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A MODIFIED KHURI SERIES AND ITS CONVERGENCE FOR A SINGLE YUKAWA POTENTIAL

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### **Author**

Ahmadzadeh, Akbar.

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University of California  
Ernest O. Lawrence  
Radiation Laboratory

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ABSTRACT

The Khuri series for the partial-wave amplitude has been modified in such a way as to explicitly single out the Born term. In deriving this modified series it is shown that one needs weaker asymptotic conditions on the partial-wave amplitude than those used by Khuri. The convergence of these series has been investigated for the case of a single Yukawa potential. It is found that the modified series converges considerably faster than the Khuri series.

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I. INTRODUCTION

It has been shown by Regge<sup>1</sup> that for a superposition of Yukawa potentials of the form

$$V(r) = \int_{m_1}^{\infty} \sigma(\mu) \frac{e^{-\mu r}}{r} d\mu \quad (1)$$

the partial-wave amplitude is meromorphic in the right-half  $\lambda$ -plane and has the asymptotic form

$$A(\lambda, s) \sim C(s) e^{-\lambda \xi_1} / \sqrt{\lambda}, \quad \text{Re } \lambda > 0, \quad |\lambda| \rightarrow \infty, \quad (2)$$

where  $\lambda = l + \frac{1}{2}$ ,  $s$  is the energy, and  $\xi_1 = \cosh^{-1}(1 + \frac{m_1^2}{2s})$ ,  $m_1$  being the lower limit of the integral in Eq. (1). From this, using the Sommerfeld-Watson transformation, Regge obtained, for the scattering amplitude, the representation

$$f(s, z) = -i \int_{-i\infty}^{i\infty} \lambda d\lambda P_{\lambda - \frac{1}{2}}(-z) \frac{A(\lambda, s)}{\cos \pi \lambda} + \pi \sum_{n=1}^N \frac{2\beta_n(s) \lambda_n P_{\lambda_n - \frac{1}{2}}(-z)}{\cos \pi \lambda_n}, \quad (3)$$

where  $\beta_n$  are the residues of the poles  $A(\lambda, s)$  at  $\lambda = \lambda_n = \alpha_n + \frac{1}{2}$ . Using the above results and considering the subclass of potentials (1) for which the amplitude is also meromorphic in the left-half  $\lambda$  plane, with the additional assumption

-2-

$$A(\lambda, s) \sim C_l(s) e^{-\lambda \xi_1} / \sqrt{\lambda}, \quad \text{Re } \lambda < 0, |\lambda| \rightarrow \infty, \quad (4)$$

Khuri<sup>2</sup> has found the following expansion for the partial-wave amplitude:

$$A(\ell, s) = \sum_{\text{all poles}} \beta_n(s) \frac{e^{-(\ell - \alpha_n) \xi_1}}{\ell - \alpha_n}, \quad \text{with } \ell \text{ an integer.} \quad (5)$$

In the remainder of this section we examine certain aspects of Eq. (5). In Section II we modify this formula, starting from weaker asymptotic conditions on  $A(\lambda, s)$ . Finally, in Sec. III, we examine the rate of convergence of Eq. (5) as well as the modified expansion of Sec. II for the case of a single Yukawa potential.

Now for the sake of simplicity let us consider Eq. (5) for a single Yukawa potential,

$$V(r) = -g \frac{e^{-m_1 r}}{r}. \quad (6)$$

The ideas can be easily generalized if the potential is of the form of Eq. (1) and behaves as  $\frac{1}{r}$  near the origin. It is well known<sup>3</sup> that as  $s \rightarrow \infty$ ,  $\alpha_n \rightarrow -n$ ,  $n$  being a positive integer, which is the asymptotic solution of the Coulomb potential case. Furthermore, as  $s \rightarrow \infty$  one needs to consider only the behavior of the potential near the origin, so that the values of the residues as well as the poles for potential (6) approach that of the Coulomb case for sufficiently large  $ss$ . The residues of the partial-wave amplitude in the case of the Coulomb potential  $V = -g/r$  are<sup>4</sup>

$$\beta_n(s) = \frac{1}{2i\sqrt{s}} \frac{(-1)^n / (n-1)!}{\Gamma(-n+1 + \frac{i g}{\sqrt{s}})} \xrightarrow{s \rightarrow \infty} \frac{g}{2s}, \quad (7)$$

so that asymptotically the residues for potential (6) are

$$\beta_n(s) \xrightarrow{s \rightarrow \infty} \frac{g}{2s}. \quad (8)$$

-3-

By Assuming that for  $s \rightarrow \infty$  the series (5) reduces to the Born term

$$A(\ell, s) = \frac{g}{2s} Q_\ell \left(1 + \frac{m_1^2}{2s}\right), \quad (9)$$

Khuri<sup>2</sup> was able to find the correct asymptotic behavior of the residues [Eq. (8)]. Thus the series (5) does indeed converge to the Born term at high energies. For practical purposes, however, the series (5) is not suitable at high energies because in that case it reduces to

$$A(\ell, s) \xrightarrow{s \rightarrow \infty} \frac{g}{2s} \sum_{n=1}^{\infty} \frac{\text{Exp}[-(\ell + n) m_1 \sqrt{s}]}{\ell + n}, \quad (10)$$

and the convergence is very slow. Khuri has suggested that, in contrast to the high-energy behavior (8), at intermediate energies the residues  $\beta_n(s)$  may decrease for poles further to the left in the  $\lambda$  plane, improving the rate of convergence. Our numerical solution of the residues, utilized in testing the convergence of the series, shows that this is not the case, and at any given energy the different residues are generally of the same order of magnitude. Aside from this difficulty, it seems plausible that, for large  $|\lambda|$ ,  $A(\lambda, s)$  should approach the Born approximation, which for negative  $\lambda$  will be dominated by the largest masses in the exponential. So, for a superposition of Yukawa potentials the asymptotic condition (4), which emphasizes the longest- rather than the shortest-range component, seems to be too strong an assumption. These arguments suggest the need for a modification of the Khuri series in such a way as to single out the Born term and also deemphasize the contribution of the pole terms further out in the left half  $\lambda$  plane. In the next section we give a derivation of such a modified formula.



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## II= MODIFICATION OF THE KHURI SERIES

In this section it is assumed that the reader is familiar with Khuri's paper. For the sake of simplicity we consider a potential of the form

$$V(r) = - \sum_{i=1}^k \frac{g_i e^{-m_i r}}{r}, \quad \text{for } m_{i+1} > m_i \quad (11)$$

Instead of assumptions (2) and (4), we make the weaker assumption that

$$A(\lambda, s) = \sum_{i=1}^k \frac{g_i}{2s} Q_{\lambda-\frac{1}{2}} \left(1 + \frac{m_i^2}{2s}\right) \underset{|\lambda| \rightarrow \infty}{\sim} C(s) e^{-\lambda \xi} / \sqrt{\lambda}, \quad (12)$$

where  $\xi = \cosh^{-1} \left(1 + \frac{m^2}{2s}\right)$ ,  $m$  being arbitrary for the moment. Starting with Eq. (3) and adding and subtracting the Born term, we obtain

$$\begin{aligned} f(s, z) &= \frac{-i}{2s} \int_{-i\infty}^{i\infty} \lambda d\lambda \frac{P_{\lambda-\frac{1}{2}}(-z)}{\cos \pi \lambda} \sum_{i=1}^k g_i Q_{\lambda-\frac{1}{2}} \left(1 + \frac{m_i^2}{2s}\right) \\ &\quad - i \int_{-i\infty}^{i\infty} \lambda d\lambda \frac{P_{\lambda-\frac{1}{2}}(-z)}{\cos \pi \lambda} \left[ A(\lambda, s) - \sum_{i=1}^k \frac{g_i}{2s} Q_{\lambda-\frac{1}{2}} \left(1 + \frac{m_i^2}{2s}\right) \right] \\ &\quad + \pi \sum_{n=1}^N \frac{2\beta_n(s) P_{\lambda_n-\frac{1}{2}}(-z) \lambda_n}{\cos \pi \lambda_n} \quad (13) \end{aligned}$$

For the first integral in Eq. (13) we close the contour to the right and immediately obtain

$$f_1(s, z) = \sum_{i=1}^k \sum_{\ell=0}^{\infty} (2\ell + 1) P_{\ell}(z) \frac{g_i}{2s} Q_{\ell} \left(1 + \frac{m_i^2}{2s}\right) = \sum_{i=1}^k \frac{g_i}{m_i^2 - t}, \quad (14)$$

where  $t = -2s(1 - z)$ . For the remaining part of Eq. (13),

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$$f_2(s, z) = -i \int_{-i\infty}^{i\infty} \lambda d\lambda \frac{P_{\lambda-\frac{1}{2}}(-z)}{\cos \pi \lambda} \left[ A(\lambda, s) - \sum_{i=1}^k \frac{g_i}{2s} Q_{\lambda-\frac{1}{2}}\left(1 + \frac{m_i^2}{2s}\right) \right] \\ + \pi \sum_{n=1}^N \frac{2\beta_n \lambda_n P_{\lambda_n-\frac{1}{2}}(-z)}{\cos \pi \lambda_n}, \quad (15)$$

we follow the same procedure as Khuri. Equation (15) can be written as

$$f_2(s, z) = \frac{-i}{\pi(2)^{3/2}} \int_{-i\infty}^{i\infty} d\lambda \left[ A(\lambda, s) - \sum_{i=1}^k \frac{g_i}{2s} Q_{\lambda-\frac{1}{2}}\left(1 + \frac{m_i^2}{2s}\right) \right] \int_{\xi}^{\infty} \frac{e^{\lambda x} \sin hx dx}{(\cosh x - z)^{3/2}} \\ - \frac{i}{\pi(2)^{3/2}} \int_{-i\infty}^{i\infty} d\lambda \left[ A(\lambda, s) - \sum_{i=1}^k \frac{g_i}{2s} Q_{\lambda-\frac{1}{2}}\left(1 + \frac{m_i^2}{2s}\right) \right] \int_{-\infty}^{\xi} \frac{e^{\lambda x} \sin hx dx}{(\cosh x - z)^{3/2}} \\ + \pi \sum_{n=1}^N \frac{2\beta_n \lambda_n P_{\lambda_n-\frac{1}{2}}(-z)}{\cos \pi \lambda_n}. \quad (16)$$

For the first term on the right-hand side, we close the contour to the left in the  $\lambda$  plane, and in addition to the pole terms of  $A(\lambda, s)$  we pick up the poles of the  $Q$  functions. For the second term we close the contour to the right in the  $\lambda$  plane, and we pick up only the poles of  $A(\lambda, s)$ . The result is

$$f_2(s, z) = \frac{1}{\sqrt{2}} \sum_{\text{left poles}} \beta_n \int_{\xi}^{\infty} \frac{e^{\lambda x} \sin hx dx}{(\cosh x - z)^{3/2}} - \frac{1}{\sqrt{2}} \sum_{i=1}^k \frac{g_i}{2s} \sum_{n=1}^{\infty} P_{n-1}\left(1 + \frac{m_i^2}{2s}\right) \\ \times \int_{\xi}^{\infty} \frac{e^{-(n-\frac{1}{2})x} \sin hx dx}{(\cosh x - z)^{3/2}} - \frac{1}{\sqrt{2}} \sum_{\text{right poles}} \beta_n \int_{-\infty}^{\xi} \frac{e^{\lambda x} \sin hx dx}{(\cosh x - z)^{3/2}} \\ + \pi \sum_{n=1}^N 2\beta_n \lambda_n \frac{P_{\lambda_n-\frac{1}{2}}(-z)}{\cos \pi \lambda_n}. \quad (17)$$

The partial-wave amplitude is given by

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$$A(\ell, s) = \frac{1}{2} \int_{-1}^1 f(s, z) P_\ell(z) dz, \quad \text{with } \ell \text{ an integer,} \quad (18)$$

where  $f(s, z) = f_1(s, z) + f_2(s, z)$ . From (18), (17), and (14) we obtain

$$A(\ell, s) = \sum_{\text{all poles}} \frac{\beta_n e^{-(\ell - \alpha_n)\xi}}{\ell - \alpha_n} + \sum_{i=1}^k \frac{g_i}{2s} Q_\ell \left(1 + \frac{m_i^2}{2s}\right) - \sum_{i=1}^k \frac{g_i}{2s} \sum_{n=1}^{\infty} P_{n-1} \left(1 + \frac{m_i^2}{2s}\right) \frac{e^{-(\ell+n)\xi}}{\ell + n}. \quad (19)$$

Now let us consider Eq. (19) for the case of a single Yukawa potential

$$V(r) = -g e^{-m_1 r} / r :$$

$$A(\ell, s) = \sum_{\text{all poles}} \frac{\beta_n e^{-(\ell - \alpha_n)\xi}}{\ell - \alpha_n} - \frac{g}{2s} \sum_{n=1}^{\infty} P_{n-1} \left(1 + \frac{m_1^2}{2s}\right) \frac{e^{-(\ell+n)\xi}}{\ell + n} + \frac{g}{2s} Q_\ell \left(1 + \frac{m_1^2}{2s}\right). \quad (20)$$

For  $\xi = \xi_1$ , the last two terms exactly cancel, so that, mathematically, series (20) is identical with Khuri's series if  $\xi = \xi_1 = \cosh^{-1} \left(1 + \frac{m_1^2}{2s}\right)$ .

In deriving Eq. (20), however, we have used a weaker assumption than the one used in Khuri's paper. Two immediate advantages of Eq. (20) over the Khuri series are immediately apparent. At high energies the first two terms on the right-hand side cancel, and we simply obtain the Born term. Also, at large values of  $\ell$  the two summations are small, and the Born term stands out as it should.

There is a one-to-one correspondence between the terms in the two summations, and in practice the first  $N$  terms of each summation are used. By considering the first  $N$  terms we obtain the approximate expression

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$$A(l, s) \approx \sum_{n=1}^N \beta_n \frac{e^{-(l-\alpha_n)\xi}}{l-\alpha_n} - \frac{g}{2s} \sum_{n=1}^N P_{n-1} \left(1 + \frac{m_1^2}{2s}\right) \frac{e^{-(l+n)\xi}}{l+n} + \frac{g}{2s} Q_l \left(1 + \frac{m_1^2}{2s}\right) \quad (21)$$

Here we make the conjecture that<sup>5</sup> in Eq. (12),  $m^2 = 4m_1^2$ . For  $\text{Re } \lambda > 0$ , this seems to be correct, because once the Born term is taken out of  $f(s, t)$ , the dispersion integral<sup>6</sup> in  $t$  starts at  $t = 4m_1^2$ . The asymptotic behavior of  $A(\lambda, s)$  for  $\text{Re } \lambda < 0$ ,  $|\lambda| \rightarrow \infty$  is not known, and Eq. (12) with  $m^2 = 4m_1^2$  is the weakest asymptotic behavior that we can afford and still be correct in the right-half  $\lambda$  plane. It is seen from the numerical calculations given at the end of this paper that for a single Yukawa potential we obtain a rapidly convergent series, which supports the above conjecture. Here we would also like to remark that  $m^2 = 4m_1^2$  correctly implies that the left-hand cut in  $s$  for  $A(l, s)$  in the region  $-m_1^2 < s < -m_1^2/4$  is entirely due to the Born term.

Finally, in the case of potential (1), Eq. (19) should be generalized to

$$A(l, s) = \sum_{\text{all poles}} \beta_n \frac{e^{-(l-\alpha_n)\xi}}{l-\alpha_n} - \frac{1}{2s} \int_{m_1}^{\infty} \sigma(\mu) d\mu Q_l \left(1 + \frac{\mu^2}{2s}\right) + \frac{1}{2s} \int_{m_1}^{\infty} \sigma(\mu) l\mu \sum_{n=1}^{\infty} P_{n-1} \left(1 + \frac{\mu^2}{2s}\right) \frac{e^{-(l+n)\xi}}{l+n},$$

where  $\xi = \cosh^{-1} \left(1 + \frac{4m_1^2}{2s}\right)$  . . . . . (22)

### III. NUMERICAL CALCULATIONS

In this section we shall present the results of our numerical calculations applied to series (5) as well as to series (21). For our purposes it

is more convenient to work with the S matrix rather than with the amplitude.

Series (5) for the S matrix, taking the first N terms, is

$$S(\ell, s) = 1 + \sum_{n=1}^N \beta_n \frac{e^{-(\ell-\alpha_n)\xi_1}}{\ell - \alpha_n}, \quad (5')$$

and instead of (21) we have

$$S(\ell, s) = 1 + \sum_{n=1}^N \frac{\beta_n e^{-(\ell-\alpha_n)\xi}}{\ell - \alpha_n} - \frac{i g}{\sqrt{s}} \sum_{n=1}^N R_{n-1} \left(1 + \frac{m_1^2}{2s}\right) \frac{e^{-(\ell+n)\xi}}{\ell + n} + \frac{i g}{\sqrt{s}} Q_\ell \left(1 + \frac{m_1^2}{2s}\right), \quad \text{since } S(\ell, s) = 1 + 2 i \sqrt{s} A(\ell, s). \quad (21')$$

In Eq. (5') and (21')  $\beta_n(s)$  are now the residues of the partial-wave S matrix rather than the partial-wave amplitude.

In Figs. 11-36, a plot of the real and imaginary parts of the S matrix vs the number of terms in the expansion for the Khuri series as well as for our series (21) for both  $m = m_1$  and  $m^2 = 4m_1^2$  is given. The horizontal lines correspond to the actual values of the S matrix. The Regge parameters used in the series as well as the actual S-matrix values have been calculated by numerical integration of the Schrödinger equation.

The fact that for  $g=5$  the agreement is not quite as good as for  $g=1.8$  may be due to small errors in the residues. For stronger potentials our numerical calculation of the residues is less accurate. And for  $g=5$  it turns out that, in some cases, only a few percent error in the residues introduces a considerable error in the values of the real or imaginary parts of the S-matrix calculated from the series.

## ACKNOWLEDGMENTS

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## REFERENCES

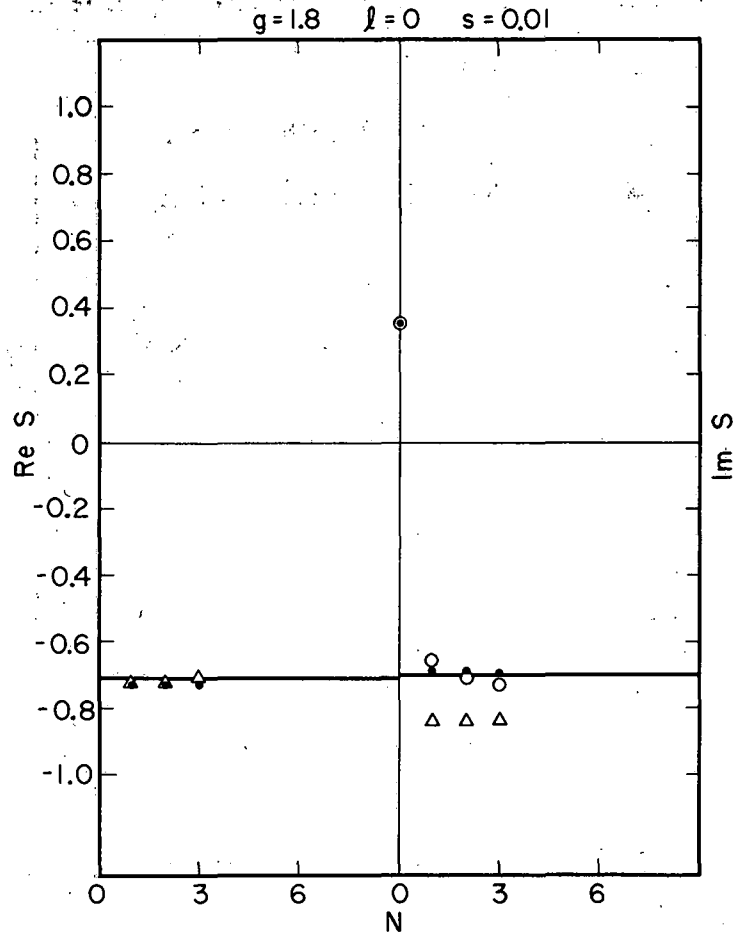
- \* Work done under the auspices of the U. S. Atomic Energy Commission.
1. A. Bottino, A. M. Longoni, and T. Regge, *Nuovo Cimento* 23, 954 (1962).
  2. N. N. Khuri, *Phys. Rev.* 130, 429 (1963).
  3. See, for example: S. Mandelstam, *Ann. Phys. (N.Y.)* 19, 254 (1962).
  4. V. Singh, *Phys. Rev.* 127, 632 (1962).
  5. Geoffrey F. Chew, (Lawrence Radiation Laboratory) Private Communication.
  6. R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, *Ann. Phys. (N.Y.)* 10, 62 (1960).

## FIGURE CAPTION

Figs. 1 through 36. Real and imaginary parts of  
S vs the number of terms in the expansion.

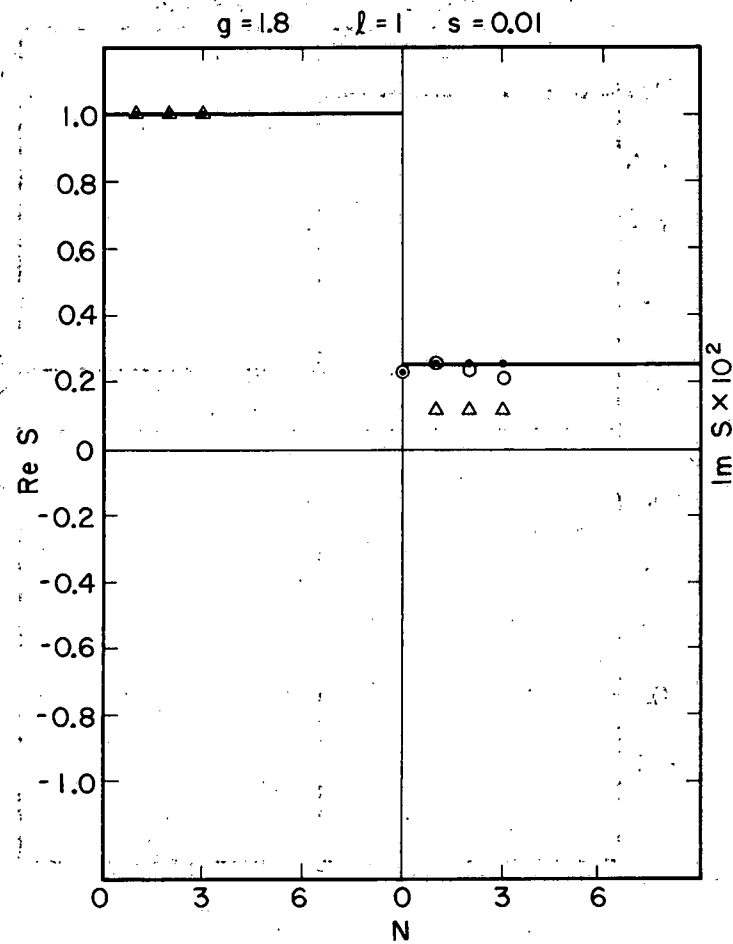
- $\Delta$  Eq. (5)
- $\circ$  Eq. (21') with  $m = m_1$
- $\bullet$  Eq. (21') with  $m = 2m_1$





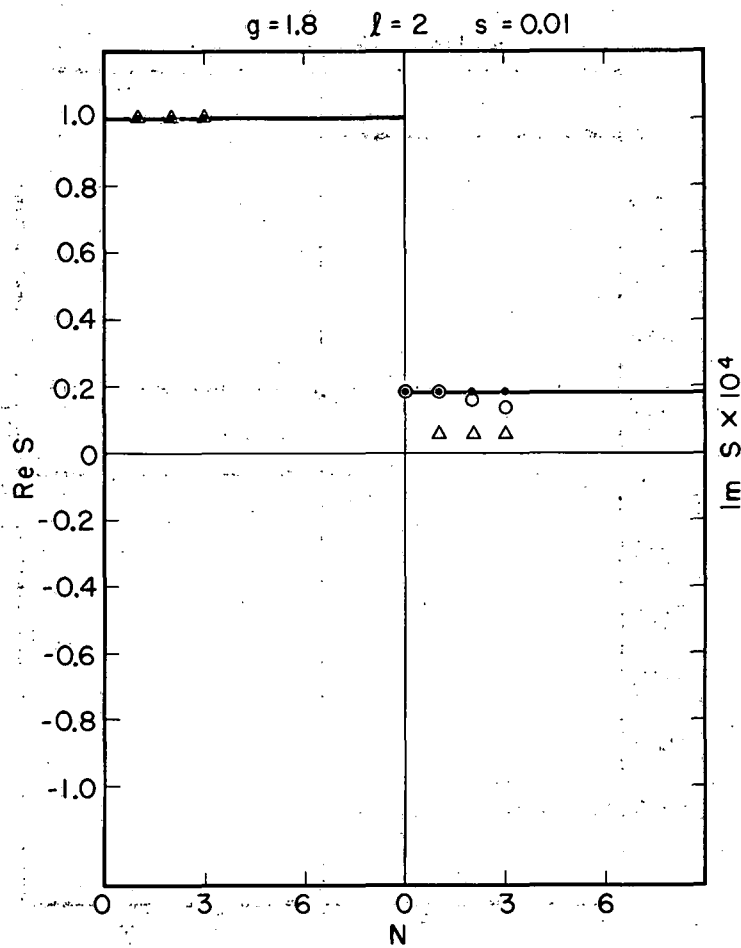
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Fig. 1



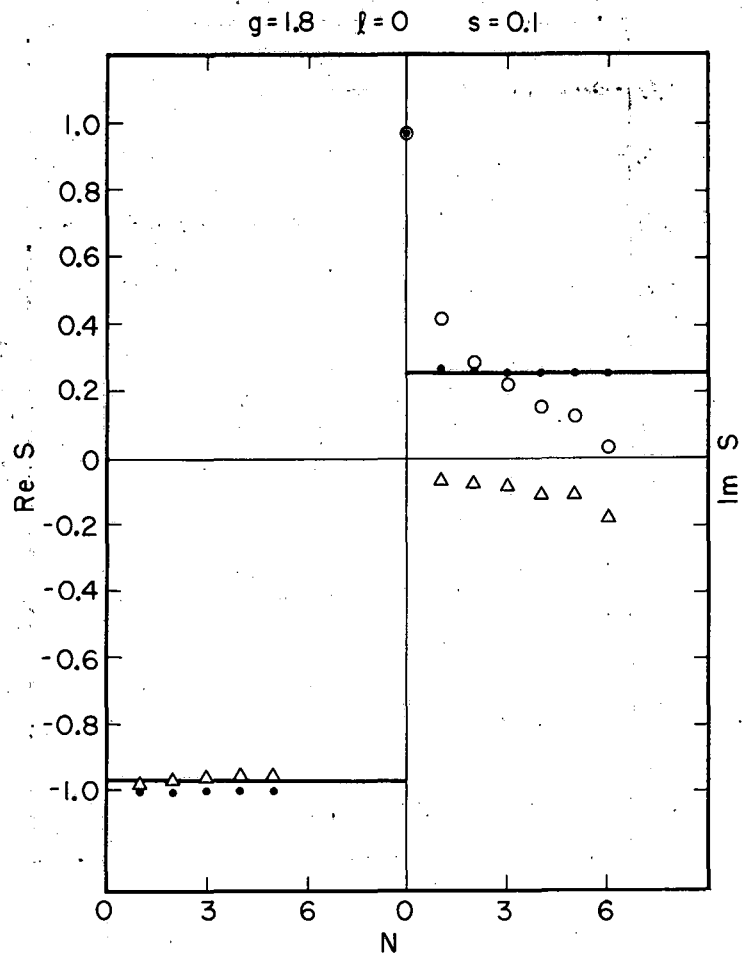
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Fig. 2



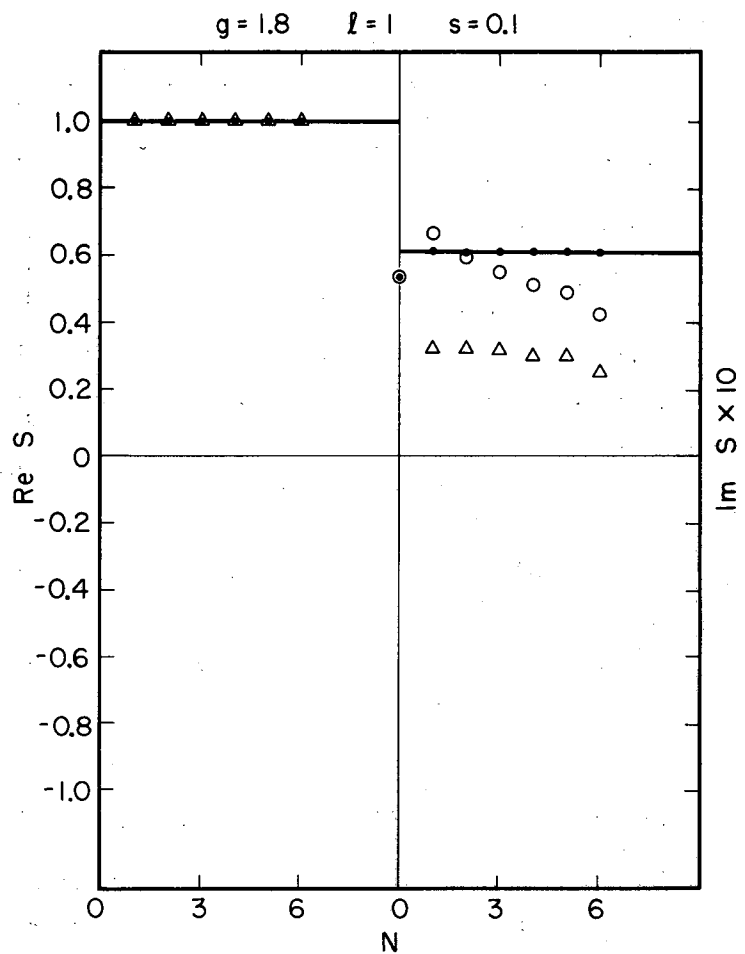
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Fig. 3



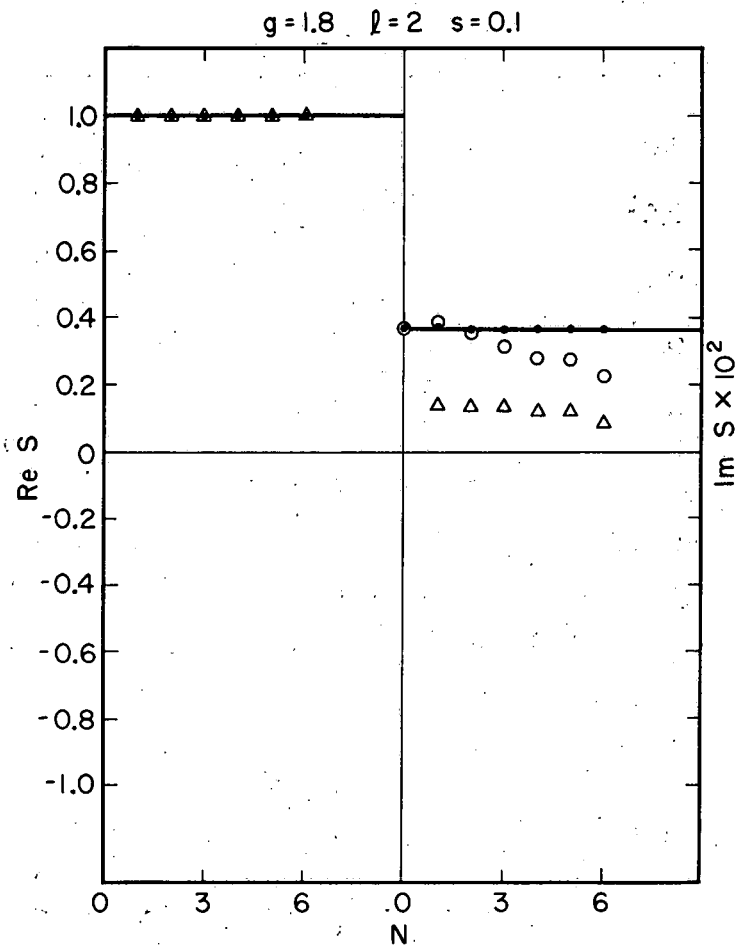
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Fig. 4



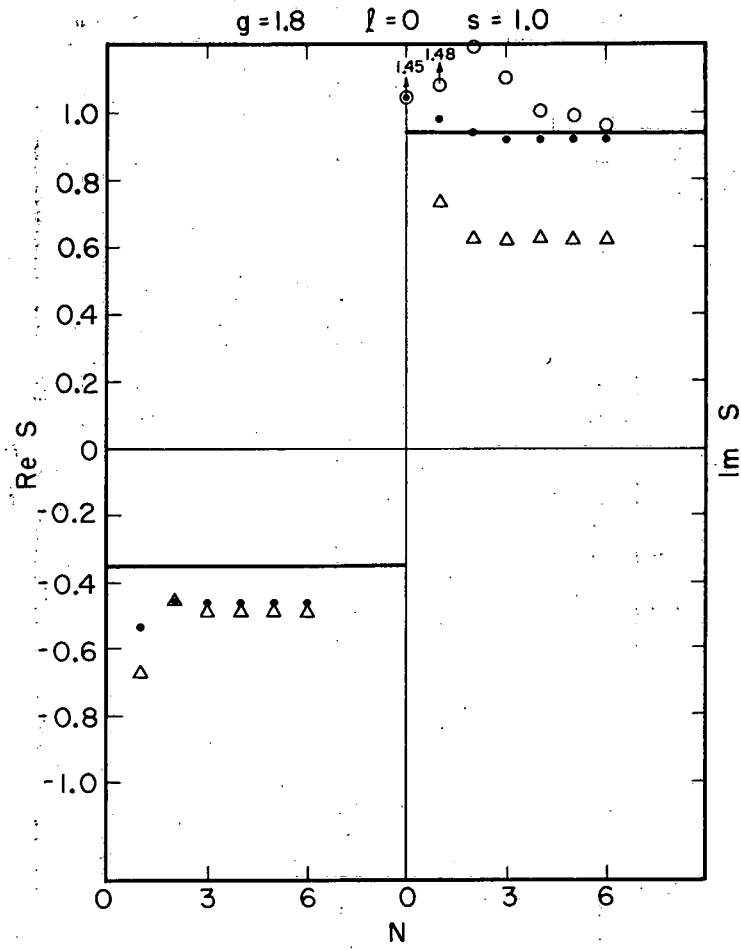
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Fig. 5



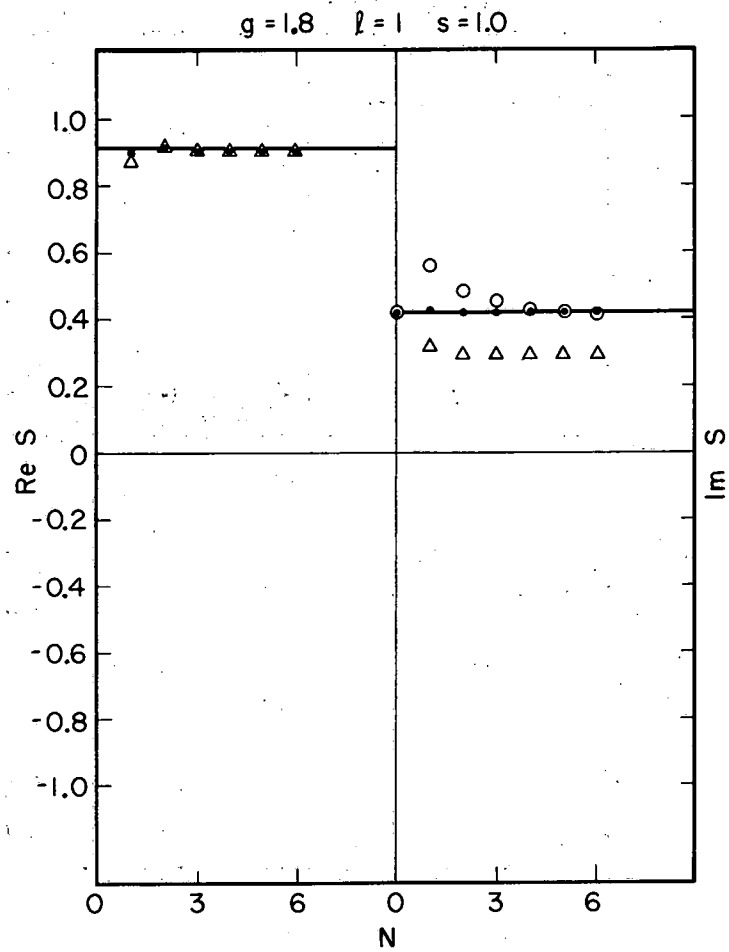
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Fig. 6



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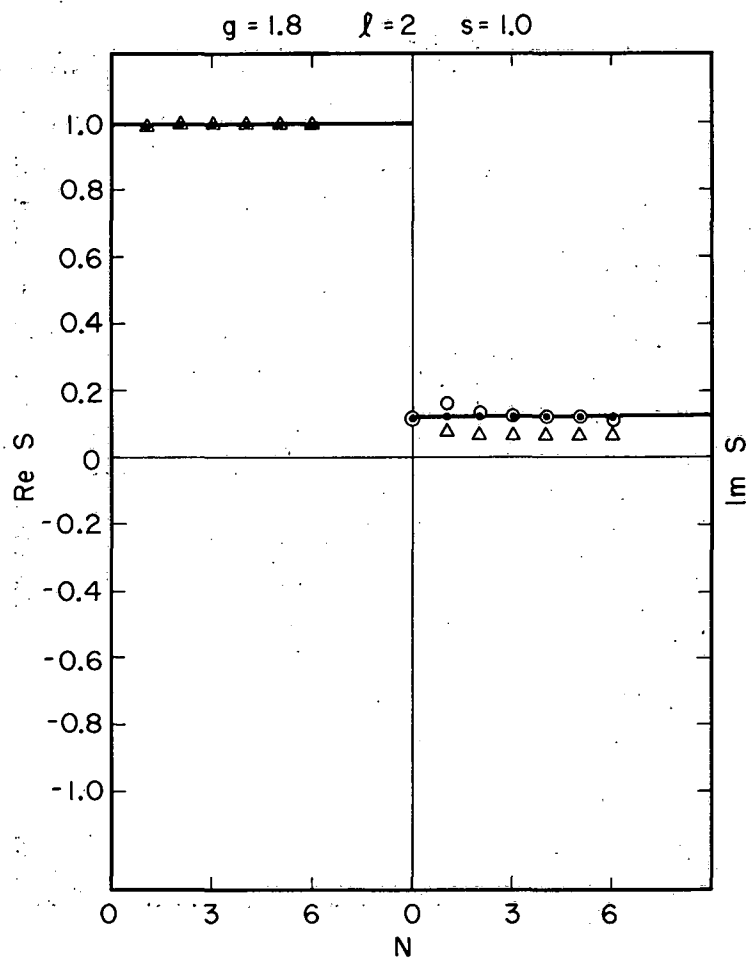
Fig. 7



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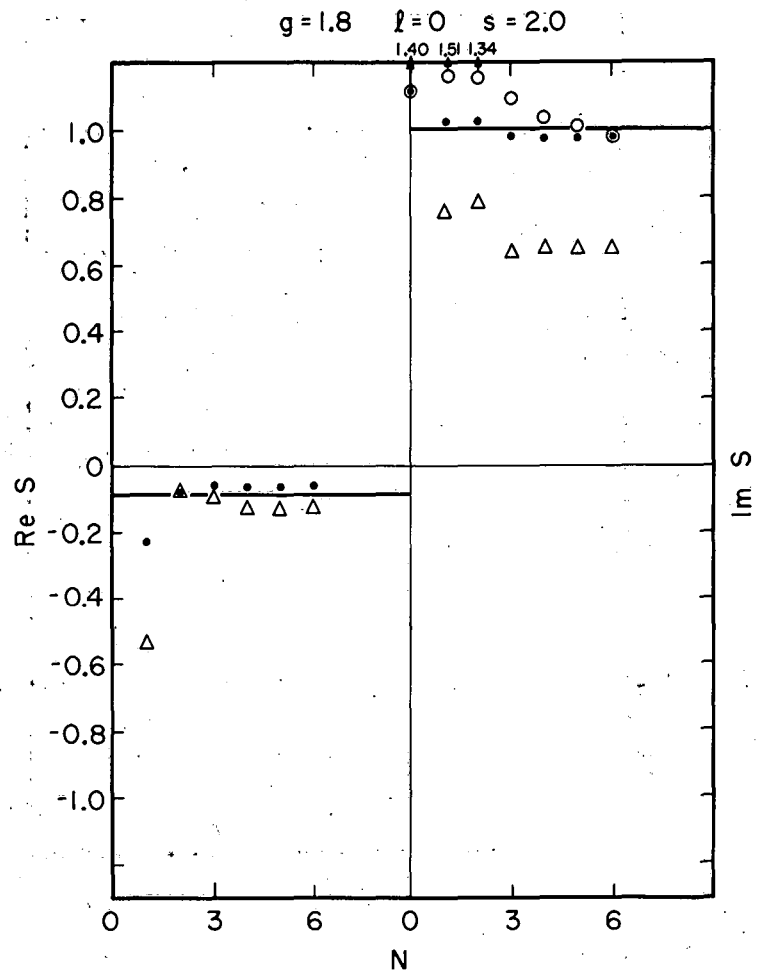
Fig. 8





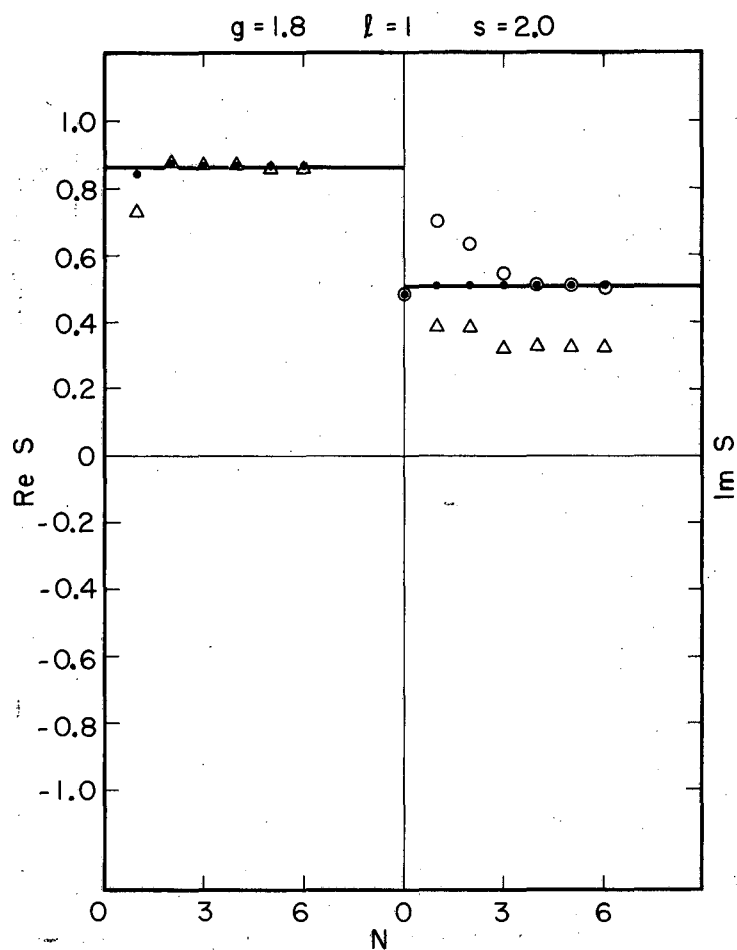
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Fig. 9



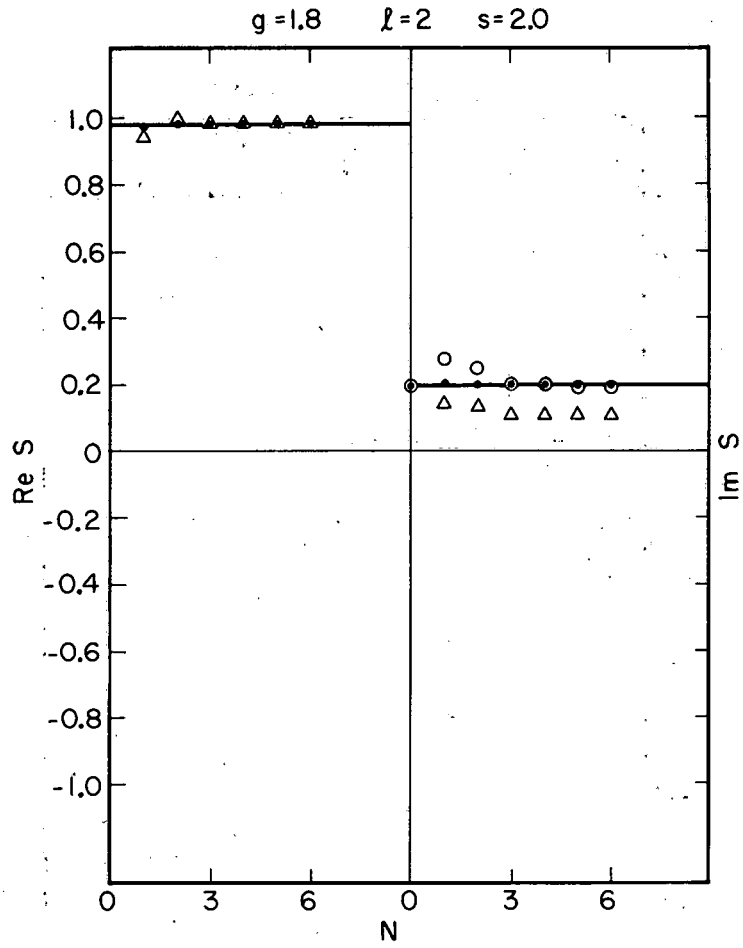
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Fig. 10



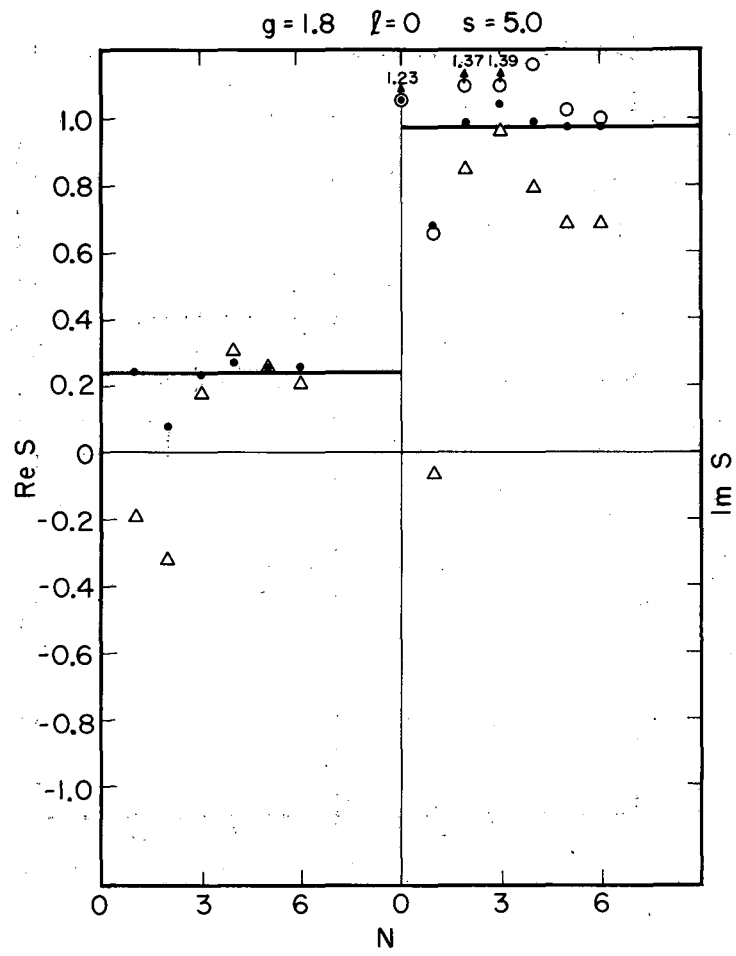
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Fig. 11



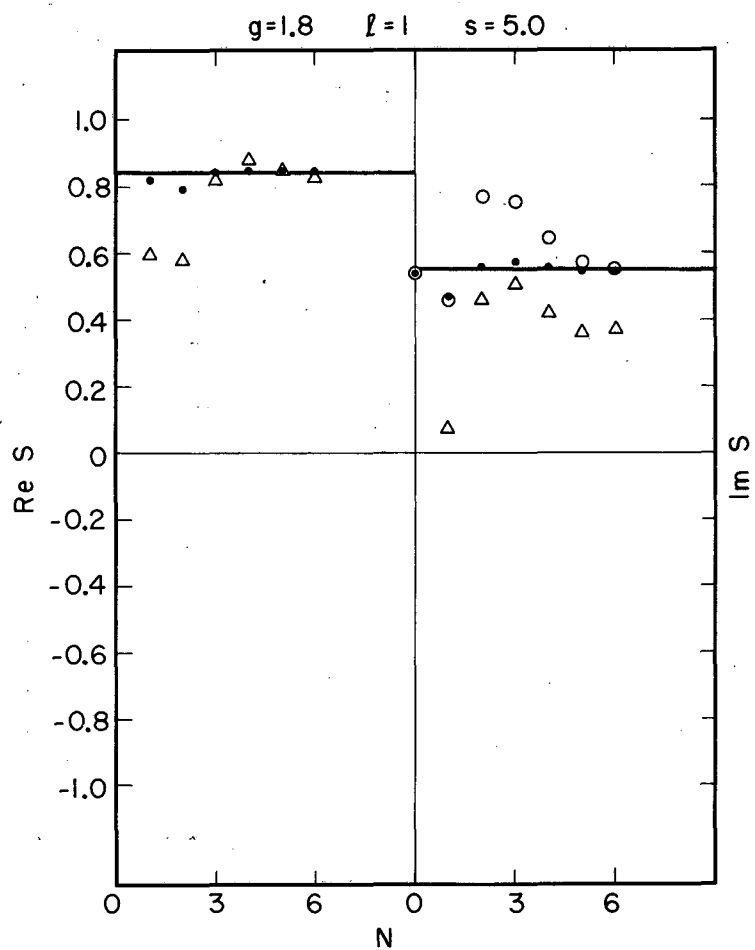
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Fig. 12



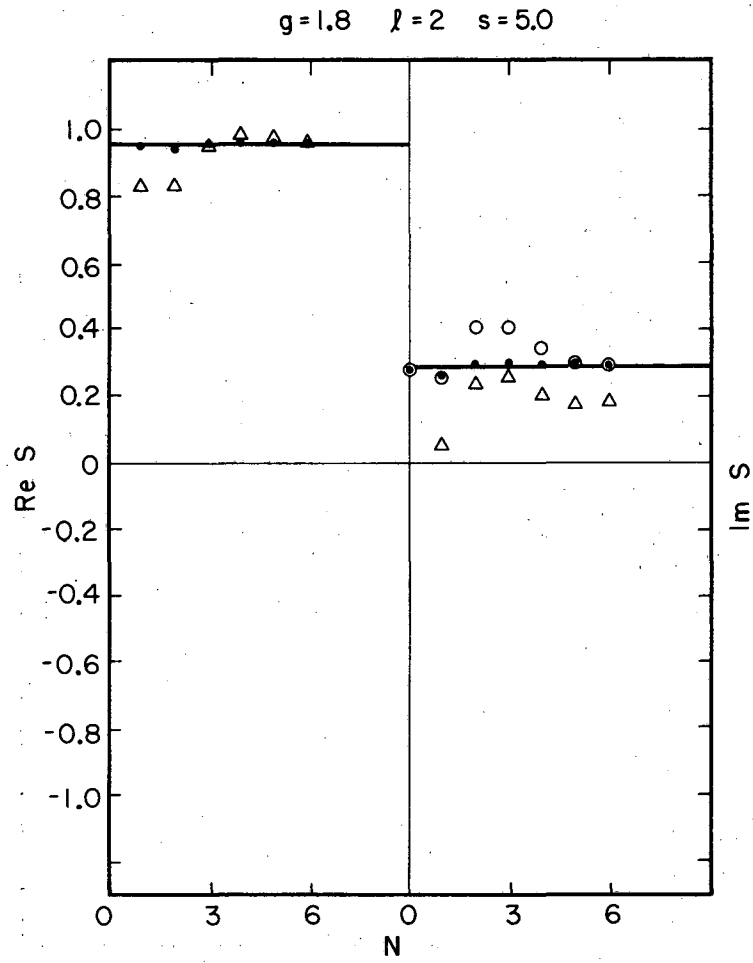
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Fig. 13



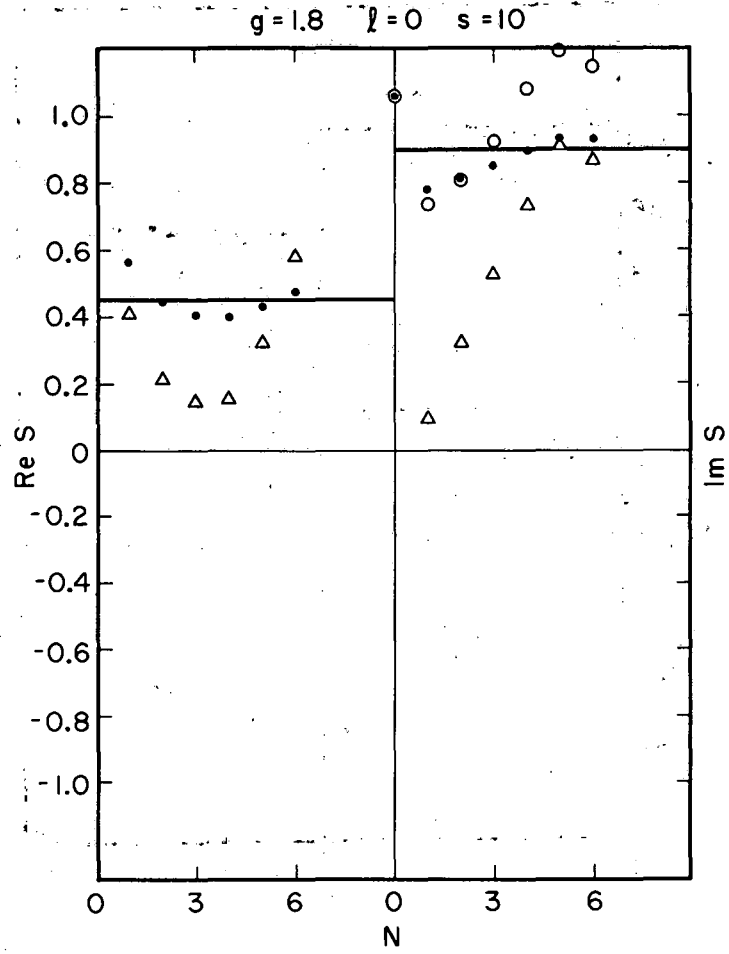
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Fig. 14



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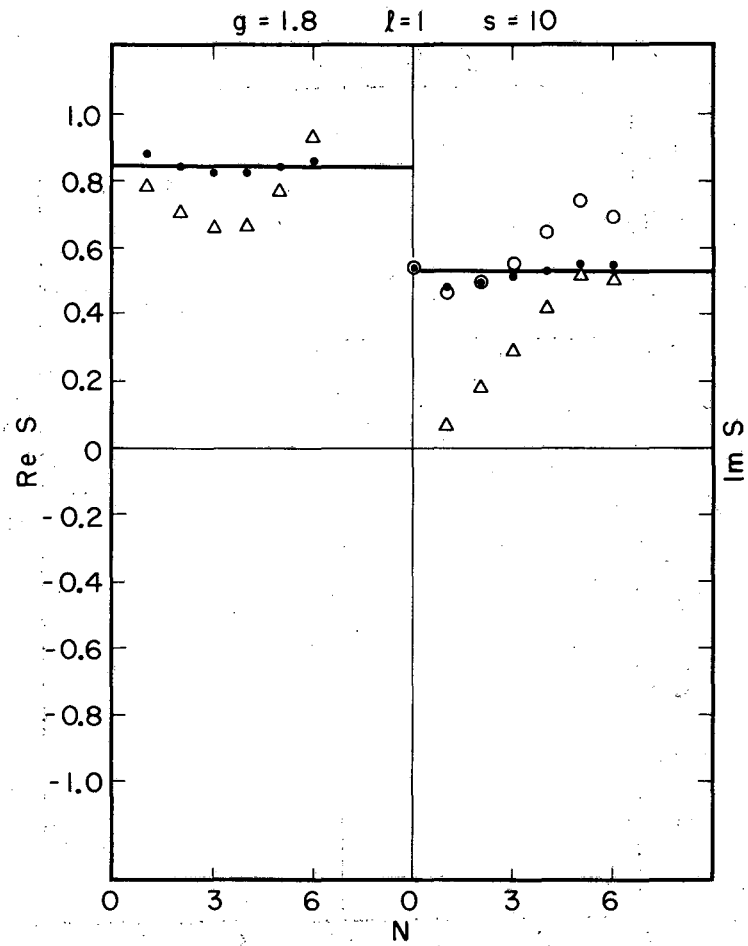
Fig. 15



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Fig. 16





MU-31567

Fig. 17

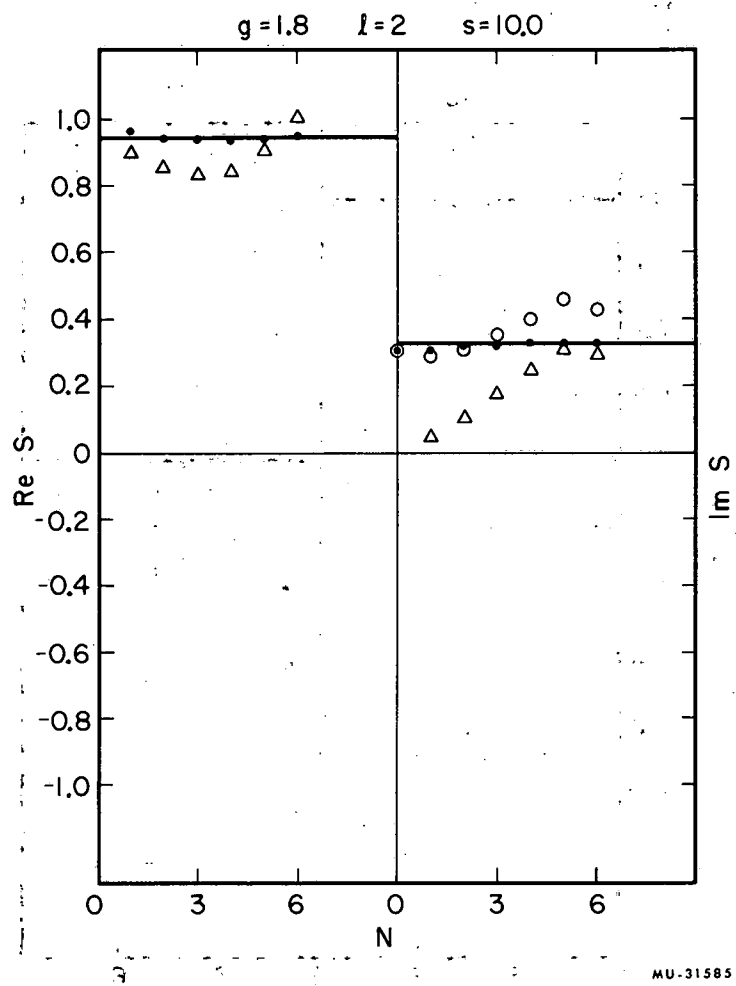
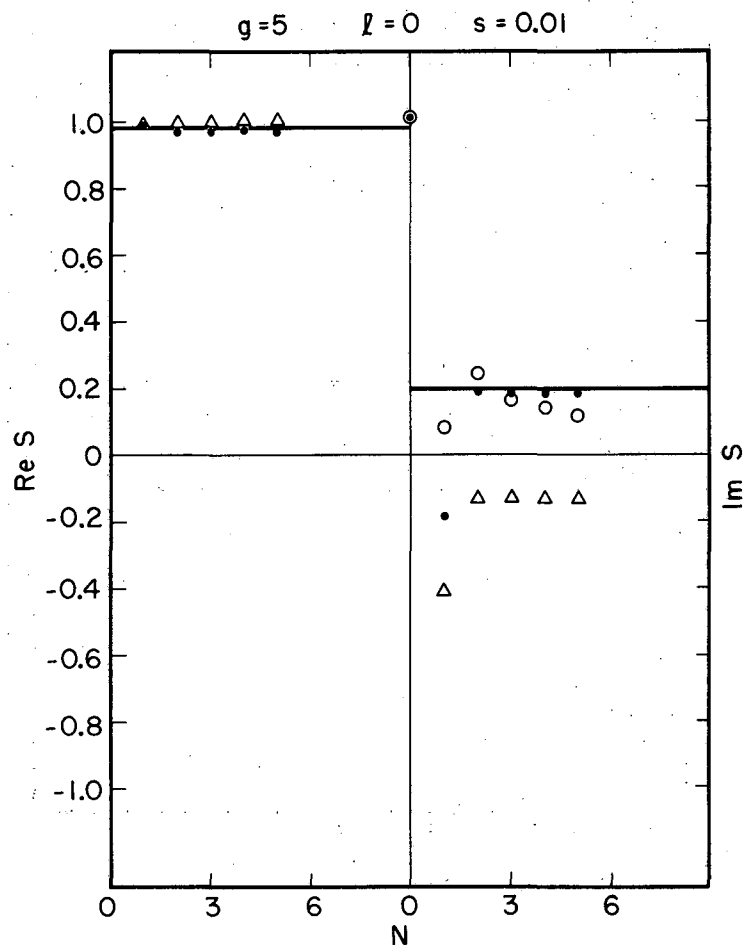
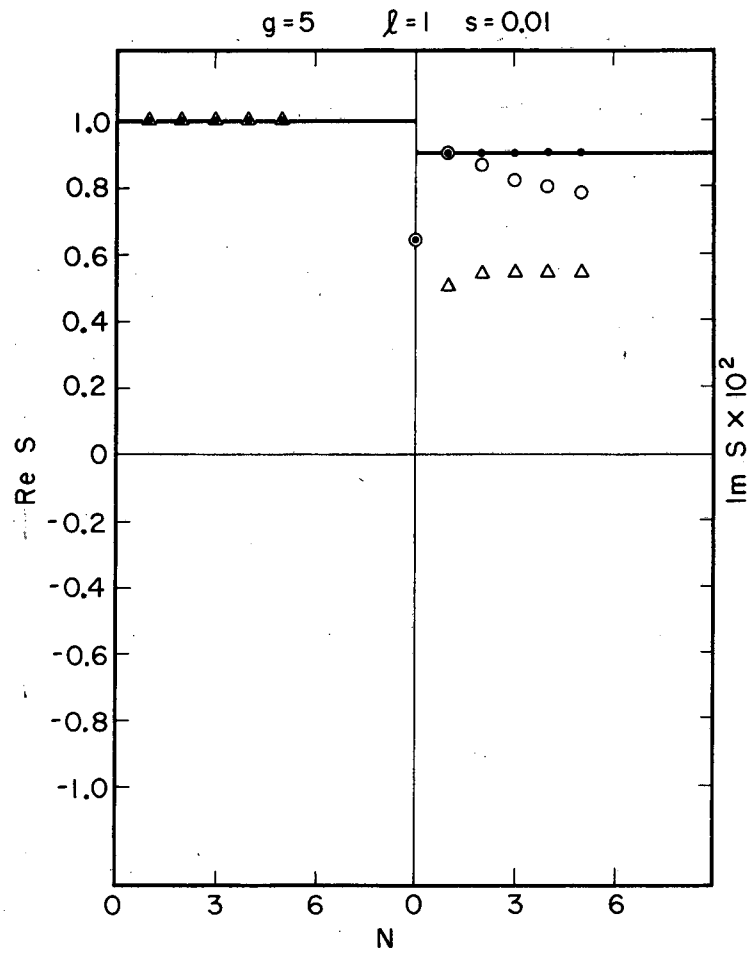


Fig. 18



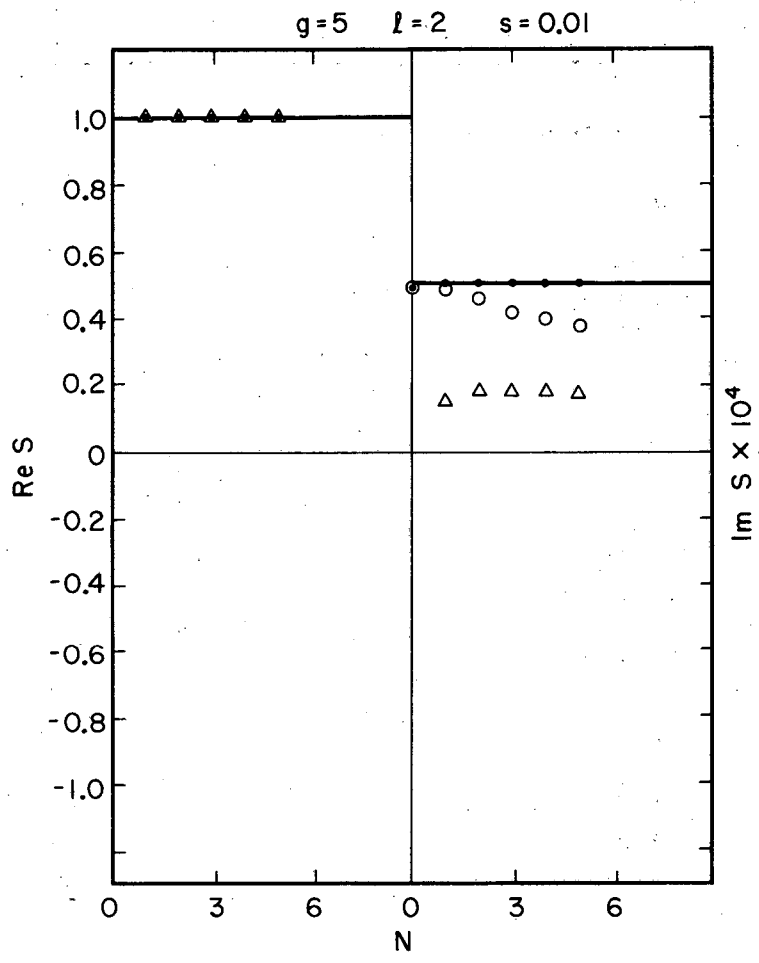
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Fig. 19



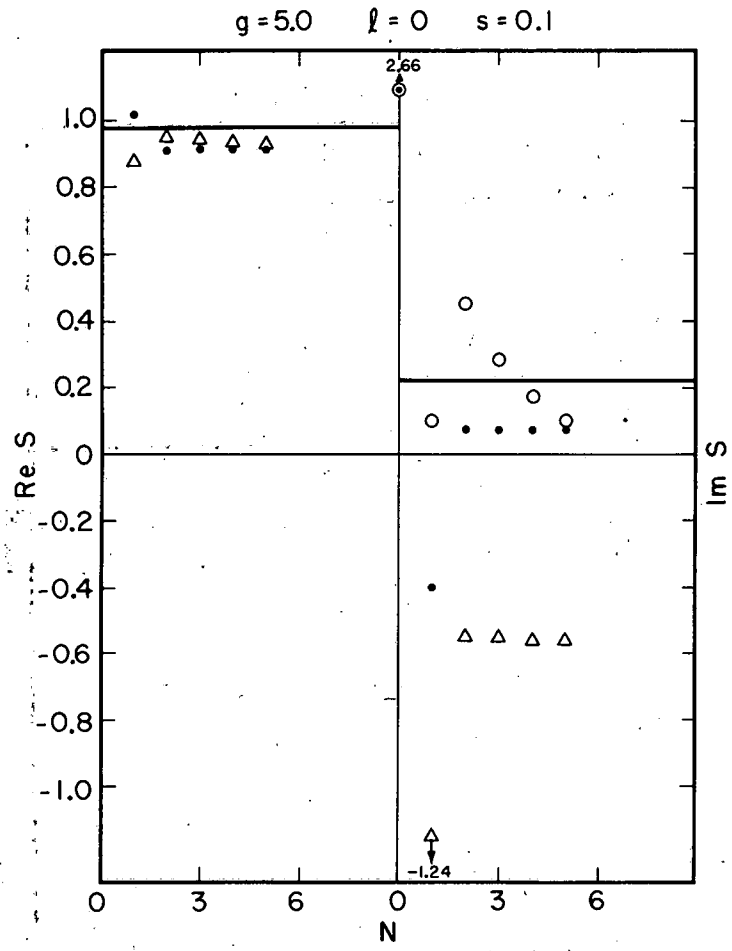
MU-31587

Fig. 20



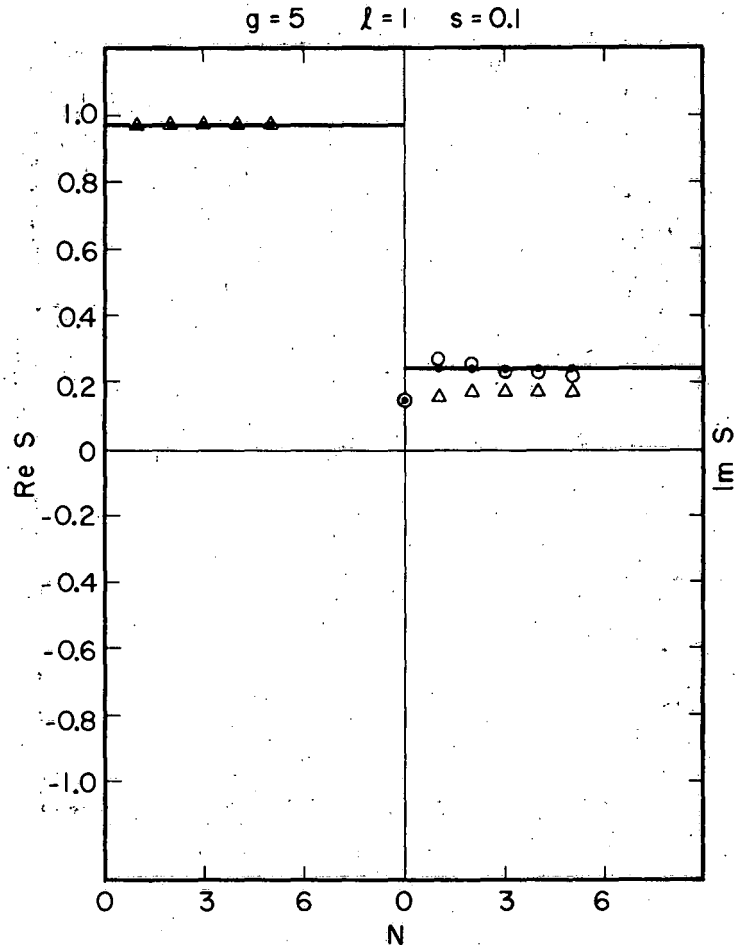
MU-31588

Fig. 21



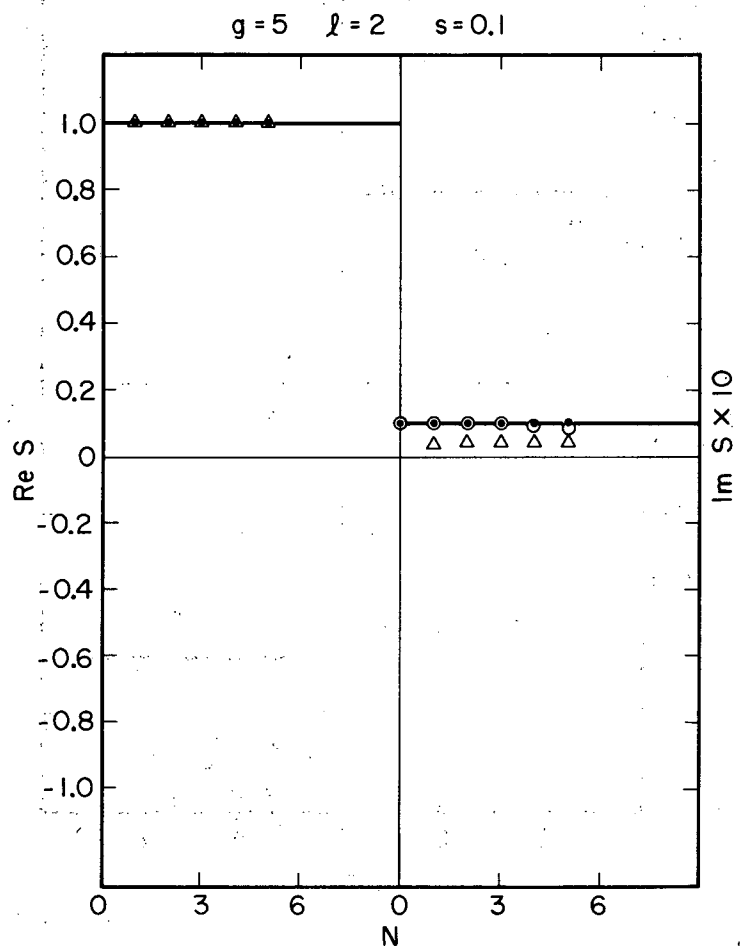
MU-31568

Fig. 22



MU-31569

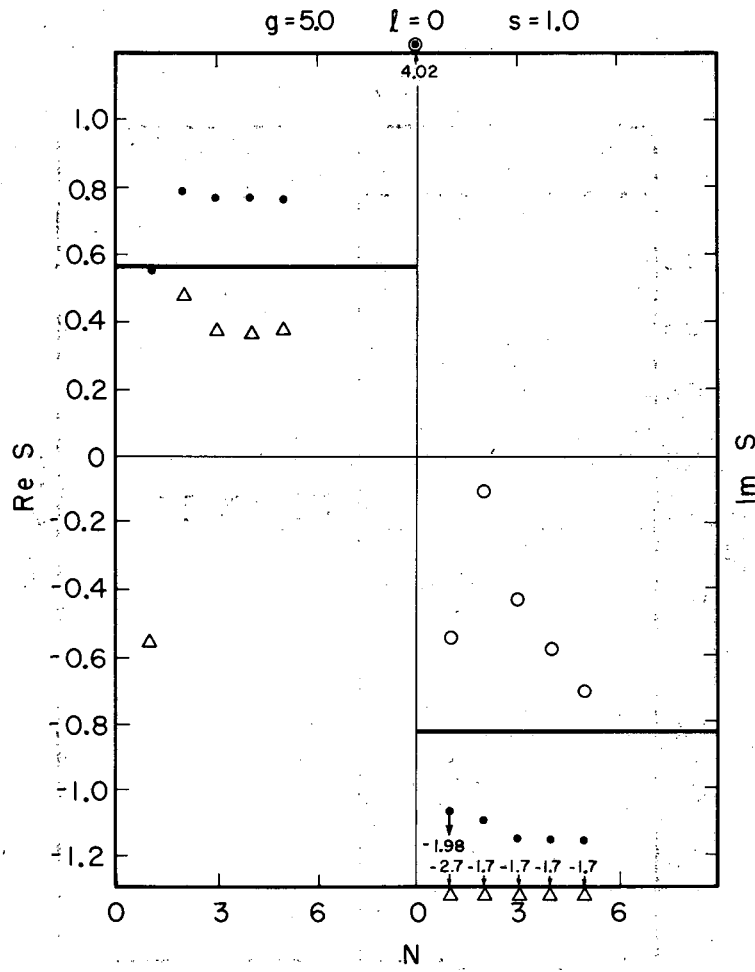
Fig. 23



MU-31595

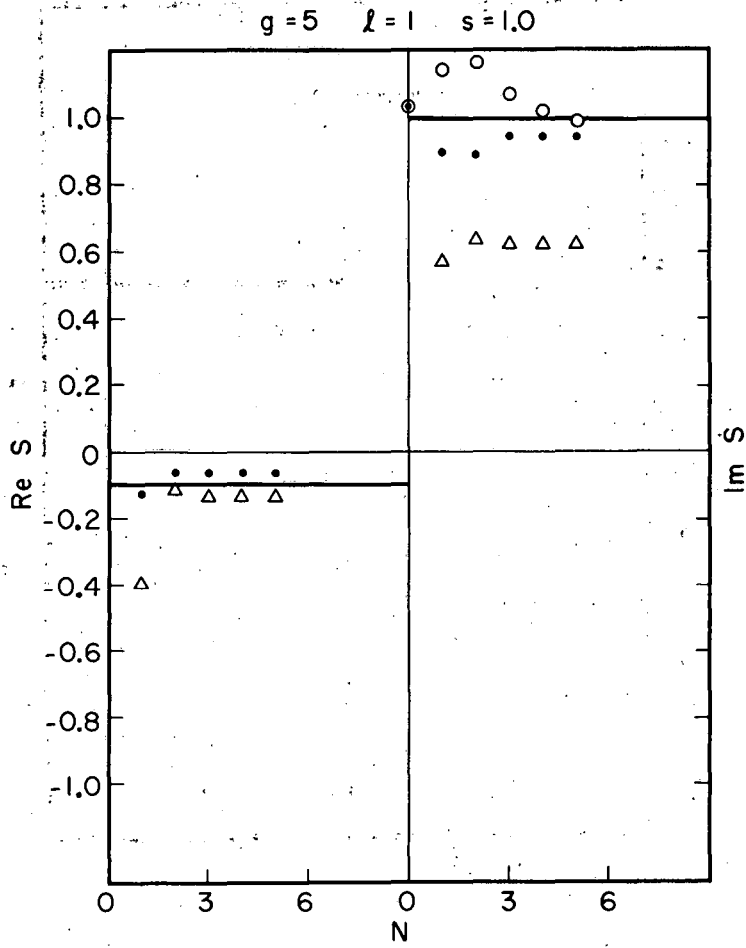
Fig. 24.





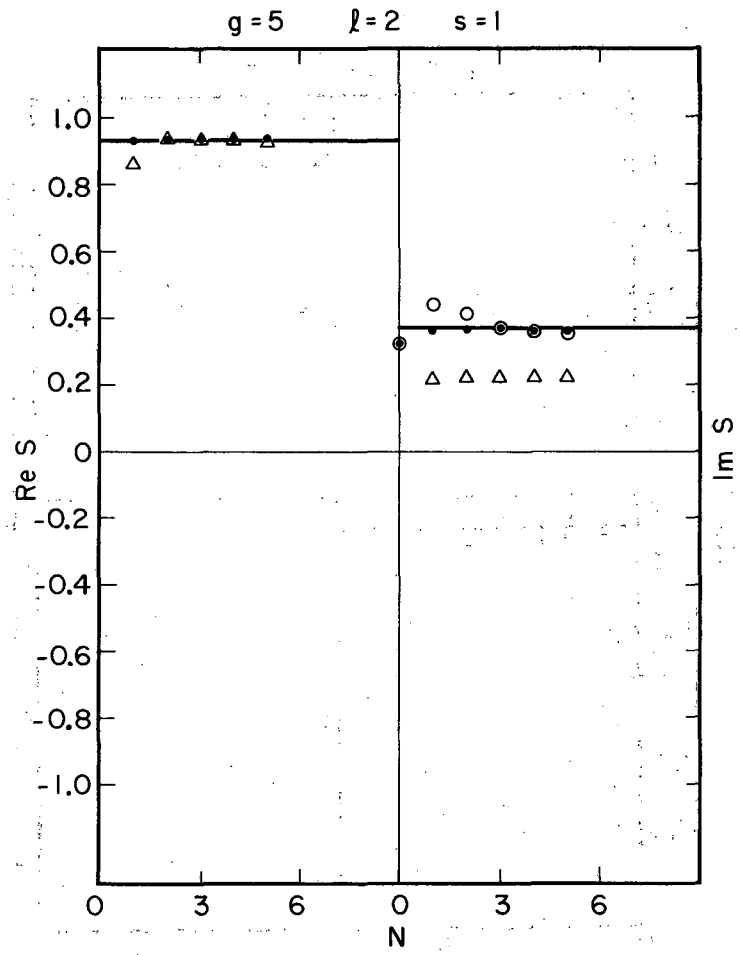
MU-31570

Fig. 25



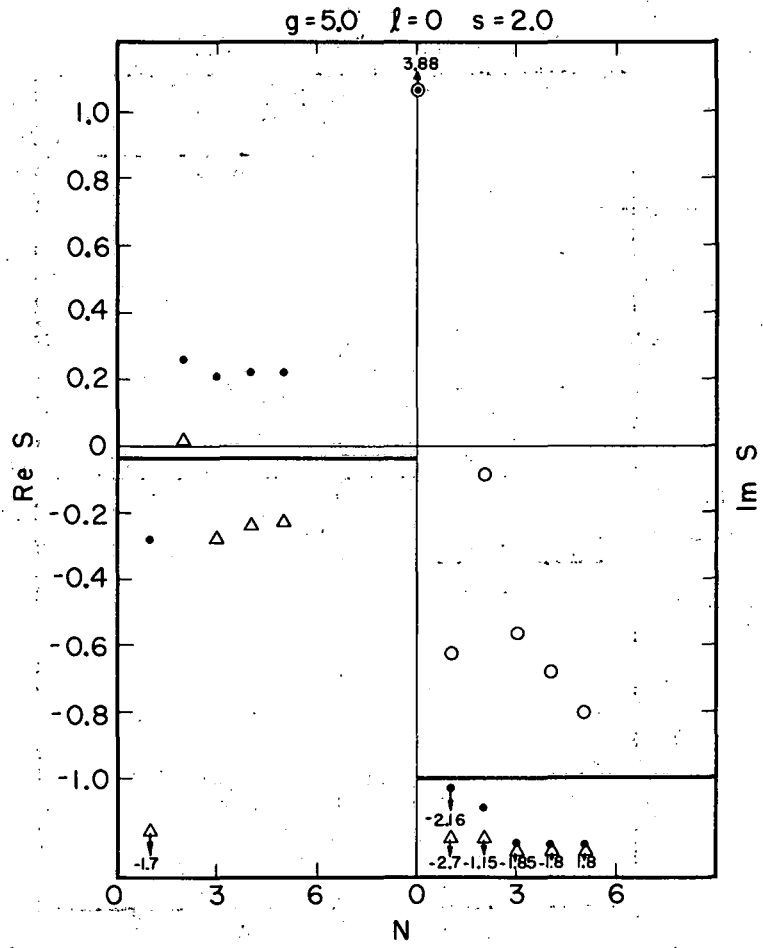
MU-31571

Fig. 26



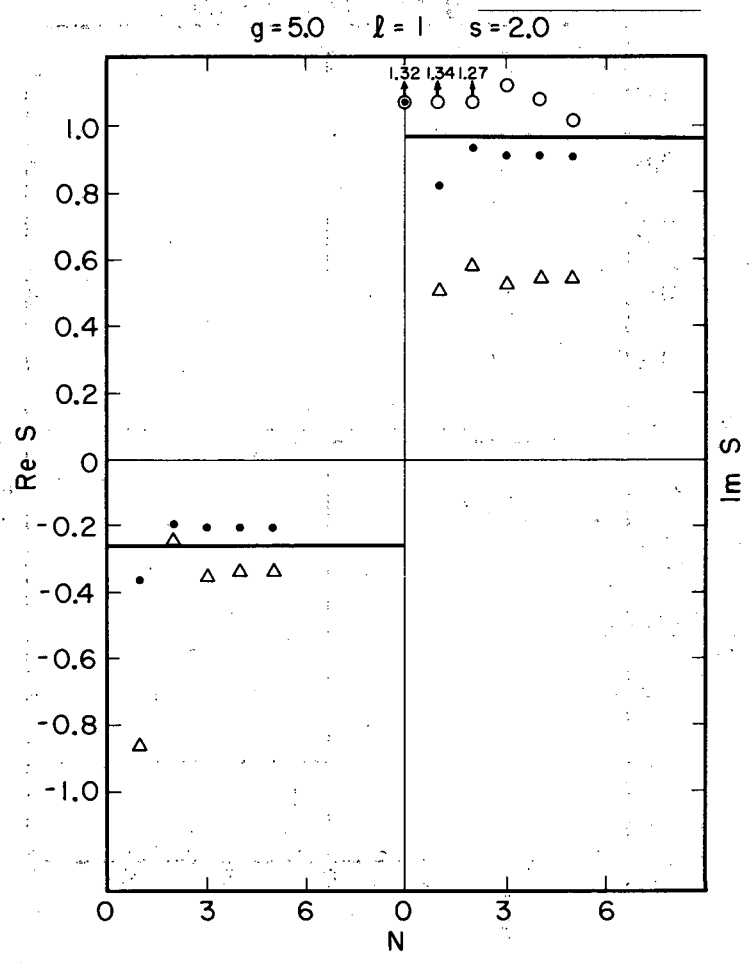
MU-31596

Fig. 27



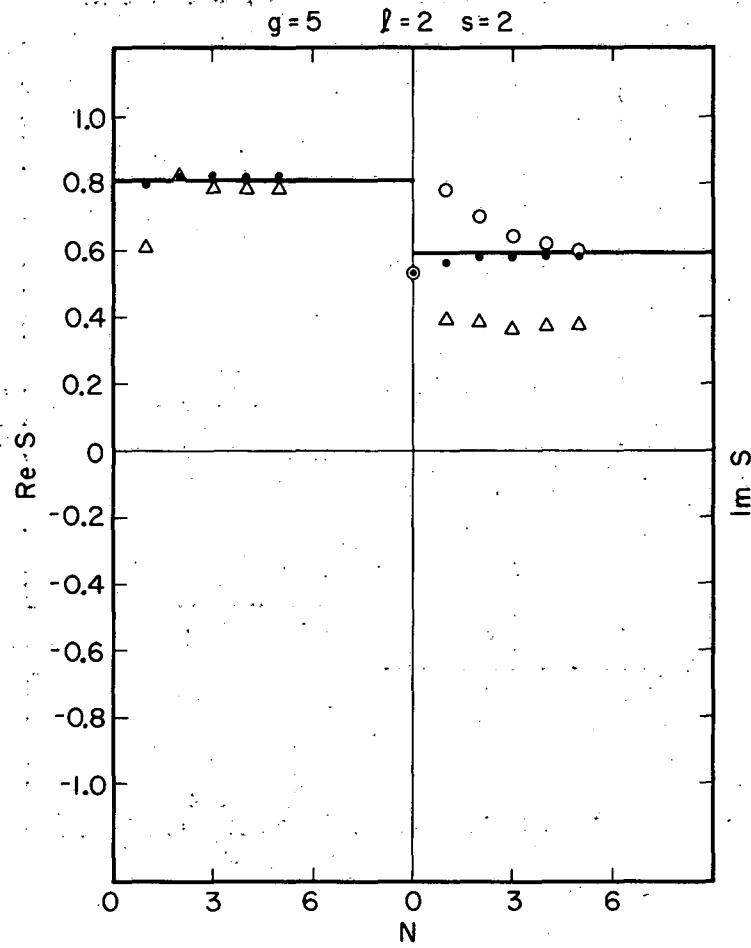
MU-31589

Fig. 28



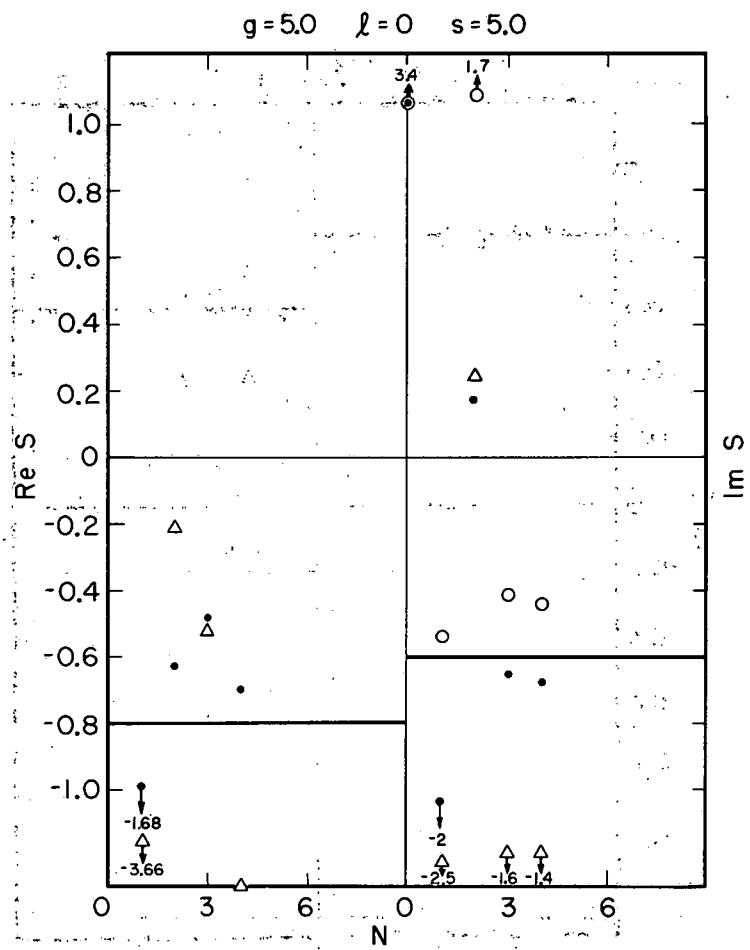
MU-31590

Fig. 29



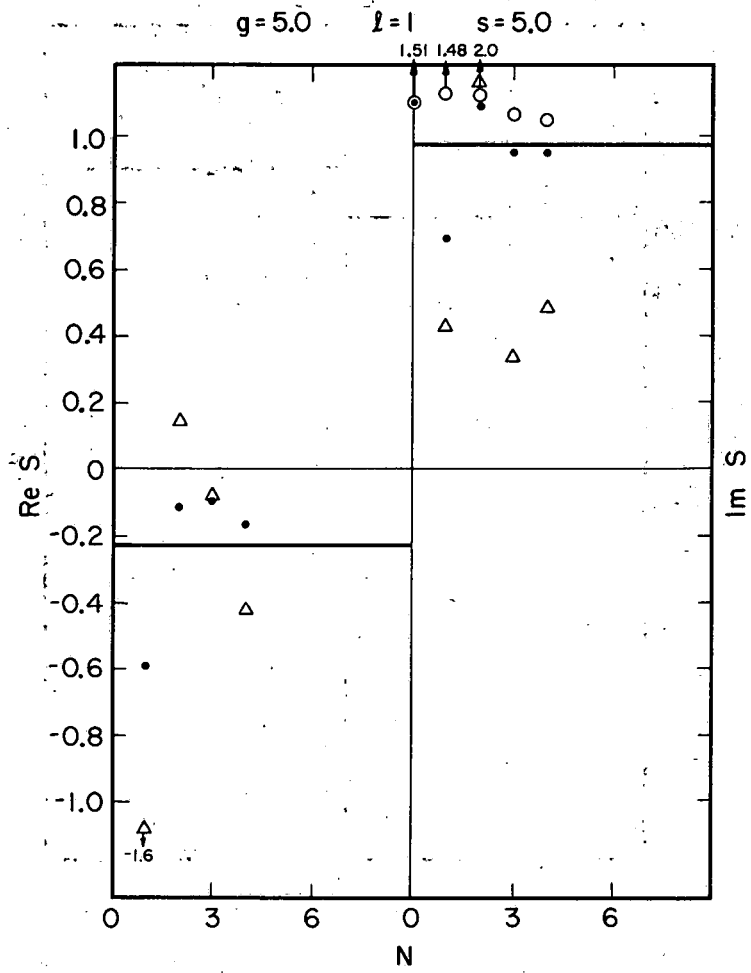
MU-31591

Fig. 30



MU-31592

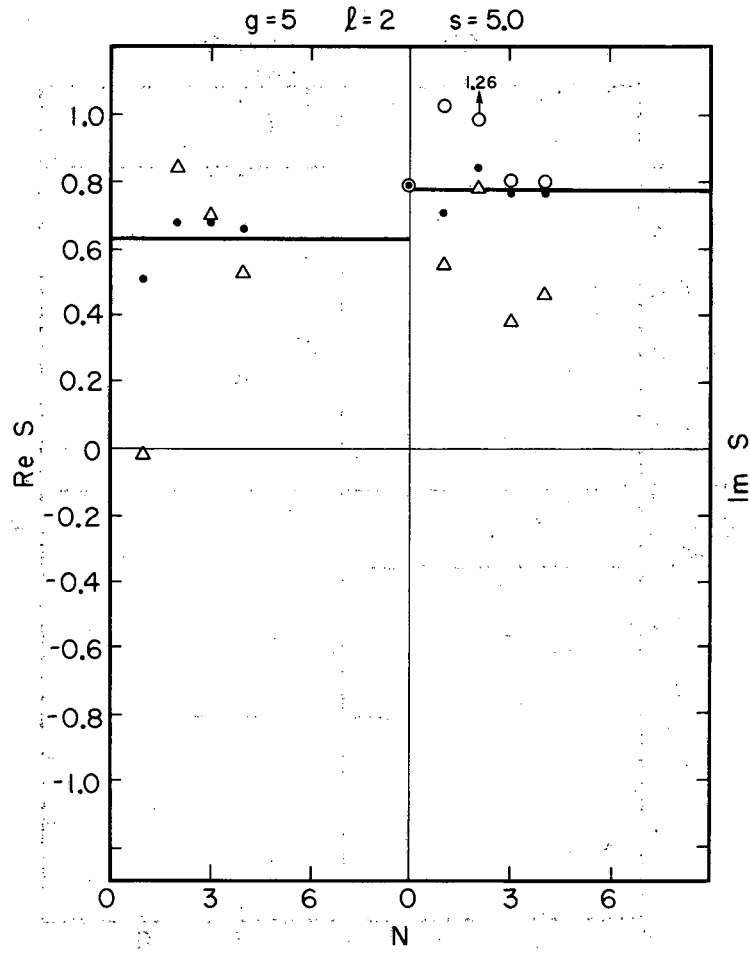
Fig. 31



MU-31593

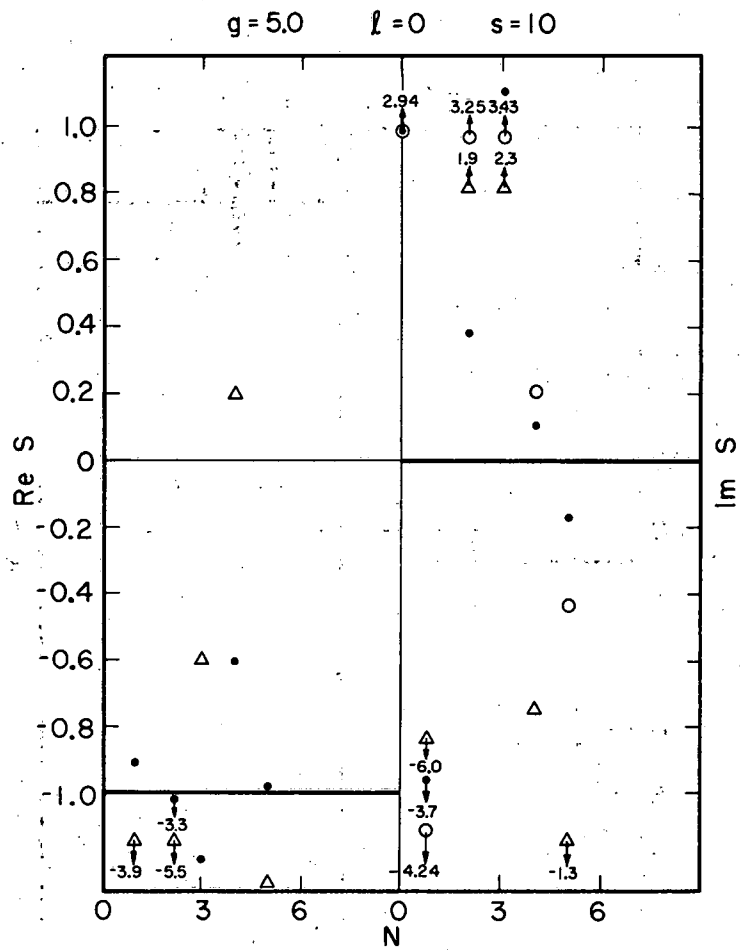
Fig. 32





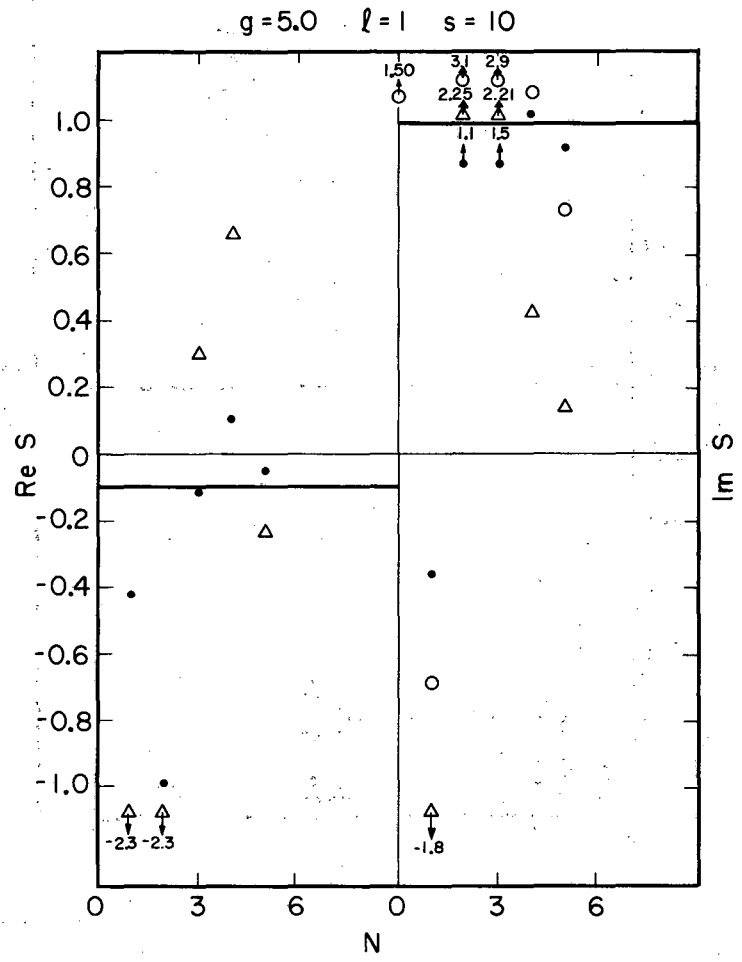
MU-31594

Fig. 33



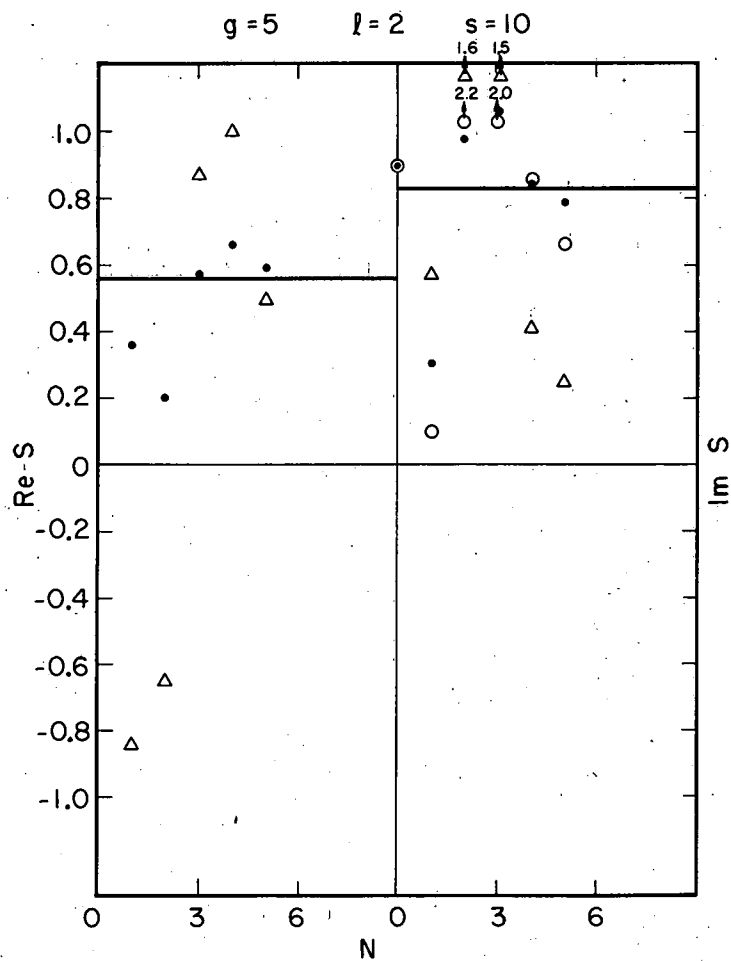
MU-31572

Fig. 34



MU-31573

Fig. 35



MU-31597

Fig. 36

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