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The Geometry of Hilbert Schemes on Projective Space

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The Geometry of Hilbert Schemes on Projective Space

by

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Abstract

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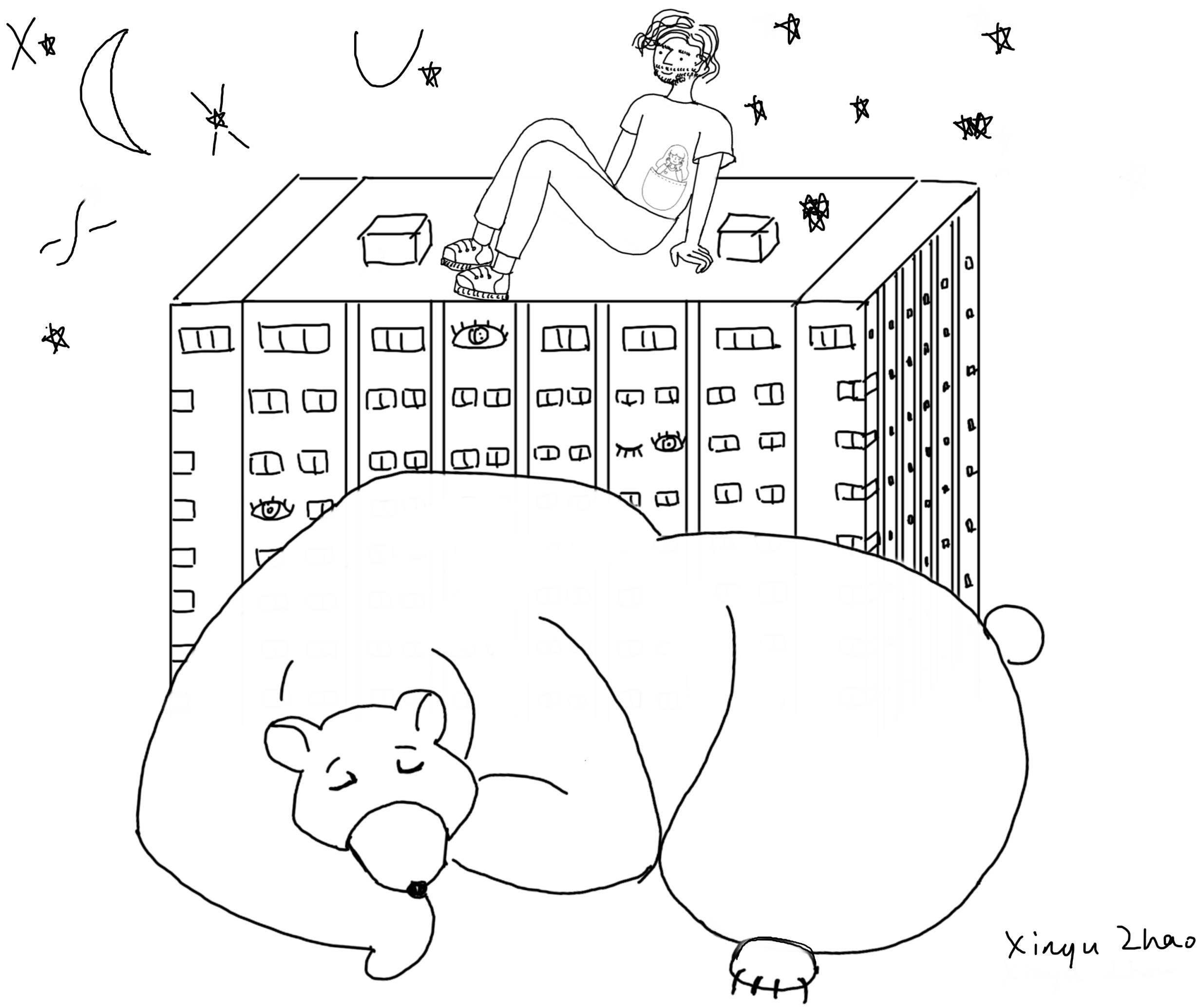
In this thesis we study singularities of Hilbert schemes and show that there are many (components) of Hilbert schemes that are smooth or mildly singular and use them to explore phenomena in birational geometry and commutative algebra. Specifically, we study the Hilbert scheme compactification of a pair of linear spaces, describe all the subschemes parameterized by this component and show that it is a smooth Mori dream space. We study Hilbert schemes with two Borel-fixed points and prove that they are reduced, and that their irreducible components have normal and Cohen-Macaulay singularities. We study the Hilbert scheme of points on a threefold and extend results on the Hilbert scheme of points of a surface to this case; we also provide bounds on the dimension of this Hilbert scheme. Finally, we generalize the Hilbert and Quot schemes to construct the fiber-full scheme, which is a fine moduli space that controls all the cohomological data of a variety instead of just the Hilbert polynomial.

To my family.



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Acknowledgments

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Chapter 1

Introduction

> "Algebraic geometry seems to have acquired the reputation of being esoteric, exclusive, and very abstract, with adherents who are secretly plotting to take over all the rest of mathematics. In one respect this last point is accurate."
>
> – David Mumford [72]

A characteristic of algebraic geometry is that the set of varieties of a given type is often itself an algebraic variety in a natural way. For example, associating a plane curve with its defining equation, up to scalars, identifies the family of plane curves of a given degree with a projective space. Explicitly, a plane curve of degree d in the complex projective plane corresponds to the vanishing locus of a homogeneous polynomial of degree d in three variables. The collection of these polynomials, up to scalars, can be identified with the projective space $\mathbf{P}^D$ where $D = \binom{d+2}{2} - 1$. Studying the geometry of certain loci in $\mathbf{P}^D$ directly leads to a deeper understanding about the geometry of the plane curves themselves.

This correspondence can be vastly generalized. To each closed subvariety of a projective variety, one can associate a numerical invariant called the Hilbert polynomial. For instance, in the case of a plane curve of degree d, the Hilbert polynomial is $P(t) = dt + 1 - \binom{d-1}{2}$. In 1961, Grothendieck [39] constructed the Hilbert scheme which is a projective variety that parameterizes all subvarieties in a given projective variety with a fixed Hilbert polynomial. It has applications in algebraic geometry: it is used in constructing other moduli spaces and in the study of deformations of curves in birational geometry. It also appears in other areas such as representation theory, combinatorics, symplectic geometry and mathematical physics.

Unfortunately, it does have some major drawbacks. It was shown by Vakil that Hilbert schemes, in general, satisfy "Murphy's law", i.e., every singularity of finite type over $\mathbf{k}$ appears on some Hilbert scheme [96]. However, this result does not decide whether most Hilbert schemes are singular or only some specially constructed (points on) Hilbert schemes are singular. For example, every Hilbert scheme in projective space contains a generically smooth component and there are many smooth or mildly singular components of Hilbert schemes. Even the very singular ones are important: the Hilbert scheme of points on a Calabi-Yau threefold plays a significant role in the computation of Donaldson-Thomas invariants.

In this thesis we find and study (components of) Hilbert schemes that have well-behaved singularities. This thesis is broadly divided into three parts:

(i) Chapter 3, Chapter 4, Chapter 5: We study singularities of classical Hilbert schemes and show that there are many (components) of Hilbert schemes that are smooth or mildly singular and use them to explore phenomena in birational geometry and commutative algebra.
(ii) Chapter 6: We initiate a detailed study of the Hilbert scheme of points on a threefold and extend results on the Hilbert scheme of points of a surface to this case.
(iii) Chapter 7: We generalize the Hilbert and Quot schemes to construct the fiber-full scheme, which is a fine moduli space that controls all the cohomological data of a variety instead of just the Hilbert polynomial.

We will now provide some background and details regarding the aforementioned topics.

1.0.1 Smooth components of Hilbert schemes

The cases when the Hilbert scheme is smooth or has smooth components has been well studied. Early on these smooth components were used to solve numerous enumerative problems [29] and recently, with major advances in the minimal model program [9], they are also a source of examples with rich birational structure. Fogarty [30] proved that $\mathrm{Hilb}^d(\mathbf{P}^2)$ is smooth and Arcara, Bertram, Coskun and Huizenga [5] proved that it is a Mori dream space and described the stable base decomposition of its effective cone in numerous cases. Piene and Schlessinger [79] showed that $\mathrm{Hilb}^{3t+1}(\mathbf{P}^3)$ has two smooth components that meet transversely and described the points of the component corresponding to twisted cubics explicitly. Chen [15] proved that the component corresponding to the twisted cubics is the flip of $\overline{\mathcal{M}}_{0,0}(\mathbf{P}^3, 3)$ over the Chow variety. Avritzer and Vainsencher [95] proved that the component corresponding to elliptic quartics in $\mathrm{Hilb}^{4t}(\mathbf{P}^3)$ is smooth and isomorphic to a double blow up of $\mathbf{Gr}(1, 9)$; Gallardo, Huerta and Schmidt [34] computed its effective cone. Chen, Coskun and Nollet [16] showed that the component corresponding to a pair of codimension two linear spaces meeting transversely is smooth and isomorphic to a blowup of $\mathrm{Sym}^2(\mathbf{Gr}(n-2, n))$. They also completely worked out its Mori theory. It is thus very interesting to find components of Hilbert schemes that are smooth and describe their birational geometry.

In Chapter 3 we show that the component of the Hilbert scheme parameterizing a pair of linear spaces meeting transversely is smooth and isomorphic to successive blowups of a product of Grassmannians. This generalizes the classical case of the Hilbert scheme of a pair of skew lines in [16]. In Chapter 4 we study the birational geometry of this component of the Hilbert scheme. In particular, we completely describe the effective and nef cones and prove that it is a Mori dream space. This provides new examples of Mori dream spaces.

1.0.2 Measuring the complexity of Hilbert schemes

The global geometry of Hilbert schemes is not well understood. The earliest results in this direction were obtained by Hartshorne [46], who showed that $\mathrm{Hilb}^P(\mathbf{P}^n)$ is connected, and Fogarty [30], who proved that $\mathrm{Hilb}^P(\mathbf{P}^2)$ is smooth. Later on, Reeves and Stillman [83] showed that every Hilbert scheme of projective space contains a smooth Borel-fixed point. As a consequence, Hilbert schemes with a single Borel-fixed point are smooth and irreducible, and Staal [89] completely classified these Hilbert schemes. In fact, most Hilbert schemes or components of Hilbert schemes that are very well understood have few Borel-fixed points. For example, the twisted cubic compactification $\mathrm{Hilb}^{3t+1}(\mathbf{P}^n)$, which has two smooth components that meet transversely [79], has three Borel-fixed points.

Thus, by restricting the structure of the Borel-fixed points one might obtain many smooth or mildly singular (components of) Hilbert schemes. In Chapter 5, we investigate the singularities of Hilbert schemes from this perspective. It turns out that if we allow at most two Borel-fixed points then the Hilbert scheme has at most two components. Moreover, the components are smooth or have normal, Cohen-Macaulay singularities. We also provide an explicit description of these singularities as cones over certain Segre embeddings.

1.0.3 The Hilbert scheme of points on a threefold

The Hilbert scheme of d points in $\mathbf{P}^n$, denoted by $\mathrm{Hilb}^d(\mathbf{P}^n)$, parameterizes closed zero-dimensional subschemes of $\mathbf{P}^n$ of degree d. We have already seen that $\mathrm{Hilb}^d(\mathbf{P}^2)$ is smooth and has a rich history from the perspective of birational geometry. It also has connections to other areas of mathematics, such as knot theory [35, 75], representation theory [73], symplectic geometry [6] and combinatorics [41]. By contrast, the Hilbert scheme is singular for $n \geq 3$ and very little is known about its geometry. The case of $\mathrm{Hilb}^d(\mathbf{P}^3)$ is of particular interest, since it lies at the boundary between the smooth cases $n \leq 2$ and the cases $n \geq 4$ which are believed to be wildly pathological [55]. In fact, $\mathrm{Hilb}^d(\mathbf{P}^3)$ is known to be rather special, as it admits a super-potential description – it is the singular locus of a hypersurface on a smooth variety [7]. For $d \leq 11$, $\mathrm{Hilb}^d(\mathbf{P}^3)$ is irreducible [23], and its general point parametrizes configurations of d points in $\mathbf{P}^3$; in particular, the Hilbert scheme is of dimension $3d$. However, Iarrobino [52, 53] proved that $\mathrm{Hilb}^d(\mathbf{P}^3)$ is reducible for $d \geq 78$. In general, the dimension of $\mathrm{Hilb}^d(\mathbf{P}^3)$ is unknown. Basic questions about the dimension of tangent spaces to $\mathrm{Hilb}^d(\mathbf{P}^3)$ are also wide open. Over forty years ago, Briançon and Iarrobino [10] established an upper bound for the dimension of $\mathrm{Hilb}^d(\mathbf{P}^3)$, and stated a conjecture regarding the largest possible dimension of its tangent spaces.

In Chapter 6 we initiate a detailed study of the tangent space to $\mathrm{Hilb}^d(\mathbf{P}^3)$. For points parametrizing monomial subschemes, we consider a decomposition of the tangent space into six distinguished subspaces, and show that a fat point exhibits an extremal behavior in this respect. This decomposition is also used to characterize smooth monomial points on the Hilbert scheme. We prove the Briançon-Iarrobino conjecture up to a factor of $\frac{4}{3}$, and improve the known asymptotic bound on the dimension of $\mathrm{Hilb}^d(\mathbf{P}^3)$. We also provide a self-contained proof of a parity theorem that was previously established using Donaldson-Thomas theory.

1.0.4 Refining the Hilbert scheme by controlling cohomology

When studying embedded varieties and their moduli, one is led to studying loci inside the Hilbert scheme that can be defined using certain cohomological data. This can be done by fixing all the cohomological data of $\mathscr{O}_X$, as seen in the works of Martin-Deschamps and Perrin in the study of curves in $\mathbf{P}^3$ [65], or it can be done by enforcing the vanishing of certain cohomology groups, giving the arithmetically Cohen-Macaulay and Gorenstein loci [28, 49, 56–58]. For this reason it is useful to express these loci as a fine moduli space of some functor. However, trying to show that the natural functor associated to the cohomological data is representable is much more subtle since (local) cohomology groups are, in general, not finitely generated.

In Chapter 7 we show that by fixing all the cohomological data, not just the Hilbert polynomial, the corresponding functor can be represented by a scheme which we call the fiber-full scheme. This provides a generalization of the Hilbert and Quot schemes and has the added benefit of having fewer irreducible components than the Hilbert scheme. As an example, we show that all the smooth Hilbert schemes are in fact fiber-full schemes. Numerous applications of the fiber-full scheme can be found in [20].

1.0.5 Concluding remarks

In the appendix we show that one of the Hilbert scheme components from Chapter 3 has radius bigger than 1. This has been included in the thesis because, to our knowledge, no such example has appeared in the literature. Chapter 3 and 4 are reproduced from [81], Chapter 5 is from [80], Chapter 6 is from [82] and is joint work with Alessio Sammartano, and Chapter 7 is from [19] and is joint work with Yairon Cid-Ruiz.

Chapter 2

Preliminaries

> "I can illustrate the second approach with the same image of a nut to be opened. The first analogy that came to my mind is of immersing the nut in some softening liquid, and why not simply water? From time to time you rub so the liquid penetrates better, and otherwise you let time pass. The shell becomes more flexible through weeks and months-when the time is ripe, hand pressure is enough, the shell opens like a perfectly ripened avocado! A different image came to me a few weeks ago. The unknown thing to be known appeared to me as some stretch of earth or hard marl, resisting penetration... the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it... yet it finally surrounds the resistant substance."
>
> – Alexander Grothendieck [68]

In this chapter we introduce the Hilbert scheme and go over some of the structural results on Hilbert schemes in projective space.

Notation 2.0.1. Let T be a locally Noetherian scheme and X a quasiprojective scheme over T with $\mathscr{O}(1)$ a very ample line bundle on X over T.

Definition 2.0.2. The Hilbert functor is a contravariant functor

$$\underline{\mathrm{Hilb}}_{X/T} : \{\text{locally Noetherian schemes over } T\} \to \{\text{Sets}\}$$

defined as follows

- For any locally Noetherian T-scheme B,

$$\underline{\mathrm{Hilb}}_{X/T}(B) = \{Z \subseteq X \times_T B,\ \text{closed and flat over } B\}.$$

- For any morphism of locally Noetherian T-schemes, $\varphi : B \to B'$ we obtain morphism

$$\underline{\mathrm{Hilb}}_{X/T}(B') \to \underline{\mathrm{Hilb}}_{X/T}(B), \quad Z \mapsto Z \times_{B'} B.$$

Let B be a connected locally Noetherian T-scheme and $Z \subseteq X \times_T B$ a closed, flat subscheme. Let $\pi_1 : Z \to X$ and $\pi_2 : Z \to B$ be the two projections. Then for any closed point $b \in B$ it is well known that the Euler characteristic

$$P_b(t) := \chi(\mathscr{O}_{Z_b}(t)) = \chi(\mathscr{O}_{Z_b} \otimes_{\mathscr{O}_Z} \pi_1^\star(\mathscr{O}(t)))$$

is a polynomial in t when $Z_b = \pi_2^{-1}(b)$ is the closed fiber [47]. Thus for any polynomial $P \in \mathbf{Q}[t]$ we can define a subfunctor of the Hilbert functor, denoted by $\underline{\mathrm{Hilb}}^P_{X/T}$, as follows

$$\underline{\mathrm{Hilb}}^P_{X/T}(B) = \{Z \in \underline{\mathrm{Hilb}}_{X/T}(B) : P_b = P \text{ for all } b \in B\}.$$

Theorem 2.0.3 ([39]). *Let X be projective over T. Then for any polynomial $P \in \mathbf{Q}[t]$, the functor $\underline{\mathrm{Hilb}}^P_{X/T}$ is representable by a projective T-scheme $\mathrm{Hilb}^P_{X/T}$. Moreover, $\underline{\mathrm{Hilb}}_{X/T}$ is represented by*

$$\mathrm{Hilb}_{X/T} = \bigsqcup_{P \in \mathbf{Q}[t]} \mathrm{Hilb}^P_{X/T}.$$

For an open subscheme $U \subseteq X$, the functor $\underline{\mathrm{Hilb}}_{U/T}$ is represented by an open subscheme

$$\mathrm{Hilb}_{U/T} \subseteq \mathrm{Hilb}_{X/T}$$

Example 2.0.4. If $T = \mathrm{Spec}(\mathbf{k})$ then the $\mathbf{k}$-points of $\mathrm{Hilb}^P(X)$ corresponds to subschemes of X with Hilbert polynomial P. Given a subscheme $Y \subseteq X$ we denote its $\mathbf{k}$-point in the Hilbert scheme by $[Y]$. The tangent space to $[Y]$, considered as a $\mathbf{k}$-point of the Hilbert scheme, is the $\mathbf{k}$-vector space

$$T_{[Y]} \mathrm{Hilb}^P(X) = H^0(X, \mathscr{N}_{Y/X}) = \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{I}_{Y/X}, \mathscr{O}_X)$$

where $\mathscr{N}_{Y/X}$ is the normal sheaf of Y in X.

The Hilbert scheme has two natural generalizations. For a more thorough discussion of these and the Hilbert scheme, see [87].

Remark 2.0.5. Let $P_1, \ldots, P_k$ be a sequence of Hilbert polynomials. Consider the functor

$$\underline{\mathrm{Hilb}}^{P_1,\ldots,P_k}_{X/T} : \{\text{locally Noetherian schemes over } T\} \to \{\text{Sets}\}$$

that maps

$$B \mapsto \{(Z_1, \ldots, Z_k) : Z_i \subseteq Z_{i+1} \text{ and } Z_i \in \underline{\mathrm{Hilb}}^{P_i}_{X/T}(B) \text{ for all } i\}.$$

If X is projective over T, then $\underline{\mathrm{Hilb}}^{P_1,\ldots,P_k}_{X/T}$ is represented by a projective scheme called the **nested Hilbert scheme**.

Remark 2.0.6. Let $\mathscr{F}$ be a coherent sheaf on X. The **Quot functor** is defined to be

$$\underline{\mathrm{Quot}}_{\mathscr{F}/X/T} : \{\text{locally Noetherian schemes over } T\} \to \{\text{Sets}\}$$

$$B \mapsto \{\text{coherent quotients } g : \mathscr{F} \times_T B \to \mathscr{G} : \mathscr{G} \text{ is flat over } B\}/\sim$$

If X is projective over T, then $\underline{\mathrm{Quot}}_{\mathscr{F}/X/T}$ is represented by a projective scheme called the **Quot scheme**. Analogous to the Hilbert scheme, this decomposes into a disjoint union of Quot schemes indexed by the Hilbert polynomial. We recover the Hilbert scheme by taking $\mathscr{F} = \mathscr{O}_X$. One can also define nested Quot schemes similar to Remark 2.0.5.

An equivalent interpretation of the Hilbert scheme $\mathrm{Hilb}^P(\mathbf{P}^n)$ is that it parameterizes saturated homogeneous ideals of $\mathbf{k}[x_0, \ldots, x_n]$ with a fixed Hilbert polynomial. To define homogeneous one needs a grading on the polynomial ring, and implicit in the latter statement is the fact that the polynomial ring is standard graded with $\deg x_i = 1$. It is quite common to come across polynomial rings that are *multigraded*, and thus it is useful to have a scheme that parameterizes ideals in such rings with a fixed Hilbert function. Haiman and Sturmfels in [42] showed that such a scheme does indeed exist.

Remark 2.0.7 ([42]). Let $S = \mathbf{k}[x_1, \ldots, x_n]$ be a polynomial ring. We can identify a monomial $x^u \in S$ with its exponent vector $u \in \mathbf{N}^n$. A **grading** of S by an abelian group A is a semigroup homomorphism $\deg : \mathbf{N}^n \to A$. This induces a decomposition

$$S = \bigoplus_{a \in A} S_a, \quad \text{satisfying} \quad S_a \cdot S_b \subseteq S_{a+b},$$

where S_a is the $\mathbf{k}$-span of all monomials x^u whose degree is equal to a. Note that for any other $\mathbf{k}$-algebra R we get an induced grading on $R \otimes_{\mathbf{k}} S$. Given a function $h : A \to \mathbf{N}$ we define a functor $\underline{H}_S^h : k$ algebras $\to$ Sets that maps

$$R \mapsto \{I \subseteq R \otimes_{\mathbf{k}} S \text{ homogeneous} : (R \otimes_{\mathbf{k}} S)_a / I_a \text{ is locally free of rank } h(a) \text{ for all } a\}$$

There is a quasiprojective scheme H_S^h, called the **multigraded Hilbert scheme**, that represents the functor $\underline{H}_S^h$. If the grading is positive i.e., the only monomial of degree 0 is x^0, then the scheme is projective.

Remark 2.0.8. The multigraded Hilbert scheme recovers the Hilbert scheme of projective space if we take the Hilbert function to be the Hilbert polynomial in sufficiently high degree. More precisely let P be a Hilbert polynomial, let m be its Gotzmann number Remark 2.0.13 and let S be standard graded with $\deg(e_i) = 1$ for all i. Let $A = \mathbf{Z}$ and $h : \mathbf{Z} \to \mathbf{N}$ is given by

$$h(i) = \begin{cases} P(i) & \text{if } i \geq m \\ \dim_{\mathbf{k}}(S_i) & \text{else.} \end{cases}$$

Then the natural map $H_S^h \to \mathrm{Hilb}^P(\mathbf{P}^n)$ is an isomorphism.

There is still a local relation between the multigraded Hilbert scheme and $\mathrm{Hilb}^P(\mathbf{P}^n)$ in many cases.

Theorem 2.0.9 (Comparison Theorem [79]). *Let $X \subseteq \mathbf{P}^n$ be a subscheme with ideal $I_X = (f_1, \ldots, f_s)$ where $\deg f_i = d_i$ satisfying, $(\mathbf{k}[x_0, \ldots, x_n]/I_X)_e \simeq H^0(\mathcal{O}_X(e))$ for $e = d_1, \ldots, d_s$. Then there is an isomorphism between the universal deformation space of I_X and that of X; the latter is an analytic neighbourhood of $\mathrm{Hilb}(\mathbf{P}^n)$ around $[X]$. In particular,*

$$T_{[I_X]} \mathrm{Hilb}(\mathbf{P}^n) = H^0(\mathbf{P}^n, \mathcal{N}_{X/\mathbf{P}^n}) = \mathrm{Hom}(I_X, S/I_X)_0.$$

Remark 2.0.10. Let $S = \mathbf{k}[x_0, \ldots, x_n]$. With notation as in the above Theorem, consider the following exact sequence in local cohomology [26, Corollary A1.12],

$$0 \longrightarrow H^0_{\mathfrak{m}}(S/I_X) \longrightarrow S/I_X \longrightarrow H^0_\star(\mathbf{P}^n, \mathscr{O}_X) \longrightarrow H^1_{\mathfrak{m}}(S/I_X) \longrightarrow 0.$$

If we show that $H^i_{\mathfrak{m}}(S/I_X)_e = 0$ for $e = e_1, \ldots, e_r$ and $i = 0, 1$, then the Comparison theorem would apply. Here are two instances in which this is true

(i) The depth of S/I_X is at least 2 [26, Corollary A1.13].

(ii) The Castlenuovo-Mumford regularity of the ideal I_X is $\min\{e_1, \ldots, e_r\}$ [26, Proposition 4.16]. Note that $\mathrm{reg}(I_X) = \mathrm{reg}(S/I_X) + 1$.

We will be primarily interested in $\mathrm{Hilb}^P_{X/T}$ where $X = \mathbf{P}^n$ and $T = \mathrm{Spec}(\mathbf{k})$. So we will fix that once and for all.

Notation 2.0.11. We use S to denote the polynomial ring $\mathbf{k}[x_0, \ldots, x_n]$ and $\mathfrak{m} := (x_0 \ldots, x_n)$ to denote its maximal ideal. We denote the monomial $x_0^{a_0} \cdots x_n^{a_n}$ by $\boldsymbol{x}^{\boldsymbol{\alpha}}$. We use S_d to denote the subspace of monomials of degree d. The *support* of a monomial is the set of all variables that divide the monomial. By lexicographic ordering we will mean the standard lexicographic ordering on S with $x_0 > x_1 > \cdots > x_n$.

All ideals are assumed to be saturated unless otherwise specified. We use $P_X(t)$ or $P_{S/I}(t)$ to denote the Hilbert polynomial of the subscheme $X = \mathrm{Proj}\,(S/I) \subseteq \mathbf{P}^n$. We sometimes call this the Hilbert polynomial of I.

We denote $\mathrm{Hilb}^P_{\mathbf{P}^n/\mathbf{k}}$ by $\mathrm{Hilb}^P(\mathbf{P}^n)$. In this case, we use $[I]$ or $[X]$ where $X = \mathrm{Proj}\,(S/I) \subseteq \mathbf{P}^n$ to denote the corresponding point on the Hilbert scheme.

We begin our study of $\mathrm{Hilb}^P(\mathbf{P}^n)$ by determining when it is non-empty. Equivalently, determining when is P a Hilbert polynomial of some closed subscheme of $\mathbf{P}^n$.

Theorem 2.0.12 ([37]). *A polynomial $P \in \mathbf{Q}[t]$ is a Hilbert polynomial if and only if there exists an integer partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\lambda_1 \geq \cdots \geq \lambda_m \geq 1$ for which*

$$P = P_\lambda := \sum_{i=1}^{m} \binom{t + \lambda_i - i}{\lambda_i - i}. \tag{2.1}$$

This is called the ***Gotzmann decomposition*** *of P.*

Remark 2.0.13 ([37])**.** The value m in the above theorem is called the **Gotzmann number** and is an upper bound on the Castelnuovo-Mumford regularity of any saturated ideal I with Hilbert polynomial P.

The dimension of the subscheme with Hilbert polynomial P_λ is $\lambda_1 - 1$. In particular, if the closed subscheme is proper and non-empty we have $1 \leq \lambda_1 \leq n$.

Notation 2.0.14. We use λ to denote the tuple $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ of weakly decreasing positive integers and call it an *integer partition*. We use P_λ to denote the Hilbert polynomial Eq. (2.1) associated to λ. Hilbert schemes are indexed by partitions λ and we will do this by writing them as $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$.

Although we stated Gotzmann's result, Macaulay was the first one who classified Hilbert polynomials. He did this by constructing a special monomial ideal called the lexicographic ideal. A monomial ideal $L \subseteq S$ is a **lexicographic** ideal if, for all integers j, the homogeneous component of I_j is the $\mathbf{k}$-vector space spanned by the $\dim_{\mathbf{k}} I_j$ largest monomials in lexicographic order.

Theorem 2.0.15 ([63])**.** *For an integer partition $\lambda = (\lambda_1, \ldots, \lambda_m)$, there is a unique saturated lexicographic ideal, denoted by $L(\lambda)$, with Hilbert polynomial P_λ. Let a_j be the number of parts in λ equal to j for all $j \in \mathbf{N}$. If $n \geq \lambda_1$ we have*

$$L(\lambda) := (x_0^{a_n+1}, x_0^{a_n} x_1^{a_{n-1}+1}, \ldots, x_0^{a_n} x_1^{a_{n-1}} \cdots x_{n-3}^{a_3} x_{n-2}^{a_2+1}, x_0^{a_n} x_1^{a_{n-1}} \cdots x_{n-2}^{a_2} x_{n-1}^{a_1}). \tag{2.2}$$

Finally,

$$P = \sum_{k=0}^{n} \left[\binom{t+k}{k+1} - \binom{t+k-m_k}{k+1} \right]$$

*where $m_i = \sum_{i=i}^{n} a_i$. This is called the **Macaulay decomposition** of P.*

Example 2.0.16 (Hypersurfaces)**.** We will now briefly explain why the Hilbert scheme parameterizing hypersurfaces is isomorphic to a projective space. It can be shown that $Z \subseteq \mathbf{P}^n$ is a hypersurface of degree d if and only if the Hilbert polynomial of Z is P_λ with $\lambda = (n^d)$ i.e.,

$$P_Z(t) = \binom{n+t}{n} - \binom{n+t-d}{n} = \sum_{i=1}^{d} \binom{t+n-i}{n-1}.$$

Thus we have a well defined, bijective morphism

$$\mathbf{P}(S_d) \longrightarrow \mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n), \quad (f) \mapsto [f].$$

To check that this is an isomorphism it suffices to show that

$$\dim T_{[f]} \mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n) = \dim \mathbf{P}(S_d) = \binom{n+d}{d} - 1$$

since this would imply $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ is smooth. By Theorem 2.0.9 we have

$$T_{[f]}\,\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n) = \mathrm{Hom}(f, S/f) = \mathrm{Hom}(f, S/f)_0.$$

It is now straightforward to check that the map $f \mapsto x^{\alpha}$ is well defined for all $x^{\alpha} \in (S/f)_d$. Thus $\dim(\mathrm{Hom}(f, S/f)_0) = \dim(S/f)_d = \binom{n+d}{d} - 1$, as required.

The first major result on the structure of Hilbert schemes of projective space was obtained by Hartshorne, who showed that the Hilbert schemes are always connected in characteristic 0. Pardue extended this to all characteristics.

Theorem 2.0.17 ([46,77]). *A non-empty Hilbert scheme* $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ *is connected.*

This theorem is proved by showing that any point on the Hilbert scheme can be joined to the lexicogrpahic point $[L(\lambda)]$ by a chain of rational curves.

To prove that the Hilbert scheme is connected the authors study the Borel-fixed points of the Hilbert scheme. Given a matrix $A = (a_{ij})_{ij} \in \mathrm{GL}(n+1)$, the map on variables $x_i \mapsto \sum a_{ij}x_j$ induces an action on the set of ideals of S with Hilbert polynomial P. Thus, the group $\mathrm{GL}(n+1)$ acts on $\mathrm{Hilb}^P(\mathbf{P}^n)$ and so does its subgroup, $\mathcal{B}$, of upper triangular matrices. A closed point (resp. ideal) is said to be **Borel-fixed** if it is fixed by the subgroup $\mathcal{B}$.

Since Borel-fixed points are fixed by the set of diagonal matrices, they must be defined by monomial ideals. A monomial ideal $I \subseteq S$ is said to be **strongly stable** if for any monomial $m \in I$ divisible by x_j we have $m\frac{x_i}{x_j} \in I$ for all $i < j$. The relation between these two concepts is given by the following theorem.

Proposition 2.0.18 ([69, Proposition 2.3]). *If char*$(\mathbf{k}) = 0$ *a monomial ideal* $I \subseteq S$ *is Borel-fixed if and only if* I *is strongly stable.*

This combinatorial criterion can be extend to all characteristics (Definition 3.4.1).

It turns out that the lexicographic point, which is Borel-fixed, is a special point on the Hilbert scheme.

Theorem 2.0.19 ([83]). *Let* λ *be an integer partition. The lexicographic point* $[L(\lambda)]$ *is a smooth point on the Hilbert scheme* $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ *and the component it lies on is called the* ***lexicographic component***.

Moreover, any subscheme Z *parameterized by the general member of the lexicographic component may be described as follows: Choose a flag*

$$\mathbf{P}^n \supseteq \mathbf{P}^{i_l+1} \supseteq \cdots \supseteq \mathbf{P}^{i_1+1}$$

Within each $\mathbf{P}^{i_j+1}$ *choose a generic hypersurface of degree* a_j *(if* $a_{i_1} = 1$*, choose* $\mathbf{P}^{i_1} \supseteq \mathbf{P}^{i_2+1}$ *in the above flag and skip the choice of a hypersurface for* a_{i_1}*). Finally choose* a_0 *generic points in* $\mathbf{P}^n$*. Then* Z *is the union of the chosen hypersurfaces and points.*

Now that we have a distinguished component on each Hilbert scheme, it is possible to refine Hartshorne's proof of the connectedness of the Hilbert scheme. To each Hilbert scheme $\mathrm{Hilb}^P(\mathbf{P}^n)$, one can associate an incidence graph as follows: to each irreducible component we assign a vertex, and we connect two vertices if the corresponding components intersect. Define the **distance** $d(C,D)$ between two components C,D to be the number of edges in the shortest path linking the corresponding vertices. Let $r_D = \max\{d(C,D) : C \text{ a component of } \mathrm{Hilb}^P(\mathbf{P}^n)\}$, and define the **radius** of the Hilbert scheme to be

$$\mathrm{rad}(\mathrm{Hilb}^P(\mathbf{P}^n)) = \min\{r_D : D \text{ a component of } \mathrm{Hilb}^P(\mathbf{P}^n)\}.$$

We identify any component D for which $\mathrm{rad}(\mathrm{Hilb}^P(\mathbf{P}^n)) = r_D$ as a **center** of the graph. By studying the lexicographic component in relation to other components Reeves established

Theorem 2.0.20 ([84, Theorem 7])**.** Consider the Hilbert scheme $\mathrm{Hilb}^P(\mathbf{P}^n)$ and let $d = \deg P$ be the dimension of the parameterized subschemes. Then the distance from any component to the lexicographic component is at most $d+1$. In particular, the radius of the Hilbert scheme is at most $d+1$.

Now that we have some understanding of the topological structure of these Hilbert schemes, the next natural thing to study would be its singularities. We have already seen that the Hilbert scheme parameterizing hypersurfaces in $\mathbf{P}^n$ is smooth. In particular, the Hilbert scheme of $\mathbf{P}^1$ is smooth. The next result generalizes this to a surface.

Definition 2.0.21. The **symmetric product** of a scheme X is the categorical quotient $\mathrm{Sym}^d(X) := X^d/S_d$ where S_d acts naturally on X^d by permutation.

Theorem 2.0.22 ([30])**.** *The Hilbert scheme* $\mathrm{Hilb}^P(\mathbf{P}^2)$ *is smooth and irreducible. If* $P = d$ *is constant, then the Hilbert-Chow morphism*

$$\mathrm{Hilb}^d(\mathbf{P}^2) \to \mathrm{Sym}^d(\mathbf{P}^2), \quad [Z] \mapsto \sum \deg(\mathscr{O}_{Z,p})[p]$$

is a crepant[1] *resolution of the symmetric product of a surface.*

Remark 2.0.23 ([64])**.** Let $S = \mathbf{k}[x,y]$ and assume that it is graded by an abelian group A. Then for any function $h : A \to \mathbf{N}$ the multigraded Hilbert scheme Hilb^h_S is smooth and irreducible.

It is natural to wonder if one can make more general statements about the smoothness of Hilbert schemes. We state two more instances of this without going into any details:

- If a subscheme $Z \subseteq \mathbf{P}^n$ is a locally complete intersection and $H^1(Z, \mathscr{N}_{Z/\mathbf{P}^n}) = 0$ then $[Z]$ is a smooth point in the Hilbert scheme [39].

[1] A resolution of singularities $\varphi : \widetilde{Y} \to Y$ is crepant if $\varphi^\star K_Y = K_{\widetilde{Y}}$

- If $Z \subseteq \mathbf{P}^n$ is an arithmetically Cohen-Macaulay subscheme of codimension 2 or an arithmetically Gorenstein subscheme of codimension 3, then $[Z]$ is a smooth point [28,58].

However, it turns out that Hilbert schemes are very far from being *well-behaved* in general. Define an equivalence relation on pointed schemes by: If $(X, p) \to (Y, q)$ is a smooth morphism, then $(X, p) \sim (Y, q)$. We call the equivalence classes singularity types, and will call pointed schemes singularities (even if the point is regular). We say that **Murphy's Law** holds for a moduli space if every singularity type of finite type over $\mathbf{Z}$ appears on that moduli space.

Theorem 2.0.24 ([96]). *The Hilbert scheme of non-singular curves in projective space satisfies Murphy's law. The Hilbert scheme of surfaces in $\mathbf{P}^4$ satisfies Murphy's law.*

On the other hand all hope is not lost, there might be still be many smooth Hilbert schemes or smooth components of Hilbert schemes. Here is a simple lemma that reduces to checking singularities at the Borel-fixed points.

Lemma 2.0.25. *The Hilbert scheme $\mathrm{Hilb}^P(\mathbf{P}^n)$ is reduced or smooth if and only if it is reduced or smooth at all the Borel-fixed points, respectively. Moreover, an integral component, H, of the Hilbert scheme is normal, Cohen-Macaulay, Gorenstein or smooth if and only if it is normal, Cohen-Macaulay, Gorenstein or smooth at all the Borel-fixed points on H, respectively.*

Proof. Given a $\mathbf{k}$-point $[Z] \in \mathrm{Hilb}^P(\mathbf{P}^n)$, write $\mathcal{B}(Z)$ for the orbit of Z under $\mathcal{B}$. By the Borel fixed-point theorem the closure, $\overline{\mathcal{B}(Z)}$, contains a Borel-fixed point. Assume that the Hilbert scheme is reduced at all the Borel-fixed points. Since the reduced locus is open, a non-empty open subset of $\overline{\mathcal{B}(Z)}$ is also reduced. Thus, some element of $\mathcal{B}(Z)$ is also non-reduced. Since $\mathcal{B}$ acts by automorphisms, Z must be a reduced point. The same proof works for smoothness as the smooth locus is also open.

The action of $\mathcal{B}$ restricts to any irreducible component of the Hilbert scheme. Since the normal, Cohen-Macaulay and Gorenstein loci are all open, the proof given in the previous paragraph also proves the second statement. □

By Theorem 2.0.19 the lexicographic point is smooth. Thus, if the Hilbert scheme has a single Borel-fixed point then it must be smooth. Staal recently classified all the Hilbert polynomials for which this is true.

Theorem 2.0.26 ([89]). *Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be an integer partition. The Hilbert scheme $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ has a unique Borel-fixed point if and only if*

(i) *$n \geq \lambda_1$ and $\lambda_m \geq 2$,*

(ii) *$\lambda = (1)$ or $\lambda = (n^{r-2}, \lambda_{r-1}, 1)$ where $r \geq 2$ and $n \geq \lambda_{r-1} \geq 1$.*

In all of these cases the Hilbert scheme is smooth.

In Chapter 5 I take the next step and classify the singularities of Hilbert scheme with two Borel-fixed points. Part of my results were used in the recent classification of all the smooth Hilbert schemes by Skjelnes and Smith.

Theorem 2.0.27 ([88]). *Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be an integer partition. The Hilbert scheme $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ is smooth if and only if*

(i) $n = 2 \geq \lambda_1$,

(ii) $n \geq \lambda_1$ *and* $\lambda_m \geq 2$,

(iii) $\lambda = (1)$ *or* $\lambda = (n^{r-2}, \lambda_{r-1}, 1)$ *where* $r \geq 2$ *and* $n \geq \lambda_{r-1} \geq 1$,

(iv) $(n^{r-s-2}, \lambda_{r-s-2}^{s+2}, 1)$ *where* $r - 3 \geq s \geq 0$ *and* $m - 1 \geq \lambda_{r-s-2} \geq 3$,

(v) $(n^{r-s-5}, 2^{s+4}, 1)$ *where* $r - 5 \geq s \geq 0$,

(vi) $(n + 1)$ *or* $r = 0$.

Chapter 3

Pair of linear spaces - Smoothness

In this chapter we show that the component of the Hilbert scheme that parameterizes a pair of linear spaces meeting transversely is smooth. We accomplish this by showing that the component is isomorphic to successive blowups $\mathrm{Sym}^2(\mathbf{Gr}(n-k,n))$. We classify the subschemes parameterized by this component and show that this component has a unique Borel-fixed point.

Let $\mathbf{k}$ be an algebraically closed field with $\mathrm{char}\,\mathbf{k} \neq 2$ and let $d \geq c \geq 2$. Let X be the union of an $(n-c)$-dimensional plane and an $(n-d)$-dimensional plane meeting transversely in $\mathbf{P}^n$. The Hilbert polynomial of X is

$$P^n_{n-c,n-d}(t) = \binom{n-c+t}{t} + \binom{n-d+t}{t} - \binom{n-c-d+t}{t}.$$

There is an integral component of $\mathrm{Hilb}^{P^n_{n-c,n-d}}(\mathbf{P}^n)$, denoted $\mathcal{H}^n_{n-c,n-d}$ or $\mathcal{H}_{n-c,n-d}(\mathbf{P}^n)$, whose general point parameterizes X (Proposition 3.1.2).

We begin with the natural rational map

$$\Xi : \mathbf{Gr}(n-c,n) \times \mathbf{Gr}(n-d,n) \dashrightarrow \mathcal{H}^n_{n-c,n-d}, \quad (\Lambda,\Lambda') \mapsto [I_\Lambda I_{\Lambda'}]. \tag{3.1}$$

If $c = d$, the rational map is $\mathfrak{S}_2$-equivariant where $\mathfrak{S}_2$ is the group of order 2. It acts on $\mathbf{Gr}(n-c,n)^2$ by interchanging the two factors and acts trivially on $\mathcal{H}^n_{n-c,n-c}$.

Definition 3.0.1. *For each $1 \leq i \leq c$ define an incidence variety*

$$\Gamma_i = \{(\Lambda,\Lambda') : \mathrm{codim}_{\mathbf{P}^n}(\Lambda \cap \Lambda') \leq d-1+i\} \subseteq \mathbf{Gr}(n-c,n) \times \mathbf{Gr}(n-d,n).$$

Note that Ξ is defined on the open subset where the two planes meet transversely. If X spans $\mathbf{P}^n$ (when $n \geq c+d-1$) then this open set is precisely the complement of Γ_c. Moreover, in this case, Ξ is also defined on the complement of Γ_{c-1} (Lemma 3.1.3).

In this thesis we will only be considering the case when $c = d$. The case when $c \neq d$ can be found in [81]. By explicitly resolving Ξ and studying the induced morphism, we obtain

Theorem 3.0.2. Let $k \geq 2$ and $n \geq 2k-1$. The component $\mathcal{H}^n_{n-k,n-k}$ is smooth and the map Ξ induces an isomorphism

$$\mathrm{Bl}_{\bar{\Gamma}_{k-1}} \cdots \mathrm{Bl}_{\bar{\Gamma}_1} \mathrm{Sym}^2 \mathbf{Gr}(n-k,n) \longrightarrow \mathcal{H}^n_{n-k,n-k}$$

where $\bar{\Gamma}_i$ is the strict transform of $\Gamma_i/\mathfrak{S}_2$.

If $n < 2k-1$, the morphism $\mathcal{H}^n_{n-k,n-k} \longrightarrow \mathbf{Gr}(2n-2k+1,n)$ that sends a scheme to its linear span is smooth; the fiber over a point Λ is $\mathcal{H}_{n-k,n-k}(\Lambda)$.

Historically, Harris [44] suggested that $\mathcal{H}^3_{1,1} \simeq \mathrm{Bl}_{\bar{\Gamma}_1} \mathrm{Sym}^2 \mathbf{Gr}(1,3)$ and that $\mathrm{Hilb}^{2t+2} \mathbf{P}^3$ is the union of $\mathcal{H}^3_{1,1}$ and another smooth component meeting transversely. The authors of [16] generalized this and proved that $\mathcal{H}^n_{n-2,n-2} \simeq \mathrm{Bl}_{\bar{\Gamma}_1} \mathrm{Sym}^2 \mathbf{Gr}(n-2,n)$ is smooth and meets exactly one other component in $\mathrm{Hilb}^{P^n_{n-2,n-2}} \mathbf{P}^n$. A major step in the proof of these statements was a computation of an analytic neighbourhood of a point in the intersection of the two components using the tangent-obstruction theory for the Hilbert scheme [16, Proposition 2.6]. Unfortunately, for general c,d there are many, sometimes singular, components meeting $\mathcal{H}^n_{n-c,n-d}$ (Remark 3.4.17). Thus a description of a neighbourhood of a point in the intersection of all these components is most likely intractable. Our proof of Theorem 3.4.7 circumvents this by using the explicit construction of Ξ and studying the induced map on tangent spaces.

In Chapter 5 we will study the idea that the complexity of a Hilbert scheme can be measured by their number of Borel fixed points. In line with our reasoning, we have the following result:

Theorem 3.0.3. The component $\mathcal{H}^n_{n-c,n-d}$ has a unique Borel fixed point.

We also give a complete description of all the subschemes parameterized by $\mathcal{H}^n_{n-c,n-d}$. In light of Theorem 3.4.7, it is enough to consider the case $n \geq 2k-1$. A **double structure** on an integral subscheme $Z \subseteq \mathbf{P}^n$ is a subscheme $Z' \subseteq \mathbf{P}^n$ such that $Z'_{\mathrm{red}} = Z$ and $\deg(Z') = 2\deg(Z)$. A double structure is said to be **pure** if it has no embedded components.

Theorem 3.0.4. Let $n \geq 2k-1$. Let Z be a subscheme parameterized by $\mathcal{H}^n_{n-k,n-k}$. Then Z is a pair of planes meeting transversely, or there exists a sequence of integers $1 \leq i_1 < \cdots < i_r \leq k$ and a flag of linear spaces $\Lambda^1 \subseteq \Lambda^2 \subseteq \cdots \subseteq \Lambda^r \subseteq \mathbf{P}^n$ with $\mathrm{codim}_{\mathbf{P}^n}(\Lambda^\ell) = (k + i_\ell - 1)$ for each ℓ, such that

(i) If $i_1 > 1$ then Z is a union of two planes meeting along Λ^1 with embedded pure double structures on Λ^ℓ for each $1 \leq \ell \leq r$.

(ii) If $i_1 = 1$ then Z is a pure double structure on Λ^1 with embedded pure double structures on Λ^ℓ for each $2 \leq \ell \leq r$.

Notation 3.0.5. For the rest of the chapter $\mathbf{k}$ will denote an algebraically closed field with $\mathrm{char}(\mathbf{k}) \neq 2$

3.1 Dimension and generic smoothness

Let X denote the union of an $(n-c)$-plane and $(n-d)$-plane meeting transversely in $\mathbf{P}^n$. Although we are primarily interested in the case of $c = d$, the results in this section hold for general c, d. It is clear that X is parameterized by an open subset of $\mathbf{Gr}(n-c, n) \times \mathbf{Gr}(n-d, n)$ of dimension $c(n-c+1) + d(n-d+1)$. If we show that the tangent space to $[X]$ on its Hilbert scheme has dimension $c(n-c+1) + d(n-d+1)$, it will follow immediately that there is an irreducible component of $\mathrm{Hilb}^{P^n_{n-c,n-d}}(\mathbf{P}^n)$ whose general member parameterizes X and whose natural scheme structure is reduced.

Since X is projectively equivalent to $Z = V(x_0, \ldots, x_{c-1}) \cup V(x_{n-d+1}, \ldots, x_n)$, it suffices to compute the tangent space to $[Z]$ on its Hilbert scheme. For the rest of this section we fix Z and $P(t) = P^n_{n-c,n-d}(t)$.

If $Z \simeq \mathbf{P}^{n-c} \sqcup \mathbf{P}^{n-d}$ is a disjoint union of linear spaces, it is smooth; this occurs if and only if $n \leq c + d - 1$. In this case we have a splitting of normals sheaves

$$\mathscr{N}_{Z/\mathbf{P}^n} = \mathscr{N}_{\mathbf{P}^{n-c}/\mathbf{P}^n} \oplus \mathscr{N}_{\mathbf{P}^{n-d}/\mathbf{P}^n} \simeq \mathscr{O}^c_{\mathbf{P}^{n-c}}(1) \oplus \mathscr{O}^d_{\mathbf{P}^{n-d}}(1).$$

Thus we obtain, $h^0(\mathbf{P}^n, \mathscr{N}_{Z/\mathbf{P}^n}) = c(n-c+1) + d(n-d+1)$ and $h^1(\mathbf{P}^n, \mathscr{N}_{Z/\mathbf{P}^n}) = 0$. It follows that $[Z]$ is a smooth point on its Hilbert scheme [48, Theorem 1.1c]. If $n > c+d-1$, we will explicitly compute the tangent space to $[Z]$ using Theorem 2.0.9 Since $n > c + d - 1$, the depth of S/I_Z is at least 2 and it follows from Remark 2.0.10 that the comparison theorem applies for Z.

Lemma 3.1.1. *We have* $\dim_{\mathbf{k}} T_{[Z]} \mathrm{Hilb}^P(\mathbf{P}^n) = c(n-c+1) + d(n-d+1)$.

Proof. We only need to consider the case $n > c + d - 1$. Moreover, it suffices to show that the tangent space dimension is at most $c(n-c+1) + d(n-d+1)$. In particular it is enough to show that any $\varphi \in \mathrm{Hom}(I_Z, S/I_Z)_0$ can be written as

$$\varphi(x_i x_j) = \sum_{\ell=0}^{n-d} a^j_\ell x_i x_\ell + \sum_{\ell=c}^{n} b^i_\ell x_j x_\ell \tag{3.2}$$

for any $0 \leq i \leq c-1$ and $n-d+1 \leq j \leq n$ with some constants, $a^i_\ell, b^i_\ell \in \mathbf{k}$.

Let us first show that $\varphi(x_i x_j)$ is supported on $\{x_i x_0, \ldots, x_i x_{n-d}, x_j x_c, \ldots, x_j x_n\}$. Let i, j be any integers satisfying $0 \leq i \leq c-1$ and $n-d+1 \leq j \leq n$. Choose j' such that $n-d+1 \leq j' \leq n$ and $j \neq j'$. Since φ is an S-module homomorphism we have, $x_{j'}\varphi(x_i x_j) = x_j \varphi(x_i x_{j'})$. This implies that x_j divides every non-zero monomial in $\varphi(x_i x_j)$ that is not annihilated by $x_{j'}$ in S/I_Z. It follows that $\varphi(x_i x_j)$ is supported on

$$\mathcal{C} = \{x_p x_q : 0 \leq p \leq c-1, 0 \leq q \leq n-d\} \cup \{x_j x_c, \ldots, x_j x_n\}.$$

Similarly, choose i' such that $0 \leq i' \leq c-1$ and $i' \neq i$. Then the equality $x_{i'}\varphi(x_i x_j) = x_i \varphi(x_{i'} x_j)$ implies x_i divides every monomial in $\varphi(x_i x_j)$ that is not annihilated by $x_{i'}$. Once again we see that $\varphi(x_i x_j)$ is supported on

$$\mathcal{C}' = \{x_i x_0, \ldots, x_i x_{n-d}\} \cup \{x_p x_q : c \le p \le n, n-d+1 \le q \le n\}.$$

Thus $\varphi(x_i x_j)$ is supported on $\mathcal{C} \cap \mathcal{C}' = \{x_i x_0, \ldots, x_i x_{n-d}, x_j x_c, \ldots, x_j x_n\}$.

For any i, j, write $\varphi(x_i x_j) = \sum_{\ell=0}^{n-d} a_\ell^{i,j} x_i x_\ell + \sum_{\ell=c}^{n} b_\ell^{i,j} x_j x_\ell$ with $b_\ell^{ij}, a_\ell^{ij} \in \mathbf{k}$. Using the relation $x_{j'}\varphi(x_i x_j) = x_j \varphi(x_i x_{j'})$ we see that $b_\ell^{i,j} = b_\ell^{i,j'}$ for each ℓ and all j, j'. Using the relation $x_{i'}\varphi(x_i x_j) = x_i \varphi(x_i' x_j)$ we obtain $a_\ell^{i,j} = a_\ell^{i',j}$ for each ℓ and all i, i'. Thus φ is of the form described in Eq. (3.2). □

We immediately deduce the following.

Proposition 3.1.2. *There is an integral component of* $\mathrm{Hilb}^P(\mathbf{P}^n)$*, denoted* $\mathcal{H}^n_{n-c,n-d}$ *or* $\mathcal{H}_{n-c,n-d}(\mathbf{P}^n)$*, whose general point parameterizes an* $(n-c)$*-plane and an* $(n-d)$*-plane meeting transversely in* $\mathbf{P}^n$*.*

In the introduction we defined a rational map (Eq. (3.1))

$$\Xi : \mathbf{Gr}(n-c, n) \times \mathbf{Gr}(n-d, n) \dashrightarrow \mathcal{H}^n_{n-c,n-d}, \quad (\Lambda, \Lambda') \mapsto [I_\Lambda I_{\Lambda'}].$$

This map is well defined along the locus where Λ, Λ' meet transversely, because in this situation $I_\Lambda I_{\Lambda'} = I_\Lambda \cap I_{\Lambda'}$. In many cases, Ξ is in fact defined on a slightly larger open set.

Lemma 3.1.3. *Let* $n \ge c + d - 1$*. The rational map* Ξ *extends to the complement of* Γ_{c-1}*.*

Proof. We need to show that Ξ is defined along $\Gamma_c \setminus \Gamma_{c-1}$. Up to projective equivalence, an element of $\Gamma_c \setminus \Gamma_{c-1}$ is of the form $V(x_0, \ldots, x_{c-1}) \cup V(x_0, x_c, \ldots, x_{c+d-2})$. It suffices to show that $J = (x_0, \ldots, x_{c-1})(x_0, x_c, \ldots, x_{c+d-2})$ has Hilbert polynomial $P(t)$. It follows by inspecting the minimal generators of J that for any $t \ge 1$, $(S/J)_t$ is spanned by

$$x_0 \mathbf{k}[x_{c+d-1}, \ldots, x_n]_{t-1} \oplus \bigoplus_{i=1}^{c-1} x_i \mathbf{k}[x_i, \ldots, x_{c-1}, x_{c+d-1}, \ldots, x_n]_{t-1} \oplus \mathbf{k}[x_c, \ldots, x_n]_t.$$

Thus the Hilbert polynomial of S/J is

$$\binom{n-c-d+t}{t-1} + \sum_{i=1}^{c-1} \binom{n-d-i+t}{t-1} + \binom{n-c+t}{t}.$$

Using the "Hockey-Stick" identity this simplifies to

$$\binom{n-c+t}{t} + \binom{n-d+t}{t} - \binom{n-c-d+t}{t} = P(t).$$

□

Lemma 3.1.4. *Let $n \geq c + d - 1$ and consider the open set*

$$\mathcal{V} = (\mathbf{Gr}(n-c,n) \times \mathbf{Gr}(n-d,n)) \setminus \Gamma_{c-1} \subseteq \mathbf{Gr}(n-c,n) \times \mathbf{Gr}(n-d,n).$$

The morphism $\Xi|_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathcal{H}^n_{n-c,n-d}$ is injective if $c \neq d$ and two-to-one if $c = d$.

Proof. Assume $\Xi|_{\mathcal{V}}(\Lambda, \Lambda') = \Xi|_{\mathcal{V}}(\tilde{\Lambda}, \tilde{\Lambda}') = [Y]$ for some scheme Y. Observe that $I_\Lambda I_{\Lambda'}$ is a saturated ideal. Indeed, up to projective equivalence, $\Lambda \cup \Lambda' = V(x_0, \ldots, x_{c-1}) \cup V(x_c, \ldots, x_{c-d-2}, x_i)$ with $i \in \{0, c-d-1\}$. In both cases, $I_\Lambda I_{\Lambda'}$ is clearly saturated. Thus we have $I_Y = I_\Lambda I_{\Lambda'}$ and taking nilradicals we obtain

$$I_{\Lambda \cup \Lambda'} = I_\Lambda \cap I_{\Lambda'} = \sqrt{I_\Lambda \cap I_{\Lambda'}} = \sqrt{I_\Lambda I_{\Lambda'}} = I_{Y_{\mathrm{red}}}.$$

Similarly, $I_{\tilde{\Lambda} \cup \tilde{\Lambda}'} = I_{Y_{\mathrm{red}}}$. Equating the two expressions we have $\Lambda \cup \Lambda' = \tilde{\Lambda} \cup \tilde{\Lambda}'$. The conclusion now follows. □

3.2 Coordinates for $\mathcal{H}^n_{n-k,n-k}$

This section is devoted to an analysis of $\mathcal{H}^n_{n-k,n-k}$. The first major goal of this section is to prove that $\mathcal{H}^n_{n-k,n-k}$ is smooth. We start with the case when the pair of planes parameterized spans $\mathbf{P}^n$. We construct a bijective morphism from a non-singular variety to $\mathcal{H}^n_{n-k,n-k}$ and deduce this is an isomorphism by proving its differential is injective (Theorem 3.4.7). For the case where the pair of planes do not span $\mathbf{P}^n$, we construct a certain fibration to reduce to the case where they do span (Corollary 3.4.8).

Let $n \geq 2k-1$ and $\mathcal{X}_0 = \mathbf{Gr}(n-k,n)^2$. For each $1 \leq v \leq k-1$, let $\mathcal{X}_v = \mathrm{Bl}_{\Gamma_v} \cdots \mathrm{Bl}_{\Gamma_1} \mathcal{X}_0$ and let $\pi_v : \mathcal{X}_v \longrightarrow \mathcal{X}_0$ be the blow-up morphism. The map given in Eq. (3.1) induces a rational map

$$\Xi : \mathcal{X}_{k-1} = \mathrm{Bl}_{\Gamma_{k-1}} \cdots \mathrm{Bl}_{\Gamma_1} \mathbf{Gr}(n-k,n)^2 \dashrightarrow \mathcal{H}^n_{n-k,n-k} \tag{3.3}$$

defined away from the strict transforms of the exceptional divisors. In order to study the structure of $\mathcal{H}^n_{n-k,n-k}$, we will begin by extending Ξ to a morphism on $\mathcal{X}_{k-1}$.

For each ordered basis $\mathbb{E} = \{e_0, \ldots, e_n\}$ of S_1 we obtain an affine neighbourhood $U_\mathbb{E} = \operatorname{Spec} \mathbf{k}[a_{i,j}, b_{i,j}]^{k \leq j \leq n}_{0 \leq i \leq k-1}$ of $\mathcal{X}_0$ such that the $\mathbf{k}$-points of $U_\mathbb{E}$ correspond to

$$(\Lambda(\mathbf{a}), \Lambda(\mathbf{b})) := (V(e_0 + \sum_{j=k}^n a_{0,j} e_j, \ldots, e_{k-1} + \sum_{j=k}^n a_{k-1,j} e_j), V(e_0 + \sum_{j=k}^n b_{0,j} e_j, \ldots, e_{k-1} + \sum_{j=k}^n b_{k-1,j} e_j)). \tag{3.4}$$

It is clear that as $\mathbb{E}$ ranges over all ordered basis of S_1, the set of $U_\mathbb{E}$ cover $\mathcal{X}_0$. In particular, it suffices to extend Ξ along each $\pi^{-1}_{k-1}(U_\mathbb{E})$ in a compatible way. For notational convenience we may assume $\mathbb{E} = \{x_0, \ldots, x_n\}$ and let $U_0 = U_{\mathbb{E}}$. Observe that the locus $\Gamma_v \cap U_0$ is cut out by the ideal generated by the $v \times v$ minors of the matrix

$$M = \begin{pmatrix} a_{0,k} - b_{0,k} & \cdots & a_{0,n} - b_{0,n} \\ \vdots & & \vdots \\ a_{k-1,k} - b_{k-1,k} & \cdots & a_{k-1,n} - b_{k-1,n} \end{pmatrix}.$$

Thus $\pi_{k-1}^{-1}(U_0)$ is obtained by blowing up U_0 along the strict transforms of the ideal generated by the $v \times v$ minors of M for $v = 1, \ldots, k-1$, in that order.

Proposition 3.2.1. *For each $1 \leq v \leq k-1$, there exists non-singular affine open subsets $U_v \subseteq \mathcal{X}_v$ such that the following hold.*

(i) *We have $U_v \subseteq \mathrm{Bl}_{\Gamma_v \cap U_{v-1}} U_{v-1} \subseteq \mathcal{X}_v$.*

(ii) *On the open set U_v, the matrix $\pi_v^{\star}(M)$ is row equivalent to the matrix*

$$\begin{pmatrix} \lambda_1 \cdots \lambda_v (T^{(v)}_{0,k} - T^{(v)}_{0,n-v+1} T^{(v)}_{k-v,k}) & \cdots & \lambda_1 \cdots \lambda_v (T^{(v)}_{0,n-v} - T^{(v)}_{0,n-v+1} T^{(v)}_{k-v,n-v}) & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \lambda_1 \cdots \lambda_v (T^{(v)}_{k-v-1,k} - T^{(v)}_{k-v-1,n-v+1} T^{(v)}_{k-v,k}) & \cdots & \lambda_1 \cdots \lambda_v (T^{(v)}_{k-v-1,n-v} - T^{(v)}_{k-v-1,n-v+1} T^{(v)}_{k-v,n-v}) & 0 & \cdots & 0 & 0 \\ \lambda_1 \cdots \lambda_v T^{(v)}_{k-v,k} & \cdots & \lambda_1 \cdots \lambda_v T^{(v)}_{k-v,n-v} & \lambda_1 \cdots \lambda_v & \ddots & \vdots & \vdots \\ \vdots & & \vdots & \vdots & \ddots & 0 & \vdots \\ \lambda_1 \lambda_2 T^{(2)}_{k-2,k} & \cdots & \lambda_1 \lambda_2 T^{(2)}_{k-2,n-v} & \lambda_1 \lambda_2 T^{(2)}_{k-2,n-v+1} & \cdots & \lambda_1 \lambda_2 & 0 \\ \lambda_1 T^{(1)}_{k-1,k} & \cdots & \lambda_1 T^{(1)}_{k-1,n-v} & \lambda_1 T^{(1)}_{k-1,n-v+1} & \cdots & \lambda_1 T^{(1)}_{k-1,n-1} & \lambda_1 \end{pmatrix}$$

where

$$\lambda_1 = a_{k-1,n} - b_{k-1,n} \text{ and } \lambda_i = T^{(i-1)}_{k-i,n-i+1} - T^{(i-1)}_{k-i,n-i+2} T^{(i-1)}_{k-i+1,n-i+1} \text{ for each } 2 \leq i \leq k-1.$$

(iii) *The strict transform of Γ_{v+1} on U_v is cut out by*

$$(T^{(v)}_{i,j} - T^{(v)}_{i,n-v+1} T^{(v)}_{k-v,j})^{0 \leq i \leq k-v-1}_{k \leq j \leq n-v}.$$

(iv) *$\Gamma_{v+1} \cap U_v$ is non-singular and the blowup along this locus is given by*

$$\mathrm{Bl}_{\Gamma_{v+1} \cap U_v} U_v := \mathrm{Proj}\, \mathbf{k}[U_v][T^{(v+1)}_{i,j}]_{i,j} / (\textit{Koszul Relations}).$$

Proof. We begin with the definition of U_1. Since Γ_1 is cut out by $(a_{i,j} - b_{i,j})_{i,j}$ on U_0, it is a non-singular subscheme and we have $\mathrm{Bl}_{\Gamma_1 \cap U_0} U_0 = \mathrm{Proj}\, \mathbf{k}[U_0][T^{(1)}_{i,j}]_{i,j} / (\text{Koszul relations})$. We define $U_1 = D(T^{(1)}_{k-1,n})$.

Let M_v denote the matrix appearing in item (ii). We will prove items (i) - (iv) inductively starting with $v = 1$. Item (i) is true for $v = 1$ by construction. On the open set U_1, the Koszul relations simplify to $a_{i,j} - b_{i,j} = \lambda_1 T^{(1)}_{i,j}$; here we have set $T^{(1)}_{k-1,n} = 1$. Substituting this into the matrix $\pi_1^\star(M)$ and subtracting appropriate multiples of the bottom row from every other row, we obtain the matrix

$$M_1 = \begin{pmatrix} \lambda_1(T^{(1)}_{0,k} - T^{(1)}_{0,n}T^{(1)}_{k-1,k}) & \cdots & \lambda_1(T^{(1)}_{0,n-1} - T^{(1)}_{0,n}T^{(1)}_{k-1,n-1}) & 0 \\ \vdots & & \vdots & \vdots \\ \lambda_1(T^{(1)}_{k-2,k} - T^{(1)}_{k-2,n}T^{(1)}_{k-1,k}) & & \lambda_1(T^{(1)}_{k-2,n-1} - T^{(1)}_{k-2,n}T^{(1)}_{k-1,n-1}) & 0 \\ \lambda_1 T^{(1)}_{k-1,k} & \cdots & \lambda_1 T^{(1)}_{k-1,n-1} & \lambda_1 \end{pmatrix}.$$

This proves item (ii) for $v = 1$. The ideal generated by the 2×2 minors of M_1 is $\lambda_1^2(T^{(1)}_{i,j} - T^{(1)}_{i,n}T^{(1)}_{k-1,j})^{0\le i\le k-2}_{0\le j\le n-1}$. Thus the ideal of the strict transform of Γ_2 is $(T^{(1)}_{i,j} - T^{(1)}_{i,n}T^{(1)}_{k-1,j})^{0\le i\le k-2}_{0\le j\le n-1}$. Since this ideal is generated by a regular sequence, the blowup along it is non-singular and equal to $\mathrm{Bl}_{\Gamma_2\cap U_1} U_1 := \mathrm{Proj}\ \mathbf{k}[U_1][T^{(2)}_{i,j}]_{i,j}/(\text{Koszul relations})$. This proves item (iii) and (iv) for $v = 1$.

Now assume items (i) - (iv) have been proved for some $1 \le v \le k-2$. Define $U_{v+1} = D(T^{(v+1)}_{k-v-1,n-v})$; equivalently let $T^{(v+1)}_{k-v-1,n-v} = 1$. Then the Koszul relations on this open simplify to $T^{(v)}_{i,j} - T^{(v)}_{i,n-v+1}T^{(v)}_{k-v,j} = \lambda_{v+1}T^{(v+1)}_{i,j}$. Once we substitute this into the matrix M_v, it is straightforward to row reduce the matrix so that it becomes M_{v+1}. Items (i) - (iv) will follow immediately as explained in the previous paragraph. □

Remark 3.2.2. It follows from Proposition 3.2.1 that a set of algebraically independent coordinates on U_{k-1} is

$$\{b_{i,j}\}^{k\le j\le n}_{0\le i\le k-1} \cup \{T^{(j)}_{i,n-j+1}\}^{0\le i\le k-1-j}_{1\le j\le k-1} \cup \{\lambda_1, \ldots, \lambda_{k-1}\} \cup \{T^{(i)}_{k-i,j}\}^{1\le i\le k-1}_{k\le j\le n-i} \cup \{T^{(k)}_{0,j}\}_{k\le j\le n-k+1}$$

with $T^{(k)}_{0,j} = T^{(k-1)}_{0,j} - T^{(k-1)}_{0,n-k+2}T^{(k-1)}_{1,j}$ for all j.

Proposition 3.2.3. *Let $n \ge 2k-1$. The rational map Ξ in Eq. (3.3) extends to a morphism $U_{k-1} \longrightarrow \mathcal{H}^n_{n-k,n-k}$.*

Proof. We will use $\mathbf{a}$ to denote the tuple $(a_{i,j})_{i,j}$ and similarly use $\mathbf{b}$ and $\mathbf{T}^{(v)}$ to denote their corresponding tuples. Moreover, we will use $\Lambda(\mathbf{a})$ to denote the $(n-k)$-plane corresponding to $\mathbf{a}$ as in Eq. (3.4). For each $0 \le i \le k-1$ let $y_i = x_i + \sum_{j=k}^n b_{i,j}x_j$. At the moment, Ξ maps

$$\begin{aligned}(\mathbf{a}, \mathbf{b}, \mathbf{T}^{(1)}, \ldots, \mathbf{T}^{(k)}) &\mapsto \left[I_{\Lambda(\mathbf{a})}I_{\Lambda(\mathbf{b})}\right] \\ &= \left[(y_0 + \sum_{j=k}^n (a_{0,j} - b_{0,j})x_j, \ldots, y_{k-1} + \sum_{j=k}^n (a_{k-1,j} - b_{k-1,j})x_j)(y_0, \ldots, y_{k-1})\right]\end{aligned} \tag{3.5}$$

and this is undefined along the strict transforms of the exceptional divisors. Although we may express $\mathbf{a}$ in terms of $\mathbf{b}$ and $\{\mathbf{T}^{(v)}\}_v$, we will still describe formulas in terms of $\mathbf{a}$ as it simplifies the exposition.

Observe that a minimal set of generators for $I_{\Lambda(\mathbf{a})}$ is given by the rows of $\left[\mathrm{Id}_{k\times k} \,|\, M\right] z^T$ where $z = \begin{bmatrix} y_0 & \cdots & y_{k-1} & x_k & \cdots & x_n \end{bmatrix}$ is a row vector. Applying row operations to $\left[\mathrm{Id}_{k\times k} \,|\, M\right]$ will produce different minimal sets of generators. In particular, applying the row operations we did to M to get M_{k-1} (Proposition 3.2.1 (ii)) to the matrix $\left[\mathrm{Id}_{k\times k} \,|\, M\right]$ we obtain a new set of generators $\alpha_0, \dots, \alpha_{k-1}$ of $I_{\Lambda(\mathbf{a})}$ where

$$\alpha_p = y_p - \sum_{j=1}^{k-1-p} T^{(j)}_{p,n-j+1} y_{k-j} + \sum_{j=k}^{n-(k-1-p)} \lambda_1 \cdots \lambda_{k-p} T^{(k-p)}_{p,j} x_j \quad \text{for} \quad 0 < p \le k-1$$

and

$$\alpha_0 = y_0 - \sum_{j=1}^{k-1} T^{(j)}_{0,n-j+1} y_{k-j} + \sum_{j=k}^{n-(k-1)} \lambda_1 \cdots \lambda_{k-1} T^{(k)}_{0,j} x_j$$

with $T^{(k)}_{0,j} = T^{(k-1)}_{0,j} - T^{(k-1)}_{0,n-k+2} T^{(k-1)}_{1,j}$ for all j. By construction, $T^{(v)}_{k-v,n-v+1} = 1$ for all $1 \le v \le k-1$.

For $0 \le p < q \le k-1$ define the following "cross terms"

$$\beta_{p,q} = \left(y_p - \sum_{j=1}^{k_p} T^{(j)}_{p,n-j+1} y_{k-j} \right) \left(\sum_{j=k}^{n-k_q} T^{(k-q)}_{q,j} x_j \right) - \lambda_{p,q} \left(y_q - \sum_{j=1}^{k_q} T^{(j)}_{q,n-j+1} y_{k-j} \right) \left(\sum_{j=k}^{n-k_p} T^{(k-p)}_{p,j} x_j \right),$$

where $k_p = k-1-p$ for all p and $\lambda_{p,q} = \begin{cases} \lambda_{k-q+1} \cdots \lambda_{k-p} & \text{if } p > 0 \\ \lambda_{k-q+1} \cdots \lambda_{k-1} & \text{if } p = 0. \end{cases}$

Note that our convention implies $\lambda_{0,1} = 1$. Extend Ξ to U_{k-1} by mapping

$$\begin{aligned} (\mathbf{a}, \mathbf{b}, \mathbf{T}^{(1)}, \dots, \mathbf{T}^{(k)}) \mapsto & \left[I_{\Lambda(\mathbf{a})}(y_0, \dots, y_{k-1}) + (\beta_{p,q})_{0 \le p < q \le k-1} \right] \\ &= \left[\left(x_i + \sum_{j=k}^{n} a_{i,j} \right)_{0 \le i \le k-1} \left(x_i + \sum_{j=k}^{n} b_{i,j} \right)_{0 \le i \le k-1} + \left(\beta_{p,q} \right)_{0 \le p < q \le k-1} \right]. \end{aligned} \tag{3.6}$$

Note that Eq. (3.6) extends the original rational map given in Eq. (3.5). Indeed, Eq. (3.5) is defined away from the strict transform of all the the exceptional divisors; this is the locus where $\lambda_1, \dots, \lambda_{k-1} \ne 0$. In this case we have

$$(y_0, \dots, y_{k-1}) I_{\Lambda(\mathbf{a})} \ni \left(y_p - \sum_{j=1}^{k_p} T^{(j)}_{p,n-j+1} y_{k-j} \right) \alpha_q - \left(y_q - \sum_{j=1}^{k_q} T^{(j)}_{q,n-j+1} y_{k-j} \right) \alpha_p = \lambda_1 \cdots \lambda_{k-q} \beta_{p,q}. \tag{3.7}$$

Thus $\beta_{p,q} \in I_{\Lambda(\mathbf{a})}(y_0, \dots, y_{k-1})$ and Eq. (3.5) and Eq. (3.6) coincide.

To show that the image of Eq. (3.6) is well defined, it is enough to show that the Hilbert polynomial of an ideal $J = I_{\Lambda(\mathbf{a})}I_{\Lambda(\mathbf{b})} + (\beta_{p,q})_{0\le p<q\le k-1}$ in this image is $P^n_{n-k,n-k}(t)$. In Lemma 3.2.5 we define a term order $>$ on S for which

$$\mathrm{in}_> J = (x_0, \dots, x_{k-1})^2 + (x_p x_{n-k_q})_{0\le p<q\le k-1}.$$

Since there is a flat degeneration from J to $\mathrm{in}_> J$ it suffices to show $\mathrm{in}_> J$ has the desired Hilbert polynomial. It is easy to see that $(S/\mathrm{in}_> J)_t$ is spanned by

$$\bigoplus_{i=0}^{k-1} x_i \mathbf{k}[x_k, \dots, x_{n-k+i+1}]_{t-1} \oplus \mathbf{k}[x_k, \dots, x_n]_t.$$

Using this and the Hockey-Stick identity we deduce that Hilbert polynomial of $S/\mathrm{in}_> J$ is

$$\binom{n-k+t}{t} + \sum_{i=0}^{k-1}\binom{n-2k+i+t}{t-1} = \binom{n-k+t}{t} + \binom{n-k+t}{t} - \binom{n-2k+t}{t} = P^n_{n-k,n-k}(t).$$

□

Prior to proving Lemma 3.2.5 we need the following auxiliary result.

Lemma 3.2.4. *The ideal $I_{\Lambda(\mathbf{a})}I_{\Lambda(\mathbf{b})} + (\beta_{p,q})_{0\le p<q\le k-1}$ in the image of Eq. (3.6) is projectively equivalent to an ideal of the form*

$$(x_p + \mu_{p,k}x_{n-k_p})_{0\le p\le k-1}(x_0, \dots, x_{k-1}) + (x_p x_{n-k_q} - \mu_{p,q}x_q x_{n-k_p})_{0\le p<q\le k-1}, \tag{3.8}$$

with $\mu_i \in \mathbf{k}$ and $\mu_{p,q} = \mu_{k-q+1}\cdots\mu_{k-p}$ for any $0 \le p < q \le k$.

Proof. Applying the projective transformation that maps $x_i \mapsto x_i - \sum_{j\ge k} b_{i,j}x_j$ if $i \le k-1$ and fixes the other x_i, we may assume $\mathbf{b} = \mathbf{0}$. For each $0 \le i \le k-1$ let τ_i denote the map that sends $x_i \mapsto x_i + \sum_{j=1}^{k-i-1} T^{(j)}_{i,n-j+1}x_{k-j}$ and fixes the other i. It is clear that $\tau_{k-1}\circ\cdots\circ\tau_0(I)$ equals,

$$\left(x_p + \sum_{j=k}^{n-k_p}\lambda_1\cdots\lambda_{k-p}T^{(k-p)}_{p,j}x_j\right)_{0\le p\le k-1}(x_0, \dots, x_{k-1}) + \left(x_p\left(\sum_{j=k}^{n-k_q}T^{(k-q)}_{q,j}x_j\right) - \lambda_{p,q}x_q\left(\sum_{j=k}^{n-k_p}T^{(k-p)}_{p,j}x_j\right)\right)_{p<q}$$

For each $0 \le i \le k-1$ let $\mu_i = \lambda_i$. If $T^{(k)}_{0,j} = 0$ for all j then let $\mu_k = 0$. If not, choose the largest index ℓ for which $T^{(k)}_{0,\ell} \neq 0$ and let $\mu_k = T^{(k)}_{0,\ell}$.

For each $1 \le i \le k-1$ consider the map τ_{n-k_i}, that maps $x_{n-k_i} \mapsto x_{n-k_i} - \sum_{j=k}^{n-k_i-1} T^{(k-i)}_{i,j}x_j$ and fixes the other x_i. As we range over all i, we obtain maps $\tau_n, \dots, \tau_{n-(k-2)}$. If $\mu_k = 0$ let

$\tau_{n-(k-1)}$ be the identity; else let $\tau_{n-(k-1)}$ denote the map that sends $x_\ell \mapsto x_{n-k_0} - \frac{1}{\mu_k}\sum_{j=k}^{\ell-1} T_{0,j}^{(k)}$, $x_{n-k_0} \mapsto x_\ell$ if $\ell < n - k_0$, and fixes the other x_i.

Using the fact that $T_{i,n-k_i}^{(k-i)} = 1$ on the open set U_{k-1}, it is straightforward to check that $\tau_{n-(k-1)} \circ \cdots \tau_n \circ \tau_{k-1} \circ \cdots \circ \tau_0(I)$ is of the desired form. □

Lemma 3.2.5. *Let $>$ denote the lexicographic ordering on S with terms ordered by $x_0 > x_1 > \cdots > x_{k-1} > x_n > x_{n-1} > \cdots > x_k$. Let $J = I_{\Lambda(\mathbf{a})} I_{\Lambda(\mathbf{b})} + (\beta_{p,q})_{0 \le p < q \le k-1}$ denote the ideal in the image of Eq. (3.6). Then we have*

$$\mathrm{in}_> J = (x_0, \ldots, x_{k-1})^2 + (x_p x_{n-k_q})_{0 \le p < q \le k-1}$$

Proof. Let J' denote the ideal in Eq. (3.8). We will first show that

$$\mathrm{in}_> J' = (x_0, \ldots, x_{k-1})^2 + (x_p x_{n-k_q})_{0 \le p < q \le k-1}. \tag{3.9}$$

Let $\gamma_{p,q} = (x_p + \mu_{p,k} x_{n-k_p}) x_q$ for $0 \le p \le q \le k-1$ and $\delta_{p,q} = x_p x_{n-k_q} - \mu_{p,q} x_q x_{n-k_p}$ for $0 \le p < q \le k-1$. Since $\mathrm{in}_> \gamma_{p,q} = x_p x_q$ and $\mathrm{in}_> \delta_{p,q} = x_p x_{n-k_q}$, to prove Eq. (3.9), it is enough to show that $G = \{\gamma_{p,q}, \delta_{p,q}\}_{p,q}$ is a Gröbner basis for J'. Note that G generates J' because for $p < q$ we have

$$\begin{aligned}(x_q + \mu_{q,k} x_{n-k_q}) x_p &= (x_p + \mu_{p,k} x_{n-k_p}) x_q + \mu_{q,k}(x_p x_{n-k_q} - \mu_{p,q} x_q x_{n-k_p}) \\ &= \gamma_{p,q} + \mu_{q,k}\delta_{p,q} \in (G).\end{aligned} \tag{3.10}$$

Notice that $\mu_{p,q}\mu_{q,k} = \mu_{p,k}$ and this will be used repeatedly in the rest of the proof.

Given $a, b \in S$ we denote their *S-pair* by $R(a,b) = (\frac{\mathrm{in}_> b}{h})a - (\frac{\mathrm{in}_> a}{h})b$ with $h = \gcd(\mathrm{in}_>(a), \mathrm{in}_>(b))$. To show that G forms a Gröbner basis we need to show that there is a *standard expression* for the S-pairs in terms of elements of G with no remainder [50, Section 2.2-2.3].

Case 1. The standard expression of $R(\gamma_{p_1,q_1}, \gamma_{p_2,q_2})$: Let $h = \gcd(\mathrm{in}_> \gamma_{p_1,q_1}, \mathrm{in}_> \gamma_{p_2,q_2})$ and we may assume $p_1 \le p_2$. If $h = 1$ then $p_1 < p_2$ and we have

$$\begin{aligned}R(\gamma_{p_1,q_1}, \gamma_{p_2,q_2}) &= x_{p_2} x_{q_2} \gamma_{p_1,q_1} - x_{p_1} x_{q_1} \gamma_{p_2,q_2} \\ &= \mu_{p_1,k} x_{p_2} x_{q_2} x_{n-k_{p_1}} x_{q_1} - \mu_{p_2,k} x_{p_1} x_{q_1} x_{n-k_{p_2}} x_{q_2} \\ &= -\mu_{p_2,k} x_{q_1} x_{q_2} \delta_{p_1,p_2}.\end{aligned}$$

This is obviously a standard expression with no remainder. If $h = x_{p_1}$ then $p_1 = p_2$ or $p_1 = q_2$; in the latter case we still have $p_1 = p_2$ as our assumptions imply $p_1 \le p_2 \le q_2$. Thus in both the situations we obtain $R(\gamma_{p_1,q_1}, \gamma_{p_2,q_2}) = x_{q_2}\gamma_{p_1,q_1} - x_{q_1}\gamma_{p_1,q_2} = 0$. If $h = x_{q_1}$ we have either $q_1 = q_2$ or $q_1 = p_2$. If $q_1 = q_2$ then as shown above we obtain

$$R(\gamma_{p_1,q_1}, \gamma_{p_2,q_2}) = x_{p_2}\gamma_{p_1,q_1} - x_{p_1}\gamma_{p_2,q_1} = \mu_{p_1,k} x_{p_2} x_{n-k_{p_1}} x_{q_1} - \mu_{p_2,k} x_{p_1} x_{n-k_{p_2}} x_{q_1} = -\mu_{p_2,k} x_{q_1} \delta_{p_1,p_2}.$$

Similarly, if $q_1 = p_2$ we obtain $R(\gamma_{p_1,q_1}, \gamma_{p_2,q_2}) = x_{q_2}\gamma_{p_1,p_2} - x_{p_1}\gamma_{p_2,q_2} = -\mu_{p_2,k} x_{q_2} \delta_{p_1,p_2}$ (if $p_1 = p_2$ this is just 0). If $h = x_{p_1} x_{q_1}$ then we have $p_1 = q_1 = p_2 = q_2$ or $p_1 = p_2 < q_1 = q_2$; in either case $R(\gamma_{p_1,q_1}, \gamma_{p_2,q_2}) = 0$.

Case 2. The standard expression of $R(\delta_{p_1,q_1}, \delta_{p_2,q_2})$: Let $h = \gcd(\mathrm{in}_> \delta_{p_1,q_1}, \mathrm{in}_> \delta_{p_2,q_2})$ and assume $p_1 \leq p_2$. If $h = 1$ we have $p_1 < p_2$ and $q_1 \neq q_2$. Then we obtain

$$\begin{aligned} R(\delta_{p_1,q_1}, \delta_{p_2,q_2}) &= x_{p_2} x_{n-k_{q_2}} \delta_{p_1,q_1} - x_{p_1} x_{n-k_{q_1}} \delta_{p_2,q_2} \\ &= -\mu_{p_1,q_1} x_{p_2} x_{n-k_{q_2}} x_{q_1} x_{n-k_{p_1}} + \mu_{p_2,q_2} x_{p_1} x_{n-k_{q_1}} x_{q_2} x_{n-k_{p_2}} \\ &= \mu_{p_2,q_2} x_{q_2} x_{n-k_{q_1}} \delta_{p_1,p_2} - x_{p_2} x_{n-k_{p_1}} (\mu_{p_1,q_1} x_{q_1} x_{n-k_{q_2}} - \mu_{p_1,p_2} \mu_{p_2,q_2} x_{q_2} x_{n-k_{q_1}}) \\ &= \begin{cases} \mu_{p_2,q_2} x_{q_2} x_{n-k_{q_1}} \delta_{p_1,p_2} - \mu_{p_1,q_1} x_{p_2} x_{n-k_{p_1}} \delta_{q_1,q_2} & \text{if } q_1 < q_2 \\ \mu_{p_2,q_2} x_{q_2} x_{n-k_{q_1}} \delta_{p_1,p_2} + \mu_{p_1,q_2} x_{p_2} x_{n-k_{p_1}} \delta_{q_2,q_1} & \text{if } q_2 < q_1. \end{cases} \end{aligned}$$

Each of the above cases is a standard expression in terms of G with no remainder [1]. If $h = x_{n-k_{q_1}}$ we have $q_1 = q_2$ and $p_1 < p_2$. Then we obtain

$$\begin{aligned} R(\delta_{p_1,q_1}, \delta_{p_2,q_2}) &= x_{p_2} \delta_{p_1,q_2} - x_{p_1} \delta_{p_2,q_2} \\ &= -\mu_{p_1,q_2} x_{p_2} x_{q_2} x_{n-k_{p_1}} + \mu_{p_2,q_2} x_{p_1} x_{q_2} x_{n-k_{p_2}} \\ &= \mu_{p_2,q_2} x_{q_2} \delta_{p_1,p_2}. \end{aligned}$$

If $h = x_{p_1}$ we have $p_1 = p_2$ and wlog we may assume $q_1 < q_2$. Then we have

$$\begin{aligned} R(\delta_{p_1,q_1}, \delta_{p_2,q_2}) &= x_{n-k_{q_2}} \delta_{p_1,q_1} - x_{n-k_{q_1}} \delta_{p_1,q_2} \\ &= -\mu_{p_1,q_1} x_{n-q_2} x_{q_1} x_{n-k_{p_1}} + \mu_{p_1,q_2} x_{n-k_{q_1}} x_{q_2} x_{n-k_{p_1}} \\ &= -\mu_{p_1,q_1} x_{n-k_{p_1}} \delta_{q_1,q_2}. \end{aligned}$$

Finally if $h = x_{p_1} x_{n-k_{q_1}}$ we have $p_1 = p_2 < q_1 = q_2$ and thus $R(\delta_{p_1,q_1}, \delta_{p_2,q_2}) = 0$.

Case 3. The standard expression of $R(\gamma_{p_1,q_1}, \delta_{p_2,q_2})$: Let $h = \gcd(\mathrm{in}_> \gamma_{p_1,q_1}, \mathrm{in}_> \delta_{p_2,q_2})$ and note that $h \in \{1, x_{p_1}, x_{q_1}\}$. If $h = x_{p_1}$ we have $p_1 = p_2$ and using Eq. (3.10) we obtain

$$\begin{aligned} R(\gamma_{p_1,q_1}, \delta_{p_2,q_2}) &= x_{n-k_{q_2}} \gamma_{p_1,q_1} - x_{q_1} \delta_{p_1,q_2} \\ &= \mu_{p_1,k} x_{n-k_{q_2}} x_{n-k_{p_1}} x_{q_1} + \mu_{p_1,q_2} x_{q_1} x_{n-k_{p_1}} x_{q_2} \\ &= \begin{cases} \mu_{p_1,q_2} x_{n-k_{p_1}} \gamma_{q_2,q_1} & \text{if } q_1 \geq q_2 \\ \mu_{p_1,q_2} x_{n-k_{p_1}} \gamma_{q_1,q_2} + \mu_{p_1,k} x_{n-k_{p_1}} \delta_{q_1,q_2} & \text{if } q_1 < q_2. \end{cases} \end{aligned}$$

Both these cases are standard expressions with no remainder. If $h = x_{q_1}$ then $q_1 = p_2$ and we obtain,

$$\begin{aligned} R(\gamma_{p_1,q_1}, \delta_{p_2,q_2}) &= x_{n-k_{q_2}} \gamma_{p_1,p_2} - x_{p_1} \delta_{p_2,q_2} \\ &= \mu_{p_1,k} x_{n-k_{q_2}} x_{n-k_{p_1}} x_{p_2} + \mu_{p_2,q_2} x_{p_1} x_{n-k_{p_2}} x_{q_2} \\ &= x_{n-k_{q_2}} \gamma_{p_1,q_2} - x_{p_1} \delta_{p_2,q_2}. \end{aligned}$$

[1] If $\mu_{p_2,q_2} \neq 0$ then $\mathrm{in}_> R(\delta_{p_1,q_1}, \delta_{p_2,q_2}) = \mu_{p_2,q_2} x_{p_1} x_{n-k_{q_1}} x_{q_2} x_{n-k_{p_2}}$. This is greater or equal to $\mathrm{in}_>(x_{q_2} x_{n-k_{q_1}} \delta_{p_1,p_2})$ and $\mathrm{in}_>(x_{p_2} x_{n-k_{p_1}} \delta_{q_1,q_2})$.

Finally consider the case $h = 1$. If we further assume $p_2 < p_1$ and $q_2 < p_1$ we have

$$\begin{aligned} R(\gamma_{p_1,q_1}, \delta_{p_2,q_2}) &= x_{p_2} x_{n-k_{q_2}} \gamma_{p_1,q_1} - x_{p_1} x_{q_1} \delta_{p_2,q_2} \\ &= \mu_{p_1,k} x_{p_2} x_{n-k_{q_2}} x_{n-k_{p_1}} x_{q_1} + \mu_{p_2,q_2} x_{p_1} x_{q_1} x_{q_2} x_{n-k_{p_2}} \\ &= \mu_{p_1,k} x_{n-k_{q_2}} x_{q_1} \delta_{p_2,p_1} + \mu_{p_2,k} x_{n-k_{q_2}} x_{q_1} x_{p_1} x_{n-k_{p_2}} + \mu_{p_2,q_2} x_{p_1} x_{q_1} x_{q_2} x_{n-k_{p_2}} \\ &= \mu_{p_1,k} x_{n-k_{q_2}} x_{p_2} \delta_{p_2,p_1} + \mu_{p_2,q_2} x_{n-k_{p_2}} x_{q_1} \gamma_{q_2,p_1} \end{aligned}$$

This is a standard expression with no remainder. We omit the other cases as their proofs use Eq. (3.10) and are very similar. We have now shown that G is a Gröbner basis for J'.

Since J' and $\mathrm{in}_> J'$ have the same Hilbert function (as graded S-modules) and J is projectively equivalent to J', J and $\mathrm{in}_> J'$ have the same Hilbert function. On the other hand, $(x_0, \ldots, x_{k-1})^2 \subseteq \mathrm{in}_> J$ and $x_p x_{n-k_q} = \mathrm{in}_>(\beta_{p,q}) \in \mathrm{in}_> J$. Thus $\mathrm{in}_> J \supseteq \mathrm{in}_> J'$. Since these ideals have the same Hilbert function they must be equal, completing the proof. □

Remark 3.2.6. For the rest of the paper, $>$ will always denote the term order from Lemma 3.2.5 and k_p will always denote $k - 1 - p$.

The following Lemma sheds some light on the structure of the subschemes in the image of the morphism, $U_{k-1} \longrightarrow \mathcal{H}^n_{n-k,n-k}$.

Lemma 3.2.7. *Let $J = I_{\Lambda(\mathbf{a})} I_{\Lambda(\mathbf{b})} + (\beta_{p,q})_{0 \le p < q \le k-1}$ denote the ideal in the image of the morphism given by Eq. (3.6). Then the following statements are true*

(i) *The ideal J is saturated.*

(ii) *If all the λ_i are non-zero and $\mathbf{T}^{(k)} \neq \mathbf{0}$ then J is the ideal of a pair of $(n-k)$-planes meeting transversely.*

(iii) *If all the λ_i are non-zero and $\mathbf{T}^{(k)} = \mathbf{0}$ then $\sqrt{J}$ is the ideal of a pair of $(n-k)$-planes meeting along an $(n - 2k + 1)$-plane.*

(iv) *Let ℓ be the smallest index for which $\lambda_\ell = 0$. Then we have*

$$J = I_{\Lambda(\mathbf{a})} I_{\Lambda(\mathbf{b})} + (\beta_{p,q})_{0 \le p < q \le k-\ell}$$

and $\sqrt{J}$ is the ideal of a pair of $(n-k)$-planes meeting along an $(n-k+1-\ell)$-plane.

Proof. Item (i) follows from the fact that $\mathrm{depth}_{\mathfrak{m}}(S/J) \ge \mathrm{depth}_{\mathfrak{m}}(S/\mathrm{in}_> J) \ge 1$ where $\mathfrak{m} = (x_0, \ldots, x_n)$. The first inequality is [50, Theorem 3.3.4] and the second inequality is true because x_k is a non-zero divisor on $S/\mathrm{in}_> J$.

Notice that $\Lambda(\mathbf{a})$ and $\Lambda(\mathbf{b})$ meet along a $(n-k+1-\ell)$-plane precisely when the matrix M (Proposition 3.2.1 (ii)) has rank $\ell - 1$. As a consequence items (ii), (iii) and the second half of (iv) follow immediately. The other half of item (iv) follows from Eq. (3.7) as it shows $\beta_{p,q} \in I_{\Lambda(\mathbf{a})} I_{\Lambda(\mathbf{b})}$ for any $q > k - \ell$. □

Proposition 3.2.8. *Let $n \geq 2k-1$. Then Ξ induces a surjective, $\mathrm{GL}(n+1)$-equivariant morphism*

$$\bar{\Xi} : \mathcal{X}_{k-1}/\mathfrak{S}_2 \simeq \mathrm{Bl}_{\Gamma_{k-1}} \cdots \mathrm{Bl}_{\Gamma_1} \mathrm{Sym}^2 \mathbf{Gr}(n-k, n) \longrightarrow \mathcal{H}^n_{n-k,n-k}.$$

Moreover, the quotient $\mathcal{X}_{k-1}/\mathfrak{S}_2$ is non-singular.

Proof. In Proposition 3.2.3 we showed that Ξ extends to a map from U_{k-1}. We will now explain how the same argument gives a morphism on all of $\pi_{k-1}^{-1}(U_0)$. Consider a pair

$$\gamma = (\gamma^1, \gamma^2) = ((\gamma^1_1, \ldots, \gamma^1_k), (\gamma^2_1, \ldots, \gamma^2_{k-1}))$$

with γ^1 an ordered k-subset of $\{0, \ldots, k-1\}$ and γ^2 an ordered $(k-1)$-subset of $\{k, \ldots, n\}$. For any such γ we can define a sequence of open sets $U^\gamma_1, \ldots, U^\gamma_{k-1}$ such that

(1) $U^\gamma_1 = D(T^{(1)}_{\gamma^1_1,\gamma^2_1}) \subseteq \mathrm{Bl}_{\Gamma_1 \cap U_0} U_0$ and let $T^{\gamma,(1)}_{i,j} = T^{(1)}_{i,j}$.

(2) For $v \geq 1$, the strict transform of Γ_{v+1} on U^γ_v is cut out by

$$\left(T^{\gamma,(v)}_{i,j} - T^{\gamma,(v)}_{i,\gamma^2_v} T^{\gamma,(v)}_{\gamma^1_v,j}\right)^{i\in\{0,\ldots,k-1\}\setminus\{\gamma^1_1,\ldots,\gamma^1_v\}}_{j\in\{k,\ldots,n\}\setminus\{\gamma^2_1,\ldots,\gamma^2_v\}}$$

(3) For $v \geq 1$, the locus $\Gamma_{v+1} \cap U^\gamma_v$ is non-singular and

$$\mathrm{Bl}_{\Gamma_{v+1}\cap U^\gamma_v} U^\gamma_v \simeq \mathrm{Proj}\, \mathbf{k}[U^\gamma_v][T^{\gamma,(v)}_{i,j}]_{i,j}/(\text{Koszul Relations}).$$

(4) For $v \geq 1$, we have $U^\gamma_v = D(T^{\gamma,(v)}_{\gamma^1_v,\gamma^2_v}) \subseteq \mathrm{Bl}_{\Gamma_v \cap U^\gamma_{v-1}} U^\gamma_{v-1}$.

Due to symmetry, the proof of Proposition 3.2.1 also establishes the above statements (note that $U_{k-1} = U^\gamma_{k-1}$ with $\gamma^1 = (k-1, k-2, \ldots, 0)$ and $\gamma^2 = (n, n-1, \ldots, n-k+2)$). It follows that $\{U^\gamma_{k-1}\}_\gamma$ is an affine cover of $\pi^{-1}_{k-1}(U_0)$ with the natural gluing maps. We omit an explicit description of the gluing maps as they will never be used.

To construct the U^γ_v and verify statement (2), we would have to row reduce M in a way analogous to Proposition 3.2.1 (each γ corresponds to a different sequence of row redutions). We will omit an explicit description of the matrix, but the corresponding lambdas are

$$\lambda^\gamma_1 = a_{\gamma^1_1,\gamma^2_1} - b_{\gamma^1_1,\gamma^2_1} \quad \text{and} \quad \lambda^\gamma_i = T^{\gamma,(i-1)}_{\gamma^1_i,\gamma^2_i} - T^{\gamma,(i-1)}_{\gamma^1_i,\gamma^2_{i-1}} T^{\gamma,(i-1)}_{\gamma^1_{i-1},\gamma^2_i} \quad \text{for each } 2 \leq i \leq k-1.$$

As in the proof of Proposition 3.2.3 we can choose a minimal generating set, $\alpha^\gamma_0, \ldots, \alpha^\gamma_{k-1}$ of $I_{\Lambda(\mathbf{a})}$ where

$$\alpha^\gamma_p = y_{\gamma^1_{k-p}} - \sum_{j=1}^{k-1-p} T^{\gamma,(j)}_{\gamma^1_{k-p},\gamma^2_j} y_{\gamma^1_j} + \sum_{j\in\{k,\ldots,n\}\setminus\{\gamma^2_1,\ldots,\gamma^2_{k-1-p}\}} \lambda^\gamma_1 \cdots \lambda^\gamma_{k-p} T^{\gamma,(k-p)}_{\gamma^1_{k-p},j} x_j$$

for $0 < p \leq k-1$ and

$$\alpha_0^\gamma = y_{\gamma_k^1} - \sum_{j=1}^{k-1} T^{\gamma,(j)}_{\gamma_k^1,\gamma_j^2} y_{\gamma_j^1} + \sum_{j \in \{k,\dots,n\} \setminus \{\gamma_1^2,\dots,\gamma_{k-1}^2\}} \lambda_1^\gamma \cdots \lambda_{k-1}^\gamma T^{\gamma,(k)}_{\gamma_k^1,j} x_j$$

with $T^{\gamma,(k)}_{\gamma_k^1,j} = T^{\gamma,(k-1)}_{\gamma_k^1,j} - T^{\gamma,(k-1)}_{\gamma_k^1,\gamma_{k-1}^2} T^{\gamma,(k-1)}_{\gamma_{k-1}^1,j}$.

For $0 \leq p < q \leq k-1$ we may define analogous "cross terms"

$$\begin{aligned}\beta_{p,q}^\gamma = &\left(y_{\gamma_{k-p}^1} - \sum_{j=1}^{k-1-p} T^{\gamma,(j)}_{\gamma_{k-p}^1,\gamma_j^2} y_{\gamma_j^1} \right) \left(\sum_{j \in \{k,\dots,n\} \setminus \{\gamma_1^2,\dots,\gamma_{k-1-q}^2\}} T^{\gamma,(k-q)}_{\gamma_{k-q}^1,j} x_j \right) \\ &- \lambda_{p,q}^\gamma \left(y_{\gamma_{k-q}^1} - \sum_{j=1}^{k-1-q} T^{\gamma,(j)}_{\gamma_{k-q}^1,\gamma_j^2} y_{\gamma_j^1} \right) \left(\sum_{j \in \{k,\dots,n\} \setminus \{\gamma_1^2,\dots,\gamma_{k-1-p}^2\}} T^{\gamma,(k-p)}_{\gamma_{k-p}^1,j} x_j \right).\end{aligned}$$

Thus we obtain a morphism

$$\Xi_{U_{k-1}^\gamma} : (\mathbf{a}, \mathbf{b}, \mathbf{T}^{\gamma,(1)}, \dots, \mathbf{T}^{\gamma,(k)}) \mapsto \left[I_{\Lambda(\mathbf{a})} I_{\Lambda(\mathbf{b})} + (\beta_{p,q}^\gamma)_{0 \leq p < q \leq k-1} \right]. \tag{3.11}$$

This is well defined as any ideal in the image of $\Xi_{U_{k-1}^\gamma}$ is still projectively equivalent to an ideal in Eq. (3.8) (the proof of Lemma 3.2.4 works with straightforward modifications). As explained in Proposition 3.2.3, $\Xi_{U_{k-1}^\gamma}$ will also extend the original rational map given by Eq. (3.5), for each γ. Thus for any γ, γ', $\Xi_{U_{k-1}^\gamma}$ and $\Xi_{U_{k-1}^{\gamma'}}$ agree on an open subset of $U_{k-1}^\gamma \cap U_{k-1}^{\gamma'}$. By uniqueness of extensions, they will agree on all of $U_{k-1}^\gamma \cap U_{k-1}^{\gamma'}$. Gluing all these maps gives us a morphism $\pi_{k-1}^{-1}(U_0) \longrightarrow \mathcal{H}^n_{n-k,n-k}$.

As mentioned in the beginning of the section, $\mathbf{Gr}(n-k,n)^2$ is covered by open sets of the form $U_\mathcal{E}$ where $\mathcal{E}$ ranges over all ordered bases of S_1. Since assuming $\mathcal{E} = \{x_0, \dots, x_n\}$ was purely notational, all the discussion in this section applies verbatim to $\pi_{k-1}^{-1}(U_\mathcal{E})$. In particular, we obtain a morphism on each $\pi_{k-1}^{-1}(U_\mathcal{E})$ that extends the original rational map given by Eq. (3.5). Thus we can glue all these maps to obtain a morphism $\Xi : \mathcal{X}_{k-1} \longrightarrow \mathcal{H}^n_{n-k,n-k}$.

Let $\mathfrak{S}_2 = \{1, g\}$ be the group on two elements and consider its natural on $\mathbf{Gr}(n-k,n)^2$ given by interchanging the two factors. Since each of the Γ_i are $\mathfrak{S}_2$ stable, the action extends to the blowup $\mathcal{X}_{k-1}$. If we consider the trivial action of $\mathfrak{S}_2$ on $\mathcal{H}^n_{n-k,n-k}$, then our construction shows that Ξ is $\mathfrak{S}_2$-equivariant. Thus, we get an induced morphism $\overline{\Xi} : \mathcal{X}_{k-1}/\mathfrak{S}_2 \longrightarrow \mathcal{H}^n_{n-k,n-k}$.

Since char $\mathbf{k} \neq 2$ and g fixes a divisor (the strict transform of the exceptional divisor of $\mathcal{X}_1$), the Chevalley-Shephard-Todd theorem [74, Theorem 7.14] implies that the quotient is non-singular. Note that

$$\mathcal{X}_{k-1}/\mathfrak{S}_2 = (\mathrm{Bl}_{\Gamma_{k-1}} \cdots \mathrm{Bl}_{\Gamma_1} \mathbf{Gr}(n-k,n)^2)/\mathfrak{S}_2 \simeq \mathrm{Bl}_{\bar{\Gamma}_{k-1}} \cdots \mathrm{Bl}_{\bar{\Gamma}_1} \mathrm{Sym}^2 \mathbf{Gr}(n-k,n).$$

Since Ξ is dominant and $\mathcal{X}_{k-1}$ is projective, $\bar{\Xi}$ is surjective.

The natural action of $\mathrm{GL}(n+1)$ on $\mathbf{P}^n$ induces an action on $\mathbf{Gr}(n-k,n)^2$ and on $\mathcal{H}^n_{n-k,n-k}$. Since the Γ_i are stable under this action, it extends to an action on $\mathcal{X}_{k-1}$. To show that Ξ is $\mathrm{GL}(n+1)$-equivariant we need to show that for any $g \in \mathrm{GL}(n+1)$ the two morphisms, $\Xi \circ g : \mathcal{X}_{k-1} \to \mathcal{H}^n_{n-k,n-k}$ given by $w \mapsto \Xi(gw)$ and $g \circ \Xi : \mathcal{X}_{k-1} \to \mathcal{H}^n_{n-k,n-k}$ given by $w \mapsto g\Xi(w)$ are identical. For any (Λ, Λ') in the open set $\mathbf{Gr}(n-k,n)^2 \setminus \Gamma_k \subseteq \mathcal{X}_{k-1}$ we have

$$(\Xi \circ g)(\Lambda, \Lambda') = \Xi(g(\Lambda), g(\Lambda')) = g(\Lambda) \cup g(\Lambda') = g(\Lambda \cup \Lambda') = (g \circ \Xi)(\Lambda, \Lambda').$$

Thus $\Xi \circ g$ and $g \circ \Xi$ must agree on all of $\mathcal{X}_{k-1}$. It follows that $\bar{\Xi}$ is also $\mathrm{GL}(n+1)$-equivariant. □

Corollary 3.2.9. *Let $n \geq 2k-1$. Any subscheme parameterized by $\mathcal{H}^n_{n-k,n-k}$ is minimally cut out by k^2 quadrics.*

Proof. By the discussion in Proposition 3.2.8 we may reduce to considering subschemes cut out by ideals in the image of morphism (Eq. (3.6)). Let J denote any such ideal and note that J, as presented, is generated by quadrics. By Lemma 3.2.7 (i), J is saturated and thus is the ideal of its corresponding subscheme. Therefore it suffices to show that $\dim_{\mathbf{k}} J_2 = k^2$. Since S/J and $S/\mathrm{in}_> J$ have the same Hilbert function we have $\dim_{\mathbf{k}} J_2 = \dim_{\mathbf{k}} (\mathrm{in}_> J)_2 = k^2$ (Lemma 3.2.5). □

Remark 3.2.10. The analogue of Lemma 3.2.7 holds verbatim for ideals in the image of Eq. (3.11). The analogue of Lemma 3.2.5 is as follows: Let J be any ideal in the image of Eq. (3.11) and let $>_\gamma$ denote a lexicographic ordering on S for which

$$x_{\gamma^1_k} > x_{\gamma^1_{k-1}} > \cdots > x_{\gamma^1_1} > x_{\gamma^2_1} > \cdots > x_{\gamma^2_{k-1}} > x_{h_1} > \cdots > x_{h_{n-2k+2}}.$$

We may choose any h_i so that $\{h_1, \ldots, h_{n-2k+2}\} = \{k, \ldots, n\} \setminus \{\gamma^2_1, \ldots, \gamma^2_{k-1}\}$. Then we have

$$\mathrm{in}_{>_\gamma} J = (x_0, \ldots, x_{k-1})^2 + (x_{\gamma^1_{k-p}} x_{\gamma^2_{k-q}})_{0 \leq p < q \leq k-1}$$

3.3 An analysis of Ξ

We split the proof of the injectivity of $\bar{\Xi}$ into two steps. Here is the first step.

Lemma 3.3.1. *For any γ, the restriction $\bar{\Xi} : U^\gamma_{k-1}/\mathfrak{S}_2 \longrightarrow \mathcal{H}^n_{n-k,n-k}$ is injective.*

Proof. It is evident from our construction that U_{k-1}^{γ} is $\mathfrak{S}_2$-stable and thus the quotient $U_{k-1}^{\gamma}/\mathfrak{S}_2$ is well defined. Without loss of generality we may assume $U_{k-1}^{\gamma} = U_{k-1}$. To prove the Lemma it suffices to show that for any $\tilde{Z}, \hat{Z} \in U_{k-1}$ satisfying $\Xi(\tilde{Z}) = \Xi(\hat{Z})$, we have $\tilde{Z} = \hat{Z}$ or $g(\tilde{Z}) = \hat{Z}$ where where g is the non-identity of $\mathfrak{S}_2$. Let $\tilde{Z} = (\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{T}}^{(1)}, \ldots, \tilde{\mathbf{T}}^{(k)})$ and $\hat{Z} = (\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{T}}^{(1)}, \ldots, \hat{\mathbf{T}}^{(k)})$ be their coordinates on U_{k-1}. The "betas" and "lambdas" corresponding to $\tilde{Z}$ are denoted by $\tilde{\beta}_{i,j}$ and $\tilde{\lambda}_i$ respectively, and the ones corresponding to $\hat{Z}$ are denoted by $\hat{\beta}_{i,j}$ and $\hat{\lambda}_i$.

We have $\Lambda(\tilde{\mathbf{a}}) \cup \Lambda(\tilde{\mathbf{b}}) = \Xi(\tilde{Z})_{\mathrm{red}} = \Xi(\hat{Z})_{\mathrm{red}} = \Lambda(\hat{\mathbf{a}}) \cup \Lambda(\hat{\mathbf{b}})$. After possibly replacing $\tilde{Z}, \hat{Z}$ by $g(\tilde{Z}), g(\hat{Z})$ respectively, we may assume $\tilde{\mathbf{a}} = \hat{\mathbf{a}}$ and $\tilde{\mathbf{b}} = \hat{\mathbf{b}}$. Thus to prove that $\bar{\Xi}$ is injective, we need to now show that $\tilde{Z} = \hat{Z}$. Since Ξ is $\mathrm{GL}(n+1)$-equivariant we may apply a projective transformation and assume $\tilde{\mathbf{b}} = \hat{\mathbf{b}} = \mathbf{0}$. For simplicity we let $\mathbf{a} := \tilde{\mathbf{a}} = \hat{\mathbf{a}}$.

By Lemma 3.2.7, $\Xi(\tilde{Z})_{\mathrm{red}} = \Xi(\hat{Z})_{\mathrm{red}}$ is a pair of $(n-k)$-planes meeting along an $(n-k+1-\ell)$-plane for some $1 \le \ell \le k+1$. If $\ell \in \{k, k+1\}$ then $\widetilde{Z}, \widehat{Z}$ lie in an open set along which Ξ was already shown to be two-to-one (Lemma 3.1.4). Thus we may assume $\ell \le k-1$. By Lemma 3.2.7 it is also the smallest index for which $\tilde{\lambda}_\ell = 0$ and, symmetrically, the smallest index for which $\hat{\lambda}_\ell = 0$.

Using Lemma 3.2.7 (iv) we get $\Xi(\tilde{Z}) = [I_{\Lambda(\mathbf{a})} I_{\Lambda(\mathbf{0})} + (\tilde{\beta}_{p,q})_{0 \le p < q \le k-\ell}]$ and $\Xi(\hat{Z}) = [I_{\Lambda(\mathbf{a})} I_{\Lambda(\mathbf{0})} + (\hat{\beta}_{p,q})_{0 \le p < q \le k-\ell}]$. Using Lemma 3.2.7 (i) we have the equality

$$I_{\Lambda(\mathbf{a})} I_{\Lambda(\mathbf{0})} + (\tilde{\beta}_{p,q})_{0 \le p < q \le k-\ell} = I_{\Lambda(\mathbf{a})} I_{\Lambda(\mathbf{0})} + (\hat{\beta}_{p,q})_{0 \le p < q \le k-\ell}.$$

I claim that $(\tilde{\beta}_{p,q})_{0 \le p < q \le k-\ell} = (\hat{\beta}_{p,q})_{0 \le p < q \le k-\ell}$. Assume $\tilde{\beta}_{p,q} = \alpha + \omega$ with $\alpha \in I_{\Lambda(\mathbf{a})} I_{\Lambda(\mathbf{0})}$ and $\omega \in (\hat{\beta}_{p,q})_{0 \le p < q \le k-\ell}$ such that α, ω are linearly independent and homogenous of degree 2. Since $\hat{\lambda}_\ell = \tilde{\lambda}_\ell = 0$, the construction in Proposition 3.2.3 implies

$$I_{\Lambda(\mathbf{a})} I_{\Lambda(\mathbf{0})} = (\alpha_0, \ldots, \alpha_{k-1})(x_0, \ldots, x_{k-1}) \subseteq (x_0, \ldots, x_{k-1}, x_{n-\ell+2}, \ldots, x_n)(x_0, \ldots, x_{k-1})$$

and

$$(\tilde{\beta}_{p,q})_{0 \le p < q \le k-\ell}, (\hat{\beta}_{p,q})_{0 \le p < q \le k-\ell} \subseteq (x_0, \ldots, x_{k-1})(x_k, \ldots, x_{n-\ell+1}).$$

This implies $\alpha = 0$ and we obtain $B = (\tilde{\beta}_{p,q})_{0 \le p < q \le k-\ell} = (\hat{\beta}_{p,q})_{0 \le p < q \le k-\ell}$. The proof will be complete once we the show that the coordinates from Remark 3.2.2 of $\widetilde{Z}$ coincide with those of $\widehat{Z}$.

It follows from the proof of Proposition 3.2.1 that the coordinate $T_{i,j}^{(v)}$ admits a formal expression

$$T_{i,j}^{(v)} = \frac{A_{i,j,v}(\mathbf{a}, \mathbf{b}, \lambda_1, \ldots, \lambda_v)}{\lambda_1^{\epsilon_1} \cdots \lambda_v^{\epsilon_v}} \tag{3.12}$$

with $A_{i,j,v}$ a polynomial in $\mathbf{a}, \mathbf{b}, \lambda_1, \ldots, \lambda_v$ and $\epsilon_1, \ldots, \epsilon_v \ge 1$. Similarly, each λ_v admits a formal expression

$$\lambda_v = \frac{B_{i,j,v}(\mathbf{a}, \mathbf{b}, \lambda_1, \ldots, \lambda_{v-1})}{\lambda_1^{\epsilon_1} \cdots \lambda_{v-1}^{\epsilon_{v-1}}} \tag{3.13}$$

with $B_{i,j,v}$ a polynomial in $\mathbf{a}, \mathbf{b}, \lambda_1, \ldots, \lambda_{v-1}$ and $\epsilon_1, \ldots, \epsilon_{v-1} \geq 1$.

(i) $\hat{\lambda}_i = \tilde{\lambda}_i$ for all $i \leq \ell$: We clearly have $\hat{\lambda}_1 = a_{k-1,n} = \tilde{\lambda}_1$. Since $\hat{\lambda}_v \neq 0$ for all $v \leq \ell - 1$ we can inductively apply Eq. (3.13) to obtain

$$\hat{\lambda}_v = \frac{B_{i,j,v}(\mathbf{a}, \mathbf{0}, \hat{\lambda}_1, \ldots, \hat{\lambda}_{v-1})}{\hat{\lambda}_1^{\epsilon_1} \cdots \hat{\lambda}_{v-1}^{\epsilon_{v-1}}} = \frac{B_{i,j,v}(\mathbf{a}, \mathbf{0}, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_{v-1})}{\tilde{\lambda}_1^{\epsilon_1} \cdots \tilde{\lambda}_{v-1}^{\epsilon_{v-1}}} = \tilde{\lambda}_v.$$

(ii) $\hat{T}_{i,j}^{(v)} = \tilde{T}_{i,j}^{(v)}$ for all $v \leq \ell - 1$ and all i, j: Analogous to item (i) above, where we instead use Eq. (3.12) to conclude

$$\hat{T}_{i,j}^{(v)} = \frac{A_{i,j,v}(\mathbf{a}, \mathbf{0}, \hat{\lambda}_1, \ldots, \hat{\lambda}_v)}{\hat{\lambda}_1^{\epsilon_1} \cdots \hat{\lambda}_v^{\epsilon_v}} = \frac{A_{i,j,v}(\mathbf{a}, \mathbf{0}, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_v)}{\tilde{\lambda}_1^{\epsilon_1} \cdots \tilde{\lambda}_v^{\epsilon_v}} = \tilde{T}_{i,j}^{(v)}.$$

(iii) $\hat{T}_{i,j}^{(v)} = \tilde{T}_{i,j}^{(v)}$ for all $k - 1 \geq v \geq \ell$ and all relevant i, j (those appearing as coordinates in Remark 3.2.2: Let r, s be any integers such that $0 \leq r < s \leq k - \ell$ and assume $\hat{\beta}_{r,s} = \sum_{0 \leq p < q \leq k-\ell} c_{p,q} \tilde{\beta}_{p,q}$ for some constants $c_{p,q} \in \mathbf{k}$. Let $p' = \min\{p : c_{p,q} \neq 0\}$ and $q' = \max\{q : c_{p',q} \neq 0\}$. Then

$$x_r x_{n-k_s} = \mathrm{in}_>(\hat{\beta}_{r,s}) = \mathrm{in}_> \left(\sum_{0 \leq p < q \leq k-\ell} c_{p,q} \tilde{\beta}_{p,q} \right) = c_{p',q'} x_{p'} x_{n-k_{q'}}.$$

It follows that $\tilde{\beta}_{r,s} = \hat{\beta}_{r,s}$. Equating the terms supported on x_r we obtain

$$\sum_{j=k}^{n-k_s} \hat{T}_{s,j}^{(k-s)} x_j = \sum_{j=k}^{n-k_s} \tilde{T}_{s,j}^{(k-s)} x_j.$$

It follows that $\hat{T}_{s,j}^{(k-s)} = \tilde{T}_{s,j}^{(k-s)}$ for all $k \leq j < n - k_s$. Similarly, equating the terms supported on x_{n-k_s} we obtain $\hat{T}_{r,n-j+1}^{(j)} = \tilde{T}_{r,n-j+1}^{(j)}$ for all $1 \leq j \leq k_r$.

(iv) $\hat{T}_{0,j}^{(k)} = \tilde{T}_{0,j}^{(k)}$ for all $k \leq j \leq n - k + 1$: Combining $\hat{\beta}_{0,1} = \tilde{\beta}_{0,1}$ and the equality of coordinates in (iii) we obtain

$$\hat{\lambda}_{0,1} \left(x_1 - \sum_{j=1}^{k-2} \hat{T}_{1,n-j+1}^{(j)} x_{k-j} \right) \left(\sum_{j=k}^{n-(k-1)} \hat{T}_{0,j}^{(k)} x_j \right) = \tilde{\lambda}_{0,1} \left(x_1 - \sum_{j=1}^{k-2} \tilde{T}_{1,n-j+1}^{(j)} x_{k-j} \right) \left(\sum_{j=k}^{n-(k-1)} \tilde{T}_{0,j}^{(k)} x_j \right).$$

Since $\hat{\lambda}_{0,1} = 1 = \tilde{\lambda}_{0,1}$, equating the coefficients of the monomials containing x_1 gives the desired result.

(v) $\hat{\lambda}_i = \tilde{\lambda}_i$ for all $i \geq \ell+1$: For each $\ell+1 \leq i \leq k-1$ we have $\tilde{\beta}_{k-i,k-i+1} = \hat{\beta}_{k-i,k-i+1}$. Note that $\hat{\lambda}_{k-i,k-i+1} = \hat{\lambda}_i$ and $\tilde{\lambda}_{k-i,k-i+1} = \tilde{\lambda}_i$. Using the equality of coordinates in (iii), the expression $\tilde{\beta}_{k-i,k-i+1} = \hat{\beta}_{k-i,k-i+1}$ reduces to

$$\hat{\lambda}_i \left(x_{k-i+1} - \sum_{j=1}^{i-2} \hat{T}^{(j)}_{k-i+1,n-j+1} x_{k-j} \right) \left(\sum_{j=k}^{n-i+1} \hat{T}^{(i)}_{k-i,j} x_j \right) = \tilde{\lambda}_i \left(x_{k-i+1} - \sum_{j=1}^{i-2} \tilde{T}^{(j)}_{k-i+1,n-j+1} x_{k-j} \right) \left(\sum_{j=k}^{n-i+1} \tilde{T}^{(i)}_{k-i,j} x_j \right).$$

Equating the coefficients of $x_{k-i+1}x_{n-i+1}$ gives the desired result. □

Lemma 3.3.2. *The fiber of Ξ over the point $[(x_0,\ldots,x_{k-1})^2 + (x_p x_{n-k_q})_{0<p<q\leq k-1}]$ consists of a single element.*

Proof. Let J denote the ideal $(x_0,\ldots,x_{k-1})^2 + (x_p x_{n-k_q})_{0<p<q\leq k-1}$. Let $X \in U_{k-1}$ be the point with all the coordinates of Remark 3.2.2 equal to 0. We clearly have $\Xi(X) = [J]$. Now assume $Z \in \mathcal{X}_{k-1}$ such that $\Xi(Z) = [J]$. Since $J_{\mathrm{red}} = (x_0,\ldots,x_{k-1})$, we must have $Z \in \pi^{-1}_{k-1}(U_0)$. In particular, $Z \in U^{\gamma}_{k-1}$ for some γ. By Remark 3.2.10 we have

$$(x_0,\ldots,x_{k-1})^2 + (x_{\gamma^1_{k-p}} x_{\gamma^2_{k-q}})_{0\leq p<q\leq k-1} = \mathrm{in}_{>_\gamma}\Xi(Z) = \mathrm{in}_{>_\gamma} J = J.$$

Comparing the monomial generators of the two ideals we deduce that $\gamma^1_{k-p} = p$ for all $0 \leq p \leq k-2$; this forces $\gamma^1_1 = k-1$. But then we also obtain $\gamma^2_{k-q} = n-k_q = n-(k-q)+1$ for all $1 \leq q \leq k-1$. Thus $U^{\gamma}_{k-1} = U_{k-1}$ and by Lemma 3.3.1, $Z = X$ or $g(Z) = X$ for the non-identity $g \in \mathfrak{S}_2$. Since $\Xi(Z)_{\mathrm{red}} = \Xi(X)_{\mathrm{red}} = V(x_0,\ldots,x_{k-1})$ we must have $g(Z) = Z$; thus $Z = X$. □

Proposition 3.3.3. *Let $n \geq 2k-1$. The morphism $\bar{\Xi} : \mathcal{X}_{k-1}/\mathfrak{S}_2 \longrightarrow \mathcal{H}^n_{n-k,n-k}$ is injective.*

Proof. Let $Y, Z \in \mathcal{X}_{k-1}$ such that $\Xi(Y) = \Xi(Z)$. Since $\Xi(Y)_{\mathrm{red}} = \Xi(Z)_{\mathrm{red}}$ we may assume wlog that $Y, Z \in \pi^{-1}_{k-1}(U_0)$. We may also assume wlog that $Y \in U_{k-1}$. By Lemma 3.3.1 we only need to show that $Z \in U_{k-1}$. Let $\ell \geq 1$ be the maximal value such that $Z \in U^{\gamma}_{k-1}$ with $\gamma^1_i = k-i$ and $\gamma^2_i = n-i+1$ for all $i < \ell$. We need to show that $\ell = k$ (then automatically, $\gamma^1_k = 0$). For the sake of a contradiction, assume that $\ell < k$. Our method is to compare certain initial ideal degenerations of $\Xi(Z)$ and $\Xi(Y)$.

Let $\mathbf{w}$ be any integral weight order corresponding to $>$ [24, Section 15]. For any $t \in \mathbf{k}^\star$ let $g_t \in \mathrm{GL}(n+1)$ denote the automorphism that maps $x_i \mapsto t^{-\mathbf{w}(i)} x_i$. Since each g_t just scales the coordinates the following facts are immediate

(1) g_t induces an action on $\mathcal{X}_0$ and extends to all the blowups $\mathcal{X}_v$.

(2) g_t fixes U^{γ}_ℓ and also fixes any closed subset of the form $V(T^{\gamma,(\ell)}_{i,j})$.

(3) For each ℓ let $\psi_\ell : \mathcal{X}_{k-1} \longrightarrow \mathcal{X}_\ell$ denote the blowdown map. Then ψ_ℓ is $\mathrm{GL}(n+1)$-equivariant and thus $\psi_\ell(g_t) = g_t(\psi_\ell)$.

Let $Y_0 = \lim_{t\to 0} g_t(Y)$ and $Z_0 = \lim_{t\to 0} g_t(Z)$. Using [24, Theorem 15.17] and Lemma 3.2.5 we obtain

$$\Xi(Y_0) = \lim_{t\to 0} g_t(\Xi(Y)) = \mathrm{in}_{>}\Xi(Y) = (x_0, \ldots, x_{k-1})^2 + (x_p x_{n-k_q})_{0<p<q\le k-1}.$$

Similarly, $\Xi(Z_0) = (x_0, \ldots, x_{k-1})^2 + (x_p x_{n-k_q})_{0<p<q\le k-1} = \Xi(Y_0)$. By Lemma 3.3.2, $Z_0 = Y_0$.

Using the notation in item (3) and our assumption on ℓ, $\psi_\ell(Z)$ and $\psi_\ell(Y)$ are $\mathbf{k}$-points of $\operatorname{Proj} \mathbf{k}[U_{\ell-1}][T^{(\ell)}_{i,j}]/(\text{Koszul}) \subseteq \mathcal{X}_\ell$. By maximality of ℓ we have $T^{(\ell)}_{k-\ell,n-\ell+1}(\psi_\ell(Z)) = 0$ i.e. $\psi_\ell(Z)$ lies in $V(T^{(\ell)}_{k-\ell,n-\ell+1})$. Then by item (2) we still have $\psi_\ell(g_t(Z)) = g_t(\psi_\ell(Z)) \in V(T^{(\ell)}_{k-\ell,n-\ell+1})$. Thus the limit $\psi_\ell(Z_0)$ also lies in there. But this contradicts the fact that $T^{(\ell)}_{k-\ell,n-\ell+1}(\psi_\ell(Y_0)) = T^{(\ell)}_{k-\ell,n-\ell+1}(Y_0) \neq 0$ (since Y_0 lies in U_{k-1}). Thus $\ell = k$ and we have $Z, Y \in U_{k-1}$, as required. □

Remark 3.3.4. It follows that the preimage $\Xi^{-1}(Z)$ is a single point precisely when Z_{red} is an $(n-k)$-plane. This occurs precisely when Z is generically non-reduced, see Theorem 3.4.13.

3.4 Smoothness of $\mathcal{H}^n_{n-k,n-k}$

We begin by showing that $\mathcal{H}^n_{n-k,n-k}$ has a unique Borel-fixed point. We begin with a combinatorial criterion for Borel-fixed points in arbitrary characteristic [24, Section 15].

Definition 3.4.1. Let $I \subseteq S$ be a monomial ideal and p a prime number. The ideal I is said to be 0-**Borel-fixed** if for any monomial generator $m \in I$ divisible by x_j, we have $\frac{x_i}{x_j} m \in I$ for all $i < j$. The ideal I is said to be p-**Borel-fixed** if for any monomial generator $m \in I$ divisible by x_j^β but no higher power of x_j, we have $(\frac{x_i}{x_j})^\alpha m \in I$ for all $i < j$ and $\alpha \preceq_p \beta$ (this means that each digit in the p-base expansion of α is less than or equal to each digit in the p-base expansion of β).

Note that a 0-Borel-fixed ideal is always p-Borel-fixed for any p.

Proposition 3.4.2. [24, Theorem 15.23] *Let* $\operatorname{char} \mathbf{k} = p \ge 0$. *Then* $I \subseteq S$ *is Borel-fixed if and only if it* I *is* p*-Borel.*

In our situation, $\operatorname{char} \mathbf{k} = p \ge 0$ with $p \neq 2$. Let I be a saturated p-Borel-fixed ideal parameterized by $\mathcal{H}^n_{n-k,n-k}$. Since I is a monomial ideal generated by quadrics (Corollary 3.2.9) and $p \neq 2$, the condition $\alpha \preceq_p \beta$ in Definition 3.4.1 reduces to the condition $\alpha \le \beta$. In particular, I is always 0-Borel.

Proposition 3.4.3. *Let $n \geq 2k-1$. Consider the ideal*

$$I^n_{n-k,n-k} = \sum_{i=0}^{k-1} x_i(x_i, \dots, x_{2k-2-i}) = (x_0, \dots, x_{k-1})^2 + (x_p x_{2k-1-q})_{0\leq p<q\leq k-1}.$$

Then $[I^n_{n-k,n-k}]$ is the unique Borel-fixed point on $\mathcal{H}^n_{n-k,n-k}$.

Proof. As noted above, Borel-fixed ideals in $\mathcal{H}^n_{n-k,n-k}$ are the same as 0-Borel-fixed ideals. Since $I^n_{n-k,n-k}$ is projectively equivalent to $(x_0, \dots, x_{k-1})^2 + (x_p x_{n-k_q})_{0\leq p<q\leq k-1}$, it lies in $\mathcal{H}^n_{n-k,n-k}$. It also clear that $I^n_{n-k,n-k}$ is Borel-fixed. Let B be any saturated 0-Borel-fixed ideal on $\mathcal{H}^n_{n-k,n-k}$. Then it is of the form $B = \sum_{i=0}^{\epsilon} x_i(x_i, \dots, x_{a_i})$ with $n-1 \geq a_0 \geq a_1 \geq \dots \geq a_\epsilon \geq \epsilon$. Since $\sqrt{B} = (x_0, \dots, x_\epsilon)$ has codimension k, we obtain $\epsilon = k-1$.

Arguing as in the end of the proof of Proposition 3.2.3 we see that the Hilbert polynomial of B is $\binom{n-k+t}{t} + \sum_{i=0}^{k-1} \binom{t+n-a_i-2}{t-1}$. Equating this with the Hilbert polynomial of $I^n_{n-k,n-k}$ we have

$$\sum_{i=0}^{k-1} \binom{n-2k+i+t}{t-1} = \sum_{i=0}^{k-1} \binom{t+n-a_i-2}{t-1}.$$

Since the set $\{\binom{t-1+a}{a}\}_{a\in\mathbf{N}}$ is a $\mathbf{Q}$-basis for $\mathbf{Q}[t]$, we obtain $a_i = 2k-i-2$ for all i; therefore $B = I^n_{n-k,n-k}$. □

Lemma 3.4.4. *Let I be a (saturated) ideal parameterized by $\mathcal{H}^n_{n-k,n-k}$. Then the Castelnuovo-Mumford regularity of I is 2 and $T_{[I]} \operatorname{Hilb}^{P^n_{n-k,n-k}(t)} \mathbf{P}^n = \operatorname{Hom}_S(I, S/I)_0$.*

Proof. Since I is generated by quadrics, the regularity is at least 2. Up to projective equivalence, we may assume I is as described by Eq. (3.8). By [50, Theorem 3.3.4] we have also $\operatorname{reg}(I) \leq \operatorname{reg}(\operatorname{in}_> I)$. Note that $\operatorname{in}_> I$ is projectively equivalent to $I^n_{n-k,n-k}$ and the regularity of a 0-Borel ideal is the highest degree of a minimal monomial generator [50, Corollary 7.2.3]. Thus $\operatorname{reg}(I) \leq \operatorname{reg}(I^n_{n-k,n-k}) = 2$, as required. The description of the tangent space follows from Remark 2.0.10 and Theorem 2.0.9. □

Definition 3.4.5. Let ζ denote the pre-image of $[I^n_{n-k,n-k}]$ in $\mathcal{X}_{k-1}$ (Remark 3.3.4) and let $\bar{\zeta}$ denote the image of ζ in $\mathcal{X}_{k-1}/\mathfrak{S}_2$.

By constructing curves passing through ζ and $\bar{\zeta}$ we will now show that the differential $d\bar{\Xi}_{\bar{\zeta}}$ is injective. This is a major portion of the proof of Theorem 3.4.7.

Lemma 3.4.6. *Let $n \geq 2k-1$. The differential $d\bar{\Xi}_{\bar{\zeta}} : T_{\bar{\zeta}}(\mathcal{X}_{k-1}/\mathfrak{S}_2) \longrightarrow T_{[I^n_{n-k,n-k}]}\mathcal{H}^n_{n-k,n-k}$ is injective.*

Proof. Note that we have a factorization

$$\begin{tikzcd} T_\zeta \mathcal{X}_{k-1} \arrow[r] \arrow[dr] & T_{\bar{\zeta}}(\mathcal{X}_{k-1}/\mathfrak{S}_2) \arrow[d] \\ & T_{[I^n_{n-k,n-k}]}\mathcal{H}^n_{n-k,n-k} \end{tikzcd}$$

By non-singularity we also have $\dim_{\mathbf{k}} T_\zeta \mathcal{X}_{k-1} = \dim_{\mathbf{k}} T_{\bar{\zeta}}(\mathcal{X}_{k-1}/\mathfrak{S}_2)$. Thus to show that $d\bar{\Xi}_{\bar{\zeta}}$ is injective it suffices to establish the following two facts

(1) $d\Xi_\zeta : T_\zeta\mathcal{X}_{k-1} \longrightarrow T_{[I^n_{n-k,n-k}]}\mathcal{H}^n_{n-k,n-k}$ has a 1 dimensional kernel

(2) The exists $\omega \in T_{\bar{\zeta}}(\mathcal{X}_{k-1}/\mathfrak{S}_2)$ for which $d\bar{\Xi}_{\bar{\zeta}}(\omega)$ does not lie in the image of $d\Xi_\zeta$.

We begin with item (1). Let $\gamma^1 = (k-1, k-2, \dots, 0)$ and $\gamma^2 = (k, k+1, \dots, 2k-2)$. Then ζ is the point $\mathbf{0}$ on U^γ_{k-1} (Proposition 3.2.8). As in Remark 3.2.2 a set of coordinates on U^γ_{k-1} is $\mathcal{N} = \mathcal{N}_1 \cup \dots \cup \mathcal{N}_5$ where

$$\mathcal{N}_1 = \{b_{i,j}\}^{k\le j\le n}_{0\le i\le k-1}, \quad \mathcal{N}_2 = \{T^{\gamma,(j)}_{i,k-1+j}\}^{0\le i\le k-1-j}_{1\le j\le k-1}, \quad \mathcal{N}_3 = \{T^{\gamma,(i)}_{k-i,j}\}^{1\le i\le k-1}_{k+i\le j\le n},$$
$$\mathcal{N}_4 = \{\lambda^\gamma_1, \dots, \lambda^\gamma_{k-1}\}, \quad \mathcal{N}_5 = \{T^{\gamma,(k)}_{0,j}\}_{2k-1\le j\le n}.$$

For each $\eta \in \mathcal{N}$ we define a curve $D_\eta : \operatorname{Spec}\mathbf{k}[t] \longrightarrow U^\gamma_{k-1}$, passing through $\mathbf{0}$, by setting $\eta = t$ and all the other coordinates in $\mathcal{N}$ to 0.

Let $\iota : \operatorname{Spec}\mathbf{k}[t]/(t^2) \longrightarrow \operatorname{Spec}\mathbf{k}[t]$ be a first order deformation of the origin. Since $\mathcal{X}_{k-1}$ is non-singular the set $\{D_\eta \circ \iota\}_{\eta\in\mathcal{N}}$ is a basis for $T_{\mathbf{0}}U^\gamma_{k-1} = T_\zeta\mathcal{X}_{k-1}$. We need to study the dimension of $\{d\Xi_\zeta(D_\eta\circ\iota)\}_\eta$. Since $d\Xi_\zeta(D_\eta\circ\iota) = (\Xi\circ D_\eta)\circ\iota$ we begin with an explicit description of each $\Xi\circ D_\eta$. The items below follow directly from the construction of the map (Eq. (3.11)).

(i) If $\eta = b_{i,j} \in \mathcal{N}_1$ then $\Xi\circ D_\eta(t)$ is

$$(x_0, \dots, x_{i-1}, x_i + tx_j, x_{i+1}, \dots, x_{k-1})^2 + (x_p x_{2k-1-q})^{0\le p<q\le k-1}_{p\ne i}$$
$$+(x_i + tx_j)(x_k, \dots, x_{2k-2-i}).$$

(ii) If $\eta = T^{\gamma,(j)}_{i,k-1+j} \in \mathcal{N}_2$ then $\Xi\circ D_\eta(t)$ is

$$(x_0, \dots, x_{k-1})^2 + (x_p x_{2k-1-q})^{0\le p<q\le k-1}_{p\ne i} + (x_i - tx_{k-j})(x_k, \dots, x_{2k-2-i}).$$

(iii) If $\eta = T^{\gamma,(i)}_{k-i,j} \in \mathcal{N}_3$ then $\Xi\circ D_\eta(t)$ is

$$(x_0, \dots, x_{k-1})^2 + (x_p x_{2k-1-q})^{0\le p<q\le k-1}_{q\ne k-i} + (x_0, \dots, x_{k-i-1})(x_{k-1+i} + tx_j).$$

(iv) If $\eta = \lambda_i^\gamma$ with $i > 1$ then $\Xi \circ D_\eta(t)$ is

$$(x_0, \dots, x_{k-1})^2 + (x_p x_{2k-1-q})_{(p,q)\neq(k-i,k-i+1)}^{0\le p<q\le k-1} + (x_{k-i}x_{k+i-2} - t x_{k-i+1}x_{k+i-1}).$$

(v) If $\eta = \lambda_1^\gamma$ then $\Xi \circ D_\eta(t)$ is

$$(x_0, \dots, x_{k-2})(x_0, \dots, x_{k-1}) + (x_{k-1} + t x_k)x_{k-1} + (x_p x_{2k-1-q})_{0\le p<q\le k-1}.$$

(vi) If $\eta = T_{0,j}^{\gamma,(k)} \in \mathcal{N}_5$ then $\Xi \circ D_\eta(t)$ is

$$(x_0, \dots, x_{k-1})^2 + (x_p x_{2k-1-q})_{(p,q)\neq(0,1)}^{0\le p<q\le k-1} + (x_0 x_{2k-2} - t x_1 x_j).$$

Let $I = I_{n-k,n-k}^n$ and under the inclusion $\mathcal{H}_{n-k,n-k}^n \subseteq \mathrm{Hilb}^{P_{n-k,n-k}^n(t)}\,\mathbf{P}^n$, we may identify $T_{[I]}\mathcal{H}_{n-k,n-k}^n$ with a subspace of $\mathrm{Hom}(I, S/I)_0$ (Lemma 3.4.4). We can explicitly describe this identification using [48, Proposition 2.3]. In particular, by re-indexing, we obtain

$$\begin{aligned}\mathrm{span}_{\mathbf{k}}\{d\Xi_\zeta(D_\eta \circ \iota)\}_{\eta\in\mathcal{N}_1\cup\mathcal{N}_2\cup\mathcal{N}_3} &= \mathrm{span}_{\mathbf{k}}\left(\left\{-x_j\frac{\partial}{\partial x_i}\right\}_{0\le i\le k-1}^{k\le j\le n} \cup \left\{x_j\frac{\partial}{\partial x_i}\right\}_{0\le i\le k-2}^{i+1\le j\le k-1} \cup \left\{-x_j\frac{\partial}{\partial x_i}\right\}_{k\le i\le 2k-2}^{i+1\le j\le n}\right)\\ &= \mathrm{span}_{\mathbf{k}}\left\{x_j\frac{\partial}{\partial x_i}\right\}_{0\le i\le 2k-2}^{i+1\le j\le n}.\end{aligned}$$

These are the *trivial deformations* i.e. the ones induced by a change of coordinates. For $i \in \{1, \dots, k-2\}$ let Δ_i be the derivation that maps $x_i x_{2k-2-i} \mapsto x_{i+1}x_{2k-1-i}$ and other generators to 0. Let Δ_{k-1} denote the derivation that maps $x_{k-1}^2 \mapsto x_{k-1}x_k$ and the other generators to 0. For $i \in \{2k-1, \dots, n\}$ let Δ_i to the derivation that maps $x_0 x_{2k-2} \mapsto x_1 x_i$. Then we have

$$\mathrm{span}_{\mathbf{k}}\{d\Xi_\zeta(D_\eta \circ \iota)\}_{\eta\in\mathcal{N}_4\cup\mathcal{N}_5} = \mathrm{span}_{\mathbf{k}}(\{\Delta_i\}_{1\le i\le k-1} \cup \{\Delta_i\}_{2k-1\le i\le n}).$$

Notice that the derivation Δ_{k-1} is a scalar multiple of $x_k\frac{\partial}{\partial x_{k-1}}$. Thus to prove (1) it suffices to show that the set $\{x_j\frac{\partial}{\partial x_i}\}_{0\le i\le 2k-2}^{i+1\le j\le n} \cup \{\Delta_i\}_{1\le i\le k-2} \cup \{\Delta_i\}_{2k-1\le i\le n}$ is linearly independent.

Assume we had a linear combination

$$\sum_{\substack{0\le i\le 2k-2\\ i+1\le j\le n}} \epsilon_{i,j}x_j\frac{\partial}{\partial x_i} + \sum_{\substack{1\le i\le k-2\\ 2k-1\le i\le n}} \epsilon_i\Delta_i \equiv 0 \bmod I \tag{3.14}$$

with some constants $\epsilon_{i,j}, \epsilon_i \in \mathbf{k}$. Assume $\epsilon_{p,q} \neq 0$ for some $p < q$. Since $x_p x_{2k-2-p} \in I$ we may evaluate Eq. (3.14) at $x_p x_{2k-2-p}$ to obtain

$$\sum_{p+1\le j\le n} \epsilon_{p,j}x_j x_{2k-2-p} + \sum_{2k-1-p\le j\le n} \epsilon_{2k-2-p,j}x_j x_p + Q \equiv 0 \bmod I \tag{3.15}$$

where

$$Q = \begin{cases} \sum_{i=2k-1}^{n} \epsilon_i x_1 x_i & \text{if } p = 0,\, 2k-2, \\ \epsilon_p x_{p+1} x_{2k-1-p} & \text{if } 1 \leq p \leq k-2, \\ 0 & \text{if } p = k-1 \\ \epsilon_{2k-2-p} x_{2k-1-p} x_{p+1} & \text{if } k \leq p \leq 2k-3. \end{cases}$$

Observe that the monomial $x_q x_{2k-2-p}$ does not appear in the support of Q. Thus, in the left hand side of Eq. (3.15), the monomial $x_q x_{2k-2-p}$ appears with a coefficient of $\epsilon_{p,q}$ if $p \neq k-1$ and a coefficient of $2\epsilon_{p,q}$ if $p = k-1$. In either case, the coefficient is non-zero. But this is a contradiction as $x_q x_{2k-2-p} \notin I$. Thus we have $\epsilon_{p,q} = 0$ for all p, q. Evaluating Eq. (3.14) at $x_p x_{2k-2-p}$ we see that $\epsilon_p = 0$ for every $p \in \{1, \ldots, k-2\}$. Finally, evaluating Eq. (3.14) at $x_0 x_{2k-2}$ we obtain $\sum_{i=2k-1}^{n} \epsilon_i x_1 x_i \equiv 0 \mod I$. Since $x_1 x_i \notin I$ for all $i \geq 2k-1$, we must have that $\epsilon_i = 0$ for all i. This completes the proof of item (1).

Let $\Delta \in \mathrm{Hom}(I, S/I)_0$ denote the derivation that maps $x_{k-1}x_k \mapsto x_k^2$ and all the other generators to 0. By evaluating at $x_{k-1}x_k$ it is easy to see that Δ does not lie in the span of $\{x_j \frac{\partial}{\partial x_i}\}_{0 \leq i \leq 2k-2}^{i+1 \leq j \leq n} \cup \{\Delta_i\}_{1 \leq i \leq k-2} \cup \{\Delta_i\}_{2k-1 \leq i \leq n}$. Consider the curve $C : \mathrm{Spec}\, \mathbf{k}[t] \longrightarrow \mathcal{H}^n_{n-k,n-k}$ given by

$$t \mapsto (x_0, \ldots, x_{k-2})(x_0, \ldots, x_{k-1}) + (x_{k-1}^2 - t x_k^2) + (x_p x_{2k-1-q})_{0 \leq p < q \leq k-1}$$

This is well defined because for any given $s \in \mathbf{k}$, $C(s)$ is the point in U^{γ}_{k-1} with $\lambda^{\gamma}_1 = -2\sqrt{s}$, $b_{k-1,k} = \sqrt{s}$ and all other coordinates equal 0. It is also clear that $C \circ \iota$ corresponds to the derivation Δ. Thus to prove item (2) it suffices to find a curve $C' : \mathrm{Spec}\, \mathbf{k}[t] \longrightarrow \mathcal{X}_{k-1}/\mathfrak{S}_2$ passing through $\bar{\zeta}$ for which $d_{\bar{\zeta}}\bar{\Xi}(C' \circ \iota) = C \circ \iota$.

Let Z denote the image of C and let Z' denote the pullback $\bar{\Xi}^{-1}(Z) \subseteq \mathcal{X}_{k-1}/\mathfrak{S}_2$. I claim that $\bar{\Xi}|_{Z'} : Z' \longrightarrow Z$ is an isomorphism. Since Z is non-singular, Z' is Cohen-Macaulay and $\bar{\Xi}$ is bijective, the morphism $\bar{\Xi}|_{Z'}$ is flat. It is clear that a finite flat degree 1 morphism is an isomorphism. Thus $C' = \bar{\Xi}|_{Z'}^{-1} \circ C : \mathrm{Spec}\, \mathbf{k}[t] \longrightarrow \mathcal{X}_{k-1}/\mathfrak{S}_2$ is the desired curve. □

We are now ready to prove the main Theorem.

Theorem 3.4.7. *Let $n \geq 2k-1$. The component $\mathcal{H}^n_{n-k,n-k}$ is smooth and isomorphic to*

$$\mathcal{X}_{k-1}/\mathfrak{S}_2 = \mathrm{Bl}_{\bar{\Gamma}_{k-1}} \cdots \mathrm{Bl}_{\bar{\Gamma}_1} \mathrm{Sym}^2\, \mathbf{Gr}(n-k, n).$$

Proof. Proposition 3.2.8 and Proposition 3.3.3 together show that $\bar{\Xi}$ is bijective and $\mathcal{X}_{k-1}/\mathfrak{S}_2$ is non-singular. Since $\bar{\Xi}$ is $\mathrm{GL}(n+1)$-equivariant, $\bar{\zeta}$ (Definition 3.4.5) is the unique Borel-fixed point on $\mathcal{X}_{k-1}/\mathfrak{S}_2$. By Borel's fixed point theorem, the closure of the Borel orbit of any point in $\mathcal{X}_{k-1}/\mathfrak{S}_2$ contains $\bar{\zeta}$. Thus to show that $\bar{\Xi}$ is an isomorphism, it suffices to show that it is an isomorphism in a neighbourhood of $\bar{\zeta}$. By the proof of [45, Theorem 14.9], this is equivalent to showing that $d\bar{\Xi}_{\bar{\zeta}} : T_{\bar{\zeta}}(\mathcal{X}_{k-1}/\mathfrak{S}_2) \longrightarrow T_{[I^n_{n-k,n-k}]}\mathcal{H}^n_{n-k,n-k}$ is injective. This is precisely the content of Lemma 3.4.6. □

When the pair of planes do not span $\mathbf{P}^n$, we obtain the following fibration

Corollary 3.4.8. *Let $n < 2k-1$. The morphism $\rho : \mathcal{H}^n_{n-k,n-k} \longrightarrow \mathbf{Gr}(2n-2k+1, n)$ that sends a scheme to its linear span is smooth; the fiber over a point Λ is $\mathcal{H}_{n-k,n-k}(\Lambda)$.*

Proof. Recall that the linear span of a subscheme $Z \subseteq \mathbf{P}^n$ is the linear space $V(H^0(\mathbf{P}^n, I_Z(1))) \subseteq \mathbf{P}^n$. Let $\mathcal{Y} \longrightarrow \mathbf{A}^1$ be a flat family such that for $t \neq 0$, $\mathcal{Y}_t$ is a disjoint pair of $(n-k)$-planes. It is clear that for any $t \neq 0$, the linear span of $\mathcal{Y}_t$ is a $(2n-2k+1)$-plane. By upper semicontunity, the limit $\mathcal{Y}_0$ also lies in a $(2n-2k+1)$-plane, which we denote by Λ. Thus $\mathcal{Y}_0$ defines a point in $\mathcal{H}^n_{n-k,n-k}(\Lambda)$ and by Corollary 3.2.9, we see that the linear span of $\mathcal{Y}_0$ is all of Λ. It follows that the linear span of any subscheme parameterized by $\mathcal{H}_{n-k,n-k}(\mathbf{P}^n)$ is of dimension $2n-2k+1$.

For each ordered basis $\mathbb{E} = \{e_0, \ldots, e_n\}$ of S_1 we obtain an open neighbourhood $U_{\mathbb{E}} = \operatorname{Spec} \mathbf{k}[f_{i,j}]^{2k-1-n \leq j \leq n}_{0 \leq i \leq 2k-2-n}$ of $\Lambda_{\mathbb{E}} = V(e_0, \ldots, e_{2k-2-n})$ in $\mathbf{Gr}(2n-2k+1, n)$. The $\mathbf{k}$-point $\mathbf{f} = (f_{i,j})_{i,j}$ is identified with

$$V(e_0 + \sum_{j=2k-1-n}^{n} f_{0,j}e_j, \ldots, e_{2k-2-n} + \sum_{j=2k-1-n}^{n} f_{2k-2-n,j}e_j).$$

Let $\mathbb{E} = \{e_i\}_i, \mathbb{E}' = \{e'_i\}_i$ be ordered bases of S_1. The isomorphism $\Lambda_{\mathbb{E}} \to \Lambda_{\mathbb{E}'}$ given by mapping $e_i \mapsto e'_i$ for all i induces an an isomorphism $\psi_{\mathbb{E},\mathbb{E}'} : \mathcal{H}_{n-k,n-k}(\Lambda_{\mathbb{E}}) \longrightarrow \mathcal{H}_{n-k,n-k}(\Lambda_{\mathbb{E}'})$. Define the following

- $\mathcal{X}_{\mathbb{E}} = \mathcal{H}_{n-k,n-k}(\Lambda_{\mathbb{E}}) \times U_{\mathbb{E}}$,
- $\mathcal{X}_{\mathbb{E},\mathbb{E}'} = \mathcal{H}_{n-k,n-k}(\Lambda_{\mathbb{E}}) \times (U_{\mathbb{E}} \cap U_{\mathbb{E}'}) \subseteq \mathcal{X}_{\mathbb{E}}$,
- $\varphi_{\mathbb{E},\mathbb{E}'} = \psi_{\mathbb{E},\mathbb{E}'} \times \mathrm{id} : \mathcal{X}_{\mathbb{E},\mathbb{E}'} \longrightarrow \mathcal{X}_{\mathbb{E}',\mathbb{E}}$.

It is clear that $\varphi^{-1}_{\mathbb{E},\mathbb{E}'} = \varphi_{\mathbb{E}',\mathbb{E}}$, $\varphi_{\mathbb{E}',\mathbb{E}''} \circ \varphi_{\mathbb{E},\mathbb{E}'} = \varphi_{\mathbb{E},\mathbb{E}''}$ on $\mathcal{X}_{\mathbb{E},\mathbb{E}'} \cap \mathcal{X}_{\mathbb{E},\mathbb{E}''}$ and $\varphi_{\mathbb{E},\mathbb{E}'}(\mathcal{X}_{\mathbb{E},\mathbb{E}'} \cap \mathcal{X}_{\mathbb{E},\mathbb{E}''}) = \mathcal{X}_{\mathbb{E}',\mathbb{E}} \cap \mathcal{X}_{\mathbb{E}',\mathbb{E}''}$. Thus the set of schemes $\{X_{\mathbb{E}}\}_{\mathbb{E}}$ glue to a smooth scheme $\mathcal{X}$ (Theorem 3.4.7).

For each $\mathbb{E}$ we obtain a natural morphism $g_{\mathbb{E}} : U_{\mathbb{E}} \longrightarrow \mathrm{GL}(n+1)$ such that for any $\mathbf{f}$, $g_{\mathbb{E}}(\mathbf{f})$ is the map that sends $e_i \mapsto e_i + \sum_{j=2k-1-n}^{n} f_{i,j}e_j$ if $i \leq 2k-2-n$ and fixes the other coordinates. Thus we may define a map

$$\mathcal{H}_{n-k,n-k}(\Lambda_{\mathbb{E}}) \times U_{\mathbb{E}} \longrightarrow \mathcal{H}_{n-k,n-k}(\mathbf{P}^n), \quad (X, \mathbf{f}) \mapsto g_{\mathbb{E}}(\mathbf{f})(X).$$

These maps glue to a morphism $\Pi : \mathcal{X} \longrightarrow \mathcal{H}^n_{n-k,n-k}$. By the first paragraph, Π is a bijective morphism. It is also clear that the differential to Π is injective at all points. As noted in Theorem 3.4.7, this implies that Π is an isomorphism. By construction, there is a smooth fibration $\rho : \mathcal{X} \longrightarrow \mathbf{Gr}(2n-2k+1, n)$ of the desired form. ◻

Theorem 3.4.9. *$\mathcal{H}^n_{n-k,n-k}$ has a unique Borel-fixed point.*

Proof. By Proposition 3.4.3 we my assume $n < 2k-1$. If X is Borel-fixed then its linear span $V((I_X)_1)$ is also Borel-fixed. Thus X lies in the fiber $\rho^{-1}(V(x_0,\ldots,x_{2k-2-n})) \simeq \mathcal{H}^{2n-2k+1}_{n-k,n-k}$. Moreover, the Borel action on $\mathcal{H}^n_{n-k,n-k}$ restricts to the Borel action on this fiber. By Proposition 3.4.3 this fiber has a unique Borel-fixed point; thus X is unique. □

We now turn our attention to the subschemes parameterized by $\mathcal{H}^n_{n-k,n-k}$. Since we are going to describe these subschemes up to projective equivalence, we may assume $n \geq 2k-1$ (Corollary 3.4.8). We begin with two Lemmas that will aid in the proof of Theorem 3.4.13.

Lemma 3.4.10. *Let $J = (x_0,\ldots,x_{k-1})^2 + (x_p x_{n-k_q} - \mu_{p,q} x_q x_{n-k_p})_{0\leq p<q\leq k-1}$ with $\mu_i \in \mathbf{k}$ and $\mu_{p,q} = \mu_{k-q+1}\cdots\mu_{k-p}$ for any $0 \leq p < q \leq k$. If all the μ_i are non-zero then the subscheme defined by J is Cohen-Macaulay; in particular, it has no embedded components. Moreover, the subscheme defined by J is double structure on $V(x_0,\ldots,x_{k-1})$.*

Proof. Applying the change of coordinates that maps $x_p \mapsto \mu_{p,k} x_p$ for all $p \leq k-1$ and fixing the other coordinates, we may assume $\mu_{p,q} = 1$ for all p,q. If $n > 2k-1$, the variables $x_k,\ldots,x_{n-k}$ form a regular sequence as they do not appear in the support of the generators of J. Thus we may quotient by the ideal $(x_k,\ldots,x_{n-k})$ to reduce to the case $n = 2k-1$; in this case $n - k_p = k + p$. Since $\mathrm{Proj}(S/J)$ is supported on $V(x_0,\ldots,x_{k-1})$, it suffices to verify the Cohen-Macaulayness on the open sets $D(x_k),\ldots,D(x_{2k-1})$.

On the open set $W = D(x_k)$ we may set $x_k = 1$. Then for all $j \neq 0$ we have $x_j - x_0 x_{k+j} = -(x_0 x_{k+j} - x_j x_k) \in J|_W$ and this implies $J|_W = (x_0^2, x_1 - x_0 x_{k+1},\ldots,x_{k-1} - x_0 x_{2k-1})$. Since $x_k,\ldots,x_{2k-1}$ forms a regular sequence on $(S/J)|_W$, $\mathrm{Proj}(S/J)|_W$ is a Cohen-Macaulay subscheme of dimension $k-1$. The argument for the other open sets is the same.

Since the Hilbert polynomial of $\mathrm{Proj}(S/J)$ is $P^n_{n-k,n-k}(t)$, its degree is 2; thus it is a double structure on the linear space $V(x_0,\ldots,x_{k-1})$ □

Remark 3.4.11. More generally, $(x_{\epsilon_1},\ldots,x_{\epsilon_2})^2 + (x_p x_{n-k_q} - \mu_{p,q} x_q x_{n-k_p})_{\epsilon_1\leq p<q\leq\epsilon_2}$ is Cohen-Macaulay for any $0 \leq \epsilon_1 \leq \epsilon_2 \leq k-1$, assuming $\mu_i \neq 0$ for all i.

Lemma 3.4.12. *Let $0 \leq \epsilon_1 \leq \epsilon_2 \leq k-1$ and let $J(\epsilon_1,\epsilon_2) = (x_{\epsilon_1},\ldots,x_{\epsilon_2})^2 + (x_p x_{n-k_q})_{\epsilon_1\leq p<q\leq\epsilon_2}$. Then we have a primary decomposition*

$$J(\epsilon_1,\epsilon_2) = \bigcap_{j=\epsilon_1}^{\epsilon_2} (x_{\epsilon_1},\ldots,x_{j-1},x_j^2,x_{j+1},\ldots,x_{\epsilon_2},x_{n-k_{j+1}},\ldots,x_{n-k_{\epsilon_2}}).$$

Proof. For the first statement we proceed by induction on ϵ_2. The base case $\epsilon_2 = \epsilon_1$ is vacuous and by induction we may assume

$$J(\epsilon_1,\epsilon_2+1) = \left[(x_{\epsilon_1},\ldots,x_{\epsilon_2})^2 + (x_p x_{n-k_q})_{\epsilon_1\leq p<q\leq\epsilon_2} + (x_{\epsilon_2+1}, x_{n-k_{\epsilon_2+1}})\right] \cap (x_{\epsilon_1},\ldots,x_{\epsilon_2},x^2_{\epsilon_2+1}).$$

The conclusion now follows from the fact that if $I_1 = (m_1,\ldots,m_{i_1}), I_2 = (m_1,\ldots,m_{i_2})$ are monomial ideals then $I_1 \cap I_2 = (\mathrm{lcm}(m_i m_j) : 1 \leq i \leq i_1, 1 \leq j \leq i_2)$. □

Theorem 3.4.13. *Let $n \geq 2k-1$. Let Z be a subscheme parameterized by $\mathcal{H}^n_{n-k,n-k}$. Then Z is a pair of planes meeting transversely, or there exists a sequence of integers $1 \leq i_1 < \cdots < i_r \leq k$ and a flag of linear spaces $\Lambda^1 \subseteq \Lambda^2 \subseteq \cdots \subseteq \Lambda^r \subseteq \mathbf{P}^n$ with $\operatorname{codim}_{\mathbf{P}^n}(\Lambda^\ell) = (k+i_\ell-1)$ for each ℓ, such that*

(i) *If $i_1 > 1$ then Z is a union of two planes meeting along Λ^1 with embedded pure double structures on Λ^ℓ for each $1 \leq \ell \leq r$.*

(ii) *If $i_1 = 1$ then Z is a pure double structure on Λ^1 with embedded pure double structures on Λ^ℓ for each $2 \leq \ell \leq r$.*

Proof. It suffices to compute a primary decomposition of the ideal

$$J = (x_p + \mu_{p,k}x_{n-k_p})_{0\leq p\leq k-1}(x_0,\ldots,x_{k-1}) + (x_p x_{n-k_q} - \mu_{p,q}x_q x_{n-k_p})_{0\leq p<q\leq k-1}$$

in Eq. (3.8). Let $\mathfrak{P}_0 = (x_p + \mu_{p,k}x_{n-k_p})_{0\leq p\leq k-1}$, $\mathfrak{P}_1 = (x_0,\ldots,x_{k-1})$ and $\delta_{p,q} = x_p x_{n-k_q} - \mu_{p,q}x_q x_{n-k_p}$ for each $0 \leq p < q \leq k-1$. Lemma 3.2.7 (ii) implies that all the μ_i are non-zero if and only if J is the ideal of a pair of $(n-k)$-planes meeting transversely. So we may assume some of the μ_i are zero. Let $i_1 < \cdots < i_r$ be all the indices i for which $\mu_i = 0$. Set $i_0 = 0$ and $i_{r+1} = k+1$. Lemma 3.2.7 (iv) implies $\sqrt{J} = \mathfrak{P}_0 \cap \mathfrak{P}_1$ and $J = \mathfrak{P}_0\mathfrak{P}_1 + (\delta_{p,q})_{0\leq p<q\leq k-i_1}$. For each $2 \leq \ell \leq r+1$ define

$$\begin{aligned}\mathfrak{P}_\ell = (x_0,\ldots,x_{k-i_\ell}) + (x_{k-i_\ell+1},\ldots,x_{k-i_{\ell-1}})^2 + (\delta_{p,q})_{k-i_\ell+1\leq p<q\leq k-i_{\ell-1}} +& \\ (x_{k-i_{\ell-1}+1},\ldots,x_{k-1},x_{n-i_{\ell-1}+2},\ldots,x_n).&\end{aligned}$$

I claim that $J = \mathfrak{P}_0 \cap \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_{r+1}$ (note that if $\mu_1 = 0$ then $\mathfrak{P}_0 = \mathfrak{P}_1$). We begin with the inclusion, $J \subseteq \mathfrak{P}_0 \cap \cdots \cap \mathfrak{P}_{r+1}$. It is enough to show that $\mathfrak{P}_0\mathfrak{P}_1$ and $\delta_{p,q}$ lie in $\mathfrak{P}_0 \cap \cdots \cap \mathfrak{P}_{r+1}$ for $0 \leq p < q \leq k-i_1$. Observe that

$$\mathfrak{P}_0\mathfrak{P}_1 = ((x_0,\ldots,x_{k-i_1}) + (x_p + \mu_{p,k}x_{n-k_p})_{k-i_1+1\leq p\leq k-1})(x_0,\ldots,x_{k-1})$$

Clearly, $(x_0,\ldots,x_{k-i_1})(x_0,\ldots,x_{k-1}) \subseteq \mathfrak{P}_j$ for all j. We also have, $x_p, x_{n-k_p} \in \mathfrak{P}_j$ for all $k-i_1+1 \leq p \leq k-1$ and all j. Thus $\mathfrak{P}_0\mathfrak{P}_1 \subseteq \mathfrak{P}_0\cap\cdots\cap\mathfrak{P}_{r+1}$. It is clear that $\delta_{p,q} \in \mathfrak{P}_0\cap\cdots\cap\mathfrak{P}_{r+1}$ if there is some ℓ such that $k-i_\ell+1 \leq p < q \leq k-i_{\ell-1}$. If this was not the case, then there is some ℓ such that $p \leq k-i_\ell < q$. This implies $\delta_{p,q} = x_p x_{n-k_q}$ and this lies in $(x_0,\ldots,x_{k-i_j})$ if $j \leq \ell$ or in $(x_{n-i_{j-1}+2},\ldots,x_n)$ if $j > \ell$; in either case, $\delta_{p,q} \in \mathfrak{P}_j$. Thus $\delta_{p,q} \in \mathfrak{P}_0 \cap \cdots \cap \mathfrak{P}_{r+1}$ and we have the desired containment.

To get the other containment it suffices to show that $\mathfrak{P}_0\cap\cdots\cap\mathfrak{P}_{r+1}$ has the same Hilbert function as J. We have

$$\operatorname{in}_> J \subseteq \operatorname{in}_> (\mathfrak{P}_0 \cap \cdots \cap \mathfrak{P}_{r+1}) \subseteq \operatorname{in}_>(\mathfrak{P}_0 \cap \mathfrak{P}_1) \cap \operatorname{in}_>\mathfrak{P}_2 \cap \cdots \cap \operatorname{in}_>\mathfrak{P}_{r+1}. \tag{3.16}$$

Our goal is to show all these containments are equalities. Using Eq. (3.10) we have

$$\begin{aligned}\mathfrak{P}_0 \cap \mathfrak{P}_1 &= ((x_0, \dots, x_{k-i_1}) + (x_p + \mu_{p,k} x_{n-k_p})_{k-i_1+1 \le p \le k-1}) \cap (x_0, \dots, x_{k-1}) \\ &= (x_0, \dots, x_{k-i_1}) + (x_p + \mu_{p,k} x_{n-k_p})_{k-i_1+1 \le p \le k-1} \cap (x_{k-i_1+1}, \dots, x_{k-1}) \\ &= (x_0, \dots, x_{k-i_1}) + (x_p + \mu_{p,k} x_{n-k_p})_{k-i_1+1 \le p \le k-1}(x_{k-i_1+1}, \dots, x_{k-1}) \\ &= (x_0, \dots, x_{k-i_1}) + ((x_p + \mu_{p,k} x_{n-k_p})x_q)_{k-i_1+1 \le p \le q \le k-1} + (\delta_{p,q})_{k-i_1+1 \le p < q \le k-1}.\end{aligned}$$

Then the proof of Lemma 3.2.5 immediately implies

$$\mathrm{in}_>(\mathfrak{P}_0 \cap \mathfrak{P}_1) = (x_0, \dots, x_{k-i_1}) + (x_{k-i_1+1}, \dots, x_{k-1})^2 + (x_p x_{n-k_q})_{k-i_1+1 \le p < q \le k-1}.$$

Similarly for $\ell \ge 2$

$$\begin{aligned}\mathrm{in}_> \mathfrak{P}_\ell = (x_0, \dots, x_{k-i_\ell}) &+ (x_{k-i_\ell+1}, \dots, x_{k-i_{\ell-1}})^2 + (x_p x_{n-k_q})_{k-i_\ell+1 \le p < q \le k-i_{\ell-1}} + \\ &(x_{k-i_{\ell-1}+1}, \dots, x_{k-1}, x_{n-i_{\ell-1}+2}, \dots, x_n).\end{aligned}$$

Using Lemma 3.4.12 we see that $\mathrm{in}_>(\mathfrak{P}_0 \cap \mathfrak{P}_1) \cap \mathrm{in}_> \mathfrak{P}_2 \cap \dots \cap \mathrm{in}_> \mathfrak{P}_{r+1}$ equals

$$\bigcap_{\ell=1}^{r+1} \bigcap_{j=k-i_\ell+1}^{k-i_{\ell-1}} (x_0, \dots, x_{j-1}, x_j^2, x_{j+1}, \dots, x_{k-1}, x_{n-k_{j+1}}, \dots, x_n).$$[2]

Applying Lemma 3.4.12 once again we see that this intersection is just $J(0, k-1) \cap (x_0, \dots, x_{k-1})$. But this ideal is precisely $(x_0, \dots, x_{k-1})^2 + (x_p x_{n-k_q})_{0 < p < q \le k-1} = \mathrm{in}_> J$. Thus all the containments in Eq. (3.16) are equalities and this shows that J has the same Hilbert function as $\mathfrak{P}_0 \cap \dots \cap \mathfrak{P}_r$.

We are left with showing $\mathfrak{P}_\ell$ is a primary component for all $\ell \ge 2$. Going modulo the linear forms it suffices to show that $(x_{k-i_\ell+1}, \dots, x_{k-i_{\ell-1}})^2 + (\beta_{p,q})_{k-i_\ell+1 \le p < q \le k-i_{\ell-1}}$ is a primary component. This is the content of Lemma 3.4.10 and Remark 3.4.11. □

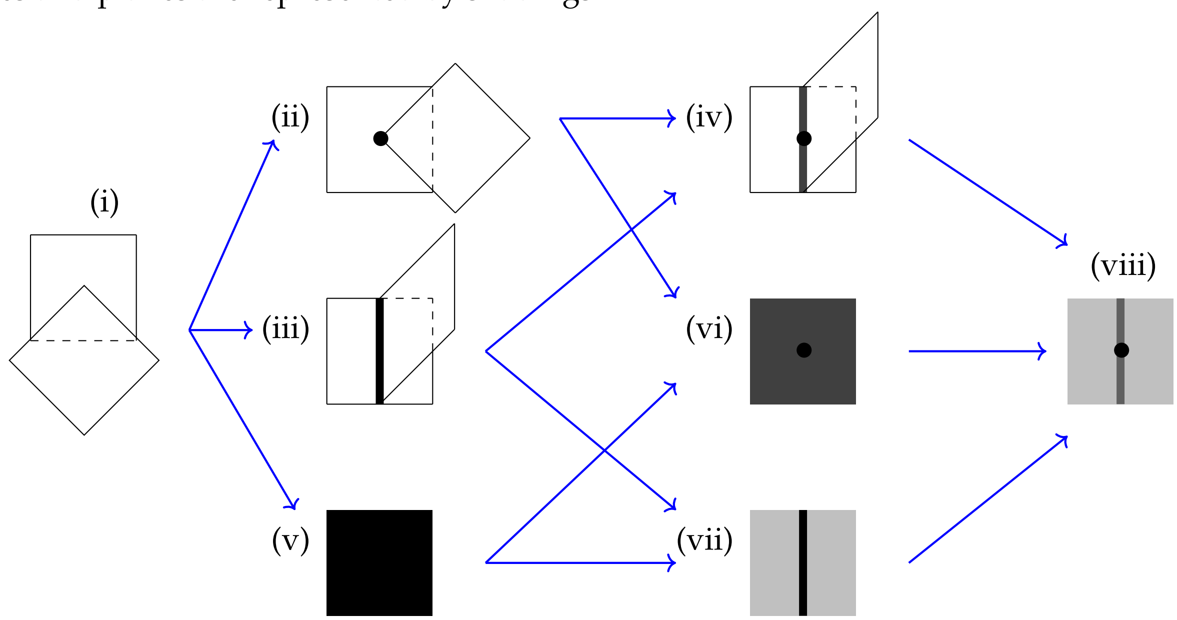
Corollary 3.4.14. *Up to projective equivalence, there are exactly 2^k schemes parameterized by $\mathcal{H}^n_{n-k,n-k}$.*

Proof. By Corollary 3.4.8 we may assume $n \ge 2k-1$. It suffices to consider ideals J as described in Eq. (3.8). Let φ denote the projective transformation that maps $x_p \mapsto \mu_{p,k} x_p$ if $\mu_{p,k} \ne 0$ and $0 \le p \le k-1$ and fixes the other coordinates. For a fixed p, note that if $\mu_{p,k} \ne 0$ then $\mu_{q,k} \ne 0$ and $\mu_{p,q} \ne 0$ for all $p < q$. Thus after applying φ we may assume that the non-zero μ_i are equal to 1. In particular, for each subset $W \subseteq \{1, \dots, k\}$ we obtain an ideal parameterized by $\mathcal{H}^n_{n-k,n-k}$ by setting $\mu_i = 0$ if $i \in W$ and 1 otherwise; this gives at most 2^k distinct ideals. On the other hand, since projective transformations preserve the dimensions of the embedded structures, each of the 2^k ideals are projectively inequivalent. □

[2] If $j = k$ the ideal $(x_0, \dots, x_{j-1}, x_j^2, x_{j+1}, \dots, x_{k-1}, x_{n-k_{j+1}}, \dots, x_n)$ is equal to $(x_0, \dots, x_{k-1})$.

Example 3.4.15. We can now determine when there is a specialization $Z \rightsquigarrow Z'$ in $\mathcal{H}^n_{n-k,n-k}$. For any subscheme $Z \in \mathcal{H}^n_{n-k,n-k}$ let $W_Z = \{\epsilon_1, \ldots, \epsilon_r\}$ be the set of dimensions of the embedded components of Z; if Z is generically non-reduced include $n-k$ in that set. Then there is a specialization $Z \rightsquigarrow Z'$ if and only if $W_Z \subseteq W_{Z'}$

Here is a diagram of specializations for $\mathcal{H}^5_{2,2}$. The non-reduced structures on points, lines and planes are represented by shadings.



Remark 3.4.16. In [94], Vainsencher uses the map $\Xi : \mathrm{Bl}_{\Gamma_2} \mathrm{Bl}_{\Gamma_1} \mathbf{Gr}(2,5)^2 \to \mathcal{H}^5_{2,2}$ to compute the degree of a family of rational cubic fourfolds in $\mathbf{P}^5$. However, he does not prove the smoothness of $\mathcal{H}^5_{2,2}$.

Remark 3.4.17. In [16] it was shown that $\mathcal{H}^n_{n-2,n-2}$ meets exactly one other component in $\mathrm{Hilb}^{P^n_{n-2,n-2}}(\mathbf{P}^n)$ and that this component is smooth. We will give two examples to show that these statements are false in general.

The component $\mathcal{H}^5_{2,2}$ will meet the component whose general member parameterizes a pair of 2-planes meeting at a point union an isolated point. It will also meet the component whose general member parameterizes a quadric union an isolated line.

In Chapter 5 we will see that $\mathrm{Hilb}^{P^n_{n-2,1}}(\mathbf{P}^n)$ is a union of $\mathcal{H}^n_{n-2,1}$ and a component $\mathcal{Y}_2$, whose general point parameterizes a line meeting an $(n-2)$-plane union an isolated point. Moreover, $\mathcal{Y}_2$ is singular and its singularity is a cone over the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^{n-2} \hookrightarrow \mathbf{P}^{2(n-1)-1}$.

Chapter 4

Pair of linear spaces - Birational Geometry

In this chapter we prove that when $\mathrm{char}(\mathbf{k}) = 0$, the Hilbert scheme of a pair of linear spaces is a Mori dream space. The main idea is to use our explicit description of Ξ obtained in Chapter 3 and the classification of ideals to completely describe the effective and nef cones of $\mathcal{H}^n_{n-k,n-k}$. We also determine the pairs (k, n) for which the component is Fano.

Notation 4.0.1. For the rest of the chapter $\mathbf{k}$ will denote an algebraically closed field of characteristic 0. Λ_m will always denote an m-dimensional linear subspace of $\mathbf{P}^n$. We begin with a description of the divisors.

Definition 4.0.2. Let Y be a smooth projective variety with $\mathrm{Cl}(Y)$ finitely generated. Then Y is a **Mori dream space** if the Cox Ring of Y is finitely generated over $\mathbf{k}$. The Cox ring of Y is defined to be

$$\bigoplus_{\mathbf{m}\in\mathbf{Z}^k} H^0\big(Y, \mathcal{O}_Y\big(\textstyle\sum_i \mathbf{m}_i D_i\big)\big)$$

where $D_1, \ldots, D_k$ are chosen to generate $\mathrm{Cl}(Y)$.

Definition 4.0.3. Let $n \geq 2k-1$. For each $1 \leq i \leq k-1$ and a choice of a flag of linear spaces $\{\Lambda_{i-1} \subseteq \Lambda_{2k-1-i}\}$, let D_i denote the divisor class of the locus of subschemes $Z \in \mathcal{H}^n_{n-k,n-k}$, for which the linear span of $\Lambda_{i-1} \cup (Z \cap \Lambda_{2k-1-i})$ has dimension less than $2k - i - 1$. Let D_k denote the divisor class of the locus of subschemes that meet a fixed Λ_{k-1}.

Definition 4.0.4. Let $n \geq 2k - 1$. Let N_1 denote the divisor class of the locus of generically non-reduced subschemes in $\mathcal{H}^n_{n-k,n-k}$. For each $2 \leq i \leq k-1$, let N_i denote the divisor class of the locus of subschemes with an embedded $(n - k + 1 - i)$-plane. If $n = 2k - 1$ let N_k denote the divisor class of the locus of subschemes with an embedded point. If $n > 2k - 1$ let N_k denote the class of the closure of the locus of pairs of planes meeting transversely, where the intersection of the two planes meets a fixed Λ_{2k-1}.

Here are the results when the pair of planes span $\mathbf{P}^n$.

Theorem 4.0.5. *Let $k \geq 2$ and $n \geq 2k-1$. The component $\mathcal{H}^n_{n-k,n-k}$ is a Mori dream space and we have,*

$$\mathrm{Eff}(\mathcal{H}^n_{n-k,n-k}) = \langle N_1, \ldots, N_k \rangle \quad \textit{and} \quad \mathrm{Nef}(\mathcal{H}^n_{n-k,n-k}) = \langle D_1, \ldots, D_k \rangle.$$

Moreover, $\mathcal{H}^n_{n-k,n-k}$ is Fano if and only if either $k = 3$ and $n = 5$, or $k \neq 3$ and $n \in \{2k-1, 2k\}$.

To state the results when the pair of planes do not span $\mathbf{P}^n$, it is more convenient to use dimension instead of codimension to index the component. In particular, the component parameterizing subschemes that do not span $\mathbf{P}^n$ are of the form $\mathcal{H}^n_{k-1,k-1}$ with $n > 2k-1$.

Definition 4.0.6. Let $n > 2k-1$. For each $1 \leq i \leq k-1$ and a choice of flag $\{\Lambda_{n-2k+i} \subseteq \Lambda_{n-i}\}$, let D'_i denote the divisor class of the locus of subschemes $Z \in \mathcal{H}^n_{k-1,k-1}$, for which the linear span of $\Lambda_{n-2k+i} \cup (\Lambda_{n-i} \cap Z)$ has dimension less than $n-i$. Let D'_k denote the divisor class of the locus of subschemes meeting a fixed Λ_{n-k}. Let F denote the divisor class of the locus of subschemes whose linear span meets a fixed Λ_{n-2k}.

Definition 4.0.7. Let $n > 2k-1$. Let N'_1 denote the divisor class of the locus of generically non-reduced subschemes in $\mathcal{H}^n_{k-1,k-1}$. For each $2 \leq i \leq k$, let N'_i denote the divisor class of the locus of subschemes with an embedded $(k-i)$-plane.

Here are the results when the pair of planes do not span $\mathbf{P}^n$.

Theorem 4.0.8. *Let $k \geq 2$ and $n > 2k-1$. The component $\mathcal{H}^n_{k-1,k-1}$ is Fano and thus a Mori dream space. Moreover we have,*

$$\mathrm{Eff}(\mathcal{H}^n_{k-1,k-1}) = \langle N'_1, \ldots, N'_k, F \rangle \quad \textit{and} \quad \mathrm{Nef}(\mathcal{H}^n_{k-1,k-1}) = \langle D'_1, \ldots, D'_k, F \rangle.$$

Analogous results for $\mathcal{H}_{n-c,n-d}$ with $c \neq d$ can be found in [81].

4.1 Divisors when the pair of planes span $\mathbf{P}^n$

In this section we study the Picard group of $\mathcal{H}^n_{n-k,n-k}$ for $n \geq 2k-1$. We give an explicit description of the divisors D_i, N_i (Remark 4.1.6, Remark 4.1.9) and describe equations for their pullback along $\Xi|_{U_{k-1}}$.

Notation 4.1.1. We will use λ_k to denote the coordinate $T^{(k)}_{0,n-k+1}$ on U_{k-1} from Remark 3.2.2. This convention will simplify the formulas for the equations we will obtain.

The proofs of Theorem 3.4.13 and Lemma 3.2.4 give explicit equations for the various loci of embedded structures.

Lemma 4.1.2. *Let $n \geq 2k-1$ and let Z be a subscheme parameterized by $\Xi(U_{k-1})$. Then*

(i) *Z is a pair of planes meeting transversely if and only if $\lambda_1, \dots, \lambda_{k-1}, \mathbf{T}^{(k)} \neq 0$.*

(ii) *Z has an embedded $(n-2k+1)$-plane if and only if $\mathbf{T}^{(k)} = 0$.*

(iii) *For each $2 \leq i \leq k-1$, Z has an embedded $(n-k+1-i)$-plane if and only if $\lambda_i = 0$.*

(iv) *Z is generically non-reduced if and only if $\lambda_1 = 0$.*

Definition 4.1.3. Consider the sequence of blowups

$$\mathcal{X}_{k-1} \xrightarrow{\psi_{k-1}} \mathcal{X}_{k-2} \xrightarrow{\psi_{k-2}} \cdots \xrightarrow{\psi_1} \mathcal{X}_0.$$

For each i, let E_i denote the strict transform in $\mathcal{X}_{k-1}$ of the exceptional divisor of ψ_i. Let E_k denote the strict transform of Γ_k.

Lemma 4.1.4. *Let $n \geq 2k-1$. Then $N^1(\mathcal{H}^n_{n-k,n-k}) = \mathrm{Cl}(\mathcal{H}^n_{n-k,n-k}) = \mathbf{Z}^k$. In particular, linear equivalence and numerical equivalence for divisors coincide.*

Proof. Since $\mathcal{H}^n_{n-k,n-k} = \mathcal{X}_{k-1}/\mathfrak{S}_2$ is a smooth rational variety, its class group is torsion free. In particular, $N^1(\mathcal{X}_{k-1}/\mathfrak{S}_2) = \mathrm{Cl}(\mathcal{X}_{k-1}/\mathfrak{S}_2)$. Thus it suffices to prove that $\mathrm{Cl}(\mathcal{X}_{k-1}/\mathfrak{S}_2)_{\mathbf{Q}} := \mathrm{Cl}(\mathcal{X}_{k-1}/\mathfrak{S}_2) \otimes \mathbf{Q}$ is isomorphic to $\mathbf{Q}^k$. By [31, Example 1.7.6] we have $\mathrm{Cl}(\mathcal{X}_{k-1}/\mathfrak{S}_2)_{\mathbf{Q}} = \mathrm{Cl}(\mathcal{X}_{k-1})^{\mathfrak{S}_2}_{\mathbf{Q}}$. Let $E_{1,0}$ and $E_{0,1}$ be the strict transform, in $\mathcal{X}_{k-1}$, of $\mathcal{O}_{\mathcal{X}_0}(1,0)$ and $\mathcal{O}_{\mathcal{X}_0}(0,1)$, respectively. By [47, Theorem 8.5], $\mathrm{Cl}(\mathcal{X}_{k-1})_{\mathbf{Q}}$ is freely generated by $E_1, \dots, E_{k-1}, E_{1,0}, E_{0,1}$. Since $\mathfrak{S}_2$ fixes E_i and interchanges $E_{1,0}$ with $E_{0,1}$, it follows that

$$\mathrm{Cl}(\mathcal{X}_{k-1})^{\mathfrak{S}_2}_{\mathbf{Q}} = \mathrm{span}_{\mathbf{Q}}\{E_1, \dots, E_{k-1}, E_{1,0} + E_{0,1}\} \simeq \mathbf{Q}^k. \quad \square$$

Definition 4.1.5. Let $(\mathcal{X}_0)^{\mathrm{trv}} = \mathcal{X}_0 \setminus \Gamma_k$ denote the open subset of $\mathcal{X}_0$ consisting of pairs of $(n-k)$-planes such that the two planes in the pair meet transversely. We say that a pair of $(n-k)$-planes meets another plane Λ transversely, if each plane in the pair meets Λ transversely.

We now describe D_i as a scheme theoretic image under Ξ.

Remark 4.1.6. For each $1 \leq i \leq k-1$ consider a flag $\mathcal{F}_i = \{\Lambda_{i-1} \subseteq \Lambda_{2k-1-i}\}$. Let $W_i \subseteq (\mathcal{X}_0)^{\mathrm{trv}}$ be the open subset consisting of pairs of planes that meet Λ_{2k-1-i} transversely. Let $\hat{D}_i$ denote the (scheme theoretic) closure of

$$\{Z \in W_i : \dim_{\mathbf{k}} \mathrm{span}(\Lambda_{i-1} \cup (Z \cap \Lambda_{2k-1-i})) < 2k-1-i\}$$

in $\mathcal{X}_0$. Then D_i is the image of the strict transform of $\hat{D}_i$ under the map Ξ.

Similarly, given a plane Λ_{k-1}, let $\hat{D}_k$ be the scheme theoretic closure of

$$\{Z \in (\mathcal{X}_0)^{\mathrm{trv}} : Z \cap \Lambda_{k-1} \neq \emptyset\}$$

in $\mathcal{X}_0$. Then D_k is the image of the strict transform of $\hat{D}_k$ under the map Ξ.

Lemma 4.1.7. *The loci D_i are divisorial. For $1 \le i \le k-1$ let D_i be defined by the flag*

$$\Lambda_{i-1} = V(x_{i-1}, x_{i+1}, \dots, x_n) \subseteq \Lambda_{2k-i-1} = V(x_k, \dots, x_{n-k_{i-2}}, x_{n-k_i} - x_{n-k_{i-1}}). \tag{4.1}$$

Then $\Xi^\star(D_i) \cap U_{k-1}$ is cut out by $T^{(k-i)}_{i-1,n-k_i} + T^{(k-i)}_{i-1,n-k_i} T^{(k-i)}_{i,n-k_{i-1}} + \lambda_{k-i+1}$.

Proof. Assume $1 \le i \le k-1$ and let D_i be defined by the flag Eq. (4.1). To show that D_i is a divisor, it suffices to show that $\hat{D}_i \cap W_i$ is a divisor in W_i (notation from Remark 4.1.6). By symmetry, it is enough to show that $\hat{D}_i \cap W_i \cap U_0$ is a divisor in $W_i \cap U_0$.

Given a point $(\Lambda(\mathbf{a}), \Lambda(\mathbf{b})) \in W_i \cap U_0$ we have $(\Lambda(\mathbf{a}) \cup \Lambda(\mathbf{b})) \cap \Lambda_{2k-1-i} = P \cup Q$ for a pair of $(k-1-i)$-planes, P and Q. For each $n - k_{i+1} \le j \le n$ let p_j (respectively q_j) denote the point in P (respectively Q) obtained by setting $x_j = 1$ and $x_\ell = 0$ for all *other* $\ell \ge k$ (there are no such points for $i = k-1$). Explicitly,

$$\begin{aligned} p_j &= (-a_{0,j} : \cdots : -a_{k-1,j} : 0 : \cdots : 0 : 1 : 0 : \cdots : 0) \\ q_j &= (-b_{0,j} : \cdots : -b_{k-1,j} : 0 : \cdots : 0 : 1 : 0 : \cdots : 0). \end{aligned}$$

Let p_{n-k_i} (respectively q_{n-k_i}) denote the point in P (respectively Q) obtained by setting $x_{n-k_i} = x_{n-k_{i-1}} = 1$ and $x_\ell = 0$ for all other $\ell \ge k$. Explicitly,

$$\begin{aligned} p_{n-k_i} &= (-a_{0,n-k_i} - a_{0,n-k_{i-1}} : \cdots : -a_{k-1,n-k_i} - a_{k-1,n-k_{i-1}} : 0 : \cdots : 0 : 1 : 1 : 0 : \cdots : 0) \\ q_{n-k_i} &= (-b_{0,n-k_i} - b_{0,n-k_{i-1}} : \cdots : -b_{k-1,n-k_i} - b_{k-1,n-k_{i-1}} : 0 : \cdots : 0 : 1 : 1 : 0 : \cdots : 0). \end{aligned}$$

For each $\ell \in \{0, \dots, i-2, i\}$ let $r_\ell = V(x_0, \dots, x_{\ell-1}, x_{\ell+1}, \dots, x_n)$.

By construction we have, $P = \operatorname{span}(p_{n-k_i}, \dots, p_n)$, $Q = \operatorname{span}(q_{n-k_i}, \dots, q_n)$ and $\Lambda_{i-1} = \operatorname{span}(r_0, \dots, r_{i-2}, r_i)$. It follows that the points in $\operatorname{span}(\Lambda_{i-1} \cup ((\Lambda(\mathbf{a}) \cup \Lambda(\mathbf{b})) \cap \Lambda_{2k-1-i}))$ are in the row span of the matrix

$$\begin{bmatrix} q_{n-k_i} & \cdots & q_n & p_{n-k_i} & \cdots & p_n & r_0 & \cdots & r_{i-2} & r_i \end{bmatrix}^T.$$

In particular, $\hat{D}_i \cap W_i \cap U_0$ is the locus where the matrix has rank less than $2k - i$. Let $\epsilon_{l,j} = a_{l,j} - b_{l,j}$ and apply the row operation

$$\begin{pmatrix} q_{n-k_i} \\ q_{n-k_{i+1}} \\ \vdots \\ q_n \\ p_{n-k_i} \\ \vdots \\ p_n \\ r_0 \\ \vdots \\ r_{i-2} \\ r_i \end{pmatrix} \longrightarrow \begin{pmatrix} q_{n-k_i} - p_{n-k_i} - \sum_l (\epsilon_{l,n-k_i} + \epsilon_{l,n-k_{i-1}}) r_l \\ q_{n-k_{i+1}} - p_{n-k_{i+1}} - \sum_l \epsilon_{l,n-k_{i+1}} r_l \\ \vdots \\ q_n - p_n - \sum_l \epsilon_{l,n} r_l \\ p_{n-k_i} \\ \vdots \\ p_n \\ r_0 \\ \vdots \\ r_{i-2} \\ r_i \end{pmatrix}$$

and swap the i-th column and $(i-1)$-st column. It follows that the locus is cut out by the determinant of the submatrix

$$\begin{pmatrix} \epsilon_{i-1,n-k_i} + \epsilon_{i-1,n-k_{i-1}} & \epsilon_{i+1,n-k_i} + \epsilon_{i+1,n-k_{i-1}} & \cdots & \epsilon_{k-1,n-k_i} + \epsilon_{k-1,n-k_{i-1}} \\ \epsilon_{i-1,n-k_{i+1}} & \epsilon_{i+1,n-k_{i+1}} & \cdots & \epsilon_{k-1,n-k_{i+1}} \\ \epsilon_{i-1,n-k_{i+2}} & \epsilon_{i+1,n-k_{i+2}} & \cdots & \epsilon_{k-1,n-k_{i+2}} \\ \vdots & \vdots & & \vdots \\ \epsilon_{i-1,n} & \epsilon_{i+1,n} & \cdots & \epsilon_{k-1,n} \end{pmatrix}.$$

Thus $\hat{D}_i \cap W_i \cap U_0$ is a divisor and this determinant also cuts out $\hat{D}_i \cap U_0$.

The strict transform of this determinant cuts out $\Xi^\star(D_i) \cap U_{k-1}$. Pulling back this matrix to U_{k-1} and column reducing as in Proposition 3.2.1 we obtain

$$\begin{pmatrix} \lambda_1 \cdots \lambda_{k-i}(T^{(k-i)}_{i-1,n-k_i} + T^{(k-i)}_{i-1,n-k_{i-1}}) & \star & \cdots & \cdots & \star & \star \\ 0 & \lambda_1 \cdots \lambda_{k-i-1} & \ddots & & & \vdots \\ 0 & 0 & \ddots & \ddots & & \vdots \\ \vdots & \cdots & \ddots & \ddots & \star & \star \\ 0 & \cdots & & 0 & \lambda_1\lambda_2 & \star \\ 0 & \cdots & & 0 & 0 & \lambda_1 \end{pmatrix}.$$

The strict transform of its determinant is $T^{(k-i)}_{i-1,n-k_i} + T^{(k-i)}_{i-1,n-k_{i-1}}$.

- If $i > 1$ we may use Proposition 3.2.1 (ii) to rewrite $T^{(k-i)}_{i-1,n-k_{i-1}} = \lambda_{k-i+1} + T^{(k-i)}_{i-1,n-k_i} T^{(k-i)}_{i,n-k_{i-1}}$.
- If $i = 1$ we may use Remark 3.2.2 to rewrite $T^{(k-1)}_{0,n-k+1} = \lambda_k + T^{(k-1)}_{0,n-k+2} T^{(k-1)}_{1,n-k+1}$.

In either case, $\Xi^\star(D_i) \cap U_{k-1}$ is cut out by the desired equation. Lastly, D_k is a divisor since $\hat{D}_k$ is the Weil divisor associated to $\mathcal{O}_{\mathcal{X}_0}(1,1) \in \operatorname{Pic} \mathcal{X}_0 \simeq \mathbf{Z}^2$. □

Corollary 4.1.8. *Let $0 \le j < i$. For $1 \le i \le k-1$ let D_i be defined by the flag*

$$\Lambda_{i-1} = V(x_j, x_{i+1}, \dots, x_n) \subseteq \Lambda_{2k-i-1} = V(x_k, \dots, x_{n-k_{j-2}}, x_{n-k_j} - x_{n-k_{j-1}}, x_{n-k_{j+1}}, \dots, x_{n-k_i})^{1} \tag{4.2}$$

and let D_k be defined by the plane

$$\Lambda_{k-1} = V(x_j + x_{n-k_j}, x_k, \dots, x_{n-k_{j-1}}, x_{n-k_{j+1}}, \dots, x_n).$$

Then $\Xi^\star(D_i) \cap U_{k-1}$ is cut out by a polynomial in the coordinates of Remark 3.2.2 that is linear in λ_{k-j}.

[1] if $j = 0$ then $k_{j-1} = k_{-1} = k$ is still consistent with our convention, see Remark 3.2.6

Proof. Assume $i \leq k-1$ and $j \neq 0$. Imitating the proof of Lemma 4.1.7 we see that $\Xi^\star(D_i) \cap U_{k-1}$ is cut out by $T^{(k-i)}_{j,n-k_j} + T^{(k-i)}_{j,n-k_{j-1}}$. To express this in terms of our desired coordinates we will use the relation $T^{(\ell)}_{p,q} = T^{(\ell)}_{p,n-\ell+1} T^{(\ell)}_{k-\ell,q} + \lambda_{\ell+1} T^{(\ell+1)}_{p,q}$ which is true for any $q \leq n-k_p$ and any $p < k-\ell$ and $\ell < k-1$ (proof of Proposition 3.2.1). Repeatedly applying this relation we obtain the following expressions

$$T^{(k-i)}_{j,n-k_j} = \sum_{\ell=k-i}^{k-j-1} \lambda_{k-i+1} \cdots \lambda_\ell T^{(\ell)}_{j,n-\ell+1} T^{(\ell)}_{k-\ell,n-k_j} + \lambda_{k-i+1} \cdots \lambda_{k-j}$$

and

$$T^{(k-i)}_{j,q} = \sum_{\ell=k-i}^{k-j-1} \lambda_{k-i+1} \cdots \lambda_\ell T^{(\ell)}_{j,n-\ell+1} T^{(\ell)}_{k-\ell,q} + \lambda_{k-i+1} \cdots \lambda_{k-j} T^{(k-j)}_{j,q} \tag{4.3}$$

for any $q < n-k_j$. Thus $T^{(k-i)}_{j,q}$, as a polynomial in the coordinates of Remark 3.2.2, is linear in λ_{k-j} for all $q \leq n-k_j$. This implies $\Xi^\star(D_i) \cap U_{k-1}$ is linear in λ_{k-j}.

Assume $i \leq k-1$ and $j = 0$. Most of the argument from the previous paragraph still applies in this case. In particular, $\Xi^\star(D_i) \cap U_{k-1}$ is cut out by $T^{(k-i)}_{0,n-k+1} + T^{(k-i)}_{0,n-k}$ and we have

$$T^{(k-i)}_{0,q} = \sum_{\ell=k-i}^{k-2} \lambda_{k-i+1} \cdots \lambda_\ell T^{(\ell)}_{0,n-\ell+1} T^{(\ell)}_{k-\ell,q} + \lambda_{k-i+1} \cdots \lambda_{k-1} T^{(k-1)}_{0,q} \tag{4.4}$$

for all $q \leq n-k+1 = n-k_0$. Notice that $T^{(k-1)}_{0,q} = T^{(k)}_{0,q} + T^{(k-1)}_{0,n-k+2} T^{(k-1)}_{1,q}$ for all $q \leq n-k+1$ and $T^{(k)}_{0,n-k+1} = \lambda_k$ (Remark 3.2.2). Substituting this into Eq. (4.4) we see that $T^{(k-i)}_{0,n-k+1} + T^{(k-i)}_{0,n-k}$ is linear in λ_k.

Finally assume $i = k$. The locus of points $(\Lambda(\mathbf{a}), \Lambda(\mathbf{b})) \in U_0$ meeting Λ_{k-1} is clearly cut out by $(a_{j,n-k_j}-1)(b_{j,n-k_j}-1)$. The pullback of this equation to U_{k-1}, which coincides with the strict transform, defines $\Xi^\star(D_k)$. If $j \neq 0$ we can use Eq. (4.3) to deduce that

$$(a_{j,n-k_j}-1)(b_{j,n-k_j}-1) = \Big(b_{j,n-k_j} + \sum_{\ell=1}^{k-j-1} \lambda_1 \cdots \lambda_\ell T^{(\ell)}_{j,n-\ell+1} T^{(\ell)}_{k-\ell,n-k_j} + \lambda_1 \cdots \lambda_{k-j} - 1\Big)(b_{j,n-k_j}-1).$$

This expression is linear in λ_{k-j}. If $j = 0$ we can argue in the previous paragraph and deduce linearity in λ_k. This completes the proof. □

Here is an alternate description of N_i.

Remark 4.1.9. For each $1 \leq i \leq k-1$, let $N_i = \Xi(E_i)$. If $n = 2k-1$ we let $N_k = \Xi(E_k)$. If $n > 2k-1$, let $\hat{N}_k$ denote the closure in $\mathcal{X}_0$, of the locus of pairs of planes in $\mathcal{X}_0^{\mathrm{trv}}$ where the intersection of the two planes meets a fixed Λ_{2k-1}. Then N_k is the image of the strict transform of $\hat{N}_k$ under Ξ.

In the next lemma we abuse notation and use "=" to mean equality as divisor classes.

Lemma 4.1.10. *Let $n \geq 2k-1$. The loci N_i are divisorial. Moreover, we have*

(i) $\Xi^\star(N_1) = 2E_1$.

(ii) $\Xi^\star(N_i) = E_i$ *for* $2 \leq i \leq k-1$.

(iii) *If* $n = 2k-1$ *then* $\Xi^\star(N_k) = E_k$ *and* $\Xi^\star(N_k) \cap U_{k-1}$ *is cut out by* λ_k.

(iv) *If* $n > 2k-1$ *let* $\Lambda_{2k-1} = V(x_k, \dots, x_{n-k})$ *be the plane defining* N_k. *Then* $\Xi^\star(N_k) \cap U_{k-1}$ *is cut out by* λ_k.

Proof. Assume $1 \leq i \leq k-1$. Remark 4.1.9 implies that the N_i are divisors. Items (i), (ii) and the first half of (iii) follow from the fact that Ξ is a finite, degree 2 map branched along N_1 (although not phrased this way, it is part of the proof of Proposition 3.2.8), see [31, Chapter 1.7]. The rest of item (iii) is a consequence of Lemma 4.1.2 (ii).

Now assume $n > 2k-1$ and let $\hat{N}_k$ be as in Remark 4.1.9. To show that N_k is a divisor it is enough to show that $\hat{N}_k \cap \mathcal{X}_0^{\mathrm{trv}} \cap U_0$ is a divisor in $\mathcal{X}_0^{\mathrm{trv}} \cap U_0$. Given a point $(\Lambda(\mathbf{a}), \Lambda(\mathbf{b})) \in \mathcal{X}_0^{\mathrm{trv}} \cap U_0$, the intersection of the two planes is $\Lambda(\mathbf{a}) \cap \Lambda(\mathbf{b}) = V(\{\sum_{j=k}^n (a_{i,j} - b_{i,j})x_j, y_i\}_{0 \leq i \leq k-1})$. Thus the locus of points in $\mathcal{X}_0^{\mathrm{trv}} \cap U_0$ satisfying $(\Lambda(\mathbf{a}) \cap \Lambda(\mathbf{b})) \cap \Lambda_{2k-1} \neq \emptyset$ is cut out by the determinant of

$$\begin{pmatrix} a_{0,n-k+1} - b_{0,n-k+1} & \cdots & a_{k-1,n-k+1} - b_{k-1,n-k+1} \\ \vdots & & \vdots \\ a_{0,n} - b_{0,n} & \cdots & a_{k-1,n} - b_{k-1,n} \end{pmatrix}$$

Column reducing as in Proposition 3.2.1 (ii) and taking the strict transform gives item (iv). $\square$

4.2 Effective and nef cones

This section is devoted to the proof of Proposition 4.2.12. For the rest of the section we will assume $n \geq 2k-1$. We begin by constructing two families of curves and computing their intersection numbers with D_i and N_i.

Roughly speaking, the first family of curves will fix a pair of planes and vary the embedded structures while the second family will vary the planes and fix the embedded structures.

Definition 4.2.1. For each $1 \leq j \leq k-1$, define the curve $C_j : \mathbf{P}^1 \to \mathcal{H}^n_{n-k,n-k}$ by

$$C_j(s:t) = I_\Lambda I_{\Lambda'} + (s x_{j-1} x_{n-k_j} - t x_j x_{n-k_{j-1}}) + \sum_{p=0}^{j-2} x_p (x_{n-k_{p+1}}, \dots, x_{n-k_j})$$

with $\Lambda = V(x_0, \dots, x_{k-1})$ and $\Lambda' = V(x_0, \dots, x_j, x_{j+1} + x_{n-k_{j+1}}, \dots, x_{k-1} + x_n)$.

Remark 4.2.2. Theorem 3.4.13 shows that $C_j(s:t)$ is projectively equivalent to Eq. (3.8) with

$$\mu_1 = \cdots = \mu_{k-j-1} = 1, \quad \mu_{k-j} = 0, \quad \mu_{k-j+1} = \begin{cases} \frac{t}{s} \text{ if } s \neq 0 \\ 0 \text{ if } s = 0 \end{cases}, \quad \mu_{k-j+2} = \cdots = \mu_k = 0.$$

It also shows that for $j \leq k-2$, the general member of C_j is a pair of $(n-k)$-planes meeting along a pencil of embedded $(n-2k+j+1)$-planes and containing fixed embedded $(n-2k+\ell)$-planes for all $1 \leq \ell \leq j-1$, while C_{k-1} is a pencil of generically non-reduced $(n-k)$-planes. If $(s:t) = (1:0), (0:1)$, the corresponding subscheme has an embedded $(n-2k+j)$-plane.

Definition 4.2.3. Let $0 \leq j \leq k-1$. Let $\Lambda = V(x_0, \ldots, x_{k-1})$ and consider the pencil of $(n-k)$-planes $\Lambda'(s:t) = V(x_0, \ldots, x_{j-1}, sx_j + tx_{n-k_j}, x_{j+1} + x_{n-k_{j+1}}, \ldots, x_{k-1} + x_n)$. Define the curve $B_j : \mathbf{P}^1 \to \mathcal{H}^n_{n-k,n-k}$ by

$$B_j(s:t) = I_\Lambda I_{\Lambda'(s:t)} + (x_p x_{n-k_q} - x_q x_{n-k_p})_{0 \leq p < q \leq j-1} + (x_0, \ldots, x_{j-1})x_{n-k_j}.$$

Remark 4.2.4. Theorem 3.4.13 shows that $B_j(s:t)$ is projectively equivalent to Eq. (3.8) with

$$\mu_1 = \cdots = \mu_{k-j-1} = 1, \quad \mu_{k-j} = \begin{cases} \frac{t}{s} \text{ if } s \neq 0 \\ 1 \text{ if } s = 0 \end{cases}, \quad \mu_{k-j+1} = 0, \quad \mu_{k-j+2} = \cdots = \mu_k = 1.$$

If $(s:t) \neq (1:0)$, then $B_0(s:t)$ is a pair of $(n-k)$-planes meeting transversely while $B_j(s:t)$ a pair of $(n-k)$-planes with a pure embedded $(n-2k+j)$-plane for $j > 0$. Moreover, the embedded $(n-2k+j)$-plane is fixed along the curve.

If $(s:t) = (1:0)$, the corresponding subscheme has an embedded $(n-2k+j+1)$-plane. Note that $B_{k-1}(1:0)$ is, more precisely, a generically non-reduced $(n-k)$-plane.

Before we determine the intersection numbers we need to compute a few linear spans. We begin with notation that will be used a great deal in the following Lemmas.

Notation 4.2.5. We use $C_j^\dagger(s:t)$ and $B_j^\dagger(s:t)$ to denote the subschemes of $\mathbf{P}^n$ cut out by $C_j(s:t)$ and $B_j(s:t)$, respectively. Given an ideal $J \subseteq S$, let $\operatorname{sat}(J)$ denote its saturation with respect to $(x_0, \ldots, x_n)$ and let $J(1)$ denote the ideal generated by the linear forms in J.

Lemma 4.2.6. *Let $1 \leq i \leq j \leq k-1$ and let $\Lambda_{2k-i-1} = V(x_k, x_{k+1}, \ldots, x_{n-k_{i-2}}, x_{n-k_i} - x_{n-k_{i-1}})$. For any $(s:t) \in \mathbf{P}^1$, if $i \neq j$ the linear span of $C_j^\dagger(s:t) \cap \Lambda_{2k-i-1}$ is*

$$V(x_0, \ldots, x_{i-1}, x_k, \ldots, x_{n-k_{i-2}}, x_{n-k_i} - x_{n-k_{i-1}})$$

and if $i = j$ the linear span of $C_i^\dagger(s:t) \cap \Lambda_{2k-i-1}$ is

$$V(x_0, \ldots, x_{i-2}, sx_{i-1} - tx_i, x_k, \ldots, x_{n-k_{i-2}}, x_{n-k_i} - x_{n-k_{i-1}}).$$

Proof. Let $\Lambda = \Lambda_{2k-i-1}$ and note that the linear span of $C_j^{\dagger}(s:t) \cap \Lambda$ is cut out by $\mathrm{sat}(C_j(s:t) + I_\Lambda)(1)$. Assume $i < j$. It is straigthtforward to see that $x_\ell(x_0, \dots, x_n) \subseteq C_j(s:t) + I_\Lambda$ for every $0 \le \ell \le i-1$. Thus we have

$$\begin{aligned}\mathrm{sat}(C_j(s:t) + I_\Lambda) \supseteq \quad & I_\Lambda + (x_0, \dots, x_{i-1}) + (x_i, \dots, x_{k-1})(x_i, \dots, x_j, x_{j+1} + x_{n-k_{j+1}}, \dots, x_{k-1} + x_n) \\ & + (s x_{j-1} x_{n-k_j} - t x_j x_{n-k_{j-1}}) + \sum_{p=i}^{j-2} x_p (x_{n-k_{p+1}}, \dots, x_{n-k_j}) \\ = \quad & \mathfrak{Q}.\end{aligned}$$

Moreover, it is clear that $\mathfrak{Q}(d) = (C_j(s:t) + I_\Lambda)(d)$ for all $d \ge 2$. Thus if we show that $\mathfrak{Q}$ is saturated then $\mathfrak{Q} = \mathrm{sat}(C_j(s:t) + I_\Lambda)$, and this would give the desired linear span. If we write $\mathfrak{Q} = I_\Lambda + (x_0, \dots, x_{i-1}) + \mathfrak{Q}'$, it suffices to show that quadratic portion, $\mathfrak{Q}'$, is saturated. But notice that $\mathfrak{Q}'$ is projectively equivalent to an ideal of the form Eq. (3.8) (for reasons similar to Remark 4.2.2). It follows from Lemma 3.2.7 that $\mathfrak{Q}$ is saturated. The case of $i = j$ is analogous. □

Remark 4.2.7. Here are two simple facts about linear spans:

(i) If Λ_p and Λ_q are disjoint linear spaces in $\mathbf{P}^n$ then $\dim_{\mathbf{k}} \mathrm{span}(\Lambda_p \cup \Lambda_q) = p + q + 1$.

(ii) $\mathrm{span}(Y_1 \cup Y_2) = \mathrm{span}(\mathrm{span}\, Y_1 \cup \mathrm{span}\, Y_2)$ for any subschemes $Y_1, Y_2 \subseteq \mathbf{P}^n$.

The first fact is clear and the second follows from the following chain of equalities,

$$I_{Y_1 \cup Y_2}(1) = (I_{Y_1} \cap I_{Y_2})(1) = (I_{Y_1}(1) \cap I_{Y_2}(1))(1).$$

Lemma 4.2.8. *Let $1 \le i \le k$ and $1 \le j \le k-1$. We have the following intersection numbers*

(i) $D_i \cdot C_j = 0$ *whenever* $i \ne j$,

(ii) $D_i \cdot C_i = 1$ *for all* $i \le k-1$.

Proof. Assume $i > j$. Since the dimension of any embedded subscheme of $C_j^{\dagger}(s:t)$ is at most $n - 2k + j + 1$, a generic $(2k-1-i)$-plane will not intersect any embedded subscheme of $C_j^{\dagger}(s:t)$. If $i < k$, the intersection of $C_j^{\dagger}(s:t)$ with a generic Λ_{2k-1-i} is a pair of skew $(k-1-i)$-planes. Moreover, these skew planes are independent of $(s:t)$ and thus

$$\mathrm{span}\,(C_j^{\dagger}(s:t) \cap \Lambda_{2k-1-i}) \simeq \mathbf{P}^{2k-2i-1}$$

is independent of $(s:t)$. As a consequence, we may choose an $(i-1)$-plane $\Lambda_{i-1} \subseteq \Lambda_{2k-1-i}$ that does not meet the $\mathbf{P}^{2k-2i-1}$. It follows from Remark 4.2.7 that

$$\dim_{\mathbf{k}} \mathrm{span}\,(\Lambda_{i-1} \cup (C_j^{\dagger}(s:t) \cap \Lambda_{2k-1-i})) = 2k - 1 - i.$$

If we use the flag $\{\Lambda_{i-1} \subseteq \Lambda_{2k-1-i}\}$ to define D_i we see that $D_i \cdot C_j = 0$. Similarly, if $i = k$ and Λ_{k-1} is generic we have that $C_j^\dagger(s:t) \cap \Lambda_{k-1} = \emptyset$. Thus $D_k \cdot C_j = 0$.

Assume $i < j$ and let $\Lambda_{2k-i-1} = V(x_k, x_{k+1}, \ldots, x_{n-k_{i-2}}, x_{n-k_i} - x_{n-k_{i-1}})$. By Lemma 4.2.6 we have that

$$\operatorname{span}(C_j^\dagger(s:t) \cap \Lambda_{2k-1-i}) = V(x_0, \ldots, x_{i-1}, x_k, x_{k+1}, \ldots, x_{n-k_{i-2}}, x_{n-k_i} - x_{n-k_{i-1}}) \simeq \mathbf{P}^{2k-2i-1}$$

is fixed and independent of $(s:t)$. As done in the previous paragraph, if we choose a general Λ_{i-1} inside Λ_{2k-1-i} to define D_i, then $D_i \cdot C_j = 0$. This completes the proof of item (i).

Assume $i = j$ and let the flag $\{\Lambda_{i-1} \subseteq \Lambda_{2k-1-i}\}$ in Eq. (4.1) define D_i. By Lemma 4.2.6 we have that

$$\operatorname{span}(C_i^\dagger(s:t) \cap \Lambda_{2k-1-i}) = V(x_0, \ldots, x_{i-2}, sx_{i-1} - tx_i, x_k, x_{k+1}, \ldots, x_{n-k_{i-2}}, x_{n-k_i} - x_{n-k_{i-1}})$$

Thus, if $t \neq 0$, the linear span of $(C_i^\dagger(1:t) \cap \Lambda_{2k-i-1}) \cup \Lambda_{i-1}$ is all of Λ_{2k-i-1}. If $t = 0$, the linear span of $(C_i^\dagger(1:0) \cap \Lambda_{2k-i-1}) \cup \Lambda_{i-1}$ is $\Lambda_{2k-i-1} \cap V(x_{i-1})$. Thus $D_i \cap C_i$ is supported on the point $Z_0 = C_i(1:0)$.

Let $\tilde{C}_i$ denote the closure in $\mathcal{X}_{k-1}$ of the curve, $\mathbf{A}^1 \hookrightarrow U_{k-1}$ obtained by setting $\lambda_1, \ldots, \lambda_{k-i-1} = 1$, $\lambda_{k-i+1} = t$ and all the other coordinates of Remark 3.2.2 to 0. Since $\Xi(\tilde{C}_i)|_{U_{k-1}} = C_i(1:t)$ it follows that $\Xi(\tilde{C}_i) = C_i$. In particular $\tilde{C}_i \cap \Xi^\star(D_i)$ is supported at a unique point $\tilde{Z}_0 \in \Xi^{-1}(Z_0)$. Since $\Xi^\star(D_i)$ is linear in λ_{k-i+1} (Lemma 4.1.7), it follows that $\Xi^\star(D_i)$ and $\tilde{C}_i$ intersect transversely at $\tilde{Z}_0$. Using the push-pull formula we conclude that $C_i \cdot D_i = \Xi_\star \tilde{C}_i \cdot D_i = \Xi_\star(\tilde{C}_i \cdot \Xi^\star(D_i)) = 1$. ◻

Lemma 4.2.9. *Let $1 \leq i \leq k$ and $0 \leq j \leq k-1$. We have the following intersection numbers*

(i) $D_i \cdot B_j = 0$ *for all* $i \leq j$,

(ii) $D_i \cdot B_j = 1$ *for all* $i > j$.

Proof. Assume $i \leq j$ and let $\Lambda_{2k-1-i} = V(x_k, \ldots, x_{n-k_{i-2}}, x_{n-k_i} - x_{n-k_{i-1}})$. Arguing as in Lemma 4.2.6 we see that

$$\operatorname{span}(\Lambda_{2k-1-i} \cap B_j^\dagger(s:t)) = V(x_0, \ldots, x_{i-1}, x_k, x_{k+1}, \ldots, x_{n-k_{i-1}}) \simeq \mathbf{P}^{2k-2i-1}$$

is independent of $(s:t)$. Arguing as in Lemma 4.2.8 we deduce item (i).

Assume that $j < i \leq k-1$ and let $\{\Lambda_{i-1} \subseteq \Lambda_{2k-1-i}\}$ be the flag Eq. (4.2) defining D_i. Then $B_j^\dagger(s:t) \cap \Lambda_{2k-i-1}$ is a disjoint pair of $(k-i-1)$-planes defined by

$$\begin{gathered}(x_0, \ldots, x_{j-1}, sx_j + tx_{n-k_j}, x_{j+1}, \ldots, x_i, x_{i+1} + x_{n-k_{i+1}}, \ldots, x_{k-1} + x_n, \\ x_k, x_{k+1}, \ldots, x_{n-k_{j-2}}, x_{n-k_j} - x_{n-k_{j-1}}, x_{n-k_{j+1}}, \ldots, x_{n-k_i}) \cap \\ (x_0, \ldots, x_{n-k_{j-2}}, x_{n-k_j} - x_{n-k_{j-1}}, x_{n-k_{j+1}}, \ldots, x_{n-k_i}).\end{gathered}$$

For $t \neq 0$, the linear span of $(B_j^\dagger(s:t) \cap \Lambda_{2k-i-1}) \cup \Lambda_{i-1}$ is all of Λ_{2k-i-1}. On the other hand if $t = 0$, the linear span of $(B_j^\dagger(s:t) \cap \Lambda_{2k-i-1}) \cup \Lambda_{i-1}$ is $\Lambda_{2k-1-i} \cap V(x_j)$. Thus $D_i \cap B_j$ is supported at the point $Z_0 = B_j(1:0)$.

Let $\tilde{B}_j$ denote the closure in $\mathcal{X}_{k-1}$ of the curve, $\mathbf{A}^1 \hookrightarrow U_{k-1}$ obtained by setting $\lambda_1 = \cdots = \lambda_{k-j-1} = 1$, $\lambda_{k-j} = t$, $\lambda_{k-j+2} = \cdots = \lambda_k = 1$ and all the other coordinates of Remark 3.2.2 to 0. Since $\Xi(\tilde{B}_j)|_{U_{k-1}} = B_j(1:t)$ we have $\Xi(\tilde{B}_j) = B_j$. Thus $\tilde{B}_j \cap \Xi^\star(D_i)$ is supported at a unique point $\tilde{Z}_0 \in \Xi^{-1}(Z_0)$. Since $\Xi^\star(D_i)$ is linear in λ_{k-j} (Corollary 4.1.8), it follows that $\Xi^\star(D_i)$ and $\tilde{B}_j$ intersect transversely at $\tilde{Z}_0$. Using the push-pull formula we conclude that $B_j \cdot D_i = \Xi_\star \tilde{B}_j \cdot D_i = \Xi_\star(\tilde{B}_j \cdot \Xi^\star(D_i)) = 1$.

Now assume $j < i = k$ and let $\Lambda_{k-1} = V(x_j + x_{n-k_j}, x_k, \ldots, x_{n-k_{j-1}}, x_{n-k_{j+1}}, \ldots, x_n)$ be the plane defining D_k. It is evident that $B_j \cap D_k$ is supported at the point $Z_{1,1} = B_j(1:1)$. Once again, $\tilde{B}_j$ (defined in the previous paragraph) and $\Xi^\star(D_k)$ will meet at a unique point $\tilde{Z}_{1,1} \in \Xi^{-1}(Z_{1,1})$. Since $\Xi^\star(D_k)$ is linear in λ_{k-j} (Corollary 4.1.8) we see that $\tilde{B}_j$ meets $\Xi^\star(D_k)$ transversely at $\tilde{Z}_{1,1}$. Once again we conclude using the push-pull formula. □

Lemma 4.2.10. *We have the following intersection numbers,*

(i) $N_i \cdot C_j = 0$ *for each* $1 \leq i \leq k-1$ *and all* $1 \leq j \leq k-i-1$,

(ii) $N_i \cdot B_j = 0$ *for each* $1 \leq i \leq k$ *and all* $j \neq k-i, k-i+1$,

(iii) $N_i \cdot C_{k-i+1} = 2$ *for each* $2 \leq i \leq k$,

(iv) $N_1 \cdot B_{k-1} = 2$ *and* $N_i \cdot B_{k-i} = 1$ *for* $2 \leq i \leq k$.

Proof. Item (i) and item (ii), except for the case of $i = k$, follow from the definition of the N_i and the description of the embedded subschemes in Remark 4.2.2 and Remark 4.2.4. We will deal with the case of $i = k$ in the last paragraph. For the rest of the proof let $Z_0 = C_{k-i+1}(1:0)$ and $Z_\infty = C_{k-i+1}(0:1)$. We will also use the curves $\tilde{C}_{k-i+1}$ and $\tilde{B}_j$ defined in Lemma 4.2.8. In particular, let $\tilde{Z}_0, \tilde{Z}_\infty \in \tilde{C}_{k-i+1}$ be such that $\Xi(\tilde{Z}_0) = Z_0$ and $\Xi(\tilde{Z}_\infty) = Z_\infty$.

Assume $2 \leq i \leq k-1$. Since N_i is the locus of subschemes containing an embedded $(n-k+1-i)$-plane, it meets the curve C_{k-i+1} at Z_0 and Z_∞. Thus $\tilde{C}_{k-i+1}$ meets E_i at $\tilde{Z}_0$ and $\tilde{Z}_\infty$. Using Lemma 4.1.10 (ii), we obtain

$$N_i \cdot C_{k-i+1} = \Xi_\star(\tilde{C}_{k-i+1} \cdot \Xi^\star(N_i)) = \tilde{C}_{k-i+1} \cdot E_i = (\tilde{C}_{k-i+1} \cdot E_i)|_{\tilde{Z}_0} + (\tilde{C}_{k-i+1} \cdot E_i)|_{\tilde{Z}_\infty}.$$

Since $\tilde{Z}_0 \in U_{k-1}$ and E_i is cut out by λ_i, $\tilde{C}_{k-i+1}$ meets E_i transversely at $\tilde{Z}_0$. Symmetrically, $\tilde{C}_{k-i+1}$ will also meet E_i transversally at $\tilde{Z}_\infty$. To see the latter statement, consider the projective transformation $g \in \mathrm{GL}(n+1)$ that interchanges x_j with x_{j-1}, interchanges x_{n-k_j} with $x_{n-k_{j-1}}$ and fixes the other coordinates. It follows from the definition that

$g(C_{k-i+1}) = C_{k-i+1}$ and g interchanges Z_0 with Z_∞. Since intersection multiplicity is invariant under automorphisms of $\mathcal{H}^n_{n-k,n-k}$ we obtain

$$(N_i \cdot C_{k-i+1})|_{Z_\infty} = (g(N_i) \cdot g(C_{k-i+1}))\,|_{g(Z_\infty)} = N_i \cdot C_{k-i+1}|_{Z_0} = (E_i \cdot \tilde{C}_{k-i+1})|_{\tilde{Z}_0} = 1.$$

This proves item (iii) for $i \neq k$.

Since N_1 is the locus of generically non-reduced subschemes, it meets the curve B_{k-1} at $B_{k-1}(1:0)$. Using Lemma 4.1.10 (i) we obtain $N_1 \cdot B_{k-1} = \Xi_\star(\tilde{B}_{k-1} \cdot \Xi^\star(N_1)) = 2\tilde{B}_{k-1} \cdot E_1 = 2$. Similarly, using Lemma 4.1.10 we obtain $N_i \cdot B_{k-i} = 1$ for all $2 \leq i \leq k-1$. This finishes item (iv) for $i \neq k$.

Finally, assume $i = k$ and let $\Lambda_{2k-1} = V(x_k, \dots, x_{n-k})$ be the plane defining N_k (if $n > 2k-1$). By Lemma 4.1.10 (iii), (iv) we see that $\Xi^\star(N_k)$ meets $\tilde{C}_1$ at Z_0 and possibly also at Z_∞ (since the latter does not lie in U_{k-1}). Moreover, $\Xi^\star(N_k)$ meets $\tilde{C}_1$ transversely at $\tilde{Z}_0$. We may argue as in the previous paragraph to show that $\Xi^\star(N_k)$ also meets $\tilde{C}_1$ transversely at $\tilde{Z}_\infty$. Indeed, the projective transformation g fixes N_k. This is clear if $n = 2k-1$ and the case of $n > 2k-1$ follows from the fact that g fixes Λ_{2k-1}. Thus $N_k \cdot C_1 = (N_k \cdot C_1)|_{Z_0} + (N_k \cdot C_1)|_{Z_\infty} = 2(N_k \cdot C_1)|_{Z_0} = 2$, completing the proof of item (iii). For items (ii) and (iv) we argue similarly using the following projective transformation: $g' \in \mathrm{GL}(n+1)$ that maps $x_{n-k_j} \mapsto x_{n-k_j} + x_j$ and fixes the other coordinates. It is straightforward to verify that $g'(B_j) = B_j$, $g'(B_j(0:1)) = B_j(1:1)$ and g' fixes N_k (since g' fixes Λ_{2k-1}). This implies

$$(N_k \cdot B_j)|_{B_j(0:1)} = \left(g'(N_k) \cdot g'(B_j)\right)|_{g'(B_j(0:1))} = (N_k \cdot B_j)|_{B_j(1:1)} = 0$$

for $j \neq 1$. Thus, we may compute $\Xi^\star(N_k) \cdot \tilde{B}_j$ along U_{k-1} to obtain the desired results. □

Proposition 4.2.11. *Let $1 \leq i \leq k$. Then we have*

- $N_1 = 2D_k - 2D_{k-1}$,
- $N_i = 2D_{k-i+1} - D_{k-i} - D_{k-i+2}$ *for all* $2 \leq i \leq k-1$,
- $N_k = 2D_1 - D_2$.

Proof. By Lemma 4.1.4, Lemma 4.2.8 and Lemma 4.2.9 we see that $N^1(\mathcal{H}^n_{n-k,n-k})$ is generated by $\{D_1, \dots, D_k\}$. This allows us to write $N_i = \sum_{\ell=1}^k \epsilon_{i,\ell} D_\ell$ for some $\epsilon_{i,\ell} \in \mathbf{Z}$. Using Lemma 4.2.8 - Lemma 4.2.10 we obtain

- $N_1 \cdot C_\ell = \epsilon_{1,\ell} = 0$ for $\ell \leq k-2$,
- $N_1 \cdot B_{k-1} = \epsilon_{1,k} = 2$ and $N_1 \cdot B_{k-2} = \epsilon_{1,k-1} + \epsilon_{1,k} = 0$.

This immediately implies $N_1 = 2D_k - 2D_{k-1}$. For each $2 \leq i \leq k$ we obtain

- $N_i \cdot B_j = \sum_{\ell=j+1}^k \epsilon_{i,\ell} = 0$ for $j \neq k-i, k-i+1$

- $N_i \cdot B_{k-i} = \sum_{\ell=k-i+1}^{k} \epsilon_{i,\ell} = 1$ and $N_i \cdot C_{k-i+1} = \epsilon_{i,k-i+1} = 2$.

If $i \neq k$, we obtain $\epsilon_{i,k-i} = -1, \epsilon_{i,k-i+1} = 2, \epsilon_{i,k-i+2} = -1$, and $\epsilon_{i,\ell} = 0$ for other ℓ. If $i = k$ we obtain $\epsilon_{k,1} = 2$, $\epsilon_{k,2} = -1$ and $\epsilon_{i,\ell} = 0$ for other ℓ. This completes the proof. □

Proposition 4.2.12. *Let $k \geq 2$ and $n \geq 2k-1$. Then we have*

$$\mathrm{Eff}(\mathcal{H}^n_{n-k,n-k}) = \langle N_1, \dots, N_k \rangle \quad \textit{and} \quad \mathrm{Nef}(\mathcal{H}^n_{n-k,n-k}) = \langle D_1, \dots, D_k \rangle.$$

Moreover, $\mathcal{H}^n_{n-k,n-k}$ is Fano if and only if either $k = 3$ and $n = 5$, or $k \neq 3$ and $n \in \{2k-1, 2k\}$.

Proof. It is clear that the divisors $N_1, \dots, N_k$ are effective and generate $N^1(\mathcal{H}^n_{n-k,n-k})$. To conclude that the effective cone is generated by $N_1, \dots, N_k$, it is enough to show that any $\mathbf{R}$-divisor $N = \sum_{i=1}^{k} \epsilon_i N_i$, with some $\epsilon_j < 0$, is not effective. Let $A_j : \mathbf{P}^1 \hookrightarrow \mathcal{H}^n_{n-k,n-k}$ denote any curve such that for $(s : t) \neq (1 : 0)$, $A_j(s : t)$ is a pair of $(n-k)$-planes meeting transversely while $A_j(1 : 0)$ it is a pair of $(n-k)$-planes with a pure embedded $(n-k+1-j)$-plane if $j > 1$ and generically non-reduced if $j = 1$. Clearly, $A_j \cdot N_i = 0$ for $i \neq j$ and $A_j \cdot N_j > 0$. Since $N \cdot A_j = \epsilon_j < 0$ and A_j is not contained in the support of N, we see that N cannot be an effective divisor.

By varying the flags it is easy to see that each of the D_i is base point free; thus it is also nef. Similar to the previous paragraph, to show that the nef cone gone is generated by $D_1, \dots, D_k$, it is enough to show that any $\mathbf{R}$-divisor $D = \sum_{i=1}^{k} \epsilon_i D_i$, with some $\epsilon_j < 0$, is not nef. If $j \neq k$, we have $D \cdot C_j = \epsilon_j < 0$ and if $j = k$ we have $D \cdot B_{k-1} = \epsilon_k < 0$. Thus D is not nef.

We will now compute the canonical divisor of $\mathcal{H}^n_{n-k,n-k}$ using the branched cover $\Xi : \mathcal{X}_{k-1} \longrightarrow \mathcal{H}^n_{n-k,n-k}$. By [47, Exercise 8.5b] and [24, Exercise 10.10] we may write

$$K_{\mathcal{X}_{k-1}} = \sum_{j=1}^{k-1} ((k-j+1)(n-k-j+2)-1)E_j - (n+1)\hat{D}_k$$

where $\hat{D}_k$ is the strict transform of $\mathcal{O}_{\mathcal{X}_0}(1,1)$ (Remark 4.1.6). Note that the canonical divisor of $\mathcal{X}_0$ is $\mathcal{O}_{\mathcal{X}_0}(-n-1,-n-1)$. Let $K_{\mathcal{H}^n_{n-k,n-k}} = \epsilon_1 N_1 + \cdots + \epsilon_{k-1} N_{k-1} + \epsilon_k D_k$ for some $\epsilon_i \in \mathbf{Q}$. Hurwitz's theorem implies that $K_{\mathcal{X}_{k-1}} = \Xi^\star(K_{\mathcal{H}^n_{n-k,n-k}}) + E_1$. Using this and Lemma 4.1.10 we obtain

$$2\epsilon_1 E_1 + \sum_{j=2}^{k-1} \epsilon_j E_j + \epsilon_k \hat{D}_k = \Xi^\star(K_{\mathcal{H}^n_{n-k,n-k}}) = (k(n-k+1)-2)E_1 + \sum_{j=2}^{k-1} ((k-j+1)(n-k-j+2)-1)E_j - (n+1)\hat{D}_k.$$

Let $\tilde{\epsilon}_j = (k-j+1)(n-k-j+2)-1$ and using Proposition 4.2.11 we obtain

$$K_{\mathcal{H}^n_{n-k,n-k}} = \frac{1}{2}(\tilde{\epsilon}_1 - 1)(2D_k - 2D_{k-1}) + \sum_{j=2}^{k-1} \tilde{\epsilon}_j(2D_{k-j+1} - D_{k-j} - D_{k-j+2}) - (n+1)D_k.$$

For $k = 2, 3$ the above expression simplifies to

$$K_{\mathcal{H}^n_{n-2,n-2}} = (4-2n)D_1 + (n-5)D_2, \quad K_{\mathcal{H}^n_{n-3,n-3}} = (7-2n)D_1 + (n-6)D_2 - 2D_3.$$

If $k \geq 4$ we can rewrite the expression as follows:

$$\begin{aligned} K_{\mathcal{H}^n_{n-k,n-k}} &= (\tilde{\epsilon}_1 - 1)(D_k - D_{k-1}) - (n+1)D_k + \sum_{j=2}^{k-3}(2\tilde{\epsilon}_{j+1} - \tilde{\epsilon}_j - \tilde{\epsilon}_{j+2})D_{k-j} \\ &\qquad - \tilde{\epsilon}_2 D_k + (2\tilde{\epsilon}_2 - \tilde{\epsilon}_3)D_{k-1} + (2\tilde{\epsilon}_{k-1} - \tilde{\epsilon}_{k-2})D_2 - \tilde{\epsilon}_{k-1}D_1 \\ &= (\tilde{\epsilon}_1 - \tilde{\epsilon}_2 - n - 2)D_k + (2\tilde{\epsilon}_2 - \tilde{\epsilon}_3 - \tilde{\epsilon}_1 + 1)D_{k-1} + \sum_{j=2}^{k-3}(2\tilde{\epsilon}_{j+1} - \tilde{\epsilon}_j - \tilde{\epsilon}_{j+2})D_{k-j} \\ &\qquad + (2\tilde{\epsilon}_{k-1} - \tilde{\epsilon}_{k-2})D_2 - \tilde{\epsilon}_{k-1}D_1. \end{aligned}$$

Since $2\tilde{\epsilon}_{j+1} - \tilde{\epsilon}_j - \tilde{\epsilon}_{j+2} = -2$ for all j we obtain

$$K_{\mathcal{H}^n_{n-k,n-k}} = (4k-5-2n)D_1 + (n-2k-1)D_2 - 2D_3 - 2D_4 - \cdots - 2D_{k-2} - D_{k-1} - 2D_k.$$

Since the ample cone is the interior of the nef cone, we see that $-K_{\mathcal{H}^n_{n-2,n-2}}$ is ample if and only if $n = 3, 4$ and that $-K_{\mathcal{H}^n_{n-3,n-3}}$ is ample precisely when $n = 5$. If $k \geq 4$, $-K_{\mathcal{H}^n_{n-k,n-k}}$ is ample if and only if $n = 2k-1, 2k$. □

4.3 Mori dream space

This section is devoted to the proof of Theorem 4.3.14. We will show that $\mathcal{H}^n_{k-1,k-1}$ is Fano, and thus a Mori dream space. By constructing a contraction from $\mathcal{H}^n_{k-1,k-1}$ to $\mathcal{H}^n_{n-k,n-k}$ (Proposition 4.3.11) we will also deduce that $\mathcal{H}^n_{n-k,n-k}$ is a Mori dream space.

Notation 4.3.1. In this section we will primarily be interested in the case when the pair of planes do not span all of $\mathbf{P}^n$. By swapping the roles of codimension and dimension, the components we are interested in are of the form $\mathcal{H}^n_{k-1,k-1}$ with $n > 2k-1$.

Corollary 3.4.8 states that for $n > 2k-1$, the morphism $\rho : \mathcal{H}^n_{k-1,k-1} \longrightarrow \mathbf{Gr}(2k-1, n)$ that sends a scheme to its linear span is smooth; the fiber over a point Λ is $\mathcal{H}_{k-1,k-1}(\Lambda)$.

Remark 4.3.2. Let $W = \operatorname{Spec} \mathbf{k}[f_{2k,j}, \ldots, f_{n,j}]_{0\le j\le 2k-1}$ be a neighbourhood of $\Lambda = V(x_{2k}, \ldots, x_n) \in \mathbf{Gr}(2k-1, n)$ such that its $\mathbf{k}$-points are identified with

$$V(x_{2k} + \sum_{j=0}^{2k-1} f_{2k,j}x_j, \ldots, x_n + \sum_{j=0}^{2k-1} f_{n,j}x_j).$$

Then the open subset $\rho^{-1}(W)$ is naturally isomorphic to $W \times \mathcal{H}_{k-1,k-1}(\Lambda)$.

Lemma 4.3.3. *Let $n > 2k-1$. Then $N^1(\mathcal{H}^n_{k-1,k-1}) = \mathbf{Z}^{k+1}$.*

Proof. As explained in Lemma 4.1.4, since $\mathcal{H}^n_{k-1,k-1}$ is rational and smooth, it suffices to compute $N^1(\mathcal{H}^n_{k-1,k-1}) \otimes \mathbf{Q}$ which equals $\operatorname{Pic}(\mathcal{H}^n_{k-1,k-1}) \otimes \mathbf{Q} = H^2(\mathcal{H}^n_{k-1,k-1}, \mathbf{Q})$. By Corollary 3.4.8 we have a smooth morphism $\mathcal{H}^n_{k-1,k-1} \longrightarrow \mathbf{Gr}(2k-1, n)$ with fibers isomorphic to $\mathcal{H}^{2k-1}_{k-1,k-1}$. Since the base of this morphism is simply connected, we may apply the Leray-Hirsch theorem [97, Theorem 7.33] and Lemma 4.1.4 to deduce that $H^2(\mathcal{H}^n_{k-1,k-1}, \mathbf{Q}) \simeq \mathbf{Q}^{k+1}$. □

Using the fibration ρ and Remark 4.3.2 one can easily verify that the loci D'_i, N'_i, F are divisorial. We now define the curves inside $\mathcal{H}^n_{k-1,k-1}$; all but two of them come from curves lying inside $\mathcal{H}^{2k-1}_{k-1,k-1}$.

Definition 4.3.4. Let $\Lambda = V(x_{2k}, \ldots, x_n)$. For each relevant j, let A'_j, B'_j, C'_j be the images of A_j, B_j, C_j (Definition 4.2.1, Definition 4.2.3, Proposition 4.2.12) under the inclusion $\rho^{-1}(\Lambda) = \mathcal{H}_{k-1,k-1}(\Lambda) \hookrightarrow \mathcal{H}^n_{k-1,k-1}$, respectively.

Definition 4.3.5. Let $\Lambda' = V(x_k, \ldots, x_n)$ and let

$$\Lambda(s:t) = V(x_0, \ldots, x_{k-1}, sx_{2k} + tx_k, x_{2k+1}, \ldots, x_n)$$

be a pencil of $(k-1)$-planes disjoint from Λ'. Define the curve $Y_1 : \mathbf{P}^1 \to \mathcal{H}^n_{k-1,k-1}$ by $(s:t) \mapsto \Lambda(s:t) \cup \Lambda'$. Explicitly

$$Y_1(s:t) = (sx_{2k} + tx_k, x_{2k+1}, \ldots, x_n) + (x_0, \ldots, x_{k-1})(x_k, \ldots, x_{2k-1}).$$

Define the curve $Y_2 : \mathbf{P}^1 \to \mathcal{H}^n_{k-1,k-1}$ by

$$\begin{aligned} Y_2(s:t) = {} & (sx_{2k} + tx_0, x_{2k+1}, \ldots, x_n) + (x_1, \ldots, x_{k-1})(x_{k+1}, \ldots, x_{2k-1}) \\ & + (x_0, x_{2k})^2 + (x_0, x_{2k})(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{2k-1}). \end{aligned}$$

Remark 4.3.6. Let $\Lambda = V(x_0, \ldots, x_{k-1}, x_{2k}, \ldots, x_n)$ and $\Lambda' = V(x_0, x_{k+1}, \ldots, x_n)$ be a pair of $(k-1)$-planes meeting along a point. Then we have

$$Y_2(s:t) = I_\Lambda \cap I_{\Lambda'} \cap ((x_0, x_{2k})^2, sx_{2k} + tx_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{2k-1}, x_{2k+1}, \ldots, x_n).$$

In particular, Y_2 is a pair of fixed $(k-1)$-planes with a pencil of embedded points.

Lemma 4.3.7. *Y_2 is a moving curve in N'_k i.e. its deformations span N'_k.*

Proof. The general subscheme parameterized by N'_k is a pair of $(k-1)$-planes meeting along an embedded point. By Corollary 3.4.8 and Theorem 3.4.13, up to projectively equivalence, such a subscheme is cut out by

$$(x_0, \dots, x_{k-1}, x_{2k}, \dots, x_n) \cap (x_0, x_{k+1}, \dots, x_n) \cap (x_0^2, x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) = Y_2(1:0)$$

In particular, the $\mathrm{GL}(n+1)$ orbit of Y_2 covers a dense subset of N'_k. □

Lemma 4.3.8. *For all pairs of relevant indices i, j (the ones appearing in Lemma 4.2.8, Lemma 4.2.9, Lemma 4.2.10), the intersection numbers of D'_i, N'_i with B'_j, C'_j are the same as the intersection numbers of D_i, N_i with B_j, C_j, respectively.*

Proof. We will only verify $D'_i \cdot C'_j = D_i \cdot C_j$ for $1 \le i, j \le k-1$; the other cases are analogous. Let $\Lambda = V(x_{2k}, \dots, x_n)$ be a fixed $(2k-1)$-plane. Let D'_i be defined by a flag $\mathcal{F}'_i = \{\Lambda_{n-2k+i} \subseteq \Lambda_{n-i}\}$, where the flag is chosen to satisfy the following two properties:

- Λ is transverse to each element of the flag $\mathcal{F}'_i$,
- Let $D_i \subseteq \mathcal{H}_{k-1,k-1}(\Lambda)$ be defined by the flag $\mathcal{F}_i = \{\Lambda_{n-2k+i} \cap \Lambda \subseteq \Lambda_{n-i} \cap \Lambda\}$. Then either $D_i \cap C_j = \emptyset$ if $i \neq j$ or D_i is transverse to C_j if $i = j$.

Let W be the open neighbourhood of Λ from Remark 4.3.2. The first bullet point implies that every element of W is transverse to the flag $\mathcal{F}'_i$. It follows that $D'_i|_{\rho^{-1}(W)} = W \times D_i$ and $C'_j = \{\Lambda\} \times C_j$. Thus we have $D'_i \cdot C'_j = D'_i|_{\rho^{-1}(W)} \cdot C'_j = D_i \cdot C_j$. □

Lemma 4.3.9. *We have the following intersection numbers*

(i) $D'_i \cdot Y_2 = N'_i \cdot Y_1 = 0$ *for all* $1 \le i \le k$,

(ii) $N'_i \cdot Y_2 = 0$ *for all* $1 \le i \le k-1$,

(iii) $D'_i \cdot Y_1 = 1$ *for all* $1 \le i \le k$,

(iv) $F \cdot Y_1 = F \cdot Y_2 = 1$.

Proof. Items (i) and (ii) are clear from the definition of the divisors.

Let $1 \le i \le k$, $\Lambda = V(x_{2k}, \dots, x_n)$ and W be as in Remark 4.3.2. We may choose a flag $\mathcal{F}'_i$ to define D'_i so that the following properties are satisfied:

- Λ is transverse to each element of the flag $\mathcal{F}'_i$,
- $D'_i \cap Y_1$ is supported at $Z_0 = Y_1(1:0)$.

Let $W' = \operatorname{Spec} \mathbf{k}[\epsilon_1, \ldots, \epsilon_{k^2}] \subseteq \mathcal{H}_{k-1,k-1}(\Lambda)$ be any affine open containing the image of Z_0 in $\mathcal{H}_{k-1,k-1}(\Lambda)$. Then $W \times W'$ is identified with an open neighbourhood of $Z_0 \in \mathcal{H}^n_{k-1,k-1}$. Along this open set, Y_1 is the curve obtained by setting $f_{2k,k} = t$, $f_{i,j} = 0$ for other i, j, and $\epsilon_i = \delta_i$ for some constants $\delta_i \in \mathbf{k}$. On the other hand, $D'_i = W \times (D_i \cap W')$ where D_i is the divisor defined by the flag $\mathcal{F}'_i \cap \Lambda$. It immediately follows that D'_i meets Y_1 transversely at Z_0 inside $W \times W'$; this proves item (iii).

For item (iv), we will only verify $F \cdot Y_1 = 1$ as the other case is similar. Let F be defined by the $(n-2k)$-plane, $V(x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{2k})$. It follows that $F \cap Y_1$ is also supported at Z_0. Moreover, along $W \times W'$, F is cut out by the function $f_{2k,k}$. Combining this with the equation of Y_1 along $W \times W'$ we see that F meets Y_1 transversely at Z_0. □

Proposition 4.3.10. *Let $k \geq 2$ and $n > 2k - 1$. Then we have,*

$$\operatorname{Eff}(\mathcal{H}^n_{k-1,k-1}) = \langle N'_1, \ldots, N'_k, F\rangle \quad \textit{and} \quad \operatorname{Nef}(\mathcal{H}^n_{k-1,k-1}) = \langle D'_1, \ldots, D'_k, F\rangle.$$

Moreover we have,

- $N'_1 = 2D'_k - 2D'_{k-1}$,
- $N'_i = 2D'_{k-i+1} - D'_{k-i} - D'_{k-i+2}$ *for all* $2 \leq i \leq k-1$,
- $N'_k = 2D'_1 - D'_2 - F$.

Proof. Using the intersection numbers with the curves $\{C'_1, \ldots, C'_k, Y_2\}$ and arguing as in Proposition 4.2.11, Proposition 4.2.12 we see that $N^1(\mathcal{H}^n_{k-1,k-1})$ and $\operatorname{Nef}(\mathcal{H}^n_{k-1,k-1})$ are both generated by $D'_1, \ldots, D'_k, F$. Using the curves $\{A'_1, \ldots, A'_k, Y_1\}$ and arguing as in Proposition 4.2.12, we see that $N'_1, \ldots, N'_k, F$ generate the effective cone.

By Proposition 4.2.11 and Lemma 4.3.8 there exists $\epsilon_i \in \mathbf{Q}$ such that

- $N'_1 = 2D'_k - 2D'_{k-1} + \epsilon_1 F$,
- $N'_i = 2D'_{k-i+1} - D'_{k-i} - D'_{k-i+2} + \epsilon_i F$ for all $2 \leq i \leq k-1$,
- $N'_k = 2D'_1 - D'_2 + \epsilon_k F$.

Intersecting these divisors with Y_1, Y_2 and using Lemma 4.3.9 we obtain $\epsilon_1, \ldots, \epsilon_{k-1} = 0$ and $\epsilon_k = -1$. □

We are now ready to relate $\mathcal{H}^n_{k-1,k-1}$ with $\mathcal{H}^n_{n-k,n-k}$.

Proposition 4.3.11. *There is a morphism $\Psi : \mathcal{H}^n_{k-1,k-1} \longrightarrow \mathcal{H}^n_{n-k,n-k}$ with exceptional locus N'_k. Moreover, N'_k is a $\mathbf{P}^{n-2k+1}$-fibration over $\Psi(N'_k)$. Geometrically, Ψ "forgets" the embedded points.*

Proof. Given an $(n+1)$-dimensional vector space V, let

$$\Gamma_i(\mathbf{P}V) = \{(\Lambda, \Lambda') : \dim(\Lambda \cap \Lambda') \geq k - i\} \subseteq \mathbf{Gr}(k-1, \mathbf{P}V)^2.$$

By [59, Theorem 6.3] the Hilbert-Chow morphism induces a birational morphism,

$$\mathcal{H}_{k-1,k-1}(\mathbf{P}V) \longrightarrow \mathrm{Sym}^2 \mathbf{Gr}(k-1, \mathbf{P}V).$$

Let $\overline{\Gamma}_i(\mathbf{P}V)$ denote the image of $\Gamma_i(\mathbf{P}V)$ in $\mathrm{Sym}^2 \mathbf{Gr}(k-1, \mathbf{P}V)$. Since the pullback of each $\overline{\Gamma}_i(\mathbf{P}V)$ is N_i', we obtain a morphism

$$\Psi_1 : \mathcal{H}^n_{k-1,k-1} \longrightarrow \mathrm{Bl}_{\overline{\Gamma}_{k-1}(\mathbf{P}V)} \cdots \mathrm{Bl}_{\overline{\Gamma}_1(\mathbf{P}V)} \mathrm{Sym}^2 \mathbf{Gr}(k-1, \mathbf{P}V).$$

There is an isomorphism $\mathbf{Gr}(k-1, \mathbf{P}V)^2 \simeq \mathbf{Gr}(n-k, (\mathbf{P}V)^\star)^2$ induced by map $\Lambda \mapsto \Lambda^\star$ that sends a linear space to its dual variety. This isomorphism maps $\Gamma_i(\mathbf{P}V)$ to Γ_i (Definition 3.0.1) and thus maps $\overline{\Gamma}_i(\mathbf{P}V)$ to $\overline{\Gamma}_i$ after quotienting by $\mathfrak{S}_2$. Therefore we obtain an isomorphism

$$\begin{aligned}\Psi_2 : \mathrm{Bl}_{\overline{\Gamma}_{k-1}(\mathbf{P}V)} \cdots \mathrm{Bl}_{\overline{\Gamma}_1(\mathbf{P}V)} \mathrm{Sym}^2 \mathbf{Gr}(k-1, \mathbf{P}V) &\xrightarrow{\simeq} \mathrm{Bl}_{\overline{\Gamma}_{k-1}} \cdots \mathrm{Bl}_{\overline{\Gamma}_1} \mathrm{Sym}^2 \mathbf{Gr}(n-k, n) \\ &= \mathcal{H}_{n-k,n-k}((\mathbf{P}V)^\star).\end{aligned}$$

Let $\Psi = \Psi_2 \circ \Psi_1$. One can directly check that $\Psi^\star(D_i) = D_i'$ for all i and $\Psi^\star(N_i) = N_i'$ for $1 \leq i \leq k-1$.

To show that Ψ contracts N_k', it is enough to show that Ψ contracts Y_2 (Lemma 4.3.7). Using Lemma 4.3.9 we obtain $\Psi_\star Y_2 \cdot D_i = \Psi_\star(Y_2 \cdot \Psi^\star(D_i)) = \Psi_\star(Y_2 \cdot D_i') = 0$ for all i. Since $D_1, \ldots, D_k$ generates the nef-cone of $\mathcal{H}^n_{n-k,n-k}$ we must have $\Psi_\star Y_2 = 0$, i.e. Ψ contracts Y_2.

Conversely, let C be any curve contracted by Ψ. If $C \cdot D_i' \neq 0$ for some i, we would have $\Psi_\star C \cdot D_i = \Psi_\star(C \cdot D_i') \neq 0$, proving that Ψ does not contract C. Thus we may assume $C \cdot D_i' = 0$ for all i. Since $\{D_i'\}_i \cup F$ generates the nef-cone of $\mathcal{H}^n_{k-1,k-1}$ we must have $F \cdot C > 0$. Using Proposition 4.3.10 we obtain $N_k' \cdot C = -F \cdot C < 0$, i.e. C lies inside N_k'.

Lastly, we need to verify that N_k' is a $\mathbf{P}^{n-2k+1}$-fibration over $\Psi(N_k')$. Up to projective equivalence, it is enough to verify that the fiber of Ψ_1 over $Z = V(x_0, \ldots, x_{k-1}, x_{2k}, \ldots, x_n) \cup V(x_0, x_{k+1}, \ldots, x_n)$ is isomorphic to $\mathbf{P}^{n-2k+1}$, see Example 4.3.12. Let $H = \mathrm{span}_{\mathbf{k}}\{x_0, x_{2k}, \ldots, x_n\}$. Similar to the proof of Lemma 4.3.7, any subscheme parameterized by $\mathcal{H}^n_{k-1,k-1}$ and supported on Z is cut out by

$$(x_0, \ldots, x_{k-1}, x_{2k}, \ldots, x_n) \cap (x_0, x_{k+1}, \ldots, x_n) \cap \left[(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{2k-1}) + (H') + (H'')^2\right] \quad (4.5)$$

where $H' \in \mathrm{Gr}(n-2k+1, H)$ and $H'' \subseteq H$ is chosen so that $H' \oplus H'' = H$. Notice that for a fixed H', all choices of H'' give the same ideal as Eq. (4.5). It follows that the $\Psi_1^{-1}(Z)$ is paramaterized by $\mathbf{Gr}(n-2k, \mathbf{P}H) \simeq \mathbf{P}^{n-2k+1}$. □

Example 4.3.12. Consider $X \subseteq \mathbf{P}^4$ cut out by $(x_0, x_1, x_4) \cap (x_0, x_3, x_4) \cap (x_0^2, x_1, x_3, x_4)$. This is a pair of lines meeting along an embedded point. Let $x_0^\star, \ldots, x_4^\star$ be the dual coordinates on $(\mathbf{P}^4)^\star$. We can trace the image of X under the map $\Psi : \mathcal{H}_{1,1}(\mathbf{P}^4) \to \mathcal{H}_{2,2}((\mathbf{P}^4)^\star)$ as follows:

$$\begin{aligned}
(x_0, x_1, x_4) \cap (x_0, x_3, x_4) \cap (x_0^2, x_1, x_3, x_4) &\overset{\Psi_1}{\mapsto} (x_0, x_1, x_4) \cap (x_0, x_3, x_4) \\
&\overset{\Psi_2}{\mapsto} \text{point in } \mathcal{H}^4_{2,2} \text{ corresponding to } (x_2^\star, x_3^\star) \cap (x_1^\star, x_2^\star) \\
&= (x_2^\star, x_3^\star) \cdot (x_1^\star, x_2^\star) \\
&= (x_2^\star, x_3^\star) \cap (x_1^\star, x_2^\star) \cap ((x_2^\star)^2, x_1^\star, x_3^\star).
\end{aligned}$$

Proposition 4.3.13. *Let $k \geq 2$ and $n > 2k - 1$. The component $\mathcal{H}^n_{k-1,k-1}$ is Fano.*

Proof. Using Proposition 4.3.11 and the canonical divisor in Proposition 4.2.12 we deduce that

$$\begin{aligned}
K_{\mathcal{H}^n_{k-1,k-1}} &= \Psi^\star K_{\mathcal{H}^n_{n-k,n-k}} + (n-2k+1)N'_k \\
&= \Psi^\star K_{\mathcal{H}^n_{n-k,n-k}} + (n-2k+1)(2D'_1 - D'_2 - F) \\
&= \begin{cases} -3D'_1 - 2D'_2 - 2D'_3 - \cdots - 2D'_{k-2} - D'_{k-1} - 2D'_k - (n-2k+1)F & \text{if } k \geq 4, \\ -3D'_1 - D'_2 - 2D'_3 - (n-5)F & \text{if } k = 3, \\ -2D'_1 - 2D'_2 - (n-3)F & \text{if } k = 2. \end{cases}
\end{aligned}$$

The first equality is a modification of [47, Exercise 8.5] combined with the fact that the codimension of $\Psi(N'_k)$ in $\mathcal{H}^n_{n-k,n-k}$ is $n - 2k + 2$. It follows from Proposition 4.3.10 that $-K_{\mathcal{H}^n_{k-1,k-1}}$ is ample in all cases; thus $\mathcal{H}^n_{k-1,k-1}$ is always Fano. □

Here is the the main theorem of the paper:

Theorem 4.3.14. *The components $\mathcal{H}^n_{k-1,k-1}$ and $\mathcal{H}^n_{n-k,n-k}$ are Mori dream spaces.*

Proof. This follows immediately from Proposition 4.2.12, Proposition 4.3.11, Proposition 4.3.13 and the subsequent two facts:

(i) A smooth Fano variety is a Mori dream space [67, Corollary 4.9],

(ii) Let $f : X \to Y$ be a surjective morphism of smooth, projective varieties. If X is a Mori dream space, then so is Y [76, Theorem 1.1]. □

Chapter 5

Hilbert schemes with two Borel-fixed points

In this chapter we study Hilbert schemes with two Borel-fixed points. We classify Hilbert schemes with two Borel-fixed points and determine when the associated Hilbert schemes or its irreducible components are smooth. In particular, we show that the Hilbert scheme is reduced and has at most two irreducible components. By describing the singularities in a neighbourhood of the Borel-fixed points, we show that the singularities that occur are cones over certain Segre embeddings of $\mathbf{P}^a \times \mathbf{P}^b$. In particular, the singularities are always Cohen-Macaulay and normal.

After the first version of this chapter was available on arXiv, Skjelnes-Smith [88] classified all smooth Hilbert schemes and described their geometry. Complementing [88], our work may be seen as a first step towards a classification of mildly singular Hilbert schemes. To state our results we use the Gotzmann decomposition of a Hilbert polynomial (Theorem 2.0.12).

Theorem 5.0.1. Assume $\mathrm{char}(\mathbf{k}) = 0$. The Hilbert scheme $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ has two Borel-fixed points precisely in the following cases:

(i) $\lambda = (n^s, 1, 1, 1)$ for $n \geq 2$: The Hilbert scheme $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ is smooth, and when $s = 0$ its general member parameterizes three isolated points.

(ii) $\lambda = (n^s, 1, 1, 1, 1)$ for $n = 2$: The Hilbert scheme $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^2)$ is smooth, and when $s = 0$ its general member parameterizes four isolated points in the plane.

(iii) $\lambda = (n^s, 2, 2, 1)$ for $n \geq 3$: The Hilbert scheme $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ is a union of two smooth irreducible components meeting transversely. When $s = 0$, the general member of one component parameterizes a plane conic union an isolated point and the general member of the other component parameterizes two skew lines.

(iv) $\lambda = (n^s, (d+1)^q, 1)$ with $n > d+1 > 2$ and $q \geq 2$: The Hilbert scheme $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ is smooth, and when $s = 0$ its general member parameterizes a hypersurface of degree q in a $\mathbf{P}^{d+1}$ union an isolated point.

(v) $\lambda = (n^s, 2^q, 1)$ with and $n > 2$ and $q \geq 4$: The Hilbert scheme $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ is smooth, and when $s = 0$ its general member parameterizes a plane curve of degree q union an isolated point.

(vi) $\lambda = (n^s, (d+1)^q, r+1, 1)$ with $n > d+1 > r+1 > 2$: The Hilbert scheme $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ is irreducible, Cohen-Macaulay, and normal. When $s = 0$, the general member parameterizes a hypersurface of degree q in a $\mathbf{P}^{d+1}$ union a r-plane inside $\mathbf{P}^{d+1}$ and an isolated point; the hypersurface meets the r-plane transversely in $\mathbf{P}^{d+1}$. If $d = n-2$ the Hilbert scheme at the non lexicographic point is étale-locally a cone over the Segre embedding $\mathbf{P}^1 \times \mathbf{P}^{n-r-1} \hookrightarrow \mathbf{P}^{2(n-r)-1}$.

(vii) $\lambda = (n^s, (d+1)^q, 2, 1)$ with $n > d+1 > 2$ and $q \geq 3$: The description of the Hilbert scheme is identical to Case (5).

(viii) $\lambda = (n^s, d+1, 1, 1)$ with $n > d+1 > 1$: The Hilbert scheme $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ is irreducible, Cohen-Macaulay and normal. If $s = 0$ the general member parameterizes a d-plane union two isolated points. If $d = n-2$ the Hilbert scheme at the non lexicographic point is étale-locally a cone over the Segre embedding $\mathbf{P}^2 \times \mathbf{P}^{n-1} \hookrightarrow \mathbf{P}^{3n-1}$. In particular, if $n = 3$ the Hilbert scheme, which parameterizes a line union two isolated points, is Gorenstein.

(ix) $\lambda = (n^s, d+1, 2, 1)$ with $n > d+1 > 3$: The Hilbert scheme $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ is reduced with two irreducible components $\mathcal{Y}_1$ and $\mathcal{Y}_2$.

- When $s = 0$ the component $\mathcal{Y}_1$ is smooth and its general member parameterizes a disjoint union of a d-plane union a line. If $d = n-2$ the component is isomorphic to the blowup of $\mathbf{G}(1, n) \times \mathbf{G}(n-2, n)$ along the locus $\{(L, \Lambda) : L \subseteq \Lambda\}$.
- When $s = 0$ the component $\mathcal{Y}_2$ is normal and Cohen-Macaulay. Its general point parameterizes a d-plane union a line and an isolated point; the d-plane meets the line at a point. If $d = n-2$ the component at the non lexicographic point is étale-locally a cone over the Segre embedding $\mathbf{P}^1 \times \mathbf{P}^{n-2} \hookrightarrow \mathbf{P}^{2(n-1)-1}$.

After the result appeared on arXiv, work of Staal [90] shows that the classification of Hilbert schemes with two Borel-fixed points extends to positive characteristics with a minor modification. In particular, [90, Theorem 1.1] states that for $\mathrm{char}(\mathbf{k}) \neq 2$ the Hilbert scheme $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ has two Borel-fixed points if and only if λ is as in one of the cases in Theorem 5.0.1. If $\mathrm{char}(\mathbf{k}) = 2$ then λ can be any of the cases of Theorem 5.0.1 except for case (2). Since our deformation computations are characteristic independent (see Section 5.3 and Section 5.4), we obtain a description of the singularities in all characteristics.

Theorem 5.0.2. Let $\mathrm{char}(\mathbf{k}) = p$. The Hilbert scheme $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ has two Borel-fixed points if and only if

- $p \neq 2$ and λ is as in case (1) - (9) of Theorem 5.0.1, or
- $p = 2$ and λ is as in case (1) or (3) - (9) of Theorem 5.0.1.

In all of these cases the description of $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ is identical to the one given in Theorem 5.0.1.

5.1 Resolutions of Borel-fixed ideals

We use $L(\lambda)$ to denote the unique saturated lexicographic ideal with Hilbert polynomial P_λ (Eq. (2.2)). If the Hilbert scheme has exactly two Borel-fixed points we will use $I(\lambda)$ to denote the non lexicographic Borel-fixed point.

The Eliahou-Kervaire resolution provides an explicit minimal free resolution of a strongly stable ideal [27]. We will mostly be interested in resolutions of ideals of the form $I = x_0(x_0, \ldots, x_{n-1}) + x_1^q(x_1, \ldots, x_p)$ with $q \geq 1$ and $n-1 \geq p \geq 0$. Note that I is strongly stable in all characteristics. Following the presentation in [78, Section 2], let $0 \to F_{n-1} \xrightarrow{\psi_{n-1}} \cdots \xrightarrow{\psi_2} F_1 \xrightarrow{\psi_1} F_0 \xrightarrow{\psi_0} I \to 0$ denote the Eliahou-Kervaire resolution of I where

$$F_0 = \left(\bigoplus_{i=0}^{n-1} S(-2)\boldsymbol{e}^\star_{0i}\right) \bigoplus \left(\bigoplus_{i=1}^{p} S(-q-1)\boldsymbol{e}^\star_{1i}\right)$$

and

$$F_1 = \left(\bigoplus_{0 \leq j < i \leq n-1} S(-3)\boldsymbol{e}^j_{0i}\right) \bigoplus \left(\bigoplus_{0 \leq j < i \leq p} S(-q-2)\boldsymbol{e}^j_{1i}\right).$$

The first two differentials are given by $\psi_0(\boldsymbol{e}^\star_{0i}) = x_0 x_i$, $\psi_0(\boldsymbol{e}^\star_{1i}) = x_1^q x_i$ and,

$$\begin{aligned} \psi_1(\boldsymbol{e}^j_{0i}) &= x_j \boldsymbol{e}^\star_{0i} - x_i \boldsymbol{e}^\star_{0j}, \quad 0 \leq j < i \leq n-1 \\ \psi_1(\boldsymbol{e}^0_{1i}) &= x_0 \boldsymbol{e}^\star_{1i} - x_1^q \boldsymbol{e}^\star_{0i}, \quad 1 \leq i \leq p \\ \psi_1(\boldsymbol{e}^j_{1i}) &= x_j \boldsymbol{e}^\star_{1i} - x_i \boldsymbol{e}^\star_{1j}, \quad 1 \leq j < i \leq p. \end{aligned}$$

This presentation also allows us to explicitly describe the first two terms of the cotangent complex [48, Chapter 3]. Let $R = S/I$ and let

$$\mathrm{Kos} := \psi_1^{-1}\left(\{\psi_0(\boldsymbol{e}^\star_{l_1 j_1})\boldsymbol{e}^\star_{l_1 j_1} - \psi_0(\boldsymbol{e}^\star_{l_2 j_2})\boldsymbol{e}^\star_{l_2 j_2}\}\right) \subseteq F_1,$$

be the pre-image of the Koszul relations in F_0. Let $\psi_1^\vee : \mathrm{Hom}_S(F_0, S) \to \mathrm{Hom}_S(F_1, S)$ denote the dual of ψ_1. The second cotangent cohomology, $T^2(R/\mathbf{k}, R)$, is the cokernel of the following map

$$\mathrm{Hom}_R(F_0 \otimes R, R) \xrightarrow{\overline{\psi_1^\vee}} \mathrm{Hom}_R\left(F_1/(\ker\psi_1 + \mathrm{Kos}), R\right).$$

5.2 Classifying Hilbert polynomials

In this section we classify Hilbert polynomials with two Borel-fixed ideals in characteristic 0 (Proposition 5.2.10 and Proposition 5.2.11). The first step is to reduce to studying Hilbert schemes corresponding to integer partitions λ with $n > \lambda_1$, equivalently Hilbert schemes parameterizing subschemes of codimension at least 2. Using the classification of Hilbert schemes with a single Borel-fixed ideal and Algorithm 5.2.3 we obtain the desired classification.

Lemma 5.2.1. *Let $\lambda = (n^s, \lambda_{s+1}, \lambda_s, \dots, \lambda_m)$ be an integer partition with $s > 0$. Then there is an isomorphism*

$$\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n) \simeq \mathbf{P}(H^0(\mathscr{O}_{\mathbf{P}^n}(s))) \times \mathrm{Hilb}^{P_{\lambda'}}(\mathbf{P}^n)$$

where $\lambda' = (\lambda_{s+1}, \dots, \lambda_m)$. This isomorphism is $\mathrm{GL}(n+1)$*-equivariant and thus induces a bijection on Borel-fixed ideals, given by $I \mapsto x_0^s I'$.*

Proof. By [30, Theorem 1.4] and [30, Remark 2, p. 514] there is an isomorphism

$$\mathbf{P}(H^0(\mathscr{O}_{\mathbf{P}^n}(s'))) \times \mathrm{Hilb}^{P'}(\mathbf{P}^n) \simeq \mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n), \quad (f, [I]) \mapsto [fI] \tag{5.1}$$

where $\deg P' < n-1$ and

$$P_\lambda(t) = \binom{t+n}{n} - \binom{t+n-s'}{n} + P'(t-s').$$

Since the morphism Eq. (5.1) is given by multiplication of ideals, it is also $\mathrm{GL}(n+1)$-equivariant. Using the well-known identity on summation of binomial coefficients we obtain

$$\sum_{i=1}^{s} \binom{t+n-i}{n-1} + \sum_{i=s+1}^{m} \binom{t+\lambda_i - i}{\lambda_i - 1} = P_\lambda(t) = \sum_{i=1}^{s'} \binom{t+n-i}{n-1} + P'(t-s').$$

Since $\deg P' < n-1$ we must have $s = s'$ and this, in turn, implies that $P' = P_{\lambda'}$. The desired bijection on Borel-fixed points follows from the $GL(n+1)$-equivariance. ◻

By Lemma 5.2.1 it suffices to classify Borel-fixed ideals in Hilbert schemes corresponding to λ with $n > \lambda_1$.

Notation 5.2.2. For the rest of this section we will assume $\mathrm{char}(\mathbf{k}) = 0$.

We begin by briefly describing a procedure that generates all the Borel-fixed ideals in characteristic 0. Following [21, 62], we fix an order on the variables so that $x_0 > x_1 > \cdots > x_n$. This induces a partial order on monomials of a fixed degree: if $x_i > x_j$ then $x_i \boldsymbol{x}^{\boldsymbol{\alpha}} > x_j \boldsymbol{x}^{\boldsymbol{\alpha}}$. This is called the Borel order and we denote it by $\geq_B$.

Let $I \subseteq S$ be a stongly stable ideal with Hilbert polynomial $P(t)$ and let $\mathcal{G}(I)$ denote the set of minimal generators of I. Given an element $\boldsymbol{x}^{\boldsymbol{\alpha}}$ of $\mathcal{G}(I)$ that is also minimal with respect to $\geq_B$ one can produce a new strongly stable ideal with Hilbert polynomial $P(t)+1$. This procedure is known as an *expansion* of I with respect to $\boldsymbol{x}^{\boldsymbol{\alpha}}$, and the new strongly stable ideal is generated by

$$(\mathcal{G}(I) \setminus \{\boldsymbol{x}^{\boldsymbol{\alpha}}\}) \cup \{\boldsymbol{x}^{\boldsymbol{\alpha}} x_r, \boldsymbol{x}^{\boldsymbol{\alpha}} x_{r+1}, \ldots, \boldsymbol{x}^{\boldsymbol{\alpha}} x_{n-1}\}$$

where $r = \max\{i : x_i | \boldsymbol{x}^{\boldsymbol{\alpha}}\}$. For our purposes, we just need the penultimate step in the recursive algorithm.

Algorithm 5.2.3. Every saturated strongly stable ideal of S with Hilbert polynomial $P(t)$ is obtained from a strongly stable ideal of $R = \mathbf{k}[x_0, \ldots, x_{n-1}]$ with Hilbert polynomial $\Delta P(t) := P(t) - P(t-1)$ via a sequence of expansions. More precisely, I is obtained by successively expanding JS c times, where J is a strongly stable ideal of R with Hilbert polynomial $\Delta P(t)$ and $c = P(t) - P_{S/JS}(t)$ is a constant.

Remark 5.2.4. An alternative algorithm to generate the strongly stable ideals is presented in [70].

Implicit in the above Algorithm is the following Lemma that will be extremely useful for us.

Lemma 5.2.5 ([62, Lemma 3.1, §4.2])**.** *Let $I \subseteq S$ be a saturated strongly stable ideal. Then we can always expand I at a minimal generator of degree e that is minimal w.r.t to $\geq_B$. Any such expansion is strongly stable with Hilbert polynomial $P_{S/I}(t) + 1$.*

Remark 5.2.6. *Integer partitions behave well with respect to the difference operator. If $\lambda = (\lambda_1, \ldots, \lambda_m, 1^s)$ then we have $\Delta^1 P_\lambda = P_{\lambda''}$ where $\lambda'' = (\lambda_1 - 1, \ldots, \lambda_m - 1)$. Indeed, we have*

$$\Delta^1 P_\lambda = \sum_{i=1}^{m+s} \binom{t + \lambda_i - i}{\lambda_i - 1} - \sum_{i=1}^{m+s} \binom{t - 1 + \lambda_i - i}{\lambda_i - 1} = \sum_{i=1}^{m+s} \binom{t + (\lambda_i - 1) - i}{(\lambda_i - 1) - 1} = P_{\lambda''}.$$

By our discussion above we see that the number of Borel-fixed points on a Hilbert scheme $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ are, to some extent, determined by the number of Borel-fixed points on $\mathrm{Hilb}^{\Delta P_\lambda}(\mathbf{P}^{n-1})$ and $\mathrm{Hilb}^{P_\lambda - 1}(\mathbf{P}^n)$. It turns out that by considering $\mathrm{Hilb}^{P_\lambda - 1}(\mathbf{P}^n)$, we can greatly restrict the partitions λ that could give rise to Hilbert schemes with two Borel-fixed points.

Lemma 5.2.7. *If* $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ *has more than one Borel-fixed point, then* $\mathrm{Hilb}^{P_\lambda - 1}(\mathbf{P}^n)$ *is non-empty.*

Proof. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$. If $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ has more than one more Borel-fixed point then [89, Theorem 1.1] implies that $\lambda_m = 1$ and $m \geq 2$. It follows that

$$P_\lambda - 1 = \sum_{i=1}^{m} \binom{t + \lambda_i - i}{\lambda_i - 1} - 1 = \sum_{i=1}^{m-1} \binom{t + \lambda_i - i}{\lambda_i - 1} = P_{\lambda'}$$

with $\lambda' = (\lambda_1, \dots, \lambda_{m-1})$. Since λ' is an integer partition with $1 \leq \lambda'_1 \leq n$, the result follows. □

We can now state a necessary condition for a Hilbert scheme to have two Borel-fixed points.

Proposition 5.2.8. *Let* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ *be an integer partition with* $\lambda_1 \leq n - 1$. *If* $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ *has two Borel-fixed points then* $\lambda = ((d+1)^q, 1)$ *or* $\lambda = ((d+1)^q, r+1, 1)$

Proof. By [89, Theorem 1.1] we may assume $\lambda_m = 1$ and $m \geq 2$. Let $\lambda' = (\lambda_1, \dots, \lambda_{m-1})$ and we have $P_\lambda = P_{\lambda'} + 1$. If the lexicographic point, $L(\lambda')$, was generated in more than two degrees then Lemma 5.2.5 would imply that $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ contains at least three Borel-fixed points; a contradiction. So we may assume that $L(\lambda')$ (Eq. (2.2)) is generated in at most two degrees. Let r be the smallest integer for which $a_{r+1} \neq 0$ and d be the largest integer for which $a_{d+1} \neq 0$. By assumption we have $a_n = 0$. If $r = d$ we must have

$$L(\lambda') = (x_0, \dots, x_{n-d-2}, x_{n-d-1}^{a_{d+1}}) \tag{5.2}$$

which implies $\lambda' = ((d+1)^{a_{d+1}})$. If $d > r$ we have $a_{d+1} + 1 = a_{d+1} + a_d + 1 = \cdots = a_{d+1} + \cdots + a_{r+2} + 1 = a_{d+1} + \cdots + a_{r+1}$. This implies $a_{r+2}, \dots, a_d = 0$ and $a_{r+1} = 1$, and we obtain

$$L(\lambda') = (x_0, \dots, x_{n-d-2}) + x_{n-d-1}^{a_{d+1}}(x_{n-d-1}, x_{n-d-2}, \dots, x_{n-r-1}) \tag{5.3}$$

and $\lambda' = ((d+1)^{a_{d+1}}, r+1)$, as required. □

We now turn our attention to eliminating some of the λ that appeared in Proposition 5.2.8. If the Hilbert scheme $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ has two Borel-fixed points then they are both on the lexicographic component. Let X_1 and X_2 denote the two Borel-fixed points. By [84, Theorem 11] the hyperplane sections $X_i \cap V(x_n)$ must be equal to the lexicographic point $V(L(\lambda'))$ where $\Delta P_\lambda = P_{\lambda'}$. Thus, if we produce a Borel-fixed point on $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ whose hyperplane section is not $L(\lambda')$, then the corresponding Hilbert scheme cannot have two Borel-fixed points. Of course, sometimes it is simpler to directly construct three Borel-fixed ideals. We use both of these methods to obtain the following Lemma.

Lemma 5.2.9. *Let* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ *be an integer partition with* $\lambda_1 \leq n-1$. *For the following partitions* λ, *the Hilbert scheme* $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ *has at least three Borel-fixed points*

(i) $\lambda = (1^b)$ *with* $b \geq 4$ *and* $n \geq 3$*,*

(ii) $\lambda = (1^b)$ *with* $b \geq 5$ *and* $n = 2$*,*

(iii) $\lambda = ((d+1)^2, 2, 1)$ *with* $d \geq 1$*,*

(iv) $\lambda = ((d+1)^q, 1^2)$ *with* $d \geq 1$ *and* $q > 1$.

Proof. For the rest of the proof let $R = \mathbf{k}[x_0, \ldots, x_{n-1}]$. In case (1) we may use Proposition 2.0.18 to verify that the following ideals are Borel-fixed with Hilbert polynomial $P_\lambda = b$:

$$(x_0, \ldots, x_{n-3}, x_{n-2}, x_{n-1}^b),\ (x_0, \ldots, x_{n-3}, x_{n-2}^2, x_{n-2}x_{n-1}, x_{n-1}^{b-1}) \text{ and}$$
$$(x_0, \ldots, x_{n-4}, x_{n-3}^2, x_{n-3}x_{n-2}, x_{n-3}x_{n-1}, x_{n-2}^2, x_{n-2}x_{n-1}, x_{n-1}^{b-2}).$$

Similarly, in case (2) we may use Proposition 2.0.18 to verify the following ideals are Borel-fixed with Hilbert polynomial $P_\lambda = b$:

$$(x_0, x_1^b),\ (x_0^2, x_0x_1, x_1^{b-1}) \text{ and } (x_0^2, x_0x_1^2, x_1^{b-2}).$$

If we are in case (3) then consider the following Borel-fixed ideal

$$J = (x_0, \ldots, x_{n-d-3}) + x_{n-d-2}(x_{n-d-2}, \ldots, x_{n-2}) + (x_{n-d-1}^2).$$

To see that J has Hilbert polynomial P_λ, it suffices to compare it to

$$L(\lambda) = (x_0, \ldots, x_{n-d-2}) + x_{n-d-1}^2(x_{n-d-1}, x_{n-d-2}, \ldots, x_{n-3}) + x_{n-d-1}^2 x_{n-2}(x_{n-2}, x_{n-1}).$$

Indeed, for $j \gg 0$ we have

$$J_j \setminus L(\lambda)_j = \{x_{n-d-1}^2 x_{n-d-2} x_n^{j-3}\} \cup \{x_{n-d-1}^2 x_{n-1}^e x_n^{j-2-e}\}_{0 \leq e \leq j-2}$$

and

$$L(\lambda)_j \setminus J_j = \{x_{n-d-2} x_{n-1}^e x_n^{j-1-e}\}_{0 \leq e \leq j-1}.$$

Since these two sets have the same cardinality j, it follows that $P_{S/L(\lambda)}(t) = P_{S/J}(t)$. The hyperplane section $V(x_n) \cap V(J)$ is defined by the saturated ideal

$$(x_0, \ldots, x_{n-d-3}) + x_{n-d-2}(x_{n-d-2}, \ldots, x_{n-2}) + (x_{n-d-1}^2).$$

Since this is different from $L(d^3) = (x_0, \ldots, x_{n-d-2}, x_{n-d-1}^3)$, the Hilbert scheme cannot have two Borel-fixed points.

Finally, if we are in case (4) we have the following Borel-fixed ideals

$$\begin{aligned}
L(\lambda) &= (x_0, \ldots, x_{n-d-2}) + x_{n-d-1}^q(x_{n-d-1}, x_{n-d-2}, \ldots, x_{n-2}) + (x_{n-d-1}^q x_{n-1}^2),\\
I &= (x_0, \ldots, x_{n-d-3}) + x_{n-d-2}(x_{n-d-2}, \ldots, x_{n-1}) + x_{n-d-1}^q(x_{n-d-1}, x_{n-d-2}, \ldots, x_{n-1}),\\
J &= (x_0, \ldots, x_{n-d-3}) + x_{n-d-2}(x_{n-d-2}, \ldots, x_{n-2}, x_{n-1}^2) + (x_{n-d-1}^q).
\end{aligned}$$

Just as we did in case (3), it is straightforward to see that the three ideals have Hilbert polynomial P_λ. For instance, consider J and note that for $j \gg 0$ we have

$$J_j \setminus L(\lambda)_j = \{x_{n-d-1}^q x_{n-1} x_n^{j-q-1}\} \cup \{x_{n-d-1} x_n^{j-q}\}$$

and

$$L(\lambda)_j \setminus J_j = \{x_{n-d-2} x_{n-1} x_n^{j-2}, x_{n-d-2} x_n^{j-1}\}. \quad \square$$

We now ready to prove the main result of this section. It will turn out that the constraints we have found on λ up until this point are sufficient. We accomplish this by studying the expansions of Borel-fixed ideals with Hilbert polynomial $\Delta P(t)$ (Algorithm 5.2.3). Since the Borel-fixed ideals naturally fit into two distinct families, we split the result into two Propositions.

Proposition 5.2.10. *Let $\lambda = ((d+1)^q, 1)$ with $n - 2 \geq d$. The Hilbert scheme* $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ *has two Borel-fixed points if and only if $n \geq 2$ and*

(i) *$d = 0$ and $q = 2$, or*

(ii) *$d = 0$, $q = 3$ and $n = 2$, or*

(iii) *$d = 1$ and $q \neq 1, 3$, or*

(iv) *$d \geq 2$ and $q \geq 2$.*

The two Borel-fixed ideals are

$$\begin{aligned} I(\lambda) &= (x_0, \dots, x_{n-d-3}) + x_{n-d-2}(x_{n-d-2}, \dots, x_{n-1}) + (x_{n-d-1}^q), \\ L(\lambda) &= (x_0, \dots, x_{n-d-2}) + x_{n-d-1}^q(x_{n-d-1}, x_{n-d-2}, \dots, x_{n-1}). \end{aligned}$$

Proof. The ideals $I(\lambda)$ and $L(\lambda)$ are expansions of a lexicographic ideal

$$(x_0, \dots, x_{n-d-2}, x_{n-d-1}^q).$$

Since the latter ideal has Hilbert polynomial $P_{((d+1)^q)}$, it follows from Lemma 5.2.5 that the Hilbert polynomial of $I(\lambda)$ and $L(\lambda)$ is P_λ. We first show that the cases are necessary. By [89, Theorem 1.1 (ii)] if $n = 1$ or $q = 1$ the Hilbert scheme has a single Borel-fixed point. The remaining conditions on λ follow from Lemma 5.2.9.

If we are in case (1) then the Hilbert scheme parameterizes subschemes of length three. Any such subscheme can be realized as $\lim_{t \to 0} Z_t = Z$ where Z_t a reduced union of three points for $t \in \mathbf{A}^1 - 0$ [17]. By upper-semicontinuity, since the union of three reduced points is contained in a $\mathbf{P}^2$, the subscheme Z is also contained in a $\mathbf{P}^2$. If Z was Borel-fixed this implies $I_Z = (x_0, \dots, x_{n-3}) + JS$ with $J \subseteq S' := \mathbf{k}[x_{n-2}, x_{n-1}, x_n]$ and $P_{S'/J}(t) = 3$. Using Proposition 2.0.18 we see that only choices are $(x_0, \dots, x_{n-3}, x_{n-2}^2, x_{n-2}x_{n-1}, x_{n-1}^2)$ and $(x_0, \dots, x_{n-3}, x_{n-2}, x_{n-1}^3)$.

If we are in case (2) then Proposition 2.0.18 shows that (x_0, x_1^4) and (x_0^2, x_0x_1, x_1^3) are the only two Borel-fixed ideals.

So we may assume that we are in case (3) or case (4) of the theorem. Let $\lambda' = ((d+1)^q)$ and $\lambda'' = (d^q)$. By Algorithm 5.2.3 we begin by computing all the Borel-fixed ideals in $R := \mathbf{k}[x_0, \ldots, x_{n-1}]$ with Hilbert polynomial, $\Delta^1 P_\lambda = P_{\lambda''}$.

For $d \geq 2$ the Hilbert scheme $\mathrm{Hilb}^{P_{\lambda''}}(\mathrm{Proj}(R))$ has a unique Borel-fixed point [89, Theorem 1.1] and it is given by $L(\lambda'') = (x_0, \ldots, x_{n-d-2}, x_{n-d-1}^q)$. The lift of $L(\lambda'')$ to S is just the lexicographic ideal, $L(\lambda')$, with Hilbert polynomial $P_{\lambda'} = P_\lambda - 1$. Thus, in the last step of the algorithm, we only need to perform *one* successive expansion. Once with the monomial x_{n-d-2} and once with the monomial x_{n-d-1}^q, giving us the two desired Borel-fixed ideals.

The last case is if $d = 1$ and $q \neq 1, 3$. In this case we have

$$P_\lambda(t) = \sum_{i=1}^{q} \binom{t+2-i}{2-1} + 1 = qt + 2 - \binom{q-1}{2}.$$

Since $\Delta^1(P_\lambda) = q$ we compute all the Borel-fixed ideals in R with Hilbert polynomial q. One such ideal is $I = (x_0, \ldots, x_{n-3}, x_{n-2}^q)$ whose lift, IS, is the ideal of a plane curve of degree q. Thus, the Hilbert polynomial of IS is $P_{\lambda'}$ and we may expand IS at x_{n-3} and x_{n-2}^q to obtain the two Borel-fixed ideals. To finish, it suffices to show that if J is a Borel-fixed ideal in R different from I then the Hilbert polynomial of the lift, JS, is bigger than P_λ. For such a J to exist we must have $q \geq 4$. In particular, we will prove that $P_{S/JS}(t) \geq P_\lambda(t) + 1 = P_{\lambda'}(t) + 2$ for all $t \gg 0$. Since $J \neq I$, we may assume that $x_{n-2}^\ell \in J$ and $x_{n-2}^{\ell-1} \notin J$ for some $1 < \ell < q$. This implies that for $j \gg 0$, $(R/J)_j$ is spanned by

$$\left\{ m_1 x_{n-1}^{j-\deg m_1}, \ldots, m_{q-\ell} x_{n-1}^{j-\deg m_{q-\ell}}, x_{n-1}^j, x_{n-2}x_{n-1}^{j-1}, \ldots, x_{n-2}^{\ell-1}x_{n-1}^{j-\ell+1} \right\}.$$

We may assume that the m_i are monomials of degree strictly less than ℓ and not divisible by x_{n-1} (applying the exchange property to x_{n-2}^ℓ, we see that J contains all monomials of degree at least ℓ supported on $x_0, \ldots, x_{n-2}$). Thus, for $j \gg 0$ the graded piece $(S/JS)_j$ contains the monomials in $x_{n-2}^p(x_{n-1}, x_n)^{j-p}$ for $0 \leq p \leq \ell - 1$ and the monomials in $m_v(x_{n-1}, x_n)^{j-\deg m_v}$ for $1 \leq v \leq q - \ell$. This implies

$$\dim_{\mathbf{k}}(S/J)_j \geq \sum_{p=0}^{\ell-1}(j-p+1) + \sum_{v=1}^{q-\ell}(j - \deg m_v + 1) \geq \sum_{p=0}^{\ell-1}(j-p+1) + \sum_{v=1}^{q-\ell}(j-\ell+1+1).$$

If we further assume $\ell < q-1$, we may rewrite the sum and obtain

$$\begin{aligned}
\dim_{\mathbf{k}}(S/J)_j &\geq \sum_{p=0}^{\ell-1}(j-p+1) + \sum_{v=1}^{q-\ell}(j-\ell+1) + (q-\ell) \\
&\geq \sum_{p=0}^{\ell-1}(j-p+1) + \sum_{v=\ell}^{q-1}(j-v+1) + (q-\ell) \\
&= \sum_{p=0}^{q-1}(j-p+1) + (q-\ell) \\
&= qj + 1 - \binom{q-1}{2} + (q-\ell) \\
&\geq \dim_{\mathbf{k}}(S/IS)_j + 2 = P_{\lambda'}(j) + 2
\end{aligned}$$

as required. Finally, if $\ell = q-1$, the exchange property forces

$$J = (x_0, \dots, x_{n-4}, x_{n-3}^2, x_{n-3}x_{n-2}, x_{n-2}^{q-1}).$$

Since $q \geq 4$, one can observe that $P_{S/JS}(t) = P_{S/IS}(t) + 2$, completing the proof. ☐

Proposition 5.2.11. *Let $\lambda = ((d+1)^q, r+1, 1)$ with $d > r$. The Hilbert scheme $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ has two Borel-fixed points if and only if $n \geq 2$ and*

(i) $r = 0, q = 1$, *or*

(ii) $r = 1, q \neq 2$, *or*

(iii) $r \geq 2$.

The two Borel-fixed ideals are

$$I(\lambda) = (x_0, \dots, x_{n-d-3}) + x_{n-d-2}(x_{n-d-2}, \dots, x_{n-1}) + x_{n-d-1}^q(x_{n-d-1}, x_{n-d-2}, \dots, x_{n-r-1}),$$
$$L(\lambda) = (x_0, \dots, x_{n-d-2}) + x_{n-d-1}^q(x_{n-d-1}, x_{n-d-2}, \dots, x_{n-r-2}) + x_{n-d-1}^q x_{n-r-1}(x_{n-r-1}, \dots, x_{n-1}).$$

Proof. Since $I(\lambda)$ and $L(\lambda)$ are expansions of the lexicographic ideal (Eq. (5.3)) it follows from Lemma 5.2.5 that their Hilbert polynomial is P_λ. By Lemma 5.2.9 these conditions are also necessary; if $n = 1$ the Hilbert scheme has a single Borel-fixed point.

Now assume that we are in case (1), (2) or (3). Let $\lambda' = ((d+1)^q, r+1)$ and $\lambda'' = (d^q, r)$. We begin by computing all the Borel-fixed ideals in $R := \mathbf{k}[x_0, \dots, x_{n-1}]$ with Hilbert polynomial $\Delta^1 P_\lambda = P_{\lambda''}$.

If $r \geq 2$ or $(r, q) = (1, 1)$ the Hilbert scheme $\mathrm{Hilb}^{P_{\lambda''}}(\mathrm{Proj}(R))$ has a unique Borel-fixed point [89, Theorem 1.1] and it is given by

$$L(\lambda'') = (x_0, \dots, x_{n-d-2}) + x_{n-d-1}^q(x_{n-d-1}, x_{n-d-2}, \dots, x_{n-r-1}).$$

The lift of $L(\lambda'')$ to S is just the lexicographic ideal, $L(\lambda')$, with Hilbert polynomial $P_{\lambda'} = P_\lambda - 1$. Thus, to obtain all the Borel-fixed ideals we only need to perform a single expansion. Once with the monomial x_{n-d-2} and once with the monomial $x_{n-d-1}^q x_{n-r-1}$, giving us the two Borel-fixed ideals.

Similarly, if $(r, q) = (0, 1)$ the Hilbert scheme $\mathrm{Hilb}^{P_{\lambda''}}(\mathrm{Proj}(R))$ has a unique Borel-fixed point [89, Theorem 1.1] and it is given by $(x_0, \ldots, x_{n-d-1})$. The lift to S has Hilbert polynomial $\binom{t+d}{d} = P_\lambda - 2$. Thus, we begin by performing an expansion with x_{n-d-1} to obtain $(x_0, \ldots, x_{n-d-2}) + x_{n-d-1}(x_{n-d-1}, \ldots, x_{n-1})$. This is the lexicographic ideal $L(\lambda')$ and we conclude as in the previous paragraph.

Assume $r = 1$ and $q \geq 3$. Then Proposition 5.2.10 (3) implies that the Hilbert scheme $\mathrm{Hilb}^{P_{\lambda''}}(\mathrm{Proj}(R))$ has two Borel-fixed ideals, $I'' := (x_0, \ldots, x_{n-d-3}) + x_{n-d-2}(x_{n-d-2}, \ldots, x_{n-2}) + (x_{n-d-1}^q)$ and $L(\lambda'')$. We first show that the Hilbert polynomial of $I''S$ is larger than P_λ. We can do this by comparing the number of generators of $(I''S)_j$ to those of $I(\lambda)_j$ for $j \gg 0$. Let $\mathfrak{C}_j$ denote the intersection of the monomials of $(I''S)_j$ with the monomials of $I(\lambda)_j$. Then it is evident that $I(\lambda)_j$ is generated by

$$\mathfrak{C}_j \cup \{x_{n-d-2}x_{n-1}x_{n-1}^a x_n^b\}_{a+b=j-2}$$

while $(I''S)_j$ is generated by

$$\mathfrak{C}_j \cup \{x_{n-d-1}^q x_{n-1}^a x_n^b\}_{a+b=j-q}$$

for all $j \gg 0$. This implies $P_{S/I(\lambda)}(t) + j - 1 = P_{S/I''S}(t) + j - q + 1$. It follows that $P_{S/I''S}(t) = P_{S/I(\lambda)}(t) + (q-2) = P_\lambda(t) + (q-2) > P_\lambda(t)$, as required. Thus, we only need to perform one successive expansion of the lexicographic ideal, $L(\lambda'')S = L(\lambda')$. This will give us the two desired Borel-fixed ideals. □

Note that Proposition 5.2.10 corresponds to case (1) - case (5) in Theorem 5.0.1 while Proposition 5.2.11 corresponds to the other cases.

5.3 Deformation Theory

In this section we compute the tangent space to the non lexicographic Borel-fixed ideal, $[I(\lambda)]$, and provide a partial basis for the second cotangent cohomology group of $S/I(\lambda)$. These are essential for the computation of the universal deformation space of $I(\lambda)$, which we carry out in Section 5.4. The general procedure to compute the universal deformation space can be found in [92, §3] and [79, §5].

From Proposition 5.2.10 and Proposition 5.2.11 we see that $I(\lambda)$ lies inside a unique $\mathbf{P}^{d+2}$. As a consequence, any embedded deformation of the $I(\lambda)$ in $\mathbf{P}^n$ can be realized as a deformation of the $I(\lambda)$ *in* $\mathbf{P}^{d+2}$ along with a deformation of $\mathbf{P}^{d+2}$ in $\mathbf{P}^n$. In other words, étale locally around $[I(\lambda)]$ we have an isomorphism

$$\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n) \simeq \mathrm{Hilb}^{P_\lambda}(\mathbf{P}^{d+2}) \times \mathbf{A}^{(d+3)(n-d-2)}. \tag{5.4}$$

As a consequence, it suffices to prove Theorem 5.0.1 assuming $n = d - 2$.

Notation 5.3.1. For the rest of this section we assume $n = d - 2$. We also assume λ is of the form $((d+1)^q, 1)$ satisfying the conditions of Proposition 5.2.10, or of the form $((d+1)^q, r+1, 1)$ satisfying the conditions of Proposition 5.2.11. In the first case the corresponding non lexicographic ideal is

$$I(\lambda) = x_0(x_0, \dots, x_{n-1}) + (x_1^q)$$

and in the second case it is

$$I(\lambda) = x_0(x_0, \dots, x_{n-1}) + x_1^q(x_1, \dots, x_{n-r-1}).$$

We start by verifying that the comparison theorem holds in all cases of interest.

Lemma 5.3.2. *If $\lambda \neq (1^4)$ then $(S/I(\lambda))_e \simeq H^0(\mathbf{P}^n, \mathcal{O}_{\mathrm{Proj}(S/I(\lambda))}(e))$ for all $e \geq 1$.*

Proof. For the purpose of this proof it will be convenient to unify notation and express

$$I(\lambda) = x_0(x_0, \dots, x_{n-1}) + x_1^q(x_1, \dots, x_p)$$

with $0 \leq p \leq n-1$. Let $X = \mathrm{Proj}(S/I(\lambda))$ and assume $p \neq n-1$. Let $J = (x_0) + x_1^q(x_1, \dots, x_p)$ and consider the exact sequence $0 \longrightarrow J/I(\lambda) \longrightarrow S/I(\lambda) \longrightarrow S/J \longrightarrow 0$. The associated long exact sequence in local cohomology of graded S-modules is

$$0 \longrightarrow H^0_{\mathfrak{m}}(J/I(\lambda)) \longrightarrow H^0_{\mathfrak{m}}(S/I(\lambda)) \longrightarrow H^0_{\mathfrak{m}}(S/J) \longrightarrow H^1_{\mathfrak{m}}(J/I(\lambda)) \longrightarrow H^1_{\mathfrak{m}}(S/I(\lambda)) \longrightarrow H^1_{\mathfrak{m}}(S/J).$$

Since x_{n-1} and x_n are nonzero divisors on S/J we have $\mathrm{depth}_{\mathfrak{m}}(S/J) \geq 2$. This implies that the local cohomology groups $H^0_{\mathfrak{m}}(S/J)$ and $H^1_{\mathfrak{m}}(S/J)$ are zero. As graded S-modules, we have $J/I(\lambda) \simeq (S/(x_0, \dots, x_{n-1}))(-1) := \bar{S}(-1)$. The associated sheaf on $\mathbf{P}^n$ is just the structure sheaf of a point. Consider the following exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(\bar{S}(-1)) \longrightarrow \bar{S}(-1) \longrightarrow H^0_{\star}(\mathcal{O}_{\mathrm{pt}}(-1)) \longrightarrow H^1_{\mathfrak{m}}(\bar{S}(-1)) \longrightarrow 0.$$

For all $e \geq 1$ we have $H^0_{\star}(\mathcal{O}_{\mathrm{pt}}(-1))_e = H^0(\mathcal{O}_{\mathrm{pt}}(e-1)) = H^0(\mathcal{O}_{\mathrm{pt}}) = \mathbf{k} \simeq \bar{S}(-1)_e$. Thus, we have $H^0_{\mathfrak{m}}(\bar{S}(-1))_e = H^1_{\mathfrak{m}}(\bar{S}(-1))_e = 0$ for all $e \geq 1$.

Combining this with the first long exact sequence we obtain $H^0_{\mathfrak{m}}(S/I(\lambda))_e = H^1_{\mathfrak{m}}(S/I(\lambda))_e = 0$ for all $e \geq 1$. The desired result now follows from using the exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(S/I(\lambda)) \longrightarrow S/I(\lambda) \longrightarrow H^0_{\star}(\mathbf{P}^n, \mathcal{O}_X) \longrightarrow H^1_{\mathfrak{m}}(S/I(\lambda)) \longrightarrow 0.$$

The remaining case is when $p = n-1$ and $q = 1$ (we excluded the case of $n = 2, q = 2$). In this case the regularity of $I(\lambda)$ is 2 [78, Corollary 3.1]. Thus Corollary 4.8 and Proposition 4.16 in [26] establish that $\dim_{\mathbf{k}}(S/I(\lambda))_e = P_{S/I(\lambda)}(e) = P_X(e) = h^0(\mathbf{P}^n, \mathcal{O}_X(e))$ for all $e \geq 1$. ◻

The next four propositions provide a basis for the tangent space to each $[I(\lambda)]$. Since their proofs are very similar we will only provide all the details for the first one.

Definition 5.3.3. *For $S = \mathbf{k}[x_0, \dots, x_n]$ and for $q \geq 1$ define the following subsets*

(i) $\mathcal{T}_1 = \{x_{i_1} \cdots x_{i_q} : 1 \leq i_1 \leq i_2 \leq \cdots \leq i_q \leq n\} \setminus \{x_1^q, x_1^{q-1}x_2, \dots, x_1^{q-1}x_n\}$.

(ii) $\mathcal{T}_2 = \{x_1^{q-1}x_2, \dots, x_1^{q-1}x_n\}$.

Proposition 5.3.4. *Let $\lambda = ((n-1)^q, r+1, 1)$ be an integer partition. Assume $n \geq 4$ and either $r \geq 2$ and $q \geq 1$, or $r = 1$ and $q \geq 3$. Then*

$$\dim_{\mathbf{k}} T_{[I(\lambda)]} \operatorname{Hilb}^{P_\lambda}(\mathbf{P}^n) = 3n - 1 + (n-r-2)(r+1) + \binom{n+q-1}{n-1}.$$

A general $\varphi \in \operatorname{Hom}(I(\lambda), S/I(\lambda))_0$ can be written as

$$\begin{aligned}
\varphi(x_0^2) &= a_0 x_0 x_n \\
\varphi(x_0 x_i) &= a_i x_0 x_n + c_1 x_1 x_i + c_2 x_2 x_i + \cdots + c_n x_n x_i, \quad 1 \leq i \leq n-1 \\
\varphi(x_1^{q+1}) &= b_1 x_0 x_n^q + \sum_{\omega \in \mathcal{T}_1} c_\omega x_1 \omega + \ell^1_{n-r} x_1^q x_{n-r} + \cdots \ell^1_n x_1^q x_n, \quad 1 \leq i \leq n-r-1 \\
\varphi(x_1^q x_i) &= b_i x_0 x_n^q + \sum_{\omega \in \mathcal{T}_1 \cup \mathcal{T}_2} c_\omega x_i \omega + \ell^i_{n-r} x_1^q x_{n-r} + \cdots \ell^i_n x_1^q x_n, \quad 2 \leq i \leq n-r-1
\end{aligned}$$

where $a_0, \dots, a_{n-1}, b_1, \dots, b_{n-r-1}, c_1, \dots, c_n$, $\{c_\omega\}_{\omega \in \mathcal{T}_1 \cup \mathcal{T}_2}$, and $\{\ell^i_j\}^{1 \leq i \leq n-r-1}_{n-r \leq j \leq n}$ are independent parameters.

Proof. By Theorem 2.0.9 and Lemma 5.3.2, $\dim_{\mathbf{k}} T_{[I(\lambda)]} \operatorname{Hilb}^{P_\lambda}(\mathbf{P}^n) = \dim_{\mathbf{k}} \operatorname{Hom}(I(\lambda), S/I(\lambda))_0$. Let $F_1 \xrightarrow{\psi_1} F_0 \xrightarrow{\psi_0} I(\lambda) \longrightarrow 0$ be the beginning of the Eliahou-Kervaire resolution from Section 5.1. We have the following exact sequence

$$0 \longrightarrow \operatorname{Hom}(I(\lambda), S/I(\lambda))_0 \longrightarrow \operatorname{Hom}(F_0, S/I(\lambda))_0 \xrightarrow{\psi_1^\vee} \operatorname{Hom}(F_1, S/I(\lambda))_0.$$

Dualizing ψ_1 we see that $\phi \in \operatorname{Hom}(I(\lambda), S/I(\lambda))_0$ if and only if the following relations hold in $S/I(\lambda)$

$$\begin{aligned}
\phi(x_0 x_i) x_j &= \phi(x_0 x_j) x_i, \quad 0 \leq i, j \leq n-1 \\
\phi(x_0 x_j) x_1^q &= \phi(x_1^q x_j) x_0, \quad 1 \leq j \leq n-r-1 \\
\phi(x_1^q x_i) x_j &= \phi(x_1^q x_j) x_i, \quad 1 \leq i, j \leq n-r-1.
\end{aligned}$$

It is straightforward to check that the family described in the statement satisfies these relations.

Conversely, given $\phi \in \operatorname{Hom}(I(\lambda), S/I(\lambda))_0$ we need to show that ϕ lies in our family. For any $i \neq n-1$, the relation $\phi(x_0x_i)x_{n-1} = \phi(x_0x_{n-1})x_i$ implies that x_i divides all the monomials in the support of $\phi(x_0x_i)$ that are not annihilated by x_{n-1}. But the only quadratic monomial that is non-zero in S/I and annihilated by x_{n-1} is x_0x_n. Thus, for $i \neq n-1$ the image $\phi(x_0x_i)$ is supported on $\{x_1x_i, x_2x_i, \dots, x_nx_i, x_0x_n\}$. Since $r \geq 2$ or $q \geq 3$, the only quadratic monomial (non-zero in $S/I(\lambda)$) annihilated by x_{n-2} is x_0x_n. Thus the relation $\phi(x_0x_{n-2})x_{n-1} = \phi(x_0x_{n-1})x_{n-2}$ implies $\phi(x_0x_{n-1})$ is also supported on $\{x_1x_{n-1}, x_2x_{n-1}, \dots, x_nx_{n-1}, x_0x_n\}$. Analogously, we may use the relation $\phi(x_1^qx_i)x_j = \phi(x_1^qx_j)x_i$ to deduce that $\phi(x_1^qx_i)$ is supported on $\{x_1^qx_{n-r} \dots, x_1^qx_n\ x_0x_n^q\} \cup x_i\mathcal{T}_1 \cup x_i\mathcal{T}_2$.

Let $\phi(x_0x_{n-1}) = a_{n-1}x_0x_n + c_2x_{n-1}x_2 + \cdots c_nx_{n-1}x_n$ for some constants c_i. Then for $j \neq n-1$, the relation $x_j\phi(x_0x_{n-1}) = x_{n-1}\phi(x_0x_j)$ implies $\phi(x_0x_j) = a_jx_0x_n + c_2x_jx_2 + \cdots + c_nx_jx_n$ for some constant a_j. Now assume

$$\phi(x_1^qx_2) = b_2x_0x_n^q + \sum_{\omega \in \mathcal{T}_1 \cup \mathcal{T}_2} c_\omega x_2\omega + \ell^2_{n-r}x_1^qx_{n-r} + \cdots + \ell^2_nx_1^qx_n.$$

with c_ω, ℓ^2_i, b_2 some constants. For $j \geq 3$ the relation $\phi(x_1^qx_2)x_j = \phi(x_1^qx_j)x_2$ implies

$$\phi(x_1^qx_j) = b_jx_0x_n^q + \sum_{\omega \in \mathcal{T}_1 \cup \mathcal{T}_2} c_\omega x_j\omega + \ell^j_{n-r}x_1^qx_{n-r} + \cdots + \ell^j_nx_1^qx_n.$$

where l^j_i, b_j are constants. Note that if $j = 1$ then the non-zero elements of $x_j\mathcal{T}_2$ are $\{x_1^qx_{n-r}, \dots, x_1^qx_n\}$. Thus, $\phi(x_1^{q+1})$ is also of the desired form and this completes the proof. □

Proposition 5.3.5. *Let $\lambda = (n-1, 2, 1)$ be an integer partition with $n \geq 4$. Then*

$$\dim_{\mathbf{k}} T_{[I(\lambda)]} \operatorname{Hilb}^{P_\lambda}(\mathbf{P}^n) = 6n - 6.$$

A general $\varphi \in \operatorname{Hom}(I(\lambda), S/I(\lambda))_0$ can be written as

$$\begin{aligned}
\varphi(x_0^2) &= a_0x_0x_n \\
\varphi(x_0x_i) &= a_ix_0x_n + c_2x_2x_i + c_3x_3x_i + \cdots + c_nx_nx_i, \quad 1 \leq i \leq n-2 \\
\varphi(x_0x_{n-1}) &= a_{n-1}x_0x_n + c_1x_1x_{n-1} + c_2x_2x_{n-1} + \cdots + c_nx_nx_{n-1} + \boxed{\alpha x_1x_n} \\
\varphi(x_1^2) &= b_1x_0x_n + \ell^1_{n-1}x_1x_{n-1} + \ell^1_nx_1x_n \\
\varphi(x_1x_i) &= b_ix_0x_n + d_2x_2x_i + \cdots + d_nx_nx_i + \ell^i_{n-1}x_1x_{n-1} + \ell^i_nx_1x_n, \quad 2 \leq i \leq n-r-1.
\end{aligned}$$

where $\alpha, a_0, \dots, a_{n-1}, b_1, \dots, b_{n-2}, c_1, \dots, c_n, d_2, \dots, d_n$ and $\{\ell^i_{n-1}, \ell^i_n\}_{1 \leq i \leq n-2}$ are independent parameters.

Proposition 5.3.6. *Let $\lambda = (n-1, 1, 1)$ be an integer partition with $n \geq 3$. Then*

$$\dim_{\mathbf{k}} T_{[I(\lambda)]} \operatorname{Hilb}^{P_\lambda}(\mathbf{P}^n) = 6n - 4.$$

A general $\varphi \in \operatorname{Hom}(I(\lambda), S/I(\lambda))_0$ can be written as

$$\begin{aligned}
\varphi(x_0^2) &= a_0^0 x_0 x_n + a_0^1 x_1 x_n \\
\varphi(x_0 x_1) &= a_1^0 x_0 x_n + a_1^1 x_1 x_n \\
\varphi(x_0 x_i) &= a_i^0 x_0 x_n + a_i^1 x_1 x_n + c_2 x_2 x_i + c_3 x_3 x_i + \cdots + c_n x_n x_i, \quad 2 \le i \le n-1 \\
\varphi(x_1^2) &= b_1^0 x_0 x_n + b_1^1 x_1 x_n \\
\varphi(x_1 x_i) &= b_i^0 x_0 x_n + b_i^1 x_1 x_n + d_2 x_2 x_i + \cdots + d_n x_n x_i, \quad 2 \le i \le n-1.
\end{aligned}$$

where $c_2, \ldots, c_n, d_2, \ldots, d_n, \{a_i^0, a_i^1\}_{0 \le i \le n-1}, \{b_i^0, b_i^1\}_{1 \le i \le n-1}$ are independent parameters.

Proposition 5.3.7. *Let $\lambda = ((n-1)^q, 1)$ be an integer partition where either $n = 3$ and $q \ge 4$, or $n \ge 4$ and $q \ge 2$. Then*

$$\dim_{\mathbf{k}} T_{[I(\lambda)]} \operatorname{Hilb}^{P_\lambda}(\mathbf{P}^n) = 2n - 1 + \binom{n+q-1}{n-1}.$$

A general $\varphi \in \operatorname{Hom}(I(\lambda), S/I(\lambda))_0$ can be written as

$$\begin{aligned}
\varphi(x_0^2) &= a_0 x_0 x_n \\
\varphi(x_0 x_i) &= a_i x_0 x_n + c_1 x_1 x_i + \cdots + c_n x_n x_i \\
\varphi(x_1^q) &= b_1 x_0 x_n^{q-1} + \sum_{\omega \in \mathcal{T}_1 \cup \mathcal{T}_2 \setminus x_n^q} c_{i,\omega} \omega,
\end{aligned}$$

where $a_0, \ldots, a_{n-1}, b_1, c_1, \ldots, c_n, c_{i,\omega}$ are independent parameters.

As we will see in Section 5.4, for $\lambda = ((n-1)^q, 1)$ the ideal $I(\lambda)$ corresponds to a smooth point on its Hilbert scheme. To understand the geometry in a neighborhood of the other $[I(\lambda)]$, we will need to compute its deformation space. To do this, we may exclude the trivial deformations, those induced by coordinate changes, as they are unobstructed. More precisely, we want to compute $T^1(R/\mathbf{k}, R)_0$ where $R = S/I(\lambda)$ [92, §3, p. 24]. A straightforward computation of the partial derivatives gives the following bases for T^1 .

Corollary 5.3.8. *Let $\lambda = ((n-1)^q, r+1, 1)$ be an integer partition and let $R = S/I(\lambda)$. Assume $n \ge 4$ and either $r \ge 2$ and $q \ge 1$, or $r = 1$ and $q \ge 3$. Then $T^1(R/\mathbf{k}, R)_0$ is spanned by*

$$\begin{aligned}
&\varphi(x_0 x_i) = a_i x_0 x_n, \qquad 0 \le i \le n-r-1 \\
&\varphi(x_0 x_i) = 0, \qquad n-r \le i \le n-1 \\
&\varphi(x_1^{q+1}) = b_1 x_0 x_n^q + \sum_{\omega \in \mathcal{T}_1} c_\omega x_1 \omega + \ell_{n-r}^1 x_1^q x_{n-r} + \cdots \ell_n^1 x_1^q x_n \\
&\varphi(x_1^q x_i) = b_i x_0 x_n^q + \sum_{\omega \in \mathcal{T}_1} c_\omega x_i \omega, \qquad 1 \le i \le n-r-1,
\end{aligned}$$

where $a_0, \ldots, a_{n-1}, b_1, \ldots, b_{n-r-1}, \ell_{n-r}^1, \ldots, \ell_n^1$ and $\{c_\omega\}_{\omega \in \mathcal{T}_1}$ are independent parameters.

Corollary 5.3.9. *Let $\lambda = (n-1,2,1)$ be an integer partition with $n \geq 4$ and let $R = S/I(\lambda)$. Then $T^1(R/\mathbf{k}, R)_0$ is spanned by*

$$\begin{aligned}
\varphi(x_0x_i) &= a_i x_0 x_n, \qquad 0 \leq i \leq n-2\\
\varphi(x_0x_{n-1}) &= \alpha x_1 x_n\\
\varphi(x_1^2) &= b_1 x_0 x_n + d_{n-1} x_1 x_{n-1} + d_n x_1 x_n\\
\varphi(x_1x_i) &= b_i x_0 x_n, \qquad 2 \leq i \leq n-r-1,
\end{aligned}$$

where $\alpha, a_0, \ldots, a_{n-2}, b_1, \ldots, b_{n-2}, d_{n-1}, d_n$ are independent parameters.

Corollary 5.3.10. *Let $\lambda = (n-1,1,1)$ be an integer partition with $n \geq 3$ and let $R = S/I(\lambda)$. Then $T^1(R/\mathbf{k}, R)_0$ is spanned by*

$$\begin{aligned}
\varphi(x_0x_i) &= a_i^0 x_0 x_n + a_i^1 x_1 x_n, \quad 0 \leq i \leq n-1\\
\varphi(x_1^2) &= b_1^0 x_0 x_n + b_1^1 x_1 x_n, \quad 0 \leq i \leq n-1\\
\varphi(x_1x_i) &= b_i^0 x_0 x_n, \qquad\qquad\quad 2 \leq i \leq n-1,
\end{aligned}$$

where a_i^0, a_i^1, b_i^0 are independent parameters.

Lemma 5.3.11. *With notation as in Section 5.1, let F denote the Eliahou-Kervaire resolution of $I(\lambda)$. Let $R = S/I(\lambda)$ and let $\boldsymbol{f}_{li}^j \in \operatorname{Hom}(F_1, R)$ denote the dual of $\boldsymbol{e}_{li}^j$.*

(i) *If $\lambda = ((n-1)^q, r+1, 1)$ then $\{x_0x_n^2\boldsymbol{f}_{0i}^j, x_0x_n^{q+1}\boldsymbol{f}_{1i}^j\}_{i,j} \subseteq T^2(R/\mathbf{k}, R)_0$ is linearly independent.*

(ii) *If $\lambda = (n-1,2,1)$ then $\{x_0x_n^2\boldsymbol{f}_{0i}^j\, x_0x_n^2\boldsymbol{f}_{1i}^j, x_1x_n^2\boldsymbol{f}_{0,n-1}^j\}_{i,j} \subseteq T^2(R/\mathbf{k}, R)_0$ is linearly independent.*

(iii) *If $\lambda = (n-1,1,1)$ then $\{x_0x_n^2\boldsymbol{f}_{0i}^j\, x_0x_n^2\boldsymbol{f}_{1i}^j, x_1x_n^2\boldsymbol{f}_{0i}^j\, x_1x_n^2\boldsymbol{f}_{1i}^j\}_{i,j} \subseteq T^2(R/\mathbf{k}, R)_0$ is linearly independent.*

Proof. We will only prove (ii) as the other two cases are analogous (and simpler). We use A_i to denote the matrix associated to ψ_i. By construction the entries in A_i are supported on $(x_0, \dots, x_{n-1})$. Dualizing the resolution F we obtain

$$\begin{aligned}
\psi_1^\vee(\boldsymbol{f}^\star_{00}) &= -x_1\boldsymbol{f}^0_{01} - \sum_{1<j\le n-1} x_j\boldsymbol{f}^0_{0j}\\
\psi_1^\vee(\boldsymbol{f}^\star_{01}) &= x_0\boldsymbol{f}^0_{01} - x_1\boldsymbol{f}^0_{11} - \sum_{1<j\le n-1} x_j\boldsymbol{f}^1_{0j}\\
\psi_1^\vee(\boldsymbol{f}^\star_{0i}) &= x_0\boldsymbol{f}^0_{0i} + x_1\boldsymbol{f}^1_{0i} - x_1\boldsymbol{f}^0_{1i} + \sum_{2\le j<i} x_j\boldsymbol{f}^j_{0i} - \sum_{i<j\le n-1} x_j\boldsymbol{f}^i_{0j}\\
\psi_1^\vee(\boldsymbol{f}^\star_{0,n-1}) &= x_0\boldsymbol{f}^0_{0,n-1} + x_1\boldsymbol{f}^1_{0,n-1} + \sum_{2\le j<n-1} x_j\boldsymbol{f}^j_{0,n-1}\\
\psi_1^\vee(\boldsymbol{f}^\star_{1i}) &= x_0\boldsymbol{f}^0_{0j} + \sum_{1\le j<i} x_j\boldsymbol{f}^j_{1i} - \sum_{i<j\le n-2} x_j\boldsymbol{f}^i_{1j}.
\end{aligned}$$

Let us first check that $x_0x_n^2\boldsymbol{f}^j_{0i}$ and $x_0x_n^2\boldsymbol{f}^j_{1i}$ are well defined elements of $T^2(R/\mathbf{k}, R)_0$. It is enough to show that $x_0x_n^2$ annihilates $\ker\psi_1 + \mathrm{Kos}$. Since the entries in A_2 are supported on $(x_0, \dots, x_{n-1})$, multiplying by $x_0x_n^2$ annihilates $\psi_2(F_2) = \ker\psi_1$. Since the Koszul relations are supported on (x_0, x_1), $x_0x_n^2$ annihilate Kos.

Since $x_1x_n^2$ also annihilates Kos, to show that that $x_1x_n^2\boldsymbol{f}^j_{0,n-1}$ is a well defined element, we only need to prove that $x_1x_n^2$ annihilates the restriction $(\ker\psi_1)|_{S(-3)\boldsymbol{e}^j_{0,n-1}}$. Let $\boldsymbol{v} \in \ker\psi_1$ and since the differentials are linear we may assume $\boldsymbol{v}$ is linear. Then $\psi_1(\boldsymbol{v}) = 0$ implies

$$\begin{aligned}
-x_1\boldsymbol{v}_{\boldsymbol{e}^0_{01}} - x_2\boldsymbol{v}_{\boldsymbol{e}^0_{02}} - \cdots - x_{n-1}\boldsymbol{v}_{\boldsymbol{e}^0_{0,n-1}} &= 0\\
x_0\boldsymbol{v}_{\boldsymbol{e}^0_{01}} - x_1\boldsymbol{v}_{\boldsymbol{e}^0_{11}} - x_2\boldsymbol{v}_{\boldsymbol{e}^1_{02}} - \cdots - x_{n-1}\boldsymbol{v}_{\boldsymbol{e}^1_{0,n-1}} &= 0\\
x_0\boldsymbol{v}_{\boldsymbol{e}^0_{0i}} + x_1\boldsymbol{v}_{\boldsymbol{e}^1_{0i}} - x_1\boldsymbol{v}_{\boldsymbol{e}^0_{1i}} + \sum_{2\le j<i} x_j\boldsymbol{v}_{\boldsymbol{e}^j_{0i}} - \sum_{i<j\le n-1} x_j\boldsymbol{v}_{\boldsymbol{e}^i_{0j}} &= 0, \quad 2 \le i \le n-2.
\end{aligned}$$

The j-th equation above is just the j-th row of A_1 multiplied with $\boldsymbol{v}$ (we can read this off from our description of $\psi_1^\vee$). From the j-th equation we can see that $\boldsymbol{v}_{\boldsymbol{e}^j_{0,n-1}}$ is supported on $(x_0, \dots, x_{n-2})$ for all $0 \le j \le n-2$. As a consequence, $x_1x_n^2$ annihilates $\boldsymbol{v}_{\boldsymbol{e}^j_{0,n-1}}$ and all of $(\ker\psi_1)|_{S(-3)\boldsymbol{e}^j_{0,n-1}}$.

We will now show that the set $\mathcal{S} = \mathrm{span}_{\mathbf{k}}\{x_0x_n^2\boldsymbol{f}^j_{0i}\, x_0x_n^2\boldsymbol{f}^j_{1i}, x_1x_n^2\boldsymbol{f}^j_{0,n-1}\}_{i,j}$ is linearly independent in $T^2(R/\mathbf{k}, R)$. In particular, we need to show that no non-zero element of $\mathcal{S}$ is a linear combination of the form $\sum_{l,i} c_{li}Q_{li}\overline{\psi_1^\vee}(\boldsymbol{f}^\star_{li})$ where $Q_{li} \in R(2)$ are quadrics and $c_{li} \in \mathbf{k}$ constants. However, since all the elements of $\mathcal{S}$ are multiples of x_n^2 and A_1 does not contain the variable x_n, it suffices to show that no non-zero element of $\mathcal{S}$ is a linear combination of the form $\sum_{l,i} c_{li}x_n^2\overline{\psi_1^\vee}(\boldsymbol{f}^\star_{li})$. From the description of $\psi_1^\vee$ in the first paragraph we see that this is indeed the case. $\square$

5.4 Proof of the main theorem

The goal of this section is to prove Theorem 5.0.1. By Lemma 5.2.1 and Eq. (5.4) we may assume that $s = 0$ and $n = d - 2$. The proof will provide a description of the universal deformation space of $I(\lambda)$ valid in all characteristics.

Proof of Theorem 5.0.1 (1) to (3). Case (1) and (2) are [30, Theorem 2.4] while case (3) is [16, Theorem 1.1]. □

Proof of Theorem 5.0.1 (4), (5). It follows from [83, Theorem 4.1] that $\dim(\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n))$ agrees with the dimension of the tangent space to $[I(\lambda)]$ (Proposition 5.3.7). Thus, $[I(\lambda)]$ is a smooth point on the Hilbert scheme. By Theorem [83, Theorem 1.4] the lexicographic point is also a smooth point. Since $\mathrm{Hilb}^{P_\lambda}(\mathbf{P}^n)$ has only two Borel-fixed points (Proposition 5.2.10), Lemma 2.0.25 implies that the Hilbert scheme is smooth. Finally, [83, Theorem 4.1] gives the description of the general member. □

Proof of Theorem 5.0.1 (6), (7). Let $\mathbf{U} = \mathbf{k}[\![u_{00}, \dots, u_{0,n-r-1}, u_{11}, \dots, u_{1n}, \{u_{2,\omega}\}_{\omega\in\mathcal{T}_1}]\!]$ and let $\mathfrak{m}_{\mathbf{U}}$ denote its maximal ideal. Consider the following perturbation of ψ_0

$$\Psi_0(\boldsymbol{e}^\star_{0i}) = x_0x_i + u_{0i}x_0x_n, \quad i \le n-r-1$$
$$\Psi_0(\boldsymbol{e}^\star_{0i}) = x_0x_i, \quad i \ge n-r$$
$$\Psi_0(\boldsymbol{e}^\star_{11}) = x_1^{q+1} + u_{11}x_0x_n^q + \sum_{l=0}^{r} u_{1,n-r+l}x_1^q x_{n-r+l} + \sum_{\omega\in\mathcal{T}_1} u_{2,\omega}x_1\omega + \sum_{l=0}^{r}\sum_{\omega\in\mathcal{T}_1} u_{1,n-r+l}u_{2,\omega}x_{n-r+l}\omega$$
$$\Psi_0(\boldsymbol{e}^\star_{1i}) = x_1^q x_i + u_{1i}x_0x_n^q + \sum_{\omega\in\mathcal{T}_1} u_{2,\omega}x_i\omega, \quad i > 1.$$

By Corollary 5.3.8 this lifts the first order deformation by non-trivial deformations. To perturb the syzygies, we need a few definitions. Let $\mathcal{U} := \{\omega \in \mathcal{T}_1 : \text{there exists } x_i|\omega \text{ with } n-r \le i \le n-1\}$, $\mathcal{V} := \{\omega \in \mathcal{T}_1 : \omega \text{ is supported on } x_1, \dots, x_{n-r-1}, x_n\} \setminus x_n^q$ and $\eta := x_n^q$. Observe that $\mathcal{T}_1 = \mathcal{U} \sqcup \mathcal{V} \sqcup \{x_n^q\}$.

For each $\omega \in \mathcal{U}$ choose some $n-r \le i \le n-1$ for which $x_i|\omega$ and let $\bar{\omega} := \frac{\omega}{x_i}$ and $\widehat{\omega} := i$. For each $\omega \in \mathcal{V}$ define the following

- Let $\omega_0 = 1$ and for $1 \le \ell \le q$ let ω_ℓ denote the lexicographically largest monomial of degree ℓ dividing ω.
- For $0 \le \ell \le q-1$ let $\lambda(\omega_\ell)$ to be the index of the variable $\frac{\omega_{\ell+1}}{\omega_\ell}$.
- For $0 \le \ell \le q-1$ let $u_{\omega_\ell} := \frac{\omega}{\omega_\ell}|_{\{x_j=u_{0j}\}_j}$.

For example, if $\omega = x_0^3x_3^3x_4$ then $\omega_4 = x_0^3x_3$, then $\lambda(\omega_3) = x_3$ and $u_{\omega_4} = u_{03}^2u_{04}$. Define

$$\Omega := \sum_{\ell=1}^{q}(-1)^{\ell-1}u_{01}^{\ell-1}x_1^{q-\ell}x_n^\ell\boldsymbol{e}^\star_{01} + \sum_{\omega\in\mathcal{U}} u_{2\omega}\bar{\omega}x_n\boldsymbol{e}^\star_{0,\widehat{\omega}} + \sum_{\omega\in\mathcal{V}} u_{2\omega}\sum_{\ell=1}^{q}(-1)^{\ell-1}u_{\omega_{q-\ell+1}}\omega_{q-\ell}x_n^\ell\boldsymbol{e}^\star_{0,\lambda(\omega_{q-\ell})}.$$

Here is the lift of the syzygies

$$\begin{aligned}
\Psi_1(e_{0i}^j) &= (x_j + u_{0j}x_n)e_{0i}^\star - (x_i + u_{0i}x_n)e_{0j}^\star, && 0 \le j < i \le n-r-1\\
\Psi_1(e_{0i}^j) &= (x_j + u_{0j}x_n)e_{0i}^\star - x_i e_{0j}^\star, && j < n-r \le i \le n-1\\
\Psi_1(e_{0i}^j) &= x_j e_{0i}^\star - x_i e_{0j}^\star, && n-r \le j < i \le n-1\\
\Psi_1(e_{11}^0) &= x_0 e_{11}^\star - x_1^q e_{01}^\star - u_{11}x_n^q e_{00}^\star\\
&\quad - \sum_{\omega\in\mathcal{T}_1} u_{2\omega}\omega e_{01}^\star - \sum_{l=0}^{r-1} u_{1,n-r+l}x_1^q e_{0,n-r+l}^\star\\
&\quad - \sum_{l=0}^{r-1}\sum_{\omega\in\mathcal{T}_1} u_{2\omega}u_{1,n-r+l}\omega e_{0,n-r+l}^\star + (u_{01}-u_{1n})\Omega\\
\Psi_1(e_{1i}^0) &= x_0 e_{1i}^\star - x_1^q e_{0i}^\star - \sum_{\omega\in\mathcal{T}_1} u_{2\omega}\omega e_{0i}^\star - u_{1i}x_n^q e_{00}^\star + u_{0i}\Omega, && 2 \le i \le n-r-1\\
\Psi_1(e_{1i}^1) &= x_1 e_{1i}^\star - x_i e_{11}^\star + u_{11}x_n^q e_{0i}^\star - u_{1i}x_n^q e_{01}^\star + \sum_{l=0}^{r} u_{1,n-r+l}x_{n-r+l}e_{1i}^\star\\
&\quad - \sum_{l=0}^{r-1} u_{1i}u_{1,n-r+l}x_n e_{0,n-r+l}^\star, && 2 \le i \le n-r-1\\
\Psi_1(e_{1i}^j) &= x_j e_{1i}^\star - x_i e_{1j}^\star + u_{1j}x_n^q e_{0i}^\star - u_{1i}x_n^q e_{0j}^\star, && 2 \le j < i \le n-r-1.
\end{aligned}$$

It will be notationally convenient to separate the cases $q > 1$ and $q = 1$. If $q > 1$, composing Ψ_0 and Ψ_1 we obtain

$$\begin{aligned}
\Psi_0\Psi_1(e_{0i}^j) &= 0, \quad 0 \le j < i \le n-1 && (5.5)\\
\Psi_0(\Psi_1(e_{1i}^j)) &= (u_{0i}u_{1j} - u_{0j}u_{1i})x_0x_n^{q+1}, \quad 2 \le j < i \le n-r-1\\
\Psi_0(\Psi_1(e_{1i}^0)) &= (u_{0i}(-u_{2\eta} + \alpha) - u_{00}u_{1i})x_0x_n^{q+1}, \quad 2 \le i \le n-r-1 && (5.6)\\
\Psi_0(\Psi_1(e_{11}^0)) &= ((-u_{2\eta} + \alpha)(u_{01} - u_{1n}) - u_{00}u_{11})x_0x_n^{q+1}\\
\Psi_0(\Psi_1(e_{1i}^1)) &= (u_{11}u_{0i} - u_{1i}(u_{01} - u_{1n}))x_0x_n^{q+1}, \quad 2 \le i \le n-r-1
\end{aligned}$$

with $\alpha = (-1)^{q-1}u_{01}^q + (-1)^{q-1}\sum_{\omega\in\mathcal{V}} u_{2\omega}u_{\omega_0}$.

To compute the obstruction space we just repeat the above computation mod $\mathfrak{m}_{\mathbf{U}}^{l+1}$. Indeed, for $l \ge 1$ let $\Psi_0^l = \Psi_0 \bmod \mathfrak{m}_{\mathbf{U}}^{l+1}$ and $\Psi_1^l = \Psi_1 \bmod \mathfrak{m}_{\mathbf{U}}^{l+1}$. Then the image of $\Psi_0^l\Psi_1^l$

in $T^2(R/\mathbf{k}, R)_0 \otimes \mathbf{U}/\mathfrak{m}_{\mathbf{U}}^{l+2}$ is

$$\begin{aligned}
\Psi_0^l\Psi_1^l(e_{0i}^j) &\equiv 0, \quad 0 \le j < i \le n-1 \\
\Psi_0^l(\Psi_1^l(e_{1i}^j)) &\equiv (u_{0i}u_{1j} - u_{0j}u_{1i})x_0x_n^{q+1}, \quad 2 \le j < i \le n-r-1 \\
\Psi_0^l(\Psi_1^l(e_{1i}^0)) &\equiv (u_{0i}(-u_{2\eta}+\alpha) - u_{00}u_{1i})x_0x_n^{q+1}, \quad 2 \le i \le n-r-1 \\
\Psi_0^l(\Psi_1^l(e_{11}^0)) &\equiv ((-u_{2\eta}+\alpha)(u_{01}-u_{1n}) - u_{00}u_{11})x_0x_n^{q+1} \\
\Psi_0^l(\Psi_1^l(e_{1i}^1)) &\equiv (u_{11}u_{0i} - u_{1i}(u_{01}-u_{1n}))x_0x_n^{q+1}, \quad 2 \le i \le n-r-1
\end{aligned}$$

Using Lemma 5.3.11 (1), the above equation allows us to directly read off the obstruction to lift our family from the $(l-1)$-th order to l-th order (beginning with $l=1$). In particular, the ideal of obstructions to lift to q-th order is the 2×2 minors of

$$\begin{pmatrix} u_{00} & u_{01}-u_{1n} & u_{02} & u_{03} & \cdots & u_{0,n-r-1} \\ -u_{2\eta}+\alpha & u_{11} & u_{12} & u_{13} & \cdots & u_{1,n-r-1} \end{pmatrix}.$$

If we denote this ideal by J, we have $\Psi_0\Psi_1 = 0$ in $\mathbf{U}/J$ (Eq. 5.6). Thus, Ψ_0 gives a versal deformation of $I(\lambda)$. Since we are working analytically, we may apply the isomorphism that maps $u_{2\eta} \mapsto -u_{2\eta}+\alpha$ and fixes the other variables. This transformation makes J the 2×2 minors of a generic matrix. Finally, adding back the trivial deformations we obtain the universal deformation space of $I(\lambda)$.

If $q=1$ we obtain

$$\begin{aligned}
\Psi_0\Psi_1(e_{0i}^j) &= 0, \quad 0 \le j < i \le n-1 \\
\Psi_0(\Psi_1(e_{1i}^j)) &= (u_{0i}u_{1j} - u_{0j}u_{1i})x_0x_n^2, \quad 2 \le j < i \le n-r-1 \\
\Psi_0(\Psi_1(e_{1i}^0)) &= (u_{0i}u_{01} - u_{00}u_{1i})x_0x_n^2, \quad 2 \le i \le n-r-1 \\
\Psi_0(\Psi_1(e_{11}^0)) &= (u_{01}(u_{01}-u_{1n}) - u_{00}u_{11})x_0x_n^2 \\
\Psi_0(\Psi_1(e_{1i}^1)) &= (u_{11}u_{0i} - u_{1i}(u_{01}-u_{1n}))x_0x_n^2, \quad 2 \le i \le n-r-1.
\end{aligned}$$

Arguing as in the $q>1$ case we see that the versal deformation space is cut out by 2×2 minors of

$$\begin{pmatrix} u_{00} & u_{01}-u_{1n} & u_{02} & u_{03} & \cdots & u_{0,n-r-1} \\ u_{01} & u_{11} & u_{12} & u_{13} & \cdots & u_{1,n-r-1} \end{pmatrix}.$$

We have obtained the desired étale-local description as the Segre embedding $\mathbf{P}^1 \times \mathbf{P}^{n-r-1} \hookrightarrow \mathbf{P}^{2(n-r)-1}$ is cut out by the ideal of 2×2 minors of a generic $2\times(n-r)$ matrix. It is well known that the Segre embedding is normal and Cohen-Macaulay [51]. It follows that the Hilbert scheme is normal and Cohen-Macaulay in a neighbourhood of $[I(\lambda)]$. Combining this with [83, Theorem 1.4] and Lemma 2.0.25 we deduce that the Hilbert scheme is normal and Cohen-Macalay. Since the Hilbert scheme is connected [46, Corollary 5.9], it must be irreducible. Finally, the description of the general member is given in [83, Theorem 4.1] and the other statements follow from Lemma 2.0.25. $\square$

Proof of Theorem 5.0.1 (8). Let $\mathbf{U} = \mathbf{k}[\![u_{00}, \ldots, u_{0,n-1}, u_{11}, \ldots, u_{1,n-1}, v_{00}, \ldots, v_{0,n-1}, v_{11}]\!]$. For convenience we will sometimes use u_{10} to denote u_{01}. Consider the following perturbation of ψ_0

$$\begin{aligned}
\Psi_0(\boldsymbol{e}^\star_{0i}) &= x_0x_i + u_{0i}x_0x_n + v_{0i}x_1x_n, \quad 0 \le i \le n-1 \\
\Psi_0(\boldsymbol{e}^\star_{11}) &= x_1^2 + u_{11}x_0x_n + v_{11}x_1x_n \\
\Psi_0(\boldsymbol{e}^\star_{1i}) &= x_1x_i + u_{1i}x_0x_n, \quad 2 \le i \le n-1
\end{aligned}$$

and a perturbation of ψ_1

$$\begin{aligned}
&\Psi_1(\boldsymbol{e}^0_{0i}) = (x_0 + u_{00}x_n)\boldsymbol{e}^\star_{0i} - (x_i + u_{0i}x_n)\boldsymbol{e}^\star_{00} + v_{00}x_n\boldsymbol{e}^\star_{1i} - v_{0i}x_n\boldsymbol{e}^\star_{01}, && 1 \le i \le n-1 \\
&\Psi_1(\boldsymbol{e}^j_{0i}) = (x_j + u_{0j}x_n)\boldsymbol{e}^\star_{0i} - (x_i + u_{0i}x_n)\boldsymbol{e}^\star_{0j} + v_{0j}x_n\boldsymbol{e}^\star_{1i} - v_{0i}x_n\boldsymbol{e}^\star_{1j}, && 1 \le j < i \le n-1 \\
&\Psi_1(\boldsymbol{e}^0_{11}) = (x_0 + v_{01}x_n)\boldsymbol{e}^\star_{11} - x_1\boldsymbol{e}^\star_{01} - u_{11}x_n\boldsymbol{e}^\star_{00} + (u_{01} - v_{11})x_n\boldsymbol{e}^\star_{01} \\
&\Psi_1(\boldsymbol{e}^0_{1i}) = x_0\boldsymbol{e}^\star_{1i} - x_1\boldsymbol{e}^\star_{0i} + v_{0i}x_n\boldsymbol{e}^\star_{11} + u_{0i}x_n\boldsymbol{e}^\star_{01} - u_{1i}x_n\boldsymbol{e}^\star_{00}, && 2 \le i \le n-1 \\
&\Psi_1(\boldsymbol{e}^1_{1i}) = (x_1 + v_{11}x_n)\boldsymbol{e}^\star_{1i} - x_i\boldsymbol{e}^\star_{11} + u_{11}x_n\boldsymbol{e}^\star_{0i} - u_{1i}x_n\boldsymbol{e}^\star_{01}, && 2 \le i \le n-1 \\
&\Psi_1(\boldsymbol{e}^j_{1i}) = x_j\boldsymbol{e}^\star_{1i} - x_i\boldsymbol{e}^\star_{1j} + u_{1j}x_n\boldsymbol{e}^\star_{0i} - u_{1i}x_n\boldsymbol{e}^\star_{0j}, && 2 \le j < i \le n-1.
\end{aligned}$$

Composing the two we obtain

$$\begin{aligned}
&\Psi_0\Psi_1(\boldsymbol{e}^0_{01}) = (u_{11}v_{00} - u_{01}v_{01})x_0x_n^2 + (v_{01}(u_{00} - v_{01}) - v_{00}(u_{01} - v_{11}))x_1x_n^2 \\
&\Psi_0\Psi_1(\boldsymbol{e}^0_{0i}) = (u_{1i}v_{00} - u_{01}v_{0i})x_0x_n^2 + (v_{0i}(u_{00} - v_{01}) - u_{0i}v_{00})x_1x_n^2, && 2 \le i \le n-1 \\
&\Psi_0\Psi_1(\boldsymbol{e}^1_{0i}) = (u_{1i}v_{01} - u_{11}v_{0i})x_0x_n^2 + (v_{0i}(u_{01} - v_{11}) - u_{0i}v_{01})x_1x_n^2, && 2 \le i < n \\
&\Psi_0\Psi_1(\boldsymbol{e}^j_{0i}) = (u_{1i}v_{0j} - u_{1j}v_{0i})x_0x_n^2 + (u_{0j}v_{0i} - u_{0i}v_{0j})x_1x_n^2, && 2 \le j < i < n \\
&\Psi_0(\Psi_1(\boldsymbol{e}^0_{11})) = (u_{01}(u_{01} - v_{11}) - u_{11}(u_{00} - v_{01}))x_0x_n^2 + (u_{01}v_{01} - u_{11}v_{00})x_1x_n^2 \\
&\Psi_0(\Psi_1(\boldsymbol{e}^0_{1i})) = (\underline{u_{11}v_{0i} + u_{01}u_{0i} - u_{1i}u_{00}})x_0x_n^2 + (\underline{u_{0i}v_{01} + v_{11}v_{0i} - u_{1i}v_{00}})x_1x_n^2, && 2 \le i \le n-1 \\
&\Psi_0(\Psi_1(\boldsymbol{e}^1_{1i})) = (u_{0i}u_{11} - u_{1i}(u_{01} - v_{11}))x_0x_n^2 + (u_{11}v_{0i} - u_{1i}v_{01})x_1x_n^2, && 2 \le i \le n-1 \\
&\Psi_0(\Psi_1(\boldsymbol{e}^j_{1i})) = (u_{ij}u_{0i} - u_{1i}u_{0j})x_0x_n^2 + (u_{ij}v_{0i} - u_{1i}v_{0j})x_1x_n^2, && 2 \le j < i \le n-1.
\end{aligned}$$

Since the lifts Ψ_0 and Ψ_1 are first order, we see that the ideal of obstructions to lift to second order is the 2×2 minors of

$$\begin{pmatrix} u_{01} & u_{11} & u_{12} & \cdots & u_{1,n-1} \\ v_{00} & v_{01} & v_{02} & \cdots & v_{1,n-1} \\ u_{00} - v_{01} & u_{01} - v_{11} & u_{02} & \cdots & u_{0,n-1} \end{pmatrix}.$$

Indeed, most of the minors show up as coefficients of $x_0x_n^2$ and $x_1x_n^2$. The other minors come from the underlined equations

$$\begin{aligned}
u_{11}v_{0i} + u_{01}u_{0i} - u_{1i}u_{00} + (u_{1i}v_{00} - u_{01}v_{0i}) &= v_{01}u_{0i} - v_{0i}(u_{01} - v_{11}) \\
u_{0i}v_{01} + v_{11}v_{0i} - u_{1i}v_{00} - (u_{11}v_{0i} - u_{1i}v_{01}) &= u_{01}u_{0i} - u_{1i}(u_{00} - v_{01}).
\end{aligned}$$

If we denote the ideal of 2×2 minors by J we have $\Psi_0\Psi_1 = 0$ in $\mathbf{U}/J$. Thus, Ψ_0 gives a versal deformation of $I(\lambda)$. Adding back the trivial deformations gives us the universal deformation space of $I(\lambda)$. This gives us the desired étale-local description as the Segre embedding $\mathbf{P}^2 \times \mathbf{P}^{n-1} \hookrightarrow \mathbf{P}^{3n-1}$ is cut out by the ideal of 2×2 minors of a generic $3 \times n$ matrix. Similar to the previous proof, the other statements follow from [51], [46, Corollary 5.9], Lemma 2.0.25 and [83, Theorem 4.1]. $\square$

Proof of Theorem 5.0.1 (9). Let $\mathbf{U} = \mathbf{k}[\![u_{00}, \dots, u_{0,n-1}, u_{11}, \dots, u_{1n}]\!]$ and let $\mathfrak{m}_{\mathbf{U}}$ denote its maximal ideal. We will sometimes use $\boldsymbol{e}^\star_{10}$ to denote $\boldsymbol{e}^\star_{01}$. This does not cause any confusion as $\boldsymbol{e}^\star_{10}$ is not part of a basis of F_0. Consider the following perturbation of ψ_0

$$\begin{aligned}
\Psi_0(\boldsymbol{e}^\star_{00}) &= x_0^2 + u_{00}x_0x_n \\
\Psi_0(\boldsymbol{e}^\star_{01}) &= x_0x_1 + u_{01}x_0x_n - u_{0,n-1}u_{1,n-1}x_1x_n \\
\Psi_0(\boldsymbol{e}^\star_{0i}) &= x_0x_i + u_{0i}x_0x_n, && 2 \le i \le n-2 \\
\Psi_0(\boldsymbol{e}^\star_{0,n-1}) &= x_0x_{n-1} + u_{0,n-1}x_1x_n \\
\Psi_0(\boldsymbol{e}^\star_{11}) &= x_1^2 + u_{11}x_0x_n + u_{1,n-1}x_1x_{n-1} + u_{1n}x_1x_n \\
\Psi_0(\boldsymbol{e}^\star_{1i}) &= x_1x_i + u_{1i}x_0x_n, && 2 \le i \le n-2.
\end{aligned}$$

and a perturbation of ψ_1

$$\begin{aligned}
\Psi_1(\boldsymbol{e}^0_{01}) &= (x_0 + u_{00}x_n)\boldsymbol{e}^\star_{01} - (x_1 + u_{01}x_n)\boldsymbol{e}^\star_{00} + u_{0,n-1}u_{1,n-1}x_n\boldsymbol{e}^\star_{01} \\
\Psi_1(\boldsymbol{e}^0_{0i}) &= (x_0 + u_{00}x_n)\boldsymbol{e}^\star_{0i} - (x_i + u_{0i}x_n)\boldsymbol{e}^\star_{00}, && 2 \le i \le n-2 \\
\Psi_1(\boldsymbol{e}^1_{0i}) &= (x_1 + u_{01}x_n)\boldsymbol{e}^\star_{0i} - (x_i + u_{0i}x_n)\boldsymbol{e}^\star_{01} - u_{0,n-1}u_{1,n-1}x_n\boldsymbol{e}^\star_{1i}, && 2 \le i \le n-1 \\
\Psi_1(\boldsymbol{e}^j_{0i}) &= (x_j + u_{0j}x_n)\boldsymbol{e}^\star_{0i} - (x_i + u_{0i}x_n)\boldsymbol{e}^\star_{0j}, && 2 \le j < i \le n-2 \\
\Psi_1(\boldsymbol{e}^j_{0,n-1}) &= (x_j + u_{0j}x_n)\boldsymbol{e}^\star_{0,n-1} - x_{n-1}\boldsymbol{e}^\star_{0j} - u_{0,n-1}x_n\boldsymbol{e}^\star_{1j}, && 0 \le j \le n-2 \\
\Psi_1(\boldsymbol{e}^0_{11}) &= x_0\boldsymbol{e}^\star_{11} - x_1\boldsymbol{e}^\star_{01} - u_{11}x_n\boldsymbol{e}^\star_{00} - u_{1,n-1}x_1\boldsymbol{e}^\star_{0,n-1} + (u_{01} - u_{1n})x_n\boldsymbol{e}^\star_{01} \\
\Psi_1(\boldsymbol{e}^0_{1i}) &= x_0\boldsymbol{e}^\star_{1i} - x_1\boldsymbol{e}^\star_{0i} + u_{0i}x_n\boldsymbol{e}^\star_{01} - u_{1i}x_n\boldsymbol{e}^\star_{00}, && 2 \le i \le n-2 \\
\Psi_1(\boldsymbol{e}^1_{1i}) &= x_1\boldsymbol{e}^\star_{1i} - x_i\boldsymbol{e}^\star_{11} + u_{11}x_n\boldsymbol{e}^\star_{0i} - u_{1i}x_n\boldsymbol{e}^\star_{01} \\
&\quad + (u_{1,n-1}x_{n-1} + u_{1n}x_n)\boldsymbol{e}^\star_{1i} - u_{1i}u_{1,n-1}x_n\boldsymbol{e}^\star_{0,n-1}, && 2 \le i \le n-2 \\
\Psi_1(\boldsymbol{e}^j_{1i}) &= x_j\boldsymbol{e}^\star_{1i} - x_i\boldsymbol{e}^\star_{1j} + u_{1j}x_n\boldsymbol{e}^\star_{0i} - u_{1i}x_n\boldsymbol{e}^\star_{0j}, && 2 \le j < i \le n-2.
\end{aligned}$$

For $l \ge 1$ let, $\Psi_0^l \equiv \Psi_0^l \bmod \mathfrak{m}_{\mathbf{U}}^{l+1}$ and $\Psi_1^l \equiv \Psi_1^l \bmod \mathfrak{m}_{\mathbf{U}}^{l+1}$. As done previously, the obstruction to lifting to second order is the image of $\Psi_0^1\Psi_1^1$ in $T^2(R/\mathbf{k}, R)_0 \otimes \mathfrak{m}_{\mathbf{U}}^2/\mathfrak{m}_{\mathbf{U}}^3$. This is

$$
\begin{aligned}
\Psi_0^1\Psi_1^1(e_{0i}^j) &\equiv 0, && 0 \le j < i \le n-2\\
\Psi_0^1\Psi_1^1(e_{0,n-1}^0) &\equiv -u_{01}u_{0,n-1}x_0x_n^2 + u_{00}u_{0,n-1}x_1x_n^2\\
\Psi_0^1\Psi_1^1(e_{0,n-1}^1) &\equiv u_{0,n-1}(u_{01}-u_{1n})x_1x_n^2 - u_{0,n-1}u_{11}x_0x_n^2\\
&\quad - \underline{u_{0,n-1}u_{1,n-1}x_1x_{n-1}x_n}\\
\Psi_0^1\Psi_1^1(e_{0,n-1}^j) &\equiv u_{0j}u_{0,n-1}x_1x_n^2 - u_{0,n-1}u_{1j}x_0x_n^2, && 2 \le j \le n-2\\
\Psi_0^1\Psi_1^1(e_{11}^0) &\equiv (u_{01}(u_{01}-u_{1n}) - u_{00}u_{11})x_0x_n^2\\
&\quad - \underline{u_{0,n-1}u_{1,n-1}x_1^2x_n}\\
\Psi_0^1\Psi_1^1(e_{1i}^0) &\equiv (u_{01}u_{0i} - u_{00}u_{1i})x_0x_n^2, && 2 \le i \le n-2\\
\Psi_0^1\Psi_1^1(e_{1i}^1) &\equiv (u_{0i}u_{11} - u_{1i}(u_{01}-u_{1n}))x_0x_n^2\\
&\quad + \underline{u_{1,n-1}u_{1i}x_0x_{n-1}x_n}, && 2 \le i \le n-2\\
\Psi_0^1\Psi_1^1(e_{1i}^j) &\equiv (u_{0i}u_{1j} - u_{0j}u_{1i})x_0x_n^2, && 2 \le j < i \le n-2.
\end{aligned}
$$

In this image, the three underlined terms are 0. Indeed, the second and third underlined term (from the top) are 0 in R and the first term is equal to $\overline{\psi_1^\vee}(u_{0,n-1}u_{1,n-1}x_1x_n\boldsymbol{f}_{01}^\star)$. After the underlined terms vanish, $\Psi_0^1\Psi_1^1$ is written in terms of our desired basis elements (Lemma 5.3.11 (2)). Thus, the ideal generated by the coefficients, which we denote by J_1, is the ideal of of obstructions to lift to second order. Let $\mathbf{U}^1 = \mathbf{U}/J_1$ and $\mathfrak{m}_{\mathbf{U}^1}$ its maximal ideal. To compute the the obstructions to third order we compute $\Psi_0^2\Psi_1^2$ in $T^2(R/\mathbf{k}, R)_0 \otimes \mathfrak{m}_{\mathbf{U}^1}^3/\mathfrak{m}_{\mathbf{U}^1}^4$. This is

$$
\begin{aligned}
\Psi_0^2\Psi_1^2(e_{0i}^j) &\equiv 0, \quad (i,j) \ne (0, n-1)\\
\Psi_0^2\Psi_1^2(e_{0,n-1}^0) &\equiv u_{0,n-1}^2u_{1,n-1}x_1x_n^2\\
\Psi_0^2\Psi_1^2(e_{1i}^j) &\equiv 0, \quad \text{for all } j, i
\end{aligned}
$$

Thus, the ideal of obstructions to lift to third order is

$$
\begin{aligned}
J_2 := \left((u_{0,n-1}) + I_2\begin{pmatrix} u_{00} & u_{01}-u_{1n} & u_{02} & u_{03} & \cdots & u_{0,n-2}\\ u_{01} & u_{11} & u_{12} & u_{13} & \cdots & u_{1,n-2}\end{pmatrix}\right) \cap\\
(u_{00} + u_{0,n-1}u_{1,n-1}, u_{01}, u_{02}, \ldots, u_{0,n-2}, u_{11}, u_{12}, \ldots, u_{1,n-2}, u_{1n}).
\end{aligned}
$$

Here $I_2(-)$ denotes the ideal of the 2×2 minors of $-$. Finally, it is easy to see that $\Psi_0\Psi_1 = 0$ in $\mathbf{U}/J_2$ (for instance, the underlined terms in $\Psi_0^1\Psi_1^1$ are cancelled by the second order terms). Thus Ψ_0 gives a versal deformation of $I(\lambda)$. Adding back the trivial deformations gives us the universal deformation space of $I(\lambda)$.

From Proposition 5.3.5 and Corollary 5.3.9 we see that there are $4n-6$ trivial deformations; denote them by $t_1, \ldots, t_{4n-6}$. Thus, the smooth component of $\mathrm{Spec}(\mathbf{U}[t_1, \ldots, t_{4n-6}]/J_2)$ has dimension $4n-4$. Since $P_\lambda = \binom{t+n-2}{n-2} + t + 1$, there is an irreducible component, $\mathcal{Y}_1$, whose general member parameterizes a line and a disjoint $(n-2)$-plane. This is birational to $\mathbf{G}(1,n) \times \mathbf{G}(1,n-2)$ and, as a consequence, has dimension $4n-4$; thus $\mathcal{Y}_1$ is the smooth component. It is shown in [81] that $\mathcal{Y}_1$ is isomorphic to a blow up of $\mathbf{G}(1,n) \times \mathbf{G}(n-2,n)$ along the locus $\{(L,\Lambda) : L \subseteq \Lambda\}$. Similar to the previous proofs, the other statements follow from [51], [46, Corollary 5.9], Lemma 2.0.25 and [83, Theorem 4.1]. $\square$

Chapter 6

On the tangent space to $\mathrm{Hilb}^d(\mathbf{P}^3)$

In this chapter we study the tangent space to the Hilbert scheme $\mathrm{Hilb}^d(\mathbf{P}^3)$, motivated by Haiman's work on $\mathrm{Hilb}^d(\mathbf{P}^2)$ and by a long-standing conjecture of Briançon and Iarrobino on the most singular point in $\mathrm{Hilb}^d(\mathbf{P}^n)$.

For an ideal I, denote by $T(I)$ the tangent space to the corresponding point $[I]$ in the Hilbert scheme. The question of finding the largest possible dimension of a tangent space to $\mathrm{Hilb}^d \mathbf{P}^n$ has been raised in many places, including e.g. [1, 10, 69, 93]. To answer this question we restrict to an affine open $\mathbf{A}^n = \operatorname{Spec} \mathbf{k}[x_1, \ldots, x_n] \subseteq \mathbf{P}^n$. It is natural to expect that a fat point subscheme $V\big((x_1, \ldots, x_n)^r\big) \subseteq \mathbf{A}^n$ yields the most singular point in its own Hilbert scheme:

Conjecture 6.0.1 ([10]). *Let $S = \mathbf{k}[x_1, \ldots, x_n]$, $\mathfrak{m} = (x_1, \ldots, x_n)$, and $d = \binom{r+n-1}{n}$ with $r \in \mathbf{N}$. For all $[I] \in \mathrm{Hilb}^d(\mathbf{A}^n)$ we have $\dim_{\mathbf{k}} T(I) \leq \dim_{\mathbf{k}} T(\mathfrak{m}^r)$.*

No progress on the conjecture has been made so far. By degeneration arguments, one reduces Conjecture 6.0.1 to monomial ideals I, and in fact to Borel-fixed ideals in characteristic 0. Inspired by Haiman's theory of $\mathrm{Hilb}^d(\mathbf{A}^2)$ [40], we decompose the tangent space $T(I)$ to a monomial ideal $I \subseteq \mathbf{k}[x_1, \ldots, x_n]$ into subspaces defined in terms of $\mathbf{Z}^n$-graded directions, as follows.

Definition 6.0.2. A **signature** is a non-constant n-tuple on the two-element set $\{\mathrm{p}, \mathrm{n}\}$, where

$$\mathrm{p} = \text{“positive or 0”}, \quad \mathrm{n} = \text{“negative”}.$$

Let $\mathfrak{S}$ denote the set of signatures, and define for each $\mathfrak{s} \in \mathfrak{S}$

$$\begin{aligned} \mathbf{Z}^n_{\mathfrak{s}} &= \big\{(\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}^n : \alpha_i \geq 0 \text{ if } \mathfrak{s}_i = \mathrm{p},\ \alpha_i < 0 \text{ if } \mathfrak{s}_i = \mathrm{n}\big\} \\ T_{\mathfrak{s}}(I) &= \bigoplus_{\alpha \in \mathbf{Z}^n_{\mathfrak{s}}} \big|T(I)\big|_{\alpha} \subseteq T(I) \end{aligned}$$

where $|T(I)|_{\alpha}$ denotes the graded component of $T(I)$ of degree $\alpha \in \mathbf{Z}^n$.

We then have $T_{\mathrm{pp}\cdots\mathrm{p}}(I) = T_{\mathrm{nn}\cdots\mathrm{n}}(I) = 0$, and therefore $T(I) = \bigoplus_{\mathfrak{s}\in\mathfrak{S}} T_{\mathfrak{s}}(I)$, cf. Proposition 6.1.8. Our first theorem establishes a symmetry between components of opposite signature.

Theorem 6.2.4. For any monomial point $[I] \in \mathrm{Hilb}^d(\mathbf{A}^3)$ we have

$$\begin{aligned} \dim_{\mathbf{k}} T_{\mathrm{ppn}}(I) &= \dim_{\mathbf{k}} T_{\mathrm{nnp}}(I) + d, \\ \dim_{\mathbf{k}} T_{\mathrm{pnp}}(I) &= \dim_{\mathbf{k}} T_{\mathrm{npn}}(I) + d, \\ \dim_{\mathbf{k}} T_{\mathrm{npp}}(I) &= \dim_{\mathbf{k}} T_{\mathrm{pnn}}(I) + d. \end{aligned}$$

This result may be regarded as a generalization of Haiman's combinatorial proof of the smoothness of $\mathrm{Hilb}^d(\mathbf{P}^2)$ [40]. In our notation, his proof shows that

$$\dim_{\mathbf{k}} T_{\mathrm{pn}}(I) = \dim_{\mathbf{k}} T_{\mathrm{np}}(I) = d \tag{6.1}$$

for any monomial point $[I] \in \mathrm{Hilb}^d(\mathbf{A}^2)$. Theorem 6.2.4 extends Eq. (6.1) to $\mathbf{A}^3$ in the sense that it implies

$$\dim_{\mathbf{k}} T_{\mathrm{pnp}}(I) + \dim_{\mathbf{k}} T_{\mathrm{pnn}}(I) = \dim_{\mathbf{k}} T_{\mathrm{npp}}(I) + \dim_{\mathbf{k}} T_{\mathrm{npn}}(I)$$

and two other similar equations. Our result may be seen as further evidence for the exceptionality of the Hilbert scheme of points in $\mathbf{P}^3$. For instance, it implies that $\dim_{\mathbf{k}} T(I)$ has the same parity as the length $d = \dim_{\mathbf{k}}(S/I)$, a fact established in [66] where it plays a crucial role in the calculation of Donaldson-Thomas theory for toric threefolds. We are not aware of any such symmetry phenomenon in higher dimension.

As a special case, Theorem 6.2.4 provides a simple criterion for smoothness of monomial points on the Hilbert scheme, in terms of the subspaces $T_{\mathfrak{s}}(I)$.

Theorem 6.2.6. A monomial point $[I] \in \mathrm{Hilb}^d(\mathbf{A}^3)$ is smooth if and only if

$$T_{\mathfrak{s}}(I) = 0 \qquad \text{for all } \mathfrak{s} \in \{\mathrm{pnn}, \mathrm{npn}, \mathrm{nnp}\}.$$

In the opposite direction, we use the subspaces $T_{\mathfrak{s}}(I)$ to provide evidence in favor of Conjecture 6.0.1. Clearly, Conjecture 6.0.1 is implied by the statement that $\dim_{\mathbf{k}} T_{\mathfrak{s}}(I) \leq \dim_{\mathbf{k}} T_{\mathfrak{s}}(\mathfrak{m}^r)$ for all $\mathfrak{s} \in \mathfrak{S}$ and all Borel-fixed points $[I]$. For $\mathrm{Hilb}^d(\mathbf{A}^3)$, we are able to establish this inequality for four out of the six signatures $\mathfrak{s}$. As a bonus, we characterize when equality holds.

Theorem 6.3.6. Let $d = \binom{r+2}{3}$ and let $[I] \in \mathrm{Hilb}^d(\mathbf{A}^3)$ be Borel-fixed, with $\mathrm{char}(\mathbf{k}) = 0$. We have

$$\begin{aligned} \dim_{\mathbf{k}} T_{\mathrm{ppn}}(I) &\leq \dim_{\mathbf{k}} T_{\mathrm{ppn}}(\mathfrak{m}^r), \qquad & \dim_{\mathbf{k}} T_{\mathrm{nnp}}(I) &\leq \dim_{\mathbf{k}} T_{\mathrm{nnp}}(\mathfrak{m}^r), \\ \dim_{\mathbf{k}} T_{\mathrm{pnp}}(I) &\leq \dim_{\mathbf{k}} T_{\mathrm{pnp}}(\mathfrak{m}^r), \qquad & \dim_{\mathbf{k}} T_{\mathrm{npn}}(I) &\leq \dim_{\mathbf{k}} T_{\mathrm{npn}}(\mathfrak{m}^r). \end{aligned}$$

Moreover, in each case equality occurs if and only if $I = \mathfrak{m}^r$.

We conjecture that $\dim_{\mathbf{k}} T_{\mathrm{npp}}(I) \le \dim_{\mathbf{k}} T_{\mathrm{npp}}(\mathfrak{m}^r)$ and $\dim_{\mathbf{k}} T_{\mathrm{pnn}}(I) \le \dim_{\mathbf{k}} T_{\mathrm{pnn}}(\mathfrak{m}^r)$ as well, but we are unable to prove this. However, we are able to prove Conjecture 6.0.1 up to a factor of $\frac{4}{3}$. This also allows us to improve the asymptotic bound on the dimension of $\mathrm{Hilb}^d(\mathbf{P}^3)$, a problem proposed by Sturmfels in [93, Problem 2.4.c].

Theorem 6.4.2. *For all $d \in \mathbf{N}$ and $[I] \in \mathrm{Hilb}^d(\mathbf{P}^3)$ we have*

$$\dim_{\mathbf{k}} T(I) \le \frac{4}{3} \dim_{\mathbf{k}} T(\mathfrak{m}^r) \approx 3.63 \cdot d^{\frac{4}{3}} + O(d)$$

whenever $d \le \binom{r+2}{3}$. In particular, $\dim \mathrm{Hilb}^d(\mathbf{P}^3) \le 3.64 \cdot d^{\frac{4}{3}}$ for $d \gg 0$.

Note that Theorem 6.4.2 also holds for Hilbert schemes of points of arbitrary smooth threefolds, since these are étale-locally isomorphic to $\mathrm{Hilb}^d(\mathbf{P}^3)$, see for instance [8, Lemma 4.4].

6.1 The tangent space

Notation 6.1.1. For this chapter $\mathbf{k}$ will denote an infinite field and $S = \mathbf{k}[x_1, \dots, x_n]$ the polynomial ring in n variables, $\mathfrak{m} = (x_1, \dots, x_n)$ the ideal of the origin in $\mathbf{A}^n = \mathrm{Spec}(S)$ (note that this is *different* from the other chapters). When $n \le 3$, we typically denote the variables by x, y, z instead of x_1, x_2, x_3. If V is a (multi)graded vector space, we use the notation $|V|_\alpha$ to denote the graded component of V of degree α.

The main object of interest is the Hilbert scheme $\mathrm{Hilb}^d(\mathbf{A}^n)$ parametrizing 0-dimensional subschemes of $\mathbf{A}^n$ of length d, equivalently ideals $I \subseteq S$ with $\dim_{\mathbf{k}}(S/I) = d$. The Zariski tangent space to a point $[I] \in \mathrm{Hilb}^d(\mathbf{A}^n)$ may be identified with the $\mathbf{k}$-vector space (Example 2.0.4)

$$T(I) = \mathrm{Hom}_S(I, S/I).$$

The well-known generic initial ideal deformation allows to reduce questions such as Conjecture 6.0.1 to the case of Borel-fixed points, see [24, 15.9] or [69, 2.2–2.3] for details.

Lemma 6.1.2. *For every $[I] \in \mathrm{Hilb}^d \mathbf{A}^n$ we have $\dim_{\mathbf{k}} T(I) \le \dim_{\mathbf{k}} T(\mathrm{gin}\, I)$. Moreover, $\mathrm{gin}\, I \subseteq S$ is Borel-fixed.*

For a monomial point $[I] \in \mathrm{Hilb}^d(\mathbf{A}^n)$ the tangent space $T(I)$ inherits a natural $\mathbf{Z}^n$-grading. Our next goal is to describe a combinatorial interpretation of $T(I)$ in terms of regions in $\mathbf{Z}^n$.

Definition 6.1.3. For a monomial ideal I, we define $\tilde{I} \subseteq \mathbf{N}^n$ to be the subset consisting of the exponent vectors of all monomials in I.

A **path** between $\alpha, \beta \in \mathbf{Z}^n$ is a sequence $\alpha = \gamma^{(0)}, \gamma^{(1)}, \ldots, \gamma^{(m-1)}, \gamma^{(m)} = \beta$ of points of $\mathbf{Z}^n$ such that $\|\gamma^{(i+1)} - \gamma^{(i)}\| = 1$ for all i, where $\|\delta\| = \sum_{j=1}^n |\delta_j|$ denotes the $1-$norm in $\mathbf{Z}^n$.

A subset $U \subseteq \mathbf{Z}^n$ is said to be **connected** if it is non-empty and for any two points $\alpha, \beta \in U$ there is a path between them contained in U. Given a subset $V \subseteq \mathbf{Z}^n$, a maximal connected subset $U \subseteq V$ is called a **connected component**.

A subset $U \subseteq \mathbf{Z}^n$ is **bounded** if $\mathrm{Card}(U) < \infty$.

Remark 6.1.4. Let $[I] \in \mathrm{Hilb}^d(\mathbf{A}^n)$ and $\alpha \in \mathbf{Z}^n$. A connected component U of $(\tilde{I} + \alpha) \setminus \tilde{I}$ is bounded if and only if $U \subseteq \mathbf{N}^n$. The condition is sufficient as $\mathrm{Card}(\mathbf{N}^n \setminus \tilde{I}) < \infty$, and necessary since if $\beta \in U$ with $\beta_i < 0$, then $\beta + m\mathbf{e}_j \in U$ for all $m \in \mathbf{N}$ and $j \neq i$, where $\mathbf{e}_j \in \mathbf{N}^n$ is the j-th basis vector.

Proposition 6.1.5. *Let $\alpha \in \mathbf{Z}^n$ and $[I] \in \mathrm{Hilb}^d(\mathbf{A}^n)$. The set of bounded connected components of $(\tilde{I} + \alpha) \setminus \tilde{I}$ corresponds to a basis of $|T(I)|_\alpha$.*

Proof. For each bounded connected component $U \subseteq (\tilde{I} + \alpha) \setminus \tilde{I}$ we define a multigraded $\mathbf{k}-$linear map $\varphi_U : I \to S/I$ by setting $\varphi_U(\boldsymbol{x}^\beta) = \boldsymbol{x}^{\alpha+\beta} \in S/I$ if $\alpha + \beta \in U$, 0 otherwise. We claim that φ_U is S-linear; it suffices to check the equation $\phi(\boldsymbol{x}^\beta \boldsymbol{x}^\gamma) = \boldsymbol{x}^\beta \phi(\boldsymbol{x}^\gamma)$ in S/I for all $\beta \in \mathbf{N}^n, \gamma \in \tilde{I}$. This is clearly true if $\alpha + \beta + \gamma \in \tilde{I}$. If $\alpha + \beta + \gamma \notin \tilde{I}$, observe that $\alpha + \beta + \gamma \in U$ if and only if $\alpha + \gamma \in U$, since the two points are connected in $(\tilde{I} + \alpha) \setminus \tilde{I}$, thus the equation holds and $\varphi_U \in |T(I)|_\alpha$. We have $\mathrm{Image}(\varphi_U) = \mathrm{span}_{\mathbf{k}}(\boldsymbol{x}^\alpha \,:\, \alpha \in U) \subseteq S/I$, hence all maps φ_U are linearly independent.

Finally, let $\psi \in |T(I)|_\alpha$ be any map. If $\beta, \gamma \in \tilde{I}$ are such that $\alpha + \beta, \alpha + \gamma$ lie in the same connected component $U \subseteq (\tilde{I} + \alpha) \setminus \tilde{I}$, then there exists $c_{\psi,U} \in \mathbf{k}$ such that $\psi(\boldsymbol{x}^\beta) = c_{\psi,U}\boldsymbol{x}^{\alpha+\beta}$ and $\psi(\boldsymbol{x}^\gamma) = c_{\psi,U}\boldsymbol{x}^{\alpha+\gamma}$: this claim follows easily by induction on $\|\beta - \gamma\|$. In particular, $c_{\psi,U} = 0$ if U is unbounded. We deduce that $\psi = \sum_U c_{\psi,U}\varphi_U$, concluding the proof. □

Remark 6.1.6. A simple but useful consequence of Proposition 6.1.5 is the fact that, for I monomial, $\dim_{\mathbf{k}} T(I)$ is independent of $\mathbf{k}$. Thus, in Conjecture 6.0.1 we may assume $\mathrm{char}\,\mathbf{k} = 0$.

Remark 6.1.7. For $n = 2$, the tangent space $T(I)$ is analyzed combinatorially in [40] in terms of "arrows", see also [69, 18.2]. That description is essentially equivalent to the one presented here, in Proposition 6.1.5. However, we find the framework of connected components to be more transparent and efficient.

Recall the distinguished subspaces of $T(I)$ introduced in Definition 6.0.2. These are the only relevant subspaces of the tangent space:

Proposition 6.1.8. *If $[I] \in \mathrm{Hilb}^d(\mathbf{A}^n)$ is a monomial point and $n \geq 2$, then $T(I) = \bigoplus_{\mathfrak{s}\in\mathfrak{S}} T_{\mathfrak{s}}(I)$.*

Proof. Let $\alpha \in \mathbf{Z}^n$. If $\alpha_i \geq 0$ for all i, then $\tilde{I} + \alpha \subseteq \tilde{I}$ and $(\tilde{I} + \alpha) \setminus \tilde{I} = \emptyset$. Suppose $\alpha_i < 0$ for all i, we claim that $(\tilde{I} + \alpha) \setminus \tilde{I}$ is connected and unbounded. To see this, notice that the "boundary" $B = \tilde{I} \setminus \big(\tilde{I} + (1,1,\ldots,1)\big)$ is connected and unbounded. Furthermore, $(B+\alpha)\cap\tilde{I} = \emptyset$, so $(B+\alpha) \subseteq (\tilde{I}+\alpha)\setminus\tilde{I}$ is connected and unbounded. However, any point of $(\tilde{I}+\alpha)\setminus\tilde{I}$ is connected to $(B+\alpha)$, since any point of $\tilde{I}$ is connected to B by a straight path, and this verifies the claim. In either case $|T(I)|_\alpha = 0$ by Proposition 6.1.5. $\square$

For a monomial point $[I] \in \mathrm{Hilb}^d(\mathbf{A}^2)$ Proposition 6.1.8 gives the decomposition

$$T(I) = T_{\mathrm{pn}}(I) \oplus T_{\mathrm{np}}(I),$$

whereas for a monomial point $[I] \in \mathrm{Hilb}^d(\mathbf{A}^3)$ we have

$$T(I) = T_{\mathrm{ppn}}(I) \oplus T_{\mathrm{pnp}}(I) \oplus T_{\mathrm{npp}}(I) \oplus T_{\mathrm{pnn}}(I) \oplus T_{\mathrm{npn}}(I) \oplus T_{\mathrm{nnp}}(I).$$

Next, we compute the components of the tangent space for the fat point $[\mathfrak{m}^r]$. For any vector $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbf{Z}^n$ we have $\alpha = \alpha^+ - \alpha^-$ for two unique vectors $\alpha^+,\alpha^- \in \mathbf{N}^n$ such that $\alpha^+\cdot\alpha^- = 0$. Moreover we denote $\omega(\alpha) = \alpha_1 + \cdots + \alpha_n \in \mathbf{Z}$.

Lemma 6.1.9. *Let $\alpha \in \mathbf{Z}^n$ and $r \in \mathbf{N}$. We have $|T(\mathfrak{m}^r)|_\alpha = 0$ if $\omega(\alpha) \neq -1$. If $\omega(\alpha) = -1$ then $\dim_{\mathbf{k}} |T(\mathfrak{m}^r)|_\alpha = \binom{n+r-\omega(\alpha^-)-1}{n-1}$ if $\omega(\alpha^-) \leq r$, $|T(\mathfrak{m}^r)|_\alpha = 0$ otherwise.*

Proof. For simplicity we denote $M = \widetilde{\mathfrak{m}^r} \subseteq \mathbf{N}^n$. If $\omega(\alpha) \geq 0$ then $\big((M+\alpha)\setminus M\big)\cap\mathbf{N}^n = \emptyset$, while if $\omega(\alpha) \leq -2$ then the whole region $(M+\alpha)\setminus M$ is connected and unbounded, as it follows by inspecting the points $\beta+\alpha \in (M+\alpha)$ with $\omega(\beta) = r, r+1$. In either case $|T(\mathfrak{m}^r)|_\alpha = 0$ by Proposition 6.1.5.

If $\omega(\alpha) = -1$ then any bounded component of $(M+\alpha)\setminus M$ consists of a single point $\beta+\alpha \in \mathbf{N}^n$ with $\omega(\beta) = r$. These points are in bijection with points $\gamma = \beta - \alpha^- \in \mathbf{N}^n$ such that $\omega(\gamma) = r - \omega(\alpha^-)$, i.e. with the monomials of degree $r - \omega(\alpha^-)$, yielding the desired formula. $\square$

Finally, we distinguish some special tangent vectors in $T(I)$. For an S-module M, we denote its **socle** by $\mathrm{soc}(M) = 0 :_M \mathfrak{m} \subseteq M$. Notice that $\mathrm{soc}(T(I)) = \mathrm{Hom}_S(I, \mathrm{soc}(S/I)) \subseteq T(I)$.

Remark 6.1.10. If $[I] \in \mathrm{Hilb}^d(\mathbf{A}^n)$ is monomial, then $\mathrm{soc}(S/I)$ and $\mathrm{soc}(T(I))$ are $\mathbf{Z}^n$−graded. Furthermore, we see from the proof of Proposition 6.1.5 that a $\mathbf{k}$-basis for $\big|\mathrm{soc}(T(I))\big|_\alpha$ is given by the maps φ_U where $U \subseteq (\tilde{I}+\alpha)\setminus\tilde{I}$ is a connected component such that $\mathrm{Card}(U) = 1$. We refer to these φ_U's as the *socle maps*.

It is easy to compute $\dim_{\mathbf{k}} \mathrm{soc}(T(I))$, using the isomorphism

$$\mathrm{soc}\left(T(I)\right) = \mathrm{Hom}_S\left(I, \mathrm{soc}\left(\frac{S}{I}\right)\right) \cong \mathrm{Hom}_{\mathbf{k}}\left(\frac{I}{\mathfrak{m}I}, \mathrm{soc}\left(\frac{S}{I}\right)\right). \tag{6.2}$$

When $I = \mathfrak{m}^r$ we have $\mathrm{soc}(T(I)) = T(I)$ by Lemma 6.1.9, but in general the inclusion is strict.

6.2 Symmetries in the tangent space and smooth points

In the rest of the paper we work with the Hilbert scheme of points in $\mathbf{A}^3$, so we fix $S = \mathbf{k}[x, y, z]$ and $\mathfrak{m} = (x, y, z)$, unless stated otherwise. We explore symmetries between the components $T_{\mathfrak{s}}(I)$ of the tangent space introduced in Definition 6.0.2. The main results of this section are Proposition 6.2.3 and Theorem 6.2.4, which parallel phenomena observed for $\mathrm{Hilb}^d(\mathbf{A}^2)$ in [40]. As a byproduct, we also prove Theorem 6.2.6, which characterizes smooth monomial points on the Hilbert scheme.

A monomial ideal $I \subseteq S$ admits direct sum decompositions, as module over the subrings of S, into smaller monomial ideals. For instance, the $\mathbf{k}[z]-$ and $\mathbf{k}[y, z]-$decompositions of I are

$$I = \bigoplus_{i,j} x^i y^j \left(z^{b_{i,j}}\right) = \bigoplus_i x^i I_i$$

where $(z^{b_{i,j}}) \subseteq \mathbf{k}[z]$ and $I_i \subseteq \mathbf{k}[y, z]$ are monomial ideals. Clearly, such decompositions exist and are unique. Since I is an ideal, we have $b_{i,j} \geq b_{i+1,j}, b_{i,j+1}$ and $I_i \subseteq I_{i+1}$. If I is $\mathfrak{m}$-primary, then $b_{i,j} = 0$ for all but finitely many pairs i, j, and $I_i = \mathbf{k}[y, z]$ for all but finitely many i. Analogous remarks hold for the $\mathbf{k}[x]-$, $\mathbf{k}[y]-$, $\mathbf{k}[x, y]-$, and $\mathbf{k}[x, z]-$decompositions of I.

Remark 6.2.1. Let $[I] \in \mathrm{Hilb}^d(\mathbf{A}^2)$ be a monomial point. In his way to proving that $\mathrm{Hilb}^d(\mathbf{A}^2)$ is smooth, Haiman [40] shows that

$$\dim_{\mathbf{k}} T_{\mathrm{pn}}(I) = \dim_{\mathbf{k}} T_{\mathrm{np}}(I) = d. \tag{6.3}$$

In fact, a more precise statement is proved. Consider the $\mathbf{k}[y]$-decomposition $I = \bigoplus x^i (y^{b_i})$. Then for each $i \in \mathbf{N}$ we have

$$\sum_{\alpha_1 = i} \dim_{\mathbf{k}} |T(I)|_\alpha = \sum_{\alpha_1 = -i-1} \dim_{\mathbf{k}} |T(I)|_\alpha = b_i. \tag{6.4}$$

Note that Eq. (6.3) and Eq. (6.4) cannot be extended directly to $\mathbf{A}^3$, since the Hilbert scheme is singular, and the dimension of $T(I)$ actually depends on I and not just on d. Nevertheless, we are going to establish versions of these equations for $\mathrm{Hilb}^d(\mathbf{A}^3)$.

We begin with a homological lemma, which we state in the general case of a polynomial ring in n variables, for simplicity.

Lemma 6.2.2. *Let $S = \mathbf{k}[x_1, \ldots, x_n]$ and M be an Artinian $\mathbf{Z}^n$-graded S module. For each $\ell = 0, \ldots, n$ there is a natural isomorphism of functors of finitely generated $\mathbf{Z}^n$-graded S modules*

$$\mathrm{Ext}^{\ell}_S(-, M) \cong \mathrm{Ext}^{n-\ell}_S(M, - \otimes \omega_S)'$$

where $-'$ denotes the Matlis dual and ω_S the $\mathbf{Z}^n$-graded canonical module. In particular, for every finitely generated $\mathbf{Z}^n$-graded module N we have

$$\mathrm{Ext}^{\ell}_S(N, M)^\vee \cong \mathrm{Ext}^{n-\ell}_S(M, N)(-1, -1, \ldots, -1)$$

as $\mathbf{Z}^n$*−graded vector spaces, where* $-^\vee$ *denotes the* $\mathbf{k}$*-dual.*

Proof. To prove the first assertion, by the universal properties of derived functors [24, A.3.9], it suffices to verify the following four properties for the functors $\mathrm{Ext}_S^{n-\ell}(M, -\otimes \omega_S)'$.

(i) Isomorphism for $\ell = 0$, that is, $\mathrm{Hom}_S(-, M) \cong \mathrm{Ext}_S^n(M, -\otimes \omega_S)'$.

(ii) The vanishing $\mathrm{Ext}_S^{n-\ell}(M, P \otimes \omega_S)' = 0$ for finitely generated projective P and $\ell > 0$.

(iii) For each short exact sequence $0 \to N' \to N \to N'' \to 0$, there is a long exact sequence of $\mathrm{Ext}_S^{n-\ell}(M, -\otimes \omega_S)'$.

(iv) Naturality of the connecting homomorphism, that is, for each map of short exact sequences of S-modules, the two long exact sequences of $\mathrm{Ext}_S^{n-\ell}(M, -\otimes \omega_S)'$ form a a commutative diagram.

For (i), observe that we have natural isomorphisms

$$\mathrm{Hom}_S(-, M) \cong \mathrm{Hom}_S\left(-, \mathrm{Ext}_S^n(M, \omega_S)'\right) \cong \left(-\otimes \mathrm{Ext}_S^n(M, \omega_S)\right)' \cong \mathrm{Ext}_S^n(M, -\otimes \omega_S)'.$$

The first one follows from the Local Duality Theorem [11, 3.6], while the second one by Hom-Tensor adjointness. To see the third one, let $F_\bullet$ be a minimal free resolution of M, then

$$\begin{aligned}
\mathrm{Ext}_S^n(M, -\otimes \omega_S) &= H^n\left(\mathrm{Hom}(F_\bullet, -\otimes \omega_S)\right) \\
&\cong H^n\left(\mathrm{Hom}(F_\bullet, \omega_S) \otimes -\right) \\
&\cong H^n\left(\mathrm{Hom}(F_\bullet, \omega_S)\right) \otimes - && \text{by right-exactness} \\
&= \mathrm{Ext}_S^n(M, \omega_S) \otimes -.
\end{aligned}$$

For (ii) it suffices to show the vanishing in the case $P = S$, which follows since M is Cohen-Macaulay of grade n, cf. [11, 3.3]. Items (iii) and (iv) follow from the corresponding properties of $\mathrm{Ext}_S^\bullet(M, -)$ combined with the exact contravariant functor $-'$. Finally, the second assertion of the theorem follows from the first since $\omega_S \cong S(-1, -1, \dots, -1)$ and $-^\vee \cong -'$, cf. [11, 3.6]. □

Proposition 6.2.3. *Let* $[I] \in \mathrm{Hilb}^d(\mathbf{A}^3)$ *be a monomial point, with* $\mathbf{k}[z]$*−decomposition* $I = \bigoplus x^i y^j (z^{b_{i,j}})$*. For every* $i, j \in \mathbf{N}$ *we have*

$$\sum_{\substack{\alpha_1 = i \\ \alpha_2 = j}} \dim_{\mathbf{k}} |T(I)|_\alpha = b_{i,j} + \sum_{\substack{\alpha_1 = -i-1 \\ \alpha_2 = -j-1}} \dim_{\mathbf{k}} |T(I)|_\alpha. \tag{6.5}$$

Proof. Fix $i, j \in \mathbf{N}$ and consider the groups $\mathrm{Ext}_S^\ell(S/I, S/I)$ for $\ell = 0, \dots, 3$. We have

$$\mathrm{Ext}_S^0(S/I, S/I) = S/I \quad \text{and} \quad \mathrm{Ext}_S^1(S/I, S/I) = T(I),$$

where the latter holds since $\mathrm{Ext}^1_S(S/I, S/I) = \mathrm{Ext}^0_S(I, S/I)$ by homological "dimension shift". By Lemma 6.2.2 we have $\mathrm{Ext}^\ell_S(S/I, S/I)^\vee \cong \mathrm{Ext}^{3-\ell}_S(S/I, S/I)(-1,-1,-1)$, hence

$$
\begin{aligned}
\sum_{\substack{\alpha_1=i\\ \alpha_2=j}} \dim_{\mathbf{k}} \left|\mathrm{Ext}^0_S(S/I, S/I)\right|_\alpha &= \sum_{\substack{\alpha_1=i\\ \alpha_2=j}} \dim_{\mathbf{k}} |S/I|_\alpha = b_{i,j},\\
\sum_{\substack{\alpha_1=i\\ \alpha_2=j}} \dim_{\mathbf{k}} \left|\mathrm{Ext}^1_S(S/I, S/I)\right|_\alpha &= \sum_{\substack{\alpha_1=i\\ \alpha_2=j}} \dim_{\mathbf{k}} |T(I)|_\alpha,\\
\sum_{\substack{\alpha_1=i\\ \alpha_2=j}} \dim_{\mathbf{k}} \left|\mathrm{Ext}^2_S(S/I, S/I)\right|_\alpha &= \sum_{\substack{\alpha_1=-i-1\\ \alpha_2=-j-1}} \dim_{\mathbf{k}} |T(I)|_\alpha,\\
\sum_{\substack{\alpha_1=i\\ \alpha_2=j}} \dim_{\mathbf{k}} \left|\mathrm{Ext}^3_S(S/I, S/I)\right|_\alpha &= \sum_{\substack{\alpha_1=-i-1\\ \alpha_2=-j-1}} \dim_{\mathbf{k}} |S/I|_\alpha = 0.
\end{aligned}
$$

Eq. (6.5) is then equivalent to

$$
\sum_{\ell=0}^{3} (-1)^\ell \sum_{\substack{\alpha_1=i\\ \alpha_2=j}} \dim_{\mathbf{k}} \left|\mathrm{Ext}^\ell_S(S/I, S/I)\right|_\alpha = 0. \tag{6.6}
$$

Let $I = (\boldsymbol{x}^{\beta^{(1)}}, \ldots, \boldsymbol{x}^{\beta^{(m)}})$ and let $F_\bullet$ be the Taylor free resolution of S/I [69, 4.3.2]. The modules in $F_\bullet$ are given by

$$
F_\ell = \bigoplus_{\substack{\mathcal{A}\subseteq\{1,\ldots,m\}\\ \mathrm{Card}(\mathcal{A})=\ell}} S\big(-\beta^{\mathcal{A}}\big) \qquad \text{where} \quad \boldsymbol{x}^{\beta^{\mathcal{A}}} = \mathrm{lcm}\big\{\boldsymbol{x}^{\beta^{(a)}} \,:\, a \in \mathcal{A}\big\}.
$$

Since $\mathrm{Ext}^\ell_S(S/I, S/I) = H^\ell\big(\mathrm{Hom}_S(F_\bullet, S/I)\big) = H^\ell\big(\mathrm{Hom}_{S/I}(F_\bullet/IF_\bullet, S/I)\big)$, we can rephrase Eq. (6.6) as

$$
\sum_{\ell=0}^{m} (-1)^\ell \sum_{\substack{\alpha_1=i\\ \alpha_2=j}} \dim_{\mathbf{k}} \left|\mathrm{Hom}_{S/I}(F_\ell/IF_\ell, S/I)\right|_\alpha = 0. \tag{6.7}
$$

Define for each $\mathcal{A} \subseteq \{1,\ldots,m\}$ the quantity

$$
t_{\mathcal{A}} = \sum_{\substack{\alpha_1=i\\ \alpha_2=j}} \dim_{\mathbf{k}} \left|\mathrm{Hom}_{S/I}\left(S/I\big(-\beta^{\mathcal{A}}\big), S/I\right)\right|_\alpha.
$$

then Eq. (6.7) is equivalent to

$$
\sum_{\mathcal{A}\subseteq\{1,\ldots,m\}} (-1)^{\mathrm{Card}(\mathcal{A})} t_{\mathcal{A}} = 0. \tag{6.8}
$$

Note that for each α and $\mathcal{A}$ we have

$$\left|\mathrm{Hom}_{S/I}\left(S/I(-\beta^{\mathcal{A}}), S/I\right)\right|_{\alpha} = \left|\mathrm{Hom}_{S/I}\left(S/I, S/I\right)\right|_{\alpha+\beta^{\mathcal{A}}} = \left|S/I\right|_{\alpha+\beta^{\mathcal{A}}}$$

so that

$$\dim_{\mathbf{k}}\left|\mathrm{Hom}_{S/I}\left(S/I(-\beta^{\mathcal{A}}), S/I\right)\right|_{\alpha} = \begin{cases} 1 & \text{if } \alpha+\beta^{\mathcal{A}} \in \mathbf{N}^3 \setminus \tilde{I}, \\ 0 & \text{otherwise.} \end{cases}$$

Adding over all $\alpha_3 \in \mathbf{Z}$ we get $t_{\mathcal{A}} = \mathrm{Card}\left\{\alpha_3 \in \mathbf{Z} \,:\, (i,j,\alpha_3)+\beta^{\mathcal{A}} \in \mathbf{N}^3 \setminus \tilde{I}\right\}$, that is, in terms of the $\mathbf{k}[z]$−decomposition of I,

$$t_{\mathcal{A}} = b_{i+\beta_1^{\mathcal{A}},\, j+\beta_2^{\mathcal{A}}}. \tag{6.9}$$

Assuming without loss of generality that $x^{\beta^{(m)}} = z^{b_{0,0}}$, the formula Eq. (6.9) immediately implies that $t_{\mathcal{A}} = t_{\mathcal{A}\cup\{m\}}$ for every $\mathcal{A}$, which in turn yields Eq. (6.8) and concludes the proof. ◻

The following consequence of Proposition 6.2.3 is the main result of the section.

Theorem 6.2.4. *Let $[I] \in \mathrm{Hilb}^d(\mathbf{A}^3)$ be a monomial point. We have*

$$\begin{aligned} \dim_{\mathbf{k}} T_{\mathrm{ppn}}(I) &= \dim_{\mathbf{k}} T_{\mathrm{nnp}}(I) + d, \\ \dim_{\mathbf{k}} T_{\mathrm{pnp}}(I) &= \dim_{\mathbf{k}} T_{\mathrm{npn}}(I) + d, \\ \dim_{\mathbf{k}} T_{\mathrm{npp}}(I) &= \dim_{\mathbf{k}} T_{\mathrm{pnn}}(I) + d. \end{aligned}$$

Proof. The first equation follows from Proposition 6.2.3 by adding over all $i, j \in \mathbf{N}$, and using Proposition 6.1.8. The other two follow from the first by permutation. ◻

Theorem 6.2.4 provides the correct generalization of Eq. (6.3) to $\mathbf{A}^3$, since it implies

$$\dim_{\mathbf{k}} T_{\mathrm{pn*}}(I) = \dim_{\mathbf{k}} T_{\mathrm{np*}}(I), \quad \dim_{\mathbf{k}} T_{\mathrm{p*n}}(I) = \dim_{\mathbf{k}} T_{\mathrm{n*p}}(I), \quad \dim_{\mathbf{k}} T_{\mathrm{*pn}}(I) = \dim_{\mathbf{k}} T_{\mathrm{*np}}(I),$$

where e.g. $T_{\mathrm{pn*}}(I) = T_{\mathrm{pnp}}(I) \oplus T_{\mathrm{pnn}}(I)$. To the best of our knowledge, Proposition 6.2.3 and Theorem 6.2.4 do not extend to higher dimensions.

Theorem 6.2.4 is also a vast generalization of the following parity result of Maulik, Nekrasov, Okounkov, and Pandharipande, which follows from [66, Theorem 2], see also [8, Lemma 4.1 (c)].

Corollary 6.2.5. *For each monomial point $[I] \in \mathrm{Hilb}^d(\mathbf{A}^3)$ we have $\dim_{\mathbf{k}} T(I) \equiv d \bmod 2$.*

Whether $\dim_{\mathbf{k}} T(I) \equiv d \bmod 2$ for every $[I] \in \mathrm{Hilb}^d(\mathbf{A}^3)$ is an open and interesting question; see [12, Remark 22] for related matters. A stronger open question is whether for any $[I] \in \mathrm{Hilb}^d(\mathbf{A}^3)$ there exists a monomial $[M] \in \mathrm{Hilb}^d(\mathbf{A}^3)$ such that $\dim_{\mathbf{k}} T(I) = \dim_{\mathbf{k}} T(M)$.

Another interesting special case of Theorem 6.2.4 occurs when each of the three equations is a small as possible: we obtain the following smoothness criterion for monomial points in $\mathrm{Hilb}^d(\mathbf{A}^3)$.

Theorem 6.2.6. *A monomial point* $[I] \in \mathrm{Hilb}^d(\mathbf{A}^3)$ *is smooth if and only if*

$$T_{\mathfrak{s}}(I) = 0 \qquad \text{for } \mathfrak{s} \in \{\mathrm{pnn}, \mathrm{npn}, \mathrm{nnp}\}.$$

Proof. It is known that a monomial point $[I]$ lies in the closure of the component of $\mathrm{Hilb}^d(\mathbf{A}^3)$ parametrizing subschemes of d distinct points, see e.g. [13, 4.15]. We deduce that $[I]$ is a smooth point if and only if $\dim_{\mathbf{k}} T(I) = 3d$, and the statement follows by Theorem 6.2.4. □

The criterion can be particularly effective in proving that a point $[I]$ is singular: it suffices to exhibit a single tangent vector with forbidden signature. In many cases, the existence of such tangent vector follows just by looking at the minimal generators of I. We give two examples.

Corollary 6.2.7. *Let* $[I] \in \mathrm{Hilb}^d(\mathbf{A}^3)$ *be a monomial point. Suppose the minimal generating set of* I *contains three monomials* $x^{\alpha_1}y^{\alpha_2}, x^{\beta_1}z^{\beta_3}, y^{\gamma_2}z^{\gamma_3}$ *with* $\alpha_1, \alpha_2, \beta_1, \beta_3, \gamma_2, \gamma_3 > 0$ *satisfying one of the following:*

- $\alpha_1 \geq \beta_1$ *and* $\alpha_2 \geq \gamma_2$;
- $\beta_1 \geq \alpha_1$ *and* $\beta_3 \geq \gamma_3$;
- $\gamma_2 \geq \alpha_2$ *and* $\gamma_3 \geq \beta_3$.

Then $[I]$ *is a singular point.*

Proof. Since $\dim_{\mathbf{k}}(S/I) < \infty$, there are also minimal generators $x^{s_1}, y^{s_2}, z^{s_3}$, and by minimality we get $s_1 > \alpha_1, \beta_1$, $s_2 > \alpha_2, \gamma_2$, $s_3 > \beta_3, \gamma_3$. It follows that there are monomials

$$x^{\delta_1}y^{\delta_2}z^{s_3-1}, \quad x^{\epsilon_1}y^{s_2-1}z^{\epsilon_3}, \quad x^{s_1-1}y^{\zeta_2}z^{\zeta_3} \quad \in \mathrm{soc}\left(\frac{S}{I}\right)$$

for some $\delta_1 \leq \beta_1 - 1, \delta_2 \leq \gamma_2 - 1, \epsilon_1 \leq \alpha_1 - 1, \epsilon_3 \leq \gamma_3 - 1, \zeta_2 \leq \alpha_2 - 1, \zeta_3 \leq \beta_3 - 1$. By Remark 6.1.10 and by Eq. (6.2) there are three maps $\varphi_1, \varphi_2, \varphi_3 \in \mathrm{soc}(T(I)) \subseteq T(I)$ such that

$$\varphi_1(x^{\alpha_1}y^{\alpha_2}) = x^{\delta_1}y^{\delta_2}z^{s_3-1}, \quad \varphi_2(x^{\beta_1}z^{\beta_3}) = x^{\epsilon_1}y^{s_2-1}z^{\epsilon_3}, \quad \varphi_3(y^{\gamma_2}z^{\gamma_3}) = x^{s_1-1}y^{\zeta_2}z^{\zeta_3}.$$

Using the hypothesis we derive $\varphi_1 \in T_{\mathrm{nnp}}(I)$, or $\varphi_2 \in T_{\mathrm{npn}}(I)$, or $\varphi_3 \in T_{\mathrm{pnn}}(I)$. □

Corollary 6.2.8. *Let* $[I] \in \mathrm{Hilb}^d(\mathbf{A}^3)$ *be a strongly stable point. Then* $[I]$ *is smooth if and only if* $x \in I$.

Proof. Assume $x \notin I$ and let $z^{s_3} \in I$ be a minimal generator. By strong stability, xy^a is a minimal generator for some $a > 0$, and moreover $xz^{s_3-1}, yz^{s_3-1} \in I$, thus $z^{s_3-1} \in \mathrm{soc}(S/I)$. By Remark 6.1.10 and by Eq. (6.2) there is a map $\varphi \in \mathrm{soc}(T(I)) \subseteq T(I)$ such that $\varphi(xy^a) = z^{s_3-1}$, so $\varphi \in T_{\mathrm{nnp}}(I) \neq 0$.

Now assume $x \in I$. Then $\gamma_1 = 0$ for all $\boldsymbol{x}^\gamma \in S/I$, and $\beta_1 = 0$ for all generators $\boldsymbol{x}^\beta \neq x$ of I. Let $\varphi \in |T(I)|_\alpha$ for some α. If $\varphi(x) \neq 0$ then $\alpha_2, \alpha_3 \geq 0$, so $\varphi \in T_{\mathrm{npp}}(I)$. Suppose $\varphi(\boldsymbol{x}^\beta) \neq 0$ for some generator $x \neq \boldsymbol{x}^\beta \in I$, then $\alpha_1 = 0$ since $\beta_1 = 0$. Assume by contradiction that $\alpha_2, \alpha_3 < 0$. Considering the "boundary" $B = \tilde{I} \setminus \left(\tilde{I} + (0,1,1)\right)$ and arguing as in Proposition 6.1.8, we see that $(\tilde{I} + \alpha) \setminus \tilde{I}$ is connected and unbounded. This contradicts Proposition 6.1.5, thus $\alpha_2 \geq 0$ or $\alpha_3 \geq 0$, and $\varphi \in T_{\mathrm{ppn}}(I) \oplus T_{\mathrm{pnp}}(I)$. We conclude that $T_{\mathrm{pnn}}(I) = T_{\mathrm{npn}}(I) = T_{\mathrm{nnp}}(I) = 0$. □

6.3 Extremality of subspaces of the tangent space

In this section we prove Theorem 6.3.6, confirming the extremal behavior predicted by Conjecture 6.0.1 for certain components $T_{\mathfrak{s}}(I)$ of the tangent space.

Proposition 6.3.1. *Let $[I] \in \mathrm{Hilb}^d(\mathbf{A}^3)$ be a monomial point with $\mathbf{k}[z]$–decomposition $I = \bigoplus x^i y^j \left(z^{b_{i,j}}\right)$. For each $\alpha_1, \alpha_2 \geq 0$ we have the inequality*

$$\sum_{\alpha_3 < 0} \dim_{\mathbf{k}} \left|T(I)\right|_{(\alpha_1,\alpha_2,\alpha_3)} \leq \sum_{\substack{i \geq \alpha_1 \\ j \geq \alpha_2}} \left(b_{i,j} - \max\{b_{i+1,j}, b_{i,j+1}\}\right).$$

Proof. Fix non-negative integers α_1, α_2, and define the sets

$$\mathcal{C} = \bigcup_{\alpha_3 < 0} \left\{\text{bounded connected components of } \left(\tilde{I} + (\alpha_1,\alpha_2,\alpha_3)\right) \setminus \tilde{I}\right\},$$
$$\mathcal{S} = \left\{(i,j,k) \notin \tilde{I} \,:\, i \geq \alpha_1, j \geq \alpha_2 \text{ and } (i+1,j,k), (i,j+1,k) \in \tilde{I}\right\}.$$

We define a map $\Psi : \mathcal{C} \to \mathcal{S}$ by choosing, for each $U \in \mathcal{C}$, a vector $\Psi(U) = (\psi_1^U, \psi_2^U, \psi_3^U) \in U$ such that ψ_3^U is the least possible among vectors in U, and $(\psi_1^U + 1, \psi_2^U, \psi_3^U), (\psi_1^U, \psi_2^U + 1, \psi_3^U) \notin U$. The choice is possible as $\mathrm{Card}(U) < \infty$. Since U is a bounded connected component of $(\tilde{I} + (\alpha_1,\alpha_2,\alpha_3)) \setminus \tilde{I}$ for some α_3, the vector $\Psi(U)$ is indeed in $\mathcal{S}$.

We claim that the map Ψ is injective. Let $U \neq U'$ be bounded components of $\left(\tilde{I} + (\alpha_1,\alpha_2,\alpha_3)\right) \setminus \tilde{I}$ and $\left(\tilde{I} + (\alpha_1,\alpha_2,\alpha_3')\right) \setminus \tilde{I}$, respectively, for some $\alpha_3, \alpha_3' < 0$. If $\alpha_3 = \alpha_3'$ then $U \cap U' = \emptyset$ by definition of connected component, hence $\Psi(U) \neq \Psi(U')$. Suppose now $\alpha_3 < \alpha_3'$, thus $\left(\tilde{I} + (\alpha_1,\alpha_2,\alpha_3')\right) \setminus \tilde{I} \subsetneq \left(\tilde{I} + (\alpha_1,\alpha_2,\alpha_3)\right) \setminus \tilde{I}$. If $U \cap U' \neq \emptyset$ then necessarily $U' \subsetneq U$, and this implies $\Psi(U') + (0,0,\alpha_3 - \alpha_3') \in U$. We conclude that $\psi_3^U \leq \psi_3^{U'} + \alpha_3 - \alpha_3' < \psi_3^{U'}$, in particular $\Psi(U) \neq \Psi(U')$ as claimed.

Note that, for each pair i,j, we have $\mathrm{Card}\left\{(i,j,k)\notin \tilde{I} \,:\, (i+1,j,k),(i,j+1,k)\in\tilde{I}\right\} = b_{i,j}-\max\{b_{i+1,j},b_{i,j+1}\}$. We deduce that

$$\mathrm{Card}(\mathcal{C}) \le \mathrm{Card}(\mathcal{S}) = \sum_{\substack{i\ge\alpha_1\\ j\ge\alpha_2}} \left(b_{i,j}-\max\{b_{i+1,j},b_{i,j+1}\}\right)$$

concluding the proof by Proposition 6.1.5. □

By combining the inequalities for all $\alpha_1,\alpha_2\ge 0$ Proposition 6.3.1 provides upper bounds for $T_{\mathrm{ppn}}(I)$ and, up to permutation, for $T_{\mathrm{pnp}}(I)$ and $T_{\mathrm{npp}}(I)$. Using the symmetries of Section 6.2, we also obtain estimates for the other three signatures. We are going to apply these bounds to Borel-fixed points. Before we can present the main result, we need some lemmas about strongly stable ideals and powers of $\mathfrak{m}$.

Lemma 6.3.2. *Let $[I]\in\mathrm{Hilb}^d(\mathbf{A}^3)$ be a strongly stable point with $\mathbf{k}[z]-$ and $\mathbf{k}[y]-$decompositions*

$$I=\bigoplus x^i y^j\left(z^{b^z_{i,j}}\right)=\bigoplus x^i z^j\left(y^{b^y_{i,j}}\right).$$

Then $\max\{b^z_{i+1,j},b^z_{i,j+1}\}=b^z_{i,j+1}$ and $\max\{b^y_{i+1,j},b^y_{i,j+1}\}=b^y_{i,j+1}$ for all i,j.

Proof. Since I is strongly stable, $x^iy^{j+1}z^{b^z_{i,j+1}}\in I$ implies $x^{i+1}y^jz^{b^z_{i,j+1}}\in I$, thus, by definition $b^z_{i+1,j}\le b^z_{i,j+1}$ i.e. $\max\{b^z_{i+1,j},b^z_{i,j+1}\}=b^z_{i,j+1}$. The other equation is proved similarly. □

Lemma 6.3.3. *Let $[I]\in\mathrm{Hilb}^d(\mathbf{A}^3)$ be a strongly stable point with $\mathbf{k}[y,z]-$decomposition $I=\bigoplus x^iI_i$. Then for every $i\ge 0$ the ideal I_i is strongly stable, and we have $I_i : y\subseteq I_{i+1}$.*

Proof. Both properties follow easily by strong stability. □

Lemma 6.3.4. *Let $[I]\in\mathrm{Hilb}^d(\mathbf{A}^3)$ be a strongly stable point with $\mathbf{k}[y,z]-$decomposition $I=\bigoplus x^iI_i$. If $d\le\dim_{\mathbf{k}}(S/\mathfrak{m}^r)$ then for all $0\le j\le r$ we have*

$$\sum_{i=j}^{r-1}\dim_{\mathbf{k}}\frac{\mathbf{k}[y,z]}{I_i}\le\sum_{i=j}^{r-1}\dim_{\mathbf{k}}\frac{\mathbf{k}[y,z]}{(y,z)^{r-i}}. \tag{6.10}$$

Moreover, if equality holds for all $0\le j\le r-1$ then $I=\mathfrak{m}^r$.

Proof. Observe that $\mathfrak{m}^r$ has $\mathbf{k}[y,z]-$decomposition $\mathfrak{m}^r=\bigoplus x^i(y,z)^{r-i}$, with the convention that $(y,z)^h=\mathbf{k}[y,z]$ if $h<0$.

Suppose first $\dim_{\mathbf{k}}\left(\mathbf{k}[y,z]/I_0\right)\ge\dim_{\mathbf{k}}\left(\mathbf{k}[y,z]/(y,z)^r\right)$. We prove the inequalities Eq. (6.10) by induction on $\ell=\min\{h : x^h\in I\}$. The case $\ell=1$ is clear, so we assume $\ell>1$. Define $I'=\bigoplus x^{i-1}I_i\subseteq S$, then $x^{\ell-1}\in I$ and

$$\begin{aligned}\dim_{\mathbf{k}}(S/I') &= \dim_{\mathbf{k}}(S/I)-\dim_{\mathbf{k}}\left(\mathbf{k}[y,z]/I_0\right)\\ &\le \dim_{\mathbf{k}}(S/\mathfrak{m}^r)-\dim_{\mathbf{k}}\left(\mathbf{k}[y,z]/(y,z)^r\right)\\ &= \dim_{\mathbf{k}}(S/\mathfrak{m}^{r-1}).\end{aligned}$$

Applying the inductive step to I' and $\mathfrak{m}^{r-1}$ we deduce

$$\sum_{i=j}^{r-1} \dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{I_i} = \sum_{i=j-1}^{r-2} \dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{I'_i} \leq \sum_{i=j-1}^{r-2} \dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{(y,z)^{r-1-i}} = \sum_{i=j}^{r-1} \dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{(y,z)^{r-i}}.$$

verifying Eq. (6.10) for all $1 \leq j \leq r-1$, while the case $j = 0$ holds by assumption.

Suppose now that $\dim_{\mathbf{k}} \left(\mathbf{k}[y,z]/I_0\right) \leq \dim_{\mathbf{k}} \left(\mathbf{k}[y,z]/(y,z)^r\right)$. We claim that

$$\dim_{\mathbf{k}} \left(\mathbf{k}[y,z]/I_i\right) \leq \dim_{\mathbf{k}} \left(\mathbf{k}[y,z]/(y,z)^{r-i}\right)$$

for all i, implying the inequalities Eq. (6.10). By Lemma 6.3.3 it suffices to verify the following statement: if $J \subseteq \mathbf{k}[y,z]$ is strongly stable and $\dim_{\mathbf{k}} \left(\mathbf{k}[y,z]/J\right) \leq \dim_{\mathbf{k}} \left(\mathbf{k}[y,z]/(y,z)^h\right)$ for some h, then $\dim_{\mathbf{k}} \left(\mathbf{k}[y,z]/(J:y)\right) \leq \dim_{\mathbf{k}} \left(\mathbf{k}[y,z]/(y,z)^{h-1}\right)$. Write

$$J = (y^a, y^{a-1}z^{c_1}, y^{a-2}z^{c_2}, \ldots, yz^{c_{a-1}}, z^{c_a}),$$

so $J : y = (y^{a-1}, y^{a-2}z^{c_1}, y^{a-3}z^{c_2}, \ldots, z^{c_{a-1}})$. If $c_a \leq h$ then $(y,z)^h \subseteq J$ by strong stability, thus $(y,z)^{h-1} = (y,z)^h : y \subseteq J : y$ and the claim follows. If $c_a > h$ then the claim follows as

$$\dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{J} - \dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{J:y} = \sum_{i=1}^{a} c_i - \sum_{i=1}^{a-1} c_i = c_a > h = \dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{(y,z)^h} - \dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{(y,z)^{h-1}}.$$

Finally, assume equality holds in Eq. (6.10) for all j, then

$$\dim_{\mathbf{k}} \left(\mathbf{k}[y,z]/I_i\right) = \dim_{\mathbf{k}} \left(\mathbf{k}[y,z]/(y,z)^{r-i}\right)$$

for all i. We show by decreasing induction on i that $I_i = (y,z)^{r-i}$. If $x^r \notin I$ then I contains no monomial of degree r, by strong stability, yielding the contradiction $I \subseteq \mathfrak{m}^{r+1}$. Thus $I_i = \mathbf{k}[y,z]$ for all $i \geq r$. Now suppose $I_i = (y,z)^{r-i}$ for some $0 < i \leq r$. Using the argument of the previous paragraph with $J = I_{i-1}$ and $h = r-i+1$, we must have $c_a \leq h$, otherwise $\dim_{\mathbf{k}} \left(\mathbf{k}[y,z]/I_i\right) \leq \dim_{\mathbf{k}} \left(\mathbf{k}[y,z]/(I_{i-1}:y)\right) < \dim_{\mathbf{k}} \left(\mathbf{k}[y,z]/(y,z)^{r-i}\right)$. But if $c_a \leq h$ then $(y,z)^{r-i+1} = (y,z)^h \subseteq J = I_{i-1}$, and equality must hold by dimension reasons. □

Lemma 6.3.5. *Let $r \in \mathbf{N}$. We have*

$$\begin{aligned} \dim_{\mathbf{k}} T_{\mathrm{ppn}}(\mathfrak{m}^r) = \dim_{\mathbf{k}} T_{\mathrm{pnp}}(\mathfrak{m}^r) = \dim_{\mathbf{k}} T_{\mathrm{npp}}(\mathfrak{m}^r) &= \binom{r+3}{4}, \\ \dim_{\mathbf{k}} T_{\mathrm{pnn}}(\mathfrak{m}^r) = \dim_{\mathbf{k}} T_{\mathrm{npn}}(\mathfrak{m}^r) = \dim_{\mathbf{k}} T_{\mathrm{nnp}}(\mathfrak{m}^r) &= \binom{r+2}{4}. \end{aligned}$$

In particular, $\dim_{\mathbf{k}} T(\mathfrak{m}^r) = \binom{r+2}{2}\binom{r+1}{2}$.

Proof. Using Lemma 6.1.9 and the "hockey-stick identity" of binomial coefficients one gets

$$\begin{aligned}\dim_{\mathbf{k}} T_{\mathrm{ppn}}(\mathfrak{m}^r) &= \sum_{\substack{\alpha_1,\alpha_2\geq 0,\alpha_3\geq -r\\ \alpha_1+\alpha_2+\alpha_3=-1}} \binom{r+2+\alpha_3}{2} = \sum_{\alpha_1=0}^{r-1}\sum_{\alpha_2=0}^{r-1-\alpha_1}\binom{r+1-\alpha_1-\alpha_2}{2} = \sum_{\alpha_1=0}^{r-1}\sum_{h=2}^{r+1-\alpha_1}\binom{h}{2}\\ &= \sum_{\alpha_1=0}^{r-1}\binom{r+2-\alpha_1}{3} = \sum_{k=3}^{r+2}\binom{k}{3} = \binom{r+3}{4}.\end{aligned}$$

The same holds for $T_{\mathrm{pnp}}(\mathfrak{m}^r), T_{\mathrm{npp}}(\mathfrak{m}^r)$ by symmetry. The other formula is proved likewise. The last formula follows from Proposition 6.1.8. □

We are now ready to state the main theorem of this section:

Theorem 6.3.6. *Let* $\mathrm{char}(\mathbf{k}) = 0$ *and* $[I] \in \mathrm{Hilb}^d(\mathbf{A}^3)$ *be Borel-fixed, with* $d = \binom{r+2}{3}$. *Then we have*

$$\begin{aligned}\dim_{\mathbf{k}} T_{\mathrm{ppn}}(I) &\leq \dim_{\mathbf{k}} T_{\mathrm{ppn}}(\mathfrak{m}^r), &\qquad \dim_{\mathbf{k}} T_{\mathrm{pnp}}(I) &\leq \dim_{\mathbf{k}} T_{\mathrm{pnp}}(\mathfrak{m}^r),\\ \dim_{\mathbf{k}} T_{\mathrm{nnp}}(I) &\leq \dim_{\mathbf{k}} T_{\mathrm{nnp}}(\mathfrak{m}^r), &\qquad \dim_{\mathbf{k}} T_{\mathrm{npn}}(I) &\leq \dim_{\mathbf{k}} T_{\mathrm{npn}}(\mathfrak{m}^r).\end{aligned}$$

Moreover, in each case equality occurs if and only if $I = \mathfrak{m}^r$.

Proof. By Theorem 6.2.4 it suffices to prove the first two inequalities. We consider the $\mathbf{k}[z]$–, $\mathbf{k}[y]$– and $\mathbf{k}[y,z]$–decompositions of I

$$I = \bigoplus x^i y^j \left(z^{b^z_{i,j}}\right) = \bigoplus x^i z^j \left(y^{b^y_{i,j}}\right) = \bigoplus x^i I_i.$$

Note that $\sum_{j\geq 0} b^z_{i,j} = \dim_{\mathbf{k}}(\mathbf{k}[y,z]/I_i)$ for each i. Recall that $I_i = \mathbf{k}[y,z]$ for all $i \geq r$, as observed in the proof of Lemma 6.3.4. We apply Proposition 6.3.1 and Lemma 6.3.2, Lemma 6.3.4, Lemma 6.3.5 to obtain

$$\begin{aligned}\dim_{\mathbf{k}} T_{\mathrm{ppn}}(I) &= \sum_{\substack{\alpha_1,\alpha_2\geq 0\\ \alpha_3<0}} \dim_{\mathbf{k}} |T(I)|_{(\alpha_1,\alpha_2,\alpha_3)} \leq \sum_{\alpha_1,\alpha_2\geq 0}\sum_{\substack{i\geq\alpha_1\\ j\geq\alpha_2}} \left(b^z_{i,j} - \max\{b^z_{i+1,j}, b^z_{i,j+1}\}\right)\\ &= \sum_{\substack{\alpha_1\geq 0\\ \alpha_2\geq 0}}\sum_{\substack{i\geq\alpha_1\\ j\geq\alpha_2}} \left(b^z_{i,j} - b^z_{i,j+1}\right) = \sum_{i,j}(i+1)(j+1)\left(b^z_{i,j} - b^z_{i,j+1}\right) = \sum_{i,j}(i+1)b^z_{i,j}\\ &= \sum_{i=0}^{r-1}(i+1)\dim_{\mathbf{k}}\frac{\mathbf{k}[y,z]}{I_i} = \sum_{i=0}^{r-1}\sum_{j=i}^{r-1}\dim_{\mathbf{k}}\frac{\mathbf{k}[y,z]}{I_j} \leq \sum_{i=0}^{r-1}\sum_{j=i}^{r-1}\dim_{\mathbf{k}}\frac{\mathbf{k}[y,z]}{(y,z)^{r-j}}\\ &= \sum_{i=0}^{r-1}\sum_{j=i}^{r-1}\binom{r-j+1}{2} = \sum_{i=0}^{r-1}\sum_{h=2}^{r-i+1}\binom{h}{2} = \sum_{i=0}^{r-1}\binom{r-i+2}{3} = \sum_{k=3}^{r+2}\binom{k}{3} = \binom{r+3}{4}\\ &= \dim_{\mathbf{k}} T_{\mathrm{ppn}}(\mathfrak{m}^r).\end{aligned}$$

The inequality $\dim_{\mathbf{k}} T_{\mathrm{pnp}}(I) \le \dim_{\mathbf{k}} T_{\mathrm{pnp}}(\mathfrak{m}^r)$ is proved in the same way, using the second part of Lemma 6.3.2 and the fact that for each i we have $\sum_{j\ge 0} b^y_{i,j} = \dim_{\mathbf{k}}(\mathbf{k}[y,z]/I_i)$.

Finally, we verify the last assertion of the theorem. Observe that, if any of the four inequalities is an equality, then all the inequalities in the application of Lemma 6.3.4 are equalities, so for every $0 \le i \le r-1$ we have

$$\sum_{j=i}^{r-1} \dim_{\mathbf{k}}(\mathbf{k}[y,z]/I_j) = \sum_{j=i}^{r-1} \dim_{\mathbf{k}}\left(\mathbf{k}[y,z]/(y,z)^{r-j}\right),$$

and this in turn forces $I = \mathfrak{m}^r$ by the second part of Lemma 6.3.4. □

Remark 6.3.7. By Lemma 6.1.2, Remark 6.1.6, and Proposition 6.1.8, Theorem 6.3.6 verifies two thirds of Conjecture 6.0.1 for $\mathrm{Hilb}^d(\mathbf{A}^3)$. In fact, we conjecture that the remaining two inequalities

$$\dim_{\mathbf{k}} T_{\mathrm{npp}}(I) \le \dim_{\mathbf{k}} T_{\mathrm{npp}}(\mathfrak{m}^r) \qquad \text{and} \qquad \dim_{\mathbf{k}} T_{\mathrm{pnn}}(I) \le \dim_{\mathbf{k}} T_{\mathrm{pnn}}(\mathfrak{m}^r)$$

are also true. However, the bounds obtained for these subspaces through Proposition 6.3.1 are not sharp enough to prove them, as the next example shows.

Example 6.3.8. Let $I = (x) + (y,z)^s$ where $s \in \mathbf{N}$. We consider its $\mathbf{k}[x]$−decomposition $I = \bigoplus y^i z^j \left(x^{b_{i,j}}\right)$. Observe that $b_{i,j} = 1$ if $i+j < s$, whereas $b_{i,j} = 0$ if $i+j \ge s$. Proceeding as in the proof of Theorem 6.3.6, we use Proposition 6.3.1 to estimate $\dim_{\mathbf{k}} T_{\mathrm{npp}}(I)$ and obtain

$$\begin{aligned}
\dim_{\mathbf{k}} T_{\mathrm{npp}}(I) &= \sum_{\alpha_2,\alpha_3\ge 0} \sum_{\alpha_1<0} \dim_{\mathbf{k}} |T(I)|_{(\alpha_1,\alpha_2,\alpha_3)} \le \sum_{\alpha_2,\alpha_3\ge 0} \sum_{\substack{i\ge\alpha_2\\ j\ge\alpha_3}} \left(b_{i,j} - \max\{b_{i+1,j}, b_{i,j+1}\}\right) \\
&= \sum_{\alpha_2,\alpha_3\ge 0} \sum_{\substack{i\ge\alpha_2, j\ge\alpha_3\\ i+j=s-1}} 1 = \sum_{\substack{\alpha_2,\alpha_3\ge 0\\ \alpha_2+\alpha_3<s}} (s-\alpha_2-\alpha_3) = \binom{s+1}{2}s - \sum_{\substack{\alpha_2,\alpha_3\ge 0\\ \alpha_2+\alpha_3<s}} (\alpha_2+\alpha_3) \\
&= \binom{s+1}{2}s - \sum_{i=1}^{s-1} i(i+1) = \binom{s+1}{2}s - \frac{(s-1)s(s+2)}{3} = \binom{s+2}{3}.
\end{aligned}$$

Choose $s = 15$ and $r = 8$, so $\dim_{\mathbf{k}}(S/I) = \binom{15+1}{2} = 120 = \binom{8+2}{3} = \dim_{\mathbf{k}}(S/\mathfrak{m}^r)$. The inequality above yields $\dim_{\mathbf{k}} T_{\mathrm{npp}}(I) \le \binom{15+2}{3} = 680$; however, $\dim_{\mathbf{k}} T_{\mathrm{npp}}(\mathfrak{m}^r) = \binom{8+3}{4} = 330$ by Lemma 6.3.5.

6.4 Global estimates

We now take a more direct approach to estimating the dimension of tangent space to a point in $\mathrm{Hilb}^d(\mathbf{A}^3)$. This section is devoted to the proof of Theorem 6.4.2.

Let R be a regular local ring of dimension 2, and denote by $\ell(\cdot)$ the length of an R-module. A key step in the proof of the smoothness of $\mathrm{Hilb}^d(\mathbf{A}^2)$ [30] is to show that $\ell(T(I)) = 2\ell(R/I)$ for all artinian ideals $I \subseteq R$. The next proposition generalizes this fact.

Proposition 6.4.1. *Let R be a 2-dimensional regular local ring, and let $I, J \subseteq R$ be ideals satisfying $\ell(R/I), \ell(R/J) < \infty$. Then*

$$\ell\big(\mathrm{Hom}_R(I, R/J)\big) = \ell\big(R/J\big) + \ell\big((I : J)/I\big).$$

Proof. Let $0 \longrightarrow R^{a_2} \longrightarrow R^{a_1} \longrightarrow R^{a_0} \longrightarrow R/I \longrightarrow 0$ be a free resolution, then the alternating sum of ranks vanishes: $a_0 - a_1 + a_2 = 0$. Setting $\chi(R/I, R/J) = \sum_{i=0}^{2}(-1)^i \ell\big(\mathrm{Ext}^i(R/I, R/J)\big)$ we have

$$\chi(R/I, R/J) = \sum_{i=0}^{2}(-1)^i \chi(R^{a_i}, R/J) = \sum_{i=0}^{2}(-1)^i \ell(R/J) \cdot a_i = \ell(R/J) \sum_{i=0}^{2}(-1)^i a_i = 0. \tag{6.11}$$

Let $\omega_{R/I}$ be the canonical module of R/I. Since R/I is a Cohen-Macaulay R-module of codimension 2, dualizing its free resolution and using right-exactness of $- \otimes R/J$ yields

$$\mathrm{Ext}^2(R/I, R/J) \cong \mathrm{Ext}^2(R/I, R) \otimes R/J = \omega_{R/I} \otimes R/J. \tag{6.12}$$

Combining equations Eq. (6.11) and Eq. (6.12) with the exact sequence

$$0 \longrightarrow \mathrm{Hom}(R/I, R/J) \longrightarrow R/J \longrightarrow \mathrm{Hom}(I, R/J) \longrightarrow \mathrm{Ext}^1(R/I, R/J) \longrightarrow 0$$

we get

$$\begin{aligned} \ell\big(\mathrm{Hom}(I, R/J)\big) &= \ell\big(R/J\big) - \ell\big(\mathrm{Hom}(R/I, R/J)\big) + \ell\big(\mathrm{Ext}^1(R/I, R/J)\big) \\ &= \ell\big(R/J\big) + \ell\big(\mathrm{Ext}^2(R/I, R/J)\big) \\ &= \ell\big(R/J\big) + \ell\big(\omega_{R/I} \otimes_R R/J\big). \end{aligned}$$

It remains to show that $\ell\big(\omega_{R/I}/J\omega_{R/I}\big) = \ell\big((I : J)/I\big)$. We have $(I : J)/I = \big(I : (I + J)\big)/I$ and $\omega_{R/I}/J\omega_{R/I} = \omega_{R/I}/(I + J)\omega_{R/I}$ (since I annihilates $\omega_{R/I}$), so we may assume that $I \subseteq J$. In this case R/J is a finite R/I-module and $\omega_{R/J} \cong \mathrm{Hom}(R/J, \omega_{R/I})$. Since $\mathrm{Hom}(-, \omega_{R/I})$ induces a duality in the category of finite R/I-modules (cf. [24, 21.1]) we obtain

$$\begin{aligned} \mathrm{Hom}(R/J, R/I) &\cong \mathrm{Hom}\big(\mathrm{Hom}(R/I, \omega_{R/I}), \mathrm{Hom}(R/J, \omega_{R/I})\big) \\ &\cong \mathrm{Hom}(\omega_{R/I}, \omega_{R/J}) \\ &= \mathrm{Hom}(\omega_{R/I}/J\omega_{R/I}, \omega_{R/J}) \end{aligned}$$

and this implies $\ell\big(\mathrm{Hom}(R/J, R/I)\big) = \ell\big(\omega_{R/I}/J\omega_{R/I}\big)$, again by duality. The proof is completed, as $(I : J)/I = \mathrm{Hom}(R/J, R/I)$. $\square$

Now we present the main result of this section, which establishes an approximation of Conjecture 6.0.1 for the Hilbert scheme of points in $\mathbf{A}^3$.

Theorem 6.4.2. *Let $d, r \in \mathbf{N}$ be such that $d \leq \binom{r+2}{3}$. For all $[I] \in \mathrm{Hilb}^d(\mathbf{A}^3)$ we have*

$$\dim_{\mathbf{k}} T(I) \leq \frac{4}{3} \dim_{\mathbf{k}} T(\mathfrak{m}^r).$$

Proof. By Remark 6.1.6 and Lemma 6.1.2 we may assume that $\operatorname{char} \mathbf{k} = 0$ and $I \subseteq S$ is Borel-fixed. Let $I = \bigoplus x^i I_i$ be the $\mathbf{k}[y,z]$-decomposition and let $p = \min\left\{i \,:\, I_i = \mathbf{k}[y,z]\right\}$. Assuming without loss of generality that $I \neq \mathfrak{m}^r$, the hypothesis $d \leq \binom{r+2}{3}$ and the fact that I is strongly stable imply that $p < r$.

We denote by $T(I)_j$ the component of $T(I)$ of x-degree j, that is, $T(I)_j = \bigoplus_{\alpha_1 = j} \left|T(I)\right|_{\alpha}$. A tangent vector $\xi \in T(I)_j$, viewed as homomorphism $\xi : I \to S/I$, is uniquely determined by its restrictions

$$\xi\big|_{x^i I_i} \,:\, x^i I_i \longrightarrow x^{i+j} \frac{\mathbf{k}[y,z]}{I_{i+j}}$$

where $i \geq 0$ and $0 \leq i + j < p$. Clearly, $T(I)_j = 0$ if $j \geq p$. On the other hand, we also have $T(I)_j = 0$ if $j < -p$, since any monomial minimal generator of I has x-degree at most p by strong stability. For the same reason, it suffices to consider the restrictions for $i \leq p$. To summarize, every x-homogeneous $\xi \in T(I)$ is determined by the induced $\mathbf{k}[y,z]$-linear homomorphisms

$$\xi\big|_{I_i} \,:\, I_i \longrightarrow \frac{\mathbf{k}[y,z]}{I_{i+j}} \qquad \text{with} \quad -p \leq j \leq p-1, \quad \max(0,-j) \leq i \leq \min(p, p-j-1) \tag{6.13}$$

where, by abuse of notation, we drop the placeholders x^i, x^{i+j}.

Now we can estimate the dimension of the tangent space:

$$\begin{aligned}
\dim_{\mathbf{k}} T(I) &\le \sum_{j=-p}^{p-1} \sum_{i=\max(0,-j)}^{\min(p,p-j-1)} \dim_{\mathbf{k}} \operatorname{Hom}\left(I_i, \frac{\mathbf{k}[y,z]}{I_{i+j}}\right) && \text{by Eq. (6.13)} \\
&= \sum_{j=-p}^{p-1} \sum_{i=\max(0,-j)}^{\min(p,p-j-1)} \left(\dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{I_{i+j}} + \dim_{\mathbf{k}} \frac{I_i : I_{i+j}}{I_i}\right) && \text{by Proposition 6.4.1} \\
&\le \sum_{j=-p}^{p-1} \sum_{i=\max(0,-j)}^{\min(p,p-j-1)} \left(\dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{I_{i+j}} + \dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{I_i}\right) \\
&= \sum_{j=-p}^{-1} \sum_{i=-j}^{p} \left(\dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{I_{i+j}} + \dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{I_i}\right) \\
&+ \sum_{j=0}^{p-1} \sum_{i=0}^{p-j-1} \left(\dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{I_{i+j}} + \dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{I_i}\right) \\
&= \sum_{i=0}^{p-1} \sum_{j=0}^{i} \dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{I_j} + \sum_{j=0}^{p-1} \sum_{i=p-j}^{p} \dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{I_i} && \text{reindexing} \\
&+ \sum_{i=0}^{p-1} \sum_{j=i}^{p-1} \dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{I_j} + \sum_{j=0}^{p-1} \sum_{i=0}^{p-j-1} \dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{I_i} \\
&= (p+1) \sum_{j=0}^{p-1} \dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{I_j} + p \sum_{i=0}^{p} \dim_{\mathbf{k}} \frac{\mathbf{k}[y,z]}{I_i} \\
&= (2p+1) \dim_{\mathbf{k}} \frac{S}{I} \le (2r-1)\binom{r+2}{3} && \text{by assumption} \\
&\le \frac{4}{3}\binom{r+2}{2}\binom{r+1}{2} = \frac{4}{3} \dim_{\mathbf{k}} T(\mathfrak{m}^r) && \text{by Lemma 6.3.5.}
\end{aligned}$$

□

Our analysis allows verifying Conjecture 6.0.1 for many monomial ideals:

Corollary 6.4.3. *Let* $[I] \in \mathrm{Hilb}^d(\mathbf{A}^3)$ *be a monomial point with* $d \le \binom{r+2}{3}$*. If* $x^p \in I$ *with* $p \le \frac{3r+1}{4}$*, then* $\dim_{\mathbf{k}} T(I) \le \dim_{\mathbf{k}} T(\mathfrak{m}^r)$*.*

Proof. As in the proof of Theorem 6.4.2, we may assume that $\operatorname{char} \mathbf{k} = 0$ and $I \subseteq S$ is Borel-fixed: in fact, if I is any monomial ideal and $x^p \in I$, then $x^p \in \operatorname{gin} I$ as well. Now, if

$p \leq \frac{3r+1}{4}$ then we can improve the estimates in the proof of Theorem 6.4.2 obtaining

$$\dim_{\mathbf{k}} T(I) \leq (2p+1)\dim_{\mathbf{k}} \frac{S}{I} \leq \frac{6r+6}{4}\binom{r+2}{3} = \binom{r+2}{2}\binom{r+1}{2} = \dim_{\mathbf{k}} T(\mathfrak{m}^r).$$

□

As observed in the proof of Theorem 6.4.2, if I is strongly stable and $d = \dim_{\mathbf{k}}(S/I) \leq \binom{r+2}{3}$ then $x^r \in I$. Hence, Corollary 6.4.3 proves Conjecture 6.0.1 for "three quarters" of the strongly stable ideals – in fact, often for a much larger fraction. For example, we use this fact in the proof of [82] where the search for the maximum tangent space dimension for $\mathrm{Hilb}^{39}(\mathbf{A}^3)$ is reduced from all 39098 strongly stable ideals to the 2654 ones that do not contain small powers of x.

Another consequence of Theorem 6.4.2 is a new bound on the dimension of the Hilbert scheme:

Corollary 6.4.4. *For $d \gg 0$ we have* $\dim \mathrm{Hilb}^d(\mathbf{A}^3) \leq 3.64 \cdot d^{\frac{4}{3}}$.

Proof. Let $r \in \mathbf{N}$ such that $\binom{r+1}{3} < d \leq \binom{r+2}{3}$, so $r-1 \leq \sqrt[3]{6d}$. Using Theorem 6.4.2 we get

$$\begin{aligned}\dim \mathrm{Hilb}^d(\mathbf{A}^3) &\leq \max_{I \in \mathrm{Hilb}^d(\mathbf{A}^3)} \dim_{\mathbf{k}} T(I) \leq \frac{4}{3}\dim_{\mathbf{k}} T(\mathfrak{m}^r) = \frac{4}{3}\binom{r+2}{2}\binom{r+1}{2} \\ &= \frac{1}{3}(r+2)(r+1)^2(r) \leq \frac{1}{3}\left(\sqrt[3]{6d}\right)^4 + O(d) \approx 3.634 \cdot d^{\frac{4}{3}} + O(d)\end{aligned}$$

implying the desired asymptotic bound. □

Remark 6.4.5. The authors in [10] proved that $\dim \mathrm{Hilb}^d(\mathbf{A}^3) \leq 19.92 \cdot d^{\frac{4}{3}}$. On the other hand, the full Conjecture 6.0.1 would imply that $\dim \mathrm{Hilb}^d(\mathbf{A}^3) \leq 2.73 \cdot d^{\frac{4}{3}}$ for $d \gg 0$.

Chapter 7

The fiber-full scheme

> "Last time, I asked: "What does mathematics mean to you?" And some people answered: "The manipulation of numbers, the manipulation of structures." And if I had asked what music means to you, would you have answered: "The manipulation of notes?"
>
> Serge Lang [61]

In this chapter we introduce a far-reaching generalization of the Hilbert and Quot schemes that controls all the cohomological data of the quotients of a coherent sheaf $\mathscr{F}$, instead of just the Hilbert polynomial. To accomplish this we develop a theory of flattening stratifications for various modules and complexes; the most important being the local cohomology modules and the higher direct image sheaves. We also develop the notion of a fiber-full sheaf.

We start with the classical example of the Hilbert scheme compactification of the space of twisted cubics that was studied by Piene and Schlessinger [79]. The motivating example below shows how this well-studied Hilbert scheme decomposes into locally closed subschemes that have constant cohomological data.

Example 7.0.1 (Theorem 7.4.9)**.** In [79], it was shown that $\mathrm{Hilb}^{3t+1}_{\mathbf{P}^3_{\mathbf{k}}} = H \cup H'$ is a union of two smooth irreducible components such that the general member of H parametrizes a twisted cubic, and the general member of H' parametrizes a plane cubic union an isolated point. It is also known that $H - H \cap H'$ is the locus of arithmetically Cohen-Macaulay curves of degree 3 and genus 0. We then have a decomposition

$$\mathrm{Hilb}^{3t+1}_{\mathbf{P}^3_{\mathbf{k}}} = (H - H \cap H') \sqcup H'.$$

Furthermore, one can show that the functions

$$h^i_X : \mathbf{Z} \to \mathbf{N}, \quad \nu \mapsto \dim_{\mathbf{k}} \left(H^i(X, \mathscr{O}_X(\nu))\right)$$

are the same for any element $[X] \in H - H \cap H'$ and the same for any element $[X] \in H'$ (for an explicit computation, see Theorem 7.4.9). It then follows that $\mathrm{Hilb}^{3t+1}_{\mathbf{P}^3_{\mathbf{k}}}$ can be decomposed into two locally closed subschemes where the cohomological functions h^i_X are constant. It should also be noted that one might be quite interested in studying $H - H \cap H'$ as it gives all the closed subschemes of $\mathbf{P}^3_{\mathbf{k}}$ with the same cohomological data as that of a twisted cubic.

As presented below, the scheme we introduce allows us to provide a unified and systematic treatment of the decomposition seen in Theorem 7.0.1. Let S be a locally Noetherian scheme, $f : X \subset \mathbf{P}^r_S \to S$ be a projective morphism and $\mathscr{F}$ be a coherent sheaf on X. We follow Grothendieck's general idea of considering a contravariant functor whose representing scheme (if it exists) is the parameter space one is interested in.

Notation 7.0.2. In this chapter S will always denote a base scheme while R will be used to denote a polynomial ring. While this is in contrast with the rest of the thesis, it is consistent with both the papers [19] and [20]. Since this chapter is taken from [19] we have chosen to use the notation appearing there.

We define the *fiber-full functor* which for an S-scheme T parametrizes all coherent quotients $\mathscr{F}_T \twoheadrightarrow \mathscr{G}$ such that all the higher direct images of $\mathscr{G}$ and its twistings are locally free over T. More precisely, for any (locally Noetherian) S-scheme T we define

$$\mathcal{F}ib_{\mathscr{F}/X/S}(T) = \left\{ \text{coherent quotient } \mathscr{F}_T \twoheadrightarrow \mathscr{G} \;\middle|\; \begin{array}{l} R^i f_{(T)_*}(\mathscr{G}(\nu)) \text{ is locally free over } T \\ \text{for all } 0 \le i \le r, \nu \in \mathbf{Z} \end{array} \right\},$$

where $\mathscr{F}_T$ is the pull-back sheaf on $X_T = X \times_S T$ and $f_{(T)} : X_T \subset \mathbf{P}^r_T \to T$ is the base change morphism $f_{(T)} = f \times_S T$. We have that

$$\mathcal{F}ib_{\mathscr{F}/X/S} : (\mathrm{Sch}/S)^{\mathrm{opp}} \to (\mathrm{Sets})$$

is a contravariant functor from the category of (locally Noetherian) S-schemes to the category of sets (see Theorem 7.4.1). We stratify this functor in terms of "Hilbert functions" for all the cohomologies. Let $\mathbf{h} = (h_0, \ldots, h_r) : \mathbf{Z}^{r+1} \to \mathbf{N}^{r+1}$ be a tuple of functions. Then, we define the following functor depending on $\mathbf{h}$:

$$\mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S}(T) = \left\{ \mathscr{G} \in \mathcal{F}ib_{\mathscr{F}/X/S}(T) \;\middle|\; \begin{array}{l} \dim_{\kappa(t)}\left(H^i(X_t, \mathscr{G}_t(\nu))\right) = h_i(\nu) \\ \text{for all } 0 \le i \le r, \nu \in \mathbf{Z}, t \in T \end{array} \right\},$$

where $\kappa(t)$ denotes the residue field of the point $t \in T$, $X_t = X_T \times_T \mathrm{Spec}(\kappa(t))$ is the fiber over $t \in T$, and $\mathscr{G}_t$ is the pull-back sheaf on X_t. The idea of this functor is to measure the dimension of *all cohomologies of all possible twistings*. We easily obtain the stratification $\mathcal{F}ib_{\mathscr{F}/X/S}(T) = \bigsqcup_{\mathbf{h}:\mathbf{Z}^{r+1}\to\mathbf{N}^{r+1}} \mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S}(T)$ when T is connected, and so it follows that

$\mathcal{F}ib_{\mathscr{F}/X/S}(T)$ is a representable functor if all the functors $\mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S}(T)$ are representable. When $\mathscr{F} = \mathscr{O}_X$, we simplify the notation by writing $\mathcal{F}ib^{\mathbf{h}}_{X/S}$ instead of $\mathcal{F}ib^{\mathbf{h}}_{\mathscr{O}_X/X/S}$.

For any numerical polynomial $P \in \mathbf{Q}[t]$, we have Grothendieck's definition of the Quot functor

$$\mathcal{Q}uot^{P}_{\mathscr{F}/X/S} : (\mathrm{Sch}/S)^{\mathrm{opp}} \rightarrow (\mathrm{Sets})$$

which for an S-scheme T parametrizes all coherent quotients $\mathscr{F}_T \twoheadrightarrow \mathscr{G}$ that are flat over T and have Hilbert polynomial equal to P along all fibers. The Hilbert functor $\mathcal{H}ilb^{P}_{\mathscr{F}/X/S}$ is the special case of $\mathcal{Q}uot^{P}_{\mathscr{F}/X/S}$ with $\mathscr{F} = \mathscr{O}_X$. Then, the fiber-full functor can be thought of as a refinement of the Hilbert and Quot functors due to the following inclusions. From the tuple of functions $\mathbf{h} = (h_0, \ldots, h_r) : \mathbf{Z}^{r+1} \rightarrow \mathbf{N}^{r+1}$, we define the function $P_{\mathbf{h}} = \sum_{i=0}^{r}(-1)^i h_i$. When $P_{\mathbf{h}} \in \mathbf{Q}[t]$ is a numerical polynomial, since the Hilbert polynomial of a sheaf coincides with its Euler characteristic, we automatically get the inclusions

$$\mathcal{F}ib^{\mathbf{h}}_{X/S}(T) \subset \mathcal{H}ilb^{P_{\mathbf{h}}}_{X/S}(T) \quad \text{and} \quad \mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S}(T) \subset \mathcal{Q}uot^{P_{\mathbf{h}}}_{\mathscr{F}/X/S}(T)$$

for any (locally Noetherian) S-scheme T. If $P_{\mathbf{h}}$ is not a numerical polynomial, then $\mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S}(T) = \emptyset$ for any S-scheme T.

The following is the main theorem of this article. Here, we show that the functor $\mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S}$ is represented by a quasi-projective S-scheme that we call the *fiber-full scheme* and we write as $\mathrm{Fib}^{\mathbf{h}}_{\mathscr{F}/X/S}$. From the definition of $\mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S}$, it follows that the fiber-full scheme $\mathrm{Fib}^{\mathbf{h}}_{\mathscr{F}/X/S}$ is the finest possible generalization of the Quot scheme $\mathrm{Quot}^{P_{\mathbf{h}}}_{\mathscr{F}/X/S}$ if one is interested in controlling all the cohomological data.

Theorem 7.0.3. Let S be a locally Noetherian scheme, $f : X \subset \mathbf{P}^r_S \rightarrow S$ be a projective morphism and $\mathscr{F}$ be a coherent sheaf on X. Let $\mathbf{h} = (h_0, \ldots, h_r) : \mathbf{Z}^{r+1} \rightarrow \mathbf{N}^{r+1}$ be a tuple of functions and suppose that $P_{\mathbf{h}}$ is a Hilbert polynomial. Then, there is a quasi-projective S-scheme $\mathrm{Fib}^{\mathbf{h}}_{\mathscr{F}/X/S}$ that represents the functor $\mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S}$ and that is a locally closed subscheme of the Quot scheme $\mathrm{Quot}^{P_{\mathbf{h}}}_{\mathscr{F}/X/S}$.

Our main tool for constructing the fiber-full scheme is given in Theorem 7.2.2 where we provide a flattening stratification theorem that deals with all the direct images of a sheaf and its possible twistings. To prove this technical theorem we utilize some techniques previously developed in the papers [14, 18]. In a related direction, we also introduce the notion of fiber-full sheaves and we give three equivalent definitions in Theorem 7.3.2. Under the above notation, we say that $\mathscr{F}$ is a *fiber-full sheaf over* S if $R^i f_*(\mathscr{F}(\nu))$ is locally free over S for all $0 \le i \le r$ and $\nu \in \mathbf{Z}$. Fiber-full sheaves serve as a sheaf-theoretic extension of the notions of *algebras having liftable local cohomology* [60] and *cohomologically full rings* [22].

It turns out there has been previous interest in stratifying the Hilbert scheme in terms of the whole cohomological data.

In the work of Martin-Deschamps and Perrin [65], they were able to control the cohomologies of a sheaf, but not all the possible twistings, as their method would yield the intersection of infinitely many (not necessarily closed) subschemes (see [65, Chapitre VI, Proposition 1.9 and Corollaire 1.10]); their approach is based on classical techniques related to the Grothendieck complex which are covered, e.g., in [47, §III.12].

In the thesis of Fumasoli [32, 33], he stratified the Hilbert scheme by bounding below the cohomological functions of the points of the Hilbert scheme, which is a consequence of the classical upper semicontinuity theorem (see [47, Theorem III.12.8]).

Our main result Theorem 7.0.3 vastly generalizes the two aforementioned approaches and shows that one can indeed stratify the Hilbert and Quot schemes by taking into account all the cohomological data. In this regard, one important part of our work is to develop the necessary tools that allow us to prove the general stratification result of Theorem 7.2.2.

Next, we describe some applications that follow from the existence of the fiber-full scheme.

There is a large literature on the study of the loci of *arithmetically Cohen-Macaulay* (ACM for short) schemes and the loci of *arithmetically Gorenstein* (AG for short) schemes within the Hilbert scheme (see [28, 49, 56–58, 65] and the references therein). As a result of considering the fiber-full scheme, we can provide a finer description of these loci and parametrize ACM and AG schemes with a fixed cohomological data. Let $d \in \mathbf{N}$ and $h_0, h_d : \mathbf{Z} \to \mathbf{N}$ be two functions, and consider the tuple of functions $\mathbf{h} : \mathbf{Z}^{r+1} \to \mathbf{N}^{r+1}$ given by $\mathbf{h} = (h_0, 0, \ldots, 0, h_d, 0, \ldots, 0)$ where $0 : \mathbf{Z} \to \mathbf{N}$ denotes the zero function. To study ACM and AG schemes, since all the intermediate cohomologies vanish in these cases, it becomes natural to consider the following two functors. For any (locally Noetherian) S-scheme T, we have

$$\mathcal{ACM}_{X/S}^{h_0,h_d}(T) \;=\; \left\{\text{closed subscheme } Z \subset X_T \;\middle|\; Z \in \mathcal{F}ib_{X/S}^{\mathbf{h}}(T) \text{ and } Z_t \text{ is ACM for all } t \in T \right\}$$

and

$$\mathcal{AG}_{X/S}^{h_0,h_d}(T) \;=\; \left\{\text{closed subscheme } Z \subset X_T \;\middle|\; Z \in \mathcal{F}ib_{X/S}^{\mathbf{h}}(T) \text{ and } Z_t \text{ is AG for all } t \in T \right\}.$$

The following theorem shows the two functors above are representable, and so it provides the natural parameter spaces for ACM and AG schemes with fixed cohomological data.

Theorem 7.0.4 (Theorem 7.4.7)**.** Let S be a locally Noetherian scheme and $f : X \subset \mathbf{P}_S^r \to S$ be a projective morphism. Let $d \in \mathbf{N}$ and $h_0, h_d : \mathbf{Z} \to \mathbf{N}$ be two functions, and consider the tuple of functions $\mathbf{h} = (h_0, 0, \ldots, 0, h_d, 0, \ldots, 0) : \mathbf{Z}^{r+1} \to \mathbf{N}^{r+1}$. Suppose that $P_{\mathbf{h}} \in \mathbf{Q}[t]$

is a numerical polynomial. Then, there exist open S-subschemes $\mathrm{ACM}_{X/S}^{h_0,h_d}$ and $\mathrm{AG}_{X/S}^{h_0,h_d}$ of $\mathrm{Fib}_{X/S}^{\mathbf{h}}$ that represent the functors $\mathcal{ACM}_{X/S}^{h_0,h_d}$ and $\mathcal{AG}_{X/S}^{h_0,h_d}$, respectively.

We end by studying examples of Hilbert schemes that we stratify in terms of fiber-full schemes. Furthermore, by using the recent classification of Skjelnes and Smith [88], we show in Theorem 7.5.4 that smooth Hilbert schemes coincide with a fiber-full scheme (i.e., cohomological data is constant for points in a smooth Hilbert scheme).

7.1 Some flattening stratification theorems in a graded category of modules

In this section, we provide several flattening stratification theorems in a graded category modules; the list includes: the case of modules, cohomology of complexes of modules, Ext modules and local cohomology modules. For organizational purposes, we divide the section into four different subsections.

7.1.1 Flattening stratification of modules

In this subsection, we concentrate on an extension for modules of the flattening stratification theorem given in [4] (also, see [71, §8]). Throughout this subsection, we shall use the following setup.

Setup. Let A be ring (always assumed to be commutative and unitary) and R be a finitely generated graded A-algebra. For any $\boldsymbol{p} \in \mathrm{Spec}(A)$, let $\kappa(\boldsymbol{p}) := A_{\mathfrak{p}}/\boldsymbol{p}A_{\mathfrak{p}}$ be the residue field of $\boldsymbol{p}$.

For a graded R-module M, we say that M has a *Hilbert function over A* if for all $\nu \in \mathbf{Z}$ the graded part $[M]_\nu$ is a finitely generated locally free A-module of constant rank on $\mathrm{Spec}(A)$; and in this case, the Hilbert function is $h_M : \mathbf{Z} \to \mathbf{N},\ h_M(\nu) = \mathrm{rank}_A\left([M]_\nu\right)$. If a graded R-module M has a Hilbert function over A, then $M \otimes_A B$ has the same Hilbert function over any A-algebra B.

Remark 7.1.1. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of graded R-modules.*

(i) *If L and N have Hilbert functions over A, then M has a Hilbert function over A given by $h_M(\nu) = h_L(\nu) + h_N(\nu)$.*

(ii) *If M and N have Hilbert functions over A, then L has a Hilbert function over A given by $h_L(\nu) = h_M(\nu) - h_N(\nu)$.*

For completeness, we recall the following flatness result.

Lemma 7.1.2. *Assume that A is Noetherian. Let M be a finitely generated graded A-module. Then, the following locus*

$$U_M := \left\{\boldsymbol{p} \in \operatorname{Spec}(A) \mid M \otimes_A A_{\mathfrak{p}} \text{ is a flat } A_{\mathfrak{p}}\text{-module}\right\}$$

is an open subset of $\operatorname{Spec}(A)$.

Proof. For a proof, see [4, Lemma 2.1] or [14, Lemma 2.5]. □

For a given graded R-module M and a function $h : \mathbf{Z} \to \mathbf{N}$, we consider the following functor for any ring B,

$$\mathcal{F}_M^h(B) := \left\{ \text{morphism } \operatorname{Spec}(B) \to \operatorname{Spec}(A) \;\middle|\; \begin{array}{l} [M \otimes_A B]_\nu \text{ is a locally free } B\text{-module} \\ \text{of rank } h(\nu) \text{ for all } \nu \in \mathbf{Z} \end{array} \right\}.$$

We now describe our first flattening stratification theorem.

Theorem 7.1.3. *Assume A is Noetherian. Let M be a finitely generated graded R-module and $h : \mathbf{Z} \to \mathbf{N}$ be a function. Then, the following statements hold:*

(i) *The functor $\mathcal{F}_M^h$ is represented by a locally closed subscheme $F_M^h \subset \operatorname{Spec}(A)$. In other words, for any morphism $g : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$, $M \otimes_A B$ has a Hilbert function over B equal to h if and only if g can be factored as*

$$\operatorname{Spec}(B) \;\to\; F_M^h \;\to\; \operatorname{Spec}(A).$$

(ii) *There is only a finite number of different functions $h_1, \ldots, h_m : \mathbf{Z} \to \mathbf{N}$ such that $F_M^{h_i} \neq \emptyset$, and so $\operatorname{Spec}(A)$ is set-theoretically equal to the disjoint union of the locally closed subschemes $F_M^{h_i}$.*

Proof. (i) For any morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$, one has that $[M \otimes_A B]_\nu$ is locally free of rank $h(\nu)$ if and only if $\operatorname{Fitt}_{h(\nu)-1}([M \otimes_A B]_\nu) = 0$ and $\operatorname{Fitt}_{h(\nu)}([M \otimes_A B]_\nu) = B$, and that $\operatorname{Fitt}_j([M \otimes_A B]_\nu) = \left(\operatorname{Fitt}_j([M]_\nu)\right) B$ (for more details on Fitting ideals, see [25, §20.2]).

Let $Z_M^h \subset \operatorname{Spec}(A)$ be the closed subscheme given by

$$Z_M^h := \operatorname{Spec}(A/(\textstyle\sum_{\nu \in \mathbf{Z}} \operatorname{Fitt}_{h(\nu)-1}([M]_\nu))).$$

We have that $\operatorname{Fitt}_{h(\nu)-1}([M \otimes_A B]_\nu) = 0$ for all $\nu \in \mathbf{Z}$ if and only if $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ factors through Z_M^h. Therefore, we can substitute $\operatorname{Spec}(A)$ by Z_M^h, and we do so.

Let $\boldsymbol{p} \in U_M$ and suppose that $M \otimes_A A_{\mathfrak{p}}$ has a Hilbert function $h_{M \otimes_A A_{\mathfrak{p}}} = h$ over $A_{\mathfrak{p}}$. By Theorem 7.1.2, there is an affine open neighborhood $\boldsymbol{p} \in \operatorname{Spec}(A_a) \subset U_M$ of $\boldsymbol{p}$ for some $a \in A$. Thus [91, Tag 00NX] implies that for all $\nu \in \mathbf{Z}$ the function $\operatorname{Spec}(A_a) \to \mathbf{N}$, $\mathfrak{q} \mapsto \dim_{\kappa(\mathfrak{q})}([M \otimes_{A_a} \kappa(\mathfrak{q})]_\nu)$ is locally constant. Consequently, there is an open connected neighborhood $\boldsymbol{p} \in V \subset \operatorname{Spec}(A_a)$ of $\boldsymbol{p}$ such that $h_{M \otimes_A A_{\mathfrak{q}}} = h$ for all $\mathfrak{q} \in V$. It then follows that the following locus

$$U_M^h := \left\{\boldsymbol{p} \in \operatorname{Spec}(A) \mid M \otimes_A A_{\mathfrak{p}} \text{ has a Hilbert function } h_{M \otimes_A A_{\mathfrak{p}}} = h \text{ over } A_{\mathfrak{p}}\right\}$$

is an open subset of $\operatorname{Spec}(A)$.

Note that $\operatorname{Fitt}_{h(\nu)}([M \otimes_A B]_\nu) = B$ if and only if $\mathfrak{P} \cap A \not\supset \operatorname{Fitt}_{h(\nu)}([M]_\nu)$ for all $\mathfrak{P} \in \operatorname{Spec}(B)$. Hence, under the condition $\operatorname{Fitt}_{h(\nu)-1}([M \otimes_A B]_\nu) = 0$ for all $\nu \in \mathbf{Z}$, it follows that $\operatorname{Fitt}_{h(\nu)}([M \otimes_A B]_\nu) = B$ for all $\nu \in \mathbf{Z}$ if and only if $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ factors through U_M^h. So, after having changed $\operatorname{Spec}(A)$ by Z_M^h, we have that $\mathcal{F}_M^h$ is represented by the open subscheme $U_M^h \subset \operatorname{Spec}(A)$. This completes the proof of this part.

(ii) For each $\boldsymbol{p} \in \operatorname{Spec}(A)$, let $h_{\mathfrak{p}}$ be the function $h_{\mathfrak{p}}(\nu) := \dim_{\kappa(\boldsymbol{p})}([M \otimes_A \kappa(\boldsymbol{p})]_\nu)$. As we have a natural morphism $\operatorname{Spec}(\kappa(\boldsymbol{p})) \to \operatorname{Spec}(A)$, it clearly follows that $\boldsymbol{p} \in F_M^{h_{\mathfrak{p}}}$. Therefore, by the Noetherian hypothesis, we can show that there is a finite number of distinct functions $h_1, \ldots, h_m$ such that set-theoretically we have the equality $\operatorname{Spec}(A) = \bigsqcup_{i=1}^m F_M^{h_i}$. $\square$

7.1.2 Flattening stratification of the cohomologies of a complex

Here we study how the process of taking tensor product with another ring affects the cohomology of a bounded complex. The notation below will be used throughout the paper.

Notation 7.1.4. For a (co-)complex of A-modules $K^\bullet : \cdots \to K^{i-1} \xrightarrow{\phi^{i-1}} K^i \xrightarrow{\phi^i} K^{i+1} \to \cdots$, one defines $\mathrm{Z}^i(K^\bullet) := \operatorname{Ker}(\phi^i)$, $\mathrm{B}^i(K^\bullet) := \operatorname{Im}(\phi^{i-1})$, $H^i(K^\bullet) := \mathrm{Z}^i(K^\bullet)/\mathrm{B}^i(K^\bullet)$, and $C^i(K^\bullet) := K^i/\mathrm{B}^i(K^\bullet) \supset H^i(K^\bullet)$ for all $i \in \mathbf{Z}$. We use analogous notation with lower indices for a complex $K_\bullet$.

Remark 7.1.5. A basic result that we shall use several times is the following: for a complex of A-modules $K^\bullet$ and an A-module N, we have a four-term exact sequence

$$0 \to H^i\big(K^\bullet \otimes_A N\big) \to C^i(K^\bullet) \otimes_A N \to K^{i+1} \otimes_A N \to C^{i+1}(K^\bullet) \otimes_A N \to 0$$

of A-modules.

The following lemma transfers the burden of studying the cohomologies of a bounded complex to considering the cokernels of the maps.

Lemma 7.1.6. *Let $K^\bullet : 0 \to K^0 \to K^1 \to \cdots \to K^p \to 0$ be a bounded complex of graded R-modules. Suppose that each K^i has a Hilbert function over A. Let* $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *be a morphism. Then, the following two conditions are equivalent:*

(i) *$H^i(K^\bullet \otimes_A B)$ has a Hilbert function over B for all $0 \le i \le p$.*

(ii) *$C^i(K^\bullet) \otimes_A B$ has a Hilbert function over B for all $0 \le i \le p$.*

Moreover, if any of the above conditions are satisfied, we have

$$h_{H^i(K^\bullet \otimes_A B)} = h_{C^i(K^\bullet)\otimes_A B} + h_{C^{i+1}(K^\bullet)\otimes_A B} - h_{K^{i+1}\otimes_A B}$$

and

$$h_{C^i(K^\bullet)\otimes_A B} = \sum_{j=i}^{p} (-1)^{j-i} \left(h_{H^j(K^\bullet \otimes_A B)} + h_{K^{j+1}\otimes_A B} \right).$$

Proof. We have the four-term exact sequence

$$0 \to H^i\left(K^\bullet \otimes_A B\right) \to C^i(K^\bullet) \otimes_A B \to K^{i+1} \otimes_A B \to C^{i+1}(K^\bullet) \otimes_A B \to 0,$$

which can be broken into short exact sequences

$$0 \to H^i\left(K^\bullet \otimes_A B\right) \to C^i(K^\bullet)\otimes_A B \to L^i \to 0 \quad \text{and} \quad 0 \to L^i \to K^{i+1}\otimes_A B \to C^{i+1}(K^\bullet)\otimes_A B \to 0$$

where L^i is some graded R-module.

By Theorem 7.1.1, if all $C^i(K^\bullet) \otimes_A B$ have a Hilbert function over B then all L^i have a Hilbert function over B and, by the same token, it follows that all $H^i(K^\bullet \otimes_A B)$ have a Hilbert function over B. This establishes the implication (2) $\Rightarrow$ (1).

Suppose that all $H^i(K^\bullet \otimes_A B)$ have a Hilbert function over B. As a consequence of Theorem 7.1.1, if $C^{i+1}(K^\bullet) \otimes_A B$ has a Hilbert function over B, we obtain that L^i and, subsequently, $C^i(K^\bullet)\otimes_A B$ have Hilbert functions over B. Since $C^p(K^\bullet)\otimes_A B = H^p(K^\bullet \otimes_A B)$, by descending induction on i, we get that all $C^i(K^\bullet) \otimes_A B$ have a Hilbert function over B. So, the other implication (1) $\Rightarrow$ (2) also holds.

The additional equations relating the Hilbert functions of $C^i(K^\bullet)\otimes_A B$ and $H^i(K^\bullet \otimes_A B)$ are straightforwardly checked. ☐

For a given bounded complex of graded R-modules $K^\bullet : 0 \to K^0 \to K^1 \to \cdots \to K^p \to 0$ such that each K^i has a Hilbert function over A and a given tuple of $p+1$ functions $\mathbf{h} = (h_0, \ldots, h_p) : \mathbf{Z}^{p+1} \to \mathbf{N}^{p+1}$, we consider the following functor for any ring B,

$$\mathcal{F}^{\mathbf{h}}_{K^\bullet}(B) := \left\{ \text{morphism } \operatorname{Spec}(B) \to \operatorname{Spec}(A) \;\middle|\; \begin{array}{l} \left[H^i(K^\bullet \otimes_A B)\right]_\nu \text{ is a locally free } B\text{-module} \\ \text{of rank } h_i(\nu) \text{ for all } 0 \le i \le p, \nu \in \mathbf{Z} \end{array} \right\}.$$

For completeness, we include a lemma which shows that, in our setting, flatness is equivalent to being locally free.

Lemma 7.1.7. *Let $K^\bullet : 0 \to K^0 \to K^1 \to \cdots \to K^p \to 0$ be a bounded complex of graded R-modules. Suppose that $[K^i]_\nu$ is a finitely generated locally free A-module for all $0 \le i \le p, \nu \in \mathbf{Z}$. Let $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ be a morphism. Then, the following two conditions are equivalent:*

(i) *$[H^i(K^\bullet \otimes_A B)]_\nu$ is a flat B-module for all $0 \le i \le p, \nu \in \mathbf{Z}$.*

(ii) $[H^i(K^\bullet \otimes_A B)]_\nu$ *is a locally free B-module for all* $0 \le i \le p, \nu \in \mathbf{Z}$.

Proof. The implication (2) $\Rightarrow$ (1) is clear. So, we assume that each $[H^i(K^\bullet \otimes_A B)]_\nu$ is a flat B-module. As in the proof of Theorem 7.1.6, we consider the short exact sequences $0 \to H^i\big(K^\bullet \otimes_A B\big) \to C^i(K^\bullet)\otimes_A B \to L^i \to 0$ and $0 \to L^i \to K^{i+1} \otimes_A B \to C^{i+1}(K^\bullet)\otimes_A B \to 0$. Note that each $[C^p(K^\bullet) \otimes_A B]_\nu = [H^p(K^\bullet \otimes_A B)]_\nu$ is a locally free B-module since it is flat of finite presentation as a B-module. Similarly to Theorem 7.1.6, by descending induction on i, we can show that $[H^i(K^\bullet \otimes_A B)]_\nu$ and $[C^i(K^\bullet) \otimes_A B]_\nu$ are locally free B-modules for all $0 \le i \le p, \nu \in \mathbf{Z}$. ◻

The following theorem deals with the stratification of the cohomologies of bounded complexes.

Theorem 7.1.8. *Assume A is Noetherian. Let $K^\bullet : 0 \to K^0 \to K^1 \to \cdots \to K^p \to 0$ be a bounded complex of finitely generated graded R-modules and $\mathbf{h} = (h_0, \ldots, h_p) : \mathbf{Z}^{p+1} \to \mathbf{N}^{p+1}$ be a tuple of functions. Suppose that each K^i has a Hilbert function over A. Then, the functor $\mathcal{F}^{\mathbf{h}}_{K^\bullet}$ is represented by a locally closed subscheme $F^{\mathbf{h}}_{K^\bullet} \subset \operatorname{Spec}(A)$. In other words, for any morphism $g : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$, each $H^i(K^\bullet \otimes_A B)$ has a Hilbert function over B equal to h_i if and only if g can be factored as*

$$\operatorname{Spec}(B) \ \to \ F^{\mathbf{h}}_{K^\bullet} \ \to \ \operatorname{Spec}(A).$$

Proof. For any morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$, Theorem 7.1.6 implies that each $H^i(K^\bullet \otimes_A B)$ has a Hilbert function over B equal to h_i if and only if each $C^i(K^\bullet)\otimes_A B$ has a Hilbert function over B equal to h_i', where $h_i' := \sum_{j=i}^{p}(-1)^{j-i}\left(h_j + h_{K^{j+1}\otimes_A B}\right)$. Therefore, by Theorem 7.1.3, $\mathcal{F}^{\mathbf{h}}_{K^\bullet}$ is represented by the locally closed subscheme $F^{\mathbf{h}}_{K^\bullet} \subset \operatorname{Spec}(A)$ given by

$$F^{h_0'}_{C^0(K^\bullet)} \cap F^{h_1'}_{C^1(K^\bullet)} \cap \cdots \cap F^{h_p'}_{C^p(K^\bullet)}. \qquad \square$$

7.1.3 Flattening stratification of Ext modules

We now focus on a flattening stratification result for certain Ext modules. During this subsection, we shall use the following setup.

Setup 1. Let A be a Noetherian ring and R be a positively graded polynomial ring $R = A[x_1, \ldots, x_r]$ over A.

First, we recall the following result from [18].

Lemma 7.1.9. *Let M be a finitely generated graded R-module and suppose that M is a flat A-module. Let $F_\bullet : \cdots \to F_i \to \cdots \to F_1 \to F_0$ be a graded free R-resolution of M by modules of finite rank. Let*

$$D^i_M := \operatorname{Coker}\big(\operatorname{Hom}_R(F_{i-1}, R) \to \operatorname{Hom}_R(F_i, R)\big)$$

for each $i \ge 0$. Then, the following statements hold:

(i) $\mathrm{Ext}^i_R(M,R) = 0$ *for all* $i \geq r+1$.

(ii) D^i_M *is a flat* A*-module for all* $i \geq r+1$.

(iii) *If* $\mathrm{Ext}^i_R(M,R)$ *is a flat* A*-module for all* $0 \leq i \leq r$*, then*

$$\mathrm{Ext}^i_R(M,R) \otimes_A B \xrightarrow{\cong} \mathrm{Ext}^i_{R\otimes_A B}(M \otimes_A B, R \otimes_A B)$$

for all $i \geq 0$ *any* A*-algebra* B.

Proof. It follows directly from [18, Lemma 2.10]. □

For a given finitely generated graded R-module M that is A-flat and a tuple of functions $\mathbf{h} = (h_0, \ldots, h_r) : \mathbf{Z}^{r+1} \to \mathbf{N}^{r+1}$, we consider the following functor for any ring B,

$$\mathcal{FExt}^{\mathbf{h}}_M(B) := \left\{ \text{morphism } \mathrm{Spec}(B) \to \mathrm{Spec}(A) \;\middle|\; \begin{array}{l} \left[\mathrm{Ext}^i_{R\otimes_A B}(M \otimes_A B, R \otimes_A B)\right]_\nu \text{ is a locally free} \\ B\text{-module of rank } h_i(\nu) \text{ for all } 0 \leq i \leq r, \nu \in \mathbf{Z} \end{array} \right\}.$$

Note that this functor controls all the Ext modules of M because, as a consequence of Theorem 7.1.9, if M is A-flat then $\mathrm{Ext}^i_{R\otimes_A B}(M \otimes_A B, R \otimes_A B) = 0$ for all $i \geq r+1$. The next theorem provides a flattening stratification for all the Ext modules.

Theorem 7.1.10. *Let* M *be a finitely generated graded* R*-module that is a flat* A*-module, and* $\mathbf{h} = (h_0, \ldots, h_r) : \mathbf{Z}^{r+1} \to \mathbf{N}^{r+1}$ *be a tuple of functions. Then, the functor* $\mathcal{FExt}^{\mathbf{h}}_M$ *is represented by a locally closed subscheme* $\mathrm{FExt}^{\mathbf{h}}_M \subset \mathrm{Spec}(A)$*. In other words, for any morphism* $g : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$*, each* $\mathrm{Ext}^i_{R\otimes_A B}(M \otimes_A B, R \otimes_A B)$ *has a Hilbert function over* B *equal to* h_i *if and only if* g *can be factored as*

$$\mathrm{Spec}(B) \to \mathrm{FExt}^{\mathbf{h}}_M \to \mathrm{Spec}(A).$$

Proof. Let $F_\bullet : \cdots \to F_i \to \cdots \to F_1 \to F_0$ be a graded free R-resolution of M by modules of finite rank. Consider the complex $F_\bullet^{\leq r+1}$ given as the truncation $F_\bullet^{\leq r+1} : 0 \to F_{r+1} \to F_r \to \cdots \to F_1 \to F_0$, and $P^\bullet := \mathrm{Hom}_R(F_\bullet^{\leq r+1}, R)$. By Theorem 7.1.9, $D^{r+1}_M = H^{r+1}(P^\bullet) = C^{r+1}(P^\bullet)$ is a flat A-module and so each $[D^{r+1}_M]_\nu$ (being finitely presented over A) is a locally free A-module. Hence [91, Tag 00NX] implies that for all $\nu \in \mathbf{Z}$ the function $\mathrm{Spec}(A) \to \mathbf{N}$, $\boldsymbol{p} \mapsto \dim_{\kappa(\boldsymbol{p})}([D^{r+1}_M \otimes_A \kappa(\boldsymbol{p})]_\nu)$ is locally constant. As a consequence, $h_{D^{r+1}_M}$ is a constant function on each connected component of $\mathrm{Spec}(A)$.

Consider the bounded complex $K^\bullet$ given by

$$K^\bullet : \quad 0 \to P^0 \to \cdots \to P^r \to P^{r+1} \to D^{r+1}_M \to 0.$$

Note that $H^i(K^\bullet \otimes_A B) = H^i(P^\bullet \otimes_A B) \cong \mathrm{Ext}^i_{R\otimes_A B}(M \otimes_A B, R \otimes_A B)$ for all $0 \leq i \leq r$ (since M is A-flat), and that $H^{r+1}(K^\bullet \otimes_A B) = H^{r+2}(K^\bullet \otimes_A B) = 0$.

To show that the functor $\mathcal{FExt}_M^{\mathbf{h}}$ is representable, we can simply restrict $\mathrm{Spec}(A)$ to one of its connected components. Thus, we now assume that $\mathrm{Spec}(A)$ is connected, and so D_M^{r+1} has a Hilbert function over A. Let $\mathbf{h}' = (h_0, \ldots, h_r, 0, 0) : \mathbf{Z}^{r+3} \to \mathbf{N}^{r+3}$ be obtained by concatenating two zero functions $0 : \mathbf{Z} \to \mathbf{N}$ to $\mathbf{h}$. Finally, by Theorem 7.1.8, it follows that $\mathcal{FExt}_M^{\mathbf{h}}$ is represented by the locally closed subscheme $\mathrm{FExt}_M^{\mathbf{h}} := F_{K_\bullet}^{\mathbf{h}'} \subset \mathrm{Spec}(A)$. This settles the proof of the theorem. $\square$

7.1.4 Flattening stratification of local cohomology modules

Next, we provide a flattening stratification theorem for local cohomology modules. The main idea is that, by using some techniques from [14,18], we can obtain a flattening stratification of local cohomology modules from the one of Ext modules given in Theorem 7.1.10.

We start with the following lemma that gives a base change of local cohomology modules over a base which is not necessarily Noetherian.

Lemma 7.1.11. *Let A be a ring, $R = A[x_1, \ldots, x_r]$ be a positively graded polynomial ring over A, $\mathfrak{m} = (x_1, \ldots, x_r) \subset R$ be the graded irrelevant ideal, and M be a graded R-module. If M is A-flat and $\mathrm{H}_\mathfrak{m}^i(M)$ is A-flat for all $0 \le i \le r$, then $\mathrm{H}_\mathfrak{m}^i(M) \otimes_A B \xrightarrow{\cong} \mathrm{H}_\mathfrak{m}^i(M \otimes_A B)$ for all $0 \le i \le r$ and any A-algebra B.*

Proof. By using [86] and the fact that $x_1, \ldots, x_r$ is a regular sequence in R, even if A is not Noetherian, we can compute $\mathrm{H}_\mathfrak{m}^i(M)$ as the i-th cohomology of $\mathcal{C}_\mathfrak{m}^\bullet \otimes_R M$ where $\mathcal{C}_\mathfrak{m}^\bullet$ denotes the Čech complex with respect to $\mathfrak{m} = (x_1, \ldots, x_r)$. Let $L_\bullet : \cdots \to L_i \to \cdots \to L_1 \to L_0$ be a graded free R-resolution of M. By considering the spectral sequences coming from the double complex $\mathcal{C}_\mathfrak{m}^\bullet \otimes_S L_\bullet \otimes_A B$, we obtain the isomorphisms

$$\mathrm{H}_\mathfrak{m}^i(M \otimes_A B) \cong H_{r-i}\big(\mathrm{H}_\mathfrak{m}^r(L_\bullet) \otimes_A B\big)$$

for any A-algebra B and all integers i (see [14, Lemma 3.4]). By the flatness condition and standard base change results (see [14, Lemma 2.8]), we obtain

$$\mathrm{H}_\mathfrak{m}^i(M) \otimes_A B \cong H_{r-i}(\mathrm{H}_\mathfrak{m}^r(L_\bullet)) \otimes_A B \xrightarrow{\cong} H_{r-i}(\mathrm{H}_\mathfrak{m}^r(L_\bullet) \otimes_A B) \cong \mathrm{H}_\mathfrak{m}^i(M \otimes_A B),$$

and so the result follows. $\square$

The following setup is now set in place for the rest of the subsection.

Setup 2. Let A be a Noetherian ring, R be a positively graded polynomial ring $R = A[x_1, \ldots, x_r]$ over A, $\mathfrak{m} = (x_1, \ldots, x_r) \subset R$ be the graded irrelevant ideal, and $\delta := \deg(x_1) + \cdots + \deg(x_r) \in \mathbf{Z}_+$.

For a graded R-module M and a morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, we consider the graded $(R \otimes_A B)$-module $M \otimes_A B$ and we denote the *B-relative graded Matlis dual* by

$$(M \otimes_A B)^{*_B} = {}^*\mathrm{Hom}_B(M \otimes_A B, B) := \bigoplus_{\nu \in \mathbf{Z}} \mathrm{Hom}_B\left([M \otimes_A B]_{-\nu}, B\right).$$

Note that $(M \otimes_A B)^{*B}$ has a natural structure of graded $(R \otimes_A B)$-module. From the canonical perfect pairing of free A-modules in "top" local cohomology $[R]_\nu \otimes_A [\mathrm{H}^r_\mathfrak{m}(R)]_{-\delta-\nu} \to [\mathrm{H}^r_\mathfrak{m}(R)]_{-\delta} \cong A$ we obtain a canonical graded R-isomorphism $\mathrm{H}^r_\mathfrak{m}(R) \cong (R(-\delta))^{*A} = {}^*\operatorname{Hom}_A(R(-\delta), A)$. Then, for a morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ and a complex $F_\bullet : \cdots \to F_i \to \cdots \to F_1 \to F_0$ of finitely generated graded free R-modules, we obtain the isomorphisms of complexes

$$\mathrm{H}^r_\mathfrak{m}(F_\bullet \otimes_A B) \cong \mathrm{H}^r_\mathfrak{m}(F_\bullet) \otimes_A B \cong \left(\operatorname{Hom}_R(F_\bullet, R(-\delta))\right)^{*A} \otimes_A B \cong \left(\operatorname{Hom}_R(F_\bullet, R(-\delta)) \otimes_A B\right)^{*B}.$$

The next proposition gives a sort of local duality theorem (see [18, Proposition 2.11]).

Proposition 7.1.12. *Let M be a finitely generated graded R-module and suppose that M is a flat A-module. Let $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ be a morphism. Then, the following two conditions are equivalent:*

(i) $\mathrm{H}^i_\mathfrak{m}(M \otimes_A B)$ *has a Hilbert function over B for all $0 \le i \le r$.*

(ii) $\operatorname{Ext}^i_{R\otimes_A B}(M \otimes_A B, R \otimes_A B)$ *has a Hilbert function over B for all $0 \le i \le r$.*

Moreover, when any of the above equivalent conditions is satisfied, we have that

$$h_{\mathrm{H}^i_\mathfrak{m}(M\otimes_A B)}(\nu) \;=\; h_{\operatorname{Ext}^{r-i}_{R\otimes_A B}(M\otimes_A B, R\otimes_A B)}(-\nu-\delta)$$

for all $i, \nu \in \mathbf{Z}$.

Proof. Let $F_\bullet : \cdots \to F_i \to \cdots \to F_1 \to F_0$ be a graded free R-resolution of M by modules of finite rank. As M is A-flat, $F_\bullet \otimes_A B$ is a resolution of $M \otimes_A B$. Then, by using the isomorphism of complexes $\mathrm{H}^r_\mathfrak{m}(F_\bullet \otimes_A B) \cong \left(\operatorname{Hom}_R(F_\bullet, R(-\delta)) \otimes_A B\right)^{*B}$ and the same proof of [18, Proposition 2.11], we obtain that conditions (1) and (2) are equivalent, and that in the case they are satisfied, we have the isomorphism $\mathrm{H}^i_\mathfrak{m}(M \otimes_A B) \cong \left(\operatorname{Ext}^{r-i}_{R\otimes_A B}(M \otimes_A B, R(-\delta) \otimes_A B)\right)^{*B}$. ◻

For a given finitely generated graded R-module M that is A-flat and a tuple of functions $\mathbf{h} = (h_0, \ldots, h_r) : \mathbf{Z}^{r+1} \to \mathbf{N}^{r+1}$, we consider the following functor for any ring B,

$$\mathcal{FLoc}^{\mathbf{h}}_M(B) \;:=\; \left\{ \text{morphism } \operatorname{Spec}(B) \to \operatorname{Spec}(A) \;\middle|\; \begin{array}{l} \left[\mathrm{H}^i_\mathfrak{m}(M \otimes_A B)\right]_\nu \text{ is a locally free } B\text{-module} \\ \text{of rank } h_i(\nu) \text{ for all } 0 \le i \le r, \nu \in \mathbf{Z} \end{array} \right\}.$$

Finally, we have below a theorem that gives a flattening stratification for local cohomology modules.

Theorem 7.1.13. *Let M be a finitely generated graded R-module that is a flat A-module, and $\mathbf{h} = (h_0, \ldots, h_r) : \mathbf{Z}^{r+1} \to \mathbf{N}^{r+1}$ be a tuple of functions. Then, the functor $\mathcal{FLoc}^{\mathbf{h}}_M$ is represented by a locally closed subscheme $\mathrm{FLoc}^{\mathbf{h}}_M \subset \operatorname{Spec}(A)$. In other words, for any morphism $g : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$, each $\mathrm{H}^i_\mathfrak{m}(M \otimes_A B)$ has a Hilbert function over B equal to h_i if and only if g can be factored as*

$$\operatorname{Spec}(B) \;\to\; \mathrm{FLoc}^{\mathbf{h}}_M \;\to\; \operatorname{Spec}(A).$$

Proof. Let $\mathbf{h}' = (h'_0, \ldots, h'_r) : \mathbf{Z}^{r+1} \to \mathbf{N}^{r+1}$ be a tuple of functions defined by $h'_i(\nu) := h_{r-i}(-\nu - \delta)$. So, it follows directly from Theorem 7.1.12 and Theorem 7.1.10 that $\mathcal{FLoc}_M^{\mathbf{h}}$ is represented by the locally closed subscheme $\mathrm{FLoc}_M^{\mathbf{h}} := \mathrm{FExt}_M^{\mathbf{h}'} \subset \mathrm{Spec}(A)$. □

7.2 Flattening stratification of the higher direct images of a sheaf and its twistings

In this section, we provide a flattening stratification theorem that deals with all the direct images of a sheaf and its possible twistings. This result is the core of our approach to show that the fiber-full scheme exists.

For completeness, we start with a base change result which is probably well-known to the experts, but we could not find it in the generality we need (cf. [43, Lemma 4.1]). Let S be a scheme and $f : X \subset \mathbf{P}_S^r \to S$ be a projective morphism. Let $g : T \to S$ be a morphism of schemes and $t \in T$ be a point. We use the notation $X_T := X \times_S T$, $f_{(T)} := f \times_S T : X_T \to T$, $X_t := X_T \times_T \mathrm{Spec}(\kappa(t))$ and $f_{(t)} := f_{(T)} \times_T \mathrm{Spec}(\kappa(t)) : X_t \to \mathrm{Spec}(\kappa(t))$, and we consider the commutative diagram

$$\begin{tikzcd}
X_t = X_T \times_T \mathrm{Spec}(\kappa(t)) \arrow[r, "1 \times_T \iota_t"] \arrow[d, "f_{(t)}"'] & X_T = X \times_S T \arrow[r, "1 \times_S g"] \arrow[d, "f_{(T)}"] & X = X \times_S S \arrow[d, "f"] \\
\mathrm{Spec}(\kappa(t)) \arrow[r, "\iota_t"] & T \arrow[r, "g"] & S.
\end{tikzcd}$$

For a quasi-coherent sheaf $\mathscr{F}$ on X, let $\mathscr{F}_T := (1 \times_S g)^* \mathscr{F}$ be the sheaf on X_T obtained by the pull-back induced by g and $\mathscr{F}_t := (1 \times_T \iota_t)^* \mathscr{F}_T$ be the sheaf on X_t obtained by taking the fiber over t. Recall that in this setting, we have the base change map $g^* R^i f_* \mathscr{F} \to R^i f_{(T)_*}(\mathscr{F}_T)$ for all $i \geq 0$.

Proposition 7.2.1. *Let S be a scheme, $f : X \subset \mathbf{P}_S^r \to S$ be a projective morphism and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Suppose that $R^i f_*(\mathscr{F}(\nu))$ is a flat $\mathscr{O}_S$-module for all $0 \leq i \leq r, \nu \in \mathbf{Z}$. Let $g : T \to S$ be a morphism of schemes. Then, $\mathscr{F}$ is flat over S and we have a base change isomorphism*

$$g^* R^i f_*(\mathscr{F}(\nu)) \xrightarrow{\cong} R^i f_{(T)_*}(\mathscr{F}_T(\nu))$$

for all $0 \leq i \leq r, \nu \in \mathbf{Z}$.

Proof. Since the first consequence is local on S and the second one is local on T, we may assume that $T = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ are affine schemes. Then, we have the identifications

$$R^i f_*(\mathscr{F}(\nu)) \cong H^i(X, \mathscr{F}(\nu))^{\sim} \cong H^i(\mathbf{P}_A^r, \mathscr{F}(\nu))^{\sim}$$

and
$$R^i f_{(T)_*}(\mathscr{F}_T(\nu)) \cong H^i(X_T, \mathscr{F}_T(\nu))^\sim \cong H^i(\mathbf{P}^r_B, \mathscr{F}_T(\nu))^\sim$$
(see [91, Tag 01XK], [47, Proposition 8.5]). Let $R := A[x_0, \ldots, x_r]$ with $\mathbf{P}^r_A = \mathrm{Proj}(R)$, $\mathfrak{m} = (x_0, \ldots, x_r)$, and M be the graded R-module given by $M := \bigoplus_{\nu \in \mathbf{Z}} H^0(\mathbf{P}^r_A, \mathscr{F}(\nu))$. Note that $\mathscr{F} \cong M^\sim$ and $\mathscr{F}_T \cong (M \otimes_A B)^\sim$. Thus, it is clear that $\mathscr{F}$ is flat over S. We have the exact sequence

$$0 \to \mathrm{H}^0_\mathfrak{m}(M \otimes_A B) \to M \otimes_A B \to \bigoplus_{\nu \in \mathbf{Z}} H^0(\mathbf{P}^r_B, \mathscr{F}_T(\nu)) \to \mathrm{H}^1_\mathfrak{m}(M \otimes_A B) \to 0$$

and the isomorphism $\mathrm{H}^{i+1}_\mathfrak{m}(M \otimes_A B) \cong \bigoplus_{\nu \in \mathbf{Z}} H^i(\mathbf{P}^r_B, \mathscr{F}_T(\nu))$ for all $i \geq 1$. In the special case $B = A$, since $M = \bigoplus_{\nu \in \mathbf{Z}} H^0(\mathbf{P}^r_A, \mathscr{F}(\nu))$, we obtain that $\mathrm{H}^0_\mathfrak{m}(M) = \mathrm{H}^1_\mathfrak{m}(M) = 0$. Finally, Theorem 7.1.11 implies that $\mathrm{H}^i_\mathfrak{m}(M) \otimes_A B \xrightarrow{\cong} \mathrm{H}^i_\mathfrak{m}(M \otimes_A B)$ for all $0 \leq i \leq r+1$, and so the proof of the proposition is complete. ◻

We fix the following setup for the rest of this section.

Setup 3. Let S be a locally Noetherian scheme and $f : X \subset \mathbf{P}^r_S \to S$ be a projective morphism.

When we take the fiber $X_t = X_T \times_T \mathrm{Spec}(\kappa(t))$ of $f_{(T)}$ over $t \in T$, we get the isomorphism

$$R^i f_{(t)_*}(\mathscr{F}_t) \;\cong\; H^i(X_t, \mathscr{F}_t)^\sim$$

for all $i \geq 0$. Our main object of study is the following functor. For a given coherent sheaf $\mathscr{F}$ on X that is S-flat and a tuple of functions $\mathbf{h} = (h_0, \ldots, h_r) : \mathbf{Z}^{r+1} \to \mathbf{N}^{r+1}$, we consider the following functor for any scheme T,

$$\mathcal{FD}ir^{\mathbf{h}}_{\mathscr{F}}(T) := \left\{ \text{morphism } T \to S \;\middle|\; \begin{array}{l} R^i f_{(T)_*}(\mathscr{F}_T(\nu)) \text{ is locally free over } T \text{ and} \\ \dim_{\kappa(t)}\left(H^i(X_t, \mathscr{F}_t(\nu))\right) = h_i(\nu) \\ \text{for all } 0 \leq i \leq r, \nu \in \mathbf{Z}, t \in T \end{array} \right\}.$$

Note that, as a consequence of Theorem 7.2.1, a morphism $T \to S$ belongs to the set $\mathcal{FD}ir^{\mathbf{h}}_{\mathscr{F}}(T)$ if and only if $R^i f_{(T)_*}(\mathscr{F}_T(\nu))$ is a locally free $\mathscr{O}_T$-module of rank $h_i(\nu)$ for all $0 \leq i \leq r, \nu \in \mathbf{Z}$. The following theorem yields the representability of the functor $\mathcal{FD}ir^{\mathbf{h}}_{\mathscr{F}}$. This result will be our main tool.

Theorem 7.2.2. *Let $\mathscr{F}$ be a coherent sheaf on X that is flat over S, and $\mathbf{h} = (h_0, \ldots, h_r) : \mathbf{Z}^{r+1} \to \mathbf{N}^{r+1}$ be a tuple of functions. Then, the functor $\mathcal{FD}ir^{\mathbf{h}}_{\mathscr{F}}$ is represented by a locally closed subscheme $\mathrm{FDir}^{\mathbf{h}}_{\mathscr{F}} \subset S$. In other words, for any morphism $g : T \to S$ of schemes, each $R^i f_{(T)_*}(\mathscr{F}_T(\nu))$ is a locally free $\mathscr{O}_T$-module of rank $h_i(\nu)$ if and only if g can be factored as*

$$T \;\to\; \mathrm{FDir}^{\mathbf{h}}_{\mathscr{F}} \;\to\; S.$$

Proof. Let $S = \bigcup_{j\in J} S_j$ be an open covering of S where each S_j is a Noetherian affine scheme. Note that the functor $\mathcal{FDir}^{\mathbf{h}}_{\mathscr{F}}$ is a Zariski sheaf and it has a Zariski covering by the open subfunctors $\{\mathcal{G}_j\}_{j\in J}$ where

$$\mathscr{G}_j(T) := \left\{ \text{morphism } T \to S_j \;\middle|\; \begin{array}{l} R^i f_{(T)_*}(\mathscr{F}_T(\nu)) \text{ is locally free over } T \text{ and} \\ \dim_{\kappa(t)}\left(H^i\left(X_t, \mathscr{F}_t(\nu)\right)\right) = h_i(\nu) \\ \text{for all } 0 \le i \le r, \nu \in \mathbf{Z}, t \in T \end{array} \right\}$$

(see [36, §8.3]). Therefore, due to [36, Theorem 8.9], in order to show that $\mathcal{FDir}^{\mathbf{h}}_{\mathscr{F}}$ is representable by a locally closed subscheme of S, it suffices to show that each $\mathscr{G}_j$ is representable by a locally closed subscheme of S_j.

As a consequence of the above reductions, we assume that A is a Noetherian ring and $S = \operatorname{Spec}(A)$. Since all the conditions that we consider on $R^i f_{(T)_*}(\mathscr{F}_T(\nu))$ are local on T, we may restrict to an affine morphism $T = \operatorname{Spec}(B) \to \operatorname{Spec}(A)$, and we do so.

Let $R := A[x_0, \ldots, x_r]$ with $\mathbf{P}^r_A = \operatorname{Proj}(R)$ and $\mathfrak{m} = (x_0, \ldots, x_r) \subset R$. By known arguments, we can choose an integer $m \in \mathbf{Z}$ such that the following conditions are satisfied:

(i) $M := \bigoplus_{\nu \ge m} H^0(\mathbf{P}^r_A, \mathscr{F}(\nu))$ is a finitely generated graded R-module that is flat over A,

(ii) $M^\sim \cong \mathscr{F}$ and $(M \otimes_A B)^\sim \cong \mathscr{F}_T$,

(iii) $M \otimes_A B \cong \bigoplus_{\nu \ge m} H^0(\mathbf{P}^r_B, \mathscr{F}_T(\nu))$, and

(iv) $H^i(\mathbf{P}^r_A, \mathscr{F}(\nu)) = 0$ for all $1 \le i \le r, \nu \ge m$

(see, e.g., [47, §III.9]). Therefore, we obtain a short exact sequence

$$0 \to M \otimes_A B \to \bigoplus_{\nu \in \mathbf{Z}} H^0(\mathbf{P}^r_B, \mathscr{F}_T(\nu)) \to \mathrm{H}^1_{\mathfrak{m}}(M \otimes_A B) \to 0$$

that splits into the isomorphisms

$$M \otimes_A B \cong \bigoplus_{\nu \ge m} H^0(\mathbf{P}^r_B, \mathscr{F}_T(\nu)) \quad \text{and} \quad \bigoplus_{\nu < m} H^0(\mathbf{P}^r_B, \mathscr{F}_T(\nu)) \cong \mathrm{H}^1_{\mathfrak{m}}(M \otimes_A B),$$

and we get the isomorphism $\mathrm{H}^{i+1}_{\mathfrak{m}}(M \otimes_A B) \cong \bigoplus_{\nu \in \mathbf{Z}} H^i(\mathbf{P}^r_B, \mathscr{F}_T(\nu))$ for all $i \ge 1$.

We have obtained that $H^i(\mathbf{P}^r_B, \mathscr{F}_T(\nu))$ is a locally free B-module of rank $h_i(\nu)$ for all $i \ge 0, \nu \in \mathbf{Z}$ if and only if the following three conditions hold:

- $[M \otimes_A B]_\nu$ is a locally free B-module of rank $h_0(\nu)$ for all $\nu \ge m$,
- $[\mathrm{H}^1_{\mathfrak{m}}(M \otimes_A B)]_\nu$ is a locally free B-module of rank $h_0(\nu)$ for all $\nu < m$, and
- $[\mathrm{H}^i_{\mathfrak{m}}(M \otimes_A B)]_\nu$ is a locally free B-module of rank $h_{i-1}(\nu)$ for all $i \ge 2, \nu \in \mathbf{Z}$.

Let $h_0', h_0'' : \mathbf{Z} \to \mathbf{N}$ be the functions

$$h_0'(\nu) := \begin{cases} h_0(\nu) & \text{if } \nu < m \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad h_0''(\nu) := \begin{cases} h_0(\nu) & \text{if } \nu \geq m \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathbf{h}' : \mathbf{Z}^{r+2} \to \mathbf{N}^{r+2}$ be the tuple of functions defined by $\mathbf{h}' := (0, h_0', h_1, \ldots, h_r)$, where $0 : \mathbf{Z} \to \mathbf{N}$ denotes the zero function.

Finally, by Theorem 7.1.3 and Theorem 7.1.13, we obtain that each $H^i(\mathbf{P}_B^r, \mathscr{F}_T(\nu))$ is a locally free B-module of rank $h_i(\nu)$ if and only if the morphism $g : T = \operatorname{Spec}(B) \to S = \operatorname{Spec}(A)$ factors through the locally closed subscheme $F_M^{h_0''} \cap \mathrm{FLoc}_M^{\mathbf{h}'} \subset S = \operatorname{Spec}(A)$. This concludes the proof of the theorem. □

7.3 Fiber-full sheaves

In this short section, we introduce the notion of fiber-full sheaf that extends the concept of fiber-full modules from [18]. Let S be a locally Noetherian scheme, $f : X \subset \mathbf{P}_S^r \to S$ be a projective morphism, and $\mathscr{F}$ be a coherent sheaf on X.

Definition 7.3.1. We say that $\mathscr{F}$ is a **fiber-full sheaf over** S if $R^i f_*(\mathscr{F}(\nu))$ is locally free over S for all $0 \leq i \leq r$ and $\nu \in \mathbf{Z}$.

For every $s \in S$ and $q \geq 1$, let $g_{s,q}$ be the natural map $g_{s,q} : \operatorname{Spec}(\mathscr{O}_{S,s}/\mathfrak{m}_s^q) \to S$ where $\mathfrak{m}_s$ denotes the maximal ideal of the local ring $\mathscr{O}_{S,s}$, $X_{s,q}$ be the scheme $X_{s,q} := X \times_S \operatorname{Spec}(\mathscr{O}_{S,s}/\mathfrak{m}_s^q)$, and $\mathscr{F}_{s,q} := (1 \times_S g_{s,q})^* \mathscr{F}$ be the sheaf on $X_{s,q}$ obtained by the pull-back induced by $g_{s,q}$. For the case $q = 1$ (i.e., when we take the fiber at a point $s \in S$), we simply write $g_s = g_{s,1}$, $X_s = X_{s,1}$ and $\mathscr{F}_s = F_{s,1}$. The following theorem gives two further equivalent definitions for the notion of a fiber-full sheaf. The name "fiber-full" is inspired by condition (3) below.

Theorem 7.3.2. *Under the above notations, the following three conditions are equivalent:*

(i) *$\mathscr{F}$ is a fiber-full sheaf over S.*

(ii) *$\mathscr{F}$ is a locally free $\mathscr{O}_S$-module and $H^i(X_{s,q}, \mathscr{F}_{s,q}(\nu))$ is a free $\mathscr{O}_{S,s}/\mathfrak{m}_s^q$-module for all $s \in S$, $0 \leq i \leq r$, $\nu \in \mathbf{Z}$ and $q \geq 1$.*

(iii) *$\mathscr{F}$ is a locally free $\mathscr{O}_S$-module and the natural map $H^i(X_{s,q}, \mathscr{F}_{s,q}(\nu)) \to H^i(X_s, \mathscr{F}_s(\nu))$ is surjective for all $s \in S$, $0 \leq i \leq r$, $\nu \in \mathbf{Z}$ and $q \geq 1$.*

Proof. Since the three conditions are local on S, we can choose a point $s \in S$ and assume that $(B, \mathbf{b}) = (\mathscr{O}_{S,s}, \mathfrak{m}_s)$ is a Noetherian local ring and $S = \operatorname{Spec}(B)$. Moreover, in each of the three above conditions one is assuming that $\mathscr{F}$ is flat over S. Let $R := B[x_0, \ldots, x_r]$ with $\mathbf{P}_B^r = \operatorname{Proj}(R)$ and $\mathfrak{m} = (x_0, \ldots, x_r) \subset R$. Then, we can choose an integer $m \in \mathbf{Z}$ such that the following conditions are satisfied:

(i) $M := \bigoplus_{\nu \geq m} H^0(\mathbf{P}^r_B, \mathscr{F}(\nu))$ is a finitely generated graded R-module that is flat over B,

(ii) $M^\sim \cong \mathscr{F}$ and $(M \otimes_B B/\mathbf{b}^q)^\sim \cong \mathscr{F}_{s,q}$, and

(iii) $M \otimes_B B/\mathbf{b}^q \cong \bigoplus_{\nu \geq m} H^0(\mathbf{P}^r_{B/\mathbf{b}^q}, \mathscr{F}_{s,q}(\nu))$.

Similar to the proof of Theorem 7.2.2, by using the relations between local and sheaf cohomologies, the equivalence of the three conditions follows directly from [18, Theorem A]. □

7.4 Construction of the fiber-full scheme

In this section, we construct the *fiber-full scheme* which can be seen as a parameter space that generalizes the Hilbert and Quot schemes and that controls all the cohomological data instead of just the corresponding Hilbert polynomial. We also construct open subschemes of the fiber-full scheme that parametrize arithmetically Cohen-Macaulay and arithmetically Gorenstein schemes.

Let S be a locally Noetherian scheme, $f : X \subset \mathbf{P}^r_S \rightarrow S$ be a projective morphism, and $\mathscr{F}$ be a coherent sheaf on X. We define the **fiber-full functor** which for an S-scheme T parametrizes all coherent quotients $\mathscr{F}_T \twoheadrightarrow \mathscr{G}$ such that all higher direct images of $\mathscr{G}$ and its twistings are locally over T. That is, we define the following map for any (locally Noetherian) S-scheme T:

$$\mathcal{Fib}_{\mathscr{F}/X/S}(T) \;:=\; \left\{ \text{coherent quotient } \mathscr{F}_T \twoheadrightarrow \mathscr{G} \;\middle|\; \begin{array}{l} R^i f_{(T)_*}(\mathscr{G}(\nu)) \text{ is locally free over } T \\ \text{for all } 0 \leq i \leq r, \nu \in \mathbf{Z} \end{array} \right\}.$$

One important basic thing about this map is the next lemma, which tells us that

$$\mathcal{Fib}_{\mathscr{F}/X/S} \;:\; (\mathrm{Sch}/S)^{\mathrm{opp}} \rightarrow (\mathrm{Sets})$$

is a contravariant functor from the category of (locally Noetherian) S-schemes to the category of sets.

Lemma 7.4.1. *Let $g : T' \rightarrow T$ be morphism of (locally Noetherian) S-schemes. Then, we have a natural map*

$$\mathcal{Fib}_{\mathscr{F}/X/S}(g) \;:\; \mathcal{Fib}_{\mathscr{F}/X/S}(T) \rightarrow \mathcal{Fib}_{\mathscr{F}/X/S}(T'), \qquad \mathscr{G} \mapsto (1 \times_T g)^* \mathscr{G},$$

where $(1 \times_T g)^* \mathscr{G}$ is the sheaf on $X_{T'}$ obtained by the pull-back induced by g.

Proof. This is a direct consequence of Theorem 7.2.1. □

We now stratify this functor in terms of "Hilbert functions" for all the cohomologies. Let $\mathbf{h} = (h_0, \ldots, h_r) : \mathbf{Z}^{r+1} \to \mathbf{N}^{r+1}$ be a tuple of functions. Then, we define the following functor depending on $\mathbf{h}$:

$$\mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S}(T) := \left\{ \mathscr{G} \in \mathcal{F}ib_{\mathscr{F}/X/S}(T) \;\middle|\; \begin{array}{l} \dim_{\kappa(t)} \left(H^i\left(X_t, \mathscr{G}_t(\nu)\right)\right) = h_i(\nu) \\ \text{for all } 0 \le i \le r, \nu \in \mathbf{Z}, t \in T \end{array} \right\}.$$

The idea of this functor is to measure the dimension of *all cohomologies of all possible twistings*. Of course, we obtain the following stratification

$$\mathcal{F}ib_{\mathscr{F}/X/S}(T) = \bigsqcup_{\mathbf{h}:\mathbf{Z}^{r+1} \to \mathbf{N}^{r+1}} \mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S}(T)$$

when T is connected. Therefore, $\mathcal{F}ib_{\mathscr{F}/X/S}(T)$ is a representable functor if all the functors $\mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S}(T)$ are representable. When $\mathscr{F} = \mathscr{O}_X$, we simplify the notation by writing $\mathcal{F}ib^{\mathbf{h}}_{X/S}$, and we obtain the following alternative description of significant interest

$$\mathcal{F}ib^{\mathbf{h}}_{X/S}(T) := \left\{ \text{closed subscheme } Z \subset X_T \;\middle|\; \begin{array}{l} R^i f_{(T)_*}\left(\mathscr{O}_Z(\nu)\right) \text{ is locally free over } T \text{ and} \\ \dim_{\kappa(t)} \left(H^i\left(Z_t, \mathscr{O}_{Z_t}(\nu)\right)\right) = h_i(\nu) \\ \text{for all } 0 \le i \le r, \nu \in \mathbf{Z}, t \in T \end{array} \right\}.$$

These functors should be thought of as a refinement of the Hilbert and Quot functors in the following sense.

Remark 7.4.2. Let $\mathbf{h} = (h_0, \ldots, h_r) : \mathbf{Z}^{r+1} \to \mathbf{N}^{r+1}$ be a tuple of functions and suppose that $P_{\mathbf{h}} := \sum_{i=0}^{r} (-1)^i h_i \in \mathbf{Q}[t]$ is a numerical polynomial. Then, we automatically obtain the following inclusions

$$\mathcal{F}ib^{\mathbf{h}}_{X/S}(T) \subset \mathcal{H}ilb^{P_{\mathbf{h}}}_{X/S}(T) \quad \text{and} \quad \mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S}(T) \subset \mathcal{Q}uot^{P_{\mathbf{h}}}_{\mathscr{F}/X/S}(T).$$

We say that $P_{\mathbf{h}}$ is the Hilbert polynomial corresponding with the prescribed "Hilbert functions" $\mathbf{h} : \mathbf{Z}^{r+1} \to \mathbf{N}^{r+1}$ of cohomologies. Note that if the function $P_{\mathbf{h}} = \sum_{i=0}^{r} (-1)^i h_i$ does not coincide with a numerical polynomial then $\mathcal{F}ib^{\mathbf{h}}_{X/S}(T) = \emptyset$ for all S-schemes T.

Our main result is the following theorem which says that the functor $\mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S}$ is represented by a quasi-projective S-scheme.

Theorem 7.4.3. *Let* $\mathbf{h} = (h_0, \ldots, h_r) : \mathbf{Z}^{r+1} \to \mathbf{N}^{r+1}$ *be a tuple of functions and suppose that* $P_{\mathbf{h}}(t) \in \mathbf{Q}[t]$ *is a numerical polynomial. Then, there is a quasi-projective* S*-scheme* $\mathrm{Fib}^{\mathbf{h}}_{\mathscr{F}/X/S}$ *that represents the functor* $\mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S}$ *and that is a locally closed subscheme of the Quot scheme* $\mathrm{Quot}^{P_{\mathbf{h}}}_{\mathscr{F}/X/S}$.

Proof. By Theorem 7.4.2, there is an injective morphism of functors

$$\Phi : \mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S} \to \mathcal{Q}uot^{P_{\mathbf{h}}}_{\mathscr{F}/X/S}.$$

We shall show that $\mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S}$ is a locally closed subfunctor of $\mathcal{Q}uot^{P_{\mathbf{h}}}_{\mathscr{F}/X/S}$. By the existence of the Quot scheme [3,39], the functor $\mathcal{Q}uot^{P_{\mathbf{h}}}_{\mathscr{F}/X/S}$ is represented by a projective S-scheme $\mathrm{Quot}^{P_{\mathbf{h}}}_{\mathscr{F}/X/S}$ and a universal quotient $\mathscr{F}_{\mathrm{Quot}^{P_{\mathbf{h}}}_{\mathscr{F}/X/S}} \twoheadrightarrow \mathcal{W}^{P_{\mathbf{h}}}_{\mathscr{F}/X/S}$ in $\mathcal{Q}uot^{P_{\mathbf{h}}}_{\mathscr{F}/X/S}\left(\mathrm{Quot}^{P_{\mathbf{h}}}_{\mathscr{F}/X/S}\right)$. Let $Q := \mathrm{Quot}^{P_{\mathbf{h}}}_{\mathscr{F}/X/S}$ and $\mathcal{W} := \mathcal{W}^{P_{\mathbf{h}}}_{\mathscr{F}/X/S}$. Thus, for each S-scheme T and for each quotient $\mathscr{F}_T \twoheadrightarrow \mathscr{G}$ in $\mathcal{Q}uot^{P_{\mathbf{h}}}_{\mathscr{F}/X/S}(T)$, there is a unique classifying S-morphism $g_{T,\mathscr{G}} : T \to Q$ such that $\mathscr{G} = (1 \times_S g_{T,G})^* \mathcal{W}$.

By using Theorem 7.2.2, let $\mathrm{Fib}^{\mathbf{h}}_{\mathscr{F}/X/S} := \mathrm{FDir}^{\mathbf{h}}_{\mathcal{W}} \subset Q$ be the locally closed subscheme of Q that represents the functor $\mathcal{FD}ir^{\mathbf{h}}_{\mathcal{W}}$. So, it follows that a quotient in $\mathscr{F}_T \twoheadrightarrow \mathscr{G}$ in $\mathcal{Q}uot^{P_{\mathbf{h}}}_{\mathscr{F}/X/S}(T)$ belongs to $\mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S}(T)$ if and only if $g_{T,\mathscr{G}}$ factors through $\mathrm{Fib}^{\mathbf{h}}_{\mathscr{F}/X/S}$. Finally, this shows that the functor $\mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S}$ is represented by the S-scheme $\mathrm{Fib}^{\mathbf{h}}_{\mathscr{F}/X/S}$ and by the universal quotient $\mathscr{F}_{\mathrm{Fib}^{\mathbf{h}}_{\mathscr{F}/X/S}} \twoheadrightarrow (1 \times_S \iota)^* \mathcal{W}$ in $\mathcal{F}ib^{\mathbf{h}}_{\mathscr{F}/X/S}\left(\mathrm{Fib}^{\mathbf{h}}_{\mathscr{F}/X/S}\right)$, where $\iota : \mathrm{Fib}^{\mathbf{h}}_{\mathscr{F}/X/S} \hookrightarrow Q$ denotes the natural locally closed immersion. Since Q is a projective S-scheme, we obtain that $\mathrm{Fib}^{\mathbf{h}}_{\mathscr{F}/X/S}$ is a quasi-projective S-scheme. ◻

Remark 7.4.4. When the base scheme S is well understood, we may simply write the fiber-full schemes as $\mathrm{Fib}^{\mathbf{h}}_{\mathscr{F}/X}$ and $\mathrm{Fib}^{\mathbf{h}}_{X}$ instead of $\mathrm{Fib}^{\mathbf{h}}_{\mathscr{F}/X/S}$ and $\mathrm{Fib}^{\mathbf{h}}_{X/S}$, respectively.

Remark 7.4.5. Since the dimensions of the cohomology groups can jump in flat families, the fiber-full scheme is usually not projective [47, Example III.12.9.2].

We now recall the following notions.

Definition 7.4.6. Let $\mathbf{k}$ be a field and $Y \subset \mathbf{P}^r_{\mathbf{k}}$ be a closed subscheme. Let R_Y be the homogeneous coordinate ring of Y. We say that Y is **arithmetically Cohen-Macaulay** (ACM for short) if R_Y is a Cohen-Macaulay ring. If R_Y is a Gorenstein ring then Y is said to be **arithmetically Gorenstein** (AG for short).

Next, we show the existence of open subschemes of the fiber-full scheme that parametrize ACM and AG schemes. Recall that a closed subscheme $Y \subset \mathbf{P}^r_{\mathbf{k}}$ is ACM if and only if the following two conditions are satisfied:

(i) $H^i(Y, \mathscr{O}_Y(\nu)) = 0$ for all $1 \le i \le \dim(Y) - 1$ and $\nu \in \mathbf{Z}$, and

(ii) the natural map $R_Y \to \bigoplus_{\nu \in \mathbf{Z}} H^0(Y, \mathscr{O}_Y(\nu))$ is bijective if $\dim(Y) > 0$, or injective if $\dim(Y) = 0$.

Let $d \in \mathbf{N}$ and $h_0, h_d : \mathbf{Z} \to \mathbf{N}$ be two functions, and consider the tuple of functions $\mathbf{h} : \mathbf{Z}^{r+1} \to \mathbf{N}^{r+1}$ given by $\mathbf{h} = (h_0, 0, \ldots, 0, h_d, 0, \ldots, 0)$ where $0 : \mathbf{Z} \to \mathbf{N}$ denotes the zero function. To study ACM and AG schemes, it then becomes natural to consider the following two functors. For any (locally Noetherian) S-scheme T, we have

$$\mathcal{ACM}_{X/S}^{h_0,h_d}(T) \;:=\; \left\{ \text{closed subscheme } Z \subset X_T \;\middle|\; Z \in \mathcal{Fib}_{X/S}^{\mathbf{h}}(T) \text{ and } Z_t \text{ is ACM for all } t \in T \right\}$$

and

$$\mathcal{AG}_{X/S}^{h_0,h_d}(T) \;:=\; \left\{ \text{closed subscheme } Z \subset X_T \;\middle|\; Z \in \mathcal{Fib}_{X/S}^{\mathbf{h}}(T) \text{ and } Z_t \text{ is AG for all } t \in T \right\}.$$

Note that, by using the base change results of Theorem 7.1.9 and Theorem 7.1.11, we can immediately deduce that $\mathcal{ACM}_{X/S}^{h_0,h_d}$ and $\mathcal{AG}_{X/S}^{h_0,h_d}$ are indeed contravariant functors from the category of (locally Noetherian) S-schemes into the category of sets. The following theorem gives the representability of these two functors.

Theorem 7.4.7. *Let $d \in \mathbf{N}$ and $h_0, h_d : \mathbf{Z} \to \mathbf{N}$ be two functions, and consider the tuple of functions $\mathbf{h} = (h_0, 0, \ldots, 0, h_d, 0, \ldots, 0) : \mathbf{Z}^{r+1} \to \mathbf{N}^{r+1}$. Suppose that $P_{\mathbf{h}}(t) \in \mathbf{Q}[t]$ is a numerical polynomial. Then, there exist open S-subschemes $\mathrm{ACM}_{X/S}^{h_0,h_d}$ and $\mathrm{AG}_{X/S}^{h_0,h_d}$ of $\mathrm{Fib}_{X/S}^{\mathbf{h}}$ that represent the functors $\mathcal{ACM}_{X/S}^{h_0,h_d}$ and $\mathcal{AG}_{X/S}^{h_0,h_d}$, respectively.*

Proof. By Theorem 7.4.3, there is a pair $(\mathrm{Fib}_{\mathscr{F}/X/S}^{\mathbf{h}}, \mathcal{I})$ representing the functor $\mathcal{Fib}_{\mathscr{F}/X/S}^{\mathbf{h}}$, where $\mathrm{Fib}_{\mathscr{F}/X/S}^{\mathbf{h}}$ is fiber-full scheme and $\mathcal{I}$ is the universal ideal sheaf on $\mathbf{P}^r_{\mathrm{Fib}_{\mathscr{F}/X/S}^{\mathbf{h}}}$. Let $F := \mathrm{Fib}_{\mathscr{F}/X/S}^{\mathbf{h}}$. This means that, for each S-scheme T and for each $Z \in \mathcal{Fib}_{\mathscr{F}/X/S}^{\mathbf{h}}(T)$, there is a unique classifying S-morphism $g_{T,Z} : T \to F$ such that $\mathcal{I}_Z = (1 \times_S g_{T,Z})^*\mathcal{I}$ is the ideal sheaf on $\mathbf{P}^r_T$ that corresponds with the closed subscheme $Z \subset \mathbf{P}^r_T$.

Fix $Z \in \mathcal{Fib}_{\mathscr{F}/X/S}^{\mathbf{h}}(T)$, $g_{T,Z} : T \to F$ and $\mathcal{I}_Z = (1 \times_S g_{T,Z})^*\mathcal{I}$. Since the conditions defining the functors $\mathcal{ACM}_{X/S}^{h_0,h_d}$ and $\mathcal{AG}_{X/S}^{h_0,h_d}$ are local on T, we can restrict the morphism $g_{T,Z}$ to affine open subschemes $\mathrm{Spec}(B) \subset T$ and $\mathrm{Spec}(A) \subset F$ with A being Noetherian. So, we assume that $T = \mathrm{Spec}(B)$ and $F = \mathrm{Spec}(A)$. Let $R := A[x_0, \ldots, x_r]$ with $\mathbf{P}^r_A = \mathrm{Proj}(R)$ and $\mathfrak{m} = (x_0, \ldots, x_r) \subset R$. Let $I \subset R$ be the saturated ideal $I := \bigoplus_{\nu \in \mathbf{Z}} H^0(\mathbf{P}^r_A, \mathcal{I}(\nu))$. The saturated ideal and homogeneous coordinate ring of Z are given by $I_Z := \bigoplus_{\nu \in \mathbf{Z}} H^0(\mathbf{P}^r_B, \mathcal{I}_Z(\nu)) \cong I \otimes_A B$ and $R_Z := B[x_0, \ldots, x_r]/I_Z \cong R/I \otimes_A B$, respectively. For all $t \in T$, let $R_t := B[x_0, \ldots, x_r] \otimes_B \kappa(t) \cong R \otimes_A \kappa(t)$ and $R_{Z,t} := R_Z \otimes_B \kappa(t) \cong R/I \otimes_A \kappa(t)$.

First, we show that $\mathcal{ACM}_{X/S}^{h_0,h_d}$ is representable. By construction, $\mathrm{H}^0_{\mathfrak{m}}(R_{Z,t}) = 0$ for all $t \in T$, and so Z_t is ACM for all $t \in T$ when $d = 0$. If $d > 0$, we have that Z_t is ACM for all $t \in T$ if and only if $\mathrm{H}^1_{\mathfrak{m}}(R_{Z,t}) = 0$ for all $t \in T$. We have that the locus $V := \{f \in F \mid \mathrm{H}^1_{\mathfrak{m}}(R/I \otimes_A \kappa(f)) = 0\}$ is an open subscheme of F. When $d > 0$, $g_{T,Z} : T = \mathrm{Spec}(B) \to F = \mathrm{Spec}(A)$ factors through V if and only if $Z \in \mathcal{ACM}_{X/S}^{h_0,h_d}(T)$. Therefore, it follows that, in both cases $d = 0$ or $d > 0$, $\mathcal{ACM}^{h_0,h_d}_{X/S}$ is represented by an open subscheme of $\mathrm{ACM}^{h_0,h_d}_{X/S} \subset F$.

We now concentrate on the representability of $\mathcal{AG}^{h_0,h_d}_{X/S}$. Since a Gorenstein ring is Cohen-Macaulay, we assume that $Z \in \mathcal{ACM}^{h_0,h_d}_{X/S}(T)$ and so $g_{T,Z}$ factors through $\mathrm{ACM}^{h_0,h_d}_{X/S} \subset F$. Therefore, as $R_{Z,t}$ is a Cohen-Macaulay ring of dimension $d+1$, it is Gorenstein if and only if its $(d+1)$-th Bass number

$$\mu_{d+1}(R_{Z,t}) \;:=\; \dim_{\kappa(t)}\left(\mathrm{Ext}^{d+1}_{R_{Z,t}}(R_{Z,t}/\mathfrak{m}R_{Z,t}, R_{Z,t})\right)$$

is equal to one (see [11, Theorem 3.2.10]). By upper semicontinuity, the locus

$$W := \left\{f \in F \mid \mu_{d+1}(R/I \otimes_A \kappa(f)) \le 1\right\}$$

is an open subscheme of F. On the other hand, if $f \in \mathrm{ACM}^{h_0,h_d}_{X/S}$, then $\mu_{d+1}(R/I \otimes_A \kappa(f)) \ge 1$. Finally, it follows that $g_{T,Z} : T = \mathrm{Spec}(B) \to F = \mathrm{Spec}(A)$ factors through $\mathrm{AG}^{h_0,h_d}_{X/S} := \mathrm{ACM}^{h_0,h_d}_{X/S} \cap W$ if and only if $Z \in \mathcal{AG}^{h_0,h_d}_{X/S}(T)$. So, the proof of the theorem is complete. $\square$

We end this section by giving two examples.

Example 7.4.8 (Points)**.** Let S be a locally Noetherian scheme and $f : X \subseteq \mathbf{P}^r_S \to S$ be a projective morphism. Let $\mathbf{h} : \mathbb{Z}^{r+1} \to \mathbf{N}^{r+1}$ be the tuple of constant functions defined by $\mathbf{h} := (c, 0, \ldots, 0)$ and let $P_{\mathbf{h}} = c$ be the associated Hilbert polynomial. For any S-scheme T and $Z \in \mathcal{Hilb}^{P_{\mathbf{h}}}_{X/S}(T)$, we have

$$\dim_{\kappa(t)} H^i(Z_t, \mathcal{O}_{Z_t}(\nu)) = \begin{cases} c & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

for all $t \in T$ and $\nu \in \mathbf{Z}$. It follows that $\mathcal{Fib}^{\mathbf{h}}_{X/S}(T) = \mathcal{Hilb}^{P_{\mathbf{h}}}_{X/S}(T)$ for all T and thus

$$\mathrm{Fib}^{\mathbf{h}}_{X/S} = \mathrm{Hilb}^{P_{\mathbf{h}}}_{X/S}.$$

In particular, $\mathrm{Fib}^{\mathbf{h}}_{\mathbf{P}^r}$ satisfies Murphy's law up to retraction for $r \ge 16$ [55, Theorem 1.3]. More generally, for any coherent sheaf $\mathscr{F}$ on X, we have $\mathrm{Fib}^{\mathbf{h}}_{\mathscr{F}/X/S} = \mathrm{Quot}^{P_{\mathbf{h}}}_{\mathscr{F}/X/S}$.

For the next two examples, let $\mathbf{k}$ be an algebraically closed field of characteristic zero.

Example 7.4.9 (Twisted cubics)**.** By the work of [79] it is known that $\mathrm{Hilb}^{3m+1}_{\mathbf{P}^3_{\mathbf{k}}} = H \cup H'$ is a union of two smooth irreducible components such that the general member of H parametrizes a twisted cubic, and the general member of H' parametrizes a plane cubic union an isolated point. It is also known that $H - H \cap H'$ is the locus of arithmetically Cohen-Macaulay curves of degree 3 and genus 0. Then we have a decomposition

$$\mathrm{Hilb}^{3m+1}_{\mathbf{P}^3_{\mathbf{k}}} = \mathrm{Fib}^{(\mathbf{h},0,0)}_{\mathbf{P}^3_{\mathbf{k}}} \sqcup \mathrm{Fib}^{(\mathbf{h}',0,0)}_{\mathbf{P}^3_{\mathbf{k}}} = (H - H \cap H') \sqcup H'$$

where $\mathbf{h} = (h_0, h_1), \mathbf{h}' = (h_0', h_1') : \mathbf{Z}^2 \to \mathbf{N}^2$ are the tuples of functions given by

$$h_0(\nu) = \begin{cases} 0 & \text{if } \nu \leq -1 \\ 3v+1 & \text{if } \nu \geq 0, \end{cases} \quad h_0'(\nu) = \begin{cases} 1 & \text{if } \nu \leq -1 \\ 2 & \text{if } \nu = 0 \\ 3v+1 & \text{if } \nu \geq 1 \end{cases} \quad \text{and} \quad \begin{array}{l} h_1(\nu) = h_0(\nu) - (3\nu+1) \\ h_1'(\nu) = h_0'(\nu) - (3\nu+1). \end{array}$$

To verify this decomposition we appeal to the classification of ideals in [79, §4]. Since

$$h^0(X, \mathscr{O}_X(\nu)) = \chi(\mathscr{O}_X(\nu)) + h^1(X, \mathscr{O}_X(\nu)) = 3\nu + 1 + h^1(X, \mathscr{O}_X(\nu))$$

for any $[X] \in \mathrm{Hilb}^{3m+1}_{\mathbf{P}^3_{\mathbf{k}}}$, it suffices to compute $h^0(X, \mathscr{O}_X(\nu))$. Any subscheme $[X] \in H - H \cap H'$ is arithmetically Cohen-Macaulay with the ideal sheaf having a resolution

$$0 \to \mathscr{O}_{\mathbf{P}^3_{\mathbf{k}}}(-3)^2 \to \mathscr{O}_{\mathbf{P}^3_{\mathbf{k}}}(-2)^3 \to \mathscr{I}_X \to 0.$$

It follows that $h^0(\mathcal{I}_X(\nu)) = 3\binom{\nu+1}{3} - 2\binom{\nu}{3}$. Using the ideal sheaf exact sequence and the fact that $h^1(\mathcal{I}_X(\nu)) = 0$ we deduce that $h^0(X, \mathscr{O}_X(\nu)) = \binom{\nu+3}{3} - 3\binom{\nu+1}{3} + 2\binom{\nu}{3} = 3\nu + 1$ for $\nu \geq 0$ and 0 otherwise, as required.

If $[X] \in H'$ then $\mathscr{I}_X = \mathscr{I}_{X'} \cap \mathscr{J}$ where X' is a plane cubic and $\mathscr{J}$ defines a, possibly embedded, 0-dimensional subscheme. Consider the exact sequence

$$0 \to \mathscr{I}_{X'}/\mathscr{I}_X \to \mathscr{O}_X \to \mathscr{O}_{X'} \to 0$$

of sheaves on X. Since $\mathscr{I}_{X'}/\mathscr{I}_X$ is 0-dimensional, we have

$$h^0(X, \mathscr{I}_{X'}/\mathscr{I}_X) = \mathrm{length}(\mathscr{I}_{X'}/\mathscr{I}_X) = (3m+1) - 3m = 1.$$

It is straightforward to show that the cohomology of a plane curve Y of degree d is given by $h^0(Y, \mathscr{O}_Y(\nu)) = \binom{\nu+2}{2} - \binom{\nu+2-d}{2}$. Thus, we deduce that $h^0(X, \mathscr{O}_X(\nu)) = h^0(X', \mathscr{O}_{X'}(\nu)) + 1 = \binom{\nu+2}{2} - \binom{\nu-1}{2} + 1$, as required.

7.5 Smooth Hilbert schemes

In this section, we study the fiber-full scheme as a subscheme of smooth Hilbert schemes, the latter were recently classified in [88]. Our main result states that if the Hilbert scheme is smooth, then it is equal to a fiber-full scheme.

Definition 7.5.1. For an integer partition λ, define the tuple of functions $\mathbf{h}_\lambda = (h_0, \ldots, h_r) : \mathbf{Z}^{r+1} \to \mathbf{N}^{r+1}$ given by $h_i(\nu) := \dim_{\mathbf{k}} \left(H^i(\mathbf{P}^r_{\mathbf{k}}, \mathscr{O}_{V(L(\lambda))}(\nu))\right)$ for all $\nu \in \mathbf{Z}$.

We begin by describing $\mathbf{h}_\lambda$ explicitly.

Lemma 7.5.2. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \neq (r+1)$ be an integer partition and $L(\lambda) = L(a_1 \ldots, a_r)$ be the associated lexicographic ideal. Then, for all $\nu \in \mathbf{Z}$ we have*

$$\dim_{\mathbf{k}} \left(H^i(\mathbf{P}^r_{\mathbf{k}}, \mathscr{O}_{V(L(\lambda))}(\nu))\right) = \begin{cases} \sum_{i=1}^n \binom{\nu+\lambda_i-i}{\nu-i+1} + \binom{a_1+\cdots+a_r-\nu-1}{1} - \binom{a_2+\cdots+a_r-\nu-1}{1} & \textit{if } i = 0 \\ \binom{a_{i+1}+\cdots+a_r-\nu-1}{i+1} - \binom{a_{i+2}+\cdots+a_r-\nu-1}{i+1} & \textit{if } i > 0. \end{cases}$$

Proof. Fix $L = L(\lambda)$. By [83, Lemma 3.2], we obtain

$$\operatorname{Ext}^i_R(R/L, R) \cong \left(R/(x_0, \ldots, x_{i-2}, x_{i-1}^{a_{r-i+1}})\right)(a_{r-i+1} + \cdots + a_r + i - 1), \quad 1 \leq i \leq r.$$

Note that a_l in the notation of [83] corresponds to a_{l+1} in our convention. Using the exact sequence

$$0 \to (R/(x_0, \ldots, x_{q-1}))(-p) \to R/(x_0, \ldots, x_{q-1}) \to R/(x_0, \ldots, x_{q-1}, x_q^p) \to 0,$$

we deduce that

$$\dim_{\mathbf{k}} \left(\left[R/(x_0, \ldots, x_{q-1}, x_q^p)\right]_\nu\right) = \binom{\nu + r - q}{r - q} - \binom{\nu - p + r - q}{r - q}.$$

Using the above formulas and the local duality theorem (see, e.g., [11, Theorem 3.6.19]), we obtain

$$\begin{aligned}
\dim_{\mathbf{k}} \left(H^i(\mathbf{P}^r_{\mathbf{k}}, \mathscr{O}_{V(L)}(\nu))\right) &= \dim_{\mathbf{k}} \left([H^{i+1}_{\mathfrak{m}}(R/L)]_\nu\right) \\
&= \dim_{\mathbf{k}} \left([\operatorname{Ext}^{r-i}_R(R/L, R)]_{-\nu-r-1}\right) \\
&= \dim_{\mathbf{k}} \left(\left[R/(x_0, \ldots, x_{r-i-2}, x_{r-i-1}^{a_{i+1}})(a_{i+1} + \cdots + a_r + r - i - 1)\right]_{-\nu-r-1}\right) \\
&= \binom{a_{i+1} + \cdots + a_r - \nu - 1}{i+1} - \binom{a_{i+2} + \cdots + a_r - \nu - 1}{i+1}.
\end{aligned}$$

for all $i > 0$. Similarly, since L is saturated, we obtain

$$\begin{aligned}
\dim_{\mathbf{k}} \left(H^0(\mathbf{P}^r_{\mathbf{k}}, \mathscr{O}_{V(L)}(\nu))\right) &= \dim_{\mathbf{k}} ([R/L]_\nu) + \dim_{\mathbf{k}} \left([H^1_{\mathfrak{m}}(R/L)]_\nu\right) \\
&= \dim_{\mathbf{k}} ([R/L]_\nu) + \dim_{\mathbf{k}} \left([\operatorname{Ext}^r_R(R/L, R)]_{-\nu-r-1}\right) \\
&= \sum_{i=1}^n \binom{\nu + \lambda_i - i}{\nu - i + 1} + \\
&\qquad \dim_{\mathbf{k}} \left(\left[R/(x_0, \ldots, x_{r-2}, x_{r-1}^{a_1})(a_1 + \cdots + a_r + r - 1)\right]_{-\nu-r-1}\right) \\
&= \sum_{i=1}^n \binom{\nu + \lambda_i - i}{\nu - i + 1} + \binom{a_1 + \cdots + a_r - \nu - 1}{1} - \binom{a_2 + \cdots + a_r - \nu - 1}{1}.
\end{aligned}$$

The formula for $\dim_{\mathbf{k}}([R/L]_\nu)$ can be found in [88, Lemma 3.3]. □

Before we can prove the main result of this section, we need a simple lemma that relates the cohomologies of $V(fI)$ to those of $V(I)$ for any subscheme $V(I) \subseteq \mathbf{P}^r_{\mathbf{k}}$ of codimension at least two.

Lemma 7.5.3. *Let $\lambda = (\underbrace{r,\dots,r}_{a_r\text{-times}},\lambda')$ be an integer partition with $a_r > 0$ and $[I] \in \mathrm{Hilb}^{P_\lambda}_{\mathbf{P}^r_{\mathbf{k}}}$. Then, we have $I = fI'$ with $[I'] \in \mathrm{Hilb}^{P_{\lambda'}}_{\mathbf{P}^r_{\mathbf{k}}}$, $\deg(f) = a_r$ and*

$$\dim_{\mathbf{k}}\left(H^i(\mathbf{P}^r_{\mathbf{k}}, \mathscr{O}_{V(I)}(\nu))\right) = \begin{cases} \dim_{\mathbf{k}}\left(H^i(\mathbf{P}^r_{\mathbf{k}}, \mathscr{O}_{V(I')}(\nu - a_r))\right) & \text{if } i \neq r-1 \\ \binom{a_r-\nu-1}{r} - \binom{-\nu-1}{r} & \text{if } i = r-1. \end{cases}$$

Proof. The first statement is Lemma 5.2.1. The second statement follows from the local duality theorem and [83, Fact 1], similar to Theorem 7.5.2. □

The next proposition provides an equality between the fiber-full scheme and the Hilbert scheme when the latter is smooth.

Proposition 7.5.4. *Let λ denote an integer partition for which $\mathrm{Hilb}^{P_\lambda}_{\mathbf{P}^r_{\mathbf{k}}}$ is smooth. Then, we have the equality*

$$\mathrm{Fib}^{\mathbf{h}_\lambda}_{\mathbf{P}^r_{\mathbf{k}}} = \mathrm{Hilb}^{P_\lambda}_{\mathbf{P}^r_{\mathbf{k}}}.$$

Proof. Since the Hilbert scheme $\mathrm{Hilb}^{P_\lambda}_{\mathbf{P}^r_{\mathbf{k}}}$ is smooth, it suffices to just check that $\mathcal{Fib}^{\mathbf{h}_\lambda}_{\mathbf{P}^r_{\mathbf{k}}}(\mathrm{Spec}(\mathbf{k})) = \mathcal{Hilb}^{P_\lambda}_{\mathbf{P}^r_{\mathbf{k}}}(\mathrm{Spec}(\mathbf{k}))$. By Theorem 2.0.27 there are seven different families of λ for which the Hilbert scheme is smooth. We can reduce to considering partitions that satisfy $a_r = 0$, i.e., Hilbert schemes parametrizing subschemes of codimension at least two. Indeed, if $a_r > 0$, Theorem 7.5.3 implies that $\mathrm{Fib}^{\mathbf{h}_\lambda}_{\mathbf{P}^r_{\mathbf{k}}} = \mathrm{Hilb}^{P_\lambda}_{\mathbf{P}^r_{\mathbf{k}}}$ if and only if $\mathrm{Fib}^{\mathbf{h}_{\lambda'}}_{\mathbf{P}^r_{\mathbf{k}}} = \mathrm{Hilb}^{P_{\lambda'}}_{\mathbf{P}^r_{\mathbf{k}}}$ where $\lambda = (\underbrace{r,\dots,r}_{a_r\text{-times}},\lambda')$. Thus, for the rest of the proof we will only study those partitions in Theorem 2.0.27 for which $a_r = 0$.

The conclusion is immediate for Case (7) as the Hilbert scheme consists of a single point. Case (1) corresponds to the Hilbert scheme of points in $\mathbf{P}^2_{\mathbf{k}}$. In this case $\lambda = (1,\dots,1)$, equivalently P_λ is constant, and this is covered by Theorem 7.4.8. Similarly, Case (6) reduces to $\lambda = (1,1,1)$ which is also covered by Theorem 7.4.8.

To deal with Case (2) and Case (3) we use the fact that they have a unique Borel-fixed point. Let λ be as in Case (2) or Case (3) and let $[I] \in \mathrm{Hilb}^{P_\lambda}_{\mathbf{P}^r_{\mathbf{k}}}$ with I saturated. Since $\mathrm{gin}(I)$ is Borel-fixed [25, Theorem 15.20], we have $\mathrm{gin}(I) = L(\lambda)$. This implies I and $L(\lambda)$ have the same Hilbert function and thus, $L(\lambda)$ is the lexicographic ideal associated to I. The result now follows from [85, Theorem 0.1]. The characteristic assumption of [85] does not pose any issue because, in our case, the generic initial ideal is strongly stable [85, proof of Theorem 0.1, page 274].

Case (4) and Case (5) correspond to Hilbert schemes with two Borel-fixed points. By Theorem 5.0.1 we have two cases

- $\lambda = ((d+1)^q, 1)$ with $d \geq 2$ and $q \geq 2$: The general member of $\mathrm{Hilb}^{P_\lambda}_{\mathbf{P}^r_{\mathbf{k}}}$ parametrizes $C \cup \{P\}$ where $C \subseteq \mathbf{P}^{d+1}_{\mathbf{k}}$ is a hypersurface of degree q and P is a point.
- $\lambda = (2^q, 1)$ with $q \geq 4$: The general member of $\mathrm{Hilb}^{P_\lambda}_{\mathbf{P}^r_{\mathbf{k}}}$ parametrizes $C \cup P$ where C is a plane curve of degree q and P is a point.

In either case, for any subscheme $[X] \in \mathrm{Hilb}^{P_\lambda}_{\mathbf{P}^r_{\mathbf{k}}}$, we have $\mathcal{I}_X = \mathcal{I}_{X'} \cap \mathcal{J}$ with $[X'] \in \mathrm{Hilb}^{P_\lambda - 1}_{\mathbf{P}^r_{\mathbf{k}}}$ and $\mathcal{J}$ defining a, possibly embedded, 0-dimensional subscheme. Arguing as in Theorem 7.4.9, we see that $\mathrm{Fib}^{P_\lambda}_{\mathbf{P}^r_{\mathbf{k}}} = \mathrm{Hilb}^{P_\lambda}_{\mathbf{P}^r_{\mathbf{k}}}$ if and only if $\mathrm{Fib}^{P_\lambda - 1}_{\mathbf{P}^r_{\mathbf{k}}} = \mathrm{Hilb}^{P_\lambda - 1}_{\mathbf{P}^r_{\mathbf{k}}}$. But the latter equality has already been established since $\lambda = ((d+1)^q)$ and $\lambda = (2^q)$, for the aforementioned d, q, have a unique Borel-fixed point. $\square$

Appendix A

Radius of the Hilbert scheme

> Many things can cause mistakes: similar symbols, sloppy handwriting, alcohol last night, teacher's advice...
>
> – Shihoko Ishi [54]

In this short appendix we give an explicit example of a Hilbert scheme whose incidence graph has radius two. The example will involve a certain Hilbert scheme of a pair of linear spaces studied in Chapter 3.

In Chapter 2 we came across the following theorem of Reeves on the radius of the Hilbert scheme

Theorem A.0.1 ([84, Theorem 7])**.** Consider the Hilbert scheme $\mathrm{Hilb}^{P}(\mathbf{P}^n)$ and let $d = \deg P$ be the dimension of the parameterized subschemes. Then the distance from any component to the lexicographic component is at most $d+1$. In particular, the radius of the Hilbert scheme is at most $d+1$.

It is natural to ask to what extent Reeves' bound on the radius is sharp. As far as we are aware, no explicit example of a Hilbert scheme with radius larger than one has appeared in the literature. It turns out that the Hilbert schemes we studied in Chapter 3 provide such an example.

Theorem A.0.2. The radius of the Hilbert scheme $\mathrm{Hilb}^{P^5_{3,3}}(\mathbf{P}^5)$ is two. Moreover, the lexicographic component is not the center of the incidence graph.

Since the lexicographic component is, in general, the best understood component, one might start by studying the components which meet the lexicographic component. However, there are two immediate obstacles. The first is that it is difficult to determine all the components of the Hilbert scheme. Secondly, it is even more difficult to prove that two components of the Hilbert scheme do not meet. Even if we succeeded in determining which components meet the lexicographic component, the lexicographic component might not be the center of the incidence graph. We overcome these problems by working with family of Hilbert schemes $\mathrm{Hilb}^{P^n_{n-2,n-2}}(\mathbf{P}^n)$ where we completely understand a component different from the lexicographic component. For simplicity, we assume $\mathbf{k}$ is an algebraically closed field of characteristic zero.

A.1 The example with radius 2

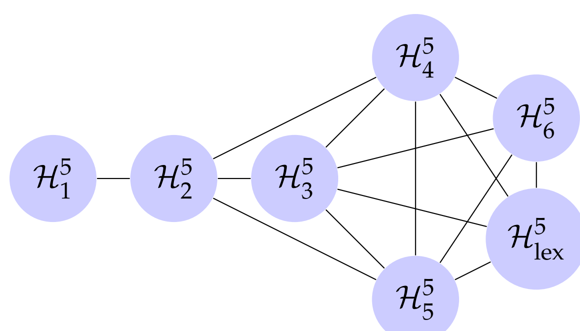
Recall from Chapter 3 that for $n \geq 3$ the Hilbert scheme

$$\mathcal{H}^n := \mathrm{Hilb}^{P^n_{n-2,n-2}}(\mathbf{P}^n)$$

has a component $\mathcal{H}^n_{n-2,n-2}$ whose general member parameterizes a pair of codimension two linear spaces meeting transversely in $\mathbf{P}^n$. For this chapter we denote this component by $\mathcal{H}^n_1$. The Hilbert scheme $\mathcal{H}^n$ has another component, denoted by $\mathcal{H}^n_2$ whose general member parameterizes $Q \cup \Lambda_{n-3}$ where Q is a quadric $(n-2)$-fold and Λ_{n-3} is a codimension three linear space such that $Q \cap \Lambda_{n-3}$ is a codimension four linear space.

Theorem A.1.1 ([16, Theorem 1.1][1]). *Let $n \geq 3$. The only component of $\mathcal{H}^n$ that $\mathcal{H}^n_1$ meets is $\mathcal{H}^n_2$.*

In the new notation, Theorem A.0.2 states that the Hilbert scheme $\mathcal{H}^5$ has radius two. With a bit more analysis, that we omit, we can describe a large portion of the incidence graph. In particular, other than the six known components of $\mathcal{H}^5$ [16, Remark 2.7] we found another component and we were able to determine how these components met one another. Moreover, we checked that all of these components are generically smooth. We believe that these are all the components, but we were unable to prove it:



Here is a description of the components appearing in the graph. For the rest of the paragraph, Λ_i will denote an i-dimensional linear space and Q will denote a quadric threefold.

[1]Our notation differs from [16]; in their paper the authors use H_n to denote the component $\mathcal{H}^n_1$.

(i) The general point of $\mathcal{H}_3^5$ parameterizes the scheme theoretic union $Q \cup \Lambda_2 \cup Z$ where Z is a double line of genus -2 embedded along Λ_2 and $Q \cap \Lambda_2$ is a conic.

(ii) The general point of $\mathcal{H}_4^5$ parameterizes $Q \cup \Lambda_2 \cup \Lambda_1$ such that Q and Λ_2 lie in a four dimensional linear subspace of $\mathbf{P}^5$, and $Q \cap \Lambda_1$ is a point.

(iii) The general point of $\mathcal{H}_5^5$ parameterizes $Q \cup \Lambda_2 \cup \Lambda_1$ such that Q and Λ_2 lie in a four dimensional linear subspace of $\mathbf{P}^5$, and $\Lambda_2 \cap \Lambda_1$ is a point.

(iv) The general point of $\mathcal{H}_6^5$ parameterizes $Q \cup \Lambda_2 \cup \Lambda_1 \cup \Lambda_0$ such that Q, Λ_2 and Λ_1 lie in a four dimensional linear subspace of $\mathbf{P}^5$, and Λ_0 is an isolated point.

(v) The general point of $\mathcal{H}_{\mathrm{lex}}^5$ parameterizes $Q \cup \Lambda_2 \cup \Lambda_1 \cup \Lambda_0 \cup \Lambda_0'$ such that Q, Λ_2 and Λ_1 lie in a four dimensional linear subspace of $\mathbf{P}^5$, $\Lambda_1 \cap \Lambda_2$ is a point, and Λ_0, Λ_0' are isolated points.

A.2 Computing the radius

Prior to analyzing $\mathcal{H}^5$ we need a sufficiently good understanding of $\mathcal{H}^4$. The general point of $\mathcal{H}_{\mathrm{lex}}^4$ parameterizes a quadric surface union a line and two isolated points, such that the line meets the quadric at two points.

Lemma A.2.1. *The Hilbert scheme $\mathcal{H}^4$ has three Borel-fixed ideals:*

$$I_1 = (x_0^2, x_0x_1, x_0x_2, x_1^2), \quad I_2 = (x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^3, x_1^2x_2), \quad I_{\mathrm{lex}} = (x_0, x_1^3, x_1^2x_2^2, x_1^2x_2x_3).$$

Moreover,

(i) *I_1 only lies in $\mathcal{H}_1^4$ and $\mathcal{H}_2^4$,*

(ii) *I_{lex} only lies in $\mathcal{H}_{\mathrm{lex}}^4$,*

(iii) *I_2 is in every component of $\mathcal{H}^4 \setminus \mathcal{H}_1^4$.*

Proof. The Borel-fixed ideals can be computed using [70, Algorithm 4.6] or using the computer algebra system *Macaulay2* [38] and the package *Strongly stable ideals* [2]. By Theorem 3.4.9, I_1 is the unique Borel-fixed ideal on $\mathcal{H}_1^4$. Since $\mathcal{H}_1^4$ meets $\mathcal{H}_2^4$ and their intersection must contain a Borel-fixed ideal, I_1 also lies in $\mathcal{H}_2^4$. Since $\mathcal{H}_1^4$ does not meet any other component (Theorem A.1.1), I_1 does not lie on any other component. We know that the lexicographic ideal I_{lex} is a smooth point and lies on its own component, $\mathcal{H}_{\mathrm{lex}}^4$. Since $\mathcal{H}^4$ is connected, every component of $\mathcal{H}^4 \setminus \mathcal{H}_1^4$ contains I_2. ◻

Proposition A.2.2. *The Hilbert scheme $\mathcal{H}^4$ has radius one while the distance between $\mathcal{H}_1^4$ and $\mathcal{H}_{\mathrm{lex}}^4$ is two.*

Proof. This is an immediate consequence of Lemma A.2.1 as every component of $\mathcal{H}^4$ meets $\mathcal{H}^4_2$ and $\mathcal{H}^4_{\mathrm{lex}}$ does not meet $\mathcal{H}^4_1$. □

This shows that even when the radius is one, the lexicographic component need not be the center of the incidence graph.

Remark A.2.3. By computing a neighbourhood of I_2 in $\mathcal{H}^4$, it can be shown that $\mathcal{H}^4_1, \mathcal{H}^4_2, \mathcal{H}^4_{\mathrm{lex}}$ are the only irreducible components of $\mathcal{H}^4$ and that $\mathcal{H}^4_2$ is smooth.

Lemma A.2.4. *The Hilbert scheme $\mathcal{H}^5$ has nine Borel-fixed ideals:*

(i) $I_1 = I_{\mathrm{lex}} = (x_0, x_1^3, x_1^2x_2^2, x_1^2x_2x_3^2, x_1^2x_2x_3x_4^2)$,

(ii) $I_2 = (x_0, x_1^3, x_1^2x_2x_3x_4, x_1^2x_2^2x_4, x_1^2x_2x_3^2, x_1^2x_2^2x_3, x_1^2x_2^3)$,

(iii) $I_3 = (x_0, x_1^4, x_1^3x_2, x_1^3x_3, x_1^3x_4, x_1^2x_2^2, x_1^2x_2x_3^2, x_1^2x_2x_3x_4)$,

(iv) $I_4 = (x_0, x_1^4, x_1^3x_2, x_1^3x_3, x_1^2x_2^2, x_1^2x_2x_3, x_1^3x_4^2)$,

(v) $I_5 = (x_0^2, x_0x_1, x_0x_2, x_0x_3, x_0x_4, x_1^3, x_1^2x_2x_3^2, x_1^2x_2x_3x_4, x_1^2x_2^2)$,

(vi) $I_6 = (x_0^2, x_0x_1, x_0x_2, x_0x_3, x_0x_4, x_1^4, x_1^3x_2, x_1^3x_3, x_1^3x_4, x_1^2x_2^2, x_1^2x_2x_3)$,

(vii) $I_7 = (x_0^2, x_0x_1, x_0x_2, x_0x_3, x_0x_4^2, x_1^3, x_1^2x_2x_3, x_1^2x_2^2)$,

(viii) $I_8 = (x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^3, x_1^2x_2)$,

(ix) $I_9 = (x_0^2, x_0x_1, x_0x_2, x_1^2)$.

Moreover, $I_1, \ldots, I_7$ are the only Borel-fixed ideals lying in the lexicographic component.

Proof. The computation of Borel-fixed ideals is similar to Lemma A.2.1. To prove the other statement we appeal to a theorem of Reeves. Given an ideal $J \subseteq S$ we define the *double saturation*, $\mathrm{sat}_{x_4,x_5}(J)$ to be the ideal obtained by setting $x_4 = 1$ and $x_5 = 1$ in J. It is shown in [84, Theorem 11] that a Borel-fixed ideal J lies in the lexicographic component if and only if $\mathrm{sat}_{x_4,x_5}(J) = \mathrm{sat}_{x_4,x_5}(I_{\mathrm{lex}})$. It is clear that the double saturation of $I_1, \ldots, I_7$ are all equal to $\mathrm{sat}_{x_4,x_5}(I_{\mathrm{lex}}) = (x_0, x_1^3, x_1^2x_2^2, x_1^2x_2x_3)$ while the double saturation of I_8 and I_9 are different. □

Notation A.2.5. Let Z_j denote the Borel-fixed points defined by the ideal I_j of Lemma A.2.4.

Lemma A.2.6. *The component $\mathcal{H}^5_2$ does not meet $\mathcal{H}^5_{\mathrm{lex}}$. Moreover, the only Borel-fixed points on $\mathcal{H}^2_5$ are Z_8 and Z_9.*

Proof. By Lemma A.2.4 it suffices to show that $\mathcal{H}_2^5$ does not contain $Z_1, \ldots, Z_7$. Assume this was not the case; then there is a flat family $\mathcal{X} \to \operatorname{Spec} \mathbf{k}[t]_{(t)}$ with generic fiber $\mathcal{X}_{\{(0)\}}$ isomorphic to a quadric threefold meeting a plane along a line and special fiber $\mathcal{X}_{\{(t)\}} = Z_i$ for some $i \leq 7$. We may choose the family so that $\mathcal{X}_{\{(0)\}}$ is transverse to the hyperplane $V(x_5)$ in $\mathbf{P}^5_{\mathbf{k}(t)}$. Since x_5 is a non-zero divisor on S/I_{Z_i}, the hyperplane section $\mathcal{X} \cap V(x_5) \to \operatorname{Spec} \mathbf{k}[t]_{(t)}$ is still flat.

Since $\mathcal{X}_{\{(0)\}} \cap V(x_5)$ is a quadric surface meeting a line at a point, $Z_i \cap V(x_5)$ must lie in the component $\mathcal{H}_2^4$. A straightforward computation shows that the (saturated) ideal of $Z_i \cap V(x_5)$ is defined by $(x_5, x_0, x_1^3, x_1^2 x_2 x_3, x_1^2 x_2^2)$. But as noted in Lemma A.2.1 (ii), this defines the lexicographic point which lies in $\mathcal{H}^4_{\text{lex}} \setminus \mathcal{H}^4_2$; a contradiction.

By Theorem 3.4.9, Z_9 is the unique Borel-fixed point in $\mathcal{H}_1^5$ and thus $Z_9 \in \mathcal{H}_1^5 \cap \mathcal{H}_2^5 \subseteq \mathcal{H}_2^5$. Since the Hilbert scheme is connected, $\mathcal{H}_2^5$ must meet a component $\mathcal{W}$ different from $\mathcal{H}_1^5$ and $\mathcal{H}^5_{\text{lex}}$. Once again using Lemma A.2.4 we see that $Z_8 \in \mathcal{H}_2^5 \cap \mathcal{W} \subseteq \mathcal{H}_2^5$. □

Proof of Theorem A.0.2. Since $\mathcal{H}_1^5$ only meets $\mathcal{H}_2^5$ (Theorem A.1.1) and $\mathcal{H}^5_{\text{lex}}$ does not meet $\mathcal{H}_2^5$ (Lemma A.2.6), the radius of $\mathcal{H}^5$ is at least two. To show that the radius of $\mathcal{H}^5$ is at most two, it is enough to establish the following two facts:

(i) The distance from $\mathcal{H}_2^5$ to $\mathcal{H}^5_{\text{lex}}$ is two,

(ii) If $\mathcal{Y}$ is a component of $\mathcal{H}^5$ that does not meet $\mathcal{H}_2^5$ then $\mathcal{W}$ meets $\mathcal{H}^5_{\text{lex}}$.

Indeed, once we know these two facts, the component connecting $\mathcal{H}_2^5$ to $\mathcal{H}^5_{\text{lex}}$ will be a center of the incidence graph. To prove (i) consider a path $\mathcal{H}_2^5 = \mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_m = \mathcal{H}^5_{\text{lex}}$ with $\mathcal{W}_i \cap \mathcal{W}_{i+1} \neq \emptyset$ and m minimal. The minimality of m implies $\mathcal{W}_3 \cap \mathcal{W}_1 = \emptyset$. Since Z_8, Z_9 lie in $\mathcal{W}_1$, the intersection $\mathcal{W}_2 \cap \mathcal{W}_3$ must contain one of $Z_1, \ldots, Z_7$. By Lemma A.2.4, $\mathcal{W}_2$ meets the lexicographic component. Thus $m = 3$ proving item (i). The proof of item (ii) is analogous. □

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That's all folks!

– Porky Pig