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THE SHAPE OF A PENDENT WATER DROP

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THE SHAPE OF A PENDENT WATER DROP

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The shape of a pendent water drop

Paul Concus and Robert Finn

in memory of
Wulf and Flora Concus

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Corrigenda and Addendum

1. p. 12, line 8, read $u_o < u \leq u_o + \beta$.

2. p. 14, equation 22, read $a_1 \frac{\ln|u_1|}{|u_1|^3}$.

3. A modification of the method of VI yields a significantly sharper estimate. Writing $T_j = -r_j u_j(2 + r_j u_j)$, we find, in any range $\frac{1}{C}|u_o|^a < u_j < C|u_o|^a$, $\frac{3}{4} < a < 1$, the estimate

$$(*)\qquad |1 - T_j|^{\frac{1}{2}} = |u_j|^{\frac{a-1}{a}} + O\left(|u_j|^{\frac{a-3}{a}} + \frac{\ln|u_j|}{|u_j|^3}\right)$$

as $|u_o| \to \infty$.

If $\frac{|u_j|^4}{\ln|u_j|} < C|u_o|$, then

$$(**)\qquad |1 - T_j|^{\frac{1}{2}} = O\left(\frac{\ln|u_j|}{|u_j|^3}\right)$$

as $|u_o| \to \infty$.

We note for reference that the horizontal displacement $\delta(u)$ between the singular solution $U(r)$ and hyperbola $ur = -1$ satisfies $\delta(u) = O\left(\frac{1}{|u|^5}\right)$. Thus, the results (*) and (**) are consistent with our conjecture on the convergence of the free surfaces to that of the singular solution.

The form of the outer surface of a symmetric water drop suspended from a circular aperture is determined by the condition that the mean curvature of the surface is proportional to the distance below a horizontal reference plane. For points near which the surface can be described by a function $u(x)$ we obtain an equation [1]

$$(1) \qquad \frac{\partial}{\partial x_i}\left\{\frac{u_{x_i}}{\sqrt{1+|\nabla u|^2}}\right\} = -\varkappa u + \lambda$$

for the height $u(x)$ above the plane.
Here $\varkappa$ is a physical constant, $\varkappa > 0$ when the water lies above the surface, and λ is a Lagrange parameter, to be determined by the constraints.

In a specific problem the determination of λ may lead to technical difficulties. Formally, however, λ can be transformed out of (1) by adding a constant to u. In the present paper we intend to characterize all symmetric solutions for the case $\lambda = 0$. A solution corresponding to given λ can then be found in this family by transforming back.

We shall also introduce the (inessential and convenient) normalization $\varkappa = 1$. We then obtain, in terms of polar radius r, the equation

$$(2) \qquad \left(r\frac{u_r}{\sqrt{1+u_r^2}}\right)_r = -ru$$

for a symmetric two dimensional surface $u(r)$.

Not all surfaces that appear physically have a simple projection on a base plane, hence for a complete description the form (2) is overly restrictive. We obtain a more suitable (parametric) form of the problem if we introduce the arc length s along a vertical section of the surface interface, measured from the vertex $(0, u_0)$. We are led to the system

$$\frac{d\Psi}{ds} = -u - \frac{1}{r}\sin\Psi$$

$$\frac{du}{ds} = \sin\Psi \qquad (3)$$

$$\frac{dr}{ds} = \cos\Psi$$

where Ψ is the angle between a tangent to the section and the r-axis, measured counterclockwise from the positively directed axis to the tangent line.

From the point of view of general theory, one would expect a solution of (3) to be determined, at least locally, by the initial data

$$r(0) = 0\ ; \quad \Psi(0) = 0\ ; \quad u(0) = u_0\ ; \qquad (4)$$

however, the system (3) is singular at $s = 0$, and because of this the second condition in (4) is superfluous (cf the discussion in [4]).

The question of local existence has been studied by Lohnstein [5], who established the convergence of a formal power series expansion. Alternatively, one could adapt the Picard method, as used by Johnson and Perko [6] for the capillary problem, to the case studied here. One obtains locally, by these methods, a non-parametric solution u(r) of the equation (2), which we may write in the form

$$(r\sin\Psi)_r = -ru\ , \qquad (5)$$

corresponding to the (single) initial condition

$$u(0) = u_0 \qquad (6)$$

The circumstance that only one initial datum is required yields an important simplification for the problem of characterizing all solutions. It suffices to describe the one-parameter family determined by u_0, and it is this approach we adopt in the present work.

In general, the solution $u(r;u_0)$ determined in this way cannot be continued indefinitely as solution of (5). We shall show however that for any u_0, the function $u(r;u_0)$ can be continued as a parametric solution of (3) for all s, yielding a surface without singularities or self-intersections.

We shall characterize quantitatively the asymptotic form of the surface in the case $|u_0| \gg 0$, and we shall characterize qualitatively the global structure of all such surfaces.

The global behavior changes qualitatively when $|u_0|$ increases beyond a critical value. If $|u_0| \gg 0$, there is an initial range for s in which the surface looks like a succession of spheres centered on the u-axis with radius $\approx 2/|u|$. In all cases, the section can be expressed for large s in the form $u(r)$ and has an oscillatory behavior as $r \longrightarrow \infty$.

If $u_0 = 0$ the unique solution of (2) is given by $u \equiv 0$. We assume throughout this paper that $u_0 < 0$; the remaining case is obtained by a simple change of sign [(2)]. We are interested particularly in what happens when $u_0 \ll 0$. The resulting surfaces are then physically unstable under most conditions of everyday experience; however, the problem has an independent mathematical interest (one specific feature of which we indicate below) and probably also a physical interest for situations in which gravity forces are small compared with those of surface tension.

We have proved in [7] the existence of a particular singular solution of (2) that can be expressed in the form $U(r)$ in $0 < r < \delta$, and such that $U(r) \sim -\frac{1}{r}$ as $r \longrightarrow 0$. In [8] we have presented numerical evidence suggesting that the symmetric solutions discussed above tend uniformly to $U(r)$ in any fixed region $u > A > -\infty$. A particular consequence of the analysis in the present paper will be a proof of a preliminary form of that conjecture, namely we shall show in section VI that the solutions converge asymptotically into a neighborhood of U(r).

We remark that we know of few other studies of the problem from a general theoretical point of view [3]. To our knowledge the first attempt to characterize the shape of a water drop appears in Bashforth and Adams [10] in which a numerical procedure is developed to calculate the sectional form up to the first vertical point Thomson [11] used a geometrical method and was able to obtain a figure corresponding, in our notation, to $u_0 \approx -7$. Computational studies were greatly facilitated by development of high speed computers and related techniques, and particular cases have now been calculated with much larger $|u_0|$, see, e.g., Hida and Miura [12] and Concus and Finn [8]. Such calculations are suggestive and instructive, but they cannot provide the unifying insight of a general formal description. The present work is intended as an initial step toward that objective.

In this work we study the formal solutions of the equations and ignore the question of physical stability of the surfaces. With regard to this related matter the reader may wish to consult recent contributions by Pitts [13, 14] and by Hida and Miura [12], where also further references can be found.

The central difficulty in the general study of the solutions of (2) lies in the failure of the maximum principle. In the particular situation studied here, a residue of this principle remains, permitting us to compare the solutions with those of a simpler equation. This circumstance, in conjunction with elementary formal manipulation of the equation, provides the central tool in our investigation. We proceed in a succession of steps, most of which are elementary and immediate; when taken together, however, they yield the requisite characterization.

We remark that the comparison technique has proved effective also in other (related) contexts, and has led in particular to new information on the behavior of solutions of (2) near isolated singular points, see, e.g., [15].

The latter author wishes to thank J. Serrin and J. Spruck for a number of stimulating conversations.

I The case of small $|u_o|$.

We shall prove :

Theorem 1 : If, in the initial value problem (5, 6) there holds [4] $u_o \geq -2$, then the solution can be continued as a (nonparametric) solution of the equation

$$(2) \qquad \left(\frac{r u_r}{\sqrt{1+u_r^2}} \right)_r = -ru$$

for all $r > 0$. It has an infinity of zeros. For any two successive extrema r_a, r_b of $u(r)$ there holds $|u(r_b)| < |u(r_a)|$. Asymptotically as $u_o \to 0$ the first zero r_o is the first zero of the Bessel function $J_o(r)$, $r_o \sim 2.405$.

We study first the portion of the trajectory preceding the first zero, and we note that (2) is equivalent to (5) on any interval on which $|u_r| < \infty$.

Ii : Let $u(r)$ satisfy (5) in $0 < r < R$ and (6) at $r = 0$. Then [5] $\sin \Psi(0) = 0$.

Proof: Integrating (5) from $\epsilon > 0$ to r, we find

$$r \sin \Psi - \epsilon \sin \Psi(\epsilon) = -\int_\epsilon^r \rho u \, d\rho ,$$

hence, using (6),

$$(7) \qquad \sin \Psi = \frac{u_r}{\sqrt{1+u_r^2}} = -\frac{1}{r} \int_0^r \rho \, u \, d\rho$$

from which we conclude $\lim_{r \to 0} u_r(r) = 0$. Hence there exists

$$u_r(0) = \lim_{r \to 0} \frac{u(r) - u_o}{r} = \lim_{r \to 0} \frac{1}{r} \int_0^r u_r(\tau)\, d\tau = 0 .$$

I ii <u>Let</u> $u(r)$ <u>satisfy</u> (5) <u>in</u> $0 < r < R$ <u>and</u> (6) <u>at</u> $r = 0$. <u>If</u> $u(r) < 0$ <u>in</u> $0 < r < R$, <u>then</u> $\sin \Psi > 0$ <u>in this interval.</u>

The proof is contained in (7).

It follows in particular that $u(r) \longrightarrow u_R \leq 0$ as $r \longrightarrow R$, that $\sin \Psi_R = \lim_{r \to R} \sin \Psi(r)$ exists, and that

$$0 < \sin \Psi_R = - \frac{1}{R} \int_0^R \rho\, u(\rho)\, d\rho \leq 1.$$

We conclude also that if the solution curve does not cross the hyperbola $ru = -1$, then $\sin \Psi_R < 1$. The following assertion covers as well the case of solution curves crossing that hyperbola.

I iii : <u>Under the hypotheses of</u> I ii, <u>if in addition</u> $u_o \geq -2$, <u>then</u> $0 < \sin \Psi < 1$ <u>in</u> $0 < r \leq R$.

<u>Proof</u>: Consider the relation

$$(8) \qquad \varkappa_m + \varkappa_\ell \equiv \frac{\sin \Psi}{r} + (\sin \Psi)_r = -u ,$$

the left side of which splits the mean curvature of the rotation surface defined by $u(r)$ into a sum of latitudinal ($\varkappa_\ell$) and meridianal ($\varkappa_m$) sectional curvatures. We note by I ii that $u(r)$ is increasing in $0 < r < R$; thus

$$(9) \qquad \frac{\sin \Psi}{r} = - \frac{1}{r^2} \int_0^R \rho\, u(\rho)\, d\rho > - \frac{u(r)}{2}$$

in that interval. Integrating (8) with respect to u yields, using (9),

$$\cos \Psi_R > 1 - \frac{1}{4} (u_o^2 - u_R^2)$$

which contains the assertion. We infer now from the general existence theorem, applied at $r = R$, that the solution curve either can be continued upward until it crosses the r-axis, or else it tends asymptotically to this axis with increasing r. We may however exclude the latter possibility.

Iiv : If $u(r) < 0$ in $a \leq r < R < \infty$, then $R < a \exp\left\{-\dfrac{u_a}{a \sin \Psi_a}\right\}$.

Proof: From (5) we find $r \sin \Psi > a \sin \Psi_a$ in $a \leq r < R$. By Iii, $\sin \Psi_a > 0$. Thus,

$$\frac{du}{dr} = \tan \Psi > \sin \Psi > \frac{a \sin \Psi_a}{r}$$

and the result follows on an integration.

We have thus established that if $u_o \geq -2$ the solution curve is in its initial trajectory monotonically increasing and can be continued till it crosses the r-axis at a point $r = a_1$. To study the further trajectory, we observe that the curve can be continued at least locally across the axis as a solution of (5), and we compare its inclination at a given height h with the inclination of the initial branch at an equal negative height.

Iv : If the curve can be continued monotonically to a height h above the r-axis, then its inclination at this height is smaller than the inclination of the initial branch at the height $-h$, that is,

$$\left.\frac{du}{dr}\right]_h < \left.\frac{du}{dr}\right]_{-h} .$$

Proof: We integrate (8) with respect to u between the height $-h$ and h, obtaining

$$\cos \Psi\Big]_h - \cos \Psi\Big]_{-h} = \int_{-h}^{h} \frac{\sin \Psi}{r}\, du > 0 .$$

Ivi : Under the conditions of Iiv, the curve is strictly convex downward when $u > 0$, and $u_{rr} < -u$.

Proof: From (8),

$$(\sin\Psi)_r = \frac{u_{rr}}{(1+u_r^2)^{3/2}} = -u - \frac{\sin\Psi}{r} < -u .$$

From Iv and Ivi we find

Ivii : The curve can be continued to a maximum height $h_1 < |u_o|$ at a point $r = m_1 > a_1$, at which point $\sin\Psi(m_1) = 0$.

We now proceed as above, comparing inclinations at corresponding heights until the curve crosses the r-axis a second time, then comparing inclinations as in Iv, and so on. We obtain the qualitative picture indicated in Theorem 1, of a curve oscillating about the r-axis with successively decreasing extrema (see Fig 1). We note also the additional information, yielded by the method:

Iviii : All inflections of the curve occur on (monotone) curve segments approaching the r-axis, in the sense of increasing s. At any two successive points α, β, at which $|u_\alpha| = |u_\beta|$, there holds

$$\left|\frac{du}{dr}\right|_\alpha > \left|\frac{du}{dr}\right|_\beta .$$

To prove the final statement of Theorem 1, we note by (7), Iii and Iviii that $|u_r(r;u_o)|$ tends uniformly to zero with u_o; thus the function $v(r;u_o) = u_o^{-1}\, u(r;u_o)$ tends uniformly to $J_o(r)$ as $u_o \longrightarrow 0$.

II Large $|u_o|$; initial arc

If $u_o << 0$ the above reasoning on the behavior of the initial segment fails, and so do the results.

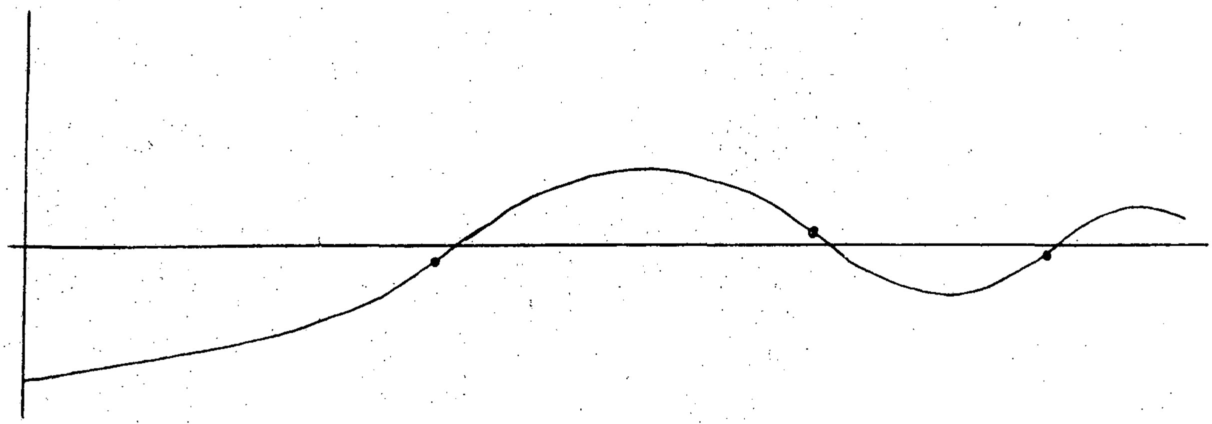


Figure 1. The case $u_o \geq -2$; inflections

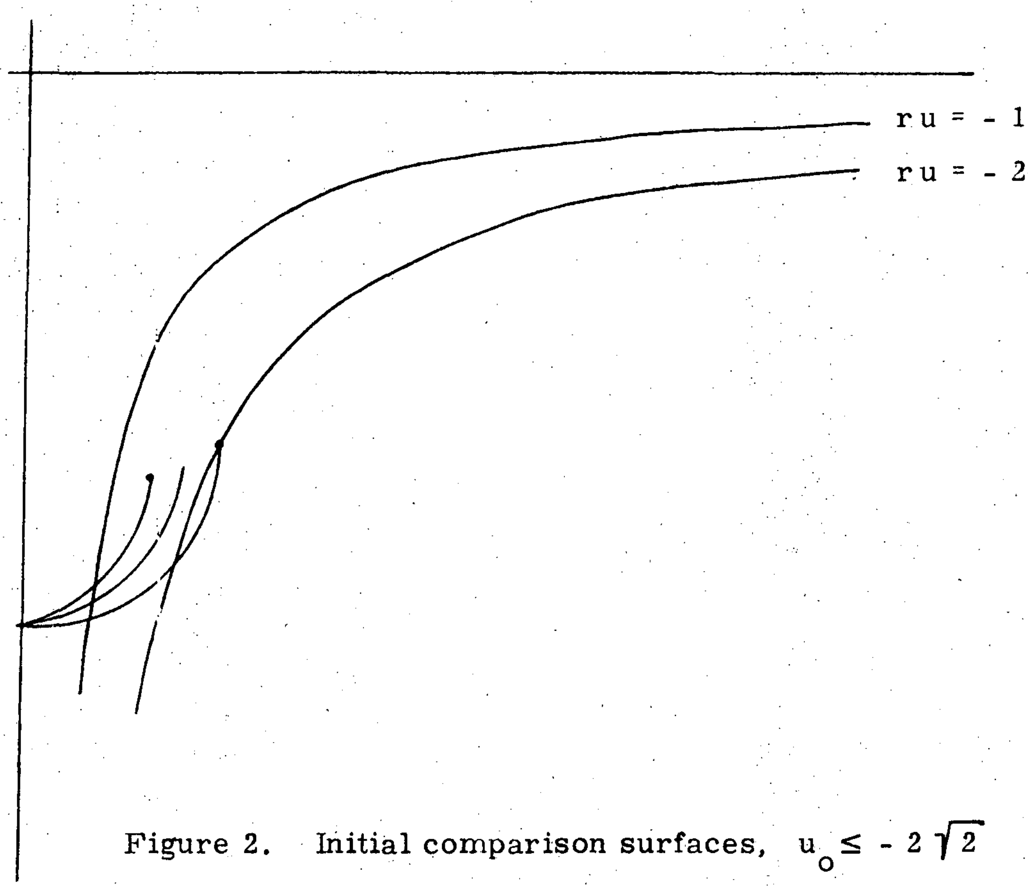


Figure 2. Initial comparison surfaces, $u_o \leq -2\sqrt{2}$

Theorem 2: If[4] $u_o \leq -2\sqrt{2}$, there exists a value r, beyond which $u(r)$ cannot be continued as a solution of (5). As $r \nearrow r_1$, $\sin\Psi \nearrow 1$.

The proof could proceed by a direct study of the equation, as in section I. We obtain more precise results and also develop techniques that will be needed later if we proceed instead via an obvious comparison principle.

II i : Let $v^{(1)}(r)$, $v^{(2)}(r)$ be functions defined in $a \leq r \leq b$ and such that $(r \sin\Psi^{(1)})_r \geq (r\sin\Psi^{(2)})_r$. Suppose $\sin\Psi^{(1)}(a) \geq \sin\Psi^{(2)}(a)$. Then $\sin\Psi^{(1)}(b) \geq \sin\Psi^{(2)}(b)$, and equality holds if and only if $v^{(1)} \equiv v^{(2)} + \text{const.}$ on $a \leq r \leq b$.

The interest in II i lies in the fact that $\frac{1}{r}(r\sin\Psi)_r$ is exactly twice the mean curvature of the rotation surface defined by $u(r)$, and this circumstance facilitates the choice of comparison surfaces. In the present case we choose as initial comparison surface the sphere of constant mean curvature $-u_o/2$, with center at the point $(r, u) = (0, u_o - 2/u_o)$. Thus, if $v(r)$ describes a vertical section of the sphere, there holds $u(0) = v(0)$, $u(r) < v(r)$ in the interval $0 < r \leq -\frac{2}{u_o}$ (see Fig 2). Using I ii, we find:

II ii : The solution $u(r)$ of (5, 6) can be continued at least until $r = -2/u_o$, and $\sin\Psi(r) < -\dfrac{u_o r}{2}$.

We need also:

II iii : A solution $u(r)$ of (5) admits no inflections in the region $r u < -1$.

Proof: From (8) follows $ru + \sin\Psi = 0$ at any inflection.

Thus, Ψ must continue to increase until either a vertical point is reached or the curve meets again the hyperbola $ru = -1$. Integrating (8) with respect to u and using II ii yields

$$1 - \cos \Psi > - \frac{1}{2} (u^2 - u_o u)$$

Noting that for $r > - 2/u_o$ there holds

$$r \frac{u_o}{2} [1 + \sqrt{1 - 8/u_o^2}] \leq - 1$$

we conclude that a vertical slope appears at a value

(10) $$u_1 < \frac{u_o}{2} (1 + \sqrt{1 - 8/u_o^2})$$

which completes the proof of Theorem 2.

We may use a similar procedure to estimate the value r_1. We note that if $w(r)$ describes a vertical section of the sphere of constant mean curvature

(11) $$\frac{1}{\beta} = - \frac{u_o}{4} (1 + \sqrt{1 - 8/ u_o^2})$$

with center at $(0, u_o + \beta)$, then there holds $u(0) = w(0)$, and by II i $u'(r) > w'(r)$ on any interval $0 < r \leq R$ along which $\beta u \leq - 2$. This condition is however satisfied at $u = u_1$ by (10), hence on the entire arc $u_o < u \leq u_1$. We conclude $u(r) > w(r)$ until the first vertical occurs at $r = r_1 < \beta$.

We note that at $r = \beta$, where $w'(r) = \infty$, the circle $w(r)$ intersects the hyperbola $r u = - 2$.

From II iii we conclude the initial solution curve is convex in the region $r u < - 1$. This property holds in fact for the entire arc; on the segment of $u(r)$ joining the initial point to the point (r_c, u_c) on the hyperbola $r u = - 1$ we obtain from (7, 8, I ii), using the comparison circle $v(r)$,

$$\varkappa_m = (\sin \Psi)_r = - u - \frac{\sin \Psi}{r} > - v + \frac{u_o}{2} > - v(r_c) + \frac{u_o}{2} > 0.$$

The last relation holds whenever $u_o \leq - \sqrt{2}$, which is the condition that $v(r)$ and the hyperbola $r u = - 1$, $u < 0$ intersect. We conclude also from I ii and the relation

$$(12) \qquad \sin\Psi = -\frac{ru}{2} + \frac{1}{2r}\int_0^r \rho^2 u_\rho \, d\rho$$

that $ru > -2$ on the arc considered.

It turns out the sectional curvatures $\varkappa_\ell$ and $\varkappa_m$ are both monotone decreasing on the initial arc. We have

$$(13) \qquad \frac{d}{dr}\varkappa_\ell = \frac{d}{dr}\frac{\sin\Psi}{r} = \frac{2}{r^3}\int_0^r \rho\, u\, du - \frac{u}{r} = \frac{1}{r}\left(-2\frac{\sin\Psi}{r} - u\right) < 0$$

by (8, 12). Also, we have from (7, 8, I ii)

$$(14) \qquad \frac{d}{dr}\varkappa_m = (\sin\Psi)_{rr} = -u_r + \frac{1}{r}\left(2\frac{\sin\Psi}{r} + u\right)$$

$$< -u_r + \frac{1}{r}(u - u_o) = -\int_0^r \rho\, u_{\rho\rho}\, d\rho < 0$$

by the convexity of u.

At the initial point $(0, u_o)$ there holds $\varkappa_\ell = \varkappa_m = -u_o/2$. From (8, 12, I ii) follows for $r > 0$ on the initial arc

$$(15) \qquad \varkappa_m < -u/2 < \varkappa_\ell$$

From (7, 8) we have also

$$(16) \qquad \varkappa_m = -u - \frac{\sin\Psi}{r} > -u + \frac{u_o}{2} \quad .$$

The inequality (15) implies $\varkappa_m < -u_o/2$, which is the meridional curvature of the comparison surface $v(r)$. Comparing the surface $u(r)$ with $v(r)$ at corresponding values of u and applying II ii now yields

$$(17) \qquad u_1 > v\left(-\frac{2}{r_o}\right) = u_o - \frac{2}{r_o}$$

(see Fig. 2).

We summarize the above results:

Theorem 3: Under the condition of Theorem 2, the initial arc of the solution curve, from (r_o, u_o) to (r_1, u_1), is convex, with sectional curvatures $\varkappa_m$, $\varkappa_\ell$ decreasing and satisfying $\varkappa_m < -u/2 < \varkappa_\ell$ in $r_o < r \leq r_1$. There holds (4)

(18)
$$- \frac{2}{u_o} < r_1 < - \frac{u_o}{2} \left(1 - \sqrt{1 - \frac{8}{u_o^2}} \right)$$
$$u_o - \frac{2}{u_o} < u_1 < u_o - \frac{u_o}{2} \left(1 - \sqrt{1 - \frac{8}{u_o^2}} \right)$$

For $0 < r \leq -2/u_o$ the arc lies below the comparison circle $v(r)$ and has smaller curvature, and for $u_o > u \geq u_o + \beta$ the arc lies above the comparison circle $w(r)$ (see Fig 2).

Further remark: The hypothesis $u_o \leq -2\sqrt{2}$ of Theorem 2 could be sharpened by using the comparison surfaces $v(r)$ and $w(r)$ in (7) and iterating. A direct numerical integration of (7) yields [16] $u_o \approx -2.5678$ as the value for which a vertical first appears. We find immediately :

II iv : Let u_{oc} be the largest value of u_o for which a vertical point appears. If $u_o = u_{oc}$ the vertical occurs at the second intersection of the solution curve with the hyperbola $ru = -1$, and is an inflection point for the solution curve (see Fig 3).

If $u_o \leq -5$, the upper bounds in (18) can be expressed more simply, yielding

(19)
$$\frac{-2}{u_o} < r_1 < - \frac{2}{u_o} - \frac{5}{u_o^3}$$
$$u_o - \frac{2}{u_o} < u_1 < u_o - \frac{2}{u_o} - \frac{5}{u_o^3}$$

These bounds could also be improved by iteration, starting with the

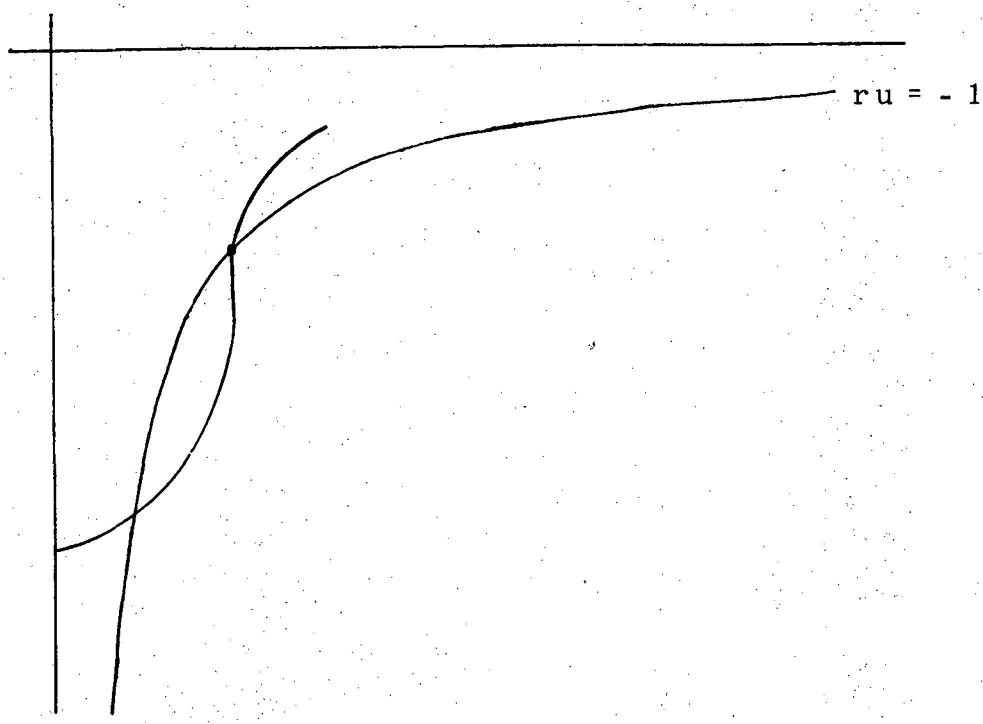


Figure 3. The case $u_o = u_{oc}$

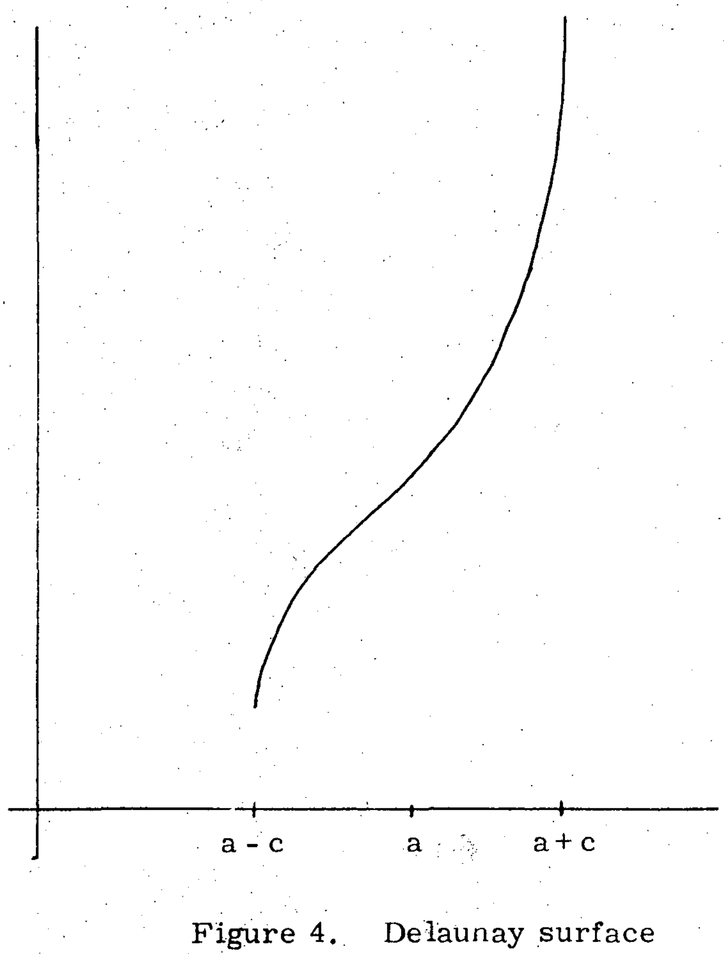


Figure 4. Delaunay surface

comparison surfaces $v(r)$ and $w(r)$. We note for reference that the asymptotic series obtained in [16] by formal perturbation expansion yields, for the normalization used here,

$$r_1 = -\frac{2}{u_o} - \frac{4}{3u_o^3} + O(u_o^{-5})$$

(20)

$$u_1 = u_o - \frac{2}{u_o} - \frac{4 + 8 \ln 2}{3 u_o^3} + O(u_o^{-5})$$

as $u_o \longrightarrow -\infty$.

III Very large $|u_o|$

If $u_o \leq -2\sqrt{2}$ then $u(r)$ cannot be continued beyond r_1 as a solution of (2). The curve can, however, be continued as a solution of the parametric system (3) as long as r remains different from zero. We study now the behavior of this family of solutions in terms of the parameter u_o, asymptotically as $u_o \longrightarrow -\infty$. We base the discussion principally on II i; to do so, we introduce as comparison functions the sections of rotation surfaces generated by the roulades of an ellipse. The following result is due to Delaunay [17]:

Let an ellipse of major axis $2a$ and distance $2c$ between focal points, roll rigidly on an axis without slipping. Let $\mathcal{C}$ be the curve swept out by one of the focal points. Then the surface generated by rotating $\mathcal{C}$ about the axis has constant mean curvature $H = (2a)^{-1}$.

We note that $\mathcal{C}$ is periodic with half-period τ satisfying $2a < \tau < \pi a$, and that each half-period can be represented in the interval $a - c \leq r \leq a + c$ by a single valued function $v(r)$ for which the equation

(21) $$(r \sin \psi)_r = 1/a$$

holds, and for which $\sin \psi = 1$ at the two end points (see Fig 4).

We proceed step by step :

The procedure of II shows that an infinite slope first appears at (r_1, u_1), with bounds on (r_1, u_1) given by (18). The system (3) is non-singular at (r_1, u_1), hence the curve can be continued beyond this point as a solution of (3, 4). From (16) we find at (r_1, u_1)

$$\varkappa_m > -u_1 + \frac{u_o}{2} > \frac{u_o}{2} + \frac{2}{u_o} > 0$$

so the curve turns back toward the u-axis, and can be described again (locally) as a solution of (5). We compare it with a roulade $v^{(1)}(r)$ whose mean curvature is $-\frac{u_1}{2}$ and for which $a_1 + c_1 = r_1$ (see Fig 3). Since $v_r^{(1)}(r_1) = -\infty$, II i yields $u_r < v_r^{(1)}$, hence $u(r) > v^{(1)}(r)$ as long as the continuation of both u and $v^{(1)}$ as single valued function is possible.

The curve $v^{(1)}(r)$ can be continued toward the u-axis only until the point $(a_1 - c_1, u_1 + \tau_1)$, with $a_1 - c_1 = -\frac{2}{u_1} - r_1 > 0$; at this point the slope is again infinite. It follows there is a value $r_2 > -\frac{2}{u_1} - r_1$ beyond which this branch of the solution curve cannot be continued as a single valued function.

From the geometrical interpretation of τ_1 as the half-circumference of an ellipse with major axis $2a_1 = -2/u_1$ and focal length $c_1 = r_1 - a_1$, one finds that for large $|u_o|$,

$$(22) \qquad \tau_1 = -\frac{2}{u_1} + \alpha_1 \frac{\ln|u_1|}{|u_1|}, \qquad \alpha_1 = -\frac{16}{3} + O\left(\frac{1}{\ln|u_1|}\right).$$

Let us estimate r_2 from above. To do so, we compare $u(r)$ with a roulade $\hat{v}_1(r)$, which is determined by the conditions

$$(23) \qquad \hat{a}_1 = -\frac{1}{u_1 + \hat{\tau}_1}$$

$$\hat{a}_1 + \hat{c}_1 = r_1$$

(23)

$$\hat{\tau}_1 = \int_0^{\pi} \sqrt{a^2 - c^2 \cos^2\theta}\, d\theta$$

A formal estimate shows such a roulade exists if $u_1 < -2\sqrt{\pi}$.

The conditions (23) are chosen so that the roulade can be placed with its lower vertical point at (r_1, u_1), (see Fig 5), and so that in that configuration its mean curvature will be exactly the one determined from the right side of (5) by the upper vertical. Applying II i we obtain $u_r > \hat{v}_r^{(1)}(r)$, $u(r) < \hat{v}^{(1)}(r)$ for all $r < r_1$ for which $u(r) < u_1 + \hat{\tau}_1$. This condition clearly holds for r near r_1; since $\hat{v}^{(1)}(r) < u_j + \hat{\tau}_1$, we conclude it holds on the entire interval $\hat{a}_1 - \hat{c}_1 < r < r_1$, thus

$$0 > v_r^{(1)}(r) > u_r(r) > \hat{v}_r^{(1)}(r) > -\infty$$

on this interval, and hence the solution can be continued to the left of r_1, at least until the value

(24)
$$r_2 < -\frac{2}{u_1 + \hat{\tau}_1} - r_1 = \beta_2$$

For large $|u_o|$ we find

(25)
$$\hat{\tau}_1 = -\frac{2}{u_1} + \hat{\alpha}_1 \frac{\ln|u_1|}{|u_1|^3}$$

with

(26)
$$\hat{\alpha}_1 = -\frac{40}{3} + O\left(\frac{1}{\ln|u_1|}\right).$$

Thus

(27)
$$r_2 < -\frac{2}{u_1} - \frac{\alpha_2}{u_1^3} - r_1$$

with

(28)
$$\alpha_2 = 4 + O(u_1^{-2} \ln|u_1|).$$

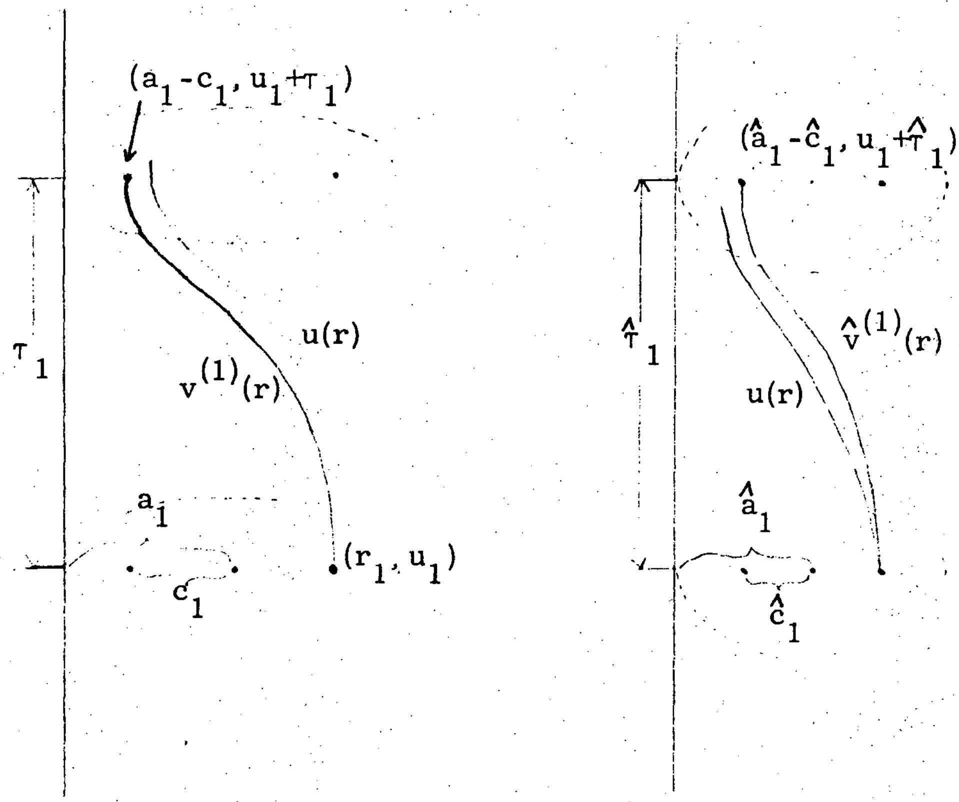


Figure 5. Comparison with roulades

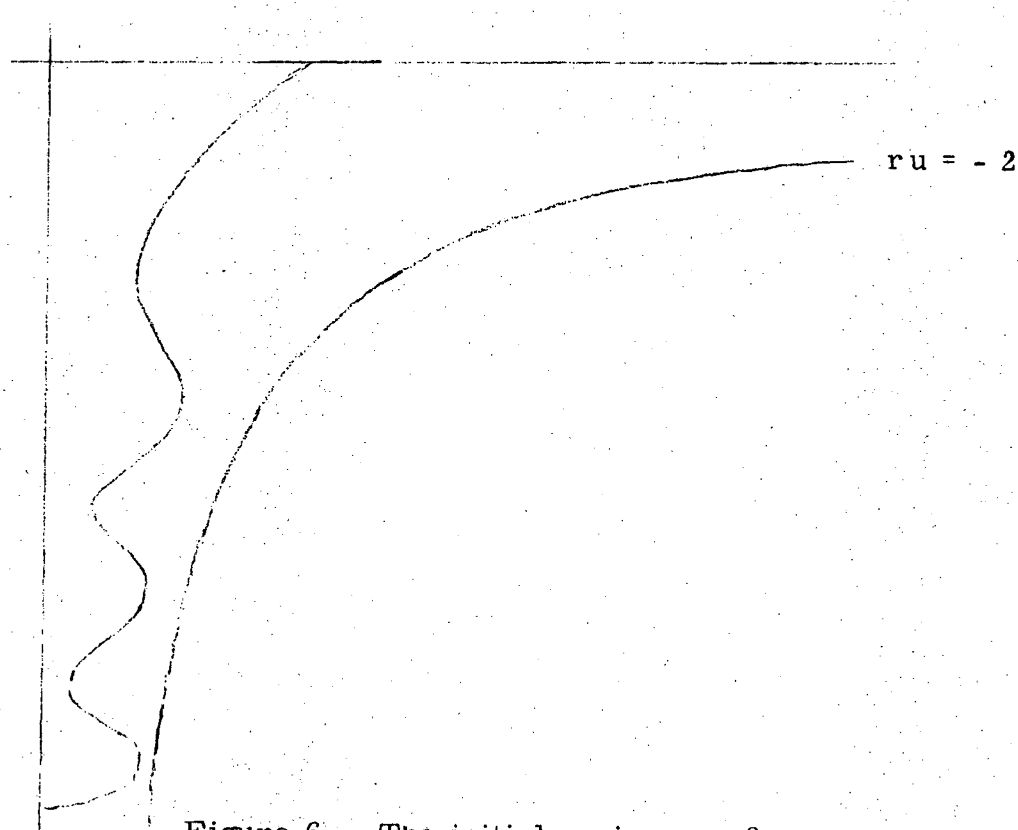


Figure 6. The initial region $u < 0$

We now proceed, essentially, as in the proof of Theorem 2. We note

$$\sin \Psi > \sin \Psi^{(1)} = - \frac{r u_1}{2} + \frac{r_1}{r}\left(1 + \frac{r_1 u_1}{2} \right);$$

thus from (24-28) we find for $r < \beta_2$,

$$\frac{\sin \Psi}{r} > - \frac{3}{50} u_1^{\ 3} + O(u_1^{-5} \ln|u_1|)$$

We integrate (8) in u from $u(\beta_2)$; using that $\cos \Psi < 0$ until a vertical is reached, and that

$$\cos \Psi(\beta_2) > \cos \Psi^{(1)}(\beta_2) = - \frac{\sqrt{21}}{5} + O(u_1^{-2} \ln|u_1|)$$

we are led to a contradiction unless the curve becomes vertical before u has increased by a value $-16 u_1^{-3}$. That is, a vertical must appear at a value

(29) $$u_2 < u_1 + \hat{\tau}_1 - 16 u_1^{-3}$$

The solution curve then turns back from the axis at (r_2, u_2) and initiates a further branch.

We summarize these results :

<u>Theorem 4</u> : <u>From</u> (r_1, u_1) <u>the solution curve continues backwards towards the</u> u-<u>axis until a second vertical is reached</u>, <u>at a point</u> (r_2, u_2) <u>with</u>

(30)
$$- \frac{2}{u_1} - r_1 < r_2 < - \frac{2}{u_1 + \hat{\tau}_1} - r_1 = \beta_2$$
$$v^{(1)}(\beta_2) < u_2 < u_1 + \hat{\tau}_1 - 16 u_1^{-3}$$

<u>In the interval</u> $r_2 < r < r_1$ <u>there holds</u> $u_r < v_r^{(1)}$, $u > v^{(1)}$;
<u>in the interval</u> $\beta_2 < r < r_1$ <u>there holds</u> $u_r > \hat{v}_r^{(1)}$, $u < \hat{v}_r^{(1)}$.

We note in particular that the horizontal distance of the second vertical from the axis exceeds that of the first vertical from the hyperbola $ru = -2$.

III i : <u>There is exactly one inflection between</u> (r_1, u_1) <u>and</u> (r_2, u_2).

<u>Proof</u>: Clearly, at least one inflection appears. Using (8), we find

$$(ru + \sin \Psi)_r = \left(\frac{r}{\cos \Psi} - \frac{1}{r}\right) \sin \Psi < 0$$

on the arc. Hence there is at most one inflection.

We indicate briefly one further step in the procedure. We construct a roulade $v^{(2)}(r)$ passing through (r_2, u_2), with major axis $2a_2 = -\frac{2}{u_2}$, and a second roulade $\hat{v}^{(2)}(r)$ with a property analogous to that introduced for $\hat{v}^{(1)}(r)$. Then there holds $\hat{v}_r^{(2)} < u_r < v_r^{(2)}$, $\hat{v}^{(2)} < u < v^{(2)}$ in the intervals for which the comparison makes sense, and (as before) still another point (r_3, u_3) is found such that $\sin \Psi (r_3) = 1$. The procedure can be continued as long as the values of $|u(r)|$ remain sufficiently large to justify the indicated steps.

We find easily:

III ii : <u>The successive horizontal distances of the vertical points, from the axis and from the hyperbola, increase monotonically.</u>

III iii : <u>On each arc segment returning from the hyperbola to the axis there is exactly one inflection. The same statement holds on the remaining arc segments for sufficiently large</u> $|u|$.

<u>Theorem 5</u>: <u>In the initial region</u> $u < 0$, <u>the entire curve is bounded (strictly) between the</u> u-<u>axis and the hyperbola</u> $ru = -2$ (see Fig 6). <u>In this region, the curve can be represented by a single valued function</u> $r = r(u)$, <u>with</u> $|r'(u)| < \infty$.

<u>Proof</u>: Since Theorem 4 and III i apply to any returning arc, we conclude the curve cannot contact the u-axis. The relation (9) shows

the curve does not meet the hyperbola on the initial arc starting from $(0, u_o)$. To show this property for any forward arc, we integrate (5) on such an arc from a vertical point (r_{2j}, u_{2j}), obtaining

$$r \sin \Psi - r_{2j} = \frac{r_{2j}^2 u_{2j} - r^2 u}{2} + \frac{1}{2} \int_{r_{2j}}^{r} \rho^2 u' \, d\rho$$

$$> - \frac{r_{2j}}{2} - \frac{r^2 u}{2}$$

from which follows $ru > -2$ on this arc. The same inequality shows $\sin \Psi > 0$ on this arc; an analogous integration establishes the same property on a returning arc.

IV Global behavior

The discussion thus far shows that the solution curve can be continued upward without self-intersections until it crosses the r-axis. For by I iii an outward branch must either achieve a vertical or cross that axis, and the comparison method of II yields readily that a returning branch has the same property. There are no horizontal points, by Theorem 5.

We show here that a returning branch cannot cross the r-axis. Precisely :

IV i : Let $r = a_1$ be the first point at which the solution curve meets the r-axis. Then $0 < u'(a_1) < \infty$.

Suppose $u'(a_1) < 0$, or equivalently, $\cos \Psi_1 < 0$. The curve could then be continued backward into the negative u-plane till a first vertical (r_α, u_α) (see Fig 7), at which, by Theorem 5,

(31) $$r_\alpha u_\alpha > -2 .$$

We write (5) in the form

$$\frac{\sin \Psi}{r} + (\sin \Psi)_r = -u$$

0 0 0 0 4 5 0 0 2 5 5

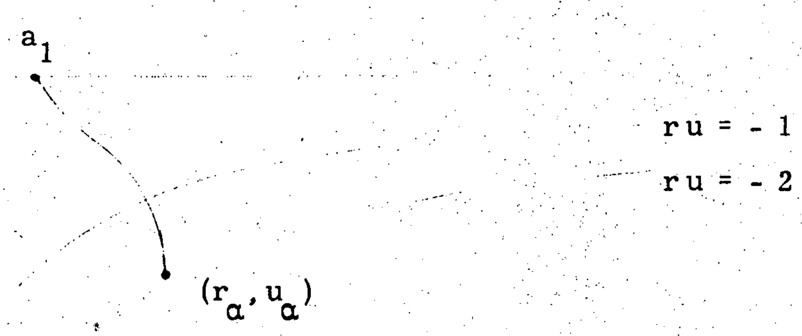


Figure 7. Proof of IV i

and integrate with respect to u, from u_α to 0 :

$$(32) \qquad \int_{u_\alpha}^{0} \frac{\sin \Psi}{r}\, du = \cos \Psi_1 + \frac{1}{2} u_\alpha^2$$

To evaluate the left side of (32) we integrate (5) in r between r and r_α :

$$r_\alpha - r \sin \Psi = - \int_r^{r_\alpha} \rho\, u\, d\rho < \frac{r^2 - r_\alpha^2}{2} u_\alpha$$
$$< \frac{r^2 u_\alpha}{2} + r_\alpha$$

by (31). Thus

$$(33) \qquad \frac{\sin \alpha}{r} > - \frac{u_\alpha}{2}$$

on the entire arc. Placing (33) into (32) yields $\cos \Psi_1 > 0$, contradicting the assumption.

Now observe from (8) that at the crossing point a_1, the meridional curvature is negative; thus, if $\cos \Psi_1 = 0$ there would again be a backward branch from a_1 into the negative u-plane, and we obtain a contradiction as above.

From IV i one sees immediately that the proof of Theorem 1 applies without change to the region $r \geq a_1$. We conclude from Theorem 1 and 5:

<u>Theorem 6:</u> <u>The solution of the parametric system</u> (3, 4) <u>defined by the data</u> u_o <u>can be continued indefinitely as a non-self-intersecting curve</u>. <u>It has the form indicated in</u> Figs 1,8, 9, 10 .

V Maximum diameter

We define the <u>diameter</u> of a (symmetric) water drop as the largest diameter of all circular sections $u = u_j$, at which the bounding surface is vertical.

From Theorem 1, 5, 6 we see that each drop has a well defined diameter. It is less obvious that there is a universal upper bound for the diameters of all possible drops, independent of u_o.

Theorem 7 : Let $\delta \sim 2.47341$ be the unique positive root of the equation

(34) $$r^3 - 3^{3/2} r - 3^{3/4} = 0$$

Then 2δ exceeds the diameter of any solution of (3, 4).

We base the proof on a lemma, which also has an independent interest.

V i : Let $u(r)$ represent a solution curve passing through (a, u_a) with $-1 \leq a u_a < 0$, and such that

(35) $$a \sin \Psi_a \geq a/2$$

Suppose $u(r) < 0$ in $a \leq r < R$. Then $\sin \Psi > 0$ on this arc segment. If the curve meets the hyperbola $ru = -1$ in a point (c, u_c) with $a < c < R$, then $c < 3^{1/4}$, and $\sin \Psi_c > 1/2$.

Proof: We integrate (5) between α and r, obtaining

(36) $$r \sin \Psi_r - \alpha \sin \Psi_\alpha = \frac{1}{2} \left(\alpha^2 u_\alpha - r^2 u(r) \right) + \frac{1}{2} \int_\alpha^r \rho^2 u'(\rho)\, d\rho$$

from which, if $\alpha = a$,

(37) $$r \sin \Psi_r \geq -\frac{1}{2} r^2 u(r) + \frac{1}{2} \int_a^r \rho^2 u'(\rho)\, d\rho .$$

For r sufficiently near a, there holds $\sin \Psi > 0$. Thus, if $\sin \Psi$ were to vanish at any points interior to $a \leq r < R$, there would be a minimum $r = r_\gamma > a$ at which this occurs. But (37) would then imply

$$0 = r_\gamma \sin \Psi_\gamma > \frac{1}{2} \int_a^{r_\gamma} \rho^2 u'(\rho)\, d\rho > 0,$$

a contradiction. Thus, $\sin \Psi > 0$ on $a \leq r \leq c$, and hence $u'(\rho) > 0$ on this interval. Setting now $r = c$ in (37) yields

(38) $\sin \Psi_c > -\frac{1}{2} cu(c) = \frac{1}{2}$.

Finally, we note that at $r = c$ the inclination of the solution curve cannot exceed that of the hyperbola. Thus, $\sin \Psi_c < \frac{1}{\sqrt{1 + c^4}}$, and $c^4 < 3$ follows from (38).

We proceed to prove Theorem 7. For any given u_o, the maximum width is attained at a point (r_{2j+1}, u_{2j+1}) with $-1 \geq r_{2j+1} u_{2j+1} > -2$, $j \geq 0$ (see section III). At the preceding point (r_{2j}, u_{2j}) there holds either $r_{2j} = 0$ (if $j = 0$), or else $\sin \Psi_{2j} = 1$. In either event, (35) holds with $a = r_{2j}$. Also $-1 < r_{2j} u_{2j} < 0$, and thus the curve crosses the hyperbola $ru = -1$ at a point (c, u_c), $r_{2j} < c < r_{2j+1}$. Setting $\alpha = c$, $r = r_{2j+1}$ in (36) and applying V i yields, using II iii,

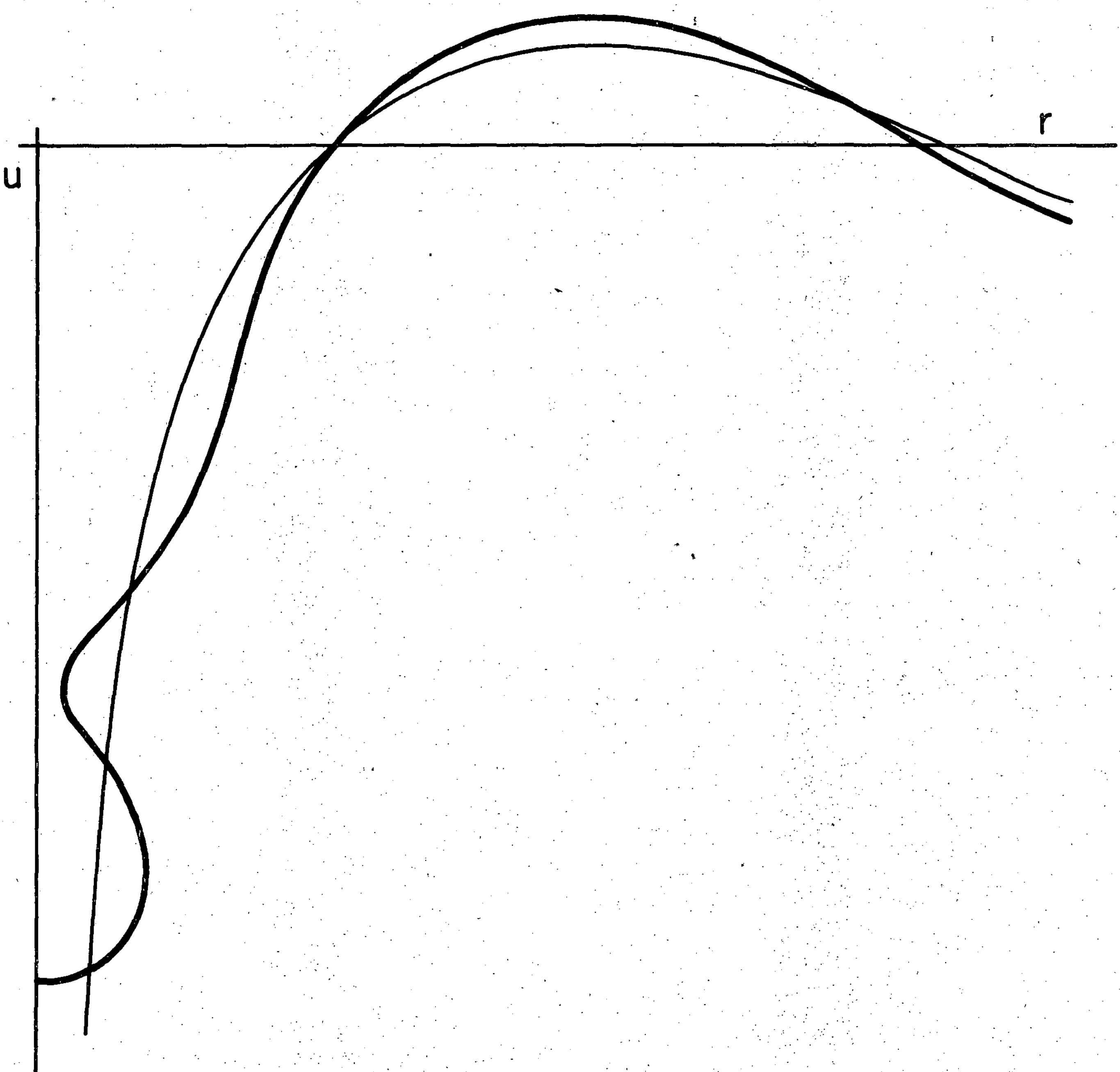
(39) $r_{2j+1}^3 - 3^{3/2} r_{2j+1} - 3^{3/4} < 0.$

The (single) positive solution of (34) exceeds any solution of (39). Since j is arbitrary, we conclude 2δ exceeds the diameter of any drop.

VI Convergence to the singular solution

The solutions discussed in this paper are apparently related to a singular solution $U(r)$ of (2), whose existence we have proved in [7]. The function $U(r)$ is defined in a deleted neighborhood of $r = 0$, and there holds asymptotically $U(r) \sim -\frac{1}{r}$, as $r \longrightarrow 0$. We have conjectured that in any interval $0 < a \leq r \leq b < \infty$, the solutions of (3, 4) admit a single valued representation $u(r; u_o)$ and converge uniformly to $U(r)$, as $u_o \longrightarrow \infty$. Figures 8, 9, 10 show the results of calculations supporting the conjecture.

In this section we derive a strengthened form of the left hand inequality in (30), and we prove as a consequence a preliminary (asymptotic) form of the conjecture. To do so, we iterate the comparison procedure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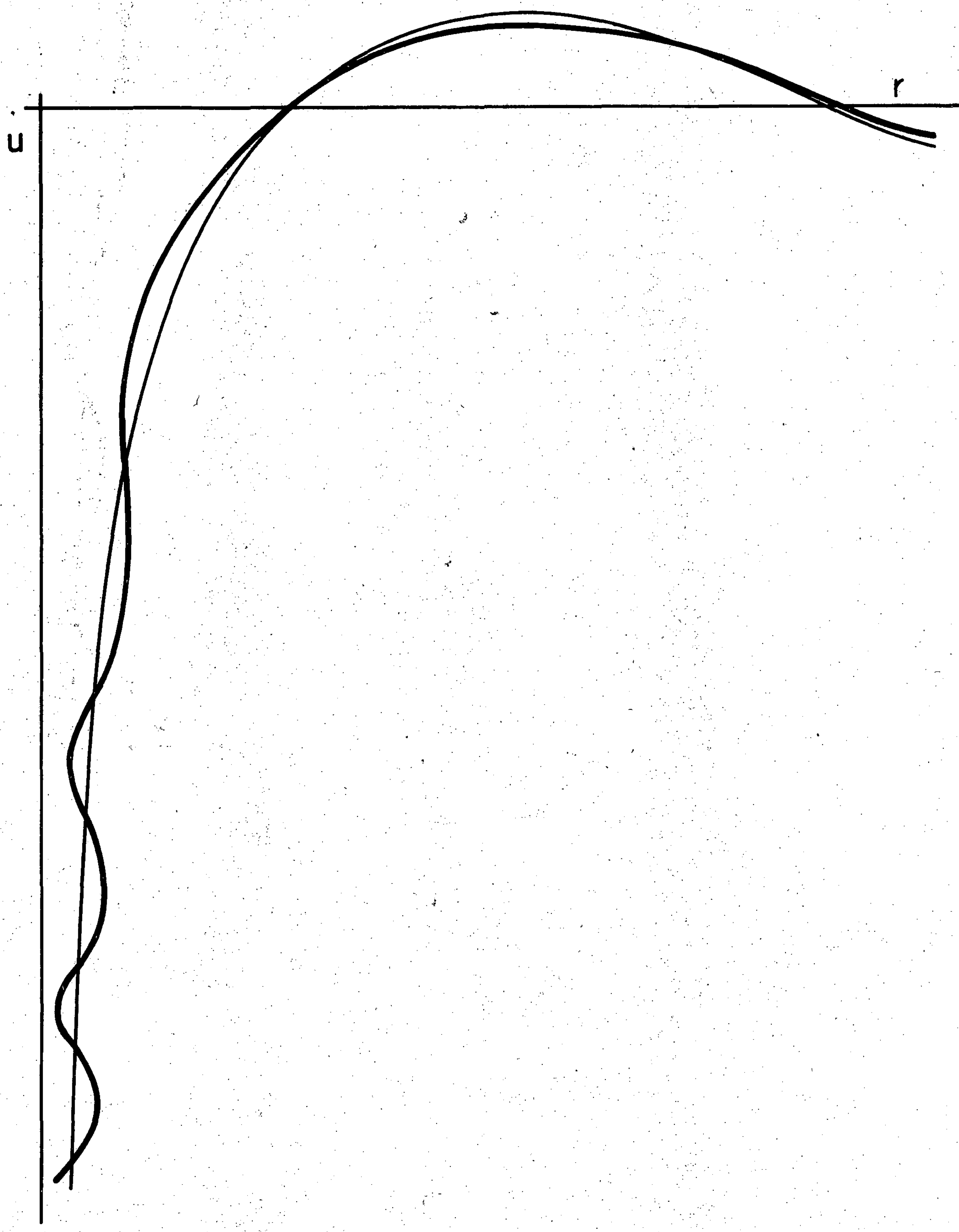

Figure 9. $u_o = -8$; singular solution

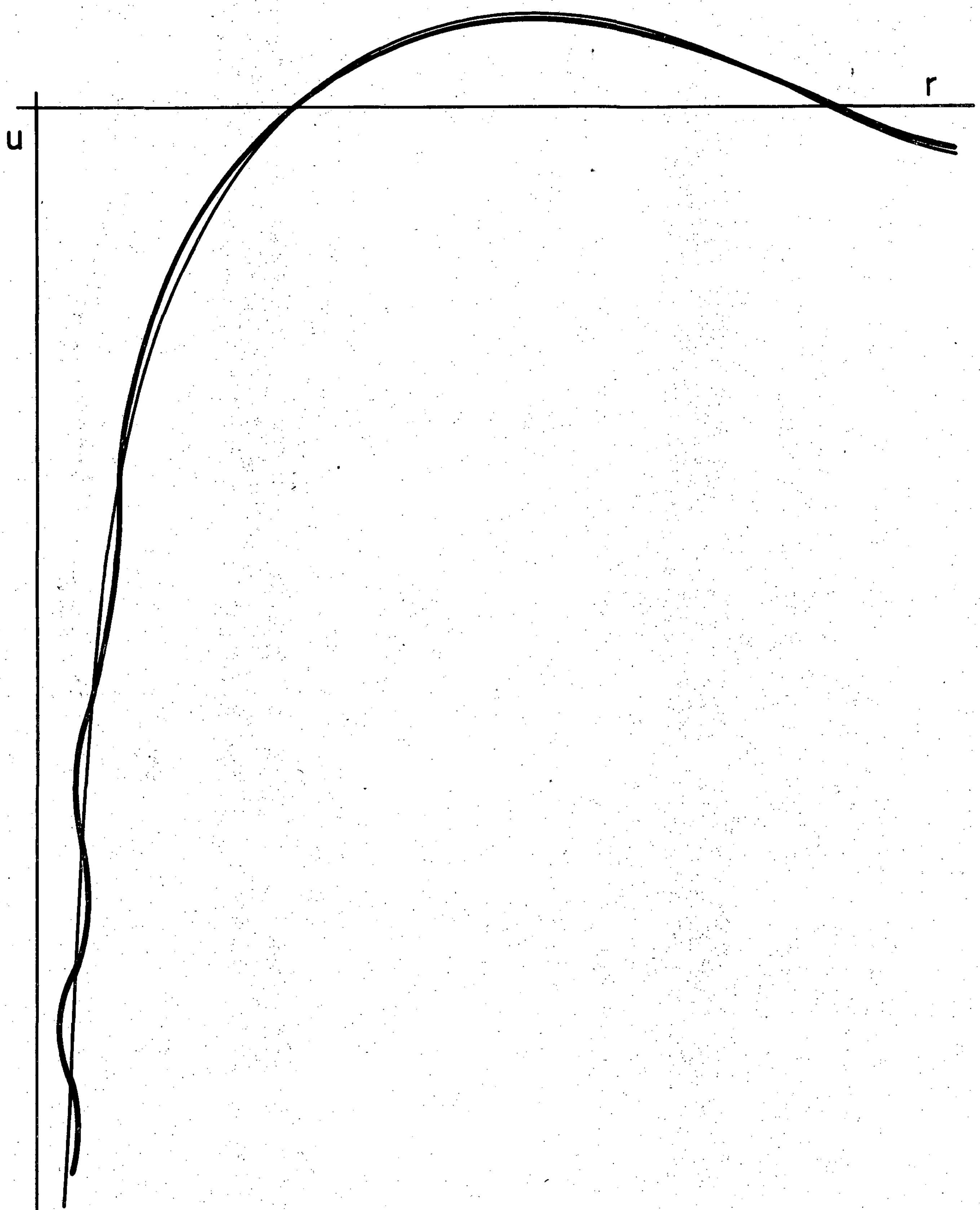

Figure 10. $u_o = -16$; singular solution

of III. Introducing again the function $v^{(1)}(r)$, and applying II i, we find from (7, 21)

$$
\begin{aligned}
r(\sin\Psi - \sin\Psi^{(1)}) &= \int_r^{r_1} \rho(u - u_1)\, d\rho \\
\text{(40)} \qquad &> \int_r^{r_1} \rho(v^{(1)} - u_1)\, d\rho \\
&= \int_r^{r_1} \rho f(\rho)\, d\rho
\end{aligned}
$$

where

(41) $$f(r) = \int_r^{r_1} \frac{\sin\Psi^{(1)}}{\sqrt{1 - \sin^2\Psi^{(1)}}}\, d\rho$$

(42) $$
\begin{aligned}
\sin\Psi^{(1)}(r) &= -\frac{ru_1}{2} + \frac{r_1}{r}\left(1 + \frac{r_1u_1}{2}\right) \\
&= -\frac{1}{2}\left\{ u_1 r + \frac{1}{u_1 r}\, T \right\}
\end{aligned}
$$

with

(43) $$T = -r_1u_1(2 + r_1u_1)$$

A formal if tedious estimation yields

$$\int_r^{r_1} \rho f(\rho)\, d\rho = -\frac{\lambda^*(T)}{u_1^3} + O(u_1^{-5})$$

where $\lambda^*(T)$ has the property

(44) $$\lim_{T \to 0} \lambda^*(T) = 1$$

Thus,

$$\sin\Psi > -\frac{1}{2}\left(u_1 r + \frac{1}{u_1 r}\, T\right) - \frac{\lambda(T)}{u_1^3\, r} .$$

We conclude $\sin\Psi = 1$ at a value

(45) $$r_2 > -\frac{2}{u_1} - r_1 - \frac{2\lambda(T)}{u_1^3} + O\left(\frac{1}{u_1^5}\right)$$

where $\lambda(T)$ again has the property (44).

Repeating the procedure yields

$$(46) \qquad r_{2j+2} > r_{2j} - \frac{4\lambda(T)}{u_{2j}^3} + O\left(\frac{1}{u_{2j}^5}\right).$$

On the other hand we find from the methods of III

$$(47) \qquad u_{2j+2} = u_{2j} - \frac{4\mu(T)}{u_{2j}} + o\left(|u_{2j}|^{-2-\beta}\right)$$

for any β, $0 < \beta < 1$, and where $\mu(T)$ has the property (44).

Integrating (47) from $j = 0$ to $j = n$, we find

$$(48) \qquad n = \frac{1}{4\mu}\left(u_o^2 - u_{2n}^2\right) + o\left(|u_o|^{1-\beta}\right).$$

We place this result in (46) and integrate, obtaining

$$(49) \qquad r_{2n} > \frac{\lambda}{4\mu} \frac{1}{\sqrt{u_{2n}^2 + o\left(|u_o|^{1-\beta}\right)}} + O\left(\frac{1}{|u_o|}\right)$$

This relation yields information if we choose n so that

$$(50) \qquad c_1 |u_o|^{\alpha} < u_{2n} < c_2 |u_o|^{\alpha}$$

for positive constants $c_1 < c_2$, and $0 < \alpha < 1$. Then (49) implies that in any interval of the type (50), there holds asymptotically as $u_o \longrightarrow \infty$,

$$(51) \qquad |r_{2n} u_{2n}| > \frac{\lambda}{4\mu} \left[1 + o\left(|u_o|^{1-\beta-2\alpha}\right) \right]$$

Choosing β sufficiently large that $1 - \beta - 2\alpha < 0$, we obtain from (51) a lower bound for $|r_{2n} u_{2n}|$ in the interval (50), depending only on T. It follows that T cannot remain arbitrarily small, for λ and μ would then both be close to unity. We have proved:

Theorem 8 : There exists a universal constant $\hat{T} > 0$, such that on any interval of the form (50), there holds $T > \hat{T}$ at all vertical points, for all sufficiently large $|u_o|$.

Our result is at least suggestive of convergence to the singular solution U(r); this latter solution is characterized asymptotically by the condition $T \sim 1$.

It seems likely that a closer examination of the asymptotic relations discussed above will yield significantly new information on the (conjectured) convergence to U(r).

Footnotes

(1) p1 For background information on the derivation of (1) see, e.g. [1, 2, 3].

(2) p3 The remaining case can be realized physically, e.g., as the lower surface of a column of water in a glass capillary tube.

(3) p4 We call attention however to a remarkable existence theorem due to Wente [9].

(4) p5, 9, 12 This improves the result announced in [8].

(5) p5 A stronger result of this type is given in [4].

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