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September 23, 1968

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Submitted to Physical Review

UCRL-18485
Preprint

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

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ABSTRACT

The Feynman-Dyson perturbation theory is applied to Schwinger's model of the monopole. The propagator for photons between electric and magnetic charges is found to be $D^{AB}_{\mu\nu}(k) = (k^2 + i\epsilon)^{-1}(\epsilon_{\mu\nu\lambda\kappa}\, n^\lambda k^\kappa)/(n \cdot k)$, n being the unit vector in the direction of the singularity line. Since the exact theory is independent of n, one might try to obtain a manifestly covariant perturbation expansion by averaging over all directions of n. However, under such a procedure the Born term fails to reduce to the known nonrelativistic limit.

I. INTRODUCTION

Schwinger[1] has constructed a field theory of magnetic monopoles and proved that it is Lorentz-invariant as a consequence of the charge quantization condition $eg/4\pi$ = integer, e being the electric charge and g the magnetic charge. But the problem of calculating the interaction between electric and magnetic charges has remained unsolved, since all the known approximation methods seem to break down. S-matrix techniques fail because the photon has zero mass, and a perturbation expansion is dubious in view of the large coupling constant. Furthermore, several authors[2-5] have pointed out that the interaction of charges with monopoles seems to violate Lorentz invariance, analyticity, and crossing symmetry. Since these authors demanded manifest Lorentz invariance of the theory, while Schwinger permits the apparent asymmetry of the singularity line, one should reexamine their conclusions on the basis of Schwinger's formalism. In this paper we investigate to what extent the Feynman-Dyson perturbation method can be applied to Schwinger's monopole theory.

After a brief summary of Schwinger's formalism we calculate, in Section III, the propagator for photon exchange between electric and magnetic charges. We use the familiar method of ordinary quantum electrodynamics;[6] that is, we choose a particular frame, analyze the transverse and the instantaneous (Coulomb) parts separately, and find that their sum becomes covariant if we use current conservation. The resulting propagator $(k^2 + i\epsilon)^{-1}[\epsilon_{\mu\nu\lambda\kappa}\, n^\lambda\, k^\kappa/(n\cdot k)]$ is covariant, apart from a dependence on n, the unit vector in the direction of the singularity line. Now we can show explicitly (in Section IIIc) why Weinberg's[2] and Zwanziger's[4] arguments against the monopole do not hold in Schwinger's theory.

Now, we might expect to obtain a manifestly covariant perturbation expansion by averaging over all directions of the singularity line. However, as we see in Section V, such a procedure fails to reproduce the correct nonrelativistic limit.[5] The discrepancy comes from the azimuthal dependence of the amplitude.

II. SCHWINGER'S THEORY

A. The Need for a Singularity Line

We follow Schwinger's theory[1] because (i) it is a relativistically covariant field theory, (ii) it is a natural generalization of conventional quantum electrodynamics, and (iii) it maintains the complete symmetry between electricity and magnetism implied by Maxwell's equations

$$\partial_\nu F^{\mu\nu} = j^\mu , \qquad \partial_\nu {}^*F^{\mu\nu} = {}^*j^\mu ,$$
$${}^*F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\kappa} F_{\lambda\kappa} , \tag{2.1}$$

where j^μ is the electric, ${}^*j^\mu$ the magnetic current, both of which are, of course, conserved. Schwinger considered a model with spin 1/2 magnetic charges, but it can be generalized to spin 0 and 1 particles.[7]

For the spin 1/2 model the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2} (\underset{\sim}{E}^2 + \underset{\sim}{H}^2) + \overline{\psi}\, \underset{\sim}{\gamma} \cdot (-i\underset{\sim}{\nabla} - e\underset{\sim}{A}^T - e\underset{\sim}{A}_g)\psi + m_e \overline{\psi}\psi$$
$$+ \overline{\chi}\, \underset{\sim}{\gamma} \cdot (-i\underset{\sim}{\nabla} - g\underset{\sim}{B}^T - g\underset{\sim}{B}_e)\chi + m_g \overline{\chi}\chi , \tag{2.2}$$

with the fields

$$\underset{\sim}{E} = \underset{\sim}{E}^T - \underset{\sim}{\nabla}\phi \quad \text{and} \quad \underset{\sim}{H} = \underset{\sim}{H}^T - \underset{\sim}{\nabla}{}^*\phi ; \tag{2.3}$$

the scalar potentials are

$$\phi(x) = \int d^3x' \,\mathcal{D}(\underset{\sim}{x}-\underset{\sim}{x}')\, j^0(x') \quad \text{and} \quad {}^*\phi(x) = \int d^3x' \,\mathcal{D}(\underset{\sim}{x}-\underset{\sim}{x}')\, {}^*j^0(x') \tag{2.4}$$

where

$$\mathcal{D}(\underset{\sim}{x}) = \frac{1}{4\pi|\underset{\sim}{x}|} ,$$

and the transverse vector potentials are defined by

$$\underset{\sim}{H}^T = \underset{\sim}{\nabla} \times \underset{\sim}{A}^T \qquad \text{and} \qquad \underset{\sim}{E}^T = -\underset{\sim}{\nabla} \times \underset{\sim}{B}^T . \tag{2.5}$$

The spin 1/2 field for the electric charge is ψ, for the magnetic charge it is χ, and the electric and magnetic currents are $j^\mu = e\,\overline{\psi}\,\gamma^\mu\,\psi$ and ${}^*j^\mu = g\,\overline{\chi}\,\gamma^\mu\,\chi$ respectively. The additional vector potentials

$$\underset{\sim}{A}_g(x) = \int d^3x'\ \underset{\sim}{a}(\underset{\sim}{x}-\underset{\sim}{x}')\ {}^*j^0(x') \quad \text{and} \quad \underset{\sim}{B}_e(x) = \int d^3x'\ \underset{\sim}{a}(\underset{\sim}{x}-\underset{\sim}{x}')\ j^0(x') \tag{2.6}$$

where

$$\underset{\sim}{a}(\underset{\sim}{x}) = \frac{1}{2}\,\mathcal{D}(\underset{\sim}{x})\hat{n} \times \underset{\sim}{x} \left(\frac{1}{|\underset{\sim}{x}| + \hat{n}\cdot\underset{\sim}{x}} - \frac{1}{|\underset{\sim}{x}| - \hat{n}\cdot\underset{\sim}{x}}\right) \tag{2.7}$$

($\hat{n}$ = unit vector in the direction of the singularity line) are needed to express the static interaction between a fixed monopole and a moving electric charge, and vice versa, as $\underset{\sim}{j}\cdot\underset{\sim}{A}_g$ and ${}^*\underset{\sim}{j}\cdot\underset{\sim}{B}_e$. We see that the interaction terms in Eq. (2.2) come from the free Hamiltonian via the familiar "minimal substitution"

$$\overline{\psi}\, p^\mu\, \psi \rightarrow \overline{\psi}(p^\mu - e\,A^\mu)\psi , \qquad \underset{\sim}{A} = \underset{\sim}{A}^T + \underset{\sim}{A}_g ,$$

$$\overline{\chi}\, p^\mu\, \chi \rightarrow \overline{\chi}(p^\mu - g\,B^\mu)\chi , \qquad \underset{\sim}{B} = \underset{\sim}{B}^T + \underset{\sim}{B}_e .$$

The need for singularity line in the static vector potentials $\underset{\sim}{A}_g$ and $\underset{\sim}{B}_e$ is easy to see. If the potentials are to satisfy

$$\underset{\sim}{E} \overset{!}{=} -\underset{\sim}{\nabla} \times \underset{\sim}{B} \qquad \text{and} \qquad \underset{\sim}{H} \overset{!}{=} \underset{\sim}{\nabla} \times \underset{\sim}{A}$$

or

$$\underset{\sim}{E}^T - \underset{\sim}{\nabla}\phi \overset{!}{=} -\underset{\sim}{\nabla} \times \underset{\sim}{B}^T - \underset{\sim}{\nabla} \times \underset{\sim}{B}_e \qquad \text{and} \qquad \underset{\sim}{H}^T - \underset{\sim}{\nabla}\,{}^*\phi \overset{!}{=} \underset{\sim}{\nabla} \times \underset{\sim}{A}^T + \underset{\sim}{\nabla} \times \underset{\sim}{A}_g$$

then we are faced with equations like $\underset{\sim}{\nabla}\phi = \underset{\sim}{\nabla} \times \underset{\sim}{B}_e$ which cannot be solved exactly because after integration over a closed surface the RHS would be zero while the LHS would yield the total charge inside the surface. There are, however, solutions which are valid almost everywhere, that is they fail only on some line from the charge to infinity. Intuitively one can think of the flux of field lines passing from the charge through the singularity line to infinity (or to another charge, which would correspond to an infinitely long and infinitely thin dipole). This singularity line is the string that Dirac attached to his monopole; it destroys manifest rotational invariance of the formalism, but it is unobservable because the charge quantization condition restores the rotational invariance of the theory.

In the following we use Schwinger's two-sided straight singularity line from $-\infty$ to the particle to $+\infty$, and the corresponding singular function $\underset{\sim}{a}(\underset{\sim}{x})$ of Eq. (2.7). It satisfies

$$\underset{\sim}{\nabla} \times \underset{\sim}{a} = -\underset{\sim}{\nabla}\mathfrak{D} + \underset{\sim}{h}\,, \tag{2.8}$$

where

$$\underset{\sim}{h} = \underset{\sim}{h}_{\hat{n}}(\underset{\sim}{x}) = -\frac{1}{2}\hat{n}\frac{\hat{n}\cdot\underset{\sim}{x}}{|\underset{\sim}{x}|}\delta_{\hat{n}}(\underset{\sim}{x}) \tag{2.9}$$

and $\delta_{\hat{n}}(\underset{\sim}{x})$ is the two-dimensional δ function in the plane orthogonal to $\hat{n}$. Now the exact relation between the fields and the potentials becomes

$$\begin{aligned} \underset{\sim}{E}(x) &= -\underset{\sim}{\nabla}\times\underset{\sim}{B}(x) - \int d^3x'\, \underset{\sim}{h}(\underset{\sim}{x}-\underset{\sim}{x}')\, j^0(x') \\ \underset{\sim}{H}(x) &= \underset{\sim}{\nabla}\times\underset{\sim}{A}(x) - \int d^3x'\, \underset{\sim}{h}(\underset{\sim}{x}-\underset{\sim}{x}')\, {}^{*}j^0(x') . \end{aligned} \tag{2.10}$$

Schwinger has proved that his theory is Lorentz invariant in spite of the singularity line provided that $eg/4\pi$ is quantized. The singularity line is necessary to formulate the theory and to carry out calculations; the physically observable results, if calculated exactly, will be independent of the singularity line.

B. Interactions

Now let us examine the various terms in the Hamiltonian Eq. (2.2). It can be broken up into $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$

where

$$\mathcal{H}_0 = \frac{1}{2}(\underset{\sim}{E}^T)^2 + \frac{1}{2}(\underset{\sim}{H}^T)^2 + \bar{\psi}(-\gamma\cdot\underset{\sim}{\nabla} + m_e)\psi + \bar{\chi}(-\gamma\cdot\underset{\sim}{\nabla} + m_g)\chi$$

is the free Hamiltonian, and

$$\begin{aligned} \mathcal{H}_I = \frac{1}{2}(\underset{\sim}{E}^L)^2 + \frac{1}{2}(\underset{\sim}{H}^L)^2 &- e\,\bar{\psi}\,\gamma\cdot\underset{\sim}{A}^T\,\psi - e\,\bar{\psi}\,\gamma\cdot\underset{\sim}{A}_g\,\psi \\ &- g\,\bar{\chi}\,\gamma\cdot\underset{\sim}{B}^T\,\chi - g\,\bar{\chi}\,\gamma\cdot\underset{\sim}{B}_e\,\chi \end{aligned} \tag{2.11}$$

contains all the interactions (the $\underset{\sim}{E}^L \cdot \underset{\sim}{E}^T$ and $\underset{\sim}{H}^L \cdot \underset{\sim}{H}^T$ terms in the energy density have been neglected; they would drop out after integration over all space). The $\frac{1}{2}(\underset{\sim}{E}^L)^2 = \frac{1}{2}(-\underset{\sim}{\nabla}\phi)^2$ term is the familiar static Coulomb energy of electric charges, and so is $\frac{1}{2}(H^L)^2 = \frac{1}{2}(-\nabla\, {}^*\phi)^2$ for the monopoles. These Coulomb terms cancel the noncovariant parts of the photon propagators $\langle A_\mu(x)\, A_\nu(y)\rangle_+ = \langle 0|T\{A_\mu(x), A_\nu(y)\}|0\rangle$ between electric and $\langle B_\mu(x)\, B_\nu(y)\rangle_+$ between magnetic charges.

The big problem is, of course, the interaction between charges and monopoles. To order eg there are two kinds of terms

(i) the exchange of transverse photons (Fig. 1a)

$$\underset{\sim}{j}\cdot\langle \underset{\sim}{A}\,\underset{\sim}{B}\rangle_+ \cdot {}^*\underset{\sim}{j}\,, \tag{2.12}$$

(ii) the instantaneous interaction (Fig. 1b)

$$-e\,\bar{\psi}\,\underset{\sim}{\gamma}\cdot\underset{\sim}{A}_g\,\psi \;-\; g\,\bar{\chi}\,\underset{\sim}{\gamma}\cdot\underset{\sim}{B}_e\,\chi\,. \tag{2.13}$$

Now we have accounted for all the terms in $\mathcal{H}_I$. In the next section we compute the terms (2.12) and (2.13) explicitly. Their sum gives us the propagator $\langle A_\mu(x)\, B_\nu(y)\rangle_+$ for photon exchange between electric and magnetic charges (Fig. 1c). While each of the interaction terms by itself is ugly and noncovariant, their sum, as a result of current conservation, turns out to be as simple and covariant as could be expected in Schwinger's theory, that is

$$D^{AB}_{+\mu\nu}(k) = \frac{1}{k^2 + i\epsilon}\;\frac{\epsilon_{\mu\nu\lambda\kappa}\, n^\lambda\, k^\kappa}{n\cdot k} \tag{2.14}$$

in momentum space. This method is familiar from the quantization of ordinary electrodynamics.[6]

3. THE COMPUTATION OF THE PROPAGATOR

A. The Transverse Part

To calculate $\langle A_m(x)B_n(y)\rangle_+$ (see Fig. 1a) we use Schwinger's technique[8], first expressing the potentials in terms of the fields and then writing the vacuum expectation values of field products in the most general form allowed by Lorentz invariance. Equation (2.10) can be written as

$$\begin{aligned}\underset{\sim}{\nabla} \times \underset{\sim}{A}(x) &= \underset{\sim}{H}(x) + \int d^3x_1\, \underset{\sim}{h}(\underset{\sim}{x} - \underset{\sim}{x}_1)\, \underset{\sim}{\nabla}_1 \times \underset{\sim}{H}(x_1) \\ &= \underset{\sim}{\nabla} \times \left(\int d^3x_1\, \underset{\sim}{h}(\underset{\sim}{x} - \underset{\sim}{x}_1) \times \underset{\sim}{H}(x_1)\right),\end{aligned}$$

since $\underset{\sim}{\nabla}_1 \cdot \underset{\sim}{H}(x_1) = {}^*j^0(x_1)$ and $\underset{\sim}{\nabla} \cdot \underset{\sim}{h}(\underset{\sim}{x} - \underset{\sim}{x}_1) = -\delta(\underset{\sim}{x} - \underset{\sim}{x}_1)$. Conforming with the gauge condition $\underset{\sim}{\nabla} \cdot \underset{\sim}{A} = 0 = \underset{\sim}{\nabla} \cdot \underset{\sim}{B}$ this yields

$$\underset{\sim}{A}(x) = \left(\int d^3x_1\, \underset{\sim}{h}(\underset{\sim}{x} - \underset{\sim}{x}_1) \times \underset{\sim}{H}(x_1)\right)^T,$$

and similarly (3.1)

$$\underset{\sim}{B}(x) = -\left(\int d^3x_1\, \underset{\sim}{h}(\underset{\sim}{x} - \underset{\sim}{x}_1) \times \underset{\sim}{E}(x_1)\right)^T .$$

We can forget about the selection of the transverse part for the moment and instead subtract the longitudinal part at the end of the whole calculation. With this proviso we can write

$$\langle A_m(x)B_n(x')\rangle_+ = -\int d^3x_1\, d^3x_1'\, \epsilon_{mij}\, h_i(\underset{\sim}{x} - \underset{\sim}{x}_1)\, \langle H_j(x_1)E_r(x_1')\rangle_+ \, h_s(\underset{\sim}{x}' - \underset{\sim}{x}_1')\, \epsilon_{nsr} .$$

Choose a frame in which $\hat{n} = \hat{z}$; then $\underset{\sim}{h}(\underset{\sim}{x}) = -\frac{1}{2}\,\delta(x)\delta(y)\epsilon(z)\hat{z}$ and

$$\langle A_m(x)B_n(x')\rangle_+ = \frac{1}{4}\,\epsilon_{m3j}\,\epsilon_{nr3}\int dz_1\, dz_1'\,\epsilon(z - z_1)\,\epsilon(z' - z_1')$$

$$\times\,\langle H_j(x,y,z_1,t)\,E_r(x',y',z_1',t')\rangle_+ \,. \qquad (3.2)$$

The integrals over dz_1 and dz_1' are convolutions. Since we are really interested in the Fourier transform of the propagator we can use the convolution theorem and conclude that

$$\text{F.T.}\,\{\langle A_m\,B_n\rangle_+\} = \frac{1}{4}\,\epsilon_{m3j}\,\epsilon_{nr3}\,(2\pi)^2\left(\text{F.T.}\{\epsilon\}\right)^2\,\text{F.T.}\{\langle H_jE_r\rangle_+\}\,. \qquad (3.3)$$

To minimize the number of factors of 2π we have chosen the convention $F(k) = \text{F.T.}\{f\} = \frac{1}{2\pi}\int dx\, e^{ikx}\, f(x)$ and in four dimensions

$$D^{AB\;T}_{+mn}(k) = \text{F.T.}\{\langle A_m\,B_n\rangle_+\} = \frac{1}{(2\pi)^4}\int d^4x\; e^{ik\cdot(x-x')}\langle A_m(x)\,B_n(x')\rangle_+ \,.$$

(This implies dropping the conventional $(2\pi)^{-4}$ in the propagators.) The transform of $\epsilon(z)$ is

$$\text{F.T.}\{\epsilon\} = \frac{1}{2\pi}\int dz\; e^{ik_3z}\,\epsilon(z) = \frac{i}{\pi}\,\frac{P}{k_3}\,, \qquad (3.4)$$

where the P means principal values; after all, we are dealing with distributions.

By Lorentz invariance the vacuum expectation value of the product of two fields must have the form

$$\langle F_{\mu\nu}(x)F_{\lambda\kappa}(x')\rangle_+ = \int d^4k\; e^{-ik\cdot(x-x')}\int_0^\infty dm^2\;\frac{F_{\mu\nu,\lambda\kappa}(k)}{k^2 - m^2 + i\epsilon}\,, \qquad (3.5)$$

with

$$F_{\mu\nu,\lambda\kappa}(k) = (k_\mu k_\lambda g_{\nu\kappa} - k_\nu k_\lambda g_{\mu\kappa} + k_\nu k_\kappa g_{\mu\lambda} - k_\mu k_\kappa g_{\nu\lambda})A(m^2)$$
$$+ (g_{\mu\lambda} g_{\nu\kappa} - g_{\nu\lambda}\, g_{\mu\kappa})m^2 A'(m^2) - \epsilon_{\mu\nu\lambda\kappa}\, m^2\, A''(m^2) \; .$$

Here we need $H_j = \frac{1}{2}\, \epsilon_{jk\ell}\, F_{k\ell}$ and $E_r = F_{0r}$

$$F_{k\ell,\,0r}(k) = (k_k k_0 g_{\ell r} - k_\ell k_0 g_{kr})\; A(m^2) - \epsilon_{k\ell\,0r}\; m^2\; A''(m^2) \; . \qquad (3.6)$$

Because of the photon pole at $m^2 = 0$ the spectral function A has the form $A(m^2) = A_0\; \delta(m^2) + A_1(m^2)$. The propagator of perturbation theory is obtained when we keep only the one-photon contribution to the spectrum and set the charge renormalization constant $(A_0)^{1/2} = 1$, that is we approximate $A(m^2) = \delta(m^2)$ and $m^2 A''(m^2) = 0$. Another way of looking at it is this: up to now we have been working in the Heisenberg picture with the exact fields $F_{\mu\nu} = F_{\mu\nu\; H}$. The Feynman-Dyson perturbation expansion is carried out in the interaction picture. The approximation of keeping only the $m^2 = 0$ part of the spectrum is equivalent to replacing $F_{\mu\nu\; H}$ by $F_{\mu\nu\; I}$ (or by the free fields $F_{\mu\nu\; in}$).

Now the integration over dm^2 in Eq. (3.5) is trivial and we obtain

$$\langle H_j(x) E_r(x')\rangle_+ \;=\; \epsilon_{jkr} \int d^4k\; e^{-ik\cdot(x-x')}\; \frac{k_0\; k_k}{k^2 + i\epsilon}$$

and its transform

$$\text{F.T. } (\langle H_j E_r\rangle_+) \;=\; \epsilon_{jkr}\; \frac{k_0\;\; k_k}{k^2\; + i\epsilon} \; . \qquad (3.7)$$

Finally we put (3.7) and (3.4) into (3.3)

$$D^{AB\,T}_{+mn}(k) = \frac{1}{4}\,\epsilon_{m3j}\,\epsilon_{nr3}\,(2\pi)^2\left\{\frac{i}{\pi}\;\frac{P}{k_3}\right\}^2\left\{\epsilon_{jkr}\;\frac{k_0\,k_k}{k^2+i\epsilon}\right\}$$

$$= -\,\epsilon_{mn3}\;\frac{P}{k_3}\;\frac{k_0}{k^2+i\epsilon}\;.$$

Since we had chosen $\hat{n} = \hat{z}$, the k_3 really means $\underset{\sim}{k}\cdot\hat{n}$ and ϵ_{mn3} means $\epsilon_{mn\ell}n^{\ell}$. Therefore

$$D^{AB\,T}_{+mn}(k) = -\,\frac{k_0}{k^2+i\epsilon}\;\frac{\epsilon_{mn\ell}\,\hat{n}^{\ell}}{k\cdot\hat{n}}\;. \qquad (3.8)$$

Now we can face the gauge condition $\underset{\sim}{\nabla}\cdot\underset{\sim}{A} = 0 = \underset{\sim}{\nabla}\cdot\underset{\sim}{B}$ as promised in the beginning. In momentum space this requires $k^m\,D^{AB\,T}_{+mn}(k) = 0 = k^n\,D^{AB\,T}_{+mn}(k)$. We subtract the longitudinal parts, maintaining the anti-symmetry in the indices m and n, and find

$$D^{AB\,T}_{+mn}(k) = \frac{1}{k^2+\epsilon}\;\frac{k_0}{\underset{\sim}{k}\cdot\hat{n}}\left\{-\,\epsilon_{mn\ell}\,\hat{n}^{\ell}+\epsilon_{rn\ell}\,\hat{n}^{\ell}\,\frac{k^r k_m}{\underset{\sim}{k}^2} - \epsilon_{rm\ell}\,\hat{n}^{\ell}\;\frac{k^r\,k_n}{\underset{\sim}{k}^2}\right\}. \qquad (3.9)$$

At this stage in ordinary quantum electrodynamics one generalizes such an expression to four-vector notation and finds that the noncovariant parts are canceled by the Coulomb term and by current conservation. As we shall see, the same things happens here. Current conservation $k^{\mu}j_{\mu} = 0 = k^{\mu *}\,j_{\mu}$ allows us to drop terms proportional to k_{μ} and k_{ν}, and to write Eq. (3.9) as

$$D^{AB\,T}_{+\mu\nu}(k) = \frac{1}{k^2+i\epsilon}\;\frac{k\cdot\eta}{k\cdot n}\left\{\epsilon_{\mu\nu\lambda\kappa}\,n^{\lambda}\,\eta^{\kappa} + \epsilon_{\sigma\nu\lambda\kappa}\,n^{\lambda}\,\eta^{\kappa}\,\frac{k^{\sigma}\eta_{\mu}(k\cdot\eta)}{k^2-(k\cdot\eta)^2} - \epsilon_{\sigma\mu\lambda\kappa}\,n^{\lambda}\,\eta^{\kappa}\;\frac{k^{\sigma}\,\eta_{\nu}(k\cdot\eta)}{k^2-(k\cdot\eta)^2}\right\}. \qquad (3.10)$$

where $\epsilon_{0123} = +1$, $n = (0,\hat{\underset{\sim}{n}})$, and we have introduced the timelike unit vector $\eta = (1,\underset{\sim}{0})$.

B. The Instantaneous Part

The instantaneous interaction (2.12) (Fig. 1b), $-\underset{\sim}{j}\cdot\underset{\sim}{A}_g - {}^*\underset{\sim}{j}\cdot\underset{\sim}{B}_e$, is due to the force between a static magnetic and a moving electric charge, and vice versa. To first order in eg its contribution to the S matrix between states $|i\rangle = |p,s;q,t\rangle$ and $\langle f| = \langle p',s';q',t'|$ is

$$-i\langle f|\int d^4x[-\underset{\sim}{j}\cdot\underset{\sim}{A}_g - {}^*\underset{\sim}{j}\cdot\underset{\sim}{B}_e]|i\rangle = -i\frac{eg}{(2\pi)^6}\int d^4x\, d^4x'\, \delta(t-t')$$

$$\times\Big\{-\bar{u}(p',s')\gamma_i\, u(p,s)e^{i(p'-p)\cdot x}\, \bar{w}(q',t')\gamma_0\, w(q,t)e^{i(q'-q)\cdot x'}$$

$$+\bar{w}(q',t')\gamma_i\, w(q,t)e^{i(q'-q)\cdot x}\, \bar{u}(p',s')\gamma_0 u(p,s)e^{i(p'-p)\cdot x'}\Big\}$$

$$\times a_i(\underset{\sim}{x}-\underset{\sim}{x}'),$$

where we have inserted the explicit form of $\underset{\sim}{A}_g$ and $\underset{\sim}{B}_e$ from Eq. (2.6); $u(p,s)$ and $w(q,t)$ are the electric and magnetic spinors. This equation integrates to

$$-i\frac{eg}{(2\pi)^2}\,\delta^4(p'+q'-p-q)\Big\{\bar{w}\,\gamma_i\, w\,\bar{u}\,\gamma_0\, u - \bar{w}\,\gamma_0\, w\,\bar{u}\,\gamma_i\, u\Big\}\tilde{a}_i(\underset{\sim}{p}-\underset{\sim}{p}') \tag{3.11}$$

where the Fourier transform of $\underset{\sim}{a}$ of Eq. (2.7) remains to be evaluated

$$\tilde{a}_i(\underset{\sim}{k}) = \int d^3x\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\, a_i(\underset{\sim}{x}).$$

With the choice $\hat{n} = \hat{z}$, $\underset{\sim}{a}(\underset{\sim}{x})$ becomes

$$a_1(\underset{\sim}{x}) = -\frac{yz}{4\pi|\underset{\sim}{x}|(x^2+y^2)}, \qquad a_2(\underset{\sim}{x}) = \frac{xz}{4\pi|\underset{\sim}{x}|(x^2+y^2)}, \qquad a_3(\underset{\sim}{x}) = 0,$$

and the transforms are

$$\tilde{a}_1(\underset{\sim}{k}) = -\frac{1}{4\pi}\int d^3x\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} \frac{yz}{|\underset{\sim}{x}|(x^2+y^2)} = \frac{1}{4\pi}\frac{\partial^2 I(\underset{\sim}{k})}{\partial k_z\,\partial k_y},$$

$$\tilde{a}_2(\underset{\sim}{k}) = -\frac{1}{4\pi}\frac{\partial^2 I(\underset{\sim}{k})}{\partial k_z\,\partial k_x}, \qquad \tilde{a}_3 = 0,$$

where

$$I(\underset{\sim}{k}) \equiv \int d^3x\, \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}}{|\underset{\sim}{x}|(x^2+y^2)}.$$

The rotational invariance of the integrand about the $\hat{z}$ axis allows us to use coordinates in which $k_x' = 0$ and $k_y' = (k_x^2 + k_y^2)^{1/2} \equiv k'$:

$$I(\underset{\sim}{k}) = \int dz\, e^{ik_z z}\int dy\, e^{ik'y}\int \frac{dx}{(x^2+y^2)(x^2+y^2+z^2)^{1/2}},$$

$$\frac{\partial^2 I(\underset{\sim}{k})}{\partial k_z\,\partial k_x} = -\frac{2k_x}{k'}\int_{-\infty}^{\infty} dz\, e^{izk_z}\int_{-\infty}^{\infty} dy\, e^{iyk'} \arctan\left(\frac{z}{y}\right)$$

$$= \frac{8k_x}{k'}\int_0^{\infty} dz\, \sin(zk_z)\int_0^{\infty} dy\, \sin(yk')\arctan\left(\frac{z}{y}\right).$$

Bateman[9], p. 87, #8 does the y integral:

$$\frac{\partial^2 I(\underset{\sim}{k})}{\partial k_z\, \partial k_x} = \frac{4\pi k_x}{(k')^2} \int_0^\infty dz\, \sin(zk_z)\left(1 - e^{-zk'}\right) .$$

Again we have to interpret the integrals as distributions,

$$\int_0^\infty dz\, \sin(zk_z) = \frac{P}{k_z} ,$$

and with Bateman, p. 72, #1, we get

$$\frac{\partial^2 I(\underset{\sim}{k})}{\partial k_z\, \partial k_x} = \frac{4\pi\, k_x}{(k')^2} \left(\frac{P}{k_z} - \frac{k_z}{\underset{\sim}{k}^2}\right) = \frac{4\pi\, k_x}{k_z\, \underset{\sim}{k}^2} .$$

The result $\tilde{a}_1(\underset{\sim}{k}) = \dfrac{k_y}{k_z\, \underset{\sim}{k}^2}$, $\tilde{a}_2(\underset{\sim}{k}) = - \dfrac{k_x}{k_z\, \underset{\sim}{k}^2}$, $\tilde{a}_3(\underset{\sim}{k}) = 0$

can be reexpressed in three-vector notation,

$$\tilde{a}_i(\underset{\sim}{k}) = \frac{\epsilon_{ijk}\, k^j\, \hat{n}^k}{(\hat{n}\cdot\underset{\sim}{k})\, \underset{\sim}{k}^2} . \tag{3.12}$$

Inserting this into Eq. (3.11), we get the matrix element

$$\frac{i\, e\, g}{(2\pi)^2}\, \delta^4(p' + q' - p - q)\, \frac{\epsilon_{ijk}\, k^j\, \hat{n}^k}{(\hat{n}\cdot\underset{\sim}{k})\, \underset{\sim}{k}^2} (-\bar{w}\, \gamma_i\, w\, \bar{u}\, \gamma_0\, u + \bar{w}\, \gamma_0\, w\, \bar{u}\, \gamma_i\, u) \tag{3.13}$$

with $k = p - p'$ = momentum transfer. To pass over to four-vector notation we use the fact that $k^\lambda - \eta^\lambda(\eta\cdot k)$ has no time component, and write Eq. (3.13) as

$$\frac{i\,e\,g}{(2\pi)^2}\,\delta^4(p' + q' - p - q)\;\frac{\epsilon_{\mu\nu\lambda\kappa}[k^\lambda - \eta^\lambda(\eta\cdot k)]n^\kappa}{(n\cdot k)[k^2 - (\eta\cdot k)^2]}\;\bar{u}\,\gamma^\mu\,u\,\bar{w}\,\gamma^\nu\,w\,.$$

Finally we leave off all the factors not belonging to the propagator and identify

$$D^{AB\ \text{instant}}_{\mu\nu}(k) = \frac{\epsilon_{\mu\nu\lambda\kappa}\,n^\lambda\,[k^\kappa - \eta^\kappa(\eta\cdot k)]}{(n\cdot k)[k^2 - (\eta\cdot k)^2]}\,. \tag{3.14}$$

Note that it does not depend on k_0, reflecting the fact that the interaction is instantaneous.

C. The Complete Propagator

The complete propagator is the sum of the transverse Eq. (3.10) and the instantaneous Eqs. (3.14) terms (see Fig. 1c):

$$D^{AB}_{+\mu\nu}(k) = D^{AB\ T}_{+\mu\nu}(k) + D^{AB\ \text{instant}}_{\mu\nu}(k)\,. \tag{3.15}$$

At first this sum looks rather bulky,

$$D^{AB}_{+\mu\nu}(k) = \frac{i}{k^2 + i\epsilon}\,\frac{k_0}{n\cdot k}\left\{\epsilon_{\mu\nu\lambda\kappa} - \epsilon_{\sigma\nu\lambda\kappa}\,\frac{k^\sigma\,\eta_\mu\,k_0}{\underset{\sim}{k}^2} + \epsilon_{\sigma\mu\lambda\kappa}\,\frac{k^\sigma\,\eta_\nu\,k_0}{\underset{\sim}{k}^2}\right\} n^\lambda\eta^\kappa$$

$$- \frac{1}{n\cdot k}\,\frac{1}{\underset{\sim}{k}^2}\left\{\epsilon_{\mu\nu\lambda\kappa}\,n^\lambda\,k^\kappa - \epsilon_{\mu\nu\lambda\kappa}\,n^\lambda\,\eta^\kappa\,k_0\right\}.$$

but it boils down to something simple when we try various values of μ and ν:

$$D^{AB}_{+ij} = \frac{1}{k^2 + i\epsilon}\,\frac{k_0}{n\cdot k}\,\epsilon_{ij\lambda 0}\,n^\lambda$$

is equal to $(k^2 + i\epsilon)^{-1} [\epsilon_{\mu\nu\lambda\kappa} n^\lambda k^\kappa / n \cdot k]$ for $\mu, \nu = 1, 2, 3$ and $D^{AB}_{+0j} = (k^2 + i\epsilon)^{-1} [\epsilon_{0j\lambda\kappa} n^\lambda k^\kappa / n \cdot k]$ also agrees with that form for $\mu = 0$, $\nu = 1, 2, 3$. Since $D^{AB}_{+\mu\nu}$ is antisymmetric in μ and ν, we can conclude

$$D^{AB}_{+\mu\nu}(k) = \frac{1}{k^2 + i\epsilon} \frac{\epsilon_{\mu\nu\lambda\kappa} n^\lambda k^\kappa}{n \cdot k} . \tag{3.16}$$

Actually we have proved cancellation of the noncovariant parts of the propagator only to first order in eg. The extension to all orders follows the method of ordinary quantum electrodynamics.[6]

The propagator (3.16) looks very reasonable. The $(k^2 + i\epsilon)^{-1}$ factor represents the propagation of a massless photon, while the polarization term $\epsilon_{\mu\nu\lambda\kappa} n^\lambda k^\kappa / n \cdot k$ is the only second-rank tensor built out of the available four-vectors, that is n and k, which is of order $(k)^0$ and $(n)^0$ and contains a $\epsilon_{\mu\nu\lambda\kappa}$. It should contain the latter in order for the theory to reproduce the nonrelativistic cross-product of three-vectors implied by the Lorentz force law $\underset{\sim}{F} = \underset{\sim}{v} \times \underset{\sim}{H}$.

Weinberg[2] calculated this propagator, using the most general representations of the Lorentz group, and found

$$D^{AB\,T}_{+\mu\nu}\Big|_{\text{Weinberg}} = \frac{1}{k^2 + i\epsilon} \frac{k \cdot \eta}{k^2 - (k \cdot \eta)^2} \epsilon_{\mu\nu\lambda\kappa} \eta^\lambda k^\kappa ,$$

where $\eta = (1, \underset{\sim}{0})$ in the frame of quantization. This is hopelessly noncovariant, i.e., there is no way of turning this into a covariant answer by using current conservation or by adding some instantaneous interaction. This is because Weinberg insists on manifest Lorentz invariance of the theory. In other words, a field theory of monopoles cannot possess manifest Lorentz invariance, a conclusion which has also been reached by other methods.[3] By contrast, with a singularity line we do find the appropriate cancellations and obtain a covariant answer when $eg/4\pi$ = integer.

With our propagator we can point out a loophole in another argument against the existence of monopoles, namely Zwanziger's claim[4] that a magnetic charge would produce pathological singularities in the S matrix. When Zwanziger solves for the general form of a charge-photon-vertex he finds (his Eq. 3) essentially the result implied by our propagator (3.16) if we identify the arbitrary four-vector a_μ which he had to introduce to solve his equations with Schwinger's singularity line n_μ. With a very special choice of a Lorentz frame and of a vector a_μ, Zwanziger obtains a vertex which appears to be both independent of a_μ and Lorentz-invariant, but which has unacceptable singularities. The point is he implicitly assumed manifest Lorentz invariance for the vertex which really depends on a_μ. This dependence on the singularity line has to be kept in all vertices until the end of the whole calculation; it will disappear only in the exact result, that is, after summing over all possible diagrams.

Incidentally we might wonder if our method reproduces the usual photon propagator $(k^2 + i\epsilon)^{-1} g_{\mu\nu}$ when we consider photon exchange between electric charges alone. It is a straightforward matter to verify that this is indeed the case.

IV. AVERAGING OVER SINGULARITY LINES

Let us assume that we can calculate an observable quantity F via a Feynman-Dyson perturbation expansion, using the propagator (3.16). The complete sum

$$F = \sum_{l,m=0}^{\infty} e^l g^m F^{(l,m)}(n) \qquad (4.1)$$

is covariant, while each perturbation term $F^{(l,m)}(n)$ depends on the singularity line. Since a meaningful approximation must not vary with n, we might try to remove the n-dependence of the individual perturbation terms by averaging over all directions of n. If we can interchange the operations $\sum_{l,m=0}^{\infty}$ and $\int d^4n\, \delta(n^2+1)$, then we obtain a manifestly covariant perturbation expansion

$$F = \langle F \rangle = \sum_{l,m=0}^{\infty} e^l g^m \langle F^{(l,m)} \rangle \; . \qquad (4.2)$$

The final answer must not depend on the Lorentz frame in which the quantization has been carried out, i.e., the frame where n is pure spacelike. Therefore we must allow n to have a time component, and the averaging has to cover all n subject to $n^2 = -1$. The noncompactness of the Lorentz group does not present an obstance if we restrict the integral $\int d^4n\, \delta(n^2+1)$ to values of $|\underset{\sim}{n}| < N$ and let $N \to \infty$ at the end. A suitable formula is

$$\langle f \rangle = \lim_{N \to \infty} \frac{1}{2\pi N^2} \; P\!\!\!\int (d^3n/n_0)\, \theta(N - |\underset{\sim}{n}|)\, f(n) \Big|_{n_0=(\underset{\sim}{n}^2-1)^{1/2}} \; . \qquad (4.3)$$

The principal value prescription applies to the poles at $n \cdot k = 0$ in the propagators.

V. THE CLASSICAL LIMIT

Let us apply our propagator to the elastic scattering to order eg of electric and magnetic charges (Fig. 1c) and take the limit of small velocities and momentum transfers. In this limit Goldhaber's calculation[5] should be exact, and if perturbation theory is meaningful, its lowest-order term should reproduce the classical result.

Since Goldhaber considered spin zero particles, we shall do the same, assuming that the propagator (3.16) applies in this case as well. The S matrix for $\langle f| = \langle p',q'|$ and $|i\rangle = |p,q\rangle$ to lowest order is

$$S_{fi}^{(eg)} = \frac{i\,e\,g}{(2\pi)^2}\;\delta^4(p' + q' - p - q)\,\frac{1}{4p_0q_0}\,(p+p')^\mu\,\frac{\epsilon_{\mu\nu\lambda\kappa}\,n^\lambda\,k^\kappa}{n\cdot k}\,(q + q')^\nu, \tag{5.1}$$

with momentum transfer $k = p - p'$. In the center-of-mass frame it becomes

$$S_{fi}^{(eg)} = \frac{i\,e\,g}{(2\pi)^2}\;\delta^4(P_f - P_i)\,\frac{(p_0 + q_0)}{p_0\,q_0}\;\frac{\underset{\sim}{n}\cdot(\underset{\sim}{k}\times\underset{\sim}{p})}{\underset{\sim}{k}^2(\underset{\sim}{n}\cdot\underset{\sim}{k})}\;.$$

To simplify things take $\underset{\sim}{p}^2 \to 0$ and the monopole mass $m_g \to \infty$

$$S_{fi}^{(eg)} \to \frac{i\,e\,g}{(2\pi)^2}\;\delta^4(P_f - P_i)\,\frac{1}{p_0}\;\frac{\underset{\sim}{n}\cdot(\underset{\sim}{k}\times\underset{\sim}{p})}{k^2(\underset{\sim}{n}\cdot\underset{\sim}{k})}\;.$$

Finally choose a frame where the charge is incident along the $\hat{z}$ axis $\underset{\sim}{p} = |\underset{\sim}{p}|\hat{z}$, and where the scattering takes place in the yz plane $\underset{\sim}{k} = |\underset{\sim}{p}|\theta\,\hat{y}$ with $\theta \ll 1$. Then $\underset{\sim}{p} \times \underset{\sim}{k} = |\underset{\sim}{p}|^2\,\theta\,\hat{x}$ and

$$\frac{\underset{\sim}{n}\cdot(\underset{\sim}{k} \times \underset{\sim}{p})}{\underset{\sim}{k}^2(\underset{\sim}{n}\cdot\underset{\sim}{k})} \rightarrow -\frac{1}{|\underset{\sim}{p}|\theta^2}\frac{n_x}{n_y} .$$

Since $p_0 = \frac{|p|}{v}$ with velocity v, we obtain in this limit

$$S_{fi}^{(eg)} = \left(\frac{-ieg\ v}{(2\pi)^2}\right)\left(\frac{1}{|\underset{\sim}{p}|^2\ \theta^2}\right)\left(\frac{n_x}{n_y}\right)\delta^4(P_f - P_i) . \qquad (5.2)$$

On the other hand, the S matrix for Coulomb scattering of electric charges under the same conditions is

$$S_{fi}^{(ee)} = \frac{-i\ e^2}{(2\pi)^2}\ \delta^4(P_f - P_i)\ \frac{1}{4p_0\ q_0}\ \frac{(p + p')^\nu\ (q + q')_\nu}{\underset{\sim}{k}^2}$$

$$\longrightarrow \left(\frac{-i\ e^2}{(2\pi)^2}\right)\left(\frac{1}{|\underset{\sim}{p}|^2\ \theta^2}\right)\delta^4(P_f - P_i) . \qquad (5.3)$$

If we forget about the factor (n_x/n_y), then the eg cross section is equal to the Coulomb cross section for two electric charges e and $e' = g\ v$. Thus far our result agrees with Goldhaber's:

$$S_{fi}^{(eg)}\Big|_{\text{Goldhaber}} = \left(\frac{-i\ e\ g\ v}{(2\pi)^2}\right)\left(\frac{1}{|\underset{\sim}{p}|^2\ \theta^2}\right)\ \delta^4(P_f - P_i)\exp[2i\ \frac{eg}{4\pi}\ \phi\ ] . \qquad (5.4)$$

The only discrepancy lies in the factor (n_x/n_y) and the phase factor $\exp[2i(eg/4\pi)\phi]$.

Now let us test whether our averaging prescription (4.2) can be applied to the S matrix. When we average the Born amplitude (5.2) we replace n_x/n_y by $\langle n_x/n_y\rangle$, which is easily found to be

$$\frac{1}{2\pi} P\int_0^{2\pi} d\phi \tan\phi = 0 ,$$

obviously not the required answer. If, on the other hand we average the cross section corresponding to (5.2) we find

$$(n_x/n_y)^2 \to \langle (n_x/n_y)^2 \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\phi \tan^2\phi = \infty ,$$

a divergent result, alas! There does not seem to be any good reason for setting $n_x/n_y \to 1$ as the classical answer would require.

VI. CONCLUSIONS

We should not be too surprised that our averaging prescription fails to yield the correct nonrelativistic limit. First of all, an expansion in terms of e and g may be doomed, since these parameters are not independent and since g is excessively large. Or the interchange of summation and integration in Eq. (4.2) may be unjustified.

The change from one singularity line n to another one n' is a gauge transformation,[1] which multiplies the fields by a phase factor of the form $\psi_n \to \psi_{n'} = \exp[ieg\alpha(n',n)]\psi_n$ thus scrambling all orders in eg. This is reflected in the n-dependence of the propagator (3.16). The Born term, for instance, can take any value between $-\infty$ and $+\infty$, while the higher-order terms adjust themselves in such a way as to render the exact cross section independent of n. All perturbation terms may be equally important, and there is hardly any hope that a finite subset will give a good approximation (an infinite subset like the ladder series in the Bethe-Salpeter equation might conceivably come close enough).

Furthermore, if we average the S matrix we have to assume that all azimuthal directions are equivalent. Let us think about this in the context of the phase factor $e^{i\Delta\phi}$ of helicity flip amplitudes, $\Delta = \lambda_i - \lambda_f$ being the difference between initial and final helicities. As nonrelativistic calculations[5,10] have shown, the S matrix for elastic charge monopole scattering has an azimuthal dependence $\propto \exp[2i(eg/4\pi)\phi]$ corresponding to a helicity reversal from $\lambda_i = +(eg/4\pi)$ to $\lambda_f = -(eg/4\pi)$ (due to the angular momentum

$$\underset{\sim}{S} = \frac{eg}{4\pi} \frac{\underset{\sim}{x}_g - \underset{\sim}{x}_e}{|\underset{\sim}{x}_g - \underset{\sim}{x}_e|}$$

of the electromagnetic field). Obviously one must not average the helicity flip phase factors, for otherwise there would be no helicity flips in this world:

$$\int_0^{2\pi} d\phi \; e^{i\Delta\phi} = 0.$$

Therefore one should rather try to average the cross sections. But, as we have seen in Section V, nonintegrable double poles spoil this approach, unless cancellations occur between different orders.

ACKNOWLEDGMENTS

I am very grateful to Professor Alfred S. Goldhaber for his constant help and encouragement. I should also like to thank Professor Martin B. Halpern and Dr. Joseph V. Lepore for some useful discussions.

FOOTNOTES AND REFERENCES

* Research supported by the U. S. Atomic Energy Commission.

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FIGURE LEGENDS

Fig. 1. (a) The exchange of transverse photons, $D_{+\mu\nu}^{AB\ T}(k)$.

(b) The instantaneous interaction, $D_{\mu\nu}^{AB\ \mathrm{instant}}(k)$.

(c) The complete photon propagator between electric and magnetic charges, $D_{+\mu\nu}^{AB}(k)$.

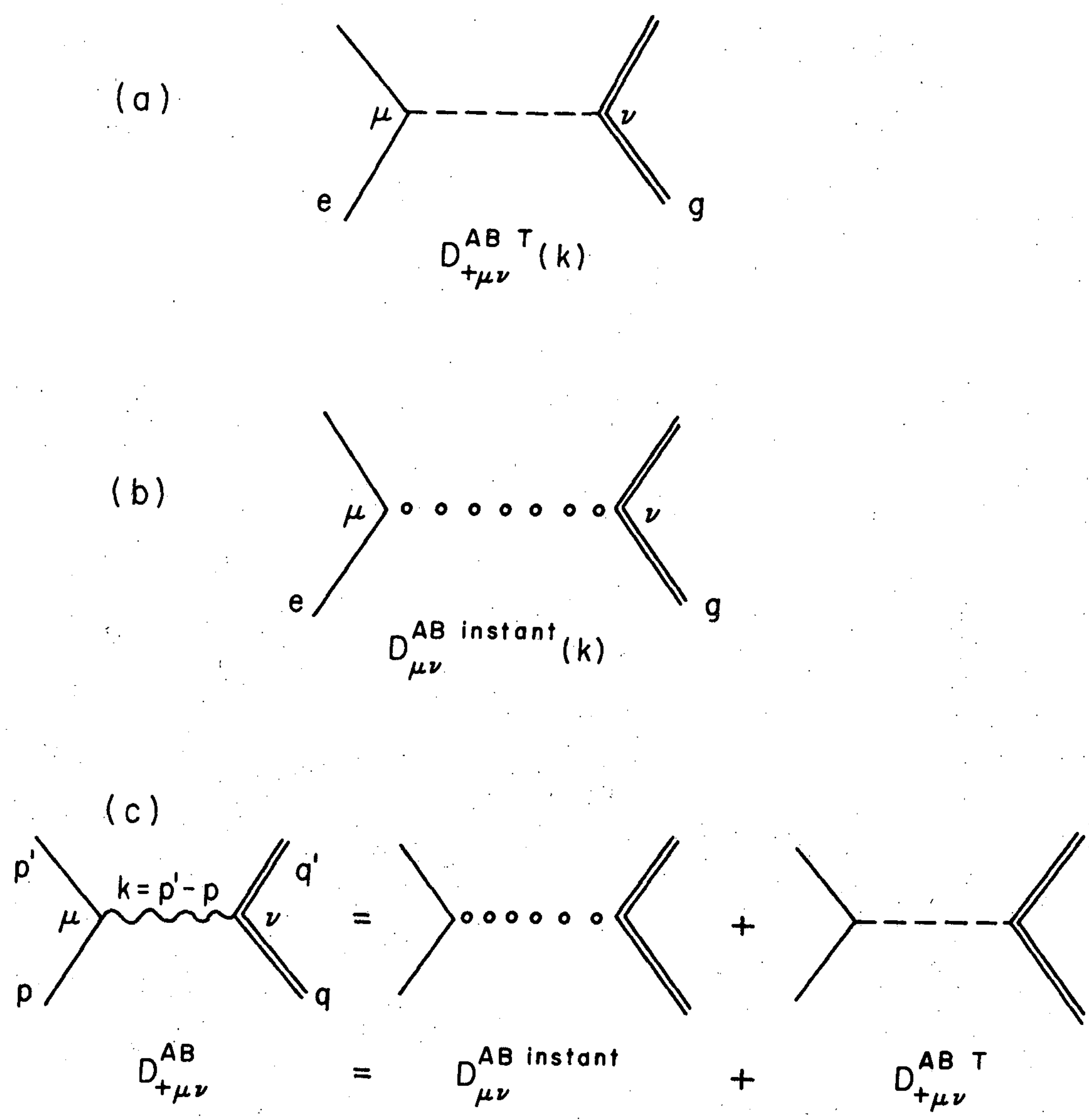

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Fig. 1.