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# ON THE SIZES OF $t$ -INTERSECTING $k$ -CHAIN-FREE FAMILIES

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**Abstract.** A set system  $\mathcal{F}$  is  $t$ -intersecting, if the size of the intersection of every pair of its elements has size at least  $t$ . A set system  $\mathcal{F}$  is  $k$ -Sperner, if it does not contain a chain of length  $k + 1$ .

Our main result is the following: Suppose that  $k$  and  $t$  are fixed positive integers, where  $n + t$  is even and  $n$  is large enough. If  $\mathcal{F} \subseteq 2^{[n]}$  is a  $t$ -intersecting  $k$ -Sperner family, then  $|\mathcal{F}|$  has size at most the size of the sum of  $k$  layers, of sizes  $(n + t)/2, \dots, (n + t)/2 + k - 1$ . This bound is best possible. The case when  $n + t$  is odd remains open.

**Keywords.** Extremal set theory, Sperner families, intersection theorems

**Mathematics Subject Classifications.** 05D05

## 1. Introduction

### 1.1. Definitions and Notation

For a positive integer  $n$ , we write  $[n] := \{1, 2, \dots, n\}$  and  $2^{[n]}$  for the power set of  $[n]$ . For a set  $S$ , we denote by  $\binom{S}{i}$  the family of all  $i$  element subsets of  $S$ .

For a family of sets  $\mathcal{F} \subseteq 2^{[n]}$ , we define  $\mathcal{F}_i := \{F \in \mathcal{F} : |F| = i\}$  and  $f_i := |\mathcal{F}_i|$ . We use  $\Delta_i$  and  $\nabla_i$  to denote the  $i$ -shadow and  $i$ -shade of  $\mathcal{F}$ , respectively, so that

$$\Delta_i \mathcal{F} := \{A : |A| = i, A \subset F \text{ for some } F \in \mathcal{F}\}$$

and

$$\nabla_i \mathcal{F} := \{A : |A| = i, A \supset F \text{ for some } F \in \mathcal{F}\}.$$

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If the subscript  $i$  is unspecified, then assuming  $\mathcal{F}$  is  $r$ -uniform,  $\Delta\mathcal{F} = \Delta_{r-1}\mathcal{F}$  and similarly  $\nabla\mathcal{F} = \nabla_{r+1}\mathcal{F}$ .

**Definition 1.1.** [ $k$ -Sperner family]

A  $(k + 1)$ -chain is a collection of  $k + 1$  sets  $A_0, A_1, \dots, A_k$  such that  $A_0 \subset A_1 \subset \dots \subset A_k$ . A family of sets  $\mathcal{F} \subseteq 2^{[n]}$  is a  $k$ -Sperner family if there is no  $(k + 1)$ -chain in  $\mathcal{F}$ . If  $k = 1$ , then  $\mathcal{F}$  is simply a Sperner family or an antichain.

**Definition 1.2.** [ $t$ -intersecting family]

A family of sets  $\mathcal{F} \subseteq 2^{[n]}$  is  $t$ -intersecting if for every pair of sets  $A, B \in \mathcal{F}$ , we have  $|A \cap B| \geq t$ . If  $t = 1$ , then  $\mathcal{F}$  is intersecting.

## 1.2. History

The maximum size of an antichain in  $2^{[n]}$  was determined by Sperner [Spe28].

**Theorem 1.3** (Sperner). *Let  $\mathcal{F} \subseteq 2^{[n]}$  be an antichain. Then,*

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Furthermore, equality holds only if  $\mathcal{F}$  is one of the largest layers in the Boolean lattice  $2^{[n]}$ .

Sperner's theorem was extended to  $k$ -Sperner families by Erdős [Erd45].

**Theorem 1.4** (Erdős). *The maximum-size  $k$ -Sperner family  $\mathcal{F} \subseteq 2^{[n]}$  is the union of the largest  $k$  layers in the Boolean lattice  $2^{[n]}$ .*

A different extension of Sperner's theorem was given by Milner [Mil68]. Milner additionally required the family  $\mathcal{F}$  to be  $t$ -intersecting.

**Theorem 1.5** (Milner). *If  $\mathcal{F} \subseteq 2^{[n]}$  is a  $t$ -intersecting antichain, then*

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n+t+1}{2} \rfloor}.$$

In a different direction, Frankl [Fra90] determined the maximum size of an intersecting  $k$ -Sperner family. Different proofs were given by Gerbner [Ger13] and by Gerbner, Methuku and Tompkins [GMT17].

**Theorem 1.6** (Frankl). *Let  $\mathcal{F} \subseteq 2^{[n]}$  be an intersecting,  $k$ -Sperner family. Then,*

$$|\mathcal{F}| \leq \begin{cases} \sum_{i=\frac{n+1}{2}}^{\frac{n+1}{2}+k-1} \binom{n}{i}, & \text{if } n \text{ is odd,} \\ \binom{n-1}{\frac{n}{2}-1} + \sum_{i=\frac{n}{2}+1}^{\frac{n}{2}+k-1} \binom{n}{i} + \binom{n-1}{\frac{n}{2}+k}, & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, if  $n$  is odd, equality holds only if

$$\mathcal{F} = \binom{[n]}{\lfloor \frac{n}{2} \rfloor + 1} \cup \binom{[n]}{\lfloor \frac{n}{2} \rfloor + 2} \cup \dots \cup \binom{[n]}{\lfloor \frac{n}{2} \rfloor + k},$$

while if  $n$  is even and  $k > 1$ , equality holds only if for some  $x \in [n]$ ,

$$\mathcal{F} = \left\{ F \in \binom{[n]}{\frac{n}{2}} : x \in F \right\} \cup \binom{[n]}{\frac{n}{2} + 1} \cup \dots \cup \binom{[n]}{\frac{n}{2} + k - 1} \cup \left\{ F \in \binom{[n]}{\frac{n}{2} + k} : x \notin F \right\}.$$

A common generalization of the theorems of Milner and Frankl would be to determine the maximum size of a  $t$ -intersecting,  $k$ -Sperner family.

Frankl [Fra21] proposed conjectures on the maximum size of a  $t$ -intersecting  $k$ -Sperner family  $\mathcal{F} \subset 2^{[n]}$  and made some progress towards proving these conjectures. The conjectured extremal family depends on the parity of  $n + t$ .

In the case when  $n + t$  is even, the conjectured maximum size of a  $t$ -intersecting,  $k$ -Sperner family is very easy to describe.

**Conjecture 1.7** (Frankl). If  $n + t$  is even,  $n > t$ , and  $\mathcal{F} \subseteq 2^{[n]}$  is a  $t$ -intersecting,  $k$ -Sperner family, then

$$|\mathcal{F}| \leq \sum_{i=0}^{k-1} \binom{n}{\frac{n+t}{2} + i}.$$

Conjecture 1.7 is clearly tight if true, as evidenced by the family  $\bigcup_{i=0}^{k-1} \binom{[n]}{\frac{n+t}{2} + i}$ .

Let us mention that Frankl proved Conjecture 1.7 when  $t \geq n - O(\sqrt{n})$ . As the main result of this paper, we settle Conjecture 1.7 if  $t$  and  $k$  are fixed and  $n$  is sufficiently large.

**Theorem 1.8.** Let  $t$  and  $k$  be positive integers. There exists  $n_0 = n_0(t, k)$  such that if  $n \geq n_0$  and  $n + t$  is even, then the following holds: if  $\mathcal{F} \subseteq 2^{[n]}$  is a  $t$ -intersecting  $k$ -Sperner family, then

$$|\mathcal{F}| \leq \binom{[n]}{\frac{n+t}{2}} + \dots + \binom{[n]}{\frac{n+t}{2} + k - 1}.$$

Furthermore equality holds if and only if  $\mathcal{F} = \bigcup_{i=0}^{k-1} \binom{[n]}{\frac{n+t}{2} + i}$ .

## 2. Proof of Theorem 1.8

### 2.1. Main ideas

The proof has two parts. In the first part we compress  $\mathcal{F}$ , a  $t$ -intersecting,  $k$ -Sperner family, into the layers of the Boolean lattice containing the sets of sizes  $\frac{n+t}{2} - k + 1, \dots, \frac{n+t}{2} + 2k - 2$ . This part of the proof also works when  $n + t$  is odd. In the second part, we use Katona's circle method, i.e., for every cyclic permutation  $\sigma$  of  $[n]$  we define  $\mathcal{F}_\sigma$  to be the collection of sets from  $\mathcal{F}$ , whose elements are consecutive on  $\sigma$ , the so-called *intervals*. For every  $\sigma$ , we show that for an appropriate weight function  $w$ , the total weight  $w(\mathcal{F}_\sigma)$  is maximized when  $\mathcal{F}_\sigma$  contains all intervals of size  $r$  for every  $\frac{n+t}{2} \leq r \leq \frac{n+t}{2} + k - 1$ . Then we reduce the general problem to this weighted version of the problem on the cycle.

## 2.2. Compression Argument

We recall the well-known Katona shadow  $t$ -intersection theorem.

**Theorem 2.1** (Katona shadow  $t$ -intersection theorem [Kat64]). *Let  $\mathcal{F}$  be an  $r$ -uniform,  $t$ -intersecting family. Then, for  $r - t \leq \ell \leq r$ ,*

$$|\Delta_\ell(\mathcal{F})| \geq \frac{\binom{2r-t}{\ell}}{\binom{2r-t}{r}} |\mathcal{F}|.$$

One can conclude the following lemma from the above theorem. We include its proof for sake of completeness.

**Lemma 2.2** ([Mil68]). *For  $i \leq \lfloor \frac{n+t-1}{2} \rfloor$ , if  $\mathcal{F} \subseteq 2^{[n]}$  is  $t$ -intersecting, then we have*

$$|\nabla_{i+1}(\mathcal{F}_i)| \geq |\mathcal{F}_i|.$$

*Proof.* Define the family of complements  $\mathcal{F}_i^C := \{F^C : F \in \mathcal{F}_i\}$ . Since  $\mathcal{F}_i$  is  $t$ -intersecting,  $\mathcal{F}_i^C$  is  $(n+t-2i)$ -intersecting. Since  $i \leq \lfloor \frac{n+t-1}{2} \rfloor$ , we have  $n+t-2i \geq 1$ , so Theorem 2.1 can be applied to  $\mathcal{F}_i^C$  with  $r := n-i$ ,  $t := n+t-2i$ , and  $\ell := n-i-1$ , yielding

$$|\Delta_{n-i-1}(\mathcal{F}_i^C)| \geq \frac{\binom{2(n-i)-(n+t-2i)}{n-i-1}}{\binom{2(n-i)-(n+t-2i)}{n-i}} |\mathcal{F}_i^C| = \frac{\binom{n-t}{n-i-1}}{\binom{n-t}{n-i}} |\mathcal{F}_i^C| = \frac{n-i}{i-t+1} |\mathcal{F}_i| \geq |\mathcal{F}_i|.$$

Since  $|\nabla_{i+1}(\mathcal{F}_i)| = |\Delta_{n-i-1}(\mathcal{F}_i^C)|$ , the desired result follows.  $\square$

The next lemma shows that we may restrict our attention to families of sets of size at least  $\frac{n+t}{2} - (k-1)$ .

**Lemma 2.3.** *Let  $\mathcal{F} \subseteq 2^{[n]}$  be a  $t$ -intersecting and  $k$ -Sperner family, where  $n+t$  is even. Then there exists a  $t$ -intersecting  $k$ -Sperner family  $\mathcal{G} \subseteq 2^{[n]}$  with  $|\mathcal{G}| \geq |\mathcal{F}|$  and  $\min\{|G| : G \in \mathcal{G}\} \geq \frac{n+t}{2} - (k-1)$ .*

*Proof.* Recall that  $f_i := |\mathcal{F}_i|$ . Assume that there is  $i < \frac{n+t}{2} - (k-1)$  such that  $f_i > 0$  and  $f_j = 0$  for every  $j < i$ . We show that there is a  $t$ -intersecting  $k$ -Sperner family  $\mathcal{F}'$  with  $|\mathcal{F}'| \geq |\mathcal{F}|$  and  $|F'| \geq i+1$  for every set  $F' \in \mathcal{F}'$ . We show the existence of such an  $\mathcal{F}'$  by using a compression operation.

We define a series of auxiliary families  $\mathcal{H}_j$  for  $j \geq i$  as follows:  $\mathcal{H}_i := \mathcal{F}_i$  and  $\mathcal{H}_j := \nabla_j(\mathcal{H}_{j-1}) \cap \mathcal{F}_j$  for  $j > i$ . Note  $\mathcal{H}_{i+k} = \emptyset$ , as otherwise there would be a  $(k+1)$ -chain in  $\mathcal{F}$ . Let  $u$  be the smallest integer such that  $i+1 \leq u \leq i+k$  and  $\mathcal{H}_u = \emptyset$ . Then we formally define the family  $\mathcal{F}' := (\mathcal{F} \setminus \mathcal{F}_i) \cup \nabla_{i+1}(\mathcal{H}_i) \cup \dots \cup \nabla_u(\mathcal{H}_{u-1})$ . Conceptually, the compression operation is as follows: we compress the sets in  $\mathcal{H}_i$  onto  $\nabla_{i+1}(\mathcal{H}_i)$ ; that is, we define an intermediate family  $\mathcal{G}_{i+1} = (\mathcal{F} \setminus \mathcal{H}_i) \cup \nabla_{i+1}(\mathcal{H}_i)$ . If  $u = i+1$ , we stop and define  $\mathcal{F}' := \mathcal{G}_{i+1}$ . Otherwise, we think of the sets in  $\mathcal{H}_{i+1}$  as appearing with multiplicity two in the family  $\mathcal{G}_{i+1}$ . Now, define a new intermediate family  $\mathcal{G}_{i+2} := (\mathcal{G}_{i+1} \setminus \mathcal{H}_{i+1}) \cup \nabla_{i+2}(\mathcal{H}_{i+1})$ . Conceptually, compress one of the copies of each set in  $\mathcal{H}_{i+1}$  onto its  $(i+2)$ -shade  $\nabla_{i+2}(\mathcal{H}_{i+1})$ , and leave the other copy on

the  $(i + 1)$ -layer, so that  $\mathcal{H}_{i+1} \subset \mathcal{G}_{i+2}$ . We repeat this compression process for every  $j < u$ . The family obtained after performing this series of compressions is  $\mathcal{F}'$ .

In each step we added elements to the sets, hence  $\mathcal{F}'$  will be  $t$ -intersecting. By Lemma 2.2, we have  $|\nabla_{j+1}(\mathcal{H}_j)| \geq |\mathcal{H}_j|$  for  $j \leq \frac{n+t}{2} - 1$ , so  $|\mathcal{F}'| \geq |\mathcal{F}|$ . It remains to be shown that  $\mathcal{F}'$  contains no  $(k + 1)$ -chains.

Let  $A_0 \subset A_1 \subset \dots \subset A_k$  be a  $(k + 1)$ -chain in  $\mathcal{F}'$  with  $|A_0| = j$ , where  $i + 1 \leq j \leq i + k$ . Note that  $|A_1| \geq j + 1, \dots, |A_{i+k-j}| \geq j + (i + k - j) = i + k$ , so  $A_{i+k-j+1}, \dots, A_k \in \mathcal{F}$ .

If all of the sets  $A_0, A_1, \dots, A_{i+k-j}$  were contained in  $\mathcal{F}$ , then the  $(k + 1)$ -chain  $A_0 \subset \dots \subset A_k$  would have already been in  $\mathcal{F}$ . Otherwise, pick the largest  $m$  such that  $A_m \notin \mathcal{F}$ , and assume that  $|A_m| = \ell$ , so that  $\ell \geq j + m$ . By construction, there must be a chain  $B_0 \subset B_1 \subset \dots \subset B_{\ell-i-1} \subset A_m$  with  $B_r \in \mathcal{F}_{i+r}$  for  $0 \leq r \leq \ell - i - 1$ . Now the chain  $B_0 \subset B_1 \subset \dots \subset B_{\ell-i-1} \subset A_{m+1} \subset \dots \subset A_k$  is contained in  $\mathcal{F}$ , and it has size  $\ell - i + k - m \geq j - i + k \geq k + 1$ , which is a contradiction.  $\square$

*Remark 2.4.* The bottleneck of the above proof is that we need to do the upshifting operation up to  $k - 1$  times, and we need the shade to be expanding, *i.e.*, that is the reason that we require  $i < \frac{n+t}{2} - (k - 1)$ . Observe that the parity of  $n + t$  was not considered, so the same proof works when  $n + t$  is odd.

We will use the following simple lemma whose proof we include for completeness.

**Lemma 2.5** (Sperner [Spe28]). *Let  $\mathcal{F} \subseteq 2^{[n]}$  be a Sperner family with  $m := \min\{|F| : F \in \mathcal{F}\} > \frac{n}{2}$ . Then for every  $\lfloor \frac{n}{2} \rfloor \leq j \leq m$ , we have  $|\Delta_j(\mathcal{F})| \geq |\mathcal{F}|$ .*  $\square$

*Proof.* Observe that if  $\mathcal{F} \subseteq \binom{[n]}{\ell}$  for some  $\ell > \frac{n}{2}$ , then  $|\Delta_{\ell-1}(\mathcal{F})| \geq |\mathcal{F}|$ . This holds because if we consider the bipartite graph  $B$  with parts  $\mathcal{F}$  and  $\Delta_{\ell-1}(\mathcal{F})$  with  $F' \in \Delta_{\ell-1}(\mathcal{F})$  connected to  $F \in \mathcal{F}$  if and only if  $F' \subset F$ , then  $d_B(F) = \ell$  for all  $F$ , while  $d_B(F') \leq n - \ell + 1 \leq \ell$ . Thus  $|\mathcal{F}| \cdot \ell \leq |\Delta_{\ell-1}(\mathcal{F})| \cdot (n - \ell + 1)$  and so  $|\mathcal{F}| \leq |\Delta_{\ell-1}(\mathcal{F})|$ .

Now the lemma follows from repeated applications of this statement.  $\square$

Our next lemma shows that we may restrict our attention to families of sets of size at most  $\frac{n+t}{2} + 2k - 2$ .

**Lemma 2.6.** *If  $\mathcal{F} \subseteq 2^{[n]}$  is a  $t$ -intersecting  $k$ -Sperner family with  $\min\{|F| : F \in \mathcal{F}\} = \frac{n+t}{2} - c$ , then there exists a  $t$ -intersecting  $k$ -Sperner family  $\mathcal{F}' \subseteq 2^{[n]}$  with  $|\mathcal{F}'| \leq |\mathcal{F}|$ , and*

$$\min\{|F| : F \in \mathcal{F}\} = \min\{|F'| : F' \in \mathcal{F}'\} \quad \text{and} \quad \max\{|F'| : F' \in \mathcal{F}'\} \leq \frac{n+t}{2} + c + k - 1.$$

*Proof.* We first partition  $\mathcal{F}$  into  $\mathcal{F}^1, \mathcal{F}^2, \dots, \mathcal{F}^k$  by letting  $\mathcal{F}^1$  consist of all minimal sets of  $\mathcal{F}$  and once  $\mathcal{F}^1, \dots, \mathcal{F}^j$  are defined, then let  $\mathcal{F}^{j+1}$  consist of all the minimal sets of  $\mathcal{F} \setminus \cup_{i=1}^j \mathcal{F}^i$ . Then for every  $1 \leq j \leq k$ , we partition  $\mathcal{F}^j$  into  $\mathcal{F}^{j>} \cup \mathcal{F}^{j\leq}$  with

$$\mathcal{F}^{j>} = \{F \in \mathcal{F}^j : |F| > \frac{n+t}{2} + c + j - 1\} \quad \text{and} \quad \mathcal{F}^{j\leq} = \{F \in \mathcal{F}^j : |F| \leq \frac{n+t}{2} + c + j - 1\}.$$

We define  $\mathcal{F}'^j := \mathcal{F}^{j\leq} \cup \Delta_{\frac{n+t}{2} + c + j - 1}(\mathcal{F}^{j>})$ .

Sets in  $\mathcal{F}^j \cap \mathcal{F}'^j$  have size at most  $\frac{n+t}{2} + c + j - 1$ , sets  $F$  in  $\mathcal{F}'^j \setminus \mathcal{F}^j$  have size exactly  $\frac{n+t}{2} + c + j - 1$  and there exists  $F^* \in \mathcal{F}^j \setminus \mathcal{F}'^j$  with  $F \subsetneq F^*$ . This implies  $\mathcal{F}'^j$  is an antichain

for all  $j = 1, 2, \dots, k$ . By Lemma 2.5, we have  $|\mathcal{F}^j| \leq |\mathcal{F}'^j|$  for all  $1 \leq j \leq k$ . Also, if  $1 \leq i < j \leq k$ , then  $\mathcal{F}'^i \cap \mathcal{F}'^j = \emptyset$  since all new sets in  $\mathcal{F}'^j \setminus \mathcal{F}^j$  are of size  $\frac{n+t}{2} + c + j - 1$ , while all sets in  $\mathcal{F}'^i$  have size at most  $\frac{n+t}{2} + c + i - 1 < \frac{n+t}{2} + c + j - 1$ . Thus for  $\mathcal{F}' := \cup_{j=1}^k \mathcal{F}'^j$ , we have  $|\mathcal{F}| \leq |\mathcal{F}'|$ . Observe that  $\mathcal{F}'$  is  $k$ -Sperner as it is the union of  $k$  antichains. Finally,  $\mathcal{F}'$  is  $t$ -intersecting as all sets in  $\mathcal{F}' \setminus \mathcal{F}$  have size at least  $\frac{n+t}{2} + c$  and all sets in  $\mathcal{F} \cap \mathcal{F}'$  have size at least  $\frac{n+t}{2} - c$ .  $\square$

Observe that starting with an arbitrary  $t$ -intersecting  $k$ -Sperner family  $\mathcal{F}$ , after applying Lemma 2.3 we obtain another one  $\mathcal{F}'$  with  $|\mathcal{F}| \leq |\mathcal{F}'|$  and  $\min\{|F| : F \in \mathcal{F}'\} \geq \frac{n+t}{2} - k + 1$ . Then applying Lemma 2.6 with  $c = \frac{n+t}{2} - \min\{|F| : F \in \mathcal{F}'\}$ , we obtain a  $t$ -intersecting  $k$ -Sperner family  $\mathcal{F}''$  with  $|\mathcal{F}| \leq |\mathcal{F}'| \leq |\mathcal{F}''|$  and  $\min\{|F| : F \in \mathcal{F}''\} = \frac{n+t}{2} - m$  for some  $0 \leq m \leq k - 1$  and  $\max\{|F| : F \in \mathcal{F}''\} \leq \frac{n+t}{2} + k - 1 + m$ . Therefore, in the next subsection, in the rest of the proof of Theorem 1.8, we will assume that  $\mathcal{F}$  has this property.

### 2.3. Proof of Theorem 1.8

Let  $\sigma$  be a cyclic permutation of  $[n]$  and  $\mathcal{F}_\sigma$  be the subfamily of those sets in  $\mathcal{F}$  that form an interval in  $\sigma$ . Note that there are  $(n - 1)!$  choices for  $\sigma$ . For a set  $G$ , let  $w(G) = \binom{n}{|G|}$  and  $w(\mathcal{G}) = \sum_{G \in \mathcal{G}} w(G)$ . Remember  $m$  is defined as  $\frac{n+t}{2} - \min\{|F| : F \in \mathcal{F}\}$ . By the above discussions, we have  $0 \leq m \leq k - 1$ . If  $m = 0$  then  $\mathcal{F}$  has the required structure, hence we assume  $m > 0$ . The aim of this subsection is to prove the following lemma.

**Lemma 2.7.** *Suppose  $n + t$  is even with  $t \leq n$  and  $n$  is large enough. For every cyclic permutation  $\sigma$  and  $t$ -intersecting  $k$ -Sperner family  $\mathcal{F} \subseteq \cup_{i=\frac{n+t}{2}-m}^{\frac{n+t}{2}+k-1+m} \binom{[n]}{i}$ , we have  $w(\mathcal{F}_\sigma) \leq n \sum_{i=0}^{k-1} \binom{n}{\frac{n+t}{2}+i}$  and equality holds if and only if  $\mathcal{F}_\sigma$  consists of all intervals of size at least  $\frac{n+t}{2}$  and at most  $\frac{n+t}{2} + k - 1$ .*

Before continuing, let us show how Lemma 2.7 implies Theorem 1.8.

*Proof of Theorem 1.8 using Lemma 2.7.* As mentioned in the last paragraph of the previous subsection, by Lemma 2.3 and Lemma 2.6, we can assume that  $\mathcal{F} \subseteq \cup_{i=\frac{n+t}{2}-m}^{\frac{n+t}{2}+k-1+m} \binom{[n]}{i}$  holds. Then using Lemma 2.7 we have:

$$\sum_{\sigma} \sum_{F \in \mathcal{F}_\sigma} w(F) \leq (n - 1)! \cdot n \sum_{i=0}^{k-1} \binom{n}{\frac{n+t}{2} + i} = n! \cdot \sum_{i=0}^{k-1} \binom{n}{\frac{n+t}{2} + i}.$$

From the other side,

$$\sum_{\sigma} \sum_{F \in \mathcal{F}_\sigma} w(F) = \sum_{F \in \mathcal{F}} |F|!(n - |F|)! \binom{n}{|F|} = n! |\mathcal{F}|,$$

which implies the required upper bound on  $|\mathcal{F}|$ .

The uniqueness of the extremal family follows from two observations. First, the proofs of Lemma 2.3 and Lemma 2.6 imply that if a  $t$ -intersecting  $k$ -Sperner family  $\mathcal{F}$  contains a set

$F$  with  $|F| \notin [\frac{n+t}{2}, \frac{n+t}{2} + k - 1]$ , then so does the output  $\mathcal{F}''$  of the compression argument. Then, for any  $\mathcal{F} \subseteq \bigcup_{i=-m}^{k-1+m} \binom{[n]}{\frac{n+t}{2}+i}$  that contains  $F$  with  $|F| \notin [\frac{n+t}{2}, \frac{n+t}{2} + k - 1]$ , there exists a cyclic permutation  $\sigma$  with  $F \in \mathcal{F}_\sigma$ , and so the uniqueness part of Lemma 2.7 implies that  $w(\mathcal{F}_\sigma) < n \sum_{i=0}^{k-1} \binom{n}{\frac{n+t}{2}+i}$ , and thus the inequality above is strict, and so is the bound on  $|\mathcal{F}|$ .  $\square$

In order to prove Lemma 2.7, we need some preparation. Let us fix a cyclic permutation  $\sigma$  of  $[n]$ . We partition all intervals, i.e., sets of consecutive elements of  $[n]$  with respect to  $\sigma$ , into  $n$  chains: the  $h$ -th chain  $C_h$  consists of  $\{\sigma(h)\}, \{\sigma(h), \sigma(h+1)\}, \dots, [n] \setminus \{\sigma(h-1)\}$  and we let  $\mathcal{C}_\sigma = \{C_h : h \in [n]\}$ . A family  $\mathcal{G}$  of intervals is  $\sigma$ - $k$ -Sperner  $t$ -intersecting if it is  $t$ -intersecting and for every  $C \in \mathcal{C}_\sigma$  we have  $|\mathcal{G} \cap C| \leq k$ . Such a family is *consecutive* if for every  $C \in \mathcal{C}_\sigma$  the chain  $C \cap \mathcal{G}$  consists of consecutive intervals, and *full consecutive* if further  $|C \cap \mathcal{G}| = k$  holds for every  $C \in \mathcal{C}_\sigma$ . Clearly, if  $\mathcal{G}$  is a  $k$ -Sperner  $t$ -intersecting family of intervals of  $[n]$  with respect to  $\sigma$ , then  $\mathcal{G}$  is  $\sigma$ - $k$ -Sperner  $t$ -intersecting, and if  $\mathcal{F} \subseteq 2^{[n]}$  is  $t$ -intersecting  $k$ -Sperner, then  $\mathcal{F}_\sigma$  is  $\sigma$ - $k$ -Sperner  $t$ -intersecting for any  $\sigma$ . As the  $t$ -intersection property depends only on the smallest intervals  $G_h \in C_h$ , one can replace any  $G \in \mathcal{G} \cap C_h$  by any  $G' \in C_h \setminus \mathcal{G}$  with  $|G'| > |G_h|$  to obtain another  $\sigma$ - $k$ -Sperner  $t$ -intersecting family. So if  $G'_h$  is the maximum interval of  $C_h \cap \mathcal{G}$  and  $G$  is an interval from  $C_h \setminus \mathcal{G}$  with  $|G_h| < |G| < |G'_h|$ , then we can proceed as follows: if  $|G| \geq \frac{n}{2}$ , then we replace  $G'_h$  by  $G$ , while if  $|G| < \frac{n}{2}$ , then we replace  $G_h$  by  $G$  to obtain a family  $\mathcal{G}'$ . By our choice, we have  $w(\mathcal{G}) < w(\mathcal{G}')$ . As the difference of the sizes of maximum and minimum intervals of  $C_h$  in the family strictly decreased, after a finite number of replacements, we obtain a consecutive  $\sigma$ - $k$ -Sperner  $t$ -intersecting family. Note that during this process, we do not care if we created a  $(k+1)$ -chain which is not in one of the  $C_h$ .

Finally, we can extend any consecutive  $\sigma$ - $k$ -Sperner  $t$ -intersecting family to a full one. More generally, the following holds.

**Observation 2.8.** *If  $\mathcal{G}$  is a  $k$ -Sperner  $t$ -intersecting family of intervals with respect to  $\sigma$  such that every interval has size between  $\frac{n+t}{2} - m$  and  $\frac{n+t}{2} + k - 1 + m$  for some  $0 \leq m \leq k - 1$ , then there exists a full consecutive  $\sigma$ - $k$ -Sperner  $t$ -intersecting family  $\mathcal{G}'$  with  $w(\mathcal{G}) \leq w(\mathcal{G}')$ .*

*Proof.* By the argument above, we can obtain a consecutive  $\sigma$ - $k$ -Sperner  $t$ -intersecting family  $\mathcal{G}^*$ . If for a chain  $C_h$ , we have  $|\mathcal{G}^* \cap C_h| < k$ , then we add the interval of  $C_h$  to  $\mathcal{G}^*$  that is one larger than the maximum interval in  $\mathcal{G}^* \cap C_h$ . We could only get into trouble if for some  $h$  the smallest interval  $G_h$  of  $C_h \cap \mathcal{G}^*$  is strictly larger than  $\frac{n+t}{2} + m$ , but then we can add  $G \in C_h$  with  $|G| = \frac{n+t}{2} + m$  to  $\mathcal{G}^*$  without violating the  $t$ -intersecting property as such  $G$   $t$ -intersects all other intervals in  $\mathcal{G}^*$  because of the size restrictions.  $\square$

To prove Lemma 2.7, it is sufficient to show the following statement.

**Lemma 2.9.** *Suppose  $n + t$  is even and  $n$  is large enough. Let  $\mathcal{G}$  be a full consecutive  $\sigma$ - $k$ -Sperner  $t$ -intersecting family of intervals on a cycle of length  $n$  such that  $\min\{|G| : G \in \mathcal{G}\} = \frac{n+t}{2} - m$  and  $\max\{|G| : G \in \mathcal{G}\} \leq \frac{n+t}{2} + k - 1 + m$  for some  $0 \leq m \leq k - 1$ . Then  $w(\mathcal{G}) \leq n \sum_{i=0}^{k-1} \binom{n}{\frac{n+t}{2}+i}$ , and equality holds if and only if  $\mathcal{G}$  consists of all intervals of size at least  $\frac{n+t}{2}$  and at most  $\frac{n+t}{2} + k - 1$ .*



The following Fact is well-known; we include its proof for completeness. For a family  $\mathcal{G}$ , let  $\Delta_{iv}(\mathcal{G})$  denote the family of those sets in  $\Delta(\mathcal{G})$  that form an interval on the cycle.

**Fact 2.10.** *Let  $\mathcal{G}$  be a family of intervals on a cycle of length  $n$ . If  $\mathcal{G}$  consists of  $i$ -intervals for some  $2 \leq i$ , then  $|\Delta_{iv}(\mathcal{G})| \geq |\mathcal{G}|$ .*

*Proof.* There is an injection  $\iota$  from  $\mathcal{G}$  to  $\Delta_{iv}(\mathcal{G})$  defined as follows:  $\iota(G)$  is the interval obtained by removing  $G$ 's clockwise rightmost element.  $\square$

The next Fact is standard. We include its proof, as it is a bit technical to see it instantly.

**Fact 2.11.** *For every fixed  $a < b$  with  $0 < b$  there exists  $n_0 = n_0(a, b)$  such that if  $n \geq n_0$ , then we have  $\binom{n}{\lfloor \frac{n}{2} \rfloor + a} + \binom{n}{\lfloor \frac{n}{2} \rfloor + b} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor + a + 1} + \binom{n}{\lfloor \frac{n}{2} \rfloor + b - 1}$  and equality holds if and only if  $b = a + 1$ .*

*Proof.* If  $a$  is negative, then  $\binom{n}{\lfloor \frac{n}{2} \rfloor + a} < \binom{n}{\lfloor \frac{n}{2} \rfloor + a + 1}$  and  $\binom{n}{\lfloor \frac{n}{2} \rfloor + b} \geq \binom{n}{\lfloor \frac{n}{2} \rfloor + b - 1}$  so the statement of the Fact holds for all values of  $n$ .

Hence, we can assume  $0 \leq a < b$ . If  $b = a + 1$ , then clearly equality holds. If  $b > a + 1$ , then dividing by  $n!$  and multiplying by  $(\lfloor \frac{n}{2} \rfloor + a + 1)! \cdot (\lceil \frac{n}{2} \rceil - a)! \cdot (\lfloor \frac{n}{2} \rfloor + b)! \cdot (\lceil \frac{n}{2} \rceil - b + 1)!$ , the desired inequality is equivalent to

$$\begin{aligned} & \left( \left\lfloor \frac{n}{2} \right\rfloor + a + 1 \right) \cdot \left( \left\lfloor \frac{n}{2} \right\rfloor + b \right)! \cdot \left( \left\lceil \frac{n}{2} \right\rceil - b + 1 \right)! \\ & + \left( \left\lfloor \frac{n}{2} \right\rfloor + a + 1 \right)! \cdot \left( \left\lceil \frac{n}{2} \right\rceil - a \right)! \cdot \left( \left\lceil \frac{n}{2} \right\rceil - b + 1 \right) < \\ & \left( \left\lceil \frac{n}{2} \right\rceil - a \right) \cdot \left( \left\lfloor \frac{n}{2} \right\rfloor + b \right)! \cdot \left( \left\lceil \frac{n}{2} \right\rceil - b + 1 \right)! \\ & + \left( \left\lfloor \frac{n}{2} \right\rfloor + a + 1 \right)! \cdot \left( \left\lceil \frac{n}{2} \right\rceil - a \right)! \cdot \left( \left\lfloor \frac{n}{2} \right\rfloor + b \right). \end{aligned}$$

Rearranging gives

$$(2a + 1) \cdot \left( \left\lfloor \frac{n}{2} \right\rfloor + b \right)! \cdot \left( \left\lceil \frac{n}{2} \right\rceil - b + 1 \right)! < (2b - 1) \cdot \left( \left\lfloor \frac{n}{2} \right\rfloor + a + 1 \right)! \cdot \left( \left\lceil \frac{n}{2} \right\rceil - a \right)!,$$

which is equivalent to

$$\frac{2a + 1}{2b - 1} < \frac{(\lceil \frac{n}{2} \rceil - a) \cdot \dots \cdot (\lceil \frac{n}{2} \rceil - b + 2)}{(\lfloor \frac{n}{2} \rfloor + b) \cdot \dots \cdot (\lfloor \frac{n}{2} \rfloor + a + 2)}.$$

The left hand side is a fixed rational number smaller than 1, while the right hand side tends to 1 as  $n$  tends to infinity.  $\square$

The next simple observation is going to be the core of our argument. For a cyclic permutation  $\sigma$  and an interval  $G$ , define  $\overline{G}^t$  as the complement of  $G$  together with the (counterclockwise) leftmost  $\lfloor \frac{t}{2} \rfloor$  and rightmost  $\lceil \frac{t}{2} \rceil$  elements of  $G$  with respect to  $\sigma$ . For example, if  $\sigma$  is  $123 \dots n$ ,  $G$  is the interval  $[7, s]$ , then  $\overline{G}^5$  is  $\{7, 8\} \cup \overline{G} \cup \{s - 2, s - 1, s\}$ .

For a family  $\mathcal{G}$  of intervals, let  $\overline{\mathcal{G}}^t = \{\overline{G}^t : G \in \mathcal{G}\}$ .

**Lemma 2.12.** *Suppose  $n + t$  is even,  $\sigma$  is a cyclic permutation of  $[n]$ . If  $\mathcal{G}$  is a full consecutive  $\sigma$ - $k$ -Sperner  $t$ -intersecting family with interval sizes between  $\frac{n+t}{2} - m$  and  $\frac{n+t}{2} + k - 1 + m$  for some  $0 \leq m \leq k - 1$ , then for any  $G \in \mathcal{G}$  no proper subinterval  $H$  of  $\overline{G}^t$  belongs to  $\mathcal{G}$ . Moreover, if  $|G| = \frac{n+t}{2} - m$ , then  $\overline{G}^t \in \mathcal{G}$ .*

*Proof.* Any proper subinterval of  $\overline{G}^t$  intersects  $G$  in less than  $t$  elements, thus  $\overline{G}^t$  cannot contain any interval from  $\mathcal{G}$ .

As  $|G| + |\overline{G}^t| = n + t$ , if  $|G| = \frac{n+t}{2} - m$ , then  $\overline{G}^t$   $t$ -intersects every element of  $\mathcal{G}$ . Also, if  $\overline{G}^t \notin \mathcal{G}$ , then for the chain  $C_h$  containing  $\overline{G}^t$ , we have  $|\mathcal{G} \cap C_h| < k$  as there are  $k - 1$  intervals in  $C_h$  that are larger than  $\overline{G}^t$ . This contradicts the full consecutive  $\sigma$ - $k$ -Sperner property.  $\square$

The next lemma establishes some inequalities on the number of intervals that a full consecutive  $\sigma$ - $k$ -Sperner  $t$ -intersecting family  $\mathcal{G}$  satisfying the assumptions of Lemma 2.12 may contain. For  $i = -m, -m + 1, \dots, k + m - 1$ , let  $\mathcal{G}_i$  be the family of intervals of length  $\frac{n+t}{2} + i$  in  $\mathcal{G}$  and let  $g_i$  denote the size of  $\mathcal{G}_i$ .

**Lemma 2.13.** *Suppose  $n+t$  is even,  $m < k$  and  $n$  is large enough. Let  $\mathcal{G}$  be a full consecutive  $\sigma$ - $k$ -Sperner  $t$ -intersecting family of intervals on a cycle of length  $n$  such that  $\min\{|G| : G \in \mathcal{G}\} = \frac{n+t}{2} - m$  and  $\max\{|G| : G \in \mathcal{G}\} \leq \frac{n+t}{2} + k - 1 + m$ . Then we have the following inequalities:*

1.  $g_{-j-1} + g_j \leq n$  for all  $0 \leq j \leq m - 1$  satisfying  $j < k - m$ .
2.  $\sum_{i=-(j+1)}^j g_i \leq (j + 1)n - \sum_{i=k-j}^m g_{-i}$  for all  $0 \leq j \leq m - 1$  such that  $j \geq k - m$ .
3.  $\sum_{i=-m}^{k-j} g_i \leq (k - j + 1)n - \sum_{i=j}^m g_{-i}$  for all  $1 \leq j \leq m$  such that  $j < k - m$ .
4.  $\sum_{i=-m}^{k-1+j-1} g_i \leq kn - \sum_{i=-m}^{-j} g_i$  for all  $1 \leq j \leq m$ .

*Proof.* To prove (1), note that Lemma 2.12 applied to  $\mathcal{G}_{-(j+1)}$  implies that  $\Delta(\overline{\mathcal{G}_{-(j+1)}}^t)$  is disjoint from  $\mathcal{G}$ . Fact 2.10 implies that  $|\Delta(\overline{\mathcal{G}_{-(j+1)}}^t)| \geq |\overline{\mathcal{G}_{-(j+1)}}^t| = g_{-j-1}$ , and since  $\Delta(\overline{\mathcal{G}_{-(j+1)}}^t)$  is a family of  $(\frac{n+t}{2} + j)$ -intervals, it follows that  $g_j + g_{-j-1} \leq g_j + |\Delta(\overline{\mathcal{G}_{-(j+1)}}^t)| \leq n$ .

The proofs of (2) and (3) are similar. We define families  $\mathcal{H}_1, \mathcal{H}_2$  of missing intervals, i.e. that are not members of  $\mathcal{G}$ , as follows:

$$\mathcal{H}_1^j = \{H \notin \mathcal{G} : \frac{n+t}{2} \leq |H| \leq \frac{n+t}{2} + j, \nexists G \in \mathcal{G} \ H \supset G\},$$

$$\mathcal{H}_2^j = \{H \notin \mathcal{G} : \frac{n+t}{2} \leq |H| \leq \frac{n+t}{2} + j, \nexists G \in \mathcal{G} \ G \supset H\}.$$

Observe that by definition and by the full consecutive property, we have  $\mathcal{H}_1 \cap \mathcal{H}_2 = \emptyset$ .

To prove (2), we consider  $\mathcal{H}_1^j$  and  $\mathcal{H}_2^j$ . First, as in the proof of (1), for any  $1 \leq i \leq j + 1$ ,  $\Delta(\overline{\mathcal{G}_{-i}}^t)$  is disjoint from  $\mathcal{G}$  by Lemma 2.12 and, by Fact 2.10, has size at least  $g_{-i}$ . Note that all these missing intervals (missing from  $\mathcal{G}$ ) are *only below* intervals of  $\mathcal{G}$ , so  $\mathcal{H}_1^j \supseteq \bigcup_{i=1}^{j+1} \Delta(\overline{\mathcal{G}_{-i}}^t)$ . On the other hand, if  $G \in \mathcal{G}_{-i}$  with  $k - j \leq i \leq m$ , then, as  $\mathcal{G}$  is consecutive, the chain  $C_h \in \mathcal{C}_\sigma$  that contains  $G$  misses all intervals that are exactly  $k$  larger than  $|G|$ , i.e. of size  $\frac{n+t}{2} + k - i$ .

Thus we obtain that  $\mathcal{H}_2^j$  contains at least  $\sum_{i=k-j}^m g_{-i}$  missing intervals each of which are *only above* some intervals of  $\mathcal{G}$ . This means that  $\cup_{i=0}^j \mathcal{G}_i$ ,  $\mathcal{H}_1^j$ ,  $\mathcal{H}_2^j$  are pairwise disjoint, have sizes  $\sum_{i=0}^j g_i$ ,  $\sum_{-(j+1)}^{-1} g_i$ , and  $\sum_{i=k-j}^m g_{-i}$ , and contain intervals of sizes between  $\frac{n+t}{2}$  and  $\frac{n+t}{2} + j$ . There are  $(j+1)n$  such intervals, therefore  $\sum_{i=0}^j g_i + \sum_{-(j+1)}^{-1} g_i + \sum_{i=k-j}^m g_{-i} \leq (j+1)n$  holds. Merging the first two terms and rearranging yields (2).

To prove (3), we consider  $\mathcal{H}_1^{k-j}$ ,  $\mathcal{H}_2^{k-j}$ . As  $j < k - m$ , this time  $\cup_{i=1}^m \Delta(\overline{\mathcal{G}}_{-i}^t)$  belongs to  $\mathcal{H}_1^{k-j}$ , and by Lemma 2.12 and Fact 2.10,  $\mathcal{H}_1^{k-j}$  has size at least  $\sum_{i=1}^m g_{-i}$ . For any  $G \in \mathcal{G}_{-i}$  with  $i \geq j$ , the intervals  $G'$  of  $C_h$  with  $G \in C_h$  and  $|G'| - |G| \geq k$  are missing by the consecutive property of  $\mathcal{G}$ . There are  $i - j + 1$  of such missing intervals. We obtain that  $\mathcal{H}_2$  contains at least  $\sum_{i=j}^m g_{-i}$  missing intervals. Again,  $\cup_{i=0}^{k-j} \mathcal{G}_i$ ,  $\mathcal{H}_1^{k-j}$ , and  $\mathcal{H}_2^{k-j}$  are pairwise disjoint, so the sum of their sizes is at most  $(k - j + 1)n$ . After rearrangement, this yields (3).

Finally, to see (4) observe first that as for a full consecutive  $\sigma$ - $k$ -Sperner  $t$ -intersecting family, we have  $|C_h \cap \mathcal{G}| = k$  for all  $h$ , we have  $\sum_{i=-m}^{k-1+m} g_i = |\mathcal{G}| = kn$ . So the statement of (4) is equivalent to the statement that the number of intervals in  $\mathcal{G}$  of size at least  $\frac{n+t}{2} + k - 1 + j$  is at least  $\sum_{i=-m}^{-j} g_i$ . Again, we apply Lemma 2.12 and observe that intervals  $G$  of  $\overline{\mathcal{G}}_{-i}^t$  do not strictly contain any interval of  $\mathcal{G}$ . Therefore, if  $i \geq j$ , then the chain  $C_h$  containing  $G$  has at least  $i - j + 1$  intervals from  $\mathcal{G}$  of size at least  $\frac{n+t}{2} + k + j - 1$ . Counting all these,  $\mathcal{G}$  contains at least  $\sum_{i=-m}^{-j} g_i$  intervals of size at least  $\frac{n+t}{2} + k + j - 1$  as desired.  $\square$

We are now ready to prove Lemma 2.9. As the proof involves lots of formulas, we sketch the main idea. As mentioned in the last paragraph of the proof of Lemma 2.13, the size of a full consecutive  $\sigma$ - $k$ -Sperner  $t$ -intersecting family  $\mathcal{G}$  is  $kn$ , so its weight  $w(\mathcal{G}) = \sum_{G \in \mathcal{G}} w(G) = \sum_{G \in \mathcal{G}} \binom{n}{|G|} = \sum_{i=-m}^{k+m-1} g_i \binom{n}{\frac{n+t}{2} + i}$  is a sum of  $kn$  binomial coefficients. If  $g_i = 0$  whenever  $i$  is negative, then we are done as  $w(|G|)$  is monotone decreasing in  $|G|$  if  $|G| > \frac{n}{2}$ , and  $g_i \leq n$  for all  $i$ . If there exists  $i < 0$  with  $g_i > 0$ , then we plan to apply Fact 2.11 to obtain another set of coefficients  $g'_i$  such that  $\sum_i g'_i = kn$ ,  $\sum_i g_i \binom{n}{\frac{n+t}{2} + i} \leq \sum_i g'_i \binom{n}{\frac{n+t}{2} + i}$ , and  $\sum_{i=0}^j g'_i \leq (j+1)n$  hold for all  $j = 0, 1, \dots, k-1$ . When applying Fact 2.11, we will match  $g_{-i}$  with  $g_{k-1+i}$  for all  $i = 1, 2, \dots, m$ , therefore we will first have to make sure that  $g_{-i} \leq g_{k-1+i}$  and then we can apply Fact 2.11.

*Proof of Lemma 2.9.* Note that if  $m = 0$  then we are done, hence we assume  $m \geq 1$ . Let us introduce the coefficients  $g'_i$ :

- for  $-m \leq i \leq k-1$ , let  $g'_i = g_i$ ,
- for  $i = 2, 3, \dots, m$  let  $g'_{k-1+i} = g_{-i}$ ,
- let  $g'_k = \sum_{i=k}^{k+m-1} g_i - \sum_{i=2}^m g_{-i}$ .

By the definition of  $g'_k$  and the fact that  $|\mathcal{G}| = \sum_{i=-m}^{k+m-1} g_i = kn$ , we have  $\sum_{i=-m}^{k+m-1} g'_i = kn$  and  $\sum_{i=k}^{k+m-1} g'_i = \sum_{i=k}^{k+m-1} g_i$ .

Lemma 2.13 (4) states  $\sum_{i=-m}^{k-1+j-1} g_i \leq kn - \sum_{i=-m}^{-j} g_i$ . Plugging in  $kn = \sum_{i=-m}^{k-1+m} g_i$  and rearranging yields

$$\sum_{i=-m}^{-j} g_i \leq \sum_{i=k-1+j}^{k-1+m} g_i \tag{*}$$

Applying (\*) with  $j = 1$ , we obtain

$$g'_k = \sum_{i=k}^{k+m-1} g'_i - \sum_{i=k+1}^{k-1+m} g'_i = \sum_{i=k}^{k+m-1} g_i - \sum_{i=-m}^{-2} g'_i \geq \sum_{i=-m}^{-1} g_i - \sum_{i=-m}^{-2} g_i = g_{-1}. \tag{**}$$

We would like to compare  $\sum_{i=k}^{k+m-1} g'_i \binom{n}{\frac{n+t}{2}+i}$  to  $\sum_{i=k}^{k+m-1} g_i \binom{n}{\frac{n+t}{2}+i}$ . As mentioned above,  $A := \sum_{i=k}^{k+m-1} g'_i = \sum_{i=k}^{k+m-1} g_i$ . Also, (\*) and  $g'_{k-1+i} = g_{-i}$  for all  $i = 2, 3, \dots, m$  imply

$$\sum_{i=k-1+j}^{k-1+m} g'_i = \sum_{i=-m}^{-j} g_i \leq \sum_{i=k-1+j}^{k-1+m} g_i \tag{\diamond}$$

for all  $j = 2, 3, \dots, m$ . Therefore, we can apply the following general statement that can be easily seen by induction:  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$  and  $d_1 \geq d_2 \geq \dots \geq d_n$  are all non-negative integers with  $\sum_{i=j}^n a_i \leq \sum_{i=j}^n b_i$  for all  $j = 2, 3, \dots, n$  and  $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ . Then  $\sum_{i=1}^n a_i d_i \geq \sum_{i=1}^n b_i d_i$ . Plugging in  $n := m, a_i := g'_i, b_i := g_i$ , and  $d_i := \binom{n}{\frac{n+t}{2}+i}$ , we obtain

$$\sum_{i=k}^{k+m-1} g'_i \binom{n}{\frac{n+t}{2}+i} \geq \sum_{i=k}^{k+m-1} g_i \binom{n}{\frac{n+t}{2}+i}.$$

Thus

$$w(\mathcal{G}) = \sum_{i=-m}^{k+m-1} g_i \binom{n}{\frac{n+t}{2}+i} \leq \sum_{i=-m}^{k+m-1} g'_i \binom{n}{\frac{n+t}{2}+i}. \tag{***}$$

Now for every  $1 \leq j \leq m$ , we apply Fact 2.11 either  $2j - 1$  times if  $\frac{n+t}{2} + j - 1 \leq \frac{n+t}{2} + k - j$  or  $k$  times if  $\frac{n+t}{2} + j - 1 > \frac{n+t}{2} + k - j$  to obtain

$$g'_j \left( \binom{n}{\frac{n+t}{2}-j} + \binom{n}{\frac{n+t}{2}+k+j-1} \right) \leq g'_j \left( \binom{n}{\frac{n+t}{2}+j-1} + \binom{n}{\frac{n+t}{2}+k-j} \right). \tag{****}$$

Based on (\*\*\*\*), we want to give the weights of intervals of length  $\frac{n+t}{2} - j$  to “imaginary” intervals of length  $\frac{n+t}{2} + j - 1$  and those of length  $\frac{n+t}{2} + k + j - 1$  to those of length  $\frac{n+t}{2} + k - j$ . As  $m \leq k - 1$ , all imaginary intervals will have length between  $\frac{n+t}{2}$  and  $\frac{n+t}{2} + k - 1$  (actually,  $m \leq k$  would suffice). Recall that  $g'_j$  is defined for  $-m \leq j \leq k + m - 1$ . Therefore, we introduce

$$g''_j = \begin{cases} g'_j + g'_{-(j+1)} & \text{if } 0 \leq j \leq m - 1 \text{ and } \frac{n+t}{2} + j < \frac{n+t}{2} + k - m, \\ g'_j + g'_{-(j+1)} + g'_{-(k-j)} & \text{if } 0 \leq j \leq m - 1 \text{ and } \frac{n+t}{2} + j \geq \frac{n+t}{2} + k - m, \\ g'_j & \text{if } j \geq m \text{ and } \frac{n+t}{2} + j < \frac{n+t}{2} + k - m, \\ g'_j + g'_{-(k-j)} & \text{if } j \geq m \text{ and } \frac{n+t}{2} + j \geq \frac{n+t}{2} + k - m, \\ 0 & \text{if } j < 0 \text{ or } j > k \\ g'_k - g_{-1} & \text{if } j = k. \end{cases}$$

Observe that  $g''_k = g'_k - g_{-1} \geq 0$ , see (\*\*). Also, the values  $g'_{-j}$  decreased to  $0 = g''_{-j}$  for  $j = 1, 2, \dots, m$ , but according to the first two cases of the definition of  $g''$ , the value of  $g'_{j-1}$  was increased by  $g_{-j}$  (and possibly something else). Also, the values of  $g'_{k-1+j} = g'_{-j}$  were erased for  $j = 2, 3, \dots, m$  and were given to  $g'_{k-j}$  according to the second and fourth cases of the definition of  $g''$ . Finally,  $g_{-1} = g'_{-1}$  from  $g'_k$  was given to  $g'_{k-1}$  according to the fourth case of the definition of  $g''$ . So  $\sum_{i=0}^k g''_i = \sum_{i=-m}^{k-1+m} g'_i = kn$ .

Now, (\*\*\*\*) implies that (\*\*) continues as

$$w(\mathcal{G}) = \sum_{i=-m}^{k+m-1} g_i \binom{n}{\frac{n+t}{2} + i} \leq \sum_{i=-m}^{k+m-1} g'_i \binom{n}{\frac{n+t}{2} + i} \leq \sum_{i=-m}^{k+m-1} g''_i \binom{n}{\frac{n+t}{2} + i}. \quad (*****)$$

We claim that for any  $j = 0, 1, \dots, k-1$ , we have  $\sum_{i=0}^j g''_i \leq (j+1)n$ .

If  $0 \leq j \leq m-1$  such that  $\frac{n+t}{2} + j < \frac{n+t}{2} + k - m$ , then more is true:

$$g''_j = g'_j + g'_{-(j+1)} = g_j + g_{-(j+1)} \leq n$$

by Lemma 2.13 (1).

If  $0 \leq j \leq m-1$  such that  $\frac{n+t}{2} + j \geq \frac{n+t}{2} + k - m$ , then

$$\begin{aligned} \sum_{i=0}^j g''_i &= \sum_{i=0}^{k-1-m} g''_i + \sum_{i=k-m}^j g''_i \\ &= \sum_{i=0}^{k-1-m} g_i + g_{-(i+1)} + \sum_{i=k-m}^j g_i + g_{-(i+1)} + g_{-(k-i)} = \sum_{i=-(j+1)}^j g_i + \sum_{i=k-j}^m g_{-i} \leq (j+1)n \end{aligned}$$

ensured by Lemma 2.13 (2).

If  $m \leq j < k - m$ , then  $g''_j = g'_j = g_j \leq n$ , so the inequality  $\sum_{i=0}^j g''_i \leq (j+1)n$  holds, as it holds in the previous two cases.

Finally, let us consider the case  $j \geq \max\{m, k - m\}$ . If  $j = k - 1$ , then there is nothing to prove as  $\sum_{i=0}^{k-1} g''_i \leq \sum_{i=0}^k g''_i = kn$ . If  $j \leq k - 2$ , then we write  $j = k - j^*$  with  $2 \leq j^* \leq m$  and obtain

$$\sum_{i=0}^{k-j^*} g''_i = \sum_{i=-m}^{k-j^*} g_i + \sum_{i=j^*}^m g_{-i} \leq (k - j^* + 1)n$$

by Lemma 2.13 (3).

To finish the proof of the lemma, observe that  $g_i = 0$  for  $i < 0$ ,  $\sum_{i=0}^k g''_i = kn$ , and  $\sum_{i=0}^j g''_i \leq (j+1)n$  imply that (\*\*\*\*\*) can be continued as

$$w(\mathcal{G}) = \sum_{i=-m}^{k+m-1} g_i \binom{n}{\frac{n+t}{2} + i} \leq \sum_{i=-m}^{k+m-1} g'_i \binom{n}{\frac{n+t}{2} + i} \leq \sum_{i=-m}^{k+m-1} g''_i \binom{n}{\frac{n+t}{2} + i} \leq n \sum_{i=0}^{k-1} \binom{n}{\frac{n+t}{2} + i},$$

as claimed.

The uniqueness of the extremal family follows from the strictness of Fact 2.11 if it is applied with  $b \neq a + 1$  which is the case in our proof. So  $\mathcal{G}$  can be extremal only if Fact 2.11 is never applied and that happens only if  $\mathcal{G}$  does not contain intervals of size smaller than  $\frac{n+t}{2}$ .  $\square$

### 3. Concluding remarks

The conjectured extremal families do not have such a simple structure when  $n + t$  is odd. We construct two plausible candidates for the maximum size  $t$ -intersecting,  $k$ -Sperner family:

$$\mathcal{A}(t, k) = \left\{ F \in \binom{[n]}{\lfloor \frac{n+t-1}{2} \rfloor} : n \notin F \right\} \cup \left\{ A : \frac{n+t-1}{2} + 1 \leq |A| \leq \frac{n+t-1}{2} + (k-1) \right\}.$$

$$\mathcal{B}(t, k) = \left\{ F \in \binom{[n]}{\lfloor \frac{n+t-1}{2} \rfloor} : [1, t] \in F \right\} \cup \left\{ A : \frac{n+t-1}{2} + 1 \leq |A| \leq \frac{n+t-1}{2} + (k-1) \right\} \\ \cup \left( \left\{ B : |B| = \frac{n+t-1}{2} + k \right\} \setminus \left\{ B : |B| = \frac{n+t-1}{2} + k, [1, t] \in B \right\} \right).$$

It is not hard to show that  $|\mathcal{B}(t, k)| \gg |\mathcal{A}(t, k)|$  for  $n$  sufficiently large (in terms of  $k$  and  $t$ ). However, it may be checked by computer that  $\mathcal{A}(t, k)$  is optimal for small values of  $n$  and specific choices of  $t$  and  $k$ , for example  $t = 2$  and  $k = 2$ . We conjecture that  $\mathcal{B}(t, k)$  is the largest such family when  $n$  is sufficiently large.

**Conjecture 3.1.** There exists a positive integer  $n_0 = n_0(k, t)$  such that if  $n + t$  is odd,  $n > n_0$ , and  $\mathcal{F} \subseteq 2^{[n]}$  is a  $t$ -intersecting,  $k$ -Sperner family, then

$$|\mathcal{F}| \leq |\mathcal{B}(t, k)| = \binom{n-t}{\lfloor \frac{n-t-1}{2} \rfloor} + \sum_{i=1}^k \binom{n}{\lfloor \frac{n+t-1}{2} \rfloor + i} - \binom{n-t}{\lfloor \frac{n-t-1}{2} \rfloor + k}.$$

Frankl [Fra21] more modestly conjectured the following (Frankl’s conjecture is formulated for  $s$ -union families rather than  $t$ -intersecting families, but our formulation is equivalent to Frankl’s after taking complements).

**Conjecture 3.2** (Frankl). Let  $g(n, t, k) := \max\{|\mathcal{G}| - |\Delta_{\lfloor \frac{n-t+1}{2} \rfloor - k}(\mathcal{G})| : \mathcal{G} \subset \binom{[n]}{\lfloor \frac{n-t+1}{2} \rfloor} \text{ is intersecting}\}$ . Then, if  $n + t$  is odd and  $\mathcal{F}$  is a  $t$ -intersecting,  $k$ -Sperner family, then

$$|\mathcal{F}| \leq g(n, t, k) + \sum_{i=1}^k \binom{n}{\lfloor \frac{n+t-1}{2} \rfloor + i}.$$

Note that Conjecture 3.1 can be interpreted as a strengthening of Conjecture 3.2, in that additionally there is a conjecture for the value of the function  $g(n, t, k)$  for sufficiently large  $n$ . The connection may be made more apparent by noting that, after taking complements, we may equivalently define  $g(n, t, k) := \max\{|\mathcal{G}| - |\nabla_{\lfloor \frac{n+t-1}{2} \rfloor + k}(\mathcal{G})| : \mathcal{G} \subset \binom{[n]}{\lfloor \frac{n+t-1}{2} \rfloor} \text{ is } t\text{-intersecting}\}$ .

Finally, let us mention that a cycle method proof for the case of odd  $n + t$  is unlikely to work, for the following reason. The conjectured extremal family  $\mathcal{B}(t, k)$  contains only those  $(\lfloor \frac{n+t}{2} \rfloor)$ -sets that are supersets of a fixed  $t$ -element set  $T$ . Suppose one can prove a lemma analogous to Lemma 2.7 stating that some appropriate weight function  $w(\mathcal{G})$  is maximized by a family of intervals that contain  $t$  fixed elements. In that case, the  $t$  fixed elements should be consecutive on the cycle. However, for many cyclic permutations  $\sigma$ ,  $T$  is not consecutive, so  $\mathcal{B}(t, k)_\sigma$  does not achieve the bound of this putative lemma. Therefore, the double-counting argument at the beginning of Section 2.3 would only give an upper bound that is larger than the size of  $\mathcal{B}(t, k)$ .

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