=1}} |L| \, |L^\perp| \, \mu(L, G/N)$$

and

$$m^{f(X)}_{(G/M,\Phi_M)} = \frac{|((G/M)/\alpha(X))^*|}{|G/M|\cdot|(G/M)^*|} \sum_{\substack{K\leqslant G/N\\ Kf(X)=G/N\\ \Phi_M|_{K^\perp}=1}} |K|\,|K^\perp|\,\mu(K,G/M)\,.$$

Note that the summand for L in the first sum is equal to the summand for $K = f(L)$ in the second sum. Thus, with $X = (K\cap M)N/N$, we obtain

$$m^{(K\cap M)N/N}_{(G/N,\Phi_N)} = m^{f((K\cap M)N/N)}_{(G/M,\Phi_M)} \tag{6.5}$$

for any $K \leqslant G$ with $KM = G = KN$ and $\Phi|_{K^\perp} = 1$. It now suffices to find an isomorphism $f_1\colon (G/M)/f((K\cap M)N/N) \xrightarrow{\sim} (G/M)/((K\cap N)M/M)$ such that

$$(\Phi_M)_{f((K\cap M)N/N)} \circ f_1^* = (\Phi_M)_{(K\cap N)M/M}\,,$$

since then, by induction, we have $m^{f((K\cap M)N/N)}_{(G/M,\Phi_M)} = m^{(K\cap N)M/M}_{(G/M,\Phi_M)}$ and together with Equation (6.5) this implies that desired equation. We define $f_1 := \eta\circ\bar{f}^{-1}$, where

$$\bar{f}\colon (G/N)/((K\cap M)N/N) \xrightarrow{\sim} (G/M)/f((K\cap M)N/N)$$

is induced by f and $\eta := \eta_{\Delta^K_{M,N}}\colon (G/N)/((K\cap M)N/N) \xrightarrow{\sim} (G/M)/((K\cap N)M/M)$ is induced by $\Delta^K_{M,N} := \{(kM,kN)\mid k\in K\}$, see 2.12. It remains to prove that

$$\Phi\circ\nu_M^*\circ\nu^*_{f((K\cap M)N/N)}\circ f_1^* = \Phi\circ\nu_M^*\circ\nu^*_{((K\cap N)M/M)}\,, \tag{6.6}$$

where the maps $\nu_M\colon G\to G/M$, $\nu_{f(K\cap M)N/N}\colon G/M\to (G/M)/f((K\cap M)N/N)$, and $\nu_{(K\cap N)M/M}\colon G/M \to (G/M)/((K\cap N)M/M)$ denote the natural epimorphisms. But $f_1^* = (\bar{f}^{-1})^*\circ\eta^*$, $\nu^*_{f((K\cap M)N/N)}\circ(\bar{f}^{-1})^* = (f^{-1})^*\circ\nu^*_{(K\cap N)M/M}$, and $\Phi\circ\nu_M^*\circ(f^{-1})^* = \Phi_M\circ(f^{-1})^* = \Phi_N = \Phi\circ\nu_N^*$. Thus, the left hand side of Equation (6.6) is equal to

$\Phi \circ \nu_N^* \circ \nu_{(K\cap M)N/N}^* \circ \eta^*$. By Lemma 6.0.2 and since $\Phi|_{K^\perp} = 1$, it now suffices to show that

$$(\eta \circ \nu_{(K\cap M)N/N} \circ \nu_N)|_K = (\nu_{(K\cap M)N/N} \circ \nu_N)|_K \,.$$

But this follows from $(kM, kN) \in \Delta_{M,N}^K$ for $k \in K$, and the proof is complete. □

Chapter 7

B^A-pairs and the subfunctors $E^N_{(G,\Phi)}$ of $B^A_{\mathbb{K}}$

Throughout this section G denotes a finite group and we assume that $\mathbb{K}$ is a field of characteristic 0 which is a splitting field for $\mathbb{K}G^*$ for all finite groups G. This is equivalent to requiring that, for any torsion element a of A, the field $\mathbb{K}$ has a root of unity whose order is the order of a.

In this section we introduce the important subfunctors $E_{(G,\Phi)}$ of $B^A_{\mathbb{K}}$ and study their properties.

Definition 7.0.1. *For any finite group G and $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$ we denote by $E_{(G,\Phi)}$ the subfunctor of $B^A_{\mathbb{K}}$ generated by $e^G_{(G,\Phi)}$. In other words, for each finite group H, one has*

$$E_{(G,\Phi)}(H) = \{x \cdot_G e_{(G,\Phi)} \mid x \in B^A_{\mathbb{K}}(H,G)\}\,.$$

Proposition 7.0.2. *For $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$, the following are equivalent:*

(i) *If H is a finite group with $E_{(G,\Phi)}(H) \neq \{0\}$ then $|G| \leqslant |H|$.*

(ii) *If* H *is a finite group with* $E_{(G,\Phi)}(H) \neq \{0\}$ *then* G *is isomorphic to a subquotient of* H*.*

(iii) *For all* $\{1\} \neq N \trianglelefteq G$ *one has* $m^N_{(G,\Phi)} = 0$.

(iv) *For all* $\{1\} \neq N \trianglelefteq G$ *one has* $\mathrm{def}^G_{G/N}(e^G_{(G,\Phi)}) = 0$.

Proof. That (ii) implies (i) and that (i) implies (iv) follows from the definitions. Moreover, that (iii) and (iv) are equivalent follows from Proposition 4.0.3(a). So, it suffices to prove that (iii) implies (ii).

Assume that (iii) holds and let H be a finite group with $E_{(G,\Phi)}(H) \neq \{0\}$. By the definition of $E_{(G,\Phi)}$ this implies that there exists $(U,\phi) \in \mathcal{M}(G \times H)$ such that $\left[\frac{H\times G}{U,\phi}\right] \cdot_G e^G_{(G,\Phi)} \neq 0$. Using the canonical decomposition of $\left[\frac{H\times G}{U,\phi}\right]$ from Theorem 2.4.4 implies that

$$\left[\frac{(P/K)\times(Q/L)}{\bar{U},\bar{\phi}}\right] \underset{p_2(U)/\ker(\phi_2)}{\cdot} \mathrm{def}^Q_{Q/L} \underset{Q}{\cdot} \mathrm{res}^G_Q \underset{G}{\cdot} e^G_{(G,\Phi)} \neq 0\,,$$

with $P := p_1(U)$, $K := \ker(\phi_1)$, $Q := p_2(U)$, $L := \ker(\phi_2)$, $\bar{U}$ corresponding to $U/(K \times L)$ via the canonical isomorphism $(P \times Q)/(K \times L) \cong (P/K) \times (Q/L)$, and $\bar{\phi} \in (\bar{U})^*$ induced by ϕ. Proposition 4.0.1(c) implies that $G = Q$, and then Proposition 4.0.3(a) implies that $L = \{1\}$. Thus, $p_1(\bar{U}) = P/K$, $p_2(\bar{U}) = G$, $k_2(\bar{U}) = \{1\}$ and

$$\left[\frac{P/K\times G}{\bar{U},\bar{\phi}}\right] \underset{G}{\cdot} e^G_{(G,\Phi)} \neq 0\,.$$

Lemma 5.0.2 implies that $(\bar{\phi})_2$ extends to G and Lemma 5.0.1 implies that $\bar{\phi}$ extends to some $\alpha \times \beta \in ((P/K) \times G)^*$ with $\alpha \in (P/K)^*$ and $\beta \in G^*$, since $p_1(\bar{U}) = P/K$ and

$p_2(\bar{U}) = G$. By (2.14), we have

$$0 \neq \left[\frac{(P/K)\times G}{\bar{U},\bar{\phi}}\right] \cdot_G e^G_{(G,\Phi)} = \mathrm{tw}_\alpha \cdot_{P/K} \left[\frac{(P/K)\times G}{\bar{U},1}\right] \cdot_G \mathrm{tw}_\beta \cdot_G e^G_{(G,\Phi)}$$

with $\mathrm{tw}_\beta \cdot_G e^G_{(G,\Phi)} e = \Phi(\beta) e^G_{(G,\Phi)}$ by Proposition 4.0.5(c). Thus, using the canonical decomposition of $\left[\frac{(P/K)\times G}{\bar{U},1}\right]$ as in Theorem 2.4.4 we obtain $\mathrm{def}^G_{G/k_2(\bar{U})} \cdot_G e^G_{(G,\Phi)} \neq 0$. Proposition 4.0.3(a) implies that $k_2(\bar{U}) = \{1\}$. But this implies that $G \cong G/\{1\} \cong p_2(\bar{U})/k_2(\bar{U}) \cong p_1(\bar{U})/k_1(\bar{U})$ is isomorphic to a subquotient of $\bar{U}$, which is isomorphic to a subquotient of H. □

Note that by the formula for $m^N_{(G,\Phi)}$ in Proposition 4.0.3, the condition in Proposition 7.0.2(iii) is independent of the choice of $\mathbb{K}$ as long as $\mathbb{K}$ has enough roots of unity.

Definition 7.0.3. *Let G and H be finite groups and let $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$ and $\Psi \in \mathrm{Hom}(H^*, \mathbb{K}^\times)$.*

(a) We call (G,Φ) and (H,Ψ) isomorphic *and write $(G,\Phi) \cong (H,\Psi)$ if there exists an isomorphism $f\colon H \xrightarrow{\sim} G$ such that $\Psi \circ f^* = \Phi$. We write $(H,\Psi) \preccurlyeq (G,\Phi)$ if there exists a normal subgroup N of G such that $(H,\Psi) \cong (G/N, \Phi_N)$.*

*(b) The pair (G,Φ) is called a B^A-*pair *if the equivalent conditions in Proposition 7.0.2 are satisfied.*

Remark 7.0.4. *(a) If $(G,\Phi) \cong (H,\Psi)$ then $E_{(G,\Phi)} = E_{(H,\Psi)}$. In fact, if $f\colon H \xrightarrow{\sim} G$ satisfies $\Psi \circ f^* = \Phi$ then $\mathrm{iso}_f(e^H_{(H,\Psi)}) = e^G_{(G,\Phi)}$, by Proposition 4.0.5(a).*

(b) The relation $\preccurlyeq$ is reflexive and transitive. Moreover, if $(H,\Psi) \preccurlyeq (G,\Phi)$

and $(G,\Phi) \preccurlyeq (H,\Psi)$ then $(G,\Phi) \cong (H,\Psi)$. It induces a partial order on the set of isomorphism classes $[G,\Phi]$ of pairs (G,Φ), where G is a finite group and $\Phi \in \mathrm{Hom}(G,\mathbb{K}^\times)$. We denote this relation again by $\preccurlyeq$. This partial order restricts to a partial order on the set $\mathcal{B}^A$ of isomorphism classes of B^A-pairs.

Proposition 7.0.5. *Let G and H be finite groups and let $\Phi \in \mathrm{Hom}(G^*,\mathbb{K}^\times)$ and $\Psi \in \mathrm{Hom}(H^*,\mathbb{K}^\times)$.*

(a) *If $(H,\Psi) \preccurlyeq (G,\Phi)$ then $E_{(G,\Phi)} \subseteq E_{(H,\Psi)}$.*

(b) *If (H,Ψ) is a B^A-pair and $E_{(G,\Phi)} \subseteq E_{(H,\Psi)}$ then $(H,\Psi) \preccurlyeq (G,\Phi)$.*

Proof. (a) Let $N \trianglelefteq G$ and let $f\colon H \xrightarrow{\sim} G/N$ be an isomorphism with $\Psi \circ f^* = \Phi_N$. Then, by Proposition 4.0.5(a) and Proposition 4.0.2(b), we have

$$e^G_{(G,\Phi)} = e^G_{(G,\Phi)} \cdot (\mathrm{inf}^G_{G/N} \underset{G/N}{\cdot} \mathrm{iso}_f \underset{H}{\cdot} e^H_{(H,\Psi)}) \in E_{(H,\Psi)}(G)\,,$$

so that $E_{(G,\Phi)} \subseteq E_{(H,\Psi)}$.

(b) Since $E_{(G,\Phi)} \subseteq E_{(H,\Psi)}$, we have $e^G_{(G,\Phi)} \in E_{(H,\Psi)}(G)$ and there exists $(U,\phi) \in \mathcal{M}(G\times H)$ such that

$$0 \neq e^G_{(G,\Phi)} \cdot \left(\left[\frac{G\times H}{U,\phi}\right] \underset{H}{\cdot} e^H_{(H,\Psi)} \right)\,. \tag{7.1}$$

Lemma 5.0.3 implies that $p_1(U) = G$, $p_2(U) = H$, ϕ extends to some $\alpha \times \beta \in (G\times H)^*$, $\Phi_{k_1(U)} = \Psi_{k_2(U)} \circ \eta_U^*$, and $m^{k_2(U)}_{(H,\Psi)} \neq 0$. Since (H,Ψ) is a B^A-pair, we obtain $k_2(U) = \{1\}$ and $\Phi_{k_1(U)} = \Psi \circ \eta_U^*$. Thus, η_U is an isomorphism $H \xrightarrow{\sim} G/k_1(U)$ with $\Phi_{k_1(U)} = \Psi \circ \eta_U^*$, so that $(H,\Psi) \preccurlyeq (G,\Phi)$. □

Chapter 8

Subfunctors of $B_{\mathbb{K}}^A$

We keep the assumptions on $\mathbb{K}$ from Section 9. In this section we prove one of our main results, Theorem 8.0.7, which describes the lattice of subfunctors of $B_{\mathbb{K}}^A$.

For any finite group G, the group $\mathrm{Aut}(G)$ acts on $\mathcal{X}(G)$ via ${}^f(K,\Psi) := (f(K), \Psi \circ (f|_K)^*)$. We will denote by $\hat{\mathcal{X}}(G) \subseteq \mathcal{X}(G)$ the set of those pairs (K,Ψ) with $K = G$. Note that $\hat{\mathcal{X}}(G)$ is $\mathrm{Aut}(G)$-invariant and that G acts trivially by conjugation on $\hat{\mathcal{X}}(G)$, so that $\hat{\mathcal{X}}(G)$ can be viewed as an $\mathrm{Out}(G)$-set.

Proposition 8.0.1. *Let F be a subfunctor of $B_{\mathbb{K}}^A$ in $\mathcal{F}_{\mathbb{K}}^A$.*

(a) *For each finite group G one has*

$$F(G) = \bigoplus_{(K,\Psi)\in[G\backslash\mathcal{X}_F(G)]} \mathbb{K}e_{(K,\Psi)}^G \,,$$

where $\mathcal{X}_F(G) := \{(K,\Psi) \in \mathcal{X}(G) \mid e_{(K,\Psi)}^G \in F(G)\}$.

(b) *For any finite group G, the set $\mathcal{X}_F(G)$ is invariant under the action of* $\mathrm{Aut}(G)$*.*

(c) *If H is a minimal group for F then $\mathcal{X}_F(H)$ contains only elements of the form (K, Ψ) with $K = H$. Each $(H, \Psi) \in \mathcal{X}_F(H)$ is a B^A-pair and one has $E_{(H,\Psi)} \subseteq F$.*

Proof. (a) For all $a \in F(G)$ and $x \in B_{\mathbb{K}}^A(G)$, (2.13) implies $x \cdot a = \Delta(x) \cdot_G a \in F(G)$. Thus, $F(G)$ is an ideal of $B_{\mathbb{K}}^A(G)$. Since the elements $e_{(K,\Theta)}^G$ with $(K, \Psi) \in [G \backslash \mathcal{X}(G)]$ form a $\mathbb{K}$-basis of $B_{\mathbb{K}}^A(G)$ consisting of pairwise orthogonal idempotents, the assertion in (a) follows.

(b) If $e_{(K,\Psi)}^G \in F(G)$ and $f \in \mathrm{Aut}(G)$ then $e_{(f(K),\Psi\circ(f|_K)^*)}^G = \mathrm{iso}_f(e_{(K,\Psi)}^G)$ is contained in $F(G)$.

(c) Assume that H is a minimal group for F and that $(K, \Psi) \in \mathcal{X}_F(H)$. Then $e_{(K,\Psi)}^H \in F(H)$ and

$$0 \neq e_{(K,\Psi)}^K = e_{(K,\Psi)}^K \mathrm{res}_K^H(e_{(K,\Psi)}^H) \in F(K)\,,$$

by Proposition 4.0.1(b). The minimality of H implies $K = H$. By Proposition 4.0.3(a), the minimality also implies that $m_{(H,\Psi)}^N = 0$ for all $\{1\} \neq N \trianglelefteq H$, since $e_{(H,\Psi)}^H \in F(H)$. Clearly, $E_{(H,\Psi)} \subseteq F$. □

Definition 8.0.2. *Let F be a subfunctor of $B_{\mathbb{K}}^A$ in $\mathcal{F}_{\mathbb{K}}^A$. If H is a minimal group for F and $\Psi \in \mathrm{Hom}(H, \mathbb{K}^\times)$ is such that $(H, \Psi) \in \mathcal{X}_F(H)$ then we call (H, Ψ) a* minimal pair *for F. By Proposition 8.0.1(c), each minimal pair for F is a B^A-pair.*

Proposition 8.0.3. *Let H be a finite group, $\Psi \in \mathrm{Hom}(H^*, \mathbb{K}^*)$, and let (G, Φ) be a minimal pair for $E_{(H,\Psi)}$. Then:*

(a) *$E_{(H,\Psi)} = E_{(G,\Phi)}$.*

(b) *There exists* $N \trianglelefteq H$ *with* $m^N_{(H,\Psi)} \neq 0$ *and* $(H/N, \Psi_N) \cong (G, \Phi)$. *In particular* $(G,\Phi) \preccurlyeq (H,\Psi)$. *Moreover, if also* $N' \trianglelefteq H$ *satisfies* $(H/N', \Psi_{N'}) \cong (G,\Phi)$ *then* $m^{N'}_{(H,\Psi)} = m^N_{(H,\Psi)} \neq 0$.

(c) *Up to isomorphism,* (G,Φ) *is the only minimal pair for* $E_{(H,\Psi)}$.

(d) *If* (H,Ψ) *is a* B^A*-pair, then, up to isomorphism,* (H,Ψ) *is the only minimal pair of* $E_{(H,\Psi)}$. *In particular,*

$$E_{(H,\Psi)}(H) = \bigoplus_{\substack{(H,\Psi') \in \hat{\mathcal{X}}(H) \\ (H,\Psi') =_{\mathrm{Out}(H)} (H,\Psi)}} \mathbb{K} e^H_{(H,\Psi')} \,.$$

Proof. (b) Since (G,Φ) is a minimal pair for $E_{(H,\Psi)}$, there exists $x \in B^A_{\mathbb{K}}(G,H)$ such that $e^G_{(G,\Phi)} = x \cdot_H e_{(H,\Psi)}$. Multiplication with $e^G_{(G,\Phi)}$ yields $e^G_{(G,\Phi)} = e^G_{(G,\Phi)} \cdot (x \cdot_H e^H_{(H,\Psi)})$. Thus, there exists $(U,\phi) \in \mathcal{M}(G \times H)$ with

$$e^G_{(G,\Phi)} \cdot \left(\left[\frac{G \times H}{U,\phi} \right] \underset{H}{\cdot} e^H_{(H,\Psi)} \right) \neq 0 \,.$$

Lemma 5.0.3 implies that $p_1(U) = G$, ϕ has an extension to $G \times H$, $\Phi_{k_1(U)} = \Psi_{k_2(U)} \circ \eta^*_U$, and $m^{k_2(U)}_{(H,\Psi)} \neq 0$. Since ϕ has an extension to $G \times H$, $\left[\frac{G \times H}{U,\phi}\right]$ factors through $q(U) \cong G/k_1(U)$ by Lemma 5.0.1. Since G is a minimal group of $E_{(H,\Psi)}$ this implies $k_1(U) = \{1\}$. Set $N := k_2(U) \trianglelefteq H$. Then $\eta_U \colon H/N \xrightarrow{\sim} G$ satisfies $\Psi_N \circ \eta^*_U = \Phi$. If also $N' \trianglelefteq G$ satisfies $(H/N', \Psi_{N'}) \cong (G,\Phi)$, then $(H/N, \Psi_{N'}) \cong (H/N, \Psi_N)$ and we obtain $m^{N'}_{(H,\Psi)} = m^N_{(H,\Psi)} \neq 0$ by Proposition 6.0.4.

(a) Note that $E_{(G,\Phi)} \subseteq E_{(H,\Psi)}$ by Proposition 8.0.1(c). The converse follows from Proposition 7.0.5(a) and Part (b).

(c) Assume that also (G', Φ') is a minimal pair for $E_{(H,\Phi)} = E_{(G,\Phi)}$. Then, by Part (b) applied to (G', Φ') and (G, Φ) in place of (G, Φ) and (H, Ψ), we have $(G', \Phi') \preccurlyeq (G, \Phi)$. Since both G and G' are minimal groups for $E_{(H,\Psi)}$, they have the same order. Thus, $(G', \Phi') \cong (G, \Phi)$.

(d) Now assume that (H, Ψ) is a B^A-pair and let (G, Φ) be a minimal pair for $E_{(H,\Psi)}$. Then, by Part (b), there exists $N \trianglelefteq G$ with $m^N_{(H,\Psi)} \neq 0$ and $(H/N, \Psi_N) \cong (G, \Phi)$. Since (H, Ψ) is a B^A-pair, this implies $N = \{1\}$ and $(H, \Psi) \cong (G, \Phi)$. □

Notation 8.0.4. *For any finite group G and any $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$ we denote by $\beta(G, \Phi)$ the class of all minimal pairs for $E_{(G,\Phi)}$. Thus $\beta(G, \Phi)$ is the isomorphism class $[H, \Psi]$ of a B^A-pair (H, Ψ), see Remark 7.0.4(b). Note that $\beta(G, \Phi) \preccurlyeq [G, \Phi]$ by Proposition 8.0.3(b).*

The following proposition is not used in this paper, but of interesting its own right. It is the analogue of [Bc10, Theorem 5.4.11].

Proposition 8.0.5. *Let G be a finite group and let $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$.*

(a) *If (H, Ψ) is a B^A-pair with $(H, \Psi) \preccurlyeq (G, \Phi)$ then $[H, \Psi] \preccurlyeq \beta(G, \Phi)$.*

(b) *For any $N \trianglelefteq G$ the following are equivalent:*

(i) $m^N_{(G,\Phi)} \neq 0$.

(ii) $\beta(G, \Phi) \preccurlyeq [G/N, \Phi_N]$.

(iii) $\beta(G, \Phi) = \beta(G/N, \Phi_N)$.

(c) *For any $N \trianglelefteq G$ the following are equivalent:*

(i) $[G/N, \Phi_N] \cong \beta(G, \Phi)$.

(ii) $(G/N, \Phi_N)$ *is a* B^A*-pair and* $m^N_{(G,\Phi)} \neq 0$.

Proof. Let $(K, \Theta) \in \beta(G, \Phi)$. Thus, $E_{(G,\Phi)} = E_{(K,\Theta)}$ by Proposition 8.0.3(a) and (K, Θ) is a B^A-pair.

(a) Let (H, Ψ) be as in the statement. Then $E_{(K,\Theta)} = E_{(G,\Phi)} \subseteq E_{(H,\Phi)}$ by Proposition 7.0.5(a). Now Proposition 7.0.5(b) implies $(H, \Psi) \preccurlyeq (K, \Theta)$.

(b) (i)⇒(ii): Since $m^N_{(G,\Phi)} \neq 0$, Proposition 4.0.3(a) implies that

$$e^{G/N}_{(G/N,\Phi_N)} = (m^N_{(G,\Phi)})^{-1} \mathrm{def}^G_{G/N}(e^G_{(G,\Phi)}) \in E_{(G,\Phi)}(G/N) = E_{(K,\Theta)}(G/N)$$

so that $E_{(G/N,\Phi_N)} \subseteq E_{(K,\Theta)}$. Proposition 7.0.5(b) implies $(K, \Psi) \preccurlyeq (G/N, \Phi_N)$.

(ii)⇒(iii): By Part (a) applied to (K, Θ) and $(G/N, \Phi_N)$ we obtain $\beta(G, \Phi) = [K, \Theta] \preccurlyeq \beta(G/N, \Phi_N)$. Conversely, we have $\beta(G/N, \Phi_N) \preccurlyeq [G/N, \Phi_N] \preccurlyeq [G, \Phi]$ and Part (a) again implies $\beta(G/N, \Phi_N) \preccurlyeq \beta(G, \Phi)$.

(iii)⇒(i): By Proposition 8.0.3(b) there exists $M \trianglelefteq G$ such that $m^M_{(G,\Phi)} \neq 0$ and $[G/M, \Phi_M] \cong \beta(G, \Phi)$. Similarly, there exists $N \leqslant M' \trianglelefteq G$ such that $m^{M'/N}_{(G/N,\Phi_N)} \neq 0$ and $[(G/N)/(M'/N), (\Phi_N)_{M'}] = \beta(G/N, \Phi_N)$. Since

$$[G/M', \Phi_{M'}] = [(G/N)/(M'/N), (\Phi_N)_{M'}] = \beta(G/N, \Phi_N) = \beta(G, \Phi) = [G/M, \Phi_M],$$

Proposition 6.0.4 implies that $m^{M'}_{(G,\Phi)} = m^M_{(G,\Phi)} \neq 0$. By Proposition 6.0.1 we have $m^{M'}_{(G,\Phi)} = m^N_{(G,\Phi)} \cdot m^{M'}_{(G/N,\Phi_N)}$ which implies that $m^N_{(G,\Phi)} \neq 0$.

(c) This follows immediately from the equivalence between (i) and (iii) in Part (b), noting that $\beta(G/N, \Phi_N) = (G/N, \Phi_N)$ if $(G/N, \Phi_N)$ is a B^A-pair and that

$\beta(G,\Phi)$ consists of B^A-pairs. $\square$

Definition 8.0.6. *A subset $\mathcal{Z}$ of the poset $\mathcal{B}^A$, ordered by the relation $\preccurlyeq$ (cf. Remark 7.0.4(b)) is called* closed *if for every $[H,\Psi] \in \mathcal{Z}$ and $[G,\Phi] \in \mathcal{B}^A$ with $[H,\Psi] \preccurlyeq [G,\Phi]$ one has $[G,\Phi] \in \mathcal{Z}$.*

Theorem 8.0.7. *Let $\mathcal{S}$ denote the set of subfunctors of $B_{\mathbb{K}}^A$ in $\mathcal{F}_{\mathbb{K}}^A$, ordered by inclusion of subfunctors, and let $\mathcal{T}$ denote the set of closed subsets of $\mathcal{B}^A$, ordered by inclusion of subsets. The map*

$$\alpha\colon \mathcal{S} \to \mathcal{T}\,, \quad F \mapsto \{[H,\Psi] \in \mathcal{B}^A \mid E_{(H,\Psi)} \subseteq F\}$$

is an isomorphism of posets with inverse given by

$$\beta\colon \mathcal{T} \to \mathcal{S}\,, \quad \mathcal{Z} \mapsto \sum_{[H,\Psi]\in\mathcal{Z}} E_{(H,\Psi)}\,.$$

Proof. Clearly, α and β are order-preserving. Let $F \in \mathcal{S}$. By Proposition 8.0.1(a) we have

$$F = \sum_{\substack{G \\ (H,\Psi)\in\mathcal{X}_F(G)}} \langle e^G_{(H,\Psi)} \rangle\,,$$

where G runs through a set of representatives of the isomorphism classes of finite groups and $\langle e^G_{(H,\Psi)} \rangle$ denotes the subfunctor of $B_{\mathbb{K}}^A$ generated by $e^G_{(H,\Psi)}$. For any finite group G and any $(H,\Psi) \in \mathcal{X}(G)$ one has $e^G_{(H,\Psi)} \in F(G)$ if and only if $e^H_{(H,\Psi)} \in F(H)$. In fact, $e^H_{(H,\Psi)} = e_{(H,\Psi)} \cdot \mathrm{res}^G_H(e^G_{(H,\Psi)})$ by Proposition 4.0.1 and $e^G_{(H,\Psi)} \in \mathbb{K} \cdot \mathrm{ind}^G_H(e^H_{(H,\Psi)})$ by Proposition 4.0.4. Thus

$$F = \sum_{\substack{H \\ (H,\Psi)\in\hat{\mathcal{X}}_F(H)}} E_{(H,\Psi)}\,,$$

where H runs again through a set of representatives of the isomorphism classes of finite groups and $\hat{\mathcal{X}}_F(H) = \hat{\mathcal{X}}(H) \cap \mathcal{X}_F(H)$. By Propositions 8.0.1(c) and 8.0.3(a), we obtain

$$F = \sum_{\substack{[H,\Psi]\in\mathcal{B}^A \\ (H,\Psi)\in\mathcal{X}_F(H)}} E_{(H,\Psi)} = \sum_{[H,\Psi]\in\alpha(F)} E_{(H,\Psi)} = \beta(\alpha(F))\,,$$

since $(H,\Psi) \in \mathcal{X}_F(H)$ if and only if $E_{(H,\Psi)} \subseteq F$.

Let $\mathcal{Z}$ be a closed subset of $\mathcal{B}^A$. By definition of α and β we have

$$\alpha(\beta(\mathcal{Z})) = \{[H,\Psi] \in \mathcal{B}^A \ \mid E_{(H,\Psi)} \subseteq \sum_{[G,\Phi]\in\mathcal{Z}} E_{(G,\Phi)}\}\,.$$

The inclusion $\mathcal{Z} \subseteq \alpha(\beta(\mathcal{Z}))$ is obvious. Conversely, assume that $[H,\Psi] \in \mathcal{B}^A$ satisfies $E_{(H,\Psi)} \subseteq \sum_{[G,\Phi]\in\mathcal{Z}} E_{(G,\Phi)}$. Evaluation at H and Proposition 8.0.1(a) imply that there exists $[G,\Phi] \in \mathcal{Z}$ with $e^H_{(H,\Psi)} \in E_{(G,\Phi)}(H)$, which implies $E_{(H,\Psi)} \subseteq E_{(G,\Phi)}$. Since (G,Φ) is a B^A-pair, Proposition 7.0.5(b) implies $[G,\Phi] \preccurlyeq [H,\Psi]$. Since $[G,\Phi] \in \mathcal{Z}$ and $\mathcal{Z}$ is closed we obtain $[H,\Psi] \in \mathcal{Z}$. Thus, $\alpha(\beta(\mathcal{Z})) \subseteq \mathcal{Z}$, and the proof is complete. □

Remark 8.0.8. *(a) If (G,Φ) is a B^A-pair, then the subfunctor $E_{(G,\Phi)}$ of $B^A_{\mathbb{K}}$ corresponds under the bijection in Theorem 8.0.7 to the subset $\mathcal{B}^A_{\succcurlyeq[G,\Phi]} := \{[H,\Psi] \in \mathcal{B}^A \mid [G,\Phi] \preccurlyeq [H,\Psi]\}$. Clearly, $\mathcal{B}^A_{\succ[G,\Phi]} := \{[H,\Psi] \in \mathcal{B}^A \mid [G,\Phi] \prec [H,\Psi]\}$ is the unique maximal closed subset of $\mathcal{B}^A_{\succcurlyeq[G,\Phi]}$.*

(b) For every element $[G,\Phi] \in \mathcal{B}^A$ there exist only finitely many elements $[H,\Psi] \in \mathcal{B}^A$ with $[H,\Psi] \preccurlyeq [G,\Phi]$. Therefore, every non-empty subset of $\mathcal{B}^A$ has a minimal element.

Chapter 9

Composition factors of $B_{\mathbb{K}}^A$

We keep the assumptions on $\mathbb{K}$ from Section 9. In this section we determine the composition factors of $B_{\mathbb{K}}^A$.

Recall from Section 2.6 that the simple A-fibered biset functors S over $\mathbb{K}$ are parametrized by isomorphism classes of quadruples (G, K, κ, V). Here G is a minimal group for S, $(K, \kappa) \in \mathcal{M}(G)$ is such that the idempotent $f_{(K,\kappa)} \in B_{\mathbb{K}}^A(G, G)$ does not annihilate $S(G)$, and $V := S(G)$ is an irreducible $\mathbb{K}\Gamma_{(G,K,\kappa)}$-module for the finite group $\Gamma_{(G,K,\kappa)}$.

Note that the idempotent $f_{(K,\kappa)}$ lies in the $\mathbb{K}$-span of standard basis elements $\left[\frac{G\times G}{U,\phi}\right]$ with $(U, \phi) \in \mathcal{M}(G, G)$ such that $k_2(U) \geqslant K$. Note also that in the case $(K, \kappa) = (\{1\}, 1)$, the group $\Gamma_{(G,\{1\},1)}$ is the set of standard basis elements $\left[\frac{G\times G}{U,\phi}\right]$ of $B_{\mathbb{K}}^A(G, G)$ with $p_1(U) = G = p_2(U)$ and $k_1(U) = \{1\} = k_2(U)$. The multiplication is given by $\cdot_G$. Thus, in this case, $\left[\frac{G\times G}{U,\phi}\right] = \mathrm{tw}_\alpha \cdot_G \mathrm{iso}_f$, where $f = \eta_U \in \mathrm{Aut}(G)$ and $\alpha \in G^*$ is given by $\alpha(g) = \phi(\eta_U(g), g)$ for $g \in G$. Mapping $\left[\frac{G\times G}{U,\phi}\right]$ to the element

$(\alpha, \bar{f}) \in G^* \rtimes \mathrm{Out}(G)$ defines an isomorphism. Here, $\mathrm{Out}(G)$ acts on G^* via ${}^{\bar{f}}\alpha := \alpha \circ f^*$. Moreover, $\Gamma_{(G,\{1\},1)}$ acts on $S(G)$ by $\cdot_G$.

Proposition 9.0.1. *Let (G, Φ) be a B^A-pair. The subfunctor $E_{(G,\Phi)}$ of $B^A_{\mathbb{K}}$ has a unique maximal subfunctor $J_{(G,\Phi)}$, given by*

$$J_{(G,\Phi)} = \sum_{[H,\Psi] \in \mathcal{B}^A_{\succ[G,\Psi]}} E_{(H,\Psi)} \,.$$

The simple functor $S_{(G,\Phi)} := E_{(G,\Phi)}/J_{(G,\Phi)}$ is isomorphic to $S_{(G,\{1\},1,V_\Phi)}$ where V_Φ is the irreducible $\mathbb{K}[G^* \rtimes \mathrm{Out}(G)]$-module

$$V_\Phi := \mathrm{Ind}_{G^* \rtimes \mathrm{Out}(G)_\Phi}^{G^* \rtimes \mathrm{Out}(G)} (\mathbb{K}_{\tilde{\Phi}}) \,,$$

with $\tilde{\Phi} \in \mathrm{Hom}(G^* \rtimes \mathrm{Out}(G)_\Phi, \mathbb{K}^\times)$ defined by $\tilde{\Phi}(\phi, \bar{f}) := \Phi(\phi)$ for $\phi \in G^*$ and $f \in \mathrm{Aut}(G)$.

Proof. By Remark 8.0.8(a) and Theorem 8.0.7, $J_{(G,\Phi)}$ is the unique maximal subfunctor of $E_{(G,\Phi)}$. Thus, the functor $S := S_{(G,\Phi)}$ is a simple object in $\mathcal{F}^A_{\mathbb{K}}$. Moreover, G is a minimal group for S, since G is a minimal group for $E_{(G,\Phi)}$ and $E_{(H,\Psi)}(G) = \{0\}$ for all $[H, \Psi] \in \mathcal{B}^A_{\succ[G,\Phi]}$.

Let $(U, \phi) \in \mathcal{M}(G \times G)$ with $k_2(U) \neq \{1\}$. Then $\left[\frac{G \times G}{U,\phi}\right]$ factors through the group $q(U)$ which has smaller order than G. Thus, $\left[\frac{G \times G}{U,\phi}\right] \cdot_G e^G_{(G,\Phi')} = 0$ for all $(G, \Phi') \in \hat{\mathcal{X}}(G)$ with $(G', \Phi') =_{\mathrm{Out}(G)} (G, \Phi)$. By Proposition 8.0.3(d) this yields $\left[\frac{G \times G}{U,\phi}\right] \cdot_G S(G) = \{0\}$. Thus, $f_{(K,\kappa)} \cdot_G S(G) = 0$ for all (K, κ) with $|K| > 1$.

This implies that S is parametrized by the quadruple $(G, \{1\}, 1, V)$, with $V = S(G)$ viewed as $\mathbb{K}\Gamma_{(G,\{1\},1)}$-module. Since $S(G)$ is the $\mathbb{K}$-span of the idempotents $e^G_{(G,\Phi')}$, with (G,Φ') running through the $\mathrm{Out}(G)$-orbit of (G,Φ), and since $\mathrm{tw}_{\alpha\cdot G}\mathrm{iso}_{f\cdot G}e^G_{(G,\Phi')} = \Phi(\alpha)\cdot e^G_{(G,\Phi'\circ f^*)}$ for all $\alpha\in G^*$ and $f\in\mathrm{Aut}(G)$ and (G,Φ'), the $\mathbb{K}\Gamma_{(G,\{1\},1)}$-module $S(G)$ is monomial. The stabilizer of the one-dimensional subspace $\mathbb{K}e^G_{(G,\Phi)}$ is equal to $G^*\rtimes\mathrm{Out}(G)_\Phi$ and this group acts on $\mathbb{K}e^G_{(G,\Phi)}$ via $\tilde{\Phi}$. Thus, $S(G)\cong V_\Phi$ as $\mathbb{K}\Gamma_{(G,\{1\},1)}$-module and the proof is complete. □

Theorem 9.0.2. *Let $F'\subset F\subseteq B^A_{\mathbb{K}}$ be subfunctors in $\mathcal{F}^A_{\mathbb{K}}$ such that F/F' is simple. Then there exists a unique $[G,\Phi]\in\mathcal{B}^A$ such that $E_{(G,\Phi)}\subseteq F$ and $E_{(G,\Phi)}\not\subseteq F'$. Moreover, $E_{(G,\Phi)}+F'=F$ and $E_{(G,\Phi)}\cap F'=J_{(G,\Phi)}$, and $F/F'\cong S_{(G,\Phi)}$.*

Proof. Since $\alpha(F')$ is a maximal subset of $\alpha(F)$ and both are closed, it follows from Theorem 8.0.7 and Remark 8.0.8 that $\alpha(F)\smallsetminus\alpha(F')=\{[G,\Phi]\}$ for a unique $[G,\Phi]\in\mathcal{B}^A$. For any $[H,\Psi]\in\mathcal{B}^A$ one has $E_{(H,\Psi)}\subseteq F$ and $E_{(H,\Psi)}\not\subseteq F'$ if and only if $[H,\Psi]\in\alpha(F)$ but $[H,\Psi]\notin\alpha(F')$. Thus, the first condition is equivalent to $[H,\Psi]=[G,\Phi]$. Further, we have $F'\subset F'+E_{(G,\Phi)}\subseteq F$ which implies $F'+E_{(G,\Phi)}=F$, since F/F' is simple. Thus, $0\neq E_{(G,\Phi)}/(E_{(G,\Phi)}\cap F')\cong(E_{(G,\Phi)}+F')/F'=F/F'$, and by Proposition 9.0.1 we obtain $E_{(G,\Phi)}\cap F'=J_{(G,\Phi)}$ so that $F/F'\cong E_{(G,\Phi)}/J_{(G,\Phi)}\cong S_{(G,\Phi)}$. □

Chapter 10

The case $A \leqslant \mathbb{K}^\times$

In this section we assume that A is a subgroup of the unit group of a field $\mathbb{K}$ of characteristic 0. Then the assumptions on A and $\mathbb{K}$ from the beginnings of Sections 3–9 are satisfied. This special case has been used for instance in the theory of canonical induction formulas, see [Bo98]. This assumption was also used in [Ba04] and [CY19]. By double duality it allows us to consider pairs $(G, gO(G))$ for a normal subgroup $O(G)$ of G instead of pairs (G, Φ) with $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$. This section makes this translation precise and also translates previously defined features for pairs (G, Φ) to features for pairs $(G, gO(G))$.

For any finite group G we have a homomorphism

$$\zeta_G \colon G \to \mathrm{Hom}(G^*, \mathbb{K}^\times)\,, \quad g \mapsto \varepsilon_g\,, \quad \text{with } \varepsilon_g(\phi) := \phi(g),$$

for $\phi \in G^*$. Note that ζ_G is functorial in G, i.e., if $f \colon G \to H$ is a group homomorphism then $\zeta_H \circ f = \mathrm{Hom}(f^*, \mathbb{K}^\times) \circ \zeta_G$. We set

$$O^A(G) := O(G) := \ker(\zeta_G) = \bigcap_{\phi \in G^*} \ker(\phi)\,,$$

which is a normal subgroup of G containing the commutator subgroup $[G, G]$ of G. Thus, we obtain an injective homomorphism $\bar{\zeta}_G \colon G/O(G) \to \mathrm{Hom}(G^*, \mathbb{K}^\times)$.

Proposition 10.0.1. *Let G be a finite group.*

(a) *The homomorphism ζ_G is surjective and $\bar{\zeta}_G \colon G/O(G) \xrightarrow{\sim} \mathrm{Hom}(G^*, \mathbb{K}^\times)$ is an isomorphism.*

(b) *The subgroup $O(G)$ is the smallest subgroup $[G, G] \leqslant M \leqslant G$ such that A has an element of order $\exp(G/M)$.*

(c) *For any normal subgroup N of G one has $O(G/N) = O(G)N/N$.*

Proof. (a) Applying the functoriality with respect to the natural epimorphism $f \colon G \to G/[G, G]$, and using that f^* is an isomorphism, it suffices to show the statement when G is abelian. Since $\mathrm{Hom}(-^*, \mathbb{K}^\times)$ preserves direct products of abelian groups, we are reduced to the case that G is cyclic. Using again the functoriality with respect to the natural epimorphism onto the largest quotient of G whose order occurs as an element order in A, we are reduced to the case that G is cyclic of order n and A has an element of order n. In this case it is easy to see that ζ_G is injective and that G and $\mathrm{Hom}(G^*, \mathbb{K}^\times)$ have the same order.

(b) First note that if M_1 and M_2 have the stated property, then also $M_1 \cap M_2$ has this property. In fact, $G/(M_1 \cap M_2)$ is isomorphic to a subgroup of $G/M_1 \times G/M_2$, whose exponent is equal to the order of an element in A. Here we use that if elements a and b in A have orders k and l respectively, then A has an element whose order is the least common multiple of k and l. Thus, there exists a smallest subgroup M with the stated property. Clearly, $\ker(\phi)$ has the property for every $\phi \in G^*$. Therefore, also $O(G)$ has the desired property. Conversely, if M has the property, then by writing G/M as a direct product of n cyclic groups whose orders are achieved as element order in A, it is easy to construct elements $\phi_1, \ldots, \phi_n \in G^*$ such that $\bigcap_{i=1}^n \ker(\phi_i) = M$, implying that $O(G) \leqslant M$.

(c) Since the exponent of $G/O(G)$ is equal to the order of an element of A also the exponent of $(G/N)/(O(G)N/N) \cong G/(O(G)N)$ is equal to the order of an element of A. Thus, $O(G/N) \leqslant O(G)N/N$. Conversely,

$$O(G) = \bigcap_{\phi \in G^*} \ker(\phi) \leqslant \bigcap_{\substack{\phi \in G^* \\ \phi|_N = 1}} \ker(\phi)\,,$$

and taking images in G/N yields the reverse inclusion. □

For any finite group G, Proposition 10.0.1(a) yields a bijection between the set of pairs of the form (G, Φ), with $\Phi \in \mathrm{Hom}(G^*, \mathbb{K}^\times)$ and the set of pairs $(G, gO(G))$ with $gO(G) \in G/O(G)$. More precisely, we identify $(G, gO(G))$ with (G, ε_g). The following proposition translates various relevant features of pairs (G, Φ) to features of the corresponding pairs $(G, gO(G))$. The proofs are straightforward and left to the reader.

Proposition 10.0.2. *Let G and H be finite groups.*

(a) *Let* $g \in G$ *and* $h \in H$*. Then* $(G, gO(G)) \cong (H, hO(H))$ *if and only if there exists an isomorphism* $f\colon G \to H$ *such that* $f(g)O(H) = hO(H)$*.*

(b) *Let* N *be a normal subgroup of* G*, let* $g \in G$*, and set* $\Phi := \varepsilon_g$*. Then* $\Phi_N = \varepsilon_{gN}$*.*

(c) *Let* $g \in G$ *and* $h \in H$*. Then* $(H, \varepsilon_h) \preccurlyeq (G, \varepsilon_g)$ *if and only if there exists a normal subgroup* N *of* G *and an isomorphism* $f\colon H \xrightarrow{\sim} G/N$ *with* $f(h) \in gO(G)N$*.*

(d) *Let* $K \leqslant G$ *and* $g \in G$*. Then* $\varepsilon_g|_{K^\perp} = 1$ *if and only if* $g \in KO(G)$*.*

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