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ABSTRACT

Results on the spin and decay parameters of the Ξ^- hyperon are presented. These have been obtained from the analysis of events of the type $K^- + p \rightarrow K^+ + \Xi^-$ produced with K^- beams of momentum 1.2 to 2.7 BeV/c incident on the 72-inch bubble chamber. Two methods of treating the data are presented, one involving the projection of coefficients in decay distributions and the other involving maximum-likelihood analysis of these distributions. The projection or moment method yields the better estimate of Ξ^- spin, with a result favoring spin 1/2 and excluding spin 3/2 to approximately 2.5 standard deviations. The maximum-likelihood method gives good estimates of decay parameters, with α_Ξ (the usual asymmetry parameter) = -0.30 ± 0.08 and with the ratio β_Ξ/γ_Ξ = 0.07 ± 0.19. The formalism of Byers and Fenster is utilized.

I. INTRODUCTION

A collection of particles with spin J can be completely described in their rest frame by the real and imaginary parts of expectation values of spin operators, the number and dimensionality of which are determined by the spin of the particles. Knowledge of these expectation values is equivalent to knowledge of the probability amplitudes for occupation of the various J, M quantum-mechanical states permitted for the particles. If the particles are unstable, the character of their original state and the transition amplitudes for decay (into final states with differing orbital angular momenta) will completely determine the angular dependence of directions and polarizations of final-state particles.

It has been customary to fit observed angular distributions (i.e., to find coefficients) with cosine series or with series of Legendre polynomials; the latter have the advantage of being orthogonal functions so that the addition of higher-order polynomials does not change lower-order coefficients in fitting data. Dependence on azimuthal angle as well as polar angle is important for decays of particles with spin higher than 1/2; thus the (orthogonal) spherical harmonics $Y_{LM}(\theta, \phi)$ or the (orthogonal) symmetrical-top functions $\mathcal{D}^L_{MM'}(\phi, \theta, 0)$ are convenient for the fitting of angular distributions (intensities, not amplitudes) of decay. (The theoretical work which nicely presents these ideas and which will be discussed below is that of Byers and Fenster.[1])

Testing data for compatibility with various spin hypotheses may be possible through the determination of the highest-order functions needed to fit the decay distributions. The L of a Y_{LM} or $\mathcal{D}^L_{MM'}$ function equals the rank of the spin operator whose expectation value multiplies the function. This expectation value describes the original set of particles; hence, L must be $\leq 2J$ where J is the spin of the particles. In general, it is impossible to say whether there has been some fortuitous cancellation of high-L terms of decay distributions; thus the complexity of the distributions gives only a lower limit for J.

Consistency checks can be made in the fitting of spin hypotheses to data, as the parameters of the initial state and of the decay amplitudes are overdetermined by data of decay distributions. In the specific case of a fermion decaying into a spinless boson plus a spin - 1/2 particle, the direction of the latter and also the θ, ϕ dependence of its three polarization components yield several estimates of each initial-state parameter as well as an evaluation of each decay parameter. However, statistical errors may be such that reasonable values for all parameters can be found for either of two spin hypotheses if terms of all orders are compared simultaneously. A conclusive determination of spin is most likely if only the well-defined low-order terms of the various distributions are compared. (At least one nonzero term will be found in each of the four decay distributions if the initial fermion sample is highly polarized and the decay parameters are nonzero.) Since the polarization components of the spin - 1/2 particle <u>transverse</u> to its flight direction contain in their leading terms the same initial-state parameter that appears in leading terms of the other distributions, and in addition contain a <u>J-dependent factor</u>, the spin J can be extracted by comparison of only these terms. The best answer for spin J will be found from a sample with the highest initial polarization of the parent fermion (given some particular statistical error).

To treat the Ξ^- data reported in this article, we made separate analyses of samples of events grouped according to momentum and production angle. None of the samples showed any large, statistically significant high-order terms characteristic of $J = 3/2$ or $5/2$. The J-dependent factor was evaluated for each sample.

The estimation of the goodness of fit of spin hypotheses to all the data required either a combination of the events at various momenta or a combination of the results for the J factor. As interpretation of the latter appeared difficult, considerable effort was expended to obtain an answer from the former. It was evident that low-polarization samples gave weak discrimination between spin hypotheses. The most conclusive answer was obtained from a combination (sample D) of positive- and negative-polarization samples, with the omission of those samples that had very low polarization and with rotation of coordinates in negative-polarization samples (to change the sign of P_Ξ).[2] This combination of events gave discrimination against the spin-3/2 hypothesis, with a confidence level of about 0.01; it produced excellent agreement with spin 1/2. The decay parameters from these data were reasonable and in good agreement with values from a maximum-likelihood analysis.

Alternative treatments of data are also presented. These include the analysis of all 749 Ξ^- events (with rotation of the negative-polarization samples) (sample C) and the separate analyses of positive-polarization and negative-polarization samples (involving neither high-polarization selection nor rotation) (samples A and B, respectively). Analysis of sample C gives a result which is less conclusive than that for sample D, but which still excludes spin 3/2. Analyses of samples A and B, without rotation, also favor spin 1/2, though not so strongly.*

The maximum-likelihood analysis of the same sets of data also favored spin 1/2, although the results were not so conclusive as those obtained by direct calculation of the $2J+1$ factor. For the high-polarization combined sample D, spin 3/2 was discriminated against with a confidence level of about 0.024. On the other hand, the maximum-likelihood treatment in general yielded smaller errors in the estimation of the Ξ decay parameters. Best values of the decay parameters obtained from analysis of sample D were $\alpha_\Xi = -0.30 \pm 0.08$, $\beta_\Xi = 0.07 \pm 0.18$, and $\gamma_\Xi = 0.95 \pm 0.02$. (These pertain to spin 1/2. The errors in β and γ are correlated.)

Many sets of fake events were generated with decay distributions corresponding to specified spin, decay parameters, and initial-state polarization parameters. The following facts were thus verified: (i) both methods of analysis correctly estimate decay parameters and polarization parameters[3]; (ii) the calculated values of any parameter have a Gaussian distribution about the expected value, with the variance of this distribution equal to the average of the squared error calculated for a single "fake" experiment.

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* A non-optimum combination (with respect to polarization) of all data gave a spin-3/2 confidence level of 0.15 or 0.015 for α_Ξ values of -0.48 or -0.34, respectively.

II. THEORY

One formalism convenient for determination of the Ξ^- properties is the treatment of Byers and Fenster, utilizing irreducible tensors as spin operators. The initial collection of Ξ^- hyperons is described by a density matrix of the form

$$\rho = (2J+1)^{-1} \sum_{L=0}^{2J} \sum_{M=-L}^{+L} (2L+1) \langle T_{LM} \rangle^* T_{LM} \tag{1}$$

for any (half-integral) value of spin J.[1] This density matrix is equivalent to $\rho = \sum_n w_n \chi_n \chi_n^+$ in terms of a spinor χ_n and a weight w_n for the state n. The number of independent $\langle T_{LM} \rangle$'s needed to describe the Ξ^- is $(1/2)(2J+1)^2 - 1$. The T_{LM} are spin-space operators that may be constructed from S_x, S_y, and S_z spin operators in a manner similar to that in which the spherical harmonics Y_{LM} are constructed from the coordinates x, y, and z.[4] With the $\langle T_{LM} \rangle$ parameters abbreviated to t_{LM}, the decay distributions describing the process $\Xi^- \to \Lambda + \pi^-$ are (see Appendix A and Fig. 1):

$$\begin{aligned}
I(\theta,\phi) &= \left(\sum_{L_e=0}^{2J-1} + \alpha_\Xi \sum_{L_o=1}^{2J} \right) \sum_M n^J_{L0}\, t_{LM}\, Y^*_{LM}(\theta,\phi) \\
I\overline{P}_\Lambda \cdot \hat{\Lambda} &= \left(\alpha_\Xi \sum_{L_e=0}^{2J-1} + \sum_{L_o=1}^{2J} \right) \sum_M n^J_{L0}\, t_{LM}\, Y^*_{LM}(\theta,\phi) \\
I\overline{P}_\Lambda \cdot \hat{x} &= \mathrm{Re}\, [(i\beta_\Xi - \gamma_\Xi)\, F(\theta,\phi)] \\
I\overline{P}_\Lambda \cdot \hat{y} &= \mathrm{Im}\, [(i\beta_\Xi - \gamma_\Xi)\, F(\theta,\phi)]
\end{aligned} \tag{2}$$

with

$$\begin{aligned}
F(\theta,\phi) &= \sum_{L_o=1}^{2J} \sum_M n^J_{L1}\, t_{LM}\, \mathscr{D}^L_{M1}(\phi,\theta,0)\, [(2L+1)/4\pi]^{1/2} \\
&= (2J+1) \sum_{L_o=1}^{2J} \sum_M n^J_{L0}\, t_{LM}\, \mathscr{D}^L_{M1}(\phi,\theta,0)\, [(2L+1)/4\pi]^{1/2} [L(L+1)]^{-1/2}
\end{aligned}$$

(L_e takes on only even values of L, and L_o takes on only odd-L values.) The decay parameters are given by $\alpha = 2\ \mathrm{Re}\ a^*b/N$, $\beta = 2\ \mathrm{Im}\ a^*b/N$, and $\gamma = (|a|^2 - |b|^2)/N$, with a defined as the $J - 1/2$ decay amplitude, with b defined as the $J + 1/2$ decay amplitude, and with N equal to $|a|^2 + |b|^2$. The term $I(\theta,\phi)$ represents the angular distribution of the Λ in the Ξ^- rest frame; each of the IP distributions represents the product of $I(\theta,\phi)$ and a component of Λ polarization. The first polarization component is along the Λ line of flight ($\hat{\Lambda}$); the last two are those in the directions perpendicular to $\hat{\Lambda}$ ($\hat{x} \propto \hat{\Lambda} \times (\hat{\Lambda} \times \hat{n})$ and $\hat{y} \propto \hat{n} \times \hat{\Lambda}$, where $\hat{n}$ represents the normal to the production

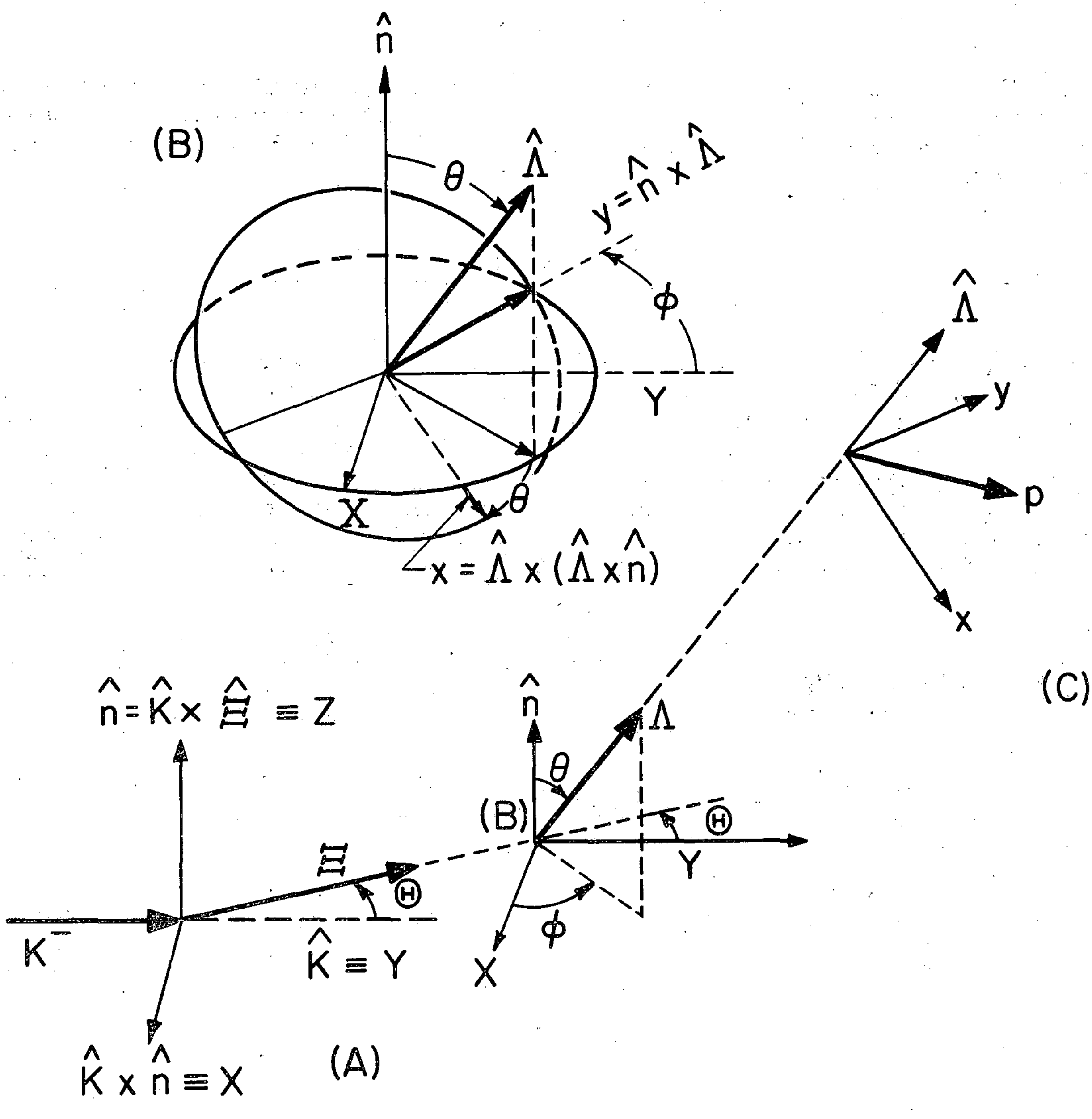

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Fig. 1. A diagram of Ξ production and decay is presented with (A) representing the c.m. production system, and defining X, Y, and Z axes; with (B) showing the Λ direction in the Ξ rest frame; and with (C) presenting the proton from Λ decay in the Λ rest frame. The x and y axes used for the analysis of polarization are shown in the blow-up of system (B).

plane). The sums are taken over L values from 0 to 2J. Only even-M values need be used if the θ, ϕ coordinate system has its polar axis along the production normal (as a consequence of parity conservation in production). The $Y_{LM}(\theta, \phi)$ are the usual spherical harmonics and the $\mathscr{D}^{L}_{M1}(\phi, \theta, 0)$ are the "symmetrical-top" functions (defined, for example, in Jacob and Wick[5]).

The n^J_{L0} and n^J_{L1} quantities appearing in the above expressions contain Clebsch-Gordan coefficients, since these must modify the single spherical harmonics [of various rank (L)] that result from the combination of two decay amplitudes. As Byers and Fenster have shown, the exact expressions are the following:

$$
\begin{aligned}
n^J_{L0} &= (-)^{J-1/2} \left[(2J+1)/4\pi\right]^{1/2} C(JJL;\ 1/2,\ -1/2) \\
n^J_{L1} &= (-)^{J-1/2} \left[(2J+1)/4\pi\right]^{1/2} C(JJL;\ 1/2,\ 1/2)
\end{aligned}
\tag{3}
$$

[where $C(j_1\, j_2\, j;\, m_1\, m_2)$ is the usual Clebsch-Gordan coefficient]. For any J and a particular L, the second quantity is proportional to the first times the factor (2J + 1). In particular,[6]

$$
n^J_{L1} = (2J+1)\left[L(L+1)\right]^{-1/2} n^J_{L0} . \tag{4}
$$

As the n^J_{L1} quantities are contained in the transverse-polarization terms and the n^J_{L0} quantities in the angular-distribution and longitudinal-polarization terms, comparison of coefficients containing a certain t_{LM} may be made to evaluate the expression 2J + 1 and hence the spin J of the Ξ.

III. EXPERIMENTAL ANALYSIS

Analysis was made of 749 events of the type

$$
K^- + p \rightarrow \Xi^- + K^+;\ \Xi^- \rightarrow \Lambda + \pi^- \tag{5}
$$

obtained at K^- momenta from 1.2 to 1.7 BeV/c (and also of higher-momentum data discussed below). Almost half of the events were those produced by 1.5 BeV/c K^-; the remainder were data samples spaced at 100-MeV/c intervals. The events treated did not include those in which either the Ξ^- or the Λ had a length of < 0.5 cm in the laboratory system; appropriate corrections were made for this cutoff. With the production normal chosen as the Z axis and the incident K^- direction chosen as the Y axis, each event was analyzed to determine the θ, ϕ angles of the Λ (in the Ξ^- rest frame) and the direction $\hat{p}$ of the decay proton (in the Λ rest frame). See Fig. 1.

A. Moment Method

The orthogonality properties of the $Y_{LM}(\theta, \phi)$ and the $\mathscr{D}^{L}_{M1}(\phi, \theta, 0)$,

$$
\begin{aligned}
&\int Y_{LM}(\theta,\phi)\, Y^*_{L'M'}(\theta,\phi)\, d\Omega = \delta_{LL'}\, \delta_{MM'} \\
&\int \mathscr{D}^{L}_{M1}(\phi,\theta,0)\, \mathscr{D}^{L'*}_{M'1}(\phi,\theta,0)\, d\Omega = \left[4\pi/(2L+1)\right] \delta_{LL'}\, \delta_{MM'} ,
\end{aligned}
\tag{6}
$$

permit the finding of moments, i.e., the projection of coefficients $n_{L0}^{J}\ t_{LM}$ or $n_{L1}^{J}\ t_{LM}$ out of each of the experimental distributions corresponding to the theoretical expressions of Eq. (2). Thus, with the index k running over all events and with L given an even or an odd value,

$$(1 \text{ or } \alpha_{\Xi})\, n_{L0}^{J}\ {\mathrm{Re} \atop \mathrm{Im}}\, t_{LM} = \sum_{k=1}^{N} {\mathrm{Re} \atop \mathrm{Im}}\, Y_{LM}(\theta_k, \phi_k)/N \tag{7a}$$

$$(\alpha_{\Xi} \text{ or } 1)\, n_{L0}^{J}\ {\mathrm{Re} \atop \mathrm{Im}}\, t_{LM} = \sum_{k=1}^{N} {\mathrm{Re} \atop \mathrm{Im}}\, Y_{LM}(\theta_k, \phi_k)\, \hat{p}_k \cdot \hat{\Lambda}_k\, (3/\alpha_{\Lambda} N) \tag{7b}$$

$$(2J+1)(i\beta - \gamma)\, n_{L0}^{J}\ t_{LM} = [(2L+1)L(L+1)/4\pi]^{1/2} [\sum_k \mathrm{Re} + i \sum_k \mathrm{Im}] \cdot [\mathcal{D}_{M1}^{L*}(\phi_k, \theta_k, 0)(\hat{p}_k \cdot \hat{x}_k + i\hat{p}_k \cdot \hat{y}_k)]\, (3/\alpha_{\Lambda} N) \tag{7c}$$

Equation (7c) holds only for odd L values. (Values of γ Re t_{LM}, β Re t_{LM}, γ Im t_{LM}, and β Im t_{LM} can be extracted from the last relation and a similar one for $t_{L,-M}$ with the use of the relation $t_{L,-M} = (-)^M t_{LM}^{*}$.) With division by n_{L0}^{J} or by $(2J+1)\, n_{L0}^{J}$, there result from the above equations four evaluations of each odd-L t_{LM} (times 1, α, β, or γ) and two evaluations of each even-L t_{LM} (times 1 or α). The spin hypothesis that would generally be considered most probable is that which leads to closest agreement among the various determinations of each t_{LM} for some choice of the α, β, and γ decay parameters (satisfying the constraint $\alpha^2 + \beta^2 + \gamma^2 = 1$).

An error matrix $G_{LM,\, L'M'}$ for the t_{LM} parameters was constructed from the data, the diagonal terms being the average of $[(\mathrm{Re}\ Y_{LM})^2 - \langle \mathrm{Re}\ Y_{LM} \rangle^2]$ or $[(\mathrm{Im}\ Y_{LM})^2 - \langle \mathrm{Im}\ Y_{LM} \rangle^2]$. With the use of this error matrix, a $\chi_J^{\,2}$ was formed to test moments higher than those appropriate for spin J for consistency with zero:

$$\chi_J^{\,2} = \sum_{L>2J}^{5} \sum_{M} t_{LM}\, G_{LM,L'M'}^{-1}\, t_{L'M'}\ . \tag{8}$$

The spin - 1/2 requirement that the t_{LM}'s of order greater than t_{10} be zero was very well satisfied; the combining of $\chi_{1/2}^2$ results from 10 sets of data (high-polarization samples totaling 440 events) from 1.2 through 1.7 BeV/c yielded a total $\chi_{1/2}^2$ value of 365 when 400 was expected. (The $\chi_{3/2}^2$ was of course also acceptable.)

Three separate χ^2's were formed to test the equality of t_{LM} and $(\alpha\ t_{LM})/\alpha'$, of t_{LM} and $(\beta\ t_{LM})/\beta'$, and of t_{LM} and $(\gamma\ t_{LM})/\gamma'$ (where the prime designates an assumed value of a parameter); these yielded estimates of α, β, and γ and also indicated the degree of consistency among the several moments containing each t_{LM}.[7] Unfortunately, neither the spin χ^2 nor the decay-parameter χ^2's afforded any discrimination between spin 1/2, 3/2, or 5/2.

It was found convenient to separate the data into various subsamples according to incident momentum and production angle. Parameter values (t_{LM}'s, α, β, and γ) are presented for a typical sample in Table I. The size of each subsample was greater than 20 events, as samples smaller than this had been observed to give abnormal results in strong-decay analyses of spin-3/2 resonances. For all analyses, the value of α_Λ was taken to be 0.62.[12]

B. Selection of Samples

The moment analysis of subsamples yielded t_{10} (or polarization P_Ξ) values which varied rapidly with the incident K^- momentum and with the Ξ production angle; further, the estimates for other polarization parameters, those appropriate for $J>1/2$, were not significantly different from zero. As shown by the character of the relations in Eq. (7), discrimination between different J_Ξ hypotheses thus depended on a comparison of the evaluations of $(2J+1)t_{10}$ in the Λ transverse polarization and t_{10} in other distributions; hence, a fairly large and well-determined t_{10} was desirable.

A list of the data subsamples and the values of t_{10} found by averaging t_{10}, $\alpha t_{10}/\alpha_W$ and $\gamma t_{10}/\gamma_W$ from moment analysis are presented in Table II. (The α_W and γ_W represent the best estimates to date on the Ξ decay parameters, -0.48 ± 0.08 and 0.85 ± 0.04. The former is independent of spin, whereas the latter changes only slightly with spin assumption.[8]) As the relative values of the t_{10} estimates vary a little with the spin assumption [because of the $(2J+1)$ factor in the γt_{10} moment], these estimates were calculated for both spin = 1/2 and 3/2 hypotheses. However, classification of subsamples by relative t_{10} values was found invariant under change of spin. (Maximum-likelihood analysis yields a single evaluation of t_{10} by handling all distributions simultaneously; but likelihood analysis of small data samples was found difficult in some cases.)

In order to enhance the polarization or t_{10} value for large numbers of events, it was necessary to treat separately a combination of subsamples with positive t_{10} estimates (called sample A below) and one with negative t_{10} estimates (sample B). In addition all data were analyzed together (sample C) after rotation of 180° about the incident K direction (changing the production-system coordinate axes from $\hat{Z} = \hat{n}$ and $\hat{X} = \widehat{K \times n}$ to $\hat{Z} = -\hat{n}$ and $\hat{X} = -\widehat{K \times n}$) for the negative-polarization subsamples; this rotation changed the sign of t_{10} in all distributions and left unchanged α, β, and γ. Finally the subsamples with large $|t_{10}|$ estimates ($|t_{10}| > 0.30$ or $|P_\Xi| > 0.52$) were combined, with rotation of all negative-t_{10} subsamples (sample D). This combination contained 440 events and yielded a high t_{10} ($t_{10} = 0.48$ or $P_\Xi = 0.83$)[9]; thus it provided good discrimination between spin hypotheses, as is shown below. The test to be described, involving the t_{10} moments, did not make use of any constraint on the magnitude of t_{10}; and thus the somewhat artificial reinforcement of the t_{10} values of subsamples by rotation and high-polarization selection should not cause a bias (in any obvious way) for the moment test. The bias problem is treated further in Section III.C. (For maximum-likelihood analysis, the use of constraints and the bias introduced by selection are important questions and are discussed in detail in Sec. III.F.)

Table I. Parameter values from moment analysis of a typical data sample (59 events at 1.3 BeV/c, $\hat{\Xi}\cdot\hat{K} > 0$).

J	L	M	t_{LM}	αt_{LM}	βt_{LM}	γt_{LM}	Decay parameters
1/2	0	0	1.00	-0.45 ± 0.36			$\alpha = -0.50 \pm 0.22$
	1	0	-0.55 ± 0.31	0.28 ± 0.12	0.26 ± 0.24	-0.38 ± 0.27	$\beta = -0.48 \pm 0.50$
							$\gamma = 0.73^{+0.27}_{-0.50}$
3/2	0	0	1.00	-0.45 ± 0.36			
	1	0	-1.23 ± 0.68	0.63 ± 0.26	0.29 ± 0.27	-0.43 ± 0.30	
	2	0	0.11 ± 0.12	-0.14 ± 0.33			$\alpha = -0.76 \pm 0.20$
(Re)[a]	2	2	-0.12 ± 0.09	0.41 ± 0.24			$\beta = -0.54 \pm 0.40$
(Im)	2	2	-0.04 ± 0.10	-0.44 ± 0.28			$\gamma = 0.37^{+0.63}_{-0.20}$
	3	0	0.05 ± 0.26	-0.03 ± 0.09	-0.06 ± 0.22	-0.42 ± 0.22	
(Re)[a]	3	2	0.05 ± 0.18	0.10 ± 0.07	-0.11 ± 0.17	0.03 ± 0.15	
(Im)	3	2	-0.01 ± 0.20	0.06 ± 0.07	0.30 ± 0.16	0.11 ± 0.17	

a. Only positive-M t_{LM} values are presented, as negative-M t_{LM} values are related by $t_{L,-M} = t^*_{LM}(-)^M$.

Table II. Data samples with t_{10} and $|2J+1|$ [Eq. (9)] evaluations from moments.[a]

Sample No.	Momentum (BeV/c)	$\hat{\Xi}\cdot\hat{K}$	No. events	$t_{10}(\equiv P_{\Xi}/\sqrt{3})$	"High" $\lvert P_{\Xi}\rvert$	$\lvert 2J+1\rvert$
1	1.2	> 0	22	-0.93± 0.30 (-0.60)[b]	x	4.7± 2.8
2	1.3	> 0	59	-0.53± 0.18 (-0.40)	x	1.8± 1.2
3	1.2, 1.3	< 0	31	0.09± 0.25 (0.15)		68± 1200
4	1.4	> 0	47	-0.39± 0.17 (-0.45)	x	2.6± 2.4
5		< 0	21	0.63± 0.29 (0.25)	x	3.1± 2.5
6	1.5	0.9-1.0	64	0.03± 0.17 (-0.10)		4.0± 4.2
7		0.8-0.9	47	-0.16± 0.22 (-0.15)		4.8± 8.5
8		0.6-0.8	69	-0.28± 0.17 (-0.25)		6.6± 9.3
9		0.4-0.6	41	-0.57± 0.21 (-0.35)	x	3.8± 3.0
10		0.0-0.4	64	-0.12± 0.18 (-0.15)		1.3± 2.2
11		-0.4-0.0	62	0.29± 0.17 (0.30)	x	2.3± 2.9
12		-1.0- -0.4	72	0.46± 0.14 (0.45)	x	12± 14
13	1.6	> 0	33	-0.06± 0.24 (-0.05)		2.5± 3.9
14		< 0	21	0.43± 0.31 (0.40)	x	1.0± 1.2
15	1.7	> 0	75	0.18± 0.16 (0.30)	x[c]	0.2± 1.1
16		< 0	21	0.60± 0.32 (0.40)	x	2.4± 1.9

Combined samples:		With α_{Ξ} = -0.48		With α_{Ξ} = -0.34	
		t_{10}	$\lvert 2J+1\rvert$	t_{10}	$\lvert 2J+1\rvert$
A (Nos. 3, 5, 6, 7, 11-16)[d]	438			0.43± 0.08	1.83± 0.70
B (Nos. 1, 2, 4, 8, 9, 10)	311			-0.25± 0.07	2.14± 1.01
C [A + B (rotated)]	749	0.30± 0.06	2.42± 0.79	0.32± 0.05	1.87± 0.59
D [Nos. 5, 11, 12, 14, 15, 16 + (1, 2, 4, 9) rotated]	440	0.48± 0.07	2.53± 0.72	0.50± 0.07	1.96± 0.52

[a] The t_{10} values are those calculated for spin 1/2; these change relatively for $J = 3/2$ because the higher spin assumption depresses γt_{10} relatively to t_{10} and αt_{10} for each set of data. However, the changes are small enough that the sample classifications as to sign and magnitude of polarization remain the same.

The errors stated for $|2J+1|$ are not standard-deviation errors.

[b] The numbers in parentheses are t_{10} estimates obtained from likelihood calculations which fitted production and decay of the Ξ; these calculations, done by J. Peter Berge, evidently give good agreement with the averages from moments. Spin 1/2 was assumed for the likelihood analysis.

[c] The likelihood calculation indicates that the t_{10} estimate from moments is low for sample 15; this sample was included in the combined, high-polarization sample D.

[d] Samples included are predominantly those with positive polarization; however, two samples with negative, but low, polarizations were made part of sample A because of earlier selection on the basis of slightly different criteria.

C. Spin Analysis with Moments

The separate projections of coefficients from the various distributions of Eq. (2) permitted evaluation of the following:

$$(2J+1)^2 = \frac{(A\beta t_{LM})^2 + (A\gamma t_{LM})^2}{(1-\alpha^2)\, t_{LM}^2}, \qquad (9)$$

where $[L(L+1)]^{1/2}\, n^J_{L1}/n^J_{L0}$ has been represented by A.[10] This relation holds for any odd-L and M combination. As the moments above $L=1$ were not significantly different from zero, no evaluation of Eq. (9) except that with $L=1$ and $M=0$ gave definitive values of $|2J+1|$. The world-average and spin-independent (from Λ helicity) value of $\alpha_{\Xi} = -0.48 \pm 0.08 \equiv \alpha_W$ was used in the denominator of Eq. (9).[8] The t_{10} of the denominator was a weighted average of t_{10} from the longitudinal polarization and $(\alpha t_{10})/\alpha_W$ from the angular distribution of the Λ. The last column of Table II(a) presents the evaluations of $|2J+1|$ made with the t_{10} moments from the various samples of data. [Errors do <u>not</u> represent standard deviations, as the experimental ratio of Eq. (9) is not a normally distributed quantity.] Figure 2(a) is a plot of the same results, $|2J+1|$ being the radius of each point. The contributions of the βt and the γt moments to each $|2J+1|$ evaluation are represented by the projections onto the two axes. The points are expected to cluster near the positive γt axis. The best-defined answers correspond very roughly to the highest markers, the height of each marker being inversely proportional to the fractional $|2J+1|$ error of Table II(a). It is apparent that the data favor $|2J+1| = 2$ (spin 1/2) rather than $|2J+1| = 4$. (The tendency for $|2J+1|$ estimates to fall slightly below the true value is explained below.) Figure 2(b) presents a histogram of $|2J+1|$ for the Ξ^- data subsamples and also a $|2J+1|$ histogram for a control experiment. (19 samples are shown in (a) and (b)).

Figure 3(a) and (b) present estimates of $2J+1$ from the same data resulting from the omission of the βt_{10} term in the numerator of Eq. (9). As β_{Ξ} is estimated to be close to zero (see Appendix B), the $2J+1$ evaluations are no more than 6% lower than they should be. The dropping of the β term permits a straightforward calculation of the $2J+1$ probability distribution and shows that some of the larger $|2J+1|$ values are <u>negative</u> fluctuations (which may more likely fit the $J=1/2$ than the $J=3/2$ hypothesis). Figure 3(a) and (b) shows the probability distributions P(A), the quantity A being that defined for Eq. (9) and experimentally having an average of $2J+1$. The distribution curves were calculated for Fig. 3(a) [Fig. 3(b)] under the assumption of spin 1/2 (spin 3/2); actual experimental errors (<u>including correlations</u>) and estimates of t_{10} (i.e., polarization) from representative data samples were used as parameters in calculating the distributions. (See Appendix C.) Correction for the neglect of the β term was found to have a scarcely discernible effect on the distribution curves.

The composite, high-polarization sample of 440 Ξ's (sample D) gave the following moments and errors, as determined by projection (with J taken as 1/2):

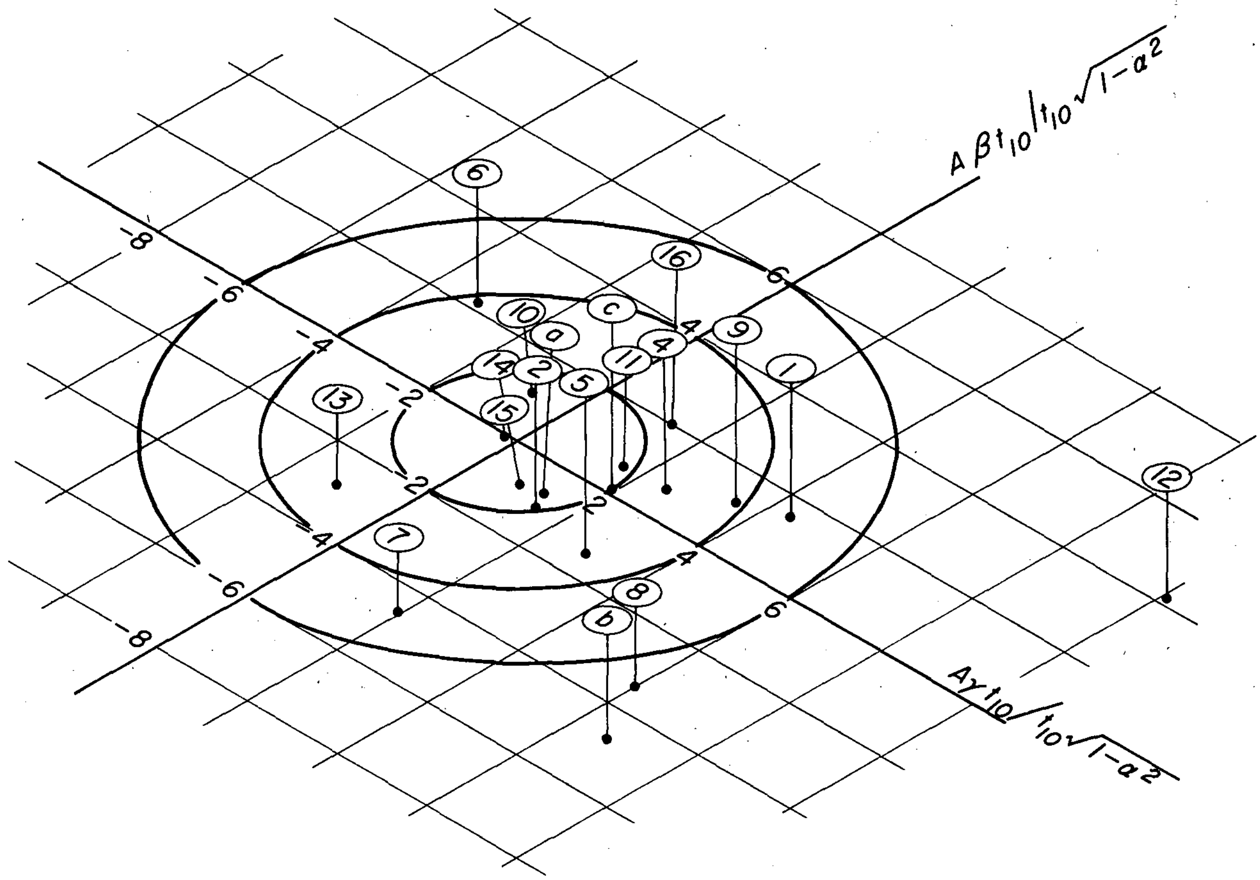

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Fig. 2(a). The quantity $|2J+1|$ is the radial distance to each of the experimental points designated by the "pins" on this plot. See Eq. (9) for identification of the components along the abscissa and the ordinate. (Error correlations and the non-Gaussian nature of the $2J+1$ evaluation tend to pull the estimates inside the expected radius.) The height of each pin was made inversely proportional to the fractional error of $|2J+1|$ given in Table II(a) or II(b).

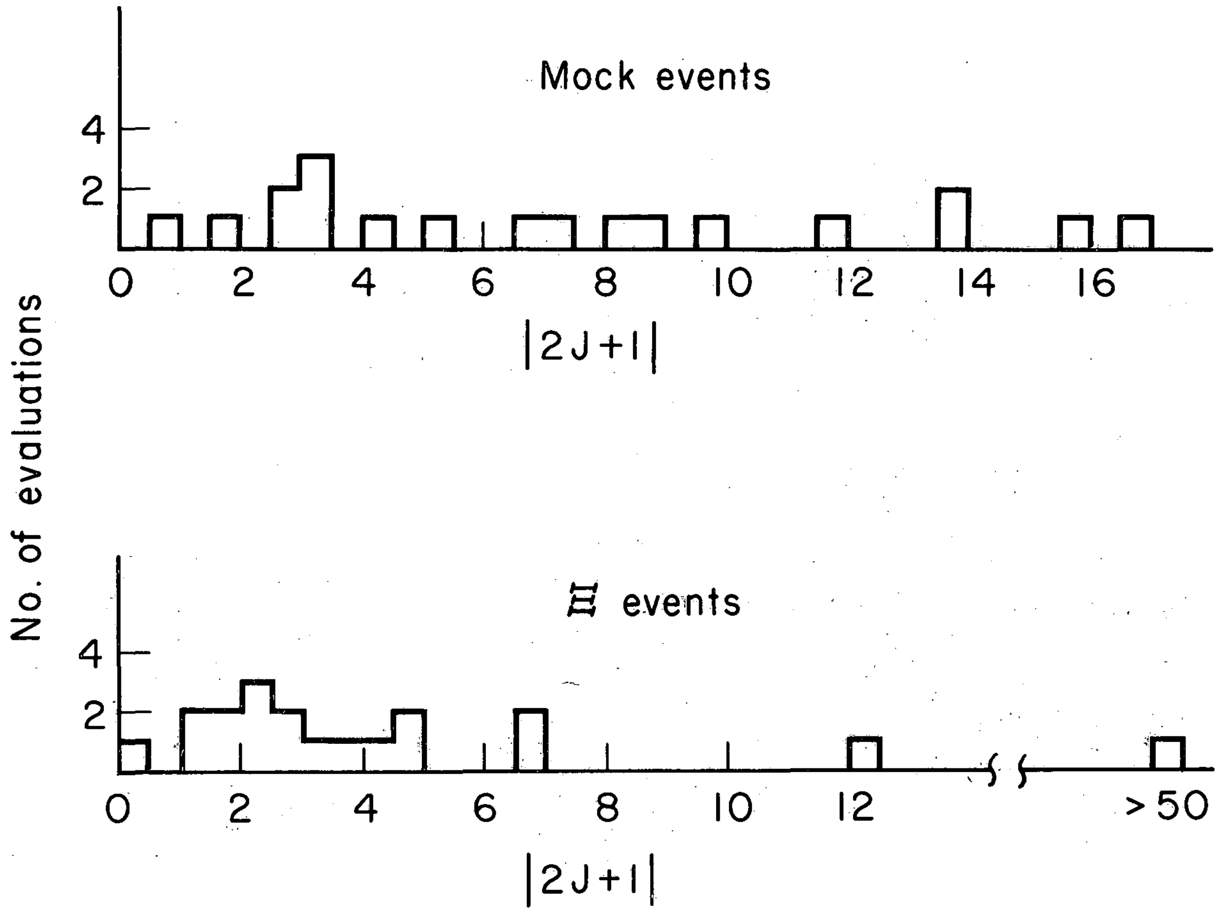

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Fig. 2(b). The lower histogram presents the distribution in $|2J+1|$ of all data subsamples; the values correspond to the radial distances of the points in Fig. 2(a).

The upper histogram presents the distribution in $|2J+1|$ for samples of randomly generated "mock events" which had polarization ≈ 0.50 (and $J = 3/2$). These samples represent a control experiment; they indicate only slight preference for $|2J+1| \approx 4$ because the smallness of the samples (each of which contained ≈ 40 events) caused $|2J+1|$ fractional errors to be fairly large.

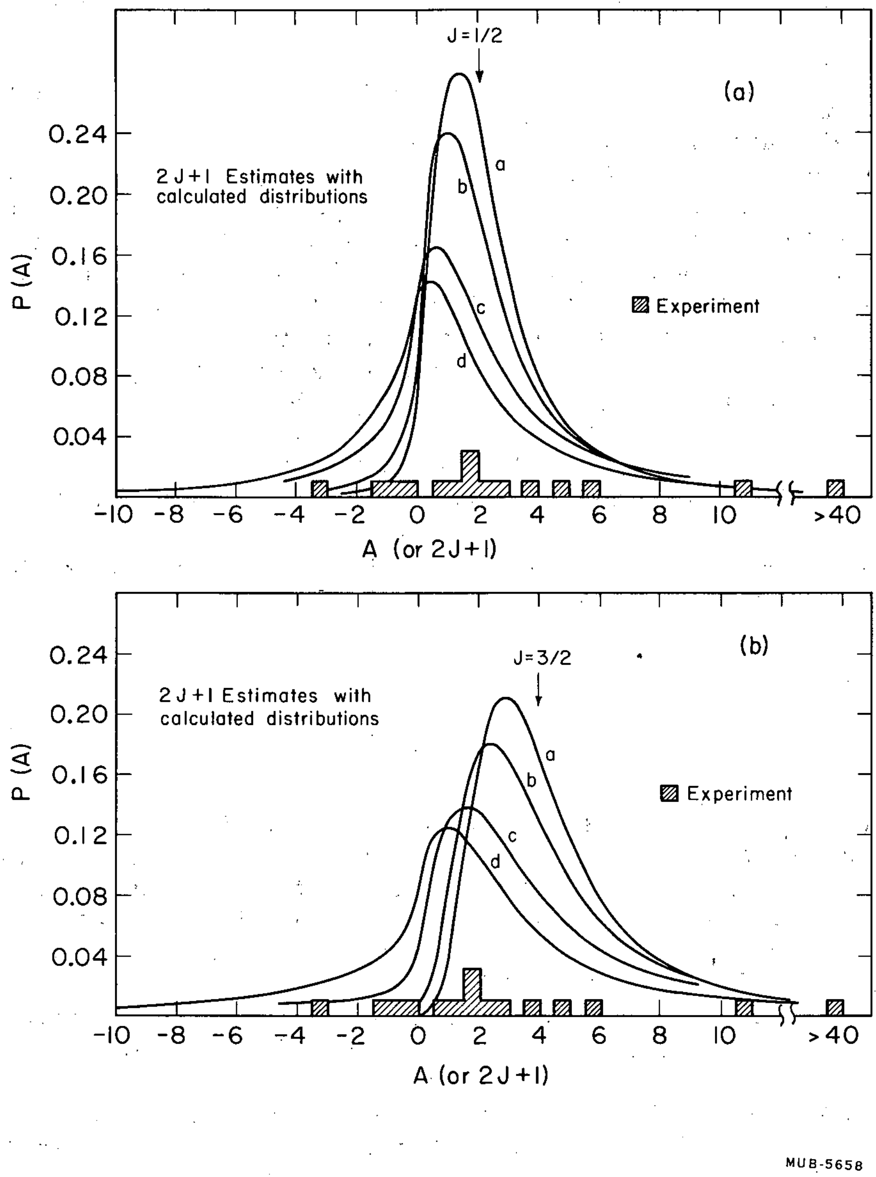

Fig. 3. Evaluations of $2J+1$ with β set zero are shown for 16 low-momentum data subsamples. Representative curves P(A) for four data samples of varying polarizations are shown to indicate the expected distribution of A(or 2J+1) estimates for each of these samples; Fig. 3(a) gives such curves for the hypothesis $J = 1/2$, and Fig. 3(b) gives the curves for $J = 3/2$. (These were calculated by the method outlined in Appendix C.) The values of t_{10} or $P_{\Xi}/\sqrt{3}$ for curves (a) through (d) were 0.51, 0.40, 0.24, and 0.15, respectively.

$$
\begin{array}{llll}
t_{10} = \ \ 0.48 \pm 0.14 & \langle \delta t \ \ \delta\alpha t \rangle = -0.00019 & \langle \delta\alpha t \ \delta\gamma t \rangle & = 0.00013 \\
\alpha t_{10} = -0.19 \pm 0.05 & \langle \delta t \ \ \delta\beta t \rangle = -0.00033 & \langle \delta\beta t \ \delta\gamma t \rangle & = 0.00069 \\
\beta t_{10} = \ \ 0.10 \pm 0.09 & \langle \delta t \ \ \delta\gamma t \rangle = -0.00062 & & \\
\gamma t_{10} = \ \ 0.47 \pm 0.09 & \langle \delta\alpha t \ \delta\beta t \rangle = -0.00010 & (1-\alpha^2)^{1/2} \langle t_{10} \rangle & = 0.37 \pm 0.11
\end{array}
$$

The $2J+1$ estimate was found sensitive to α_{Ξ}. With $\alpha_{\Xi} = -0.48$, the evaluation of $2J+1$ from sample D (with the βt_{10} term neglected) yielded $2J+1 = 2.52$.[11] A probability distribution for $2J+1$ from this large sample was calculated for both spin hypotheses; the parameters were the $(1-\alpha^2)^{1/2}\ t_{10}$ and the errors given above. (See Fig. 4.) The area under the tail of the $J=3/2$ curve lying below the experimental value (determined after a 6% correction for the β term) was found to be 0.024 times the total area of the distribution curve. The data thus indicate rather a small probability for the hypothesis of spin 3/2 even though the quantity β has probably been overestimated. (See Appendix B.)

The α_{Ξ} value required by the t_{00} moment (Λ helicity) of our total data was -0.34. With this α_{Ξ}, sample D analysis yielded $2J+1=1.92$, or a spin-3/2 confidence level of 0.0005.*

The decay parameters found for the high-polarization (sample D) events by moment analysis were (assuming $J = 1/2$)

$$
\begin{aligned}
\alpha &= -0.30 \pm 0.09 \\
\beta &= \ \ 0.20 \pm 0.22 \ . \\
\gamma &= \ \ 0.93^{+0.07}_{-0.30}
\end{aligned}
$$

Finally, possible biases may have been introduced in the calculations for the 440-event high-polarization sample through the selection of the events on the basis of high polarization or through the rotation of the "negative-polarization" subsamples. (The former probably does not affect the $2J+1$ calculation, as the experimental errors of the numerator and denominator quantities are similar and selection of high-polarization samples does not seem to prefer "large-denominator" samples to "large-numerator" samples or vice versa. (This is a statement of the "average situation" for $J = 1/2$ and $J = 3/2$; both spin hypotheses led to the same sample classification.) The problem of bias through rotation does not seem serious because all rotated subsamples of sample D had the same (negative) sign for numerator and denominator quantities.

To investigate further the possibility of bias, we examine spin estimates from samples A, B, and C. Results were as follows (for $\alpha_{\Xi} = -0.34$):

No rotation or selection:
- $|2J+1| = 1.83\ (\pm 0.70)$ for 438 positive-polarization events (sample A)
- $|2J+1| = 2.14\ (\pm 1.01)$ for 311 negative-polarization events (sample B)

No polarization selection:
- $|2J+1| = 1.87\ (\pm 0.59)$ for 749 events with rotation of negative-polarization samples (sample C).

---

* With the βt_{10} term included, sample D analysis yielded $|2J+1| = 2.53$ or 1.96 for $\alpha_{\Xi} = -0.48$ or -0.34, respectively.

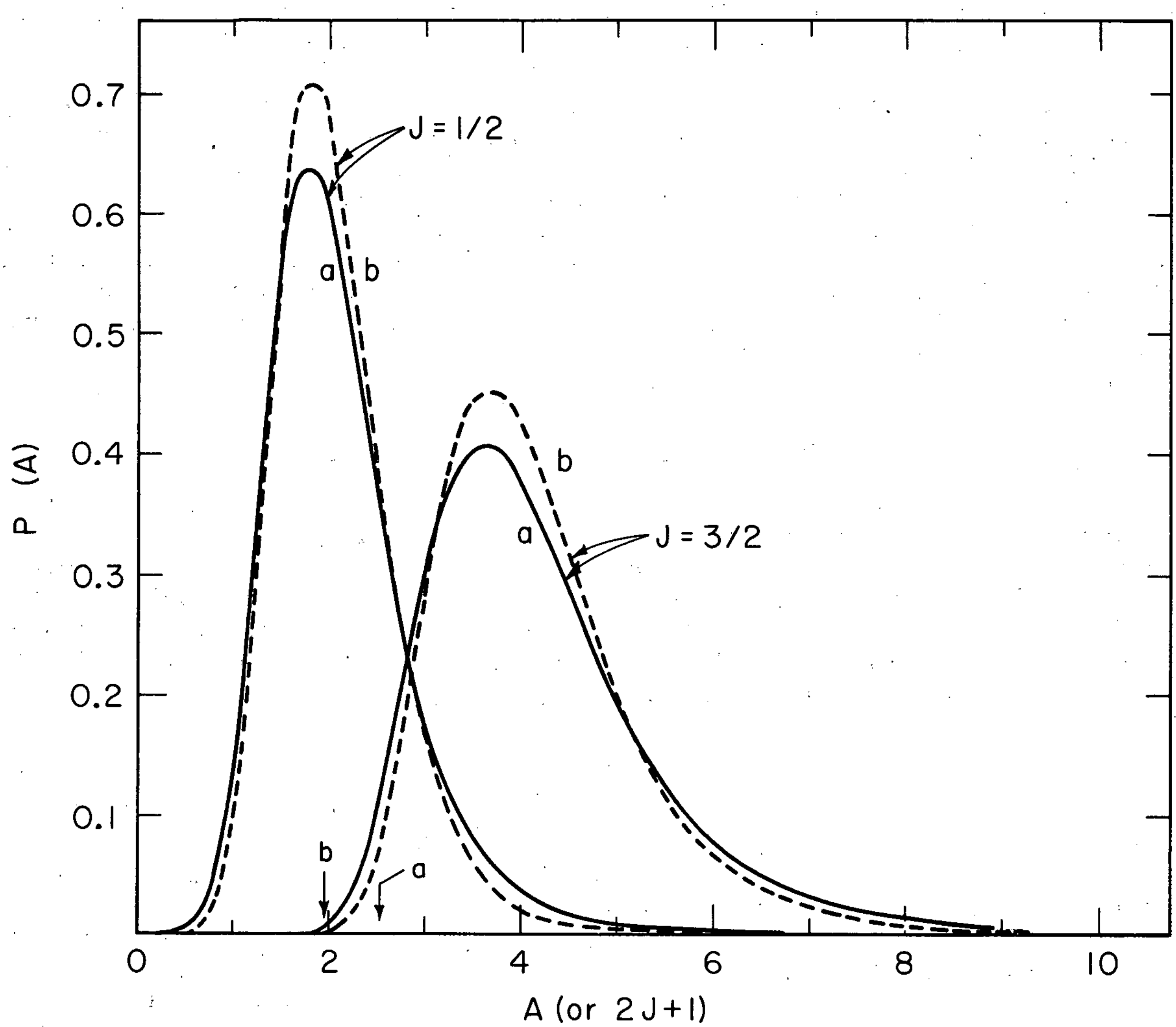

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Fig. 4. The P(A) probability distribution for the 2J+1 estimate from the high-polarization sample D is shown for the spin hypotheses 1/2 and 3/2. The actual estimate from the analysis of sample D is indicated by the arrow. The two sets of curves and the two arrows, labelled a and b, correspond to different assumed values of α_{Ξ}: a) $\alpha_{\Xi} = -0.48$ and b) $\alpha_{\Xi} = -0.34$.

These $|2J+1|$ results include the β term (and do not have Gaussian errors). The corresponding results for $2J+1$ with $\beta \equiv 0$ are 1.83, 1.59, and 1.72. The confidence level for the spin-3/2 hypothesis obtained from the sample C (749 events) analysis was 0.022 (the area under the probability curve below the $2J+1$ estimate) with $\alpha_\Xi = -0.48$; it was < 0.0005 with $\alpha_\Xi = -0.34$.

An alternative method of estimating $2J+1$ from the bulk of the Ξ data is to average the $2J+1$ estimations from all samples (all 749 events), with each weighted by the inverse of its error squared. (Such averaging requires that each error squared be equivalent to the variance of each distribution, an assumption that is only approximately valid.) The weighted average of the $|2J+1|$ estimates [the values of Table II(a), also presented as radii in Fig. 2] is 1.64(±0.53).* An average of the $2J+1$ estimates with βt set equal to zero (and hence with algebraic sign of $2J+1$ included) gives a still smaller answer. These averages are consistent with the single evaluation from the composite sample D given above.

D. Maximum-Likelihood Method

In general, the likelihood for a set of events having certain angular configurations is the product of the individual probabilities for the angular configuration of each event. The probability that the ith event will have a decay configuration given by a set of measured angles $\overline{x_i}$ is the distribution function $f_i = f(x_i, \alpha)$, where α represents a set of parameters that may be varied according to an assumed theoretical model. The likelihood L for N events having measured angles $x_i\,(i = 1, 2, \ldots N)$ is given by

$$L = \prod_{i=1}^{N} f_i . \qquad (10)$$

A maximum-likelihood analysis consists of varying the parameters α to achieve a maximum in L, at which point the final values α_0 of the parameters α constitute a description of the experimental data. In this experiment, the measured angles x describe $\hat{\Lambda}$, the momentum vector (normalized to unit length) of the Λ in the Ξ rest frame, and $\hat{p}$, the momentum vector of the decay proton in the Λ rest frame. The parameters α entering into the distribution function f_i are J, the assumed spin of the Ξ; α_Ξ, β_Ξ, and γ_Ξ, the assumed decay parameters of the Ξ; α_Λ, the decay asymmetry parameter of the Λ; and the initial-state polarization parameters t_{LM} of the Ξ.

The joint distribution function representing the probability that an event with $\hat{\Lambda}$ within a solid angle $d\Omega_\Lambda$ and $\hat{p}$ within a solid angle $d\Omega_p$ may be observed is

$$f(\hat{\Lambda}, \hat{p}) \propto I(\hat{\Lambda}) \left[1 + \alpha_\Lambda \vec{P}_\Lambda(\hat{\Lambda}) \cdot \hat{p}\right] = I(\hat{\Lambda}) + \alpha_\Lambda I\vec{P}_\Lambda(\hat{\Lambda}) \cdot \hat{p} . \qquad (11)$$

The term $I(\hat{\Lambda})$ is the Λ angular distribution $I(\theta,\phi)$ given in Eq. (2), and $I\vec{P}_\Lambda(\hat{\Lambda})$ is the Λ polarization distribution given by the three components $I\vec{P}_\Lambda \cdot \hat{\Lambda}$, $I\vec{P}_\Lambda \cdot \hat{x}$, and

* A weighted average of $|2J+1|$ with the use of fractional errors yields $|2J+1| = 3.6\,(\pm 0.90)$.

$\mathrm{I}\vec{P}_\Lambda \cdot \hat{y}$ in Eq. (2). The Λ decay distribution $(1 + \alpha_\Lambda \vec{P}_\Lambda(\hat{\Lambda}) \cdot \hat{p})$ follows from the fact that the Λ has spin 1/2. Throughout this analysis we have used $\alpha_\Lambda = +0.62$.[12]

Various simplifications may be made to ease the computation of $f(\hat{\Lambda}, \hat{p})$ for a given event. These reductions are outlined in Appendix D.

A maximum-likelihood search program was used to calculate all the maximum-likelihood results quoted in this paper.[13] A search for the maximum likelihood in the space of all variable parameters is done by successive iterations with Newton's method:

Let w be the logarithm of the likelihood function L, as a function of m variable parameters $a_i (i = 1, 2, \ldots m)$. A maximum in w (and hence a maximum in L itself) is achieved when $\partial w/\partial a_1 = \partial w/\partial a_2 = \cdots = \partial w/\partial a_m = 0$. If we are near the maximum in L, then the parameters a_i should be changed by

$$(\Delta a)_i = -\sum_j (A^{-1})_{ij} \frac{\partial w}{\partial a_j},$$

$$\text{where} \qquad A_{ij} = \frac{\partial^2 w}{\partial a_i\, \partial a_j} = \frac{\partial^2 w}{\partial a_j\, \partial a_i}, \tag{12}$$

to bring us closer to the point of maximum w, where $a_i = a_{oi}$ $(i = 1, 2, \ldots m)$.[14] Near the maximum, by a Taylor's expansion,

$$w \approx \text{const.} - \frac{1}{2} \sum_{i,j} (a_i - a_{oi})\, A_{ij} (a_j - a_{oj}) \tag{13}$$

if partial derivatives higher than those of second order are small. The errors δa_i on the parameters a_i are given by $(\delta a_i)^2 = G_{ii}$, where $G_{ij} = (A^{-1})_{ij}$.

We see that L has the form

$$L \propto e^{-1/2\chi^2} = \exp\left[-(1/2) \sum_{ij} (a_i - a_{oi})\, G_{ij}^{-1} (a_j - a_{oj})\right] \tag{14}$$

where χ^2 tests the consistency of the parameters a_i with the solution $a_i = a_{oi}$. If a large number of experiments are performed to determine the true parameters a'_{oi} describing a decaying state, we expect the individual determinations of a_{oi} to be normally distributed about a'_{oi}. The standard deviation of this Gaussian distribution should be approximately equal to the error calculated from a single experiment. (These facts have been borne out by an analysis of random events generated by a Monte Carlo program.[15])

It was desired to incorporate into the likelihood-function analysis the constraint $\alpha^2 + \beta^2 + \gamma^2 = 1$. Because β and γ (entering into the transverse polarization terms) were found to be strongly correlated with each other and only weakly correlated with the longitudinal-polarization parameter α, the independent parameters in the search were chosen to be α and Φ, where

$$\Phi \equiv \tan^{-1}(\beta/\gamma)$$
$$\beta = \sqrt{1-\alpha^2}\ \sin\Phi \quad . \qquad (15)$$
$$\gamma = \sqrt{1-\alpha^2}\ \cos\Phi$$

E. Decay Parameters from Likelihood Analysis

With α_Λ fixed at +0.62, and with spin 1/2 assumed for the Ξ, three independent parameters enter into the likelihood function, namely α, Φ, and t_{10}.

Six additional parameters enter into the likelihood function for spin 3/2, namely t_{20}, $^{\mathrm{Re}}_{\mathrm{Im}}\, t_{22}$, t_{30}, and $^{\mathrm{Re}}_{\mathrm{Im}}\, t_{32}$. If we wish, we may hold $^{\mathrm{Re}}_{\mathrm{Im}}\, t_{22}$ and $^{\mathrm{Re}}_{\mathrm{Im}}\, t_{32}$ equal to zero; this corresponds to averaging over ϕ_Λ, the azimuthal angle of the Λ.

Values of t_{LM}, α, and Φ are presented in Table III for various samples of data under the assumptions of spin 1/2 and 3/2. (Unlike the moment method discussed earlier, with the maximum-likelihood analysis, all information is used simultaneously so that one value for each t_{LM} parameter may be obtained.) In one case, (sample D) the parameters determined by the moment method are given for comparison.

As discussed earlier, sample C contains all events from samples A and B, but with those from sample B rotated 180° about the beam axis. Sample C has a large value of t_{10} (corresponding to 59% polarization for spin 1/2); this high polarization is essential for an accurate measurement of the decay parameters and of the spin J. In sample D, the average polarization has been raised to 90% (for spin 1/2) by omission of those samples having low $|t_{10}|$; this yields an even more accurate measurement of Φ, as shown in Table III.

With the assumption $\alpha_\Lambda = 0.62$ and spin 1/2 for the Ξ, the best values of the decay parameters from the maximum-likelihood analysis are (for sample D)

$$\alpha = -0.30 \pm 0.08$$

and $$\Phi = 0.07 \pm 0.19$$

(yielding $\beta = 0.07 \pm 0.18$, $\gamma = 0.95 \pm 0.02$). These evaluations are in good agreement with those given above for the moment analysis and have somewhat smaller errors.

For each sample in Table III, there exists a secondary local maximum of w, yielding a value of w much smaller than that at the primary maximum. For example, with the spin - 1/2 hypothesis for sample D (the highly polarized sample of 440 events), we find solutions at

(primary solution) $\alpha = -0.30 \pm 0.08$, $\Phi = 0.07 \pm 0.19$, $t_{10} = 0.52 \pm 0.06$, $w = 28.01$

and (secondary solution) $\alpha = +0.07 \pm 0.16$, $\Phi = -3.00 \pm 0.30$, $t_{10} = -0.20 \pm 0.10$, $w = 2.44$.

Table III. Values for t_{LM} and decay parameters for various data samples.[a]

Spin	Parameter	From likelihood: Sample A (438 events)	Sample B (311 events)	Sample C (749 events)	Sample D (440 events)	From moments[b], Sample D (440 events): t_{LM}	αt_{LM}	βt_{LM}	γt_{LM}
J = 1/2	t_{00}	1.00	1.00	1.00	1.00	1.00	-0.18± 0.14		
	t_{10}	+0.27± 0.07	-0.45± 0.08	+0.34± 0.05	+0.52± 0.06	0.48± 0.14	-0.19± 0.05	0.10± 0.09	0.47± 0.09
	α	-0.34± 0.11	-0.31± 0.10	-0.33± 0.08	-0.30± 0.08	α = -0.30± 0.09			
	Φ	+0.49± 0.38	+0.01± 0.28	+0.21± 0.23	+0.07± 0.19				
	β	+0.44± 0.32	+0.01± 0.27	+0.20± 0.21	+0.07± 0.18	β = 0.20± 0.22			
	γ	+0.83± 0.17	+0.95± 0.05	+0.92± 0.06	+0.95± 0.02	$\gamma = 0.93^{+0.07}_{-0.30}$			
J = 3/2	t_{00}	1.00	1.00	1.00	1.00	1.00	-0.18± 0.14		
	t_{10}	+0.38± 0.10	-0.56± 0.11	+0.46± 0.08	+0.71± 0.09	1.07± 0.31	-0.43± 0.11	0.11± 0.10	0.52± 0.10
	t_{20}	-0.01± 0.05	-0.01± 0.06	-0.01± 0.04	-0.03± 0.05	-0.04± 0.05	-0.01± 0.14		
	Re t_{22}	-0.04± 0.03	+0.05± 0.04	-0.01± 0.03	+0.02± 0.03	0.01± 0.03	0.05± 0.10		
	Im t_{22}	+0.01± 0.03	+0.02± 0.04	-0.00± 0.03	+0.01± 0.03	-0.01± 0.03	0.01± 0.10		
	t_{30}	-0.03± 0.06	-0.08± 0.07	+0.02± 0.05	+0.06± 0.06	-0.08± 0.11	-0.08± 0.04	-0.01± 0.08	0.08± 0.08
	Re t_{32}	-0.01± 0.04	+0.03± 0.05	-0.02± 0.03	-0.01± 0.04	-0.01± 0.07	0.04± 0.03	0.04± 0.06	0.05± 0.06
	Im t_{32}	-0.02± 0.04	-0.04± 0.05	-0.03± 0.03	-0.02± 0.04	0.08± 0.07	0.01± 0.03	-0.06± 0.06	-0.01± 0.06
	α	-0.36± 0.12	-0.37± 0.13	-0.39± 0.09	-0.37± 0.10	α = -0.38± 0.09			
	Φ	+0.56± 0.36	+0.05± 0.29	+0.24± 0.23	+0.04± 0.17				
	β	+0.49± 0.28	+0.05± 0.27	+0.22± 0.21	+0.04± 0.16	β = 0.09± 0.10			
	γ	+0.79± 0.18	+0.93± 0.05	+0.89± 0.07	+0.93± 0.04	$\gamma = 0.92^{+0.08}_{-0.35}$			

Sample A: Positive-polarization sample, selected to include chiefly subsamples having $t_{10} > 0$.

Sample B: Negative-polarization sample. $t_{10} < 0$.

Sample C: Sample A and sample B combined, with events of sample B rotated 180° about the beam axis. (Effect is to leave α, β, and γ unchanged and to change only the sign of t_{10}, Im t_{22}, t_{30}, and Re t_{32} of sample B.)

Sample D: High-polarization sample. Bins having $|t_{10}| \lesssim 0.30$ removed.

a. All low-momentum data, 1.2 – 1.7 BeV/c, with corrections for short Λ's and short Ξ's; $\alpha_\Lambda = 0.62$. $t_{00} = 1.00$ required by normalization.

b. A weighted average of t_{LM} estimates from all given moments should be compared with the single t_{LM} of the maximum-likelihood analysis.

(The likelihood function has been arbitrarily normalized to yield $w = 0$ for an isotropic distribution function, i. e. for $\alpha = 0$ and t_{LM} (other than t_{00}) = 0. Only <u>differences</u> in values of w are of significance.)

F. Spin Analysis by Likelihood Techniques

The most probable spin hypothesis can be determined by comparison of logarithms of the likelihood function for different spin assumptions. We see that the spin 1/2 and 3/2 likelihood functions differ in three respects: (i) the coefficients n_{L0}^J multiplying the t_{LM} contain J-dependent factors; (ii) the spin 3/2 function contains six additional parameters t_{20}, ${}^{Re}_{Im}\,t_{22}$, t_{30} and ${}^{Re}_{Im}\,t_{32}$; (iii) the coefficients n_{L1}^J multiplying the transverse polarization terms contain an extra factor of $(2J+1)$. The change in the J-dependent coefficients is of no consequence, since the t_{LM} always occur in the combination $n_{L0}^J\, t_{LM}$; that is, a change in the n_{L0}^J will not affect the value of the likelihood function at its maximum, since the t_{LM} will merely be readjusted to leave the products $n_{L0}^J\, t_{LM}$ unchanged.

In order to determine the most probable spin hypothesis, the factor $(2J+1)$ multiplying the transverse polarization terms in the distribution function was varied in finite steps. The logarithm of the likelihood function, after maximization at each value of $(2J+1)$, is plotted in Fig. 5 (for sample D) as a function of $(2J+1)$. The lowest curve is that for the spin-1/2 form of the distribution function, containing only α, Φ, and t_{10} as free parameters. Point A on this curve has the $(2J+1)$ factor appropriate for spin 1/2 and corresponds to the spin-1/2 solution for sample D in Table III.

The uppermost curve is that for the spin-3/2 form of the distribution function, containing nine free parameters, namely α, Φ, t_{10}, t_{20}, ${}^{Re}_{Im}\,t_{22}$, t_{30}, and ${}^{Re}_{Im}\,t_{32}$. Point B on this curve has the $(2J+1)$ factor appropriate for spin 3/2 and corresponds to the spin-3/2 solution for sample D in Table III.

The intermediate curve is obtained by setting ${}^{Re}_{Im}\,t_{22} = {}^{Re}_{Im}\,t_{32} = 0$ in the spin-3/2 form of the distribution function, thereby effectively ignoring the azimuthal distribution of the Λ. The resulting function has five free parameters, namely α, Φ, t_{10}, t_{20}, and t_{30}.

Table III indicates that the moments t_{20}, t_{30}, ${}^{Re}_{Im}\,t_{22}$, and ${}^{Re}_{Im}\,t_{32}$ for the Ξ may be zero, a necessary but not sufficient condition for proof that the spin of the Ξ is 1/2. (Similar results were obtained for smaller samples at all momenta and production angles.) If the spin is really 1/2, one expects that the χ^2 defined (with error correlations neglected) by

$$\chi_A^2 = \sum_{\substack{L,M \\ (L>1)}} \frac{(t_{LM} - 0)^2}{(\delta t_{LM})^2} \tag{16}$$

should be about 6.0 for an "average" experiment, where the sum includes t_{20}, ${}^{Re}_{Im}\,t_{22}$, t_{30}, and ${}^{Re}_{Im}\,t_{32}$. Noting that $L \propto \exp\left[-(1/2)\chi_A^2\right]$ (ignoring error correlations), one expects $w = \log L$ to increase by about 3.0, on the average, as one goes from 3 to 9 independent parameters.[16]

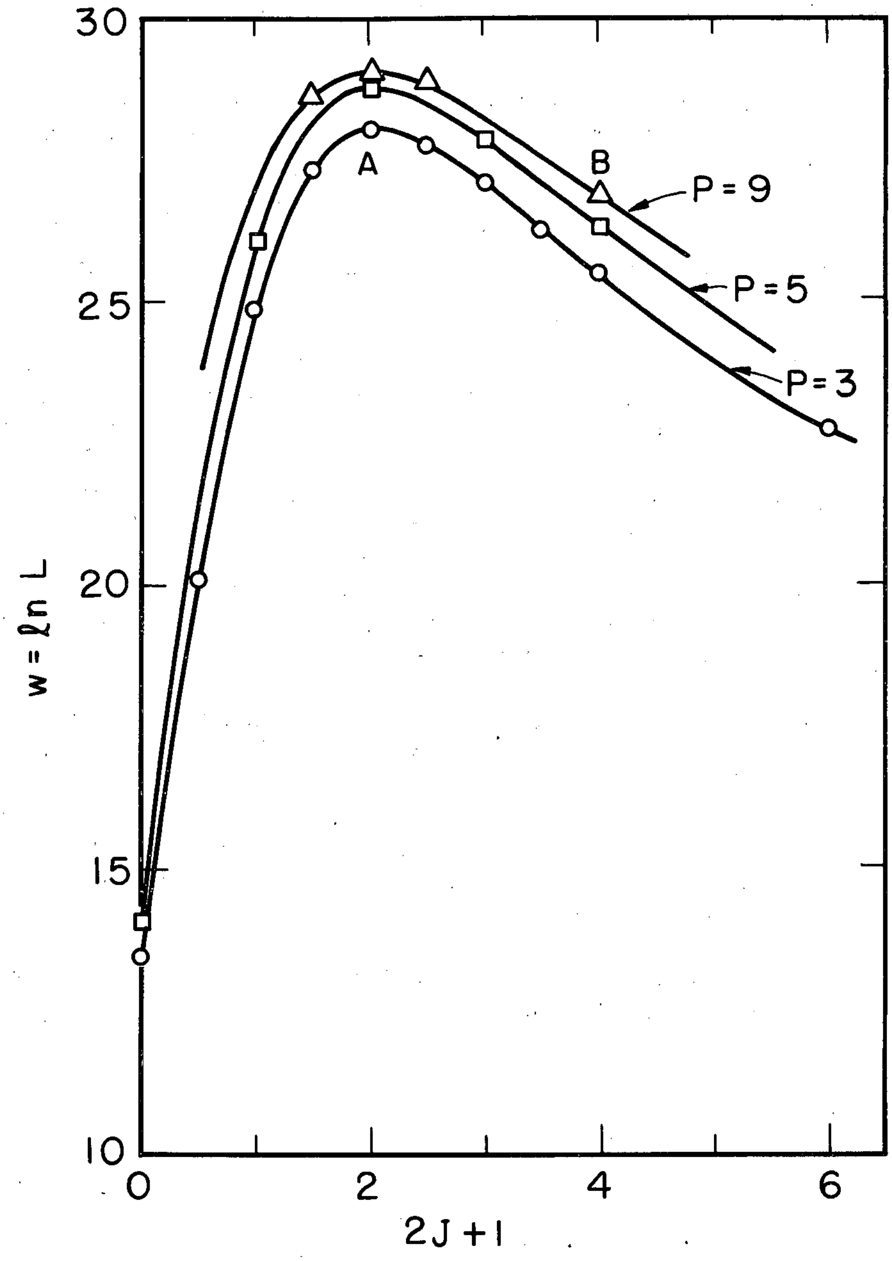

Fig. 5. Several plots are shown of $w = \ell n\ L$ as a function of $2J + 1$ for the high-polarization sample D. The symbol P represents the number of free parameters (t_{LM}'s and decay parameters) used in each treatment of data. The lowest curve was that obtained with the usual spin-1/2 form of the likelihood function; the higher two included parameters required to be zero for spin 1/2. See text for identification of points A and B, as well as further description of the likelihood functions.

Evaluation of χ_A^2 with the parameters of sample D, Table III (with $J = 3/2$) yields 2.9 when 6.0 is expected, whereas the values of $w = \ln L$ (for spin 3/2) with and without the six extra parameters differ by 1.35 when 3.0 is expected. Hence t_{20}, $\genfrac{}{}{0pt}{}{\text{Re}}{\text{Im}}\, t_{22}$, t_{30}, and $\genfrac{}{}{0pt}{}{\text{Re}}{\text{Im}}\, t_{32}$ are "very" consistent with zero for sample D.

If these six moments are actually zero, the separation (measured for many different experiments) between the upper and lower curves in Fig. 5 should be distributed about an "average" value near 3.0. The actual separation for a given experiment, however, tells us little about the spin J unless the separation is significantly greater than 3.0 (proving inconsistency with spin 1/2). Accordingly, no conclusions should be drawn from comparison of $w = \ln L$ at points A and B.

Interpreting $(2J + 1)$ as a variable parameter, we obtain from Fig. 5 the value $(2J + 1) = 2.0^{-0.4}_{+0.7}$. (We note that this result is nicely consistent with that from moment analysis, given above.) The value of $(2J + 1)$ is not strongly dependent on whether the 3-, 5-, or 9-parameter form of the likelihood function is used. A more direct comparison between the two hypotheses is obtained by measuring w, with a fixed number of parameters, at the two points $(2J + 1) = 2$ and $(2J + 1) = 4$. We obtain $[w(2J + 1 = 2) - w(2J + 1 = 4)] = 2.54$, 2.46, and 2.20 for the 3-, 5-, and 9-parameter function respectively. This corresponds to rejection of the spin-3/2 hypothesis with a confidence level of about 0.015, or 2.2 standard deviations.

One more important fact has not yet been utilized; namely, that the values of t_{LM} may not exceed certain limits and these limits are dependent upon spin.

First, the t_{LM} must be such that the distribution function is positive for all values of $\hat{\Lambda}$ and $\hat{p}$. The solutions listed in Table III have this character without further constraining the t_{LM}.

Second, Byers and Fenster[1] have defined the t_{LM} such that t_{10} for any spin J is related to the expectation value of the spin operator S_z by $t_{10} = [1/\sqrt{J(J+1)}]\, \langle S_z \rangle$, so that

$$|t_{10}| \leq \frac{J}{\sqrt{J(J+1)}} . \tag{17}$$

This constraint upon t_{10} is equivalent to the inequality derived by Lee and Yang:[17]

$$|\langle \cos\theta_\Lambda \rangle| \equiv |\langle \hat{\Lambda}\cdot\hat{n} \rangle| \leq \frac{1}{2J+2} . \tag{18}$$

An even more stringent constraint is imposed by the requirement that the diagonal elements of the density matrix (which are the occupation probabilities of the various spin states) be real and nonnegative. The density matrix ρ is of the form given in Eq. (1), where

$$T_{00}^{J} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \text{ and } T_{10}^{J} = \frac{S_z}{\sqrt{J(J+1)}} = \frac{1}{\sqrt{J(J+1)}} \begin{pmatrix} J & & & \\ & J-1 & & \\ & & \ddots & \\ & & & -J \end{pmatrix},$$

If the t_{LM} values are equal to zero for $L \geqslant 2$, the nonnegative conditions on the two corner matrix elements of ρ become

$$1 \pm 3 \frac{J}{\sqrt{J(J+1)}} t_{10} \geqslant 0, \quad \text{whereby}$$

$$|t_{10}| \lesssim \frac{1}{3J} \sqrt{J(J+1)} \quad \text{if } t_{LM} \approx 0 \text{ for } L \geqslant 2.$$

Hence

$$|t_{10}| \leqslant \sqrt{\frac{1}{3}} = 0.578 \text{ for spin } 1/2, \text{ and}$$

$$|t_{10}| \lesssim \frac{1}{3}\sqrt{\frac{5}{3}} = 0.430 \text{ for spin } 3/2.$$

This constraint upon t_{10} is equivalent to a second inequality of Lee and Yang:[17] $|\langle\cos\theta_\Lambda\rangle| \equiv |\langle\hat{\Lambda}\cdot\hat{n}\rangle| \leqslant 1/6J$ if no powers higher than $\cos\theta_\Lambda$ appear in the lambda angular distribution.

From Table III, we see that t_{10} for the spin-1/2 assumption does not exceed the limit of 0.578 for any of the samples A, B, C, or D. However, for the spin 3/2 assumption, the limit of 0.430 is exceeded for samples B, C, and D. For these samples, application of the density-matrix constraint will obviously result in reduced values of $w = \ln L$ for the spin 3/2 hypothesis.

In Table IV we present values of $\Delta w \equiv [w(2J+1=2) - w(2J+1=4)]$, with and without density-matrix constraints. (These results are for the 3-parameter likelihood function in each case; values of Δw from the 5- and 9-parameter functions are nearly identical in every case.) See Fig. 6 for distributions.

Table IV. Values of $\Delta w \equiv [w(2J+1=2) - w(2J+1=4)]$ for 3-parameter likelihood function.[a]

Sample	Without density-matrix constraint	With density-matrix constraint (diagonal elements $\geqslant 0$)
Positive-polarization sample A(438 events)	0.50	0.50
Negative-polarization sample B(311 events)	2.36	3.12
Combined, with rotation of negative-polarization events. Sample C (749 events)	2.32	2.39
Combined, with rotation and with selection of high-polarization events. Sample D(440 events)	2.54	6.78[b]

a. Values of Δw from the 5- and 9-parameter form of the likelihood function are almost the same as those presented here.

b. The value $\Delta w = 6.78$ is believed to be biased. (See Sec. III.F.)

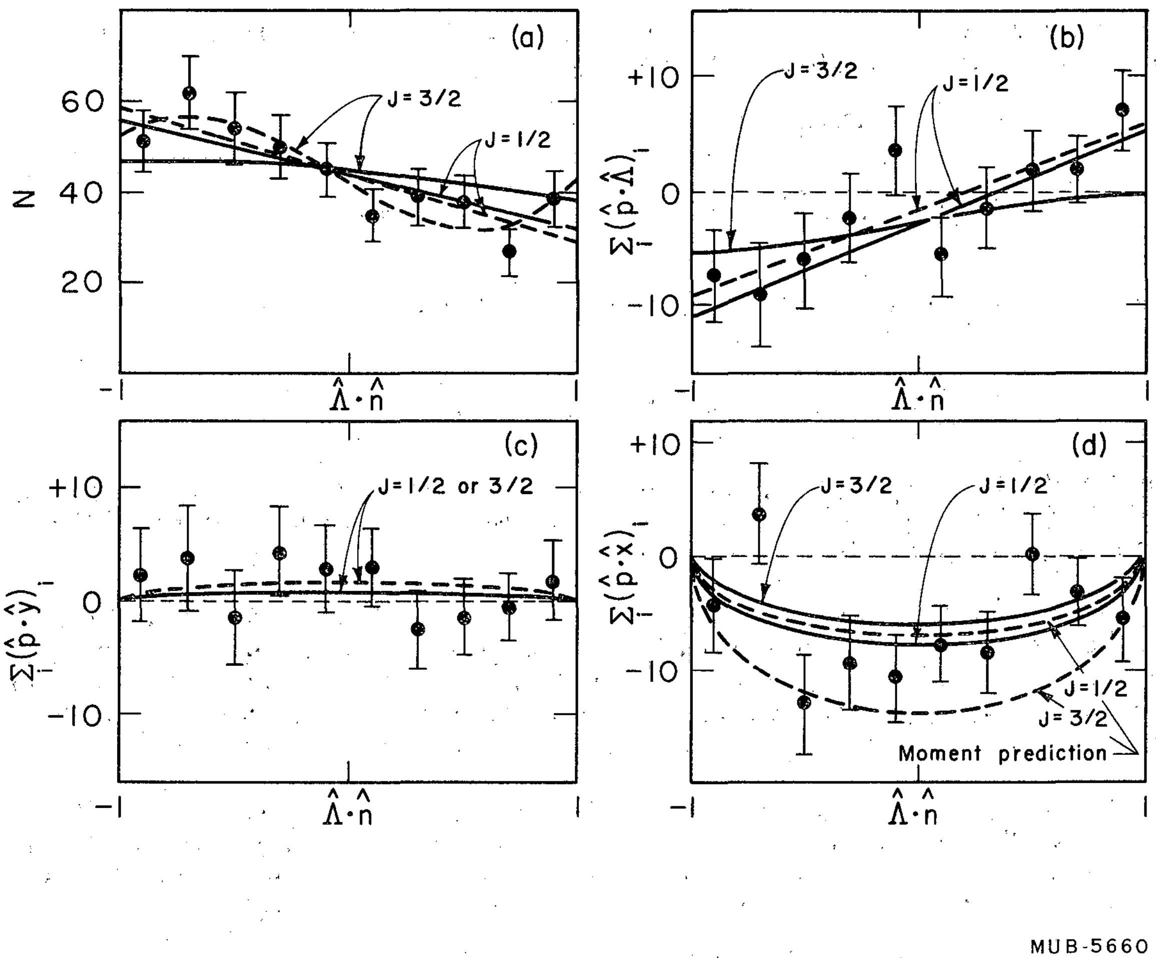

Fig. 6. (a) Angular distribution $[I(\theta)]$ for sample D events. The solid curves represent likelihood solutions (with density-matrix constraints imposed); the dashed curves are constructed from moments.

(b) Longitudinal polarization $[(I\overline{P}\cdot\hat{\Lambda})\times \alpha_\Lambda/3]$ vs $\hat{\Lambda}\cdot\hat{n}$. Only the J = 1/2 curve (dashed) is shown from moment analysis.

(c) One component of transverse polarization $[(I\overline{P}\cdot\hat{x})\times\alpha_\Lambda/3]$ vs $\hat{\Lambda}\cdot\hat{n}$. The ordinate is proportional to β_Ξ and to 2J+1. Dashed curve same as in Fig. 6(b).

(d) The other component of transverse polarization $[(I\overline{P}\cdot\hat{y})\times\alpha_\Lambda/3]$ containing γ_Ξ and 2J+1. The dashed curves are predictions from moment analysis [based on t_{10} moments of Figs. 6(a) and 6(b)], curve (a) being the J = 1/2 prediction and curve (b) being the J = 3/2 prediction. (The spin-3/2 likelihood solutions have been adjusted to satisfy constraints, whereas the moment curves have not.)

Application of the density-matrix constraint may possibly give biased results where events have been selected or rotated to yield a high value of average polarization. Through statistical fluctuations, it may be possible for a selected and/or rotated sample (even if really of spin 3/2) to have a high enough average value of t_{10} to favor strongly spin 1/2. For this reason, we choose to disregard the value of $\Delta w = 6.78$ in Table IV (for sample D with the density-matrix constraint). (The value of $\Delta w = 2.39$ for sample C may be biased as well, although the rotation of negative-polarization events alone probably cannot bias likelihood calculations if the signs of polarizations of the subsamples are reasonably well defined.) We are using randomly generated events to investigate thoroughly the possible biases introduced through rotation and selection of events.

At present it seems that neither rotation nor selection introduces significant biases if the density-matrix constraint is not imposed in the likelihood calculation. Accordingly, the most definitive results for spin from the maximum-likelihood method are those from sample D, for which the difference between ℓn L for spin 1/2 and spin 3/2 is 2.54; if $L(J = 3/2)/L(J = 1/2)$ can be interpreted as having a Gaussian distribution, then this difference corresponds to discrimination against spin 3/2 by 2.25 standard deviations (or a confidence level of 0.024).[18]

Output from the random-event generator has been used to test the likelihood function; answers agreed well with the known characteristics of faked events and also compared very well with results on the faked events from moment analysis. The same fake-event generator is being used to investigate the distribution of values of the maximum-likelihood function itself, as a function of decay parameters and number of events.

IV. ADDITIONAL DATA

Analysis was made of additional data, 224 events, from the $K^- + p \rightarrow \Xi^- + K^+$ reaction obtained at momenta of 2.4 to 2.7 BeV/c. Polarization of the Ξ^- was higher at these momenta than in the 1.2 to 1.7 BeV/c range; the low number of events, however, led to somewhat inconclusive results.

The data were subdivided for moment analysis into the three samples presented in Table II(b). The $|2J + 1|$ evaluations from moments have been added to Fig. 2; they give further support to the $2J + 1 = 2$ or spin-1/2 hypothesis. Weighted averaging of the three evaluations yields $|2J + 1| = 1.87 \pm 0.99$. However, the analysis of all data samples (high-momentum) combined yields $2J + 1 = 3.4$ and $|2J + 1| = 3.8 \pm 2.1$ (with β omitted and included, respectively). These could satisfy either the $J = 1/2$ or 3/2 hypothesis. The decay parameters found by moments are $\alpha = -0.22$, $\beta = 0.47$, and $\gamma = 0.86$. (For the evaluation of $|2J + 1|$, this unusually small value of $|\alpha|$ and the large value of β manifest themselves in a higher $|2J + 1|$ value than that found for other data yielding more nearly normal estimates of decay parameters.) Thus, the confidence level for $J = 3/2$ is not much reduced by consideration of the 2.4 to 2.7 BeV/c data.

For the high-momentum data, (all 224 events, with no selection and no rotation) the decay parameters for spin 1/2 as found by maximum-likelihood analysis are

$$\alpha = -0.21 \pm 0.12$$

$$\Phi = 0.62 \pm 0.30 \,(\beta = 0.57 \pm 0.24,\ \gamma = 0.80 \pm 0.17).$$

The initial polarization was such that $t_{10} = 0.44 \pm 0.10$ ($P_\Xi = 0.76 \pm 0.17$). Calculation of the difference in the logarithms of likelihoods (spin-1/2 form) for spins 1/2 and 3/2 yields $w(2J+1=2) - w(2J+1=4) = 0.39$; thus no discrimination against spin 3/2 is obtained. (The 3- and 9-parameter curves analogous to those of Fig. 4 are separated by 2.5, when 3.0 is expected; thus the data are consistent with spin 1/2.)

V. CONCLUSIONS

The best values for the decay parameters (spin 1/2) were those obtained by maximum-likelihood analysis from the high-polarization data (sample D):

$$\alpha = -0.30 \pm 0.08$$

$$\Phi = 0.07 \pm 0.19,$$

yielding $\beta = 0.07 \pm 0.18$, $\gamma = 0.95 \pm 0.02$. Results from the moment analysis were in close agreement.

The conclusion that may be drawn from the calculation of $2J+1$ in the moment analysis is that the Ξ spin is 1/2 rather than 3/2, the latter having a confidence level of perhaps 0.01. The spin-1/2 hypothesis does not give a poor fit for any of the data at various K^- momenta. The maximum-likelihood analysis rejects spin 3/2 with a confidence level of 0.024; and this result can possibly be improved.[18, 19]

Another study that supports the $J = 1/2$ hypothesis is that by the UCLA group.[20] From some 187 Ξ^-K^+ and 169 $\Xi^-K\pi$ events at 1.8 and 1.95 BeV/c K^- momenta, UCLA physicists obtain the value $2J+1 = 1.53 \pm 0.88$ and interpret this as approximately three standard deviations away from spin 3/2.[20]

It has recently become clear that the strong discrimination against spin 3/2 obtained by moment analysis of the combined samples (C and D), with the $\alpha_\Xi = -0.34$ demanded by the data cannot be duplicated through the likelihood treatment. There are of course several characteristics that make the moment and likelihood spin tests nonequivalent. (For the former, the constraint on decay parameters is not included and an "external" value of α_Ξ may be introduced. Also, the only moment involved is t_{10}, whereas the likelihood analysis demands also the t_{00} normalization term.) However, the strength of the moment answer must be somewhat discounted for the following reason. Not all the t_{10} estimates for the various subsamples (Table II) were well defined. Selecting the samples as to the magnitude or the sign of t_{10} may have distorted the distributions of the γt_{10}, t_{10}, or αt_{10}, which were assumed Gaussian; further, the true value of polarization (especially for sample D) may have been somewhat less than that calculated. The P(A) distributions constructed for the combined data samples thus could be inadequate for estimation of confidence levels.

A more naive selection of data samples has been used. This ignored even the polarization information from likelihood production-decay fits made earlier (by J. P. Berge) and separated data into four samples: 1.2-1.4 BeV/c, yielding $t_{10} = -0.29 \pm 0.10$; 1.5 BeV/c, forward production, with $t_{10} = -0.20 \pm 0.09$; 1.5 BeV/c, backward production, with $t_{10} = 0.40 \pm 0.11$; and 1.6-1.7 BeV/c, with $t_{10} = 0.23 \pm 0.12$. The 2J+1 estimates made from Eq. (9) for these samples and the appropriate probability distributions are shown in Figs. 7(a) and 7(b). (The βt_{10} moments are found to have nearly zero values.) Only the 1.6-1.7 BeV/c data give good discrimination against spin 3/2. (These calculations were done with $\alpha_{\Xi} = -0.48$ and would give lower spin-3/2 confidence levels with $\alpha_{\Xi} = -0.34$.) As all the t_{10} estimates were fairly well defined, the two negative-t_{10} samples were treated with the rotation technique and combined with the positive-t_{10} samples. For this combination of 749 events, a value of $|2J+1|$ equal to 2.86 (2.18) was obtained with $\alpha_{\Xi} = -0.48$ (-0.34). Again, it is questionable whether the usual sort of probability distribution of Appendix III should be used. However, if this P(A) distribution is applicable, it indicates for spin 3/2 a confidence level of 0.15 if $\alpha_{\Xi} = -0.48$ and one of 0.015 if α_{Ξ} is -0.34. (See Fig. 7(c).) The confidence levels for spin 1/2 are good (0.22 and 0.42); those for spin 5/2 are poor (0.003 and < 0.0002).

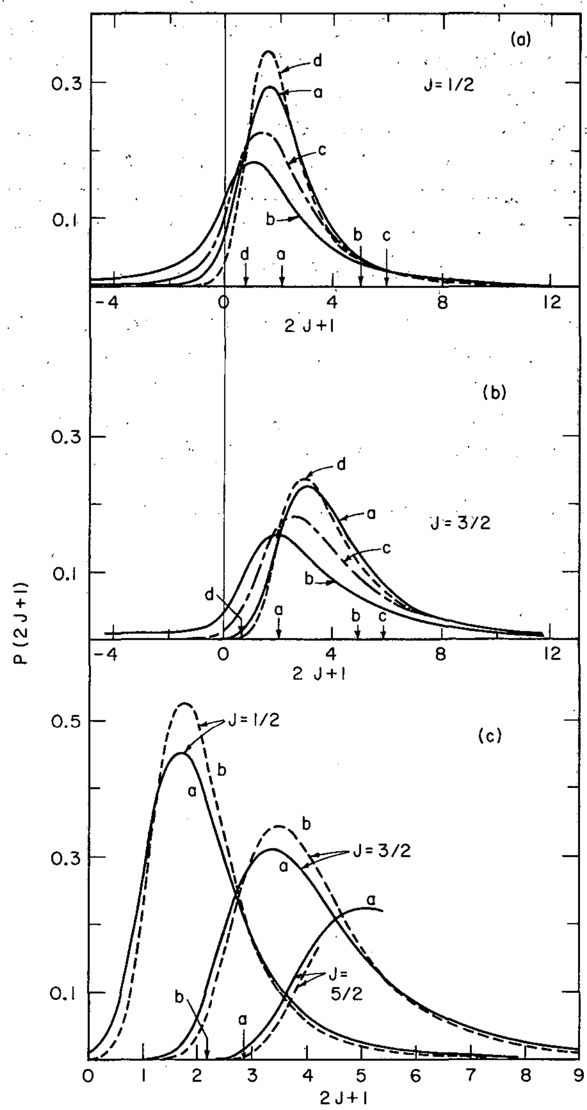

Fig. 7. (a) Evaluations (arrows) and probability distributions of 2J+1 for four data samples simply selected: (a) 1.2 through 1.4 BeV/c; (b) 1.5 BeV/c, forward production; (c) 1.5 BeV/c, backward production; and (d) 1.6 through 1.7 BeV/c. A true spin of 1/2 is assumed for the calculated distributions.

(b) Data used are the same as for Fig. 7(a). Spin 3/2 is assumed for probability distributions.

(c) Evaluation and probability distributions of 2J+1 from the total data, treated as a combination of the four samples used for Figs. 7(a) and (b). (See text.) Spins 1/2, 3/2, and 5/2 are assumed for the probability distributions. All curves are normalized so as to have the same area.

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Finally, we are grateful for the encouragement of Professor Luis W. Alvarez.

APPENDICES

A. Decay Distributions and T_{LM} Operators

The decay distributions presented in Eq. (2) of the text can readily be derived by the use of helicity states for the decay Λ. Since the Λ has spin 1/2, there can be only two states; and the decay matrix for the transition from the initial Ξ spin state to the final Λ state can be only a two-by-two matrix (regardless of the magnitude of the Ξ spin) if the Ξ spin functions are expressed in the representation that has $\hat{\Lambda}$ as its quantization axis. A rotation of the Ξ state must be made from the J, M system (usually having the production normal as quantization axis) to the $(\hat{\Lambda})$ helicity system; as pointed out by Jacob and Wick,[5] the rotation function $\mathcal{D}^{J*}_{M\lambda}(\phi, \theta, 0)$ takes the amplitude with quantum numbers J, M into the helicity state with quantum number $\lambda (= +1/2 \text{ or } -1/2)$ if the new quantization axis is related to the old by the polar angles θ and ϕ. After this rotation, the decay matrix is diagonal; as its form is $a + b\overline{\sigma}\cdot\hat{\Lambda}$, the two "helicity amplitudes" that are the diagonal elements are $a+b$ and $a-b$. (As usual, a is the $J-1/2$ amplitude and b the $J+1/2$ amplitude for a parity-violating decay.)

With density-matrix formalism, it is readily seen that the final Λ density matrix will contain products of two rotation functions and two helicity amplitudes.[21] By the use of a relation that replaces each product of $\mathcal{D}^J_{M\lambda}$ functions with a single $\mathcal{D}$ function, the diagonal terms $[I(\theta,\phi)$ and $I\overline{P}\cdot\hat{\Lambda}(\theta,\phi)]$ are shown to contain $\mathcal{D}^L_{M0}$ or $Y_{LM}(\theta,\phi)$, whereas the off-diagonal terms $[I\overline{P}\cdot\hat{x}(\theta,\phi)$ and $I\overline{P}\cdot\hat{y}(\theta,\phi)]$ contain $\mathcal{D}^L_{M1}(\phi,\theta,0)$. (See Appendix ii of Byers and Fenster unpublished report, Dept. of Physics, UCLA, May 27, 1963. The published article of reference 1 presents an expression for transverse polarization moments, which expression contains sums of tensor polarization components times Y_{LM} functions; this is equivalent to the $\langle \mathcal{D}^{L*}_{M1} \; P_1{}^1\rangle$ or $\langle \mathcal{D}^{L*}_{M1}(\overline{P}\cdot\hat{x} + i\,\overline{P}\cdot\hat{y})\rangle$ implied by Eq. (7) of this article.)

In matrix representation, the T_{LM} have the following forms for spin 1/2 and spin 3/2: $T_L{}^{M})_{mm'} = C(JLJ;\, m'M)$ with $m'+M=m$ (in the representation where T_{L0} is diagonal)

For spin 3/2, $$T_{10} = \frac{1}{\sqrt{15}}\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$ For spin 1/2, $$T_{10} = \sqrt{3}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T_{20} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_{22} = \sqrt{2/5}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{30} = \frac{1}{\sqrt{35}}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad T_{32} = \sqrt{2/7}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$T_{2,-2} = T_{22}^{\dagger}$ and $T_{3,-2} = T_{32}^{\dagger}$

B. Effect of Setting $\beta = 0$ in $2J+1$ Expression [Eq. (9)]

The values of β previously reported from experimental data give an average of

$$\beta = +0.14 \pm 0.14. \tag{19}$$

(This number represents the averaging of Φ determinations made at CERN, LRL, UCLA, EP+ (Ecole Polytechnique), and BNL+S (Brookhaven National Laboratory and Syracuse University), as reported by H. K. Ticho.[22] The expression of Eq. (9) with β set equal to 0 would be correct if the $(1-\alpha^2)$ term in the denominator were replaced by $(1-\alpha^2-\beta^2)$. Taking an overestimated β value of 0.30 instead of 0.0 decreases this factor by 0.09/0.77 and hence increases the $2J+1$ evaluation by 6 percent. Leaving the denominator unchanged, but compensating for the neglect of the β term in the numerator of Eq. (9) requires modification of two experimentally determined parameters used to calculate the $2J+1$ distribution curves; these modifications have a very small effect.

Neglecting the β term in Eq. (9) is reasonable from an experimental point of view, whether the true spin of the Ξ^- is 1/2, 3/2, or higher; the evaluation of the moment $\beta n^J_{11}t_{10}$ is independent of J value and is certainly much smaller than the $\gamma n^J_{11}t_{10}$ moment (also in the numerator) in the experiment reported here. From a theoretical point of view, β can be estimated by use of the expression $\beta/\alpha = \tan(\delta_p - \delta_s)$ or $\tan(\delta_d - \delta_p)$ for spin 1/2 or spin 3/2, with the experimental form of α and with estimated phase shifts. (The quantities δ_ℓ denote phase shifts for ℓ-wave scattering of the Λ and π in the final state of the Ξ decay.) As no spin-1/2 resonances are known for the $\Lambda-\pi$ system, it seems unlikely that either δ_s or δ_p for a spin-1/2 system would be large. The $Y_1^*(1385\text{ MeV})$ (a P-3/2 $\Lambda-\pi$ system) could influence the $\Lambda-\pi$ final state from a spin-3/2 Ξ; but with a width of about 55 MeV, it could account for a $|\beta|$ value of no more than $0.4\,\alpha$ or 0.16, as the phase shift δ_p for Y^* decay at a $\Lambda-\pi$ energy of 1320 MeV can be no more than 23° (and the d-wave phase shift is probably zero).

C. Calculation of $2J+1$ Distributions

With knowledge of experimental errors for the $\beta n^J_{L1}\, t_{LM}$ and the $\gamma n^J_{L1}\, t_{LM}$ moments of Eq. (9), it is possible to calculate the distribution in the quantity $(2J+1)'$ given by the expression [approximately equivalent to that of Eq. (9) for the experiment reported here]

$$(2J+1)' = \frac{\sqrt{L(L+1)}\;|\gamma n^J_{L1}\, t_{LM}|}{(1-\alpha^2)^{1/2}\;|n^J_{L0}\, t_{LM}|} \equiv \frac{|Y|}{|X|}\;. \tag{20}$$

(The discussion below also yields the distribution for Y/X, with signs included.) It is assumed that the numerator moment is normally distributed (this being a safe assumption for most experiments and a convenient one for calculation); it is also assumed that the denominator quantity X is normally distributed. The latter is not so rigorously true as the former, but is quite good if $(1-\alpha^2)^{1/2}$ is very accurately known in comparison with the (normally distributed) moment $n^J_{L0}\, t_{LM}$.

As discussed by Byers,[23] the probability of finding A in some interval dA, if A is determined from the ratio of two experimental quantities

$$A = Y/X, \tag{21}$$

is given by

$$P(A)\,dA = dA \int P'(A, X)\,dX = dA \int Q(AX)\,R(X)\,\frac{\partial(AX, X)}{\partial(A, X)}\,dX\,. \tag{22}$$

The quantities Y (or AX) and X are assumed to be independently determined. Further, they are taken to be normally distributed, with standard deviations σ_X and σ_Y, so that the probability distribution for X is

$$R(X) = \frac{1}{\sqrt{2\pi}\,\sigma_X} \exp -[(X - X_0)^2/2\,\sigma_X^2] \tag{23}$$

and that for Y has a similar form.

For the case of interest to which the above P(A) distribution is applicable, the expected value of A is the $(2J+1)'$ quantity defined in Eq. (20).

It is convenient to take absolute values of $Y(=\sqrt{L(L+1)}\,|\gamma n^J_{11}\,t_{10}|)$ and of $X(=(1-\alpha^2)^{1/2}\,|n^J_{10}\,t_{10}|)$. The expression for P(A) of Eq. (22) is valid also for $P(|A|)$ [if all arguments are taken to have absolute values] except that the forms for Q(Y) and R(X) must now include contributions from positive and negative arguments; thus R(X) becomes

$$R(X) = \frac{1}{\sqrt{2\pi}\,\sigma_X} \cosh\left(\frac{XX_0}{\sigma_X^2}\right) e^{-(X^2+X_0^2)/2\sigma_X^2}$$

and Q(Y) takes a similar form. Now, with all variables taking only positive values, $P(|A|)$ is given by the following:

$$P(|A|) = \frac{1}{2\pi\,\sigma_X\,\sigma_Y} \int_0^\infty X\,dX(e^F + e^{-F} + e^G + e^{-G}) \exp\left\{-\frac{1}{2}\,X^2\left(\frac{1}{\sigma_X^2} + \frac{A^2}{\sigma_Y^2}\right) - \frac{1}{2}\,X_0^2\left[\frac{1}{\sigma_X^2} + \frac{(2J+1)^2}{\sigma_Y^2}\right]\right\}$$

with $$F = \frac{XX_0}{\sigma_X^2} + \frac{YY_0}{\sigma_Y^2} = XX_0\,\frac{[\sigma_Y^2 + A(2J+1)\,\sigma_X^2]}{\sigma_X^2\,\sigma_Y^2}$$

and $$G = -\frac{XX_0}{\sigma_X^2} + \frac{YY_0}{\sigma_Y^2}\,.$$

The expression AX has been substituted for Y, and $(2J+1)\,X_0$ for Y_0. The sum of the four exponential terms of course derives from the possibilities of producing a value A in four ways: Y and X both positive, Y and X both negative, Y positive and X negative, and Y negative and X positive.

The above $P(|A|)$ expression is readily evaluated analytically; however, with the inclusion of error correlations such evaluation becomes rather tedious. Upon introduction of the terms ϵ_{ij}^{-1} of the inverse error matrix for X and Y, the exponent in the $P(|A|)$ integral for the positive-Y-and-positive-X contribution or negative-Y-and-negative-X contribution only is

$$-\frac{1}{2}\,\epsilon_{XX}^{-1}\,[X^2 + X_0^{\,2} \mp 2XX_0] \qquad (\equiv \phi_\pm)$$

$$-\frac{1}{2}\,\epsilon_{YY}^{-1}\,[A^2\,X^2 + (2J+1)^2\,X_0^{\,2} \mp 2A\,(2J+1)\,XX_0] \qquad (\equiv \psi_\pm)$$

$$-\,\epsilon_{XY}^{-1}\,[AX^2 + (2J+1)\,X_0^{\,2} \mp (A+2J+1)\,XX_0] \qquad (\equiv \Sigma_\pm)$$

Appropriate sign changes are made for the other contributions. The final expression may be written as $\{\mathcal{P}[\,|A|, -(2J+1)]$ being the probability for negative A $\}$

$$P(|A|) = \mathcal{P}\,[|A|,\,(2J+1)] + \mathcal{P}\,[|A|,\,-(2J+1)]$$

where
$$\mathcal{P}\,[|A|,\,(2J+1)] \equiv \frac{\sqrt{\det\,\epsilon^{-1}}}{2\pi}\int_0^\infty X\,dX\,\{\exp(\phi_+ + \psi_+ + \Sigma_+) + \exp(\phi_- + \psi_- + \Sigma_-)\}\ .$$

Numerical integration of this result gave the curves of Figs. 3 and 4. Analytical integration was used to check some of the program-calculated points; agreement was obtained to three or four significant figures.

D. The Distributions of Eqs. (2) and (11)

The relations in Eq. (2) of the text are used both for the moment method of analysis and for the maximum-likelihood treatment [Eq. (11)]. The expressions as given follow directly from the helicity-amplitude derivation of decay distributions. They may be written more explicitly in terms of calculable quantities.

With the symmetry relation for the t_{LM} parameters

$$t_{L,-M} = (-)^M\,t_{LM}^{*}$$

and an identical one for the spherical harmonics, it is evident that

$$t_{LM}\,Y_{LM}^{*} + t_{L,-M}\,Y_{L,-M}^{*} = 2\,\mathrm{Re}\,(t_{LM}\,Y_{LM}^{*})\ .$$

Thus all negative-M terms in $I(\theta,\phi)$ and $I\overline{P}\cdot\hat{\Lambda}(\theta,\phi)$ may be combined with positive-M terms.

The transverse polarization expressions of Eq. (2) require

$$I\overline{P}\cdot\hat{x} + iI\overline{P}\cdot\hat{y} = (i\beta - \gamma)\, F(\theta,\phi)$$

where $F(\theta,\phi)$ is a complex function:

$$F(\theta,\phi) = X + iY = (2J+1) \sum_{L_o} \sum_{M} n_{L0}^{J}\, t_{LM} \mathcal{D}_{M1}^{L}(\phi,\theta,0)\, [(2L+1)/4\ ]^{1/2} \times [L(L+1)]^{-1/2} \ .$$

To simplify this function, we found the following definitions and relations useful:

$$\mathcal{D}_{M\lambda}^{L}(\phi,\theta,0) = e^{-iM\phi}\, d_{M\lambda}^{L}(\theta) \qquad \text{(Ref. 24)}$$

$$d_{M0}^{L}(\theta) = [4\pi/(2L+1)]^{1/2}\, Y_{LM}(\theta,0) \qquad \text{(Ref. 24)}$$

$$d_{M1}^{L}(\theta) = [L(L+1)]^{-1/2} \{-M(1+\cos\theta)\, d_{M0}^{L}(\theta)/\sin\theta - \sqrt{(L-M)(L+M+1)}\, d_{M+1,0}^{L}(\theta)\}. \qquad \text{(Ref. 25)}$$

Also, in addition to the symmetry relation for the $t_{L,-M}$, the following symmetry property of the $d(\theta)$ functions is useful:

$$d_{-M\lambda}^{L}(\theta) = (-)^{L+\lambda}\, d_{M\lambda}^{L}(\pi-\theta) \qquad \text{(Ref. 25)}$$
$$= d_{M\lambda}^{L}(\pi-\theta) \quad \text{for odd } L \text{ and odd } \lambda.$$

The expression for $F(\theta,\phi)$ can be written now as a sum over positive M only (for $M \neq 0$ and even):

$$F(\theta,\phi) = (2J+1) \sum_{L} \sum_{|M|} n_{L0}^{J} [\mathcal{D}_{M1}^{L}(\phi,\theta,0)\, t_{LM} + \mathcal{D}_{M1}^{L*}(\phi,\pi-\theta,0)\, t_{LM}^{*}] \times [(2L+1)/4\pi]^{1/2}\, [L(L+1)]^{-1/2} \ .$$

Thus, the real and imaginary parts of the function $F(\theta,\phi)$ become (for $M \neq 0$ and even)

$$(2J+1) \sum_{L_o} \sum_{|M|} n_{L0}^{J}\, \mathrm{Re}\,(t_{LM}\, e^{-iM\phi})\, [d_{M1}^{L}(\theta) + d_{M1}^{L}(\pi-\theta)] \times [(2L+1)/4\pi]^{1/2}\, [L(L+1)]^{-1/2}$$

$$(2J+1) \sum_{L_o} \sum_{|M|} n_{L0}^{J}\, \mathrm{Im}\,(t_{LM}\, e^{-iM\phi})\, [d_{M1}^{L}(\theta) - d_{M1}^{L}(\pi-\theta)] \times [(2L+1)/4\pi]^{1/2}\, [L(L+1)]^{-1/2}$$

and with substitution of the expression for $d_{M1}^{L}(\theta)$ in terms of spherical harmonics,

$$X(\theta,\phi) = (2J+1)\left\{-2\sum_{L_o}\sum_{|M|\neq 0} n^J_{L0}\ \mathrm{Re}(t_{LM}\, e^{-iM\phi})\right.$$

$$\cdot\left[M\cot\theta\, Y_{LM}(\theta,0) + \sqrt{(L-M)(L+M+1)}\, Y_{L,M+1}(\theta,0)\right][L(L+1)]^{-1/2}$$

$$\left. - \sum_{L_o} n^J_{L0}\ t_{L0}\ Y_{L1}(\theta,0)\right\}[L(L+1)]^{-1/2} .$$

$$Y(\theta,\phi) = (2J+1)\left\{-2\sum_{L_o}\sum_{|M|\neq 0} n^J_{L0}\ \mathrm{Im}(t_{LM}\, e^{-iM\phi})\ M\, Y_{LM}(\theta,0)/\sin\theta\right\}[L(L+1)]^{-1}$$

FOOTNOTES AND REFERENCES

*Work done under the auspices of the U. S. Atomic Energy Commission.

1. N. Byers and S. Fenster, Phys. Rev. Letters 11, 52 (1963), and unpublished report of Byers and Fenster, Dept. of Physics, University of California, Los Angeles, May 27, 1963.

2. That is, if the J factor was given by the ratio of measurements A and A' from one sample and by the ratio of measurements B and B' from another, etc., and if the best estimate of polarization for only the second sample was negative, the experimental estimate of the spin factor was changed from $(A+B+C+\ldots)/(A'+B'+C'+\ldots)$ to $(A-B+C+\ldots)/(A'-B'+C'+\ldots)$. This rotation of negative-polarization samples was also performed for maximum-likelihood analysis.

3. This had previously been verified for moment and likelihood analyses by treating the already-analyzed $Y_1^*(1385)$ events (Phys. Rev. Letters 10, 179 (1963) and Phys. Rev. 134, B1372 (1964); by checking special cases with hand calculations; and by comparing likelihood results with those of Philippe Eberhard (who analyzed Ξ^- decays with somewhat different formalism from that reported here).

   Samples of fake events are being analyzed to determine if it is possible, by sample selection or rotation, to bias the analysis of events actually having spin 3/2 in such a way that spin 1/2 is favored. Results to date indicate that such a bias is rather unlikely.

4. For example, $T_{11} \propto S_x + iS_y$. (See Appendix A for further description of the T_{LM} operators.)

5. M. Jacob and G. C. Wick, Ann. Phys. 7, 404 (1959).

6. This is easily proved by use of recursion relations for Clebsch-Gordan coefficients. See A. R. Edmonds, Angular Momentum in Quantum Mechanics, (Princeton University Press, Princeton, New Jersey, 1957).

7. For evaluation of the α, β, and γ parameters, we could have formed a total χ^2 which received equal contributions from all moments. However, the construction and minimization would have been rather complicated; probably no better information on the Ξ^- spin and in some cases not much better determinations of decay parameters would have been obtained.

8. D. D. Carmony, G. M. Pjerrou, P. E. Schlein, W. E. Slater, D. H. Stork, and H. K. Ticho, Phys. Rev. Letters 12, 482 (1964). The world-average value of α_Ξ is somewhat larger than the Berkeley value (which would give a lower estimation of $2J+1$ if used).

9. The use of α_W in obtaining this t_{10} estimate as an average of moments suppresses the estimate; as is discussed below, maximum-likelihood analysis of the same data yields an α_Ξ of -0.30 and a t_{10} of 0.52.

10. Equation (9) reduces to an identity if the relation connecting n_{L1}^{J} and n_{LO}^{J} [Eq. (4)] and also that connecting α^2, β^2, and γ^2 are substituted. See Ref. 1 or M. Ademollo and R. Gatto, Phys. Rev. 133, B531 (1964).

11. From the 440-event sample, the estimates of β_{Ξ} for spin 1/2 and 3/2 are 0.20 ± 0.22 and 0.09 ± 0.10, respectively.

12. The value $\alpha_{\Lambda} = +0.62 \pm 0.07$ is that determined by J. W. Cronin and O. E. Overseth, Phys. Rev. 129, 1795 (1963).

13. See Alvarez Physics Memo 533 by J. Button-Shafer for development of the likelihood function for weak decay. A complete description of the computer program, called LIKEF and written by D. W. Merrill and J. R. Morris, may be obtained from an Alvarez Group Programming memo, to be completed shortly. Other similar programs have been written by P. Eberhard and J. P. Berge. The computation of the distribution function and the methods of arriving at a maximum are done differently in all three programs, but the results obtained are in perfect agreement.

14. Safeguards are taken in the program to ensure that w actually increases with each successive step. Depending upon the starting values of the variable parameters, between 3 and 8 iterations are usually sufficient to determine all parameters to 3 significant figures.

    First and second derivatives are calculated numerically from finite differentials. Although more time-consuming than an analytical calculation method, this numerical method permits one to impose easily inequalities and linear constraints on parameters.

15. An early version of this program, called SPIN-P-MOCK, is described in a Lawrence Radiation Laboratory Programming Memo P-54, by J. Morris and J. Button-Shafer (2/18/64). The program has since been modified to include weak decay. (Memo to be completed shortly.)

16. This is reasonable if w is taken to have the form $w \approx a - 1/2\,\chi^2$, with a equal to a constant that shows little or no dependence on the parameters α. Adding parameters reduces the number of degrees of freedom and thus lowers χ^2 and raises w.

17. T. D. Lee and C. N. Yang, Phys. Rev. 109, 1755 (1958). To see the equivalence, note that the Λ angular distribution given in Eq. (2) reduces to

$$I(\theta,\phi) \propto \left[1 + \alpha \frac{3}{2\sqrt{J(J+1)}}\, t_{10} \cos\theta + \ldots \text{(terms of order } \cos^2\theta \text{ and higher)}\right]$$

and that this requires $|\langle \cos\theta \rangle| \leq \left| \frac{\alpha}{2\sqrt{J(J+1)}}\, t_{10\,\max} \right|$.

18. Recently a more powerful maximum-likelihood search program has been developed by J. Peter Berge. In this program, all the available events can be sorted into a large number of bins according to momentum and production angle. In the maximum-likelihood search, a fit is made to a single value of α and Φ as before, but distinct values of t_{LM} are calculated separately for each bin. This method promises to yield more definitive maximum-likelihood results for the Ξ spin and decay parameters.

19. It is not too surprising that answers from moment and likelihood analyses should differ as to confidence levels. The ratio test in the moment method ignores the t_{00} (normalization) term and all other non-t_{10} terms; further it requires the use of the world-average α and does not allow, as formulated, any adjustment of the α or other decay parameters.

20. The UCLA data required no rotation or selection of data, as polarization was high. The values of UCLA decay parameters differ somewhat from those reported by most other groups; in particular, β is rather large.

21. $\rho_\Lambda = M\mathcal{D}\rho_i^*\mathcal{D}M^\dagger$, where ρ_i is the density matrix of the initial state, $\mathcal{D}$ represents the rotation matrix taking ρ_i to the helicity state, and M is the decay matrix in the helicity representation.

22. H. K. Ticho, Proceedings of the International Conference on Fundamental Aspects of Weak Interactions (Brookhaven National Laboratory, September, 1963), p. 410.

23. N. Byers, (UCLA), private communication, May 1964.

24. M. Jacob and G. C. Wick, Ann. Physics 7, 404 (1959), Eq. (17), footnote 11, and Table I.

25. These follow from recursion relations, Ref. 24, Eqs. (A1), (A2), and (A5).