Title
On the non-triviality of arithmetic invariants modulo p

Author
Burungale, Ashay A.

Publication Date
2015

University of California
Los Angeles

On the Non-Triviality of Arithmetic Invariants Modulo p

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Ashay Burungale

2015

Abstract of the Dissertation

On the Non-Triviality of Arithmetic Invariants Modulo p

by

Ashay Burungale
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2015
Professor Haruzo Hida, Chair

Arithmetic invariants are often naturally associated to motives over number fields. One of the basic questions is the non-triviality of the invariants. One typically expects generic non-triviality of the invariants as the motive varies in a family. For a prime p, the invariants can often be normalised to be p-integral. One can thus further ask for the generic non-triviality of the invariants modulo p. The invariants can often be expressed in terms of modular forms. Accordingly, one can try to recast the non-triviality as a modular phenomenon. If the phenomena can be proven, the non-triviality typically follows in turn. This principle can be found in the work of Hida and Vatsal among a few others.

We have been trying to explore a strategy initiated by Hida in the case of central critical Hecke L-values over the $\mathbf{Z}_p$-anticyclotomic extension of a CM-field. The strategy crucially relies on a linear indepedence of mod p Hilbert modular forms. Several arithmetic invariants seem to admit modular expression analogous to the case of Hecke L-values. This includes the case of Katz p-adic L-function, its cyclotomic derivative and p-adic Abel-Jacobi image of generalised Heegner cycles. We approach the non-triviality of these invariants based on the independence. An analysis of the zero set of the invariants suggests finer versions of the independence. We approach the versions based on Chai's theory of Hecke stable subvarieties of a mod p Shimura variety. We formulate a conjecture regarding the analogue of the independence for mod p modular forms on other Shimura varieties. We prove the analogue in the case of quaternionic Shimura varieties over a totally real field.

The dissertation of Ashay Burungale is approved.

Don Blasius

Chandrashekhar Khare

Weng Kee Wong

Haruzo Hida, Committee Chair

University of California, Los Angeles

2015

To my teachers ...

Table of Contents

Acknowledgments

I am grateful to Don Blasius, Chandrashekhar Khare, Haruzo Hida and Weng Kee Wong for being part of the thesis committee. I am also grateful to late Carol Newton, an ex-committee member. Weng Kee Wong kindly agreed to be her replacement.

I am indebted to my advisor Haruzo Hida for continuous guidance, encouragement and support. I feel fortunate enough for his suggestion of the thesis topic. The thesis builds on his ideas on the topic. My naive perception of the mathematical world is rather heavily influenced by his lectures, books and research articles. He has been very generous in sharing his ideas and intuition. His mathematical work will continue to be an inspiration. I am also indebted to my other advisor Chandrashekhar Khare for kind advice and encouragement. He has introduced me to new worlds, mathematical and otherwise, generously sharing his ideas and view points. My interest in Galois representations and literature owes heavily to him. In the near future, I hope to closely take up his mathematical suggestions. For the time to come, conversations with my advisors will continue to be a source of inspiration. I hope that opportunity comes again and again to work with them.

I am grateful to my coauthor Ming-Lun Hsieh. At the early stages, he patiently answered questions regarding his work on the non-triviality and his assistance was critical in providing an entry point to the topic. I am grateful to Ching-Li Chai for a number of instructive comments. The thesis crucially relies on his theory of Hecke stable subvarieties of a mod p Shimura variety. I am also grateful to Christopher Skinner for suggestions and encouragement. The questions treated in chapter 4 was essentially suggested by him. I thank Mahesh Kakde and Jacques Tilouine for suggestive conversations and encouragement over the years. I thank Francesc Castella and Marc-Hubert Nicole for helpful conversations. I also thank Miljan Brakocevic, Ernest Hunter Brooks, Henri Darmon, Ben Howard, Dimitar Jetchev, Jan Nekovář, Kartik Prasanna, Romyar Sharifi, Richard Taylor, Burt Totaro, Vinayak Vatsal and Xin Wan for their interest and suggestions. I have benefited tremendously from various courses in UCLA, especially the ones by Paul Balmer, Don Blasius, Haruzo Hida, Chandrashekhar Khare, Ciprian Manolescu, Alexander Merkurjev and Burt Totaro.

Films continue to be a major influence. I would like to express my gratitude to late Robert Bresson, Luis Bunuel, Federico Fellini, Akira Kurosawa, Kenji Mizoguchi, Yasujiro Ozu, Jean Renoir, Roberto Rossellini, Andrei Tarkovsky, Satyajit Ray and also Michael Haneke, Hou Hsiao-Hsien, Abbas Kiarostami, Wong Kar-Wai.

I am indebted to my undergraduate advisors Bala Sury and Sujatha Ramdorai. Bala Sury was very caring and supportive since my arrival in Indian Statistical Institute, Bangalore. At critical stages, he offered precious advice. Sujatha Ramdorai introduced me to Iwasawa Theory and modern Number Theory, in general. She was very generous in sharing her enthusiasm and time. Her encouragement was critical in various ways. I would like to thank Jishnu Biswas and Eknath Ghate for their guidance during my undergraduate days.

I am indebted to my high school teacher Hemant Deshpande. It was his lectures and curious sense which first drew me into Mathematics. He guided and motivated me in various ways. He also introduced me to Mathematics Olympiad. With generous guidance and encouragement from Vinaykumar Acharya, I participated in the Olympiads. The enriching experience was critical in the decision to pursue Mathematics.

I have been fortunate enough to have excellent friends over the years - Anand, Aniket, Asilata, Kaustubh, Nachiket, Prateek, Pushkaraj, Soumitra and Vaidehee in Satara/ Pune; Gaurish, Issan, Jyoti, Nanda, Nishant, Santosh, Shaunak and Sravan in Bangalore; (another) Anand, Ben, Bregje, Davide, Erik, Giang, Ha, Himadri, Humberto, Jackie, Jacques, Jianfeng, Kevin, Neil, Nhui, Parag, Patrick, Philip, Sam, Shagnik, Xiaojing, Yajing, Yi, Yingkun and Zuhair in Los Angeles. I would like to write separately about each one of them. For reasons of space, let me only say that they continue to teach me a lot and offer different perspectives.

I would like to thank my brother Vishu. Finally, no words can express the gratitude I feel towards my parents.

Vita

2009 B. Math (Hon) with distinction, Indian Statistical Institute, Bangalore. 2009–2014

2011 M. A., Mathematics, UCLA.

Publications

A. Burungale and M.-L. Hsieh, *The vanishing of μ-invariant of p-adic Hecke L-functions for CM fields*, Int. Math. Res. Not. IMRN 2013, no. 5, 1014–1027.

A. Burungale, *On the μ-invariant of the cyclotomic derivative of a Katz p-adic L-function*, J. Inst. Math. Jussieu 14 (2015), no. 1, 131–148.

A. Burungale, *On the non-triviality of the p-adic Abel-Jacobi image of generalised Heegner cycles modulo p, II: Shimura curves*, J. Inst. Math. Jussieu, to appear.

A. Burungale, *A conjectural linear independence of mod p modular forms*, preprint 2012.

A. Burungale and H. Hida, *$\mathfrak{p}$-rigidity and Iwasawa μ-invariants*, preprint 2013, submitted.

A. Burungale, *An $l \neq p$-interpolation of genuine p-adic L-functions*, preprint 2013, submitted.

A. Burungale, *$\mathfrak{p}$-rigidity and $\mathfrak{p}$-independence of quaternionic modular forms modulo p*, preprint 2014, submitted.

A. Burungale, *On the non-triviality of the p-adic Abel-Jacobi image of generalised Heegner cycles modulo p, I: modular curves*, preprint 2014, submitted.

A. Burungale, *Non-triviality of generalised Heegner cycles over anticyclotomic towers: a survey*, submitted to the proceedings of the ICTS program 'p-adic aspects of modular forms', 2014.

A. Burungale and H. Hida, *André-Oort conjecture and non-vanishing of central L-values over Hilbert class fields*, preprint 2015, submitted.

CHAPTER 1

Introduction

Arithmetic invariants are often naturally associated to motives over number fields. The invariants are typically defined in an algebraic or an analytic way and reflect an intrinsic aspect of the motive.

One of the basic questions is the non-triviality of the invariants. For a specific motive, the invariant may well be trivial. However, one typically expects generic non-triviality of the invariants as the motive varies in a family. For a prime p, the invariants can often be normalised to be p-integral. One can thus further ask for the generic non-triviality of the invariants modulo p. The zero set is expected to be thin. For an instructive discussion of the non-triviality, we refer to the preface of [37].

The invariants can often be expressed in terms of modular forms. Accordingly, one can try to recast the non-triviality as a modular phenomena. If the phenomena can be proven, the non-triviality typically follows in turn. This principle can be found in the work of Hida and Vatsal among a few others.

Hida considered the non-vanishing of Hecke L-values modulo p over anticyclotomic extensions of CM fields. Here underlying arithmetic object is the motive corresponding to a Hecke character over the CM field. An Iwasawa theoretic family arises from l-power order anticyclotomic twists of the Hecke character for a prime l. Shimura found that Hecke L-values can be expressed as a sum of evaluation of Hilbert modular forms at CM points. The expression over the family suggested an Ax-Lindemann type functional independence of mod p Hilbert modular forms. Based on Chai's theory of Hecke-stable subvarieties of a mod p Shimura variety (cf. [19], [20] and [21]), Hida proved the independence (cf. [34]).

We have been trying to explore the strategy initiated by Hida. Several arithmetic invariants seem to admit modular expression analogous to the case of Hecke L-values. The non-triviality can thus be approached based on the independence. An analysis of the zero set leads to finer versions of the independence. Another relevant question is the analogue of the independence for mod p modular forms on other Shimura varieties. The analogue is bound to have applications to the non-triviality.

In §1.1, we describe results regarding Iwasawa μ-invariants mainly arising from Katz p-adic L-functions. In §1.2, we describe results regarding the independence of mod p modular forms. In §1.3, we describe results regarding the non-triviality of the p-adic Abel-Jacobi image of generalised Heegner cycles modulo p over anticyclotomic extensions of the underlying imaginary quadratic field.

1.1 Iwasawa μ-invariants

In this section, we describe results regarding Iwasawa μ-invariants mainly arising from Katz p-adic L-functions. The results are partly a joint work with Hida and Hsieh.

Let F be a totally real field of degree d and O the ring of integers. Let p be an odd prime unramified in F. Let $\mathfrak{p}_1, ..., \mathfrak{p}_r$ be the primes above p. Fix two embeddings $\iota_\infty \colon \overline{\mathbf{Q}} \to \mathbf{C}$ and $\iota_p \colon \overline{\mathbf{Q}} \to \mathbf{C}_p$. Let v_p be the p-adic valuation of $\mathbf{C}_p$ normalised such that $v_p(p) = 1$. Let $\mathfrak{m}_p$ be the maximal ideal of $\overline{\mathbf{Z}}_p$. Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_p$.

Let K be a totally imaginary quadratic extension of F. Let $h_?$ denote the class number of ?, for ? $= K, F$. Let c denote the complex conjugation on $\mathbf{C}$ which induces the unique non-trivial element of $Gal(K/F)$ via the complex embedding ι_∞. We assume the following hypothesis:

(ord) Every prime of F above p splits in K.

The condition (ord) guarantees the existence of a p-adic CM type Σ i.e. Σ is a CM type of K such that, p-adic places induced by elements in Σ via ι_p are disjoint from those induced by Σc. We fix such a CM type. Let K_∞^- (resp. K_∞^+) be the anticyclotomic $\mathbf{Z}_p^d$-extension (resp. cyclotomic $\mathbf{Z}_p$-extension) of K and and $K_{\mathfrak{p},\infty}^- \subset K_\infty^-$ be the $\mathfrak{p}$-anticyclotomic subextension i.e. the maximal subextension unramified outside the primes above $\mathfrak{p}$ in K. Let $K_{\mathfrak{p},\infty} = K_{\mathfrak{p},\infty}^- K_\infty^+$. Let $\Gamma^\pm := Gal(K_\infty^\pm/K)$, $\Gamma_\mathfrak{p}^- = Gal(K_{\mathfrak{p},\infty}^-/K)$. and $\Gamma_\mathfrak{p} = Gal(K_{\mathfrak{p},\infty}/K)$.

Let $\mathfrak{C}$ be a prime-to-p integral ideal of K. Decompose $\mathfrak{C} = \mathfrak{C}^+\mathfrak{C}^-$, where $\mathfrak{C}^+$ (respectively $\mathfrak{C}^-$) is a product of split primes (respectively ramified or inert primes) over F. Let λ be a Hecke character of K. Suppose that $\mathfrak{C}$ is the prime-to-p conductor of λ. Associated to this data, we have a natural $(d+1)$-variable Katz p-adic L-function $L_{\Sigma,\lambda} = L_{\Sigma,\lambda}(T_1, ..., T_d, S) \in \overline{\mathbf{Z}}_p[\![\Gamma]\!]$ (cf. [45] and [30]). Here T_i's are the anticyclotomic variables and S is the cyclotomic variable.

For a power series $g \in \overline{\mathbf{Z}}_p[\![\Gamma]\!]$, let

$$\mu(g) = \liminf_{(n_1,\dots,n_d,n_{d+1})} v_p(a_{n_1,\dots,n_d,n_{d+1}}(g))$$

for the coefficients $a_{n_1,\dots,n_d,n_{d+1}}(g)$.

The μ-invariant of $L_{\Sigma,\lambda}$ is given by the following theorem (cf. [35, Thm. I] and [7, Thm. A]).

Theorem 1.1.1.

$$\mu(L_{\Sigma,\lambda}) = 0.$$

Let $L^{-}_{\Sigma,\lambda} \in \overline{\mathbf{Z}}_p[\![\Gamma^{-}]\!]$ (resp. $L^{-}_{\Sigma,\lambda,\mathfrak{p}} \in \overline{\mathbf{Z}}_p[\![\Gamma^{-}_{\mathfrak{p}}]\!]$) be the anticyclotomic (resp. $\mathfrak{p}$-anticyclotomic) projection obtained from the projection $\pi^{-} : \overline{\mathbf{Z}}_p[\![\Gamma]\!] \twoheadrightarrow \overline{\mathbf{Z}}_p[\![\Gamma^{-}]\!]$ (resp. $\pi^{-}_{\mathfrak{p}} : \overline{\mathbf{Z}}_p[\![\Gamma]\!] \twoheadrightarrow \overline{\mathbf{Z}}_p[\![\Gamma^{-}_{\mathfrak{p}}]\!]$). Let $L_{\Sigma,\lambda,\mathfrak{p}} \in \overline{\mathbf{Z}}_p[\![\Gamma_{\mathfrak{p}}]\!]$ be obtained from the projection $\pi_{\mathfrak{p}} : \overline{\mathbf{Z}}_p[\![\Gamma]\!] \twoheadrightarrow \overline{\mathbf{Z}}_p[\![\Gamma_{\mathfrak{p}}]\!]$. We call $L_{\Sigma,\lambda,\mathfrak{p}}$ as the Katz $\mathfrak{p}$-adic L-function.

We have the following $\mathfrak{p}$-version of Theorem 1.1.1 (cf. [11, Thm. D]).

Theorem 1.1.2.

$$\mu(L_{\Sigma,\lambda,\mathfrak{p}}) = 0.$$

The μ-invariant of $L^{-}_{\Sigma,\lambda,\mathfrak{p}}$ is given by the following theorem (cf. [11, Thm. B]).

Theorem 1.1.3.

$$\mu(L^{-}_{\Sigma,\lambda,\mathfrak{p}}) = \mu(L^{-}_{\Sigma,\lambda}).$$

In most of the cases, $\mu(L^{-}_{\Sigma,\lambda})$ has been explicitly determined (cf. [34] and [40]). Thus, we obtain a formula for $\mu(L^{-}_{\Sigma,\lambda,\mathfrak{p}})$. A result analogous to the theorem also holds for a class of Rankin-Selberg anticyclotomic $\mathfrak{p}$-adic L-functions.

When λ is self-dual with the root number -1, all the Hecke L-values appearing in the interpolation property of $L^{-}_{\Sigma,\lambda}$ vanish. Accordingly, $L^{-}_{\Sigma,\lambda}$ identically vanishes. We can consider the cyclotomic derivative

$$L'_{\Sigma,\lambda} = \frac{\partial}{\partial S} L_{\Sigma,\lambda}(T_1, ..., T_d, S)\Big|_{S=0}.$$

We suppose λ is of infinity type $k\Sigma$, for a positive integer k. To state our regarding the μ-invariant of $L'_{\Sigma,\lambda}$, we introduce further notation. For each local place v of F, choose a uniformiser ϖ_v and let $|\cdot|_v$ denote the corresponding absolute value normalised so that $|\varpi_v|_v = |N(\varpi_v)|_l$ and $|l|_l = \frac{1}{l}$, where N is the norm, $v \cap \mathbf{Q} = (l)$ and $l > 0$. For each v dividing $\mathfrak{C}^-$ and λ as above, the local invariant $\mu_p(\lambda_v)$ is defined by

$$\mu_p(\lambda_v) = \inf_{x \in K_v^\times} v_p(\lambda_v(x) - 1).$$

Let us also define

$$\mu'_{p,v}(\lambda) = v_p\left(\frac{\log_p(|\varpi_v|_v)}{\log_p(1+p)}\right) + \sum_{w \neq v, w | \mathfrak{C}^-} \mu_p(\lambda_w)$$

and

$$\mu'_p(\lambda) = \sum_{v | \mathfrak{C}^-} \mu_p(\lambda_v).$$

Our result regarding the non-triviality of $L'_{\Sigma,\lambda}$ is the following (cf. [8, Thm. A]).

Theorem 1.1.4. *Let $h_K^- = h_K/h_F$ be the relative class number and $p \nmid h_K^-$. Then, we have*

$$\mu(L'_{\Sigma,\lambda}) = \min_{v | \mathfrak{C}^-} \{\mu'_p(\lambda), \mu'_{p,v}(\lambda)\}.$$

We have the following $\mathfrak{p}$-version of Theorem 1.1.4 (cf. [11, Thm. C]).

Theorem 1.1.5. *Suppose that $p \nmid h_K^-$. Then, we have*

$$\mu(L'_{\Sigma,\lambda,\mathfrak{p}}) = \mu(L'_{\Sigma,\lambda}).$$

The results are based on the strategy initiated by Hida (cf. §1.1). We refer to the introduction of the above articles for a sketch of the proofs and §1.2 for a flavour of the underlying geometric results.

Let $\mathfrak{p}_i$ and $\mathfrak{p}_j$ be primes above p as above. Theorem 1.1.3 implies an intriguing equality $\mu(L^{-}_{\Sigma,\lambda,\mathfrak{p}_i}) = \mu(L^{-}_{\Sigma,\lambda,\mathfrak{p}_j})$ of Iwasawa μ-invariants. This is rather surprising as theses μ-invariants could be non-zero and one in general does not expect any relation between the p-adic L-functions $L_{\Sigma,\lambda,\mathfrak{p}_i}$'s. These p-adic L-functions depend on independent variables whose number may vary with i. As per as we know, Theorem 1.1.3 is a first phenomena possibly suggesting a relation.

The results have several arithmetic applications. Here we restrict to some applications of Theorem 1.1.4. When F equals the rationals, the theorem implies the non-vanishing of the anticyclotimic regulator of elliptic units (cf. [8, §4]). The theorem also plays a crucial role in the proof of the Iwasawa main conjectures for self-dual Hecke characters with the root number -1 over CM fields (cf. [41, Intro. and §8]).

In [9, Thm. B], we prove the vanishing of the μ-invariant for a class of anticyclotomic Rankin-Selberg p-adic L-functions (cf. §4.1).

1.2 $\mathfrak{p}$-rigidity

In this section, we describe results regarding $\mathfrak{p}$-rigidity of mod p modular forms. The results are a joint work with Hida.

Unless otherwise stated, let the notation and hypotheses be as in §1.1.

Let $Sh_{/\mathbb{F}}$ be the prime to p Hilbert modular Shimura variety associated to F. Let $x \in Sh$ be a closed ordinary point. Recall, from Serre-Tate deformation theory, a p^∞-level structure on x induces a canonical isomorphism

$$Spf(\widehat{\mathcal{O}}_{Sh,x}) \simeq \prod_i \widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}_i}.$$

Let $\mathfrak{p} = \mathfrak{p}_i$, for some i. Let f be a mod p Hilbert modular form. In view of the irreducibility of the connected components of Sh, the form f is determined by its restriction to $Spf(\widehat{\mathcal{O}}_{Sh,x})$. In fact, we have the following rigidity result (cf. [11, Thm. A]).

Theorem 1.2.1. *($\mathfrak{p}$-rigidity) Let f be a non-zero mod p Hilbert modular form. Then, f does not vanish identically on the partial Serre-Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$. In particular, a mod p Hilbert modular form is determined by its restriction to the partial Serre-Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$.*

We now describe our result regarding an Ax-Lindemann type functional independence of mod p modular forms restricted to the partial Serre-Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$. Via Hecke action, recall that the group $(Res_{O_{(p)}/\mathbf{Z}_{(p)}} GL_2)(\mathbf{Z}_{(p)})$ acts on Sh. Here $\mathbf{Z}_{(p)}$ is the localisation of the integers $\mathbf{Z}$ at the prime ideal (p) and $O_{(p)} = O \otimes \mathbf{Z}_{(p)}$. Let $H_x(\mathbf{Z}_{(p)}) \subset (Res_{O_{(p)}/\mathbf{Z}_{(p)}} GL_2)(\mathbf{Z}_{(p)})$ be the stabiliser of x. Let $\pi : Ig \to Sh$ be the Igusa tower. Let I be the irreducible component above V and $\tilde{x}$ a point in $I_{/\mathbb{F}}$ over x. As π is *étale*, we have a canonical isomorphism $\widehat{\mathcal{O}}_{V,x} \simeq \widehat{\mathcal{O}}_{I,\tilde{x}}$. Recall that $H_x(\mathbf{Z}_{(p)})$ stabilises x. Thus, $H_x(\mathbf{Z}_p)$ acts on $\widehat{\mathcal{O}}_{V,x}$. This is a local analog of the Hecke symmetries. We let $H_x(\mathbf{Z}_p)$ act on $\widehat{\mathcal{O}}_{I,\tilde{x}}$ via the isomorphism induced by π. Recall that a mod p modular form is a formal function on the Igusa tower. For $a \in H_x(\mathbf{Z}_{(p)})$, let $a_{\mathfrak{p}} \in H_x(\mathbf{Z}_{(p)})_{\mathfrak{p}}$ be as in [11, §3.2]. We have the following functional independence (cf. [11, §4]).

Theorem 1.2.2. *(Multiple $\mathfrak{p}$-rigidity) Let x be a closed ordinary point on the Hilbert modular Shimura variety Sh and $H_x(\mathbf{Z}_{(p)})$ the stabiliser. For $1 \leq i \leq n$, let $a_i \in H_x(\mathbf{Z}_p)$ such that*

$(a_i a_j^{-1})_{\mathfrak{p}} \notin H_x(\mathbf{Z}_{(p)})_{\mathfrak{p}}$ for all $i \neq j$. Let $f_1, ..., f_n$ be n non-constant mod p modular forms on Sh. Then, $(a_{i,\mathfrak{p}} \circ (f_i|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}}))_i$ are linearly independent in the partial Serre-Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$.

These rather surprising results were suggested by an attempt to adopt Hida's strategy for the anticyclotomic Katz $\mathfrak{p}$-adic L-function $L^{-}_{\Sigma,\lambda,\mathfrak{p}}$. The proofs crucially rely on on Chai's theory of Hecke-stable subvarieties of a mod p Shimura variety (cf. [19], [20] and [21]).

The results underly the determination of several Iwasawa μ-invariants in §2 and are perhaps of independent interest as well.

In [13], we prove an analogue of the results for mod p modular forms on quaternionic Shimura varieties over totally real fields. In [10], we formulate a conjecture regarding an analogue of the multiple $\mathfrak{p}$-rigidity for the entire deformation space and mod p modular forms on a Shimura variety.

1.3 Generalised Heegner cycles

In this section, we describe results regarding the non-triviality of the p-adic Abel-Jacobi image of generalised Heegner cycles modulo p over anticyclotomic extensions of an imaginary quadratic field.

For simplicity, we mostly restrict to the case of Heegner points.

Unless otherwise stated, let the notation and hypotheses be as in §1.1.

We suppose that F equals the rationals. Let $\mathfrak{p}$ be a prime above p in K induced via the p-adic embedding ι_p. For an integral ideal $\mathfrak{n}$ of K, let $H_{\mathfrak{n}}$ be the ring class field of K of conductor $\mathfrak{n}$. Let H be the Hilbert class field.

Let N be a positive integer such that $p \nmid N$. For $k \geq 2$, let $S_k(\Gamma_0(N), \epsilon)$ be the space of elliptic modular forms of weight k, level $\Gamma_0(N)$ and neben-character ϵ. Let $f \in S_2(\Gamma_0(N), \epsilon)$ be an elliptic newform. Let $N_\epsilon | N$ be the conductor of ϵ. Let E_f be the Hecke field of f and $\mathcal{O}_{E_f}$ the ring of integers. Let $\mathfrak{P}$ be a prime above p in E_f induced by the p-adic embedding ι_p. Let $\rho_f : Gal(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\mathcal{O}_{E_{f,\mathfrak{P}}})$ be the corresponding p-adic Galois representation.

We assume the following Heegner hypothesis:

(Hg) $\mathcal{O}$ contains a cyclic ideal $\mathfrak{N}$ of norm N.

From now, we fix such an ideal $\mathfrak{N}$. Let $\mathfrak{N}_\epsilon | \mathfrak{N}$ be the unique ideal of norm N_ϵ.

Let $\mathbf{N}$ denote the norm Hecke character over $\mathbf{Q}$ and $\mathbf{N}_K := \mathbf{N} \circ N^K_{\mathbf{Q}}$ the norm Hecke character over K. For a Hecke character λ of K, let $\mathfrak{f}_\lambda$ (resp. ϵ_λ) denote its conductor (resp. the restriction $\lambda|_{\mathbf{A}_{\mathbf{Q}}^\times}$, where $\mathbf{A}_{\mathbf{Q}}$ denotes the adele ring over $\mathbf{Q}$). We say that λ is central critical for f if it is of infinity type (j_1, j_2) with $j_1 + j_2 = 2$ and $\epsilon_\lambda = \epsilon_f \mathbf{N}^2$.

Let b be a positive integer prime to pN. Let $\Sigma_{cc}(b, \mathfrak{N}, \epsilon)$ be the set of Hecke characters λ such that:

(C1) λ is central critical for f,

(C2) $\mathfrak{f}_\lambda = b \cdot \mathfrak{N}_\epsilon$ and

(C3) The local root number $\epsilon_q(f, \lambda^{-1}) = 1$, for all finite primes q.

Let χ be a finite order Hecke character such that $\chi\,\mathbf{N}_K \in \Sigma_{cc}(b,\mathfrak{N},\epsilon)$. Let $E_{f,\chi}$ be the finite extension of E_f obtained by adjoining the values of χ.

Let $X_1(N)$ be the modular curve of level $\Gamma_1(N)$, ∞ a cusp of $X_1(N)$ and $J_1(N)$ the corresponding Jacobian. Let B_f be the abelian variety associated to f by the Eichler-Shimura correspondence and $T_f \subset E_f$ an order such that B_f has T_f-endomorphisms. Let $\Phi_f : J_1(N) \to B_f$ be the associated surjective morphism. Let ω_f be the differential form on $X_1(N)$ corresponding to f. We use the same notation for the corresponding one form on $J_1(N)$. Let $\omega_{B_f} \in \Omega^1(B_f/E_f)^{T_f}$ be the unique one form such that $\Phi_f^*(\omega_{B_f}) = \omega_f$. Here $\Omega^1(B_f/E_f)^{T_f}$ denotes the subspace of 1-forms given by

$$\Omega^1(B_f/E_f)^{T_f} = \big\{\omega \in \Omega^1(B_f/E_f) | [\lambda]^*\omega = \lambda\omega, \forall \lambda \in T_f\big\}.$$

Let A_b be an elliptic curve with endomorphism ring $\mathcal{O}_b = \mathbf{Z} + b\mathcal{O}$, defined over the ring class field H_b. Let t be a generator of $A_b[\mathfrak{N}]$. We thus obtain a point $(A_b, A_b[\mathfrak{N}], t) \in X_1(N)(H_{bN})$. Let $\Delta_b = [A_b, A_b[\mathfrak{N}], t] - (\infty) \in J_1(N)(H_{bN})$ be the corresponding Heegner point on the modular Jacobian. We regard χ as a character $\chi : Gal(H_{bN}/K) \to E_{f,\chi}$. Let $G_b = Gal(H_{bN}/K)$. Let H_χ be the abelian extension of H cut out by the character χ. To the pair (f,χ), we associate the Heegner point $P_f(\chi)$ given by

$$P_f(\chi) = \sum_{\sigma \in G_b} \chi^{-1}(\sigma)\Phi_f(\Delta_b^\sigma) \in B_f(H_\chi) \otimes_{T_f} E_{f,\chi}.$$

The restriction of the p-adic formal group logarithm gives a homomorphism $\log_{\omega_{B_f}} : B_f(H_\chi) \to \mathbf{C}_p$. We extend it to $B_f(H_\chi) \otimes_{T_f} E_{f,\chi}$ by $E_{f,\chi}$-linearity.

We now fix a finite order Hecke character η such that $\eta\mathbf{N}_K \in \Sigma_{cc}(c,\mathfrak{N},\epsilon)$, for some c. For $v|c^-$, let $\Delta_{\eta,v}$ be the finite group $\eta(\mathcal{O}_{K_v}^\times)$. Here $\mathcal{O}_{K_v}$ denotes the integer ring of the local field K_v. Let $l \neq p$ be an odd prime unramified in K and prime to cN. Let $H_{cNl^\infty} = \bigcup_{n\geq 0} H_{cNl^n}$ be the ring class field of conductor cNl^∞. Let $G_n = Gal(H_{cNl^n}/K)$ and $\Gamma_l = \varprojlim G_n$. Let $\mathfrak{X}_l$ denote the set of l-power order characters of Γ_l. As $\nu \in \mathfrak{X}_l$ varies, we consider the non-triviality of

$\log_{\omega_{B_f}}(P_f(\eta\nu))/p$ modulo p.

Our result is the following (cf. [14, Thm. A]).

Theorem 1.3.1. *Let $f \in S_2(\Gamma_0(N), \epsilon)$ be an elliptic newform and η a finite order Hecke character such that $\eta \boldsymbol{N}_K \in \Sigma_{cc}(c, \mathfrak{N}, \epsilon)$, for some c. In addition to the hypotheses (ord) and (Hg) hold, suppose that*

(1). The residual representation $\rho_f|_{G_K}$ mod $\mathfrak{m}_p$ is absolutely irreducible and
(2). $p \nmid \prod_{v|c^-} \Delta_{\eta,v}$.

Then, for all but finitely many $\nu \in \mathfrak{X}_l$ we have

$$v_p\left(\frac{\log_{\omega_{B_f}}(P_f(\eta\nu))}{p}\right) = 0.$$

In particular, for all but finitely many $\nu \in \mathfrak{X}_l$ the Heegner points $P_f(\eta\nu)$ are non-zero in $B_f(H_{\eta\nu}) \otimes_{T_f} E_{f,\eta\nu}$.

Our approach is modular. It is based on the p-adic Waldspurger formula due to Bertolini-Darmon-Prasanna (cf. [2, Thm. 5.13]) and Hida's strategy regarding non-triviality of anticyclotomic toric periods of a p-adic modular form modulo p (cf. [29], [32] and [14]).

Note that "In particular" part of the theorem involves only the prime l in its formulation. The part was conjectured by Mazur and proven by Cornut and Vatsal, independently (cf. [23] and [64]). Our approach gives a finer information regarding the p-adic logarithm. It also allows a rather smooth transition to the higher weight case.

Let (f, η) be a pair of an elliptic Hecke eigenform and a Hecke character over K as in [14, §3.3]. The corresponding Rankin-Selberg convolution is self-dual with root number -1. The Bloch-Beilinson conjecture implies the existence of a non-torsion null-homologous cycle in the Chow realisation of the motive associated to the convolution. The Bloch-Kato conjecture implies the non-triviality of the p-adic Abel-Jacobi image of the cycle. In this situation, a natural candidate for a non-trivial null-homologous cycle is the generalised Heegner cycle. The construction is due to Bertolini-Darmon-Prasanna (cf. [2, §2.3]) and generalises the one of classical Heegner cycles. The cycle lives in a middle dimensional Chow group of a fiber product of a Kuga-Sato variety arising from a modular curve curve and a self product of a CM elliptic curve. In the case of weight two, the cycle coincides with a Heegner point. For a prime l, twists of the theta series by l-power order anticyclotomic characters of K give rise to an Iwasawa theoretic family of generalised Heegner cycles. In [14, Thm. 3.5], we prove an analog of Theorem 4.1 for the p-adic Abel-Jacobi image of these cycles.

For other applications of the non-triviality, we refer to [14, Intro. and §4].

In [9], we prove an analogue of the theorem for Shimura curves over the rationals in the case $l = p$ (cf. §4.3). The analog is an input in the ongoing work of Jetchev-Skinner-Wan on the p-adic Birch and Swinnerton-Dyer conjecture. The result in turn is expected to play a key role in the ongoing work of Bhargava-Skinner-Zhang on the Birch and Swinnerton-Dyer conjecture for a large proportion of elliptic curves over the rationals.

1.4 Organisation

In this thesis, we restrict only to certain aspects of the non-triviality. In chapter 2, we describe a conjectural linear independence of mod p modular forms on Shimura varieties. In chapter

3, we prove the conjecture in the case of Shimura curves over the rationals. In chapter 4, we prove the mod p non-triviality of the p-adic Abel-Jacobi image of generalised Heegner cycles over $\mathbf{Z}_p$-anticyclotomic extension of the underlying imaginary quadratic extension. We refer to the introduction of the individual chapters for their organisation.

We refer to the articles [7], [8] and [11]-[16] for our other results regarding the non-triviality.

1.5 Notation

We use the following notation unless otherwise stated.

Let D_K be the different (resp. discriminant) of $K/\mathbf{Q}$. Let v be a place of $\mathbf{Q}$ and w be a place of K above v. Let $\mathbf{Q}_v$ be the completion of $\mathbf{Q}$ at v, ϖ_v an uniformiser and $K_v = \mathbf{Q}_v \otimes_{\mathbf{Q}} K$.

For a number field L, let $\mathbf{A}_L$ be the adele ring, $\mathbf{A}_{L,f}$ the finite adeles and $\mathbf{A}_{L,f}^{\square}$ the finite adeles away from a finite set of places $\square$ of L. Let G_L be the absolute Galois group of L and $rec_L : \mathbf{A}_L^{\times} \to G_L^{ab}$ the geometrically normalized reciprocity law.

CHAPTER 2

Conjectural Linear Independence of Mod p Modular Forms

In this chapter, we describe a conjectural linear independence of mod p modular forms on Shimura varieties.

2.1 Introduction

Zeta values seem to suggest deep phenomena in Mathematics. They seem to encode mysterious arithmetic information. Sometimes, they are a sum of evaluation of modular forms at CM points. Somewhat surprisingly, this seems to suggest a linear independence of mod p modular forms.

Such an expression for the critical Hecke L-values suggested (to Hida) a linear independence of mod p Hilbert modular forms. Based on Chai's theory of Hecke-stable subvarieties of a Shimura variety ([19], [20] and [21]), Hida proved this in [34, §3]. In this speculative chapter, we consider a linear independence of mod p modular forms on a PEL Shimura variety. We also consider a linear independence of relative mod p modular forms on a special subvariety of a PEL Shimura variety. Chai's global rigidity conjecture guided us to the current formulation of conjectures on linear independence.

We now give a rough formulation of the conjectures. For the precise hypothesis and formulation, we refer the reader to §2.2. Let p be a prime and $\mathbb{F}$ be an algebraic closure of

$\mathbb{F}_p$. Let $Sh_{/\mathbb{F}}$ be a PEL Shimura variety associated to a PEL datum unramified at p. Let G be the corresponding reductive group. Suppose that the ordinary locus $Sh^{ord}_{/\mathbb{F}}$ of $Sh_{/\mathbb{F}}$ is non-empty. Let x be an ordinary closed point in $Sh_{/\mathbb{F}}$. The group $G(\mathbf{Z}_{(p)})$ acts on Sh. Let $H_{x,Sh}$ be the stabiliser of x in $G(\mathbf{Z}_{(p)})$. Let V be an irreducible component of Sh containing x. Let $\pi : I \to V_{/\mathbb{F}}$ be the Igusa tower over V. Let $\tilde{x}$ be a point in $I_{/\mathbb{F}}$ over x. Let $f_1, ..., f_n$ be non-constant elements in $H^0(I, \mathcal{O}_{I_{/\mathbb{F}}})$ i.e. n non-constant mod p modular forms on V. For $1 \leq i \leq n$, let $a_i \in H_{x,Sh}(\mathbf{Z}_p)$ such that $a_i a_j^{-1} \notin H_{x,Sh}(\mathbf{Z}_{(p)})$ for all $i \neq j$. As π is étale, $\widehat{\mathcal{O}}_{V,x} \simeq \widehat{\mathcal{O}}_{I,\tilde{x}}$. Since a_i acts on $\widehat{\mathcal{O}}_{V,x}$, it acts on $\widehat{\mathcal{O}}_{I,\tilde{x}}$ via the previous isomorphism.

Our basic question is the following.

(Q) Are $(a_i(f_i))_i$ linearly independent in $\widehat{\mathcal{O}}_{I,\tilde{x}}$ over $\mathbb{F}$?

While trying to consider (Q), we are led to the notion of a strongly symmetric point on a Shimura variety. It is as follows.

Let $\Sigma_{x,Sh}$=$\{$ V': $V' \subset V$ closed geometrically irreducible, V' a special subvariety of Sh, $x \in V'$ and V' stable under the action of $H_{x,Sh}(\mathbf{Z}_{(p)})$ $\}$.

By definition, $V \in \Sigma_{x,Sh}$. We say that x is strongly symmetric in Sh if $\Sigma_{x,Sh} = \{V\}$. This notion seems to be interesting in its own right. Let $Z(H_{x,Sh})(\mathbf{Z}_p)$ be the center of $H_{x,Sh}(\mathbf{Z}_p)$.

Conjecture 1. If x is strongly symmetric in Sh and $a_i \in Z(H_{x,Sh})(\mathbf{Z}_p)$, then $(a_i(f_i))_i$ are linearly independent in $\widehat{\mathcal{O}}_{I,\tilde{x}}$ over $\mathbb{F}$.

We can also consider a relative version of the linear independence as follows. Let V^* be an irreducible special subvariety of Sh containing x corresponding to a Shimura subvariety

Sh^* and $\gamma \in G(\mathbf{A}_{\mathbf{Q},f}^{(p)})$. Let $\pi^*_{Sh} : I^*_{Sh} \to V^*$ be the relative Igusa tower given by the Cartesian diagram

$$\begin{array}{ccc} I^*_{Sh} & \hookrightarrow & I \\ \downarrow{\scriptstyle \pi^*_{Sh}} & & \downarrow{\scriptstyle \pi} \\ V^* & \hookrightarrow & V. \end{array}$$

Let $g_1, ..., g_n$ be non-constant elements in $H^0(I^*_{Sh/\mathbb{F}}, \mathcal{O}_{I^*_{Sh/\mathbb{F}}})$ i.e. n non-constant relative mod p modular forms on V^*. For $1 \le i \le n$, let $b_i \in H_{\gamma^{-1}x,Sh^*}(\mathbf{Z}_p)$ such that $b_i b_j^{-1} \notin H_{\gamma^{-1}x,Sh^*}(\mathbf{Z}_{(p)})$ for all $i \neq j$.

Our basic relative question is the following.

(RQ) Are $(b_i(g_i))_i$ linearly independent in $\widehat{\mathcal{O}}_{I^*_{Sh},\tilde{x}}$ over $\mathbb{F}$?

While trying to consider (RQ), we are lead to the notion of a relative strongly symmetric point on a special subvariety. It is as follows. Let $V' \in \Sigma_{x,Sh}$.

Let $\Sigma_{x,Sh} \cap V' = \{ V'' \colon V'' \subset V'$ closed geometrically irreducible, V'' a special subvariety of Sh, $x \in V''$ and V'' stable under the action of $H_{x,Sh}(\mathbf{Z}_{(p)}) \}$.

For definition of a special subvariety, we refer to Definition 2.2.1. By definition, $V' \in \Sigma_{x,Sh} \cap V'$. We say that x is strongly symmetric in V' relative to Sh if $\Sigma_{x,Sh} \cap V' = \{V'\}$. By definition, x is relatively strongly symmetric in V' for any V' of smallest positive dimension in $\Sigma_{x,Sh}$. Let $Z(H_{\gamma^{-1}x,Sh^*})(\mathbf{Z}_p)$ be the center of $H_{\gamma^{-1}x,Sh^*}(\mathbf{Z}_p)$.

Conjecture 2. (1). If x is strongly symmetric in V^* and $b_i \in Z(H_{\gamma^{-1}x,Sh^*})(\mathbf{Z}_p)$, then $(b_i(g_i))_i$ are linearly independent in $\widehat{\mathcal{O}}_{I^*_{Sh},\tilde{x}}$ over $\mathbb{F}$.

(2). If x is relatively strongly symmetric in V^*, $b_i \in Z(H_{\gamma^{-1}x,Sh^*})(\mathbf{Z}_p)$ and ϕ_{V^*} is $H_{x,Sh}(\mathbf{Z}_{(p)})$-equivariant, then $(b_i(g_i))_i$ are linearly independent in $\widehat{\mathcal{O}}_{I^*_{Sh},\tilde{x}}$ over $\mathbb{F}$.

The notation ϕ_{V^*} and the corresponding hypothesis in part (2) will be explained in §2.3.2.

In some cases, Conjecture 2 implies Conjecture 1 for the closed point $\gamma^{-1}x$. Thus, we expect that the answer to (Q) might be affirmative even if the hypothesis in Conjecture 1 is not satisfied.

Note that Conjecture 1 and Conjecture 2 are local in nature. While trying to consider question (Q) (resp. (RQ)), we are naturally led to a $H_{x,Sh}(\mathbf{Z}_{(p)})$-stable (resp. $H_{\gamma^{-1}x,Sh^*}(\mathbf{Z}_p)$-stable) subvariety of V^n (resp. $(V^*)^n$) containing x^n. In view of Chai's work, being stable under the action of a local algebraic stabiliser seems to be a rigid condition for a subvariety of a Shimura variety. We formulate a conjecture, namely Conjecture 3, about the global structure of this type of subvariety. This conjecture is inspired by Chai's global rigidity conjecture and Hida's result in the Hilbert modular case. We will refer a special case of Conjecture 3 as Conjecture 3*. It implies Conjecture 1. When Sh^* is PEL, Conjecture 3* for Sh^* implies part (1) of Conjecture 2. We do not know a local or any other way of approaching Conjecture 1 and Conjecture 2 besides Conjecture 3*. In the Hilbert modular case, Conjecture 3* has been proven by Hida. However, even in the Hilbert modular case, Conjecture 3 seems to be open. We also formulate relative version of Conjecture 3 (resp. Conjecture 3*) as Conjecture 4 (resp. Conjecture 4*). Conjecture 4* implies part (2) of Conjecture 2.

Somewhat weaker versions of the rigidity type conjectures are already present in Chai's work. However, from linear independence point of view it seems necessary to consider a stronger formulation.

It seems likely that some cases of the above conjectures can be proven based on Chai's theory.

We prove Conjecture 3^* for quaternionic Shimura varieties in [13]. In the future, we hope to consider new cases. In the Hilbert modular case, refined versions of the linear independence are considered in [11] (cf. §1.2). In the future, we also hope to consider conjectural generalisation of the refinement for other Shimura varieties.

The above formulation can also be done using the torus part $T_{x,Sh}(\mathbf{Z}_{(p)})$ of the local algebraic algebraic stabiliser $H_{x,Sh}(\mathbf{Z}_{(p)})$ (cf. Lemma 2.2.5). This formulation might be apt for the study of μ-invariant. As the formulation is analogous, we skip the details.

It would be interesting to consider the general case i.e. when x is not necessarily ordinary.

The chapter is organised as follows. In §2.2, we formulate the linear independence conjectures. In §2.2.1-2.2.8, we describe certain notions related to Shimura varieties. Some of these notions are motivated by the questions (Q) and (RQ). In §2.2.9 (resp. §2.2.10), Conjecture 1 (resp. Conjecture 2) is formulated. In §2.3, we consider locally stable subvarieties. In §2.3.1, we define a locally stable subvariety and give an example which arises while considering Conjecture 1. In §2.3.3, Conjecture 3 and Conjecture 3^* are formulated. Relative version is considered in §2.3.2 and §2.3.4. In §2.3.5, we describe a relation between these conjectures and Chai's global rigidity conjecture. We hope that this subsection gives some conceptual motivation behind the Conjectures.

2.2 Linear independence

In this section, we give more details of Conjecture 1 and Conjecture 2 formulated in the introduction. In §2.2.1-2.2.8, we describe certain notions related to Shimura varieties. Some of these notions are motivated by the questions (Q) and (RQ). In §2.2.9 (resp. §2.2.10), Conjecture 1 (resp. Conjecture 2) is formulated.

2.2.1 Shimura varieties

In this subsection, we briefly recall the notion of a Shimura variety and a special subvariety of a Shimura variety.

Let $G_{/\mathbf{Q}}$ be a reductive group. Let Z be the center of G. Let $h : Res_{\mathbf{C}/\mathbf{R}}\mathbb{G}_m \to G_{/\mathbf{R}}$ be an $\mathbf{R}$-group scheme homomorphism. Let X be the $G(\mathbf{R})$-conjugacy classes of h. Suppose that the pair (G, X) is a Shimura datum (cf. [31, §7.2.1]).

It gives rise to a smooth quasi-projective pro-algebraic variety $Sh(G, X)$ over a number field E (cf. [31, §7.2]), where E is the reflex field associated to (G, X) (cf. [31, §7.2.2]). Let $\overline{E}$ be an algebraic closure of E. We fix a complex embedding $\iota_\infty : \overline{E} \hookrightarrow \mathbf{C}$.

The complex points of Sh are given by

$$Sh(G, X)(\mathbf{C}) = G(\mathbf{Q})\backslash X \times G(\mathbf{A}_{\mathbf{Q},f})/\overline{Z(\mathbf{Q})}. \tag{2.2.1}$$

Here $\overline{Z(\mathbf{Q})}$ is the closure of the center $Z(\mathbf{Q})$ in $G(\mathbf{A}_{\mathbf{Q},f})$ under the adélic topology. From (2.1), $G(\mathbf{A}_{\mathbf{Q},f})$ acts on $Sh(G, X)(\mathbf{C})$.

(AT) By the general theory of Shimura varieties, $G(\mathbf{A}_{\mathbf{Q},f})$ also acts on $Sh(G, X)_{/E}$.

Let p be a prime. We can ask whether there exists a p-integral model $Sh^{(p)}_{/O_{E,(p)}}$ of $Sh/G(\mathbf{Z}_p)_{/E}$ extending the action of $G(\mathbf{A}^{(p)}_{\mathbf{Q},f})$, where $\mathbf{A}^{(p)}_{\mathbf{Q},f}$ denotes the finite adeles away from p. In [50], Milne has defined the notion of such a p-integral canonical model $Sh^{(p)}_{/O_{E,(p)}}$. Recently, Kisin has proven the existence of $Sh^{(p)}$ in many cases (cf. [46]). Under less restrictive conditions, Vasiu has announced the existence of $Sh^{(p)}$. In this article, we suppose that such model exists.

Let Sh' be a Shimura subvariety of Sh and E' be the corresponding reflex field. One of the characterising properties of a p-integral model implies that there exists a canonical closed immersion of $Sh'^{(p)}_{/O_{EE',(p)}}$ in $Sh^{(p)}_{/O_{EE',(p)}}$. We often identify $Sh'^{(p)}_{/O_{EE',(p)}}$ with its image and consider the image as a Shimura subvariety.

The following definition is basically due to Oort.

Definition 2.2.1. (Oort) A special subvariety of $Sh^{(p)}$ is a geometrically irreducible component of $\gamma Sh'^{(p)}$, for some $\gamma \in G(\mathbf{A}^{(p)}_{\mathbf{Q},f})$ and positive dimensional Shimura subvariety $Sh'^{(p)}$ of $Sh^{(p)}$.

For generalities and characterisations of special subvarieties, we refer the reader to [52] and [53].

2.2.2 PEL Shimura varieties

In this subsection, we briefly recall the notion of a Shimura variety associated to a PEL datum unramified at p. We closely follow [31, Ch. 7] and [33, §1].

Let D be a central simple algebra over a number field F with a positive involution ρ. Let F_0 be the subfield of F fixed by ρ. Note, F_0 is totally real and F equals F_0 or is a CM quadratic extension of F_0. Let $F_+^\times$ denote the totally positive elements in F_0. Let O be the ring of integers of F and $O_{(p)+}^\times$ be the intersection of $F_+^\times$ with $O_{(p)}$, where $O_{(p)} := O \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$. Let O_D be a maximal order of D stable under the action of ρ.

Let L be a projective O_D-module. Let $\langle .,. \rangle : V \times V \to F_0$ be a non-degenerate F_0-linear alternating form such that $\langle bx, y\rangle = \langle x, b^\rho y\rangle$ for all $b \in D$ and $x, y \in V$. Here $V = L_{\mathbf{Q}} := L \otimes_{\mathbf{Z}} \mathbf{Q}$.

Let C be the opposite algebra of $End_D(V)$. Note, C is a central simple algebra over F with an involution $*$ given by $\langle cx, y\rangle = \langle x, c^* y\rangle$ for all $c \in C$ and $x, y \in V$.

Let $G_{/\mathbf{Q}}$ be an algebraic group given by

$$G(R) := \{g \in C \otimes_{\mathbf{Q}} R : gg^* \in (F_0 \otimes_{\mathbf{Q}} R)^\times\}. \tag{2.2.2}$$

Here R is a $\mathbf{Q}$-algebra.

Let G_1 be the derived group of G.

Suppose that there exists an $\mathbf{R}$-group scheme homomorphism $h : Res_{\mathbf{C}/\mathbf{R}}\mathbb{G}_m \to G_{/\mathbf{R}}$ such that

(H1) $h(\bar{z}) = h(z)^*$.

Here z is a non-zero complex number and $\bar{z}$ is the complex conjugate of z.

Let X be the $G(\mathbf{R})$-conjugacy classes of h. The pair (G, X) is a Shimura datum (cf. [31, 7.2.1]). It gives rise to a smooth quasi-projective pro-algebraic variety $Sh(G, X)$ over a number field E (cf. §2.2.1 and [31, §7.1]).

Actually, $Sh_{/E}$ is the moduli variety of a functor $\mathcal{F}$ which classifies abelian schemes having multiplication by O_D along with additional structure (cf. [31, §7.1]).

Let us consider the following conditions.

(H2) The derived group G_1 is simply connected.
(H3) The prime p is unramified in $F/\mathbf{Q}$.

(H4) The pairing $\langle .,. \rangle$ induces $L_p := L \otimes_{\mathbf{Z}} \mathbf{Z}_p \simeq Hom(L_p, O_{0,p})$, where $O_{0,p} := O_0 \otimes_{\mathbf{Z}} \mathbf{Z}_p$.

(H5) $O_{D,p} := O_D \otimes_{\mathbf{Z}} \mathbf{Z}_p \simeq M_n(O_p)$, for some integer n, where $O_p := O \otimes_{\mathbf{Z}} \mathbf{Z}_p$.

Definition 2.2.2. The six-tuple $(D, \rho, O_D, V, \langle .,. \rangle, h)$ satisfying (H1) and (H3)-(H5) is called as a PEL datum unramified at p.

From now, we suppose the datum is unramified at p.

A p-integral moduli interpretation $\mathcal{F}^{(p)}$ of $\mathcal{F}$ leads to a smooth p-integral model $Sh^{(p)}_{/O_{E,(p)}}$ of $Sh/G(\mathbf{Z}_p)_{/E}$ (cf. [31, 7.1.3]). Here is a description of $\mathcal{F}^{(p)}$.

The functor $\mathcal{F}^{(p)}$ is given by

$$\mathcal{F}^{(p)} : SCH_{/O_{E,(p)}} \to SETS$$
$$S \mapsto \{(A, \iota, \bar{\lambda}, \eta^{(p)}, \det)_{/S}\}/\simeq . \tag{2.2.3}$$

Here,

(M1) A is abelian scheme over S of dimension of $\frac{1}{2} rank_{\mathbf{Z}} L$.

(M2) $\iota : O_D \hookrightarrow End_S A$ is an algebra embedding.

(M3) $\bar{\lambda}$ is the polarisation class of a polarisation λ up to scalar multiplication by $\iota(O^{\times}_{(p)+})$ which induces the Riemann form $\langle .,. \rangle$ on L upto scalar multiplication by $O^{\times}_{(p)+}$.

(M4) Let $\mathcal{T}(A)$ be the Tate module $\varprojlim A[N]$ of A. Let $\mathcal{T}^{(p)}(A) = \mathcal{T}(A) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$. $\eta^{(p)}$ is a prime-to-p level structure given by an O_D-linear isomorphism $\eta^{(p)} : L \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}^{(p)} \simeq \mathcal{T}^{(p)}(A)$, where $\widehat{\mathbf{Z}}^{(p)} = \prod_{l \neq p} \mathbf{Z}_l$.

(M5) Let $\Omega_{A/S}$ be the relative sheaf of differentials. The notation det is a condition on $\Omega_{A/S}$ as an $O_D \otimes \mathcal{O}_S$-module (cf. [31, 7.1.1]).

The notation $\simeq$ denotes up to a prime-to-p isogeny.

In the rest of the article, we drop *det* from the notation.

2.2.3 Local algebraic stabiliser

In this subsection, we describe the notion of the local algebraic stabiliser of a closed point in a Shimura variety.

Let $Sh_{/E}$ be a Shimura variety $Sh(G,X)_{/E}$ as in §2.2.1. Suppose G has a p-integral integral model $G_{/\mathbf{Z}_{(p)}}$. Also, suppose $Sh^{(p)}_{/O_{E,(p)}}$ is smooth.

Let $\overline{\mathbf{Q}}_p$ be an algebraic closure of $\mathbf{Q}_p$. We fix a p-adic embedding $\iota_p : \overline{E} \hookrightarrow \overline{\mathbf{Q}}_p$. Let $\mathcal{W}$ be the strict Henselisation inside $\overline{E}$ of the local ring of $O_{E,(p)}$ corresponding to ι_p. Let $\mathbb{F}$ be the residue field of $\mathcal{W}$. Note, $\mathbb{F}$ is an algebraic closure of $\mathbb{F}_p$.

Let $Sh^{(p)}_{/\mathcal{W}} = Sh^{(p)} \times_{O_{E,(p)}} \mathcal{W}$ and $Sh^{(p)}_{/\mathbb{F}} = Sh^{(p)}_{/\mathcal{W}} \times_{\mathcal{W}} \mathbb{F}$.

From now, let Sh denote $Sh^{(p)}_{/\mathbb{F}}$. By a Shimura subvariety (resp. special subvariety) of Sh, we simply mean the reduction mod p of a Shimura subvariety (resp. special subvariety) of $Sh^{(p)}_{/\mathcal{W}}$.

Let x be a closed point in Sh. Recall, $G(\mathbf{Z}_{(p)})$ acts on Sh.

Definition 2.2.3. The stabiliser of x in $G(\mathbf{Z}_{(p)})$ is called the local algebraic stabiliser of x in Sh. It will be denoted by $H_{x,Sh}(\mathbf{Z}_{(p)})$.

Note that $H_{x,Sh}(\mathbf{Z}_{(p)})$ is just an abuse of notation and $H_{x,Sh}(\mathbf{Z}_{(p)})$ need not equal the $\mathbf{Z}_{(p)}$-points of a group scheme. However, we will see below that in the PEL case $H_{x,Sh}(\mathbf{Z}_{(p)})$ equals the $\mathbf{Z}_{(p)}$-points of a group scheme $H_{x,Sh/\mathbf{Z}}$.

(PS) Let $H_{x,Sh}(\mathbf{Z}_p)$ be the p-adic closure of $H_{x,Sh}(\mathbf{Z}_{(p)})$ in $G(\mathbf{Z}_p)$.

Let V'^s be a special subvariety of Sh corresponding to a Shimura subvariety Sh' and $\gamma \in G(\mathbf{A}^{(p)}_{\mathbf{Q},f})$ (cf. Definition 2.2.1). We have an isomorphism $Sh' \simeq \gamma Sh'$. Via this isomorphism, $G'(\mathbf{A}_{\mathbf{Q},f})$ acts on $\gamma Sh'$. Let y be a closed point in V'^s. By definition, $H_{\gamma^{-1}y,Sh'}(\mathbf{Z}_{(p)})$ acts on V'^s. In fact, it is the stabiliser of y in $G'(\mathbf{A}^f)$.

We have an inclusion $H_{\gamma^{-1}y,Sh'}(\mathbf{Z}_{(p)}) \hookrightarrow H_{\gamma^{-1}y,Sh}(\mathbf{Z}_{(p)})$.

(CH) In view of the conjectural interpretation of Shimura varieties as moduli of motives (cf. [51]), it seems that under some conditions we have a homomorphism $\varphi_{Sh'} : H_{\gamma^{-1}y,Sh}(\mathbf{Z}_{(p)}) \to H_{\gamma^{-1}y,Sh'}(\mathbf{Z}_{(p)})$.

We now suppose that Sh arises from a PEL datum unramified at p as in §2.2.2. Let x be a closed point of Sh corresponding to $(A_x, \iota_x, \bar{\lambda}_x, \eta_x^{(p)})_{/\mathbb{F}}$. In view of the moduli interpretation of the $G(\mathbf{Z}_{(p)})$-action, it follows that $H_{x,Sh}(\mathbf{Z}_{(p)})$ equals the $\mathbf{Z}_{(p)}$-points of an algebraic group $H_{x,Sh/\mathbf{Z}}$ defined as follows.

Definition 2.2.4. (Chai) The unitary group associated to $(End_{O_D}(A_x) \otimes_{\mathbf{Z}} \mathbf{Q}, *_x)$ is called the local stabiliser of x. Here $*_x$ is the Rosati involution associated to x. It will be denoted by $H_{x,Sh}$.

Note that $H_{x,Sh}$ has a natural integral structure coming from $End_{O_D} A_x$. The following lemma follows from the definitions.

Lemma 2.2.5. *Suppose $P \subset End_{O_D}(A_x) \otimes_{\mathbf{Z}} \mathbf{Q}$ is a maximal commutative semi-simple F-*

subalgebra stable under the Rosati involution $*_x$. *Then,* $H_{x,Sh}(\mathbf{Z}_{(p)})$ *contains a torus* $T_{x,Sh}(\mathbf{Z}_{(p)}) := \{g \in O^{\times}_{P,(p)} : gg^{*_x} = 1\}$. *Here* $O^{\times}_{P,(p)} := P \cap (End_{O_D}(A_x) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)})$.

By definition, $T_{x,Sh}$ is a maximal torus in $H_{x,Sh}$. In general, the local stabiliser $H_{x,Sh}$ can be non-commutative. For example, we refer to [19, pp. 443].

2.2.4 Distinguished points

In this subsection, we firstly recall Chai's notion of a hypersymmetric point on a PEL Shimura variety. Then, we introduce the notion of a strongly symmetric point (resp. relatively strongly symmetric point) on a Shimura variety (resp. on a special subvariety).

In characteristic zero, we have the notion of a special point on a Shimura variety (cf. [31, §7.2.2]). In the PEL case, this corresponds to CM abelian varieties. By a theorem of Tate, every abelian variety over $\mathbb{F}$ is a CM abelian variety. However, there are other notions of a distinguished point on a mod p Shimura variety. In the PEL case, Chai's hypersymmetric point corresponds to a closed point x having as large local stabiliser $H_{x,Sh}$ as allowed by the slope constraint. We introduce the notion of a strongly symmetric point on a mod p Shimura variety. Roughly speaking, this corresponds to a closed point x such that the Shimura variety Sh has no proper positive dimensional special subvariety containing x which is stable under the action of local algebraic stabiliser $H_{x,Sh}(\mathbf{Z}_{(p)})$.

2.2.4.1 Hypersymmetric points

In this subsection, we briefly recall Chai's notion of a hypersymmetric point on a PEL Shimura variety. We closely follow [21, §5] and [22].

Let Sh be a PEL Shimura variety as in §2.2.3 and x be a closed point of Sh corresponding to $(A_x, \iota_x, \bar{\lambda}_x, \eta_x^{(p)})_{/\mathbb{F}}$.

Definition 2.2.6. (Chai) We say that x is hypersymmetric in Sh if the canonical map

$$End_{O_D}(A_{x/\mathbb{F}}) \otimes_{\mathbf{Z}} \mathbf{Z}_p \to End_{O_D}(A_x[p^\infty]_{/\mathbb{F}})$$

is an isomorphism.

It is expected that we can often find hypersymmetric points on an irreducible component of Sh (cf. [21, §5] and [22]). If the semisimple $\mathbf{Q}$-rank of G equals one, then every closed point of Sh is hypersymmetric (cf. [21, §5] and [22]).

As $H_{x,Sh}(\mathbf{Z}_{(p)})$ acts on Sh stabilising x, $H_{x,Sh}(\mathbf{Z}_p)$ acts on $\widehat{\mathcal{O}}_{Sh,x}$. Suppose that A_x is ordinary. By Serre-Tate deformation theory, $Spf(\widehat{\mathcal{O}}_{Sh,x})$ has a natural structure of a formal torus (cf. [31, §8.2]). The following property can be considered to be a distinguishing feature of ordinary hypersymmetric points.

Theorem 2.2.7. *(Chai) The action of $H_{x,Sh}(\mathbf{Z}_p)$ on the co-character group $X_*(Spf(\widehat{\mathcal{O}}_{Sh,x}))$ underlies an absolutely irreducible representation of $H_{x,Sh}(\mathbf{Q}_p)$.*

2.2.4.2 Strongly symmetric points

In this subsection, we introduce the notion of a strongly symmetric point on a Shimura variety. The notion is motivated by the linear independence question (Q).

Let Sh be a positive-dimensional Shimura variety as in §2.2.3 and x be a closed point.

Supposing the existence of a smooth toroidal compactification of $Sh^{(p)}_{/O_{E,(p)}}$, from Zariski's connectedness theorem it follows that every connected component of Sh is irreducible. Let V be a geometrically irreducible component of Sh containing x.

Definition 2.2.8. Let $\Sigma_{x,Sh}$=$\{$ V': $V' \subset V$ closed geometrically irreducible, V' a special subvariety of Sh, $x \in V'$ and V' stable under the action of $H_{x,Sh}(\mathbf{Z}_{(p)})\}$.

By definition, $V \in \Sigma_{x,Sh}$. Note that $\Sigma_{x,Sh}$ is not necessarily closed under intersection as an intersection of positive dimensional special subvarieties might be a zero dimensional special subvariety.

Definition 2.2.9. We say that x is strongly symmetric in Sh if $\Sigma_{x,Sh} = \{V\}$.

The following lemma gives a supply of strongly symmetric points on PEL Shmiura varieties with non-empty ordinary locus.

Lemma 2.2.10. *Every ordinary hypersymmetric point on a PEL Shimura variety is strongly symmetric.*

Proof. Let x be such a point. Suppose that it is not strongly symmetric. Thus, there exists $V' \in \Sigma_{x,Sh}$ such that $V' \neq V$. By the local stabiliser principle (cf. [21, Prop. 6.1]), $Spf(\widehat{\mathcal{O}}_{V',x})$ is a proper positive-dimensional subscheme of $Spf(\widehat{\mathcal{O}}_{Sh,x})$ stable under the action of $H_{x,Sh}(\mathbf{Z}_p)$. This contradicts Theorem 2.2.7. $\square$

In the PEL case, we often expect to find strongly symmetric points on an irreducible component of Sh.

It can be shown that every closed point on a Hilbert modular Shimura variety is strongly symmetric. Thus, a strongly symmetric point is not necessarily hypersymmetric.

The following question seems to be interesting.

(SSP) Can we always find a strongly symmetric point on a Shimura variety?

We close this subsection with an example of a point on a Siegel modular variety which is not strongly symmetric.

Let $g \geq 2$ be an integer. Let K be a CM field of dimension $2g$ and O_K be its ring of integers. Let K_0 be a maximal totally real subfield of K and O_{K_0} be its ring of integers. Let Sh be the Siegel modular Shimura variety corresponding to abelian schemes of dimension g and Sh' be the Hilbert modular variety corresponding to the totally real field K_0. In view of the moduli problems, we can regard Sh' as a proper Shimura subvariety of Sh. Let x be a closed point in Sh such that $End(A_x) = O_K$. As $O_{K_0} \hookrightarrow End(A_x)$, we can also regard x as a point in Sh'. Note, Sh' is stable under the action of the local stabiliser $H_{x,Sh}(\mathbf{Z}_{(p)})$. Thus, x is not strongly symmetric in Sh.

Roughly speaking, the above example suggests that if $H_{x,Sh}$ is much smaller than G, then x is less likely to be strongly symmetric in Sh.

2.2.4.3 Relatively strongly symmetric points

In this subsection, we introduce the notion of a relatively strongly symmetric point on a special subvariety of a Shimura variety. The notion is motivated by the relative linear independence question (RQ).

Let the notation be as in §2.2.4.2. Let $V' \in \Sigma_{x,Sh}$.

Definition 2.2.11. Let $\Sigma_{x,Sh} \cap V' = \big\{ V''\colon\ V'' \subset V'$ closed geometrically irreducible, V'' a special subvariety of Sh, $x \in V''$ and V'' stable under the action of $H_{x,Sh}(\mathbf{Z}_{(p)})\big\}$.

By definition, $V' \in \Sigma_{x,Sh} \cap V'$.

Definition 2.2.12. We say that x is strongly symmetric in V' relative to Sh if $\Sigma_{x,Sh} \cap V' = \{V'\}$.

By definition, x being strongly symmetric in Sh is equivalent to x being strongly symmetric in Sh relative to itself. Note that x is strongly symmetric in V' relative to Sh for any V' of smallest dimension in $\Sigma_{x,Sh}$.

2.2.5 Igusa tower

In this subsection, we briefly recall the definition of an Igusa tower over a PEL Shimura variety. We closely follow [33, §1].

Let Sh be a PEL Shimura variety. Suppose that the Hasse-invariant does not vanish on Sh. Thus, it does not vanish on every connected component of Sh. For a group theoretic characterisation of such Shimura varieties, we refer the reader to [67]. Let Sh^{ord} be the subscheme of Sh on which the Hasse-invariant does not vanish. From Grothendieck's specialisation theorem for crystals (cf. [28]), it follows that Sh^{ord} is an open subscheme of Sh. From the existence of a smooth toroidal compactification of $Sh^{(p)}_{/O_{E,(p)}}$ (cf. [47]) and Zariski's connectedness theorem, it follows that every connected component of Sh is irreducible. Thus, Sh^{ord} is dense in Sh.

Lemma 2.2.13. *If the Hasse invariant does not vanish on Sh, then Sh^{ord} is dense in Sh.*

Let $\mathcal{A}$ be the universal abelian scheme over Sh. Over Sh^{ord}, the connected part $\mathcal{A}[p^m]^\circ$ of $\mathcal{A}[p^m]$ is étale-locally isomorphic to $\mu_{p^m} \otimes_{\mathbf{Z}_p} L^\circ$ as an $O_{D,p}$-module, where L° is an $O_{D,p}$-direct summand of L_p. L° is a maximally isotropic subspace of L_p such that $rank_{\mathbf{Z}_p} L^\circ = \frac{1}{2} rank_{\mathbf{Z}} L$. For an explicit description of L°, we refer the reader to [33, pp. 9].

We now define the Igusa tower. For $m \in \mathbb{N}$, the m^{th}-layer of the Igusa tower over Sh^{ord} is defined by

$$Ig_m = \underline{\mathrm{Isom}}_{O_{D,p}}(\mu_{p^m} \otimes_{\mathbf{Z}_p} L^\circ, \mathcal{A}[p^m]^\circ). \tag{2.2.4}$$

Note that the projection $\pi_m : Ig_m \to Sh^{ord}$ is finite and étale. The full Igusa tower over Sh^{ord} is defined by

$$Ig = Ig_\infty = \varprojlim Ig_m = \underline{\mathrm{Isom}}_{O_{D,p}}(\mu_{p^\infty} \otimes_{\mathbf{Z}_p} L^\circ, \mathcal{A}[p^\infty]^\circ). \tag{2.2.5}$$

(Et) The projection $\pi : Ig \to Sh^{ord}$ is étale.

Let V be an irreducible component of Sh and V^{ord} be $V \cap Sh^{ord}$. Let I be the inverse image of V^{ord} under π. In [33], under the hypothesis (H2) Hida has shown that

(Ir) I is an irreducible component of Ig.

If I is not geometrically irreducible, we pick a geometrically irreducible component still denoting it by the same notation.

2.2.6 Relative Igusa tower

In this subsection, we introduce the notion of a relative Igusa tower over a special subvariety of a PEL Shimura variety.

Let the notation be as in §2.2.5. Let V^* be a geometrically irreducible special subvariety of Sh contained in V.

Let $\pi^*_{Sh} : I^*_{Sh} \to V^*$ be the relative Igusa tower given by the Cartesian diagram

$$\begin{array}{ccc} I^*_{Sh} & \hookrightarrow & I \\ \downarrow{\scriptstyle \pi^*_{Sh}} & & \downarrow{\scriptstyle \pi} \\ V^* & \hookrightarrow & V. \end{array}$$

If I^*_{Sh} is not geometrically irreducible, we pick a geometrically irreducible component still denoting it by the same notation.

If V^* is corresponds to a PEL Shimura subvariety Sh^*, we have two notions of Igusa tower over V^* namely the intrinsic Igusa tower I^* as in §2.2.5 and the relative Igusa tower I^*_{Sh}.

(C) Are I^* and I^*_{Sh} related?

In general, they do not seem to be the same.

2.2.7 Mod p modular forms

In this subsection, we define mod p modular forms on an irreducible component of a PEL Shimura variety.

Let V and I be as in §2.2.2. Let B be an $\mathbb{F}$-algebra. The space of mod p-modular forms on V over B is defined by

$$M(V,B) = H^0(I_{/B}, \mathcal{O}_{I_{/B}}). \tag{2.2.6}$$

Here $I_{/B} := I \times_{\mathbb{F}} B$. In view of (2.2.4), we have the following geometric interpretation of mod p modular forms.

A mod p modular form is a function f of isomorphism classes of $\tilde{x} = (x, \eta_p^\circ)_{/B'}$ where B' is a B-algebra, $x = (A, \iota, \bar{\lambda}, \eta^{(p)})_{/B'} \in cf.^{(p)}(B')$ and $\eta_p^\circ : \mu_{p^\infty} \otimes_{\mathbf{Z}_p} L^\circ \simeq A[p^\infty]^\circ$ is an $O_{D,p}$-linear isomorphism, such that the following conditions are satisfied.

(G1) $f(\tilde{x}) \in B'$.
(G2) If $\tilde{x} \simeq \tilde{x}'$, then $f(\tilde{x}) = f(\tilde{x}')$, where $\tilde{x} \simeq \tilde{x}'$ means $x \simeq x'$ and the corresponding isomorphism between A and A' induces an isomorphism between η_p° and $\eta_p'^\circ$.
(G3) $f(\tilde{x} \times_{B'} B'') = q(f(\tilde{x}))$ for any B-algebra homomorphism $q : B' \to B''$.

2.2.8 Relative mod p modular forms

In this subsection, we define relative mod p modular forms on a special subvariety of a PEL Shimura variety.

Let the notation be as in §2.2.6. Let B be an $\mathbb{F}$-algebra. The space of mod p-modular forms on V^* over B relative to Sh is defined by

$$M_{Sh}(V^*,B) = H^0(I^*_{Sh/B}, \mathcal{O}_{I^*_{Sh/B}}). \tag{2.2.7}$$

Here $I^*_{Sh/B} := I^*_{Sh} \times_{\mathbb{F}} B$.

If V^* corresponds to a PEL Shimura subvariety, we have two notions of mod p modular forms over V^* namely intrinsic mod p modular forms as in §2.2.7 and relative mod p modular forms as above. The question (C) basically asks whether these two notions are related. In general, these notions seem to be different.

2.2.9 Linear independence

In this subsection, we formulate a conjecture on a linear independence of mod p modular forms on an irreducible component of a PEL Shimura variety.

Let x be a closed point in Sh^{ord} carrying $(A_x, \iota_x, \bar{\lambda}_x, \eta_x^{(p)})_{/\mathbb{F}}$. Let V be the irreducible component of Sh containing x. Let $\pi : I \to V$ be the Igusa tower over V.

Let η_p° be a level p^∞-structure on $A_x[p^\infty]^\circ$. This gives rise to a closed point $\tilde{x} = (x, \eta_p^\circ)$ in I over x. As $H_{x,Sh}(\mathbf{Z}_{(p)})$ acts on V stabilising x, $H_{x,Sh}(\mathbf{Z}_p)$ acts on $\widehat{\mathcal{O}}_{V,x}$. From (Et), $\widehat{\mathcal{O}}_{V,x} \simeq \widehat{\mathcal{O}}_{I,\tilde{x}}$. Via this isomorphism, we let $H_{x,Sh}(\mathbf{Z}_p)$ act on $\widehat{\mathcal{O}}_{I,\tilde{x}}$.

Let $f_1, ..., f_n$ be non-constant elements in $H^0(I_{/\mathbb{F}}, \mathcal{O}_{I_{/\mathbb{F}}})$ i.e. n non-constant mod p modular forms on V. For $1 \leq i \leq n$, let $a_i \in H_{x,Sh}(\mathbf{Z}_p)$ such that $a_i a_j^{-1} \notin H_{x,Sh}(\mathbf{Z}_{(p)})$ for all $i \neq j$.

Our basic question is the following.

(Q) Are $(a_i(f_i))_i$ linearly independent in $\widehat{\mathcal{O}}_{I,\tilde{x}}$ over $\mathbb{F}$?

Let $Z(H_{x,Sh})(\mathbf{Z}_p)$ be the center of $H_{x,Sh}(\mathbf{Z}_p)$.

Conjecture 1. *If x is strongly symmetric in Sh and $a_i \in Z(H_{x,Sh})(\mathbf{Z}_p)$, then $(a_i(f_i))_i$ are linearly independent in $\widehat{\mathcal{O}}_{I,\tilde{x}}$ over $\mathbb{F}$.*

Some of the reasons behind the hypotheses will be explained in §2.3 (cf. (EQ) and (E)).

Recall that in the Hilbert modular case every closed point is strongly symmetric. In this case, Conjecture 1 has been proven by Hida (cf. [34, §3.5]).

2.2.10 Relative linear independence

In this subsection, we formulate a conjecture on a linear independence of relative mod p modular forms on a special subvariety of a PEL Shimura variety.

Let the notation be as in §2.2.6 and suppose $x \in V^*$.

Let $g_1, ..., g_n$ be non-constant elements in $H^0(I^*_{Sh/\mathbb{F}}, \mathcal{O}_{I^*_{Sh/\mathbb{F}}})$ i.e. n non-constant relative mod p modular forms on V^*. For $1 \leq i \leq n$, let $b_i \in H_{\gamma^{-1}x,Sh^*}(\mathbf{Z}_p)$ such that $b_i b_j^{-1} \notin H_{\gamma^{-1}x,Sh^*}(\mathbf{Z}_{(p)})$ for all $i \neq j$.

Our basic relative question is the following.

(RQ) Are $(b_i(g_i))_i$ linearly independent in $\widehat{\mathcal{O}}_{I^*_{Sh},\tilde{x}}$ over $\mathbb{F}$?

Conjecture 2. *(1). If x is strongly symmetric in V^* and $b_i \in Z(H_{\gamma^{-1}x,Sh^*})(\mathbf{Z}_p)$, then $(b_i(g_i))_i$ are linearly independent in $\widehat{\mathcal{O}}_{I^*_{Sh},\tilde{x}}$ over $\mathbb{F}$.*
(2). If x is relatively strongly symmetric in V^*, $b_i \in Z(H_{\gamma^{-1}x,Sh^*})(\mathbf{Z}_p)$ and ϕ_{V^*} is $H_{x,Sh}(\mathbf{Z}_{(p)})$-equivariant, then $(b_i(g_i))_i$ are linearly independent in $\widehat{\mathcal{O}}_{I^*_{Sh},\tilde{x}}$ over $\mathbb{F}$.

The notation ϕ_{V^*} and the corresponding hypothesis in part (2) will be explained in §2.3.2. Some of the reasons behind the other hypotheses will be explained in §2.3 (cf. (EQ) and (E)).

Let a_i's be as in §2.2.9. If x is strongly symmetric in V^* or relatively strongly symmetric in Sh and $b_i = \varphi_{Sh^*}(a_i) \in Z(H_{\gamma^{-1}x,Sh^*})(\mathbf{Z}_p)$ such that $b_i b_j^{-1} \notin H_{\gamma^{-1}x,Sh^*}(\mathbf{Z}_{(p)})$ for all $i \neq j$, then Conjecture 2 implies that the answer to (Q) is affirmitive for the closed point $\gamma^{-1}x$. Recall, x is relative strongly symmetric for V^* of smallest positive dimension in $\Sigma_{x,Sh}$. Thus, we expect that the answer to (Q) might be affirmative for the closed point $\gamma^{-1}x$ if (CH) exists for Sh^*, $b_i = \varphi_{Sh^*}(a_i)$'s satisfy the condition in Conjecture 2 and ϕ_{V^*} is $H_{x,Sh}(\mathbf{Z}_{(p)})$-equivariant.

2.3 Locally stable subvarieties

In this section, we consider locally stable subvarieties of Sh^n. While trying to consider question (Q) (resp. (RQ)), we are naturally led to a $H_{x,Sh}(\mathbf{Z}_{(p)})$-stable subvariety of V^n (resp. $(V^*)^n$) containing x^n. In §2.3.1, we introduce the notion of a locally stable subvariety. In view of Chai's theory, being stable under the local algebraic stabiliser seems to be rigid condition. In §2.3.3, we formulate this as Conjecture 3. We call a special case of Conjecture 3 as Conjecture 3*. Conjecture 3* seems to be closely related to Conjecture 1 and part (1) of Conjecture 2. Relative versions are considered in §2.3.2 and §2.3.4. In §2.3.5, we describe a relation between these conjectures and Chai's global rigidity conjecture.

2.3.1 Definition

In this subsection, we first define a locally stable subvariety of an irreducible component of a self-product of a Shimura variety. Then, we give an example. In this subsection, any tensor-product is taken n-times.

Let Sh be a Shimura variety as in §2.2.3 and V be a geometrically irreducible component of Sh.

Definition 2.3.1. A subvariety Y of V^n is said to be locally stable if there exists a closed point $y^n = (y, ..., y) \in Y$ such that Y is stable under the diagonal action of $H_{y,Sh}(\mathbf{Z}_{(p)})$.

We now suppose that Sh is PEL and let the notation be as in §2.2.9.

While trying to consider the linear independence question (Q), this type of subvariety arises as follows.

Consider an $\mathbb{F}$-algebra homomorphism

$$\phi_I : \mathcal{O}_{I,\tilde{x}} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{I,\tilde{x}} \to \widehat{\mathcal{O}}_{I,\tilde{x}} \tag{2.3.1}$$

given by

$$f_1 \otimes ... \otimes f_n \mapsto \prod_{i=1}^{i=n} a_i(f_i). \tag{2.3.2}$$

As we are interested in the linear independence of $(a_i(f_i))_i$, we consider $\mathfrak{b}_I := ker(\phi_I)$.

Similarly, consider an $\mathbb{F}$-algebra homomorphism

$$\phi = \phi_V : \mathcal{O}_{V,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V,x} \to \widehat{\mathcal{O}}_{V,x} \tag{2.3.3}$$

given by

$$h_1 \otimes ... \otimes h_n \mapsto \prod_{i=1}^{i=n} a_i(h_i). \tag{2.3.4}$$

(EQ) Note that ϕ is equivariant with the $H_{x,Sh}(\mathbf{Z}_{(p)})$-action if and only if $a_i \in Z(H_{x,Sh})(\mathbf{Z}_p)$ for all $1 \leq i \leq n$. This is basically the reason behind the hypothesis on a_i's in Conjecture 1.

From now, we suppose that $a_i \in Z(H_{x,Sh})(\mathbf{Z}_p)$, for all $1 \leq i \leq n$. Let $\mathfrak{b} = ker(\phi_V)$.

Lemma 2.3.2. *We have* $\mathfrak{b}_I = 0$ *if and only if* $\mathfrak{b} = 0$.

Proof. In view of (Et), we have an étale morphism

$$\pi^m : \mathcal{O}_{V,x} \otimes_{\mathbb{F}} \ldots \otimes_{\mathbb{F}} \mathcal{O}_{V,x} \to \mathcal{O}_{I,\tilde{x}} \otimes_{\mathbb{F}} \ldots \otimes_{\mathbb{F}} \mathcal{O}_{I,\tilde{x}}.$$

Note, $\mathfrak{b}_I$ is the unique prime ideal of $\mathcal{O}_{I,\tilde{x}} \otimes_{\mathbb{F}} \ldots \otimes_{\mathbb{F}} \mathcal{O}_{I,\tilde{x}}$ over $\mathfrak{b}$. This finishes the proof. □

As ϕ is equivariant with the $H_{x,Sh}(\mathbf{Z}_{(p)})$-action, $\mathfrak{b}$ is a prime ideal of $\mathcal{O}_{V,x} \otimes_{\mathbb{F}} \ldots \otimes_{\mathbb{F}} \mathcal{O}_{V,x}$ stable under the diagonal action of $H_{x,Sh}(\mathbf{Z}_{(p)})$. Let X be the Zariski closure of Spec $(\mathcal{O}_{V,x} \otimes_{\mathbb{F}} \ldots \otimes_{\mathbb{F}} \mathcal{O}_{V,x}/\mathfrak{b})$ in V^n. Thus, X is a closed irreducible subscheme of V^n containing x^n stable under the diagonal action of $H_{x,Sh}(\mathbf{Z}_{(p)})$. In particular, X is a closed irreducible locally stable subscheme of V^n.

2.3.2 Relative definition

In this subsection, we introduce the notion of a relative locally stable subvariety of a self-product of a subvariety in $\Sigma_{x,Sh}$. Then, we give an example. In this subsection, any tensor-product is taken n-times.

Let the notation be as in §2.2.10 and pick $V^* \in \Sigma_{x,Sh}$.

Definition 2.3.3. A subvariety Y^* of $(V^*)^n$ is said to be relative locally stable if there exists a closed point $y^n = (y, \ldots, y) \in Y^*$ such that Y^* is stable under the diagonal action of

$H_{y,Sh^*}(\mathbf{Z}_{(p)})$ and $H_{y,Sh}(\mathbf{Z}_{(p)})$.

While trying to consider the relative linear independence question (RQ), this type of subvariety arises as follows.

Proceeding as in §2.3.1, we define $\phi_{I^*_{Sh}}$ and ϕ_{V^*}. Let $\mathfrak{b}_{I^*_{Sh}} = ker(\phi_{I^*_{Sh}})$ and $\mathfrak{b}_{V^*} = ker(\phi_{V^*})$. By the same argument as in Lemma 2.3.2, $\mathfrak{b}_{I^*_{Sh}} = 0$ if and only if $\mathfrak{b}_{V^*} = 0$.

Let $b_i \in Z(H_{x,Sh^*})(\mathbf{Z}_p)$, for all $1 \leq i \leq n$. We suppose that ϕ_{V^*} is equivariant under the action of $H_{x,Sh}(\mathbf{Z}_{(p)})$. Thus, $\mathfrak{b}_{V^*}$ is a prime ideal of $\mathcal{O}_{V^*,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V^*,x}$ stable under the diagonal action of $H_{x,Sh^*}(\mathbf{Z}_{(p)})$ and $H_{x,Sh}(\mathbf{Z}_{(p)})$.

Let X^* be the Zariski closure of $\operatorname{Spec}(\mathcal{O}_{V^*,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V^*,x}/\mathfrak{b}_{V^*})$ in $(V^*)^n$. Thus, X^* is a closed irreducible subscheme of $(V^*)^n$ containing x^n stable under the diagonal action of $H_{x,Sh^*}(\mathbf{Z}_{(p)})$ and $H_{x,Sh}(\mathbf{Z}_{(p)})$. In particular, X^* is a closed irreducible relative locally stable subscheme of $(V^*)^n$.

2.3.3 Global structure of locally stable subvarieties

In this subsection, we formulate a conjecture on the global structure of a $H_{x,Sh}(\mathbf{Z}_{(p)})$-stable subscheme of V^n.

In this subsection, we do not suppose that Sh is necessarily PEL.

We first axiomatise the example X in §2.3.1.

(N1) Let $S_{/\mathbb{F}}$ be $\operatorname{Spec}(\mathcal{O}_{V,x})$.

(N2) Let $\mathfrak{b}$ be a prime ideal of $\mathcal{O}_{V,x}\otimes_{\mathbb{F}}...\otimes_{\mathbb{F}}\mathcal{O}_{V,x}$ stable under the diagonal action of $H_{x,Sh}(\mathbf{Z}_{(p)})$. Let $\mathcal{X}_{/\mathbb{F}}$ be Spec $(\mathcal{O}_{V,x}\otimes_{\mathbb{F}}...\otimes_{\mathbb{F}}\mathcal{O}_{V,x}/\mathfrak{b})$.

(N3) Let X be the Zariski closure of $\mathcal{X}$ in V^n.

Note that X is positive dimensional as $\mathfrak{b}$ is a prime ideal.

Let l be a positive integer. For $1 \leq i \leq l$, let $\alpha_i \in H_{x,Sh}(\mathbf{Z}_{(p)})$.

Definition 2.3.4. The skewed diagonal $\Delta_{\alpha_1,...,\alpha_l}$ of V^l is defined by

$$\Delta_{\alpha_1,...,\alpha_l} = \big\{(\alpha_1(y), ..., \alpha_l(y)) : y \in V\big\}.$$

If $\alpha_i \in Z(H_{x,Sh})(\mathbf{Z}_p)$, then $\Delta_{\alpha_1,...,\alpha_l}$ is a $H_{x,Sh}(\mathbf{Z}_p)$-stable subscheme of V^l. In general, $\Delta_{\alpha_1,...,\alpha_l}$ is not necessarily a $H_{x,Sh}(\mathbf{Z}_p)$-stable subscheme of V^l.

The global structure of X is predicted by the following conjecture.

Conjecture 3*. Let X be as in (N1)-(N3) and suppose that x is strongly symmetric in the Shimura variety Sh.

(1). If $n = 1$, then $X = V$.

(2). Let $n \geq 2$. Let S' be the product of the first $n-1$ factors of S^n and $S'' = S$ be the last factor of S^n. Suppose that the projections of $\mathcal{X}$ to S' and S'' induce dominant morphisms $\pi'_{\mathcal{X}} : \mathcal{X} \to S'$ and $\pi''_{\mathcal{X}} : \mathcal{X} \to S''$, respectively. Then, X equals V^n or $V^{n-2} \times \Delta_{\alpha_1,\alpha_2}$ for some $\alpha_1, \alpha_2 \in H_{x,Sh}(\mathbf{Z}_{(p)})$.

Remark. We will see in §2.3.5 that part (1) of Conjecture 3* follows from Chai's global rigidity conjecture and another conjecture of Chai (§2.3.3). Also, it can be shown that the case of Conjecture 3* when n is greater than two follows from the cases when n equals one and n equals two (cf. similar to [34, pp. 95-98]). We skip the details. When Sh is a quaternionic Shimura variety, the details can be found in [13, §5.6].

(E) In general, Conjecture 3^* does not hold if x is not strongly symmetric in Sh. This is basically the reason behind the hypothesis on x in Conjecture 1 and part (1) of Conjecture 2.

Here is a relation between Conjecture 3^*, Conjecture 1 and part (1) of Conjecture 2.

Proposition 2.3.5. *(1). Conjecture 3* implies Conjecture* 1.
(2). Let Sh^* be as in §2.10. If Sh^* is PEL, Conjecture 3 for Sh^* implies part (1) of Conjecture 2.*

Proof. We only prove part (1). The proof of part (2) is similar to that of part (1).

Let X be as in §2.3.1. It can be shown that X satisfies the hypothesis in Conjecture 3^* up to a permutation of the factors (cf. similar to [34, Prop. 3.11]).

When $n = 1$, part (1) of Conjecture 3^* implies that $X = V$. Thus, $\mathfrak{b} = 0$ and we are done.

When $n \geq 2$, upto a permutation of the factors part (2) of Conjecture 3^* implies that X equals V^n or $V^{n-2} \times \Delta_{\alpha_1,\alpha_2}$ for some $\alpha_1, \alpha_2 \in H_{x,Sh}(\mathbf{Z}_{(p)})$. If $X = V^n$, then we are done. Suppose $X = V^{n-2} \times \Delta_{\alpha_1,\alpha_2}$. From Serre-Tate deformation theory of the ordinary abelian scheme corresponding to x, it can be shown that there exist $1 \leq i \neq j \leq n$ such that $a_i a_j^{-1} = \alpha_1 \alpha_2^{-1}$ (similar to [34, pp. 98-99]). We skip the details. When Sh is a quaternionic Shimura variety, the details can be found in [14, §4.5]. Thus, $a_i a_j^{-1} \in H_{x,Sh}(\mathbf{Z}_{(p)})$. A contradiction. □

Remark. The above proof shows that Conjecture 3^* implies if $(f_i)_i$'s are algebraically independent in $\mathcal{O}_{I,\tilde{x}}$ over $\mathbb{F}$, then $(a_i(f_i))_i$'s are algebraically independent in $\widehat{\mathcal{O}}_{I,\tilde{x}}$ over $\mathbb{F}$.

When Sh^* is not PEL, we do not know whether the above proposition holds.

Hida has proven Conjecture 3^* in the Hilbert modular case, thereby proving linear independence (cf. [34, §3.5]). In the quaternionic case, Conjecture 3^* is proven in [14].

The hypothesis in part (2) of Conjecture 3^* is not satisfied in general. Here is a general conjecture about the global structure of $H_{x,Sh}(\mathbf{Z}_{(p)})$-stable subvarieties.

Conjecture 3. Let X be a closed, irreducible and $H_{x,Sh}(\mathbf{Z}_{(p)})$-stable subvariety of V^n containing x^n. Suppose x is strongly symmetric in Sh. Then, there exist non-negative integers k, l, m and $\alpha_i \in H_{x,Sh}(\mathbf{Z}_{(p)})$, for all $1 \le i \le l$, such that upto a permutation of the factors, X equals $V^k \times \Delta_{\alpha_1,\ldots,\alpha_l} \times x^m$ with $k + l + m = n$.

Note that Conjecture 3 evidently implies Conjecture 3^*.

2.3.4 Global structure of relative locally stable subvarieties

In this subsection, we formulate a conjecture on the global structure of a $H_{x,Sh^*}(\mathbf{Z}_{(p)})$ and $H_{x,Sh}(\mathbf{Z}_{(p)})$-stable subscheme of $(V^*)^n$, for V^* as in §2.3.2.

Let the notation be as in §2.3.2.

We first axiomatise the example X^* in §2.3.2.

(RN1) Let $S^*_{/\mathbb{F}}$ be $\operatorname{Spec}(\mathcal{O}_{V^*,x})$.
(RN2) Let $\mathfrak{b}^*$ be a prime ideal of $\mathcal{O}_{V^*,x} \otimes_{\mathbb{F}} \ldots \otimes_{\mathbb{F}} \mathcal{O}_{V^*,x}$ stable under the diagonal action of

$H_{x,Sh^*}(\mathbf{Z}_{(p)})$ and $H_{x,Sh}(\mathbf{Z}_{(p)})$. Let $\mathcal{X}^*_{/\mathbb{F}}$ be $\mathrm{Spec}\,(\mathcal{O}_{V^*,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V^*,x}/\mathfrak{b}^*)$.
(RN3) Let X^* be the Zariski closure of $\mathcal{X}^*$ in $(V^*)^n$.

Note, X^* is positive dimensional as $\mathfrak{b}^*$ is a prime ideal.

The global structure of X^* is predicted by the following conjecture.

Conjecture 4*. Let X^* be as in (RN1)-(RN3) and suppose that x is relative strongly symmetric in Sh^*.
(1). If $n = 1$, then $X^* = V^*$.
(2). Let $n \geq 2$. Let $(S^*)'$ be the product of the first $n-1$ factors of $(S^*)^n$ and $(S^*)'' = S^*$ be the last factor of $(S^*)^n$. Suppose that the projections of $\mathcal{X}^*$ to $(S^*)'$ and $(S^*)''$ induce dominant morphisms $\pi'_{\mathcal{X}^*} : \mathcal{X}^* \to (S^*)'$ and $\pi''_{\mathcal{X}^*} : \mathcal{X}^* \to (S^*)''$, respectively. Then, X^* equals $(V^*)^n$ or $(V^*)^{n-2} \times \Delta_{\beta_1,\beta_2}$ for some $\beta_1, \beta_2 \in H_{x,Sh^*}(\mathbf{Z}_{(p)})$.

Remark. In §2.3.5, we will see that part (1) of Conjecture 4* follows from Chai's global rigidity conjecture and another conjecture of Chai (§2.3.5). Also, it can be shown that the case of Conjecture 4* when n is greater than two follows from the cases when n equals one and n equals two (cf. similar to [34, pp. 95-98]). We skip the details.

Here is a relation between Conjecture 4* and part (2) of Conjecture 2.

Proposition 2.3.6. *If Sh^* is PEL, Conjecture 4* implies part (2) of Conjecture 2.*

The proof is similar to Proposition 2.3.5. We skip the details.

When Sh^* is not PEL, we do not know whether the above proposition holds.

The hypothesis in part (2) of Conjecture 4^* is not satisfied in general. Here is a general conjecture about the global structure of $H_{x,Sh^*}(\mathbf{Z}_{(p)})$ and $H_{x,Sh}(\mathbf{Z}_{(p)})$-stable stable subvarieties.

Conjecture 4. Let X^* be a closed, irreducible, $H_{x,Sh^*}(\mathbf{Z}_{(p)})$ and $H_{x,Sh}(\mathbf{Z}_{(p)})$-stable subvariety of $(V^*)^n$ containing x^n. Suppose x is relative strongly symmetric in Sh. Then, there exist non-negative integers k, l, m and $\beta_i \in H_{x,Sh^*}(\mathbf{Z}_{(p)})$, for all $1 \leq i \leq l$, such that up to a permutation of the factors, X^* equals $(V^*)^k \times \Delta_{\beta_1,\ldots,\beta_l} \times x^m$ with $k + l + m = n$.

Note that Conjecture 4 evidently implies Conjecture 4^*.

2.3.5 Chai's global rigidity conjecture

In this subsection, we describe a relation between Chai's global rigidity conjecture and our conjectures. We hope that this subsection gives some conceptual motivation behind the conjectures.

In this subsection, we suppose that Sh is PEL with non-empty ordinary locus. We first recall a couple of definitions due to Chai (cf. [20, §5]).

Let Z be a closed irreducible subscheme of $(V^{ord})^n$. Let z be a closed point of Z. From Serre-Tate deformation theory, $Spf(\widehat{\mathcal{O}}_{V^n,z})$ has a natural structure of a formal torus (cf. [31, §8.2]).

Definition 2.3.7. (Chai) Z is said to be Tate-linear at z if $Spf(\widehat{\mathcal{O}}_{Z,z})$ is a formal subtorus of $Spf(\widehat{\mathcal{O}}_{V^n,z})$. Z is said to be Tate-linear if it is Tate-linear at every closed point.

Definition 2.3.8. (Chai) Let $f : Y \to Z$ be the normalisation of Z. Z is said to be weakly

Tate-linear at z if $Spf(\widehat{\mathcal{O}}_{Y,y})$ is a formal subtorus of $Spf(\widehat{\mathcal{O}}_{V^n,z})$ for every point y in Y above z. Z is said to be weakly Tate-linear if it is weakly Tate-linear at every closed point.

(TL) Chai has conjectured that a weakly Tate-linear subvariety is Tate-linear.

Special subvarieties give rise to Tate-linear subvarieties in the following way. Let $V^\star$ be a special subvariety of Sh^n contained in V^n. Noot has proven that $(V^\star)^{ord}$ is a Tate-linear subvariety of V^n (cf. [58] and [53]), where $(V^\star)^{ord}$ denotes the inverse image of $V^\star$ in $(V^{ord})^n$. Chai has conjectured that all Tate-linear subvarieties arise in this way (cf. [20, §5]).

Conjecture. *(Global rigidity) If Z is a Tate-linear subvariety of V^n, there exists a special subvariety $V^\star$ of Sh^n such that $(V^\star)^{ord} = Z$.*

In characteristic zero, this statement has been proven by Moonen in [53] and is in fact one of the motivations behind the previous Conjecture.

Here is a relation between $n = 1$ case of part (1) of Conjecture 3^* and the global rigidity conjecture.

Proposition 2.3.9. *Suppose (TL) holds. Then, the global rigidity conjecture implies part (1) of Conjecture 3^* and part (1) of Conjecture 4^*.*

Proof. We only prove that the global rigidity conjecture implies part (1) of Conjecture 3^*. The proof of the remaining proposition is similar.

Recall, $H_{x,Sh}(\mathbf{Z}_{(p)})$ acts on X stabilising x. Thus, $Spf(\widehat{\mathcal{O}}_{X,x})$ is a formal subscheme of $Spf(\widehat{\mathcal{O}}_{V,x})$ stable under the action of $H_{x,Sh}(\mathbf{Z}_{(p)})$. From Chai's local rigidity (cf. [20, §6]), it follows that $Spf(\widehat{\mathcal{O}}_{X,x})$ is a finite union of formal subtori of $Spf(\widehat{\mathcal{O}}_{V,x})$. Proceeding as in [34, Prop. 3.8], it can be shown that $Spf(\widehat{\mathcal{O}}_{X,x})$ is in fact a formal subtorus of $Spf(\widehat{\mathcal{O}}_{V,x})$. Summing up, $Spf(\widehat{\mathcal{O}}_{X,x})$ is a formal subtorus of $Spf(\widehat{\mathcal{O}}_{V,x})$ stable under the action of $H_{x,Sh}(\mathbf{Z}_{(p)})$.

In particular, X^{ord} is a Tate-linear subvariety of V at x. As this property propagates (cf. [20, Prop. 5.3]), X is a weakly Tate-linear subvariety of V. From (TL), X is a Tate-linear subvariety of V.

By the global rigidity conjecture, there exists a special subvariety $V^\star \subset Sh$ such that $(V^\star)^{ord} = X^{ord}$. The action of $H_{x,Sh}(\mathbf{Z}_{(p)})$ preserves the isomorphism class of the p-divisible group of abelian varieties on Sh. As X is $H_{x,Sh}(\mathbf{Z}_{(p)})$-stable, it follows that $X^{ord} = (V^\star)^{ord}$ is $H_{x,Sh}(\mathbf{Z}_{(p)})$-stable. In particular, $V^\star$ is $H_{x,Sh}(\mathbf{Z}_{(p)})$-stable

Thus, $V^\star \in \Sigma_{Sh,x}$. As x is strongly symmetric, $V^\star = V$. □

Remark. Let X be as in Conjecture 3. If (TL) holds, then the above argument shows that X is a Tate-linear subvariety of V^n. Thus, in view of the global rigidity conjecture, Conjecture 3 gives a classification of the special subvarieties of V^n stable under the diagonal action of $H_{x,Sh}(\mathbf{Z}_{(p)})$ when x is strongly symmetric. A similar remark applies in the relative version. It would be interesting to try to verify this directly.

CHAPTER 3

Linear Independence of Mod p Modular Forms on Shimura Curves

In this chapter, we prove a linear independence of mod p modular forms on Shimura curves arising from indefinitie quaternion algebras over the rationals.

3.1 Introduction

Chai-Oort rigidity principle predicts that a Hecke stable subvariety of a mod p Shimura variety is a Shimura subvariety. This is a mod p analog of André-Oort conjecture. It has been proven by Chai for PEL Shimura varieties of type C (cf. [19], [20] and [21]).

In [34, §3.], Hida found a rather unexpected application of Chai's theory in the shape of an Ax-Lindemann type functional independence of mod p Hilbert modular forms. Hida basically adapts Chai's theory for self-products of the mod p Hilbert modular Shimura variety and local analog of the Hecke symmetries. Based on the independence, Hida determined the Iwasawa μ-invariant of anticyclotomic Katz p-adic L-functions (cf. [34, Thm. I]) and pave the way for related results (cf. §1.1). In [11, §3], another surprising application of Chai's theory was found in the shape that a mod p Hilbert modular form is determined by its restriction to the partial Serre-Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$.

In this chapter, we consider the analogue of the geometric results in [34] for mod p modular forms on Shimura curves arising from indefinitie quaternion algebras over the rationals.

In the next chapter, we consider the applications to a class of Iwasawa μ-invariants and generalised Heegner cycles.

Let p be an odd prime. We fix two embeddings $\iota_\infty \colon \overline{\mathbf{Q}} \to \mathbf{C}$ and $\iota_p \colon \overline{\mathbf{Q}} \to \mathbf{C}_p$. Let v_p be the p-adic valuation induced by ι_p so that $v_p(p) = 1$.

Let $K/\mathbf{Q}$ be an imaginary quadratic extension and $\mathcal{O}$ the ring of integers. As K is a subfield of the complex numbers, we regard it as a subfield of the algebraic closure $\overline{\mathbf{Q}}$ via the embedding ι_∞. Let c be the complex conjugation on $\mathbf{C}$ which induces the unique non-trivial element of $Gal(K/\mathbf{Q})$ via ι_∞. We assume the following:

(ord) p splits in K.

Let $\mathfrak{p}$ be a prime above p in K induced by the p-adic embedding ι_p. For a positive integer m, let H_m be the ring class field of K with conductor m and $\mathcal{O}_m = \mathbf{Z} + m\mathcal{O}$ the corresponding order. Let H be the Hilbert class field.

Let N be a positive integer such that $p \nmid N$. We fix a decomposition $N = N^+N^-$ where N^+ (resp. N^-) is only divisible by split (resp. ramified or inert) primes in $K/\mathbf{Q}$. We assume the following hypotheses:

(h1) The level N is square-free and prime to the discriminant of K.
(h2) The number of primes dividing N^- is positive and even.
(h3) The conductor N_ϵ divides N^+.

Let B be an indefinite quaternion algebra over $\mathbf{Q}$ of conductor N^-. In this chapter, we consider a linear independence of mod p modular forms on Shimura curves arising from B.

The chapter is organised as follows. In §3.2, we describe generalities regarding the Shimura curve. In §3.3, we consider the Serre-Tate deformation space of an ordinary closed point on the Shimura curve. Some parts of §3.3 perhaps logically come before §3.2. For example, the conclusion of §3.3.1 is needed in the beginning of §3.2.4. We suggest the reader to proceed accordingly. For convenience, we maintain the current ordering. In §3.4, we prove the independence of mod p modular forms.

3.2 Shimura curves

In this section, we describe generalities regarding Shimura curves arising from indefinite quaternion algebras over the rationals.

3.2.1 Setup

In this subsection, we recall a basic setup regarding Shimura curves. We occasionally follow [6]

Let the notation and hypotheses be as in the introduction. In particular, B denotes the indefinite quaternion algebra over $\mathbf{Q}$ of conductor N^-. Let $\mathcal{O}_B$ be a maximal order in [6, §2.1].

Let p_0 be an auxiliary prime such that:

(A1) For a prime l, the Hilbert symbol $(p_0, N^-)_l = 1$ if and only if $l|N^-$.
(A2) All primes dividing pN^+ split in the real quadratic field $M_0 := \mathbf{Q}(\sqrt{p_0})$.

The choice of p_0 determines a Hashimoto model for B as follows. The algebra B has a basis $\{1, t, j, tj\}$ as a vector space with $t^2 = -N^-$, $j^2 = p_0$ and $tj = -jt$ (cf. [6, §2.1]). Moreover, the $\mathbf{Z}$-span of the basis is contained in a unique maximal order in $\mathcal{O}_B$. Let $\dagger$ be an involution on B given by $b \mapsto b^\dagger = t^{-1}\bar{b}t$ for $b \in B$. Here $b \mapsto \bar{b}$ is the main involution of B.

Let $G_{/\mathbf{Q}}$ be the algebraic group $B^\times$ and $h_0 : Res_{\mathbf{C}/\mathbf{R}}\mathbb{G}_m \to G_{/\mathbf{R}}$ be the morphism of real group schemes arising from

$$a + bi \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

where $a + bi \in \mathbf{C}^\times$. Let X be the set of $G(\mathbf{R})$-conjugacy classes of h_0. We have a canonical isomorphism $X \simeq \mathbf{C} - \mathbf{R}$. The pair (G, X) satisfies Deligne's axioms for a Shimura variety. It gives rise to a tower $(Sh_K = Sh_K(G, X))_K$ of smooth proper curves over $\mathbf{Q}$ indexed by open compact subgroups K of $G(\mathbf{A}_{\mathbf{Q},f})$. The complex points of these curves are given by

$$Sh_K(\mathbf{C}) = G(\mathbf{Q})\backslash X \times G(\mathbf{A}_{\mathbf{Q},f})/K. \tag{3.2.1}$$

In what follows, we consider the case when K arises from the congruence subgroup $\Gamma_0(N^+)$. Here $\Gamma_0(N^+)$ denotes the norm one elements in an Eichler order of level N^+ in $\mathcal{O}_B$. We use the notation Sh_B to denote the corresponding Shimura curve.

3.2.2 p-integral model

In this subsection, we briefly recall p-integral smooth models of the Shimura curves. We occasionally follow [6].

Let the notation and hypotheses be as in §3.2.1. The Shimura curve $Sh_{B/\mathbf{Q}}$ represents a functor $\mathcal{F}$ classifying abelian surfaces having multiplication by $\mathcal{O}_B$ along with additional structure (cf. [6, §2.2]). A p-integral interpretation of $\mathcal{F}$ leads to a p-integral smooth model of $Sh_{B/\mathbf{Q}}$. A closely related functor $\mathcal{F}^{(p)}$ gives rise to the relevant Shimura variety of level prime to N^-p i. e. a tower of Shimura curves of level prime to N^-p. For later purposes, we consider the

Shimura variety.

The functor $\mathcal{F}.^{(p)}$ is given by

$$\mathcal{F}^{(p)} : SCH_{/\mathbf{Z}_{(p)}} \to SETS$$
$$S \mapsto \{(A, \iota, \bar{\lambda}, \eta^{(p)})_{/S}\}/\sim . \tag{3.2.2}$$

Here,

(PM1) A is an abelian surface over S.

(PM2) $\iota : \mathcal{O}_B \hookrightarrow End_S A$ is an algebra embedding.

(PM3) $\bar{\lambda}$ is a $\mathbf{Z}_{(p)}^{+}$-polarisation class of a homogeneous polarisation λ such that the Rosati involution of $End_S A$ maps $\iota(l)$ to $\iota(l^{\dagger})$ for $l \in \mathcal{O}_B$.

(PM4) Let $\mathcal{T}^{(N^{-}p)}(A)$ be the prime to $N^{-}p$ Tate module $\varprojlim_{(M,N^{-}p)=1} A[M]$. The notation $\eta^{(p)}$ denotes a full level structure of level prime to $N^{-}p$ given by an $\mathcal{O}_B$-linear isomorphism

$$\eta^{(p)} : \mathcal{O}_B \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}^{(N^{-}p)} \simeq \mathcal{T}^{(N^{-}p)}(A)$$

for $\widehat{\mathbf{Z}}^{(N^{-}p)} = \prod_{l \nmid N^{-}p} \mathbf{Z}_l$.

The notation $\sim$ denotes up to a prime to $N^{-}p$ isogeny.

Theorem 3.2.1 (Morita, Kottwitz). *The functor $\mathcal{F}^{(p)}$ is represented by a smooth proper pro-scheme $Sh^{(p)}_{/\mathbf{Z}[\frac{1}{N}]}$. (cf. [6, §2.2]).*

Let $\mathcal{A}$ be the universal abelian surface.

3.2.3 Idempotent

In this subsection, we describe generalities regarding an idempotent in the ring of algebraic correspondences on the universal abelian surface.

Let the notation and hypotheses be as in §3.2.1. Let $\epsilon \in \mathcal{O}_B \otimes \mathcal{O}_{M_0}[\frac{1}{2p_0}]$ be the non-trivial idempotent given by

$$\epsilon = \frac{1}{2}\left(1 \otimes 1 + \frac{1}{p_0} j \otimes \sqrt{p_0}\right).$$

Based on the hypotheses (h1) and (h2), we have an optimal embedding with respect to the Eichler order given by

$$\iota_K : K \hookrightarrow B$$

(cf. [6, §2.4]).

In view of the moduli interpretation of the Shimura variety $Sh^{(p)}$, we can let ϵ naturally act on the test objects as a self-isogeny. It follows that ϵ can be regarded as a self-isogeny on the universal abelian surface $\mathcal{A}$.

Via the p-adic embedding ι_p, we often regard the idempotent ϵ as an element in $\mathcal{O}_B \otimes \mathbf{Z}_p$. In view of the moduli interpretation, the p-divisible group $\mathcal{A}[p^m]$ has a natural structure of an $\mathcal{O}_B \otimes \mathbf{Z}_p$-module for a positive integer m. We thus have a natural action of the idempotent ϵ on the p-divisible group $\mathcal{A}[p^m]$. The action plays a key role in the article.

Heuristically, the idempotent often reduces the complexity arising from the non-commutative nature of the quaternion algebra to a commutative one. Such situations occasionally arise in the article.

Remark. An analogous idempotent arises in the case of an unitary model of a quaternionic Shimura variety over a totally real field (cf. [17] and [13]).

3.2.4 CM points

In this subsection, we introduce notation regarding CM points on the Shimura variety.

Let the notation and hypotheses be as in §3.2.2. Let b be a positive integer prime to N and recall that $\mathcal{O}_b$ denote the order $\mathbf{Z} + b\mathcal{O}$. We also recall that $Pic(\mathcal{O}_b)$ denotes the corresponding ring class group of conductor b.

Associated to an ideal class $[\mathfrak{a}] \in Pic(\mathcal{O}_b)$, we have the corresponding CM point $x(\mathfrak{a})$ on the Shimura variety $Sh^{(p)}$ (cf. [6, §2.4] and [48, §1.1]). Moreover, these CM points are Galois conjugates (cf. [6, §2.4]).

3.2.5 Igusa tower

In this subsection, we briefly recall the notion of p-ordinary Igusa tower over the Shimura variety. We follow [31, Ch. 8] and [33, §1].

Let the notation and hypotheses be as in §3.2.2. Recall that p is a prime such that $p \nmid N$. Let $\mathcal{W}$ be the strict Henselisation inside $\overline{\mathbf{Q}}$ of the local ring of $\mathbf{Z}_{(p)}$ corresponding to ι_p. Let $\mathbb{F}$ be the residue field of $\mathcal{W}$. Note that $\mathbb{F}$ is an algebraic closure of $\mathbb{F}_p$.

Let $Sh^{(p)}_{/\mathcal{W}} = Sh^{(p)} \times_{\mathbf{Z}[\frac{1}{N}]} \mathcal{W}$ and $Sh^{(p)}_{/\mathbb{F}} = Sh^{(p)}_{/\mathcal{W}} \times_{\mathcal{W}} \mathbb{F}$.

From now, let Sh denote $Sh^{(p)}_{/\mathbb{F}}$. Recall that $\mathcal{A}$ denotes the universal abelian surface over

Sh.

Let Sh^{ord} be the subscheme of Sh i.e. the locus on which the Hasse-invariant does not vanish. It is an open dense subscheme. Over Sh^{ord}, the connected part $\mathcal{A}[p^m]^\circ$ of $\mathcal{A}[p^m]$ is étale-locally isomorphic to $\mu_{p^m} \otimes_{\mathbf{Z}_p} \mathcal{O}_B^\circ$ as an $\mathcal{O}_{B,p}$-module. Here $\mathcal{O}_{B,p} = \mathcal{O}_B \otimes \mathbf{Z}_p$ and $\mathcal{O}_B^\circ$ is an $\mathcal{O}_{B,p}$-irreducible left ideal of $\mathcal{O}_{B,p}$ stable under the action of the idempotent ϵ. Moreover, $\mathcal{O}_B^\circ$ is a maximally isotropic subspace of $\mathcal{O}_{B,p}$ with respect to the bilinear form associated to the quadratic form arising from the reduced norm such that $rank_{\mathbf{Z}_p}\mathcal{O}_B^\circ = 2$. Let $\mathcal{O}_B^{ét}$ be the orthogonal complement of $\mathcal{O}_B^\circ$. For an explicit description of the submodules, we refer to the end of §3.3.1.

We now define the Igusa tower. For $m \in \mathbb{N}$, the m^{th}-layer of the Igusa tower over Sh^{ord} is defined by

$$Ig_m = \underline{\mathrm{Isom}}_{\mathcal{O}_{B,p}}(\mu_{p^m} \otimes_{\mathbf{Z}_p} \mathcal{O}_B^\circ, \mathcal{A}[p^m]^\circ). \tag{3.2.3}$$

Note that the projection $\pi_m : Ig_m \to Sh^{ord}$ is finite and étale.

In view of the description of the moduli functor $\mathcal{F}^{(p)}$, we have the following isomorphism

$$Ig_m \simeq \underline{\mathrm{Isom}}_{\epsilon\mathcal{O}_{B,p}}(\mu_{p^m} \otimes_{\mathbf{Z}_p} \epsilon\mathcal{O}_B^\circ, \epsilon(\mathcal{A}[p^m]^\circ)). \tag{3.2.4}$$

The Cartier duality and the polarisation $\bar{\lambda}_x$ induces an isomorphism

$$Ig_m \simeq \underline{\mathrm{Isom}}_{\mathcal{O}_{B,p}}(\mathcal{O}_B^{ét}/p^m\mathcal{O}_B^{ét}, \mathcal{A}[p^m]^{ét}) \simeq \underline{\mathrm{Isom}}_{\epsilon\mathcal{O}_{B,p}}(\epsilon(\mathcal{O}_B^{ét}/p^m\mathcal{O}_B^{ét}), \epsilon(\mathcal{A}[p^m]^{ét})). \tag{3.2.5}$$

The full Igusa tower over Sh^{ord} is defined by

$$Ig = \varprojlim Ig_m = \underline{\mathrm{Isom}}_{\mathcal{O}_{B,p}}(\mu_{p^\infty} \otimes_{\mathbf{Z}_p} \mathcal{O}_B^\circ, \mathcal{A}[p^\infty]^\circ) \simeq \underline{\mathrm{Isom}}_{\epsilon\mathcal{O}_{B,p}}(\mu_{p^\infty} \otimes_{\mathbf{Z}_p} \epsilon\mathcal{O}_B^\circ, \epsilon(\mathcal{A}[p^\infty]^\circ)). \tag{3.2.6}$$

In view of (3.2.5) and (3.2.6), it suffices to study the projection by ϵ of the p-divisible group.

(Et) Note that the projection $\pi : Ig \to Sh^{ord}$ is étale.

Let x be a closed ordinary point in Sh and A_x the corresponding abelian surface. We recall that Sh denotes the Shimura variety over the chosen algebraic closure $\mathbb{F}$. We have the following description of the p^∞-level structure on $A_x[p^\infty]$.

(PL) Let η_p° be a p^∞-level structure on $A_x[p^\infty]^\circ$. It is given by an $\mathcal{O}_{B,p}$-module isomorphisms $\eta_p^\circ : \mu_{p^\infty} \otimes_{\mathbf{Z}_p} \mathcal{O}_B^\circ \simeq A_x[p^\infty]^\circ$. The Cartier duality and the polarisation $\bar{\lambda}_x$ induces an isomorphism $\eta_p^{ét} : O_{B,p}^{ét} \simeq A_x[p^\infty]^{ét}$.

Let V be a geometrically irreducible component of Sh and V^{ord} the intersection $V \cap Sh^{ord}$. Let I be the inverse image of V^{ord} under π. In [31, Ch.8] and [33], it has been shown that

(Ir) I is a geometrically irreducible component of Ig.

3.2.6 Mod p modular forms

In this subsection, we briefly recall the notion of mod p modular forms on an irreducible component of the Shimura variety.

Let V and I be as in §3.2.5. Let C be an $\mathbb{F}$-algebra. The space of mod p modular forms on V over C is defined by

$$M(V,C) = H^0(I_{/C}, \mathcal{O}_{I_{/C}}) \tag{3.2.7}$$

for $I_{/C} := I \times_{\mathbb{F}} C$.

In view of §3.2.2 and §3.2.5, we have the following geometric interpretation of mod p modular forms.

A mod p modular form is a function f of isomorphism classes of $\tilde{x} = (x, \eta_p^\circ)_{/C'}$ where C' is a C-algebra, $x = (A, \iota, \bar{\lambda}, \eta^{(p)})_{/C'} \in cf.^{(p)}(C')$ and $\eta_p^\circ : \mu_{p^\infty} \otimes_{\mathbf{Z}_p} \mathcal{O}_B^\circ \simeq A[p^\infty]^\circ$ is an $\mathcal{O}_{B,p}$-linear isomorphism, such that the following conditions are satisfied.

(G1) $f(\tilde{x}) \in C'$.

(G2) If $\tilde{x} \simeq \tilde{x}'$, then $f(\tilde{x}) = f(\tilde{x}')$. Here $\tilde{x} \simeq \tilde{x}'$ denotes that $x \simeq x'$ and the corresponding isomorphism between A and A' induces an isomorphism between η_p° and $\eta_p'^\circ$.

(G3) $f(\tilde{x} \times_{C'} C'') = h(f(\tilde{x}))$ for any C-algebra homomorphism $h : C' \to C''$.

We have an analogous notion of p-adic (resp. classical) modular forms (resp., cf. [31, Ch. 8] and [6, §3.1]). In this thesis, we often regard classical modular forms as p-adic modular forms in the usual way.

3.3 Serre-Tate deformation space

In this section, we describe certain aspects of the Serre-Tate deformation space of the Shimura variety. In §3.3.1-3.3.2, we describe generalities regarding the Serre-Tate deformation theory of an ordinary closed point on the Shimura variety. In §3.3.3, we consider certain Hecke operators and determine their action on the Serre-Tate expansion of classical modular forms around CM points on the Shimura variety.

The deformation theory plays a foundational role in Chai's theory of Hecke stable subvarieties of a Shimura variety. This section thus plays a key role in the thesis.

3.3.1 Serre-Tate deformation theory

In this subsection, we briefly recall Serre-Tate deformation theory of an ordinary closed point on the Shimura variety. We follow [31, §8.2], [34, §2] and [36, §1].

Let the notation and assumptions be as in §3.2.5. In particular, Sh denotes the Shimura variety of level prime to N^-p and Sh^{ord} the p-ordinary locus. Let x be a closed point in Sh^{ord} carrying $(A_x, \iota_x, \bar{\lambda}_x, \eta_x^{(p)})_{/\mathbb{F}}$. Let V be the geometrically irreducible component of Sh containing x.

Let CL_W be the category of complete local W-algebras with residue field $\mathbb{F}$. Let $\mathcal{D}_{/W}$ be the fiber category over CL_W of deformations of $x_{/\mathbb{F}}$ defined as follows. Let $R \in CL_W$. The objects of $\mathcal{D}_{/W}$ over R consist of $x'^{*} = (x', \iota_{x'})$, where $x' \in \mathcal{F}^{(p)}(R)$ and $\iota_{x'} : x' \times_R \mathbb{F} \simeq x$. Let x'^{*} and x''^{*} be in $\mathcal{D}_{/W}$ over R. By definition, a morphism ϕ between x'^{*} and x''^{*} is a morphism (still denoted by) ϕ between x' and x'' satisfying [31, (7.3)] and the following condition.

(M) Let ϕ_0 be the special fiber of ϕ. The automorphism $\iota_{x''} \circ \phi_0 \circ \iota_{x'}^{-1}$ of x equals the identity.

Let $\widehat{\mathcal{F}}_x$ be the deformation functor given by

$$\begin{aligned} \widehat{\mathcal{F}}_x &: CL_{/W} \to SETS \\ R &\mapsto \{x'^{*}_{/R} \in \mathcal{D}\}/\simeq . \end{aligned} \tag{3.3.1}$$

The notation $\simeq$ denotes up to an isomorphism.

As $R \in CL_W$, by definition R is a projective limit of local W-algebras with nilpotent maximal ideal. We can (and do) suppose that R is a local Artinian W-algebra with the nilpotent maximal ideal m_R. Let $x'^{*}_{/R} \in \mathcal{D}$ and A denote $A_{x'}$. By Drinfeld's theorem (cf. [31, §8.2.1]), $A[p^\infty]^\circ(R)$ is killed by p^{n_0} for sufficiently large n_0. Let $y \in A(\mathbb{F})$ and $\tilde{y} \in A(R)$ such that

$\tilde{y}_0 = y$ for $\tilde{y}_0$ being the special fiber of $\tilde{y}$. As $A_{/R}$ is smooth, such a lift always exists. By definition, $\tilde{y}$ is determined modulo $ker(A(R) \mapsto A(\mathbb{F})) = A[p^\infty]^\circ(R)$. It thus follows that for $n \geq n_0$, "p^n"$y_0 := p^n\tilde{y}$ is well defined. From now, we suppose that $n \geq n_0$. If $y \in A[p^n](\mathbb{F})$, then "p^n"$y \in A[p^\infty]^\circ(R)$ by definition. We now consider the ϵ-components (cf. §3.2.3). Recall that it suffices to consider the ϵ-components in view of the isomorphisms (3.2.5) and (3.2.6).

We thus have a homomorphism

$$\text{“}p^n\text{”} : \epsilon A[p^n](\mathbb{F}) \to \epsilon A[p^\infty]^\circ(R). \tag{3.3.2}$$

We also have the commutative diagram

$$\begin{array}{ccccc} \epsilon A[p^{n+1}]^{\acute{e}t}(R) & \xrightarrow{\simeq} & \epsilon A[p^{n+1}](\mathbb{F}) & \xrightarrow{\text{“}p^{n+1}\text{”}} & \epsilon A[p^\infty]^\circ(R) \\ \downarrow p & & \downarrow p & & \downarrow = \\ \epsilon A[p^n]^{\acute{e}t}(R) & \xrightarrow{\simeq} & \epsilon A[p^n](\mathbb{F}) & \xrightarrow{\text{“}p^n\text{”}} & \epsilon A[p^\infty]^\circ(R). \end{array}$$

Passing to the projective limit, this gives rise to a homomorphism

$$\text{“}p^\infty\text{”} : \epsilon A[p^\infty](\mathbb{F}) \to \epsilon A[p^\infty]^\circ(R). \tag{3.3.3}$$

(CC) Let $y = \varprojlim y_n \in \epsilon A[p^\infty](\mathbb{F}) \simeq \epsilon A_x[p^\infty]^{\acute{e}t}$, where $y_n \in \epsilon A[p^n](\mathbb{F})$ and the later isomorphism is induced by $\iota_{x'}$. Let $q_{n,p}(y_n) =$ "p^n"y_n and $q_p(y) = \lim q_{n,p}(y_n)$. By definition,

$$q_p(y) \in \epsilon A[p^\infty]^\circ(R) \simeq Hom(\epsilon^t A_x[p^\infty]^{\acute{e}t}, \widehat{\mathbb{G}}_m(R)).$$

Let $q_{A,p}$ be the pairing given by

$$q_{A,p} : \epsilon A_x[p^\infty]^{\acute{e}t} \otimes_{\mathbf{Z}_p} \epsilon^t A_x[p^\infty]^{\acute{e}t} \to \widehat{\mathbb{G}}_m(R)$$
$$q_{A,p}(y,z) = q_p(y)(z). \tag{3.3.4}$$

We have the following fundamental result.

Theorem 3.3.1 (Serre-Tate)**.** *Let the notation be as above.*

(1). There exists a canonical isomorphism

$$\widehat{\mathcal{F}}_x(R) \simeq Hom_{\mathbf{Z}_p}(\epsilon A_x[p^\infty]^{\acute{e}t} \otimes_{\mathbf{Z}_p} \epsilon^t A_x[p^\infty]^{\acute{e}t}, \widehat{\mathbb{G}}_m(R)) \qquad (3.3.5)$$

given by $x'^* \mapsto q_{A_{x'},p}$.

(2). The deformation functor $\widehat{\mathcal{F}}_x$ *is represented by the formal scheme* $\widehat{S}_{/W} := Spf(\widehat{\mathcal{O}}_{V,x})$. *A* p^∞*-level structure as in (PL), gives rise to a canonical isomorphism of the deformation space* $\widehat{S}_{/W}$ *with the formal torus* $\widehat{\mathbb{G}}_m$ *(cf. [36, Prop. 1.2]).*

Proof. We first prove (1). It follows from an argument similar to the proof of [31, Thm. 8.9]. We sketch the details for convenience.

A fundamental result of Serre and Tate states that deforming x over CL_W is equivalent to deform the corresponding Barsotti-Tate $\mathcal{O}_{B,p}$-module $A_x[p^\infty]_{/\mathbb{F}}$ over CL_W (cf. [45, §1] and [31, §8.2.2]). In view of the isomorphisms (3.2.5) and (3.2.6), deforming the Barsotti-Tate $\mathcal{O}_{B,p}$-module $A_x[p^\infty]_{/\mathbb{F}}$ is equivalent to deform the Barsotti-Tate $\epsilon\mathcal{O}_{B,p}$-module $\epsilon A_x[p^\infty]_{/\mathbb{F}}$.

Let R be as above. Consider the sheafification $\underline{Hom}_{R_{fppf}}(\epsilon A[p^n]^\circ, \epsilon A[p^n]^{\acute{e}t})$ (resp. $\underline{Ext^1}_{R_{fppf}}(\epsilon A[p^n]^\circ, \epsilon A[p^n]^{\acute{e}t})$) of the presheaf $U \mapsto Hom_U(\epsilon A[p^n]^\circ_{/U}, \epsilon A[p^n]^{\acute{e}t}_{/U})$ (resp. $U \mapsto Ext^1_U(\epsilon A[p^n]^\circ_{/U}, \epsilon A[p^n]^{\acute{e}t}_{/U})$). Recall that a connected-étale extension $\epsilon A[p^n]^\circ \to X \twoheadrightarrow \epsilon A[p^n]^{\acute{e}t}$ in the category of finite flat $\mathbf{Z}/p^n\mathbf{Z}$-modules over R splits over an *fppf*-extension R'/R. We thus have

$$\underline{Ext^1}_{R_{fppf}}(\epsilon A[p^n]^\circ, \epsilon A[p^n]^{\acute{e}t}) = 0$$

and a splitting

$$X = \epsilon A[p^n]^\circ \oplus \epsilon A[p^n]^{\acute{e}t}.$$

Choice of a section in the splitting gives rise to a homomorphism $\phi_{R'} \in Hom_{R'}(\epsilon A[p^n]^\circ_{/R'}, \epsilon A[p^n]^{\acute{e}t}_{/R'})$. By construction, $R' \mapsto \phi_{R'}$ satisfies the descent datum. We thus have a morphism

$$Ext^1_{R_{fppf}}(\epsilon A[p^n]^\circ, \epsilon A[p^n]^{\acute{e}t}) \to H^1(R_{fppf}, \underline{Hom}_{R_{fppf}}(\epsilon A[p^n]^\circ, \epsilon A[p^n]^{\acute{e}t})).$$

By an *fppf*-descent, this is in fact an isomorphism. We conclude that

$$Ext^1_{R_{fppf}}(\epsilon A[p^\infty]^\circ, \epsilon A[p^\infty]^{\acute{e}t}) \simeq \varprojlim_n Ext^1_{R_{fppf}}(\epsilon A[p^n]^\circ, \epsilon A[p^n]^{\acute{e}t}) \simeq Hom_{\mathbf{Z}_p}(\epsilon A_x[p^\infty]^{\acute{e}t} \otimes_{\mathbf{Z}_p} \epsilon^t A_x[p^\infty]^{\acute{e}t}, \widehat{\mathbb{G}}_m(R)).$$

The last isomorphism follows from the additional fact that $\widehat{\mathbb{G}}_m \simeq \varprojlim_n R^1\pi_*\mu_{p^n} \simeq \widehat{\mathbb{G}}_m$. Here $\pi : R_{fppf} \to R_{\acute{e}t}$ is the projection for the small étale site $R_{\acute{e}t}$.

We conclude that the deformation functor $\widehat{\mathcal{F}}_x$ is represented by $Hom_{\mathbf{Z}_p}(\epsilon A_x[p^\infty]^{\acute{e}t} \otimes_{\mathbf{Z}_p} \epsilon^t A_x[p^\infty]^{\acute{e}t}, \widehat{\mathbb{G}}_m(R))$ and this finishes the proof of (1).

We now prove (2). The first part of (2) follows from general deformation theory. The polarisation λ_x induces a canonical $\mathcal{O}_B$-linear isomorphism $A_x \simeq^t A_x$. For a given choice of p^∞-level structure $\eta_p^{\acute{e}t}$ as in (PL), we have a canonical isomorphism

$$Hom_{\mathbf{Z}_p}(\epsilon A_x[p^\infty]^{\acute{e}t} \otimes_{\mathbf{Z}_p} \epsilon A_x[p^\infty]^{\acute{e}t}, \widehat{\mathbb{G}}_m) \simeq \widehat{\mathbb{G}}_m.$$

This finishes the proof. □

Let x_{ST} be the universal deformation. A p^∞-level structure η_p° of x gives rise to a canonical p^∞-level structure $\eta_{p,ST}^\circ$ of the universal deformation x_{ST}.

We now recall a few definitions.

Definition 3.3.2. Recall that a p^∞-level structure as in (PL), gives rise to a canonical isomorphism of the deformation space $\widehat{S}_{/W}$ with the formal torus $\widehat{\mathbb{G}}_m$ (cf. part (2) of Theorem 3.3.1). Under this identification, let t be the co-ordinate of the deformation space $\widehat{S}_{/W}$ for t being the usual co-ordinate of $\widehat{\mathbb{G}}_m$. We call t the Serre-Tate co-ordinate of the deformation space $\widehat{S}_{/W}$.

Definition 3.3.3. The deformation corresponding to the identity element in the deformation space is said to be the canonical lift of x.

We have $\widehat{S} = Spf(\widehat{W[X(S)]})$ for $S = \mathbb{G}_m$ and $\widehat{W[X(S)]}$ being the completion along the augmentation ideal. Here $X(S)$ is the character group of S. Note that $W[X(S)]$ is the ring consisting of formal finite sums $\sum_{\xi \in \mathbf{Z}} a(\xi) t^{\xi}$ for $a(\xi) \in W$ and t being the usual co-ordinate of $\mathbb{G}_m$.

Definition 3.3.4. Let f be a mod p modular form over $\mathbb{F}$ (cf. §3.2.6). We call the evaluation $f((x_{ST}, \epsilon\eta^{\circ}_{p,ST})) \in \widehat{\mathbb{F}[X(S)]}$ as the t-expansion of f around x.

We have the following t-expansion principle.

(t-expansion principle) The t-expansion of f around x determines f uniquely (cf. (Ir)).

We have an analogous t-expansion principle for p-adic modular forms (cf. [31, Ch. 8]).

We end this subsection with a description of the modules $\mathcal{O}_B^{\circ}$ and $\mathcal{O}_B^{\acute{e}t}$ (cf. §3.2.5).

We follow [33, pp. 9-10]. We fix an identification of $\mathcal{O}_{B,p}$ with $\mathbf{Z}_p^4 = (\mathbf{Z}_p^2)^2$. Let $\mathcal{O}_{B,p}$ act on itself by the left multiplication on $\mathbf{Z}_p^2$. Let x be a closed point in Sh^{ord} carrying $(A_x, \iota_x, \bar{\lambda}_x, \eta_x^{(p)})_{/\mathbb{F}}$. Let $\tilde{x} = (A_x, \iota_x, \bar{\lambda}_x, \eta_x^{(p)})_{/W}$ be the canonical lift of x. Here and in the rest of the paragraph, we denote objects corresponding to subscript $\tilde{x}$ with subscript x by an abuse of notation. As $A_{x/W}$ has CM, it descends to $A_{x/\mathcal{W}}$ (cf. [62]). Let $A_{x/\mathbf{C}} = A_x \times_{\mathcal{W},\iota_\infty} \mathbf{C}$. Note that $\mathcal{T}(A_x)_{/\overline{E}} \simeq H_1(A_x(\mathbf{C}), \mathbf{A}_{\mathbf{Q},f}) = H_1(A_x(\mathbf{C}), \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{A}_{\mathbf{Q},f}$. Let $\mathcal{T}_p A_{x/\mathbf{C}}$ be the p-adic Tate-module of $A_{x/\mathbf{C}}$. We choose an identification of $\mathcal{T}_p A_{x/\mathbf{C}}$ with $\mathcal{O}_{B,p} = H_1(A_x(\mathbf{C}), \mathbf{Z}_p)$. The connected-étale exact sequence $A_x[p^\infty]^\circ \hookrightarrow A_x[p^\infty] \twoheadrightarrow A_x[p^\infty]^{\acute{e}t}$ of the p-divisible group $A_x[p^\infty]_{/\mathbb{F}}$ induces an exact sequence $\mathcal{O}_B^{\circ} \hookrightarrow \mathcal{O}_{B,p} \twoheadrightarrow \mathcal{O}_B^{\acute{e}t}$ of $\mathcal{O}_{B,p}$-modules. As $A_{x/\mathcal{W}}$ is the canonical lift, it follows that $\mathcal{O}_{B,p}$ splits as $\mathcal{O}_B^{\circ} \oplus \mathcal{O}_B^{\acute{e}t}$ such that the lift of the Frobenius has unit eigenvalues on $\mathcal{O}_B^{\acute{e}t}$ and non-unit eigenvalues on $\mathcal{O}_B^{\circ}$.

3.3.2 Reciprocity law

In this subsection, we describe the action of the local algebraic stabiliser of an ordinary closed point on the Serre-Tate co-ordinates of the deformation space. This can be considered as an infinitesimal analogue of Shimura's reciprocity law.

Let the notation and hypotheses be as in §3.3.1. In particular, Sh denotes the Shimura variety of level prime to $N^{-}p$. Let $g \in G(\mathbf{Z}_{(p)})$ acts on Sh through the right multiplication on the prime to $N^{-}p$ level structure i.e.,

$$(A, \iota, \bar{\lambda}, \eta^{(p)})_{/S} \mapsto (A, \iota, \bar{\lambda}, \eta^{(p)} \circ g)_{/S}$$

(cf. §3.2.2).

Recall that x is a closed ordinary point in Sh with a p^{∞}-level structure η_p^{ord}. We suppose that x has CM by K and the existence of an embedding $\iota_x : \mathcal{O} \hookrightarrow End(A)$. Let $H_x(\mathbf{Z}_{(p)})$ be the stabiliser of x in $G(\mathbf{Z}_{(p)})$. It is explicitly given by

$$H_x(\mathbf{Z}_{(p)}) = (Res_{\mathcal{O}_{(p)}/\mathbf{Z}_{(p)}} \mathbb{G}_m)(\mathbf{Z}_{(p)}) = \mathcal{O}_{(p)}^{\times} \tag{3.3.6}$$

for $\mathcal{O}_{(p)} = \mathcal{O} \otimes \mathbf{Z}_{(p)}$ (cf. [34, §3.2] and [62]). We call $H_x(\mathbf{Z}_{(p)})$ the local algebraic stabiliser of x.

Recall that c denotes the complex conjugation of K.

Let $\alpha \in H_x(\mathbf{Z}_{(p)})$. Let $R \in CL_W$, $x' \in \widehat{\mathcal{F}}_x(R)$ and $A = A_{x'}$. We have the following

$$\begin{array}{ccccc}
Hom(\epsilon A_x[p^{\infty}]^{\acute{e}t}, \hat{\mathbb{G}}_m(R)) & \hookrightarrow & \epsilon A[p^n] & \to & \epsilon A_x[p^n]^{\acute{e}t}(R) \\
\uparrow \alpha & & \downarrow id & & \downarrow \alpha^{-c} \\
Hom(\epsilon A_x[p^{\infty}]^{\acute{e}t}, \hat{\mathbb{G}}_m(R)) & \hookrightarrow & \epsilon A[p^n] & \to & \epsilon A_x[p^n]^{\acute{e}t}(R)
\end{array}$$

commutative diagram.

Let $u = \varprojlim u_n \in \epsilon A_x[p^\infty]^{ét}$ and $q_p(u) \in \epsilon A_x[p^\infty]^\circ(R) \simeq Hom(\epsilon A_x[p^\infty]^{ét}, \hat{\mathbb{G}}_m(R))$ be as in (CC). We recall that the last isomorphism is induced by the Cartier duality composed with the polarisation λ_x. Thus, if α is prime to p, then it acts on $q_p(u)$ by

$$q_p(u) \mapsto \varprojlim \alpha(\text{``}p^n\text{''}\alpha^{-c_x}(u_n)) = q_p(\alpha)^{\alpha^{1-c_x}}.$$

In other words, α acts on q_p by $q_p \mapsto q_p^{\alpha^{1-c}}$. Fix a p^∞-level structure as in (PL). This gives rise to the Serre-Tate co-ordinate t of the deformation space $\widehat{S}_{/W}$ (cf. (CC)).

In view of the above discussion and Theorem 3.3.1, we have the following reciprocity law.

Lemma 3.3.5. *Let the notation and the assumptions be as above. If $\alpha \in H_x(\mathbf{Z}_{(p)})$ is prime to p, then it acts on the Serre-Tate co-ordinate t by $t \mapsto t^{\alpha^{1-c}}$.*

The above simple lemma plays a key role in the proof of a linear independence of mod p modular forms (cf. §3.4).

3.3.3 Hecke operators

In this subsection, we describe certain Hecke operators on the space of classical modular forms. In the case of modular curves, such operators are considered in [37, §1.3.5].

Let the notation and hypotheses be as in §3.3.1. Let $(\zeta_{p^n} = \exp(2\pi i/p^n))_n \in \overline{\mathbf{Q}}$ be a compatible system of p-power roots of unity. Via ι_p, we regard it as a compatible system in $\mathbf{C}_p$. Let

$W_n = W[\mu_{p^n}]$.

Let g be a classical modular form on a quotient Shimura curve Sh_K of the Shimura variety Sh over $\mathcal{W}$ (cf. §3.2.1). Via ι_∞, we regard it as a modular form over $\mathbf{C}$. Let z denote the complex variable of the complex Shimura curve or that of $\mathfrak{H}$ for $\mathfrak{H}$ being the upper half complex plane.

Let $\phi : \mathbf{Z}/p^r\mathbf{Z} \to \mathcal{W}$ be an arbitrary function with the normalised Fourier transform ϕ^* given by

$$\phi^*(y) = \frac{1}{p^{r/2}} \sum_{u \in \mathbf{Z}/p^r\mathbf{Z}} \phi(u) \exp(yu/p^r) \tag{3.3.7}$$

for $y \in \mathbf{Z}/p^r\mathbf{Z}$.

Let $g|\phi$ be the classical modular form given by

$$g|\phi(z) = \sum_{u \in \mathbf{Z}/p^r\mathbf{Z}} \phi^*(-u) g(z + u/p^r). \tag{3.3.8}$$

We now regard the above classical modular forms as p-adic. The action of the Hecke operator on the t-expansion around a CM point (cf. Definition 3.3.4) is the following.

Proposition 3.3.6. *Let the notation be as above. Let $g(t) = \sum_{\xi \in \mathbf{Z}_{\geq 0}} a(\xi, g) t^\xi$ be the t-expansion of g around x. We have*

$$g|\phi(t) = \sum_{\xi \in \mathbf{Z}_{\geq 0}} \phi(\xi) a(\xi, g) t^\xi.$$

Proof. Let $u \in \mathbf{Z}$ and $\alpha(u/p^r) \in G(\mathbf{A}_{\mathbf{Q},f})$ such that

$$\alpha(u/p^r)_p = \begin{bmatrix} 1 & u/p^m \\ 0 & 1 \end{bmatrix}$$

and $\alpha(u/p^r)_l = 1$, for $l \neq p$.

We have

$$g|\phi = \sum_{u \in \mathbf{Z}/p^r\mathbf{Z}} \phi^*(-u) g|\alpha(u/p^r),$$

as $\alpha(u/p^r)$ acts on $\mathfrak{H}$ by $z \mapsto z + u/p^r$. Here $\mathfrak{H}$ denotes the complex upper half plane as before.

The isogeny action of $\alpha(u/p^r)$ on the Igusa tower $\pi : Ig \to Sh_{/W_r}$ preserves the deformation space $\widehat{S}$ and induces $t \mapsto \zeta_{p^r} t^u$ (cf. [6, Lem. 4.14]).

In view of the Fourier inversion formula, this finishes the proof. □

The above Hecke operators naturally arise during the determination of the μ-invariant of a class of anticyclotomic Rankin-Selberg p-adic L-functions. Above proposition accordingly helpful in the determination of the μ-invariants.

3.4 Linear independence

In this section, we consider a linear independence of mod p modular forms based on Chai's theory of Hecke stable subvarieties of a Shimura variety. In §3.4.1, we describe the formulation. In §3.4.2-3.4.6, we prove the independence.

3.4.1 Formulation

In this subsection, we give a formulation of the linear independence of mod p modular forms.

Let the notation and hypotheses be as in §3.2.5. Recall that x is a closed ordinary point in V with p^∞-level structure η_p°. This gives rise to a closed point $\tilde{x} = (x, \eta_p^\circ)$ in the Igusa tower I over x. Recall that we have a canonical isomorphism $\widehat{\mathcal{O}}_{V,x} \simeq \widehat{\mathcal{O}}_{I,\tilde{x}}$ (cf. (Et)). Thus, we have a natural action of the p-adic stabiliser $H_x(\mathbf{Z}_p)$ on $\widehat{\mathcal{O}}_{I,\tilde{x}}$ (cf. §3.3.2).

Our formulation of the independence is the following.

Theorem 3.4.1 (Linear independence)**.** *Let x be a closed ordinary point on the Shimura variety with the local stabiliser H_x. For $1 \leq i \leq n$, let $a_i \in H_x(\mathbf{Z}_p)$ such that $a_i a_j^{-1} \notin H_x(\mathbf{Z}_{(p)})$ for all $i \neq j$ (cf. §3.3.2). Let $f_1, ..., f_n$ be non-constant mod p modular forms on V. Then, $(a_i \circ f_i)_i$ are linearly independent in the Serre-Tate deformation space $Spf(\widehat{\mathcal{O}}_{V,x})$.*

In general, note that $a_i \circ f_i$ is not a mod p modular form.

We actually prove the algebraic independence of $(a_i \circ f_i)_i$. The theorem is an analogue of Ax-Lindemann-Weierstrass conjecture for the mod p reduction of the Shimura variety.

The theorem is proven in §3.4.6.

3.4.2 Locally stable subvarieties

In this subsection, we describe the notion and the results regarding locally stable subvarieties of a self-product of the Shimura variety.

Let the notation and hypotheses be as in §3.2.5. Let n be a positive integer. In this subsection, any tensor product is taken n times.

Let Sh be as before and V a geometrically irreducible component of Sh.

Definition 3.4.2. A subvariety Y of V^n is said to be locally stable if there exists a closed point $y^n = (y, ..., y) \in Y$ such that Y is stable under the diagonal action of the local algebraic stabilser $H_y(\mathbf{Z}_{(p)})$.

While considering the problem of linear independence, this type of subvariety arises as follows.

We consider an $\mathbb{F}$-algebra homomorphism

$$\phi_I : \mathcal{O}_{I,\tilde{x}} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{I,\tilde{x}} \to \widehat{\mathcal{O}}_{I,\tilde{x}} \tag{3.4.1}$$

given by

$$f_1 \otimes ... \otimes f_n \mapsto \prod_{i=1}^{n} a_i \circ f_i. \tag{3.4.2}$$

As we are interested in the independence of $(a_i \circ f_i)_i$, we consider $\mathfrak{b}_I := ker(\phi_I)$.

Similarly, we consider an $\mathbb{F}$-algebra homomorphism

$$\phi = \phi_V : \mathcal{O}_{V,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V,x} \to \widehat{\mathcal{O}}_{V,x} \tag{3.4.3}$$

given by

$$h_1 \otimes ... \otimes h_n \mapsto \prod_{i=1}^{n} a_i \circ h_i. \tag{3.4.4}$$

The following simple observation is crucial in what follows.

(EQ) The homomorphism ϕ is equivariant with the $H_x(\mathbf{Z}_{(p)})$-action.

Let $\mathfrak{b} = ker(\phi_V)$.

Lemma 3.4.3. *Let the notation be as above. We have $\mathfrak{b}_I = 0$ if and only if $\mathfrak{b} = 0$.*

Proof. In view of (Et), we have an étale morphism

$$\pi^n : \mathcal{O}_{V,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V,x} \to \mathcal{O}_{I,\tilde{x}} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{I,\tilde{x}}.$$

As $\mathfrak{b}_I$ is the unique prime ideal of $\mathcal{O}_{I,\tilde{x}} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{I,\tilde{x}}$ over $\mathfrak{b}$, this finishes the proof. □

In view of (EQ), it follows that $\mathfrak{b}$ is a prime ideal of $\mathcal{O}_{V,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V,x}$ stable under the diagonal action of $H_x(\mathbf{Z}_{(p)})$. Let X be the Zariski closure of $\mathrm{Spec}\,(\mathcal{O}_{V,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V,x}/\mathfrak{b})$ in V^n . Thus, X is a closed irreducible subscheme of V^n containing x^n stable under the diagonal action of $H_x(\mathbf{Z}_{(p)})$. In particular, X is a closed irreducible locally stable subvariety of V^n.

We now axiomatise the above example.

(LS1) Let $S_{/\mathbb{F}}$ be $\mathrm{Spec}\,(\mathcal{O}_{V,x})$.
(LS2) Let $\mathfrak{b}$ be a prime ideal of $\mathcal{O}_{V,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V,x}$ stable under the diagonal action of $H_x(\mathbf{Z}_{(p)})$ and $\mathcal{X}_{/\mathbb{F}} = \mathrm{Spec}\,(\mathcal{O}_{V,x} \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} \mathcal{O}_{V,x}/\mathfrak{b})$.
(LS3) Let X be the Zariski closure of $\mathcal{X}$ in V^n.

Note that X is positive dimensional as $\mathfrak{b}$ is a prime ideal.

A natural class of locally stable subvarieties is the following.

Definition 3.4.4. Let l be a positive integer. For $1 \leq i \leq l$, let $\alpha_i \in H_x(\mathbf{Z}_{(p)})$. The skewed diagonal $\Delta_{\alpha_1,...,\alpha_l}$ of V^l is defined by $\Delta_{\alpha_1,...,\alpha_l} = \big\{(\alpha_1(y), ..., \alpha_l(y)) : y \in V\big\}$.

The global structure of a class of locally stable subvarieties is given by the following result.

Theorem 3.4.5. *Let $n \geq 2$ and X be locally stable as in (LS1)-(LS3). Let S' be the product of the first $n-1$ factors of S^n and $S'' = S$ be the last factor of S^n. Suppose that the projections $\pi'_{\mathcal{X}} : \mathcal{X} \to S'$ and $\pi''_{\mathcal{X}} : \mathcal{X} \to S''$ are dominant. Then, X equals V^n or $V^{n-2} \times \Delta_{\alpha_1,\alpha_2}$ for some $\alpha_1, \alpha_2 \in H_x(\mathbf{Z}_{(p)})$ up to a permutation of the first $n-1$ factors.*

The theorem is an instance of Chai-Oort principle that a Hecke stable subvariety of a Shimura variety is a Shimura subvariety (cf. [19], [20] and [21]). The principle is an analogue of Andre-Oort conjecture for mod p Shimura varieties.

The theorem is proven in §3.4.3-§3.4.5.

In [34], an analogue of the theorem is proven in the Hilbert modular case. For the proof, we follow the same strategy as in [19], [20] and [34, §3]. We first consider the structure of $Spf(\widehat{X}_{x^n})$ as a formal subscheme of $Spf(\widehat{\mathcal{O}}_{V^n,x^n})$. Recall that the Serre-Tate deformation space $Spf(\widehat{\mathcal{O}}_{V^n,x^n})$ has a natural structure of a formal torus. As $Spf(\widehat{X}_{x^n})$ is a formal subscheme of $Spf(\widehat{\mathcal{O}}_{V^n,x^n})$ stable under the diagonal action of the stabiliser $H_x(\mathbf{Z}_{(p)})$ and (pretending) X is smooth, it follows from Chai's local rigidity that $Spf(\widehat{X}_{x^n})$ is in fact a formal subtorus of $Spf(\widehat{\mathcal{O}}_{V^n,x^n})$. In view of Theorem 3.3.1, we have a description of $Spf(\widehat{\mathcal{O}}_{V^n,x^n})$ as a formal torus. Based on it, we obtain a description of the formal subtori of $Spf(\widehat{\mathcal{O}}_{V^n,x^n})$ stable under the diagonal action of the stabiliser $H_x(\mathbf{Z}_{(p)})$. We thus obtain an explicit description of the possibilities for the structure of $Spf(\widehat{X}_{x^n})$ as a formal torus. Based on de Jong's result on Tate conjecture in [26], we can eliminate all but one possibilities. Strictly speaking, the proof is a bit more involved as we cannot directly suppose that X is smooth. Instead, we apply the strategy to the normalisation of X and later show that the normalisation equals X itself.

3.4.3 Local structure

In this subsection, we consider the local structure of a class of locally stable subvarieties of a self-product of the Shimura curve.

Let the notation and hypotheses be as in §3.4.2. However, we do not suppose that the projection $\pi''_{\mathcal{X}}$ is dominant. Let $\Pi_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{X}$ be the normalisation of $\mathcal{X}$.

The following is an analogue of [34, Prop. 3.11] to our setting.

Proposition 3.4.6. *Let the notation and assumptions be as above. Suppose that the projection $\pi'_{\mathcal{X}} : \mathcal{X} \to S'$ is dominant. Then, the following holds.*

(1). The normalisation $\mathcal{Y}$ has only finitely many points y over x^n. For any such y, we have $\widehat{\mathcal{Y}}_y = \widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} L_y$ for an $\mathbf{Z}_p$-direct summand L_y of $X_*(\widehat{S^n})$. Moreover, the isomorphism class of L_y as an $\mathbf{Z}_p$-module is independent of y.
(2). The normalisation $\mathcal{Y}$ is smooth over $\mathbb{F}$ and flat over S'.
(3). Either $\mathcal{X} = S^n$ or $\mathcal{X}$ is finite over S'. In the later case, the normalisation $\mathcal{Y}$ is finite flat over S'.
(4). If $\pi'_{\mathcal{X}} \circ \Pi_{\mathcal{Y}}$ induces a surjection of the tangent space at y onto S' at x^n for some y over x^n, then $\pi'_{\mathcal{X}} \circ \Pi_{\mathcal{Y}} : \mathcal{Y} \to S'$ is étale.

Proof. The proof is similar to the proof of Proposition 3.11 in [34]. Here we only prove (1).

Let $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}_{x^n}$. By the local stabiliser principle (cf. [21, Prop. 6.1]), $\widehat{\mathcal{X}}$ is a formal subscheme of $\widehat{S^n}$ stable under the diagonal action of the p-adic stabiliser $H_x(\mathbf{Z}_p)$. In view of Chai's local rigidity (cf. [20, §6]), it follows that $\widehat{\mathcal{X}} = \bigcup_{L \in I} \widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} L$. Here L is an $\mathbf{Z}_p$-direct summand of $X_*(\widehat{S^n})$ (cf. Theorem 3.3.1) and the union is finite.

Note that the semigroup $End_{SCH}(\mathcal{X})$ naturally acts on $\mathcal{Y}$. In particular, the stabiliser $H_x(\mathbf{Z}_{(p)})$ acts on $\mathcal{Y}$. As $\widehat{\mathcal{Y}}_y$ is integral, the points y are indexed by the irreducible components of $\widehat{\mathcal{X}}$. Let y_L be the point corresponding to L. As the morphism $\pi'_{\mathcal{X}} \circ \Pi_{\mathcal{Y}}$ is dominant, there exists at least one $L_0 \in I$ such that the image of L_0 is of finite index in $X_*(\widehat{S}')$.

To arrive at the desired conclusion, we consider a global argument as follows. Let $\Pi : Y \to X$ be the normalisation of X. As before, the local algebraic stabiliser $H_x(\mathbf{Z}_{(p)})$ naturally acts on Y. Recall that the order $\mathcal{O}_B$ acts on X and hence on Θ_X. This action extends to Θ_Y. In view of the Kodaira-Spencer isomorphism for the universal abelian scheme $\mathcal{A}_{/Y}$ over Y and the identification $\widehat{Y}_{y_L} = \widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} L$, it follows that $rank_{\mathbf{Z}_p} L = rank_{\mathcal{O}_Y} \Theta_Y$. This finishes the proof of (1). □

3.4.4 Global structure I

In this subsection, we study global structure of a class of locally stable subvarieties of a 2-fold self-product of the Shimura curve.

Let the notation and hypotheses be as in §3.4.2 and $n = 2$. In view of Proposition 3.4.6, we conclude that either $\dim \mathcal{X} = \dim V$ or $X = V^2$. From now, we suppose that $\dim \mathcal{X} = \dim V$. In the rest of the subsection, we let $i = 1, 2$.

We need to show that X is a graph of an automorphism of V arising from $H_x(\mathbf{Z}_{(p)})$. In other words, we need to show that the projection $\pi_i : X \to V_i$ is an isomorphism. Here π_i is the projection to the i^{th}-factor and we view $V_i = V$ as the i^{th} factor of $V \times V$. Our strategy is the following.

Step 1. The morphism $\Pi_i : Y \to X \to V$ is étale over an open dense subscheme of V. This is proven based on Serre-Tate deformation theory and Proposition 3.4.6.

Step 2. Let $\mathcal{A}_i = \Pi_i^* \mathcal{A}$ and $\eta_i^{(p)} = \Pi_i^* \eta^{(p)}$. There exists an isogeny $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$. This is proven based on de Jong's theorem in [26]. In particular, there exists $g \in G(\mathbf{A}_{\mathbf{Q},f}^{(p)})$ such that $\varphi \circ \eta_1^{(p)} = \eta_2^{(p)} \circ g$. We consider the fiber at x and conclude that $g \in H_x(\mathbf{Z}_{(p)})$. Finally, we deduce that $Y = X$ and $X = \Delta_{1,g}$.

We now describe each step. In view of Proposition 3.4.6 and our assumption $\dim \mathcal{X} = \dim V$, it follows that the morphism $\Pi_i : Y \to V$ is finite at any point $y \in Y$ above x^2. If Π_i is not étale at y, then $\Pi_{i,*}(X_*(\widehat{Y}_y)) \subset X_*(\widehat{S})$ is a $\mathbf{Z}_p$-submodule of finite index. Let $\alpha \in H_x(\mathbf{Z}_{(p)})$ such that $\alpha^{1-c} X_*(\widehat{S}) = \Pi_{i,*}(X_*(\widehat{Y}_y))$. The morphism $\alpha^{-1} \circ \Pi_i$ is étale. Thus, from now we suppose that

(E) the morphism $\Pi_i : Y \to V$ is étale finite at any point $y \in Y$ over x^2.

Let $V^{\acute{e}t}$ be the maximal subscheme over which both Π_1 and Π_2 are étale. Let Π_i also denote the projection $\mathcal{Y} \to S$.

Lemma 3.4.7. *Let the notation and assumptions be as above. If the projections $\Pi_1 : \mathcal{Y} \to S$ and $\Pi_2 : \mathcal{Y} \to S$ are both étale, then $V^{\acute{e}t}$ is an open dense subscheme of V^2 containing x^2 and stable under the diagonal action of the stabiliser $H_x(\mathbf{Z}_{(p)})$.*

Proof. The later part follows from the definition. The former part can be proven in a way similar to the proof of [34, Prop. 3.14]. In fact, the proof is simpler in our case as Sh is proper. □

Let $Y^{ord} = Y \times_{V^2} (V^{ord})^2$. Let $y' \in Y^{ord}$ be a closed point above $(x_1, x_2) \in (V^{ord})^2$.

Definition 3.4.8. Y is said to be $\mathcal{O}_B$-linear at y' if $\Pi_1 \times \Pi_2$ embeds $\widehat{Y}_{y'}$ into $\widehat{V}_{x_1} \times \widehat{V}_{x_2}$ and the equation defining $\widehat{Y}_{y'}$ is given by $t_1^a = t_2^b$, where t_i is the Serre-Tate co-ordinate of $\widehat{V}_{x_i}$ (cf. §3.3.1) and $a, b \in End_{\epsilon\mathcal{O}_{B,p}}(\widehat{S})$.

Let $Y^{lin} \subset Y^{ord}$ be the subset of all $\mathcal{O}_B$-linear points. Here is the Step 1 outlined above.

Proposition 3.4.9. *Let the notation and assumptions be as above. The subset Y^{lin} contains the closed points of an open dense subscheme U of Y^{ord}.*

Proof. We first describe the main steps of the proof.

$\mathcal{O}_B$-linearity at y : In view of Proposition 3.4.6, it follows that Y is $\mathcal{O}_B$-linear at y. In fact, we can suppose that $b = 1$ and $a = a_x$ is a unit in $End_{\epsilon\mathcal{O}_B}(\widehat{S})$. We regard a_x as an isomorphism $a_x : \epsilon\mathcal{A}_{1,x}[p^\infty] \to \epsilon\mathcal{A}_{2,x}[p^\infty]$ of Barsotti-Tate groups. *Existence of U via extension of a homomorphism* : Based on Serre-Tate deformation theory, we reduce the existence of U to the existence of an étale covering $\widetilde{U}$ of an open dense subscheme of Y^{ord} containing y such that a_x extends to a homomorphism $\tilde{a} : \epsilon\mathcal{A}_1[p^\infty]_{/\widetilde{U}} \to \epsilon\mathcal{A}_2[p^\infty]_{/\widetilde{U}}$ of Barsotti-Tate groups over $\widetilde{U}$. *Extension of a homomorphism* : Based on Serre-Tate deformation theory, we first show that a_x extends to a homomorphism $\widehat{a} : \epsilon\mathcal{A}_1[p^\infty]_{/\widehat{Y}_y} \to \epsilon\mathcal{A}_2[p^\infty]_{/\widehat{Y}_y}$ of Barsotti-Tate groups over $\widehat{Y}_y$. The existence of $\tilde{U}$ with the desired property then follows by an *fpqc*-descent.

We now describe each step.

$\mathcal{O}_B$-linearity at y : By part (1) of Proposition 3.4.6, the formal torus $\widehat{\mathcal{Y}}_y$ is canonically isomorphic to a formal subtorus $\widehat{\mathbb{G}}_m \otimes L$ of $\widehat{S}^2 = \widehat{\mathbb{G}_m^2}$ for a $\mathbf{Z}_p$-free direct summand L of $\mathbf{Z}_p^2$. As $\dim \mathcal{X} = \dim V$, we conclude that the equation of $\widehat{\mathcal{Y}}_y$ in $\widehat{S}^2$ is given by $t_1^a = t_2^b$, for some $a, b \in \mathbf{Z}_p$ such that $a\mathbf{Z}_p + b\mathbf{Z}_p = \mathbf{Z}_p$. Moreover,

$$L = \left\{ (c, d) \in \mathbf{Z}_p^2 : ac = bd \right\}. \tag{3.4.5}$$

In particular, Y is $\mathcal{O}_B$-linear at y. In view of (E), we replace (a,b) by $(a/b,1)$ and suppose that $(a,b)=(a_x,1)$ with $a_x \in \mathbf{Z}_p^\times$. We now normalise the action of $a \in \mathbf{Z}_p$ on $\widehat{S}$ as follows. We modify a by an element in $\mathbf{Z}_p$ if necessary, such that a acts as identity on $\epsilon A_x[p^\infty]^\circ$ without affecting the original action of a on $\widehat{S}$. In other words, a acts on $\widehat{S}$ via the action on $\epsilon A_x[p^\infty]^{ét}$. We identify $\mathcal{A}_{i,y}$ with A_x. In this way, we regard a_x as an isomorphism $a_x : \epsilon\mathcal{A}_{1,x}[p^\infty] \to \epsilon\mathcal{A}_{2,x}[p^\infty]$ of Barsotti-Tate groups.

Existence of U via extension of a homomorphism : Suppose that there exists an open dense subscheme $U \subset Y^{ord}$ containing y having an irreducible étale cover $\tilde{U}$ such that a_x extends to an isomorphism $\tilde{a} : \epsilon\mathcal{A}_1[p^\infty]_{/\widetilde{U}} \to \epsilon\mathcal{A}_2[p^\infty]_{/\widetilde{U}}$ of Barsotti-Tate groups over $\widetilde{U}$. We show that Y is $\mathcal{O}_B$-linear at the closed points of U. In view of (E) and Lemma 3.4.7, shrinking the neighbourhood U of y if necessary, we suppose that the projection $\Pi_i : U \to V$ is étale. Let $u \in U$ and $(\Pi_1(u),\Pi_2(u)) = (u_1,u_2)$. In view of Serre-Tate deformation theory (cf. §3.3.1) and the isomorphism $\pi_i : \widehat{Y}_u \simeq \widehat{V}_{u_i}$, it follows that $\widehat{Y}_u$ is isomorphic to the deformation space of Barsotti-Tate $\epsilon\mathcal{O}_{B,p}$-module $\epsilon A_{u_i}[p^\infty]$ via Π_i. Note that the universal deformation of the Barsotti-Tate $\epsilon\mathcal{O}_{B,p}$-module $\epsilon A_{u_i}[p^\infty]$ is the Barsotti-Tate $\epsilon\mathcal{O}_{B,p}$-module $\epsilon\mathcal{A}_i[p^\infty]_{/\widehat{Y}_u} \simeq \epsilon\mathcal{A}[p^\infty]_{/\widehat{V}_{u_i}}$. We choose a p^∞-level structure $\eta_{i,p}$ on $\epsilon A_{u_i}[p^\infty]$ for $i=1,2$ (cf. (PL)). Accordingly, we have the Serre-Tate co-ordinates t_i (cf. (CC) and (STC)). The composition $\tilde{a} \circ \eta_{1,p}$ gives rise to a possibly new p^∞-level structure on $\epsilon A_{u_2}[p^\infty]$. In view of the fact that $Aut_{\epsilon\mathcal{O}_{B,p}}\widehat{Y}_u = \mathbf{Z}_p^\times$ and Definition 3.3.2, it follows that the new p^∞-level structures give rise to the Serre-Tate co-ordinate $t_2^{a_u}$ for some $a_u \in \mathbf{Z}_p^\times$. Thus, we have $t_1 = t_2^{a_u}$. In other words, $\widehat{Y}_u$ is contained in the formal subscheme of $\widehat{V}_{u_1} \times \widehat{V}_{u_2}$ defined by $t_1 = t_2^{a_u}$. As $\widehat{Y}_u$ is a smooth formal subscheme with an isomorphism $\pi_i : \widehat{Y}_u \simeq \widehat{V}_{u_i}$, we conclude that $\widehat{Y}_u$ is itself defined by the equation $t_1 = t_2^{a_u}$.

Extension of a homomorphism : We first extend a_x to a homomorphism $\widehat{a} : \epsilon\mathcal{A}_1[p^\infty]_{/\widehat{Y}_y} \to \epsilon\mathcal{A}_2[p^\infty]_{/\widehat{Y}_y}$ of Barsotti-Tate groups over $\widehat{Y}_y$. Let n be a positive integer. Let $\alpha_n \in H_x(\mathbf{Z}_{(p)})$ be a prime to p element such that $\alpha_n^{1-c} \equiv a \pmod{p^n}$. The homomorphism α_n also induces a homomorphism $\alpha_n : \mathcal{A}_{/V} \to \mathcal{A}_{/V}$ and hence, a homomorphism

$$\alpha_n : \epsilon\mathcal{A}_1[p^n]_{/\widehat{Y}} \xrightarrow[\Pi_1]{\simeq} \epsilon\mathcal{A}[p^n]_{/\widehat{V}_x} \xrightarrow[\alpha_n]{} \epsilon\mathcal{A}[p^n]_{/\widehat{V}_x} \xrightarrow[\Pi_2^{-1}]{\simeq} \epsilon\mathcal{A}_2[p^n]_{/\widehat{Y}}.$$

Thus, we have a homomorphism $\alpha_n : \epsilon\mathcal{A}_1[p^n]_{/\widehat{Y}_n} \to \epsilon\mathcal{A}_2[p^n]_{/\widehat{Y}_n}$ for an infinitesimal neighbourhood $\widehat{Y}_n$ of y. In view of Lemma 3.3.5, it follows that the homomorphism α_n equals $a|_{\epsilon\mathcal{A}_1[p^n]}$ over $\widehat{Y}_n$. Let $\widehat{a} = \varprojlim \alpha_n|_{\epsilon\mathcal{A}_1[p^n]_{/\widehat{Y}_n}}$. By definition, $\widehat{a} : \epsilon\mathcal{A}_1[p^\infty]_{/\widehat{Y}_y} \to \epsilon\mathcal{A}_2[p^\infty]_{/\widehat{Y}_y}$ is a homomorphism of Barsotti-Tate groups over $\widehat{Y}_y$. The existence of $\tilde{U}$ with the desired can be proven from the existence of $\widetilde{a}$ by an $fpqc$-descent by an argument similar to [20, Prop. 8.4] and [34, pp. 92-93]. □

Here is the Step 2 outlined in the introduction of this subsection.

Theorem 3.4.10. *Let X be as before. Then, X is smooth and there exist $\alpha, \beta \in H_x(\mathbf{Z}_{(p)})$ such that $X = \Delta_{\alpha,\beta}$. Moreover, if (E) holds, then we can take $(\alpha,\beta) = (1,\beta)$ with β being a p-unit.*

Proof. Recall that we have dominant projections $\Pi_i : Y \to V_i$. It follows that $End(\mathcal{A}_i)\otimes\mathbf{Q} = B$. In view of the discussion before (E), we suppose that (E) holds.

Let $\mathcal{A}_{/Y} = \mathcal{A}_1 \times_Y \mathcal{A}_2$ and $End^{\mathbf{Q}}(\mathcal{A}) = End(\mathcal{A}_{/Y}) \otimes \mathbf{Q}$. We now have two possibilities for $End^{\mathbf{Q}}(\mathcal{A})$, namely B^2 or $M_2(B)$. Based on de Jong's theorem in [26], we first show that $End^{\mathbf{Q}}(\mathcal{A}) = M_2(B)$. Along with Serre-Tate deformation theory, we later finish the proof.

We first suppose that $End^{\mathbf{Q}}(\mathcal{A}) = B^2$. We proceed in a way similar to the proof of Theorem 3.4.6. Let U be as in Proposition 3.4.9. In view of (E) and (PL), it follows that

$$End_{\epsilon\mathcal{O}_{B,p}}\epsilon\mathcal{A}[p^\infty]_{/U} = M_2(\mathcal{O}_{M_0}). \tag{3.4.6}$$

By Theorem 2.6 of [26], we have

$$End_{\mathcal{O}_{B,p}}(\mathcal{A}[p^\infty]_{/U}) = End_{\mathcal{O}_B}(\mathcal{A}_{/U}) \otimes \mathbf{Z}_p. \tag{3.4.7}$$

This contradicts our assumption that $End^{\mathbf{Q}}(\mathcal{A}) = B^2$. Thus, $End^{\mathbf{Q}}(\mathcal{A}) = M_2(B)$.

Let $e_i \in End^{\mathbf{Q}}(\mathcal{A})$ be two commuting idempotents such that $e_i(\mathcal{A}) = \mathcal{A}_{i/Y}$. As $End^{\mathbf{Q}}(\mathcal{A}) = M_2(B)$, there exists $\tilde{\beta} \in GL_2(\mathcal{O}_{B,(p)})$ such that $\tilde{\beta} \circ e_1 = e_2$. Thus, we have as isogeny $\tilde{\beta} : \mathcal{A}_1 \to \mathcal{A}_{2/Y}$. In particular, there exists $g \in G(\mathbf{A}^{(p)}_{\mathbf{Q},f})$ such that $\varphi \circ \eta_1^{(p)} = \eta_2^{(p)} \circ g$. Specialising to the fiber of $\mathcal{A}_i$ at y, we conclude that g is induced by an element $\beta \in H_x(\mathbf{Z}_{(p)})$. Thus, the equation defining $\widehat{Y}_y \subset \widehat{S} \times \widehat{S}$ is given by $t_2 = t_1^{\beta^{1-c_x}}$. In view of (E) and the argument in $\mathcal{O}_B$-linearity step of the proof of Proposition 3.4.9, we conclude that $\beta^{1-c_x} \in \mathbf{Z}_{(p)}^{\times}$. Here c_x denotes the non-trivial element in the Galois group of the imaginary quadratic extension over the rationals associated to x. Thus, we suppose that $\beta \in \mathbf{Z}_p^{\times}$. Recall that in the proof of (1) of Proposition 3.4.6 we have $\widehat{X} = \bigcup_{L \in I} \widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} L$, where L is an $\mathbf{Z}_p$-direct summand of $X_*(\widehat{S^2})$ and the union is finite. Moreover, the points of Y above x^2 are indexed by $L \in I$. More precisely, if y corresponds to L, then $\widehat{Y}_y = \widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} L$. Thus, $\widehat{\Delta_{1,\beta}}_{,x^2} = \widehat{Y}_y \subset \widehat{X}$. By an *fpqc*-descent, it follows that $\Delta_{1,\beta} \subset X$. As X is irreducible, we conclude that $\Delta_{1,\beta} = X$. $\square$

Thus, we have proven Theorem 3.4.5 for $n = 2$.

3.4.5 Global structure II

In this subsection, we study a class of global structure of locally stable subvarieties of n-fold self-product of the Shimura curve for $n \geq 2$.

Let the notation and hypotheses be as in §3.4.2.

Moreover, we suppose that $\mathcal{X} \neq S^n$.

Theorem 3.4.11. *Let X be as before. Then, X is smooth and there exist $\alpha, \beta \in H_x(\mathbf{Z}_{(p)})$ such that $X = V^{n-2} \times \Delta_{\alpha,\beta}$ up to a permutation of the first $n-1$ factors.*

Proof. The proof of Proposition 3.4.9 also works for $n \geq 2$.

Let $\mathcal{A}_{/Y} = \mathcal{A}^n \times_{V^n} Y$ and $End^{\mathbf{Q}}(\mathcal{A}) = End(\mathcal{A}_{/Y}) \otimes \mathbf{Q}$. We now have two possibilities for $End^{\mathbf{Q}}(\mathcal{A})$, namely B^n or $B^{n-2} \times M_2(B)$. In a same way as in the proof of Theorem 3.4.10, it can be shown that $End^{\mathbf{Q}}(\mathcal{A}) = B^{n-2} \times M_2(B)$. Thus, there exists $i < n$ such that i^{th} factor $\mathcal{A}_i$ of $\mathcal{A}_{/Y}$ is isogenous to the last factor $\mathcal{A}_n$ of $\mathcal{A}_{/Y}$.

Based on Serre-Tate deformation theory, we finish the proof in the same way as the proof of Theorem 3.4.10. □

This finishes the proof of Theorem 3.4.5.

Theorem 3.4.11 can also be proven by induction on n (cf. similar to the proof of [34, Cor. 3.19]).

3.4.6 Linear independence

In this subsection, we prove the linear independence of mod p modular forms as formulated in §3.4.1 based on the global structure of locally stable subvarieties.

As before, we have the following independence.

Theorem 3.4.12. *Let x be a closed ordinary point on the Shimura variety with the local stabiliser H_x. For $1 \leq i \leq n$, let $a_i \in H_x(\mathbf{Z}_p)$ such that $a_i a_j^{-1} \notin H_x(\mathbf{Z}_{(p)})$ for all $i \neq j$*

(cf. §3.3.2). Let $f_1, ..., f_n$ be non-constant mod p modular forms on V. Then, $(a_i \circ f_i)_i$ are algebraically independent in the Serre-Tate deformation space $Spf(\widehat{\mathcal{O}}_{V,x})$.

Proof. From the definition of $\mathfrak{b}$ (cf. §3.4.1), it suffices to show that $\mathfrak{b} = 0$. This is equivalent to show that $X = V^n$.

When $n = 1$, this follows from the definition. Let $n \geq 2$. We first note that there exists y over x^n such that

$$\widehat{\mathcal{Y}}_y \supset \widehat{\Delta} := \left\{(t^{a_1}, t^{a_2}, ..., t^{a_n}) | t \in \widehat{S}\right\}$$

(cf. similar to [34, pp.85]). When $n = 2$, this verifies the hypothesis in Theorem 3.4.5. When $n \geq 2$, the hypothesis in the theorem can be verified by induction on n up to a permutation of the first $n-1$ factors.

In view of the theorem, the locally stable subscheme X equals V^n or $V^{n-2} \times \Delta_{\alpha,\beta}$ for some $\alpha, \beta \in H_x(\mathbf{Z}_{(p)})$ up to a permutation of the first $n-1$ factors. We first suppose that $X = V^{n-2} \times \Delta_{\alpha,\beta}$. Let $\mathcal{X}'$ be the projection of $\mathcal{X}$ to the last two factors of S^n. As $X = V^{n-2} \times \Delta_{\alpha,\beta}$, the equation of $\widehat{\mathcal{X}'}_{x^2} \subset \widehat{S^2}$ is given by $t^{\beta^{1-c_x}} = (t')^{\alpha^{1-c_x}}$ for t (resp. t') being the Serre-Tate co-ordinate of the second last (resp. last) factor of $\widehat{S^n}$. On the other hand, it follows from the definition of X that the equation is also given by $t^{a_n} = t'^{a_{n-1}}$. Thus, $a_n a_{n-1}^{-1} = (\beta\alpha^{-1})^{1-c_x} \in H_x(\mathbf{Z}_{(p)})$. This contradicts our hypothesis on a_i's and finishes the proof. □

CHAPTER 4

Non-Triviality of Generalised Heegner Cycles Modulo p

In this chapter, we consider the non-triviality of the p-adic Abel-Jacobi image of generalised Heegner cycles modulo p.

4.1 Introduction

When a pure motive over a number field is self-dual with root number -1, the Bloch-Beilinson conjecture implies the existence of a non-torsion homologically trivial cycle in the Chow realisation. For a prime p, the Bloch-Kato conjecture implies the non-triviality of the p-adic étale Abel-Jacobi image of the cycle. A natural question is to further investigate the non-triviality of the p-adic Abel-Jacobi image of the cycle.

An instructive set up arises from a self-dual Rankin-Selberg convolution of an elliptic Hecke eigenform and a theta series over an imaginary quadratic extension K with root number -1. In this situation, a natural candidate for a non-torsion homologically trivial cycle is the generalised Heegner cycle. It lives in a middle dimensional Chow group of a fiber product of a Kuga-Sato variety arising from an indefinite Shimura curve and a self product of a CM abelian surface. In the case of weight two, the cycles coincide with the Heegner points. Twists of the theta series by p-power order anticyclotomic characters of K give rise to an Iwasawa theoretic family of generalised Heegner cycles. Under mild hypotheses, we prove the generic non-triviality of the p-adic Abel-Jacobi image of these cycles modulo p. In particular, this implies the generic non-triviality of the cycles in the top graded piece of the coniveau filtration along the $\mathbf{Z}_p$-anticyclotomic extension of K.

In the introduction, for simplicity we mostly restrict to the case of Heegner points.

Let p be an odd prime. We fix two embeddings $\iota_\infty\colon \overline{\mathbf{Q}} \to \mathbf{C}$ and $\iota_p\colon \overline{\mathbf{Q}} \to \mathbf{C}_p$. Let v_p be the p-adic valuation induced by ι_p so that $v_p(p) = 1$.

Let $K/\mathbf{Q}$ be an imaginary quadratic extension as above and $\mathcal{O}$ the ring of integers. As K is a subfield of the complex numbers, we regard it as a subfield of the algebraic closure $\overline{\mathbf{Q}}$ via the embedding ι_∞. Let c be the complex conjugation on $\mathbf{C}$ which induces the unique non-trivial element of $Gal(K/\mathbf{Q})$ via ι_∞. We assume the following:

(ord) p splits in K.

Let $\mathfrak{p}$ be a prime above p in K induced by the p-adic embedding ι_p. For a positive integer m, let H_m be the ring class field of K with conductor m and $\mathcal{O}_m = \mathbf{Z} + m\mathcal{O}$ the corresponding order. Let H be the Hilbert class field.

Let N be a positive integer such that $p \nmid N$. Let f be an elliptic newform of weight 2, level $\Gamma_0(N)$ and neben-character ϵ. Let $N_\epsilon | N$ be the conductor of ϵ. Let E_f be the Hecke field of f and $\mathcal{O}_{E_f}$ the ring of integers. Let $\mathfrak{P}$ be a prime above p in E_f induced by the p-adic embedding ι_p. Let $\rho_f : Gal(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\mathcal{O}_{E_{f,\mathfrak{P}}})$ be the corresponding p-adic Galois representation.

We fix a decomposition $N = N^+N^-$ where N^+ (resp. N^-) is only divisible by split (resp. ramified or inert) primes in $K/\mathbf{Q}$. As in Chapter 3, we assume the following hypotheses:

(h1) The level N is square-free and prime to the discriminant of K.

(h2) The number of primes dividing N^- is positive and even.

(h3) The conductor N_ϵ divides N^+.

As K satisfies the Heegner hypothesis for N^+, the integer ring $\mathcal{O}$ contains a cyclic ideal $\mathfrak{N}^+$ of norm N^+. From now, we fix such an ideal $\mathfrak{N}^+$. Let $\mathfrak{N}_\epsilon|\mathfrak{N}^+$ be the unique ideal of norm N_ϵ.

Let $\mathbf{N} : \mathbf{A}_{\mathbf{Q}}^\times/\mathbf{Q}^\times \to \mathbf{C}^\times$ be the norm Hecke character over $\mathbf{Q}$ given by

$$\mathbf{N}(x) = ||x||.$$

Here $||\cdot||$ denotes the adelic norm. Let $\mathbf{N}_K := \mathbf{N} \circ N_{\mathbf{Q}}^K$ be the norm Hecke character over K for the relative norm $N_{\mathbf{Q}}^K$. For a Hecke character $\lambda : \mathbf{A}_K^\times/K^\times \to \mathbf{C}^\times$ over K, let $\mathfrak{f}_\lambda$ (resp. ϵ_λ) denote its conductor (resp. the restriction $\lambda|_{\mathbf{A}_{\mathbf{Q}}^\times}$). We say that λ is central critical for f if it is of infinity type (j_1, j_2) with $j_1 + j_2 = 2$ and $\epsilon_\lambda = \epsilon\mathbf{N}^2$.

Let b be a positive integer prime to N. Let $\Sigma_{cc}(b, \mathfrak{N}^+, \epsilon)$ be the set of Hecke characters λ such that:

(C1) λ is central critical for f,

(C2) $\mathfrak{f}_\lambda = b \cdot \mathfrak{N}_\epsilon$.

Let χ be a finite order Hecke character such that $\chi\mathbf{N}_K \in \Sigma_{cc}(b, \mathfrak{N}^+, \epsilon)$. Let $E_{f,\chi}$ be the finite extension of E_f obtained by adjoining the values of χ.

Let B be an indefinite quaternion algebra over $\mathbf{Q}$ of conductor N^-. Let Sh_B be the corresponding Shimura curve of level $\Gamma_1(N^+)$ and J_B the Jacobian of Sh_B. Let f_B be a normalised Jacquet-Langlands transfer of f to Sh_B (cf. §4.2.1). Let B_f be the abelian variety associated to f_B by the Eichler-Shimura correspondence and $T_f \subset E_f$ an order such that B_f has T_f-endomorphisms. Let $\Phi_f : J_B \to B_f$ be the associated surjective morphism. Let ω_{f_B} be the differential form on Sh_B corresponding to f_B. We use the same notation for the corresponding one form on J_B. Let $\omega_{B_f} \in \Omega^1(B_f/E_f)^{T_f}$ be the unique one form such that $\Phi_f^*(\omega_{B_f}) = \omega_{f_B}$.

Here $\Omega^1(B_f/E_f)^{T_f}$ denotes the subspace of 1-forms given by

$$\Omega^1(B_f/E_f)^{T_f} = \left\{\omega \in \Omega^1(B_f/E_f) | [\lambda]^*\omega = \lambda\omega, \forall \lambda \in T_f\right\}.$$

Let A_b be an abelian surface with endomorphisms by $\mathcal{O}_b = \mathbf{Z} + b\mathcal{O}$, defined over the ring class field H_b. Let ι_{N^+} denote a level structure of $\Gamma_1(N^+)$-type on $A_b[\mathfrak{N}^+]$. In view of the moduli interpretation of the Shimura curve (cf. §3.2.2), we thus obtain a point $(A_b, \iota_{N^+}) \in Sh_B(H_{N^+b})$. Here we suppress additional data needed in the moduli interpretation for simplicity. Let $\Delta_b \in J_B(H_{N^+b})$ be the corresponding Heegner point on the modular Jacobian. Here we cohomologically trivialise the Heegner point as in Zhang's work, for example [69]. We regard χ as a character $\chi : Gal(H_{N^+b}/K) \to E_{f,\chi}$. Let $G_b = Gal(H_{N^+b}/K)$. Let H_χ be the abelian extension of K cut out by the character χ. To the pair (f, χ), we associate the Heegner point $P_f(\chi)$ given by

$$P_f(\chi) = \sum_{\sigma \in G_b} \chi^{-1}(\sigma)\Phi_f(\Delta_b^\sigma) \in B_f(H_\chi) \otimes_{T_f} E_{f,\chi}. \tag{4.1.1}$$

To consider the non-triviality of the Heegner points $P_f(\chi)$ as χ varies, we can consider the non-triviality of the corresponding p-adic formal group logarithm. The restriction of the p-adic formal group logarithm arising from the one form ω_{B_f} gives a homomorphism $\log_{\omega_{B_f}} : B_f(H_\chi) \to \mathbf{C}_p$. We extend it to $B_f(H_\chi) \otimes_{T_f} E_{f,\chi}$ by $E_{f,\chi}$-linearity.

We now fix a finite order Hecke character η such that $\eta\mathbf{N}_K \in \mathit{\Sigma}_{cc}(1, \mathfrak{N}^+, \epsilon)$. Let $H_{N^+p^\infty} = \bigcup_{n \geq 0} H_{N^+p^n}$ be the ring class field of conductor N^+p^∞. Let $K_\infty \subset H_{N^+p^\infty}$ be the anticyclotomic $\mathbf{Z}_p$-extension of K. In the above notation, we have $G_{p^n} = Gal(H_{N^+p^n}/K)$. Let Γ be the $\mathbf{Z}_p$-quotient of $\varprojlim G_{p^n}$. Let $\mathfrak{X}_0$ be the subgroup of finite order characters of the group $Gal(K_\infty/K) \simeq \mathbf{Z}_p$. Let $\mathfrak{X}$ denote the set of anticyclotomic Hecke characters over K factoring through Γ. For $\nu \in \mathfrak{X}_0$, let $G(\nu)$ denote the Gauss sum associated to ν considered as a primitive character. We consider the non-triviality of $G(\nu^{-1}) \log_{\omega_{B_f}}(P_f(\eta\nu))$ modulo p, as $\nu \in \mathfrak{X}_0$ varies.

Our result is the following.

Theorem A. Let the notation be as above. Let $f \in S_2(\Gamma_0(N), \epsilon)$ be an elliptic newform and η a finite order unramified Hecke character over K such that $\eta \mathbf{N}_K \in \Sigma_{cc}(1, \mathfrak{N}^+, \epsilon)$. In addition to the hypotheses (ord), (h1), (h2) and (h3), suppose that

(irr) the residual Galois representation ρ_f modulo p is absolutely irreducible.

Then, we have

$$\liminf_{\nu \in \mathfrak{X}_0} v_p\left(G(\nu^{-1}) \log_{\omega_{B_f}}(P_f(\eta\nu)) \right) = 0.$$

In particular, for $\nu \in \mathfrak{X}_0$ with sufficiently large p-power order the Heegner points $P_f(\eta\nu)$ are non-zero in $B_f(H_{\eta\nu}) \otimes_{T_f} E_{f,\eta\nu}$.

In fact, we show that the same conclusions hold when $\mathfrak{X}_0$ is replaced by any of its infinite subset. For analogous non-triviality of the p-adic Abel-Jacobi image of generalised Heegner cycles modulo p, we refer to §4.3.2.

The proof of Theorem A is based on the vanishing of the μ-invariant of an anticyclotomic Rankin-Selberg p-adic L-function and a recent p-adic Waldspurger formula due to Brooks.

We now describe the result regarding the μ-invariant. Associated to the pair (f, η), an anticyclotomic Rankin-Selberg p-adic L-function $L_p(f, \eta) \in \overline{\mathbf{Z}}_p[\![\Gamma]\!]$ is constructed in [6]. It is characterised by the interpolation formula

$$\widehat{\lambda}(L_p(f, \eta)) \doteq L(f, \eta\lambda\mathbf{N}_K, 0). \tag{4.1.2}$$

Here λ is an unramified Hecke character over K with infinity type $(m, -m)$ for $m \geq 0$ and $\widehat{\lambda}$ its p-adic avatar. The notation "$\doteq$" denotes that the equality holds up to well determined periods. Here we only mention that the periods crucially depend on the Jacquet-Langlands transfer f_B and the underlying Shimura curve.

Our result regarding the non-triviality of the p-adic L-function $L_p(f,\eta)$ is the following.

Theorem B. Let the notation be as above. Let $f \in S_2(\Gamma_0(N),\epsilon)$ be an elliptic newform and η a finite order unramified Hecke character over K such that $\eta\mathbf{N}_K \in \Sigma_{cc}(1,\mathfrak{N}^+,\epsilon)$. Suppose that the hypotheses (ord), (h1), (h2), (h3) and (irr) hold. Then, we have

$$\mu(L_p(f,\eta)) = 0.$$

We now describe the strategy of the proofs. Some of the notation used here is not followed in the rest of the article. The characters in $\mathfrak{X}_0$ are outside the range of interpolation for the p-adic L-function $L_p(f,\eta)$ and these values basically equal the p-adic formal group logarithm of Heegner points. This is based on the p-adic Waldspurger formula in [6]. A phenomena of this sort was first found by Rubin in the CM case (cf. [60]) and recently by Bertolin-Darmon-Prasanna in the general case (cf. [2]). As $\mathfrak{X}_0$ is a dense subset of characters of Γ, the p-adic Waldspurger formula reduces Theorem A to Theorem B. We prove the later based on a strategy of Hida. This strategy was introduced in [34]. Hida proves the vanishing of the μ-invariant of a class of anticyclotomic Katz p-adic L-functions in [34]. Let $G_{f,\eta} \in \overline{\mathbf{Z}}_p[\![T]\!]$ be the power series expansion of the measure $L_p(f,\eta)$ regarded as a p-adic measure on $\mathbf{Z}_p$ with support in $1+p\mathbf{Z}_p$, i.e.

$$G_{f,\eta} = \int_{1+p\mathbf{Z}_p} (1+t)^y dL_p(f,\eta)(y) = \sum_{k \in \mathbf{Z}_{\geq 0}} \Big(\int_{1+p\mathbf{Z}_p} \binom{y}{k} dL_p(f,\eta)(y) \Big) t^k. \tag{4.1.3}$$

The starting point is the fact that there are modular forms $(f_i)_{i=1}^m$ on the Shimura curve Sh_B such that

$$G_{f,\eta} = \sum_{i=1}^m a_i \circ (f_i(t)). \tag{4.1.4}$$

Here $f_i(t)$ is the t-expansion of f_i around a well chosen CM point x corresponding to the trivial ideal class in $Pic(\mathcal{O})$ on the Shimura curve Sh_B. Moreover, a_i is an automorphism of the deformation space of x in Sh_B such that the a_i's are mutually irrational. The modular forms f_i's are closely related to the Jacquet-Langlands transfer f_B. Based on Chai's study of Hecke-stable subvarieties of a Shimura variety, we prove the linear independence of $(a_i \circ f_i)_{i=1}^m$ modulo p. The independence is an analogue of Ax-Lindemann-Weierstrass conjecture for the mod p reduction of the Shimura curve Sh_B. The proof relies on Chai-Oort rigidity principle that a Hecke stable subvariety of a mod p Shimura variety is a Shimura subvariety. The principle is a mod p analogue of Andre-Oort conjecture for self products of the Shimura curve. An analogue of Ax-Lindemann-Weierstrass of this sort was originally found by Hida in the Hilbert modular case (cf. [34]). We closely follow Hida's approach. The assumption (irr) plays a key role in the independence as it implies the non-constancy of f_i's modulo p. In view of the independence and (4.1.4), it follows that $\mu(L_p(f,\eta)) = \min_i \mu(f_i(t)) = \min_i \mu(f_i)$. Based on our optimal choice of the Jacquet-Langlands transfer f_B and the p-integrality criterion in [59], we deduce that $\min_i \mu(f_i) = 0$. This finishes the proof. We would like to emphasize our perception that the independence lies at the heart of the proof of Theorem B.

“In particular" part of the Theorem A was conjectured by Mazur in the early 1980's (cf. [49]). It was proven by Cornut-Vatsal and Aflalo-Nekovář in the mid and late 2000's, respectively (cf. [23], [64], [24], [25] and [1]). We give a new approach and as far as we know the theorem is a first result regarding the non-triviality of the p-adic formal group logarithm of the Heegner points modulo p. It seems suggestive to compare our approach with the earlier approach. In the earlier approach, Ratner's theorem on ergodicity of torus actions is fundamental. As indicated above, our approach fundamentally relies on Chai's theory of Hecke stable subvarieties of a mod p Shimura variety. It is rather surprising that we have these quite different approaches for the same characteristic zero non-triviality. A speculation along these lines was expressed in [65]. It seems interesting that the ergodic nature of the earlier approach is still present in our approach albeit in a more geometric form. For a more detailed comparison, we refer to [15]. Before the p-adic Waldspurger formula, the ergodic approach and Hida's strategy appeared to be complementary. The formula also allows a rather smooth transition to the higher weight case.

As far as we know, higher weight analogue of Theorem A is a first general result regarding the non-triviality of generalised Heegner cycles. In particular, the non-triviality implies that the Griffiths group of the fiber product of a Kuga-Sato variety arising from the Shimura curve and self product of a CM abelian surface has infinite rank over $\overline{\mathbf{Q}}$ (cf. remark (2) following Theorem 4.3.2). An analogue of such a result for the Griffiths group of the Kuga-Sato variety is due to Besser (cf. [4]). In [4], the approach is based on the generic non-triviality of classical Heegner cycles over a class of varying imaginary quadratic extensions. Regarding generalised Heegner cycles in the case of modular curves, the only earlier known result seems to be the non-triviality of several examples of such families in [3]. Based on the Cornut-Vatsal approach, Howard has proven a related characteristic zero non-triviality of classical Heegner cycles in the case of modular curves in [38] and [39].

Theorem A and its higher weight analogue has various arithmetic applications. The non-triviality of the p-adic Abel-Jacobi image along the $\mathbf{Z}_p$-anticyclotomic extension is a typical hypothesis while working with an Euler system, for example [54] and [27]. The hypothesis typically plays a key role in bounding the size of relevant Selmer groups. Cornut-Vatsal's non-triviality results have been applied by Nekovář in his proof of the parity conjecture for a class of Selmer groups (cf. [55], [56], [57] and references therein). It seems that the higher weight analogue of Theorem A can have potential applications to the parity conjecture for Selmer groups associated to a more general Rankin-Selberg convolution. The non-triviality also provides an evidence for the refined Bloch-Beilinson conjecture. Some other arithmetic consequences of the "In particular" part of Theorem A are well documented in the literature, for example [23], [64], [1] and [27].

Theorem B is perhaps of independent interest as well. It is an input in the ongoing work of Jetchev, Skinner and Wan on the p-adic Birch and Swinnerton-Dyer conjecture (cf. [43]).

The result in turn is expected to play a key role in the ongoing work of Bhargava, Skinner and Zhang on the Birch and Swinnerton-Dyer conjecture for a large proportion of elliptic curves over the rationals.

Hida's strategy has been influential in the study of non-triviality of various arithmetic invariants modulo p. For example, the non-triviality results in [34], [7], [40], [42], [8] and [14] are based on the Hilbert modular independence in [34]. The strategy and especially its geometric aspects have been further explored and refined in [11].

Recently, the p-adic Waldspurger formula has been generalised to modular forms on Shimura curves over a totally real field (cf. [48]). In [13], we consider an analogue of the independence for quaternionic Shimura varieties over a totally real field. In the near future, we hope to consider an analogous non-triviality of generalised Heegner cycles over a CM field. In a forthcoming article, we consider an (l, p)-analogue of the results. Such an analogue in the case of modular curves is considered in [14].

The article is perhaps a follow up to [2] and particularly, [34]. We refer to them for a general introduction. In the exposition, we often suppose that the reader is familiar with them, particularly [34].

The chapter is organised as follows. In §4.2, we prove the vanishing of the μ-invariant of a class of anticyclotomic Rankin-Selberg p-adic L-functions. In §4.3, we prove the non-triviality of the p-adic Abel-Jacobi image of generalised Heegner cycles modulo p over the $\mathbf{Z}_p$-anticyclotomic extension.

4.2 μ-invariant of Anticyclotomic Rankin-Selberg p-adic L-functions

In this section, we prove the vanishing of the μ-invariant of a class of anticyclotomic Rankin-Selberg p-adic L-functions. In §4.2.1, we consider the p-depletion of a normalised Jacquet-Langlands transfer of an elliptic Hecke eigenform. In §4.2.2, we describe generalities regarding the anticyclotomic Rankin-Selberg p-adic L-functions. In §4.2.3, we prove the vanishing of the μ-invariant.

4.2.1 p-depletion

In this subsection, we consider the p-depletion of a normalised Jacquet-Langlands transfer of an elliptic Hecke eigenform.

Let the notation and hypotheses be as in the introduction. Let $f \in S_k(\Gamma_0(N), \epsilon)$ be an elliptic newform such that:

(irr) the residual Galois representation ρ_f modulo p is absolutely irreducible.

The Jacquet-Langlands correspondence implies the existence of a classical modular form f_B on Sh_B such that the following holds:

(JL1) f_B is a classical modular form on the Shimura curve Sh_B of weight k and neben-character ϵ.
(JL2) For positive integer n such that $(n, N^-) = 1$, f_B is a Hecke eigenform for the Hecke operator T_n with the same eigenvalue as f.

We normalise f_B by requiring that

$$\mu(f_B) = 0. \tag{4.2.1}$$

Here $\mu(f_B)$ denotes the μ-invariant of the t-expansion of f_B around any CM point as defined on page 3 of §1.1.

In what follows, we consider f_B as being defined over $\mathcal{O}_{E_f,\mathfrak{P}}$ and regard it as a p-adic modular form on the Shimura variety Sh. Here E_f is the Hecke field and $\mathfrak{P}$ a prime above p induced by ι_p.

We have the following useful lemma.

Lemma 4.2.1. *Let the notation and assumptions be as above. Then, the Hecke eigenform f_B is non-constant modulo p.*

Proof. If the Hecke eigenform f_B is constant modulo p, then the Hecke eigensystem is Eisenstein. This contradicts (irr). □

A class of anticyclotomic Rankin-Selberg p-adic L-functions is constructed via toric periods of the p-depletion of f_B.

Let

$$f_B^{(p)} = f_B|_{VU-UV} \tag{4.2.2}$$

be the p-depletion of f_B. Here V and U are Hecke operators in [6, §3.6].

We recall the following lemma.

Lemma 4.2.2. *Let the notation be as above. Let $f_B(t) = \sum_{\xi \in \mathbf{Z}_{\geq 0}} a(\xi, f_B)t^\xi$ be the t-expansion of f_B around x. We have*

$$f_B^{(p)}(t) = \sum_{\xi \in \mathbf{Z}_{\geq 0}, p \nmid \xi} a(\xi, f_B)t^\xi.$$

Proof. This readily follows from [6, Prop. 4.17]. □

We have the following proposition regarding the p-integrality of the p-depletion.

Proposition 4.2.3. *Let the notation and assumptions be as above. We have*

$$\mu(f_B^{(p)}) = 0.$$

Moreover, the p-depletion is non-constant modulo p.

Proof. The proof is based on the p-integrality criterion in [59, Prop. 2.9].

In view of the criterion and (4.2.1), we have

$$\min_{M,\chi} \mu\left(\frac{L_\chi(f_B)}{\Omega_M^{2k}}\right) = 0. \tag{4.2.3}$$

Here M is an imaginary quadratic extension of $\mathbf{Q}$ such that it has an embedding into the indefinite quaternion algebra B, p splits in M and p does not divide the class number of M. Moreover, χ is an unramified Hecke character over M with infiinity type $(k, 0)$, $L_\chi(f_B)$ the toric period of the pair (f, χ) and Ω_M the CM period (cf. [59, §2.3]).

As f_B is a Hecke eigenform, we have

$$L_\chi(f_B^{(p)}) = (1 - \chi^{-1}(\overline{\mathfrak{p}})a_p + \chi^{-2}(\overline{\mathfrak{p}})\epsilon(p)p^{k-1})L_\chi(f_B) \tag{4.2.4}$$

(cf. [6, Prop. 8.9]). Here $\mathfrak{p}$ is a prime above p induced by the p-adic embedding ι_p as before and a_p denotes the T_p-eigenvalue of f.

It thus follows that

$$\min_{M,\chi} \mu\left(\frac{L_\chi(f_B^{(p)})}{\Omega_M^{2k}}\right) = 0. \tag{4.2.5}$$

In view of the criterion, this finishes the proof.

"Moreover" part now immediately follows from the previous lemma. □

For later purposes, we introduce the following modular forms related to the p-depletion.

Let $\mathcal{U}_p$ be the torsion subgroup of $\mathbf{Z}_p^\times$. For $u \in \mathcal{U}_p$, let $\phi_u : \mathbf{Z}/p\mathbf{Z} \to \mathcal{W}$ be the indicator function corresponding to u.

Let $f_{B,u}$ be the modular form given by

$$f_{B,u} = f_B|\phi_u. \tag{4.2.6}$$

Lemma 4.2.4. *Let the notation and assumptions be as above. We have*

$$f_B^{(p)} = \sum_{u \in \mathcal{U}_p} f_{B,u}. \tag{4.2.7}$$

Proof This follows from the t-expansion principle, Proposition 3.3.6 and Lemma 4.2.2. □

4.2.2 Anticyclotomic Rankin-Selberg p-adic L-functions

In this section, we describe generalities regarding a class of anticyclotomic Rankin-Selberg p-adic L-functions. This is a slight reformulation of the results in [6, §8]. We adapt the formulation in [37, §1.3.7], [5, §8] and [42].

Let the notation and hypotheses be as in the introduction. Let $\Gamma_{N^+} = \varprojlim G_{p^n}$, where $G_{p^n} = Gal(H_{N^+p^n}/K)$. Recall that Γ denotes the $\mathbf{Z}_p$-quotient of Γ_{N^+}. We fix a section of the projection $\pi : \Gamma_{N^+} \twoheadrightarrow \Gamma$ stable under the action of complex conjugation c. Let $\mathcal{C}(\Gamma, \overline{\mathbf{Z}}_p)$ be the space of continuous functions on Γ with values in $\overline{\mathbf{Z}}_p$.

Let Σ_{cc} denote the set of Hecke character over K central critical for f i.e. the set of Hecke characters λ such that λ of infinity type (j_1, j_2) with $j_1 + j_2 = k$ and $\epsilon_\lambda = \epsilon_f \mathbf{N}^k$. Let $\Sigma_{cc}^{(1)}$ be the subset of Hecke character of K with infinity type (l_1, l_2) such that $1 \leq l_1, l_2 \leq k-1$. Let $\Sigma_{cc}^{(2)}$ be the subset of Hecke character over K with infinity type (l_1, l_2) such that $l_1 \geq k$ and $l_2 \leq 0$. Recall that $\mathfrak{X}$ denotes the set of anticyclotomic Hecke characters over K factoring though Γ.

For $\chi \in \Sigma_{cc}^{(1)}$ (resp. $\Sigma_{cc}^{(2)}$), the global root number of the Rankin-Selberg convolution $L(f, \chi^{-1}, s)$ equals -1 (resp. 1). From now, we fix an unramified Hecke character $\chi \in \Sigma_{cc}^{(2)}$ with infinity type $(k, 0)$. For $\eta \in \Gamma$, note that $\chi\eta \in \Sigma_{cc}^{(2)}$.

Let

$$Cl_- := K^\times \mathbf{A}_{\mathbf{Q},f}^\times \backslash \mathbf{A}_{K,f} / U_K$$

and Cl_-^{alg} the subgroup of Cl_- generated by ramified primes. Here $U_K = (K \otimes_{\mathbf{Q}} \mathbf{R})^\times \times (\mathcal{O} \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}})^\times$. Let $\mathcal{R}$ be the subgroup of $\mathbf{A}_K^\times$ generated by $K_v^\times$ for ramified v. Let $Cl_-^{alg} \subset Cl_-$ be the subgroup generated by $\mathcal{R}$. Let $\mathcal{D}_1$ be a set of representatives for Cl_-/Cl_-^{alg} in $(\mathbf{A}_{K,f}^{(pN)})^\times$. Let $\mathcal{U}^{alg} = U_K \cap (K^\times)^{1-c}$.

For $a \in \mathcal{D}_1$, let $\mathcal{F}_B^{(p)}(x(a))$ be the p-adic measure on Γ such that

$$\int_\Gamma \binom{x}{n} d\mathcal{F}_B^{(p)}(x(a)) = \binom{d}{n} f_B^{(p)}(x(a)).$$

Here n is a non-negative integer and d denotes the Katz p-adic differential operator defined in [44].

We have the following useful result.

Lemma 4.2.5. *The power series expansion of the measure $\mathcal{F}_B^{(p)}(x(a))$ regarded as a p-adic measure on $\mathbf{Z}_p$ with support in $1 + p\mathbf{Z}_p$ equals the t-expansion of the p-depletion $f_B^{(p)}$ around* *the CM point* $x(a)$.

Proof. This follows from a similar argument as in the proof of [5, Prop. 8.1]. □

Let $L_p(f,\chi)$ be the p-adic measure on $\Gamma_{\mathfrak{N}^+}$ such that for $\varphi \in \mathcal{C}(\Gamma, \overline{\mathbf{Z}}_p)$, we have

$$\int_{\Gamma_{N^+}} \varphi dL_p(f,\chi) = \sum_{a \in \mathcal{D}_1} \chi(a) \int_{\Gamma} \varphi|[a] d\mathcal{F}_B^{(p)}(x(a)). \tag{4.2.8}$$

The operator $|[a] \in End(\mathcal{C}(\Gamma, \overline{\mathbf{Z}}_p))$ is given by

$$\varphi \mapsto (\varphi|[a])(\sigma) = \varphi(\sigma rec_K(a)|_{\Gamma}).$$

Here rec_K is the Artin reciprocity map.

To state the interpolation property of the measure, we need further notation.

Let

$$\alpha(f, f_B) = \frac{\langle f, f \rangle}{\langle f_B, f_B \rangle}. \tag{4.2.9}$$

Here $\langle \cdot, \cdot \rangle$ denotes the normalised Petersson inner product in [59, §1].

For an unramified Hecke character $\lambda \in \Sigma_{cc}^{(2)}$ with infinity type $(k+j, -j)$ for $j \geq 0$, let

$$C(f,\lambda) = \frac{1}{4}\pi^{k+2j-1} \cdot \Gamma(j+1)\Gamma(k+j) \cdot w_K \sqrt{|d_K|} 2^{|S_f|} \cdot \prod_{l|N^-} \frac{l-1}{l+1}. \tag{4.2.10}$$

Here S_f denotes the set of primes which ramify in K that divide N^+ but do not divide the conductor of ϵ.

Let $\mathfrak{b} \subset \mathcal{O}_K$ be an ideal, $b_N \in \mathcal{O}_K$ and $W_f \in \mathbf{C}^\times$ as in [6, Prop. 8.3] and following Corollary 8.4 of [6], respectively.

Let

$$W(f,\lambda) = W_f \cdot \epsilon_f(\mathbf{N})(\mathfrak{b})^{-1} \lambda \mathbf{N}^{-j}(\mathfrak{b}) \cdot (-N)^{k/2+j} b_N^{-k-2j}. \tag{4.2.11}$$

Let $(\Omega, \Omega_p) \in \mathbf{C}^\times \times \mathbf{C}_p^\times$ be the complex and p-adic CM periods in the beginning of [6, §8.4].

We have the following result regarding the p-adic variation of central critical Rankin-Selberg L-values over the $\mathbf{Z}_p$-anticyclotomic extension of K.

Theorem 4.2.6. *(Brooks) Let the notation be as above. Let $f \in S_k(\Gamma_0(N), \epsilon)$ be an elliptic newform and $\chi \in \Sigma_{cc}^{(2)}$ an unramified Hecke character over K with infinity type $(k, 0)$. For an unramified Hecke character $\nu \in \mathfrak{X}$ with infinity type $(m, -m)$, we have*

$$\frac{\widehat{\nu}(L_p(f,\chi))}{\Omega_p^{2(k+2(a+m))}} = (1 - (\chi\nu)^{-1}(\overline{\mathfrak{p}})a_p + (\chi\nu)^{-2}(\overline{\mathfrak{p}})\epsilon(p)p^{k-1}) \cdot t_K \cdot \frac{C(f,\chi\nu)}{\alpha(f, f_B)W(f,\chi\nu)} \cdot \frac{L(f,\chi\nu,0)}{\Omega^{2(k+2m)}}.$$

Here

$$t_K = \frac{|\mathcal{U}^{alg}|}{[\mathcal{O}^\times : \mathbf{Z}^\times]|Cl_-^{alg}|}$$

Proof. This is essentially proven in [6, Prop. 8.9] based on the Waldspurger formula on the Shimura curve. The extra factor t_K arises from the definition (4.2.8). □

Remark. (1). Note that t_K equals a power of 2.

(2). We have an analogous construction of the p-adic L-function for unramified Hecke characters χ over K with infinity type $(k + a, -a)$ for $a \geq 0$.

4.2.3 μ-invariant

In this subsection, we prove the vanishing of the μ-invariant of a class of antiyclcotomic Rankin-Selberg p-adic L-function.

Let the notation and hypotheses be as in §4.2.2. Let Γ' be the open subgroup of Γ generated by the image of $1+p\mathbf{Z}_p$ via rec_K. Let $\pi_- : (\mathbf{A}_{K,f}^{(pN)})^\times \to \Gamma$ be the map arising from the reciprocity law. Let $Z' := \pi_-^{-1}(\Gamma')$ be the subgroup of $(\mathbf{A}_{K,f}^{(pN)})^\times$ and let $Cl'_- \supset Cl_-^{alg}$ be the image of Z' in Cl_- and let $\mathcal{D}'_1$ (resp. $\mathcal{D}''_1$) be a set of representatives of Cl'_-/Cl_-^{alg} (resp. Cl_-/Cl'_-) in $(\mathbf{A}_{K,f}^{(pN)})^\times$. Let $\mathcal{D}_1 := \mathcal{D}''_1\mathcal{D}'_1$ be a set of representatives of Cl_-/Cl_-^{alg}. Let $\mathcal{D}_0$ be a set of representatives of $\mathcal{U}_p/\mathcal{U}^{alg}$ in $\mathcal{U}_p$.

We have the following theorem regarding the μ-invariant of the anticyclotomic Rankin-Selberg p-adic L-functions.

Theorem 4.2.7. *Let $f \in S_k(\Gamma_0(N), \epsilon)$ be an elliptic newform and $\chi \in \Sigma_{cc}^{(2)}$ an unramified Hecke character over K with infinity type $(k, 0)$. Suppose that the hypotheses (ord), (h1), (h2), (h3) and (irr) hold. Then, we have*

$$\mu(L_p(f,\chi)) = 0.$$

Proof. Let t denote the Serre-Tate co-ordinate of the deformation space of the CM point $x(1)$ corresponding to the trivial ideal class. For $a \in \mathcal{D}'_1$, let $\langle a\rangle_\Sigma$ be the unique element in $1+p\mathbf{Z}_p$ such that $rec_{\Sigma_p}(\langle a\rangle_\Sigma) = \pi_-(rec_K(a)) \in \Gamma'$.

For $(a,b) \in \mathcal{D}_1 \times \mathcal{D}''_1$, let

$$\widetilde{\mathcal{F}}(t) = \sum_{u\in\mathcal{U}_p} f_{B,u}(t^{u^{-1}}) \tag{4.2.12}$$

and

$$\mathcal{F}^b(t) = \sum_{a\in b\mathcal{D}'_1} \chi(ab^{-1})\widetilde{\mathcal{F}}|[a](t^{\langle ab^{-1}\rangle}). \tag{4.2.13}$$

Here $|[a]$ is the Hecke action induced by a.

Let $L_p^b(f,\chi)$ be the restriction of the measure $L_p(f,\chi)$ to $\pi_-(b)\Gamma'$. By definition, we have

$$\mu(L_p(f,\chi)) = \min_{b\in\mathcal{D}_1''} \mu(L_p^b(f,\chi)). \tag{4.2.14}$$

In view of the Lemma 4.2.5, we have

$$d^q\mathcal{F}^b\big|_{t=1} = \int_{\Gamma'} \nu_q dL_p^b(f,\chi). \tag{4.2.15}$$

Here q is a non-negative integer and ν_q the p-adic character of Γ' such that $\nu_q(rec_K(y)) = y^q$ for $y \in 1 + p\mathbf{Z}_p$.

It follows that the formal t-expansion $\mathcal{F}^b(t)$ equals the power series expansion of the measure $L_p^b(f,\chi)$ regarded as a p-adic measure on $\mathbf{Z}_p$ with support in $1+p\mathbf{Z}_p$ (cf. (4.1.3)).

Note that

$$\widetilde{\mathcal{F}}(t) = |\mathcal{U}^{alg}| \cdot \sum_{u\in\mathcal{U}_p/\mathcal{U}^{alg}} f_{B,u}(t^{u^{-1}}). \tag{4.2.16}$$

Thus, we have

$$\mathcal{F}^b(t) = |\mathcal{U}^{alg}| \cdot \sum_{(u,a)\in\mathcal{D}_0\times b\mathcal{D}_1'} \chi(ab^{-1}) f_{B,u}|[a](t^{\langle ab^{-1}\rangle u^{-1}}). \tag{4.2.17}$$

Note that $p \nmid |\mathcal{U}^{alg}|$. From [40, Lemma 5.3], the linear independence of mod p modular forms (cf. Theorem 3.4.1) and the t-expansion principle of p-adic modular forms, it follows that

$$\mu(L_p^b(f,\chi)) = \min_{u\in\mathcal{D}_0} \mu(f_{B,u}). \tag{4.2.18}$$

In view of Proposition 4.2.3 and Lemma 4.2.4, this finishes the proof. □

Remark. (1). The hypothesis that the Hecke character χ is unramified is present in [6]. It mainly arises as Prasanna's explicit version of the Waldspurger formula in [59] is conditional on the hypothesis. It seems likely that the hypothesis can be removed from the above theorem once we have an explicit version of the Waldspurger formula in the ramified case. Under mild hypotheses, such a Waldspurger formula is perhaps available (cf. [68]).

(2). The theorem has a similar flavour as the results on the vanishing of the μ-invariant in [64] and [42]. In these articles, the hypothesis (irr) is essential for the vanishing. For a discussion of the necessity, we refer to the introduction of these articles.

(3). A closely related p-adic L-function is constructed in [48]. Our strategy applies to this p-adic L-function as well and we can deduce the vanishing of its μ-invariant under the above hypotheses.

(4). The theorem can be used as an input in the proof of Perrin-Riou's conjecture on Heegner points under mild hypotheses in [66]. The result on the μ-invariant in [42] is originally used in [66].

4.3 Non-triviality of generalised Heegner cycles modulo p

In this section, we consider the non-triviality of the p-adic Abel-Jacobi image of generalised Heegner cycles modulo p. In §4.3.1, we describe the p-adic Waldspurger formula due to Brooks. In §4.3.2, we prove the non-triviality.

4.3.1 p-adic Waldspurger formula

In this subsection, we describe the p-adic Waldspurger formula due to Brooks relating certain values of the anticyclotomic Rankin-Selberg p-adic L-function outside the range of interpolation to the p-adic Abel-Jacobi image of generalised Heegner cycles.

Unless otherwise stated, let the notation and hypotheses be as in the introduction of this chapter. Let f be a normalised elliptic newform of even weight $k \geq 2$, level $\Gamma_0(N)$ and nebencharacter ϵ. We also denote the Jacquet-Langlands transfer as in §4.2.1 by the same notation. Let ω_f be the corresponding differential.

We first recall that the construction of generalised Heegner cycles in [6] requires the weight being even (cf. [6, §6]).

Let

$$r = \frac{k-2}{2}. \tag{4.3.1}$$

Let A be the CM abelian surface corresponding to the trivial ideal class in $Pic(\mathcal{O})$ defined over the Hilbert class field H of K (cf. §3.2.4). Let $\mathcal{A}_r$ be the Kuga-Sato variety given by r-fold fiber product of the universal abelian surface. For an extension F/K containing the real quadratic field M_0 (cf. §3.2.1), let W_r be the variety over F given by $W_r = \mathcal{A}_r \times A^r$. By the abuse of notation, let ϵ also denote the idempotent in the ring of correspondences on W_r defined in [6, §6.1].

For an integer j such that $0 \leq j \leq 2r$, let $\omega_f \wedge \omega_A^j \eta_A^{2r-j} \in Fil^{2r+1} \epsilon H_{dR}^{4r+1}(W_r/F)$ be as in [6, §6.4]. Let n be a positive integer. For an ideal $\mathfrak{a} \subset \mathcal{O}_{N^+p^n}$, let $\Delta_{\mathfrak{a}} \in \epsilon CH^{2r+1}(W_r \otimes L)_{0,\mathbf{Q}}$ be the codimension-$(2r+1)$ homologous to zero generalised Heegner cycle defined in [6, §6.2]. Here L is the field of definition of the cycle and $CH^{2r+1}(W_r \otimes L)_{0,\mathbf{Q}}$ is the Chow group of codimension-$(2r+1)$ homologous to zero cycles over L with rational coefficients.

Let

$$AJ_p : \epsilon CH^{2r+1}(W_r)_{0,\mathbf{Q}} \rightarrow (Fil^{2r+1} \epsilon H_{dR}^{4r+1}(W_r/F)(r))^{\vee} \tag{4.3.2}$$

be the p-adic Abel-Jacobi map in [6, §6.3].

We have the following p-adic Waldspurger formula.

Theorem 4.3.1. *(Brooks) Let the notation be as above. Let* $f \in S_k(\Gamma_0(N), \epsilon)$ *be an elliptic newform and* $\chi \in \Sigma_{cc}^{(2)}$ *an unramified Hecke character over* K *with infinity type* $(k, 0)$*. Let* η *be a Hecke character such that* $\chi\eta \in \Sigma_{cc}^{(1)}$ *is an unramified Hecke character over* K *with infinity type* $(k-1-j, 1+j)$ *for* $0 \leq j \leq 2r$*. Let* $\nu \in \mathfrak{X}_0$ *be a primitive Hecke character of conductor* p^n*, where* $n \geq 1$*. Suppose that the hypotheses (ord), (h1), (h2) and (h3) hold. Then, we have*

$$\frac{\widehat{\eta\nu}(L_p(f,\chi))}{\Omega_p^{2(2r-2j)}} = \left(\frac{G(\nu^{-1})}{j!} \cdot \sum_{[\mathfrak{a}] \in G_{p^n}} (\chi\eta\nu)^{-1}(\mathfrak{a}) \boldsymbol{N}_K(\mathfrak{a}) \cdot AJ_p(\Delta_{\mathfrak{a}})(\omega_f \wedge \omega_A^j \eta_A^{r-j})\right)^2.$$

Proof. This follows from the argument in the proof of Theorem 8.11 in [6]. As η is of p-power conductor, we obtain the extra factor of the Gauss sum. For a related argument, we refer to the proof of [18, Thm. 4.9]. □

4.3.2 Non-triviality

In this subsection, we consider the non-triviality of the p-adic Abel-Jacobi image of generalised Heegner cycles modulo p.

Let the notation and hypotheses be as in §4.3.1. We have the following result regarding the non-triviality.

Theorem 4.3.2. *Let* $f \in S_k(\Gamma_0(N), \epsilon)$ *be an elliptic newform of even weight and* $\chi \in \Sigma_{cc}^{(2)}$ *an unramified Hecke character over* K *with infinity type* $(k, 0)$*. Let* η *be a Hecke character such* *that $\chi\eta \in \Sigma_{cc}^{(1)}$ is an unramified Hecke character over K with infinity type $(k-1-j, 1+j)$ for $0 \leq j \leq 2r$. Suppose that the hypotheses (ord), (h1), (h2), (h3) and (irr) hold. Then, we have*

$$\liminf_{\nu \in \mathfrak{X}_0} v_p\left(\frac{G(\nu^{-1})}{j!} \cdot \sum_{[\mathfrak{a}] \in G_{p^n}} (\chi\eta\nu)^{-1}(\mathfrak{a}) \boldsymbol{N}_K(\mathfrak{a}) \cdot AJ_p(\Delta_{\mathfrak{a}})(\omega_f \wedge \omega_A^j \eta_A^{r-j})\right) = 0,$$

where p^n is the conductor of ν. Moreover, the same conclusion holds when $\mathfrak{X}_0$ is replaced by any of its infinite subset.

Proof. This follows readily from Theorem 4.2.7 and Theorem 4.3.1. □

Remark. (1). It follows that the generalised Heegner cycles are non-trivial in the top graded piece of the coniveau filtration on the Chow group over the $\mathbf{Z}_p$-anticyclotomic extension. The non-triviality can be seen as an evidence for the refined Bloch-Beilinson conjecture as follows. Recall that the Rankin-Selberg convolution in consideration is self-dual with root number -1. In view of [61] and the Jacquet-Langlands correspondence, the corresponding Galois representation contributes to an étale cohomology of W_r. The conjecture thus predicts the existence of a non-trivial cycle in the top graded piece of the coniveau filtration on the Chow group. Generalised Heegner cycles are a natural source of cycles in the setup and in the case of weight two, they coincide with classical Heegner points. We can thus expect a generic non-triviality of generalised Heegner cycles. For the details and the role of coniveau filtration, we refer to [3, §1 and §2] and [15].

(2). In view of the theorem and the construction of generalised Heegner cycles, it follows that the Griffiths group $Gr^{r+1}(W_{r/\overline{\mathbf{Q}}}) \otimes \mathbf{Q}$ has infinite rank. An analogous result for the Griffiths group of the Kuga-Sato variety $\mathcal{A}_r$ is due to Besser (cf. [4]). The approach in [4] is via consideration of generic non-triviality of classical Heegner cycles over a class of varying imaginary quadratic extensions.

Note that Theorem A is a special case of Theorem 4.3.2 of weight 2.

References

[1] E. Aflalo and J. Nekovář, *Non-triviality of CM points in ring class field towers*, With an appendix by Christophe Cornut. Israel J. Math. 175 (2010), 225–284.

[2] M. Bertolini, H. Darmon and K. Prasanna, *Generalised Heegner cycles and p-adic Rankin L-series*, Duke Math Journal, Vol. 162, No. 6, 1033-1148.

[3] M. Bertolini, H. Darmon and K. Prasanna, *p-adic L-functions and the coniveau filtration on Chow groups*, preprint, 2013, available at "http://www.math.mcgill.ca/darmon/pub/pub.html".

[4] A. Besser, *CM cycles over Shimura curves*, J. Algebraic Geom. 4 (1995), no. 4, 659–691.

[5] M. Brakocevic, *Anticyclotomic p-adic L-function of central critical Rankin-Selberg L-value*, IMRN, Vol. 2011, No. 21, (2011), 4967-5018.

[6] E. H. Brooks, *Shimura curves and special values of p-adic L-functions*, to appear in IMRN (2014), doi: 10.1093/imrn/rnu062.

[7] A. Burungale and M.-L. Hsieh, *The vanishing of μ-invariant of p-adic Hecke L-functions for CM fields*, Int. Math. Res. Not. IMRN 2013, no. 5, 1014–1027.

[8] A. Burungale, *On the μ-invariant of the cyclotomic derivative of a Katz p-adic L-function.*, J. Inst. Math. Jussieu 14 (2015), no. 1, 131–148.

[9] A. Burungale, *On the non-triviality of the p-adic Abel-Jacobi image of generalised Heegner cycles modulo p, II: Shimura curves*, J. Inst. Math. Jussieu, to appear.

[10] A. Burungale, *A conjectural linear independence of mod p modular forms*, preprint 2012.

[11] A. Burungale and H. Hida, *$\mathfrak{p}$-rigidity and Iwasawa μ-invariants*, preprint, 2013, available at http://www.math.ucla.edu/$\sim$ ashay/ .

[12] A. Burungale, *An $l \neq p$-interpolation of genuine p-adic L-functions*, preprint 2013, submitted.

[13] A. Burungale, *$\mathfrak{p}$-rigidity and $\mathfrak{p}$-independence of quaternionic modular forms modulo p*, preprint, 2014.

[14] A. Burungale, *On the non-triviality of generalised Heegner cycles modulo p, I: modular curves*, preprint, 2014, available at http://www.math.ucla.edu/$\sim$ ashay/ .

[15] A. Burungale, *Non-triviality of generalised Heegner cycles over anticyclotomic towers: a survey*, submitted to the proceedings of the ICTS program 'p-adic aspects of modular forms', 2014.

[16] A. Burungale and H. Hida, *André-Oort conjecture and non-vanishing of central L-values over Hilbert class fields*, preprint 2015, submitted.

[17] H. Carayol, *Sur la mauvaise réduction des courbes de Shimura*, Compositio Math. 59 (1986), no. 2, 151-230.

[18] F. Castella, *Heegner cycles and higher weight specializations of big Heegner points*, Math. Annalen 356 (2013), 1247-1282.

[19] C.-L. Chai, *Every ordinary symplectic isogeny class in positive characteristic is dense in the moduli*, Invent. Math. 121 (1995), 439-479.

[20] C.-L. Chai, *Families of ordinary abelian varieties: canonical coordinates, p-adic monodromy, Tate-linear subvarieties and Hecke orbits*, preprint, 2003. Available at http://www.math.upenn.edu/∼ chai/papers.html .

[21] C.-L. Chai, *Hecke orbits as Shimura varieties in positive characteristic*, International Congress of Mathematicians. Vol. II, 295-312, Eur. Math. Soc., Zurich, 2006.

[22] C.-L. Chai and F. Oort, *Hypersymmetric abelian varieties*, Pure Appl. Math. Q. 2 (2006), no. 1, part 1, 1-27.

[23] C. Cornut, *Mazur's conjecture on higher Heegner points*, Invent. Math. 148 (2002), no. 3, 495-523.

[24] C. Cornut and V. Vatsal, *CM points and quaternion algebras*, Doc. Math. 10 (2005), 263–309.

[25] C. Cornut and V. Vatsal, *Nontriviality of Rankin-Selberg L-functions and CM points*, L-functions and Galois representations, 121–186, London Math. Soc. Lecture Note Ser., 320, Cambridge Univ. Press, Cambridge, 2007.

[26] A. J. de Jong, *Homomorphisms of Barsotti-Tate groups and crystals in positive characteristic*, Invent. Math. 134 (1998), 301-333.

[27] Olivier Fouquet, *Iwasawa theory of nearly ordinary quaternionic automorphic forms*, Compos. Math. 149 (2013), no. 3, 356–416.

[28] A. Grothendieck, *Groupes de Barsotti-Tate et cristaux de Dieudonné*, Séminaire de Mathématiques Supérieures, No. 45 (Été, 1970). Les Presses de l'Université de Montréal, Montreal, Que., 1974. 155 pp.

[29] H. Hida, *Non-vanishing modulo p of Hecke L-values*, In "Geometric Aspects of Dwork Theory" (A. Adolphson, F. Baldassarri, P. Berthelot, N. Katz and F. Loeser, eds.), Walter de Gruyter, Berlin, 2004, 735-784.

[30] H. Hida and J. Tilouine, *Anticyclotomic Katz p-adic L-functions and congruence modules*, Ann. Sci. Ecole Norm. Sup., (4) 26 (1993), no. 2, 189-259.

[31] H. Hida, *p-Adic Automorphic Forms on Shimura Varieties*, Springer Monogr. in Math., Springer-Verlag, New York, 2004.

[32] H. Hida, *Non-vanisihng modulo p of Hecke L-values and applications*, London Mathematical Society Lecture Note, Series 320 (2007), 207-269.

[33] H. Hida, *Irreducibility of the Igusa tower*, Acta Math. Sin. (Engl. Ser.) 25 (2009), 1-20.

[34] H. Hida, *The Iwasawa μ-invariant of p-adic Hecke L-functions*, Ann. of Math. (2) 172 (2010), 41-137.

[35] H. Hida, *Vanishing of the μ-Invariant of p-Adic Hecke L-functions*, Compositio Math. 147 (2011), 1151-1178.

[36] H. Hida, *Local indecomposability of Tate modules of non CM abelian varieties with real multiplication*, J. Amer. Math. Soc. 26 (2013), no. 3, 853–877.

[37] H. Hida, *Elliptic Curves and Arithmetic Invariants*, Springer Monogr. in Math., Springer, New York, 2013, xviii+449 pp.

[38] B. Howard, *Special cohomology classes for modular Galois representations*, J. Number Theory 117 (2006), no. 2, 406–438.

[39] B. Howard, *Variation of Heegner points in Hida families*, Invent. Math. 167 (2007), no. 1, 91–128.

[40] M.-L. Hsieh, *On the μ-invariant of anticyclotomic p-adic L-functions for CM fields*, J. Reine Angew. Math. 688 (2014), 67–100.

[41] M.-L. Hsieh, Eisenstein congruence on unitary groups and Iwasawa main conjecture for CM fields, J. Amer. Math. Soc., 27 (2014), no. 3, 753-862.

[42] M.-L. Hsieh, *Special values of anticyclotomic Rankin-Selbeg L-functions*, Doc. Math. 19 (2014), 709–767.

[43] D. Jetchev, C. Skinner and X. Wan, *The Birch-Swinnerton-Dyer Formula For Elliptic Curves of Analytic Rank One and Main Conjectures*, in preparation.

[44] N. M. Katz, *p-adic L-functions for CM fields*, Invent. Math., 49(1978), no. 3, 199-297.

[45] N. M. Katz, *Serre-Tate local moduli*, in Algebraic Surfaces (Orsay, 1976-78), Lecture Notes in Math. 868, Springer-Verlag, New York, 1981, pp. 138-202.

[46] M. Kisin, *Integral models for Shimura varieties of abelian type*, JAMS 23 (4) (2010), 967-1012.

[47] K.-W. Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, London Mathematical Society Monographs Series, 36. Princeton University Press, Princeton, NJ, 2013. xxvi+561 pp.

[48] Y. Liu, S. Zhang and W. Zhang, *On p-adic Waldspurger formula*, preprint, 2014, available at http://www.math.mit.edu/∼ liuyf/ .

[49] B. Mazur, *Modular curves and arithmetic*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 185–211, PWN, Warsaw, 1984.

[50] J. Milne, *The points on a Shimura variety modulo a prime of good reduction*, The zeta functions of Picard modular surface, Univ. Montreal, Montreal, QC, 1992, 151-253.

[51] J. Milne, *Shimura varieties and motives*, Proc. Sympos. Pure Math., 55, Part 2 (1994), Amerc. Math. Soc., Providence, RI, 447-523.

[52] B. Moonen, *Linearity properties of Shimura varieties I*, J. Algebraic Geom. 7 (1998), no. 3, 539-567.

[53] B. Moonen, *Linearity properties of Shimura varieties II*, Compositio Math. 114 (1998), no. 1, 3-35.

[54] J. Nekovář, *Kolyvagin's method for Chow groups of Kuga-Sato varieties*, Invent. Math. 107 (1992), no. 1, 99–125.

[55] J. Nekovář, *On the parity of ranks of Selmer groups. II*, C. R. Acad. Sci. Paris Sér. I Math. 332 (2001), no. 2, 99–104.

[56] J. Nekovář, *Growth of Selmer groups of Hilbert modular forms over ring class fields*, Ann. Sci. Éc. Norm. Supér. (4) 41 (2008), no. 6, 1003–1022.

[57] J. Nekovář, *On the parity of ranks of Selmer groups. IV*, With an appendix by Jean-Pierre Wintenberger. Compos. Math. 145 (2009), no. 6, 1351–1359.

[58] R. Noot, *Models of Shimura varieties in mixed characteristic*, J. Algebraic Geom. 5 (1996) 187-207.

[59] K. Prasanna, *Integrality of a ratio of Petersson norms and level-lowering congruences*, Ann. of Math. (2) 163 (2006), no. 3, 901–967.

[60] K. Rubin, *p-adic L-functions and rational points on elliptic curves with complex multiplication*, Invent. Math. 107 (1992), no. 2, 323–350.

[61] A. Scholl, *Motives for modular forms*, Invent. Math. 100 (1990), no. 2, 419-430.

[62] G. Shimura, *Abelian varieties with complex multiplication and modular functions*, Princeton Mathematical Series, 46. Princeton University Press, Princeton, NJ, 1998.

[63] V. Vatsal, *Uniform distribution of Heegner points*, Invent. Math. 148, 1-48 (2002).

[64] V. Vatsal, *Special values of anticyclotomic L-functions*, Duke Math J., 116, 219-261 (2003).

[65] V. Vatsal, *Special values of L-functions modulo p*, International Congress of Mathematicians. Vol. II, 501–514, Eur. Math. Soc., Zürich, 2006.

[66] X. Wan, *Heegner point Kolyvagin system and Iwasawa main conjecture*, preprint, 2014, available at http://www.math.columbia.edu/∼ xw2295.

[67] T. Wedhorn, *Ordinariness in good reductions of Shimura varieties of PEL type*, Ann. Scient. École Norm. Sup. (4) 32 (1999), 575-618.

[68] X. Yuan, S. Zhang and W. Zhang, *The Gross-Zagier formula on Shimura curves*, Annals of Mathematics Studies, vol 184. (2013) viii+272 pages.

[69] S. Zhang, *Heights of Heegner points on Shimura curves*, Ann. of Math. (2) 153 (2001), no. 1, 27–147.