Title
COLLECTIVE OSCILLATIONS OF A SYSTEM OF INTERACTING FERMIONS

Author
Glassgold, A.E.

Publication Date
1959-09-02

UCRL-8889

UNIVERSITY OF CALIFORNIA

Ernest O. Lawrence

Radiation Laboratory

BERKELEY, CALIFORNIA

DISCLAIMER

UCRL-8889

UNIVERSITY OF CALIFORNIA
Lawrence Radiation Laboratory
Berkeley, California

Contract No. W-7405-eng-48

COLLECTIVE OSCILLATIONS OF A SYSTEM OF INTERACTING FERMIONS

A. E. Glassgold

September 2, 1959

Printed for the U. S. Atomic Energy Commission

COLLECTIVE OSCILLATIONS OF A SYSTEM OF INTERACTING FERMIONS*

A. E. Glassgold

Lawrence Radiation Laboratory
University of California
Berkeley, California

September 2, 1959

ABSTRACT

Collective oscillations for an infinite system of fermions are discussed for the general case where the force between two particles depends on the internal variables as well as on their relative separation and momentum. Two theories are reviewed, one by Watson, Heckrotte, and Glassgold which is based on Sawada's treatment of the electron gas, and the other, a semiclassical treatment by Landau.

COLLECTIVE OSCILLATIONS OF A SYSTEM OF INTERACTING FERMIONS*

A. E. Glassgold

Lawrence Radiation Laboratory
University of California
Berkeley, California

September 2, 1959

I. INTRODUCTION

The quantum mechanical problem of the degenerate electron gas has received a great deal of attention in recent years. Significant progress has been made in calculating the ground state energy and properties of the excited states, particularly the collective aspects of this many-body system. There are, of course, other infinite systems of particles obeying Fermi-Dirac statistics which are of interest. As examples, we may mention the low-temperature behavior of liquid He^3 or the popular abstraction called nuclear matter. In connection with the collective motions, the novel aspect of such systems is that the internal variables, such as spin and isotopic spin, are now important. This occurs because the force between two helium atoms or between two nucleons depends strongly on the internal variables. On the other hand, the only force between two nonrelativistic electrons is the familiar spin-independent Coulomb repulsion. We are thus led to expect new kinds of collective oscillations for general fermion systems besides the familiar plasma oscillations of the electron gas.

---

*Paper read at the 1959 International Plasma Physics Institute held at the University of Washington, Seattle, August 31 to September 5, 1959. This work was performed under the auspices of the U. S. Atomic Energy Commission.

Oscillations of this type were first considered by Klimontovich and Silin[1] and by Ferentz, Gell-Mann, and Pines[2] in 1952. Starting in 1956, Landau and his school developed the theory of the Fermi liquid, with He^3 in mind as the important application.[3] As will be discussed below, this theory has marked classical and phenomenological aspects. A review of this work by Abrikosov and Khalatnikov has recently appeared in translation.[4] Independently, Watson, Heckrotte, and the author[5] developed a theory for the collective oscillations of fermion systems and applied it to the problem of nuclear matter. This treatment is a generalization of the theory of the electron gas as formulated by Sawada.[6] As such, it is more of a microscopic treatment than is Landau's approach. In the rest of this report, the essential steps and results of this procedure will be reviewed. Then Landau's theory will also be discussed briefly. A detailed discussion of the application of these theories to He^3 and nuclear matter will not be presented here. Such discussion may be found in the literature.[4, 5]

II. THE COMPLETE HAMILTONIAN

The introduction of more general kinds of forces leads to interesting new phenomena, but naturally at the expense of more complicated calculations. The difficulties arise from the simple increase in the number of degrees of freedom and from the complicated nature of the force law. To be more specific, the interaction between two particles in He^3 or nuclear matter, for example, involves a repulsive core for small separations with a relatively weak attractive region outside. In other words, the interaction is strongly momentum-dependent and also depends on spin (and isotopic spin).

To handle this difficult situation, we adopt the following popular point of view. We assume that the states of the system bear some resemblance to the ideal Fermi gas in spite of the strong interactions. To

first approximation, we now have a system of noninteracting quasiparticles whose effective mass is different than the particle mass. The interaction between the quasiparticles is considerably weaker and better behaved than the actual force law between the particles. With reference to nuclear matter we are, of course, thinking of Brueckner's theory.[7]

We will let $P = (\underset{\sim}{p}, \lambda)$ label the state of a quasiparticle, where $\underset{\sim}{p}$ is the momentum and λ is the set of internal variables such as spin, isotopic spin, etc. In the language of second quantization, a_P^+ and a_P are the operators which create and destroy a quasiparticle in state P. These operators obey the anticommutation rules for fermions

$$\left[a_P, a_{P'}^+\right]_+ = \delta_{P,P'} \qquad \left[a_P, a_{P'}\right]_+ = 0$$

Here $P = P'$ requires every quantum number of the two states to be the same. In terms of these operators, it is assumed that the Hamiltonian can be written as

$$H = E_B + \sum_P \epsilon_P a_P^+ a_P - \sum_{P<P_0} \epsilon_P$$

$$+ \frac{1}{2\Omega} \sum_{P_1 P_2 P_1' P_2'} (P_1' P_2' \left| K \right| P_1 P_2)\, a_{P_1'}^+ a_{P_2'}^+ a_{P_2} a_{P_1}, \qquad (1)$$

where Ω is simply the volume of the quantization box, and $P < P_0$ means that the sum is restricted to $p = \left|\underset{\sim}{p}\right| < p_0$. Here p_0 is the Fermi momentum

$$\frac{4\pi}{3} p_0^3 \Omega = \frac{N}{S},$$

where N is the total number of particles, S is the number of internal variables (S = 2 for He^3, S = 4 for nuclear matter, etc.), and Planck's constant has been set equal to unity.

The interaction between the quasiparticles is given by the operator K, and its matrix elements depend on the states of the two particles before (P_1, P_2) and after (P_1', P_2') "scattering". The form of the matrix elements is restricted by various conservation laws, e. g., momentum conservation requires $\underset{\sim}{p}_1 + \underset{\sim}{p}_2 = \underset{\sim}{p}_1' + \underset{\sim}{p}_2'$. (We shall not discuss here the way this operator is determined in the Brueckner theory, and we may even regard it phenomenologically.) The constant E_B is the average value of H for the ground state ϕ_0 of the system of noninteracting quasiparticles, i. e. the state in which $a_P^+ a_P = 1$ for $P < P_0$ and 0 for $P > P_0$. Thus we have

$$E_B = \sum_{P<P_0} \epsilon_P + \frac{1}{2\Omega} \sum_{P_1P_2<P_0} \left[(P_1P_2 \left| K \right| P_1P_2) - (P_1P_2 \left| K \right| P_2P_1) \right] \tag{2}$$

The prime on the second summation in Eq. (1) means that the terms diagonal for ϕ_0 have been excluded. Finally the energy of the quasiparticles is

$$\epsilon_{P_1} = \epsilon_{p_1} = p_1^2/2M + \frac{1}{\Omega} \sum_{P_2<P_0} \left[P_1P_2 \left| K \right| P_1P_2) - (P_1P_2 \left| K \right| P_2P_1) \right].$$

It will be convenient to write ϵ_{p_1} near p_0 approximately as

$$\epsilon_{p_1} \approx p_1^2/2M^*,$$

where M* is the effective mass at the Fermi surface.

III. SAWADA'S APPROXIMATION

The procedure adopted for solving the above Hamiltonian is to break the interaction into two parts. One part can easily be diagonalized and has collective eigenstates. The other part presumably will cause only small effects when treated as a perturbation.

In order to define this separation we introduce particle and hole operators:

$$a_P = \begin{cases} a_P & P > P_0 \\ \beta_P^+ & P < P_0 \end{cases} \qquad a_P^+ = \begin{cases} a_P^+ & P > P_0 \\ \beta_P & P < P_0 \end{cases} .$$

The momentum associated with a particle operator is always greater than p_0, the momentum associated with a hole is always less than p_0. Next we combine these into pair-creation and annihilation operators:

$$\gamma^+_{P'P} = a^+_{P'} \beta^+_P \qquad \gamma_{P'P} = \beta_P a_{P'} .$$

The first momentum associated with a pair operator is always greater than p_0, the second is always less than p_0. (These conditions, associated with the definition of particle and holes, will not usually be written explicitly in what follows.)

If we now examine the potential energy of interactions of the quasiparticles,

$$\frac{1}{2\Omega} \sum_{P_1 P_2 P_1' P_2'} (P_1' P_2' \left| K \right| P_1 P_2)\, a_{P_1'}^+ a_{P_2'}^+ a_{P_2} a_{P_1}$$

we can consider 16 classes of terms. These 16 classes arise from the

two possibilities available for each of the four momenta, p'_1, $p'_2 p_2$, and p_1, i.e. each momentum is either less than or greater than p_0. Of these 16 classes there are 6 that refer only to pairs as just defined, i.e. terms in which two momenta are less than p_0 and two are greater than p_0. If V is the part of the interaction which is to be treated seriously and V' the part to be considered as a perturbation, then the 10 classes of terms which are <u>not</u> pair terms are certainly included in V'. Of the pair terms we include the four which can be written as

$$V = \frac{1}{2\Omega} \sum_{P_1 P_2 P_1' P_2'} (P_1' P_2' \left| K \right| P_1 P_2) \times (\gamma_{P_1' P_1}^{+} + \gamma_{P_1 P_1'}) (\gamma_{P_2' P_2}^{+} + \gamma_{P_2 P_2'}) \tag{3}$$

Each term includes operators which (a) create two pairs ($\gamma^{+}_{P_1' P_1}\ \gamma^{+}_{P_2' P_2}$) (b) destroy two pairs ($\gamma_{P_1 P_1'}\ \gamma_{P_2 P_2'}$) and (c) transform one pair into another ($\gamma_{P_1 P_1'}\ \gamma^{+}_{P_2' P_2} + \gamma^{+}_{P_1' P_1}\ \gamma_{P_2 P_2'}$). The exchange terms for this last type (in which P_2 and P_2' are interchanged) have not been included and appear instead in V'. (A certain amount of reordering has also been done in obtaining the symmetrical form of Eq. (3), and certain trivial additions have been made to V' to compensate.) Finally, the kinetic energy is

$$K = \sum_P \epsilon_P (\alpha_P^{+} \alpha_P - \beta_P^{+} \beta_P) \tag{4}$$

The complete Hamiltonian is

$$H = E_B + K + V + V', \tag{5}$$

where H, E_B, K, and V are given by Eqs. (1) to (4), and

$V' = H - (E_B + K + V)$. We now solve the truncated Hamiltonian

$$H_0 = E_B + K + V \tag{6}$$

by making the further assumption that the pair operators $\gamma^+_{P'P}$ and $\gamma_{P'P}$ satisfy boson commutation relations:

$$\left[\gamma_{P_1'P_1}, \gamma^+_{P_2'P_2}\right]_- = \delta_{P_1P_2}\delta_{P_1'P_2'} \left[\gamma_{P_1'P_1}, \gamma_{P_2'P_2}\right]_- = 0. \tag{7}$$

With this assumption it can be shown that

$$\left[\gamma^+_{P_1'P_1}, K\right]_- = -(\epsilon_{P_1'} - \epsilon_{P_1})\gamma^+_{P_1'P_1}$$

and

$$\left[\gamma^+_{P_1'P_1}, V\right]_- = -\frac{1}{\Omega}\sum_{P_2P_2'} (P_1P_2'|K|P_1'P_2)(\gamma^+_{P_2'P_2} + \gamma_{P_2P_2'}).$$

From this result it follows that the excitation energies (and the operators creating the excited states from the ground state) can be determined exactly with standard matrix procedures.

If ψ_0 is the ground state of H_0, then it can be shown that the operator B_q^+, which creates the excited state $B_q^+\psi_0$ with momentum q and excitation energy $\Delta E_q = E_q - E_0$, has the following properties:

$$\left[H_0, B_q^+\right]_- = \Delta E_q B_q^+$$

$$\left[B_{q_1} B_{q_2}^+\right]_- = \delta_{q_1q_2} \qquad \left[B_{q_1}, B_{q_2}\right]_- = 0$$

$$B_q^+ = \sum_{\overline{P}\overline{P}'} \delta_{\underset{\sim}{p}' = \underset{\sim}{p} + \underset{\sim}{q}} \left[G^{(+)}_{\overline{P}'\overline{P}} \gamma^+_{P'P} - G^{(-)}_{\overline{P}'\overline{P}} \gamma_{PP'}\right].$$

The new notation here is that $\overline{P} = (\underset{\sim}{p}, -\lambda)$. The expansion coefficients satisfy the eigenvalue equation (for $\underset{\sim}{p}_2' = \underset{\sim}{p}_2 + \underset{\sim}{q}$):

$$\left[\Delta E_q - (\epsilon_{P_2'} - \epsilon_{P_2})\right] G^{(\pm)}_{P_2'P_2} = \pm \frac{1}{\Omega} \sum_{P_1P_1'} (P_1'P_2 | K | P_1P_2')(G^{(+)}_{P_1'P_1} + G^{(-)}_{P_1'P_1}).$$

Here $G^{(+)}_{P'P}$ is defined only for $p < p_0$ and $p' > |\underset{\sim}{p} + \underset{\sim}{q}|$, whereas $G^{(-)}_{P'P}$ is defined only for $p > p_0$ and $p' < |\underset{\sim}{p} + \underset{\sim}{q}|$.

The properties of the excited states have been investigated in detail by Sawada and Brout and others. There are, first of all, simple pair excitations with excitation energy:

$$L_{pq} = \epsilon_{p+q} - \epsilon_p \simeq \left[(\underset{\sim}{p} + \underset{\sim}{q})^2 - p^2\right]/2M^* .$$

In addition there are the genuine collective excitations whose excitation energy will be represented by ΔE_q. Some properties of these states will be discussed in detail in the next section.

Before proceeding, we note that if a wave packet is formed from the collective states, the state of the system has the form

$$\psi = e^{-iE_0t}\psi_0 + \sum_q c_q e^{-i(E_0+\Delta E_q)t} B_q^+ \psi_0 .$$

The density of particles at $(\underset{\sim}{x}, t)$ is given by the expectation value of the density operator,

$$n_\lambda = \frac{1}{\Omega} \sum_{pp'} a^+_{p'\lambda} a_{p\lambda} e^{i(\underset{\sim}{p}-\underset{\sim}{p'})\cdot\underset{\sim}{x}} .$$

For small deviations from the average density, this is

$$\rho_\lambda(x,t) = (\psi, n_\lambda \psi) = \rho_0 + \rho_\lambda'(\underset{\sim}{x}, t) ,$$

where we have

$$\rho_\lambda'(\underset{\sim}{x}, t) = \sum_q c_q e^{i(\underset{\sim}{q}\cdot\underset{\sim}{x} - \Delta E_q t)}$$

$$\times 2\mathrm{Re}\left[\frac{1}{\Omega} \sum_p \left\{G^{(+)}_{p+q,-q;-\lambda-\lambda} + G^{(-)}_{p+q,-q;-\lambda-\lambda}\right\}\right] \qquad (8)$$

This shows that, for small-amplitude oscillations, the $G^{(\pm)}$ functions are linearly related to the density fluctuations.

IV. PROPERTIES OF THE COLLECTIVE MODES

In order to study the properties of these solutions it is useful to consider simple forms for the interaction K between quasiparticles. As a simple example we consider

$$(P_1' P_2' | K | P_1 P_2) = \delta_{\lambda_1 \lambda_1'} \delta_{\lambda_2 \lambda_2'} \delta_{\underset{\sim}{p}_1' + \underset{\sim}{p}_2, \underset{\sim}{p}_1 + \underset{\sim}{p}_2'} f_{\lambda_1 \lambda_2} v((\underset{\sim}{p}_1 - \underset{\sim}{p}_1')^2).$$

Because v depends only on the momentum transfer, this is an ordinary local interaction. Note that it is not an exchange interaction, but simply a force where the strength depends on the spin state of the particles. We adopt the following notation for the index λ for the case of nuclear matter: $\lambda = 1, 2$ corresponds to protons with spin up and spin down, respectively; $\lambda = 3, 4$ corresponds to neutrons with spin up and down, respectively. The $f_{\lambda_1 \lambda_2}$ then form a matrix

$$f = \begin{pmatrix} c & b & a & b \\ b & c & b & a \\ a & b & c & b \\ b & a & b & c \end{pmatrix} ,$$

where a describes the interaction of particles with unlike spin; and b and c describe the interaction of particles with the same spin but unlike and like isotopic spin, respectively.

This model of nuclear matter might be called a "four-fluid" model. One might expect, a priori, that in the normal modes of oscillation of the fluids, the density ratio would be

$$\rho_1' : \rho_2' : \rho_3' : \rho_4' = \begin{cases} 1 : 1 : (-1) : (-1) \\ 1 : (-1) : (-1) : 1 \\ 1 : (-1) : 1 : (-1) \\ 1 : 1 : 1 : 1 \end{cases} . \tag{9}$$

In the last case, the four fluids oscillate in phase, whereas in the first case the two proton fluids move out of phase with the two neutron fluids. In case III the two spin-up fluids move out of phase with the two spin-down fluids. In case II, the two fluids with opposite spin and opposite isotopic spin are out of phase. We may then give these modes the appropriate names:

I	Isotopic-spin wave
II	Spin--isotopic-spin wave
III	Spin wave
IV	Compression wave.

(10)

Mode I is reminiscent of the Goldhaber-Teller explanation of the giant dipole resonance in the nuclear photoeffect.

The actual solution of the problem gives just these normal modes. In the long-wavelength limit, i.e. $q \ll p_0$, the excitation energies are determined by the following equation for $A_q > 1$:

$$1 - \frac{A_q}{2} \log \frac{A_q + 1}{A_q - 1} = Q^{-1}(q) . \tag{11}$$

Here we have $A_q = \Delta E_q / \frac{p_0 q}{M^*}$, i.e. A_q is the excitation energy divided by the maximum pair excitation energy, and $Q(q)$ is an eigenvalue with the following values for the four normal modes:

$$Q(q) = \left[-\frac{3}{8}\frac{N}{\Omega}\frac{V(q)}{\epsilon_0} \right] \begin{cases} c - a & \text{I} \\ c - a & \text{II} \\ c + a - 2b & \text{III} \\ c + a + 2b & \text{IV} \end{cases} . \qquad (12)$$

Figure 1 shows Q^{-1} as a function of A in the range $1 < A < \infty$. We see that Q^{-1} increases monotonically from $-\infty$ to 0 in this range. Thus if Eq. (12) yields a negative value for Q^{-1}, A is greater than 1, and the mode will be a stable oscillation with an excitation energy determined by Eq. (11). On the other hand, if Q^{-1} is positive, the corresponding mode will be unstable since A will be complex.

Although carried out for a very special case, this analysis has a number of interesting features. First we note that the finite range of the forces under consideration implies that V(0) is finite. Thus for small q, the eigenvalues Q^{-1}, and therefore A, are independent of q. Therefore we have

$$\Delta E_q = (A\ v_0)q , \qquad (13)$$

where $v_0 = p_0/M^*$ is the Fermi velocity. Thus the collective excitations have a phonon energy spectrum, i.e., the excitation energy is proportional to the momentum. Of course we mentioned before that the operators which create and destroy these excitations have boson commutation rules.

This result differs from that for the plasma oscillations of the electron gas and depends on the short range of the forces. On the other hand, the dispersion relation is similar to the one appropriate to the electron gas. The function Q is proportional to the Fourier transform of the potential; the increase in the number of degrees of freedom manifests itself through a different proportionality constant for each mode.

According to Eq. (13), A is simply the velocity of the waves in units of the Fermi velocity. In the limit of very weak forces, the excitation energies and velocities of all the modes become the same and equal to the Fermi velocity:

$$A = v_0 \qquad |Q| << 1.$$

For "strong" forces we have

$$A^2 = \left(\frac{\omega_0}{2\mu}\right)^2 \begin{cases} c - a & \text{I} \\ c - a & \text{II} \\ c + a - 2b & \text{III} \\ c + a + 2b & \text{IV} \end{cases} \qquad |Q| >> 1.$$

In this discussion we have considered V to be a Yukawa potential of range μ^{-1} and strength g^2 and therefore we can write

$$V(q^2) = \frac{4\pi g^2}{\mu^2 + q^2} .$$

The frequency ω_0 is a characteristic frequency:

$$\omega_0^2 = \left(4\pi \frac{N}{\Omega} g^2 \frac{1}{M}\right) .$$

We have also investigated the properties of the collective oscillations for the following nonlocal exchange interaction which is more general than the above simple example:[5]

$$(P_1' P_2' \left| K \right| P_1 P_2) = v\left((\underset{\sim}{p}_1 - \underset{\sim}{p}_1')^2\right) w\left((\underset{\sim}{p}_1 - \underset{\sim}{p}_2')^2\right)$$
$$\times (\lambda_1' \lambda_2' \left| 1 + C_1 \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{\sigma}_2 + C_2 \underset{\sim}{\tau}_1 \cdot \underset{\sim}{\tau}_2 + C_3 \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{\sigma}_2 \underset{\sim}{\tau}_1 \cdot \underset{\sim}{\tau}_2 \right| \lambda_1 \lambda_2) .$$

The inclusion of the exchange mixture gives rise to 16 rather than four modes of oscillation, although there are only four distinct eigenfrequencies. If $G_{pq}^{(\pm)}$ represents the 4 by 4 matrix formed by the elements $G_{p'p,\,\lambda'\lambda}^{(\pm)}$ with $\underset{\sim}{p}' = \underset{\sim}{p} + \underset{\sim}{q}$, then the oscillating amplitudes are sums of quantities like:

$$
\begin{array}{lll}
T\left[\tau_i \; G_{pq}^{(\pm)}\right] & d = 3 & I \\
T\left[\sigma_i \tau_i \; G_{pq}^{(\pm)}\right] & d = 9 & II \\
T\left[\sigma_i \; G_{pq}^{(\pm)}\right] & d = 3 & III \\
T\left[G_{pq}^{(\pm)}\right] & d = 1 & IV .
\end{array} \qquad (14)
$$

Here $T\left[\;\right]$ means the diagonal sum in the λ space, and d is the degeneracy. We have kept the same labels for the modes as in the previous example because the density ratios of Eq. (9) and the descriptions of Eq. (10) are also suitable here.

A more detailed discussion of these waves is given in reference 5. We should note that Eq. (13) is preserved and follows from the short-range nature of the forces. The form of the dispersion relation is modified, however, by the nonlocal character of the force. In addition to the function on the left side of Eq. (11), some of its derivatives are involved.

Even though A_q is greater than 1, there is still damping from the terms V' in the original Hamiltonian left out of the discussion so far. Perturbation calculations of their effect on the lifetime of these states indicate that the condition A_q is greater than 1 is actually sufficient for stability.

The compressional mode IV in this simple model of nuclear matter is characterized by $A < 1$ and, furthermore, is unstable in a growing sense. This is associated with the dominance of the attractive forces in this state. Brueckner has suggested that, in this case, the pairing procedure used in the theory of superconductivity may remove this instability.[8]

V. LANDAU'S THEORY OF THE FERMI LIQUID

One of the basic assumptions in Landau's theory[3] is that the classification of energy levels for a system of interacting fermions, i.e. a "Fermi liquid," is the same as for the ideal Fermi gas. Thus one introduces a distribution $n(\underset{\sim}{p})$ for the occupation of these levels which is a matrix in spin space. With $d\tau = (2\pi)^{-3}\Omega\ d^3p$, the total number of particles N and the total momentum $\underset{\sim}{P}$ are

$$N = T\int d\tau\ n(\underset{\sim}{P}) \tag{15a}$$

and

$$\underset{\sim}{P} = T\int d\tau\ n(\underset{\sim}{p})\underset{\sim}{p} \tag{15b}$$

The symbol T indicates, of course, a trace in spin space. Landau's next essential step is to introduce first and second functional derivatives of the total energy E with respect to the distribution function:

$$\delta E = T\int d\tau\ \delta n\ (\underset{\sim}{p})\ \epsilon(\underset{\sim}{p}) \tag{16a}$$

$$\delta\epsilon(\underset{\sim}{p}) = T'\int d\tau'\ f(\underset{\sim}{p},\underset{\sim}{p}')\ \delta n\ (\underset{\sim}{p}')\ . \tag{16b}$$

In the Fermi gas, the particles are in states of definite momentum p and energy $\epsilon^{(0)} = p^2/2M$; the equilibrium distribution at temperature $kT=\beta^{-1}$ is

$$n_0 = \left[e^{\beta(\epsilon_0 - p^2/2M)} + 1\right]^{-1}\ . \tag{17}$$

In the Fermi liquid, $n(\underset{\sim}{p})$ is the distribution of quasiparticles which obey Fermi-Dirac statistics. The energy $\epsilon(\underset{\sim}{p})$ of a quasiparticle with momentum $\underset{\sim}{p}$ is given by the change in the total energy E when a single quasiparticle is added in that state. The interaction between quasiparticles is specified by the function $f(\underset{\sim}{p},\underset{\sim}{p}')$, and is related to the change in the energy of one quasiparticle when another is added to the system. Thus a quasiparticle's energy $\epsilon(\underset{\sim}{p})$ depends on the distribution of the other quasiparticles.

By varying the distribution such that we have $\delta N = 0$, $\delta E = 0$, and $\delta S = 0$, where the entropy is

$$S = - k T \int d\tau \left[n \log n + (1-n) \log (1-n) \right] ,$$

the equilibrium distribution of quasiparticles is found to be

$$n = \left[e^{\beta(\epsilon - \mu)} + 1 \right]^{-1} .$$

The chemical potential μ is determined from the condition $N = T \int d\tau n$. This is formally the same as the Fermi gas except that ϵ and μ depend implicitly on the distribution function n.

The Fermi energy ϵ_0 is defined as the chemical potential at absolute zero. One also defines the velocity of the quasiparticles as $\underset{\sim}{v} = \partial \epsilon / \partial \underset{\sim}{p}$ and the effective mass at the Fermi surface as $M^* = p_0 / v_0$.

With these few assumptions it is possible to study the properties of the system in great detail.[4] To a large extent the quasiparticle interaction is regarded phenomenologically, subject, of course, to general invariance requirements. A correlation of this model with the observed properties of He^3 is given by Abrikosov and Khalatnikov.[4]

One of the first results deduced by Landau was the speed of ordinary sound at absolute zero, which was calculated from the compressibility,

$$c^2 = \frac{1}{3} v_0^2 + \frac{1}{6M} (p_0/2\pi)^3 \frac{1}{4\pi} \int d\Omega \, (1-\cos\theta) \, TT' \left[f/\Omega \right] . \qquad (18)$$

The first term is appropriate to the ideal Fermi gas; the second gives corrections from the mutual interactions between particles. Here f is the interaction between two quasiparticles on the Fermi surface and depends on the angle θ between their momenta and on their spins.[9]

With respect to other collective oscillations, Landau introduced the transport equation

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \underset{\sim}{r}} \frac{\partial \epsilon}{\partial \underset{\sim}{p}} - \frac{\partial m}{\partial \underset{\sim}{p}} \frac{\partial \epsilon}{\partial \underset{\sim}{r}} = I(n) .$$

We consider n and ϵ to deviate slightly from the quasiparticles' ground-state distribution:

$$n = n_0(\underset{\sim}{p}) + \delta n(\underset{\sim}{r}, \underset{\sim}{p}, t)$$
$$\epsilon = \epsilon^{(0)}(\underset{\sim}{p}) + \delta\epsilon(\underset{\sim}{r}, \underset{\sim}{p}, t) .$$

Neglecting the quasiparticle collision term I(n), we obtain the following linear equation:

$$\frac{\partial}{\partial t}(\delta n) + \frac{\partial}{\partial \underset{\sim}{r}}(\delta n)\frac{\partial \epsilon^{(0)}}{\partial \underset{\sim}{p}} - \frac{\partial}{\partial \underset{\sim}{r}}(\delta\epsilon)\frac{\partial n_0}{\partial \underset{\sim}{p}} = 0.$$

Assuming solutions of the form $\exp\left[i(\underset{\sim}{q}\cdot\underset{\sim}{r} - \omega t)\right]$, we find

$$(\underset{\sim}{q}\cdot\underset{\sim}{v} - \omega)\,\delta n = \underset{\sim}{q}\cdot\underset{\sim}{v}\left(\frac{\partial n_0}{\partial \epsilon}\right)\delta\epsilon , \tag{19}$$

where $\underset{\sim}{v} = \partial\epsilon/\partial\underset{\sim}{p}$ is the quasiparticle velocity. Because of the derivative $\partial n_0/\partial\epsilon$ on the right side, the deviations δn and $\delta\epsilon$ are restricted to the Fermi surface. By making some appropriate definitions this equation can be rewritten in a familiar form. Let $u = \omega/q$ be the wave velocity, $\eta = u/v$ the ratio of wave to quasiparticle velocity, χ the angle between two quasiparticle momenta $\underset{\sim}{p}$ and $\underset{\sim}{p}'$, and (θ, ϕ) the polar coordinates of $\underset{\sim}{p}$ with respect to $\underset{\sim}{q}$. Then we find the dispersion relation

$$\nu(\theta,\phi) = \frac{\cos\theta}{\eta - \cos\theta}\,\frac{1}{4\pi}\int d\Omega'\ \nu(\theta',\phi')F(\chi) . \tag{20}$$

The new functions are

$$\nu = \int d\epsilon\ \delta n(\underset{\sim}{p})$$

and

$$F = 4\pi(2\pi)^{-3}\left[p^2\,\frac{dp}{d\epsilon}\ T'\ f(\underset{\sim}{p},\underset{\sim}{p}')\right]_{p'=p=p_0} .$$

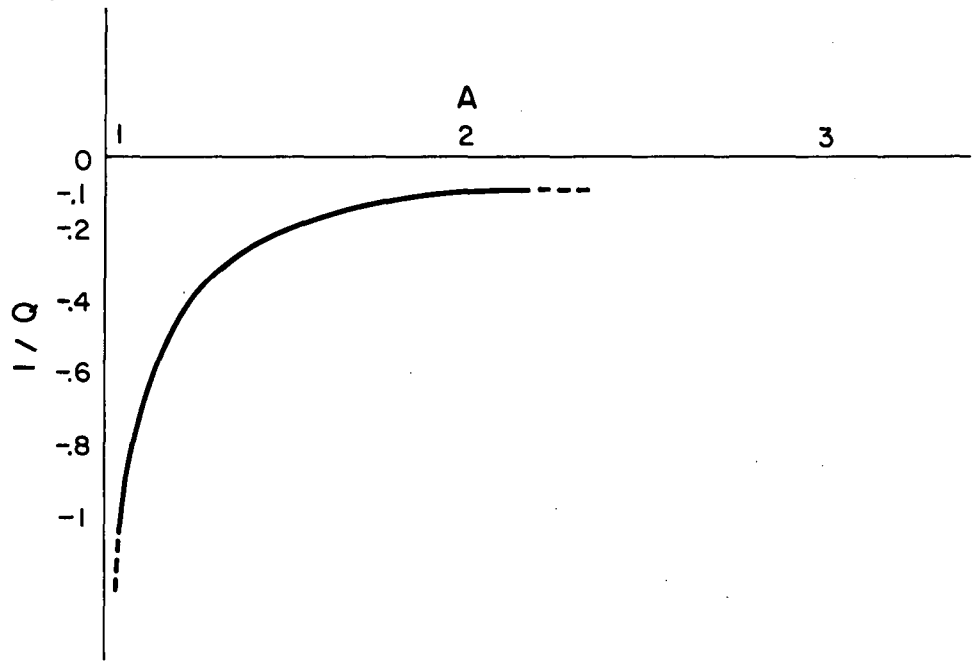
These functions are analogous to the functions $G_{pq}^{(\pm)}$ and K introduced in previous sections. For example, if the momentum dependence of F is neglected, i.e. if F is the constant $+F_0$, we obtain the dispersion relation

$$-F_0^{-1} = 1 - \frac{\eta}{2} \log \frac{\eta+1}{\eta-1} \quad . \tag{21}$$

This is identical to Eq. (11) if $-F_0$ is replaced by Q. The similarities between Landau's treatment and the microscopic theory given above can be demonstrated in much greater detail. Certainly this brief treatment does not do full justice to Landau's simple theory. It is certainly very impressive that he obtains these results with much less effort than the completely quantum mechanical study. An understanding of the relation between the two approaches would be helpful in understanding the physical basis of both methods.[10]

REFERENCES

1. Iu. L. Klimontovich and V. P. Silin, J. Exptl. Theoret. Phys. (U. S. S. R.) 23, 151 (1952); V. P. Silin, ibid., 23, 641 (1952).
2. Ferentz, Gell-Mann, and Pines, Phys. Rev. 92, 836 (1953).
3. L. D. Landau, J. Exptl. Theoret. Phys. (U. S. S. R.) 30, 1058 (1956); ibid 32, 59 (1957) and translated in Soviet Physics, JETP 3, 920 (1957), ibid. 5, 101 (1957), ibid 8, 70 (1958). Other references are given in reference 4.
4. A. A. Abrikosov and I. M. Khalatnikov, Soviet Physics, Uspekhi 1, 68 (1958).
5. Glassgold, Heckrotte, and Watson, Annals of Physics 6, 1 (1959).
6. K. Sawada, Phys. Rev. 106, 372 (1957); Sawada, Brueckner, Fukuda, and Brout, ibid 108, 507 (1957); R. Brout, ibid 108, 515 (1957); G. Wentzel, Phys. Rev. 108, 1593 (1957).
7. K. A. Brueckner and J. L. Gammel, Phys. Rev. 109, 1023 (1958).
8. Private communication from K. A. Brueckner.
9. Near T = 0 the damping of ordinary sound is strongly damped; the only kind of waves which can be propagated are those governed by Eq. (11) or Eq. (20).
10. The relation between Sawada's theory and the method of self-consistent fields has recently been studied by H. Ehrenreich and M. H. Cohen, "Self-Consistent Field Approach to the Many Electron Problem" (to be published in the Physical Review).



MU-15640

Fig. 1. The disperson function Q^{-1} of Eq. (11) plotted as a function of A for $1 < A < \infty$.