Title
The Economics of Product-Line Restrictions With an Application to the Network Neutrality Debate



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# The Economics of Product-Line Restrictions
## With an Application to the Network Neutrality Debate

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# 1 Introduction

Firms often offer product lines—several variants of the same product. Examples include different programming packages offered by cable television companies and different versions of software (*e.g.,* standard or professional) offered by software manufacturers. Typically, a firm's decision to offer a product line is viewed as unexceptional. In some circumstances, however, there are calls for public policy to limit the range of products offered. At the time we write this paper, for example, there is an intense debate taking place in the halls of Congress regarding "network neutrality" regulation. One of the central issues in this debate is whether providers of "last mile" Internet access services (typically a local telephone company offering DSL service or a cable company offering cable modem service) should be allowed to offer more than one grade of service. Proponents of regulation argue that offering multiple grades is unfair and results in some consumers' being provided unduly low-quality service.[1] For instance, Senator Olympia Snowe warned that, absent regulation, "Consumers will have all the selections of a former Soviet Union supermarket. We are going to create a two-tier Internet, for the haves who can pay the price, and the have nots who will be relegated to the Internet dirt road."[2]

In this paper, we examine the effects of product-line restrictions. Our results suggest that product-line restrictions affect welfare through several mechanisms. We model two situations.

In Section 2, we analyze a monopoly provider of a product that can be offered in a continuum of vertically differentiated variants. We compare the levels of profits, consumer surplus, and total surplus when the firm can offer a full range of products with the corresponding levels when the firm can offer only one product. We find that, as a result of the single-product restriction: (a) consumers who would otherwise have consumed a low-quality variant are excluded from the market; (b) consumers "in the middle" of the market consume a higher and more efficient quality; and (c) consumers at the top of the market consume a lower and less efficient quality. Effects (a)

[1] Similar concerns arise in other sectors of the economy. For example, there is typically strong resistance to having premium lanes on toll bridges or highways that allow travel in a less congested lane in return for payment of a fee.

[2] Ira Teinowitz, "Senate Panel Kills 'Net Neutrality' Proposal: Web Access Providers Free to Charge More for Better Service," TVWeek.com, June 28, 2006, available at http://www.tvweek.com/news.cms?newsId=10287, site visited July 24, 2006.

and (c) reduce total surplus, while effect (b) raises it. Although we find that the negative effects frequently dominate, there are situations in which restricting a monopolist raises welfare. That said, it should be observed that consumers at the bottom of the market—the ones that single-product restrictions typically are intended to aid—are almost always harmed by the restriction.

In Section 3, we examine the effects of product-line restrictions in a duopoly. Here, a new force comes into play: restriction of the number of products that each firm can offer may lead firms to choose non-overlapping products where they would otherwise have engaged in head-to-head competition across all product variants. The resulting loss of competition can harm both consumers and economic efficiency. Our analysis also illustrates the fact that a public policy that forces any given firm to offer at most one quality level may not result in all households' consuming the same quality of service—suppliers can *collectively* offer a range of products even if each firm offers only one.

The paper closes with a brief conclusion.

# 2 Monopoly

We begin by considering a monopoly service provider. We do so for two reasons. First, the lack of competition simplifies the analysis and allows us to identify forces that are also at work in settings with imperfect competition. Second, proponents of network neutrality regulation argue that market power lies at the heart of the problem. Thus, this is a useful setting to examine in its own right.

## 2.1 Preliminaries

We consider a market in which firms sell some good or service to households.[3] Each household consumes at most one unit of the product. A given household has quasi-linear utility $\theta q + y$, where $\theta$ is the household's type, $q$ is the quality of the product (*e.g.,* speed of the Internet connection) consumed by the household, and $y$ is the amount of the numeraire good consumed. Let

[3] In practice, the buyers could be firms (*e.g.,* web sites purchasing Internet connectivity), but we refer to buyers as households for simplicity.

$q = 0$ denote non-consumption. Each household knows its type. Sellers of the good or service do not know any individual household's type, but they do know the population distribution. Household types are distributed over a set $\Theta \subset \mathbb{R}_+$. Let $F(\cdot)$ denote the cumulative distribution function for types. In our baseline model, $\Theta$ is the interval $[0, \bar{\theta}]$ and $F(\cdot)$ is differentiable everywhere, with a continuous density $f(\cdot)$ such that $0 < f(\theta) < \infty$ for all $\theta \in \Theta$.[4] The last assumptions ensure that the inverse hazard rate exists for all $\theta$ and is bounded away from 0 for all $\theta < \bar{\theta}$.

Technology is such that the monopolist cannot sell products with quality less than some minimum quality $\underline{q}$, where $\underline{q} > 0$. (Alternatively, the firm could produce lower-quality products but no household would be willing to buy them even if priced at cost.) Denote the monopolist's choice space for quality by

$$\mathcal{Q} \equiv \{0\} \cup \{q | q \geq \underline{q}\} .$$

A firm incurs a cost of $c(q)$ to supply one unit of quality $q$. We assume $c(\cdot)$ is at least twice differentiable, with $c'(\cdot) > 0$ and $c''(\cdot) > 0$. These properties imply that marginal cost is strictly increasing in $q$ and positive for all $q > 0$. We also assume that $\underline{q}c'(\underline{q}) > c(\underline{q})$. Given the convexity of $c(\cdot)$, this last assumption implies that $\frac{c(q)}{q}$ is increasing in $q$ (*i.e.,* the average cost of quality rises with the quality level). Lastly, in keeping with our notational convention that $q = 0$ corresponds to non-consumption, $c(0) = 0$.

To ensure that some trade is always desirable, assume that $\bar{\theta}\underline{q} > c(\underline{q})$. To eliminate the uninteresting scenario in which $\underline{q}$ is the only efficient level of quality to offer, assume that $\bar{\theta} > c'(\underline{q})$.

Absent any fixed costs for providing a given quality, welfare (*i.e.,* total surplus) is

$$W = N \int_{\Theta} \Big( \theta q(\theta) - c\big(q(\theta)\big) \Big) dF(\theta) \, , \tag{1}$$

where $q(\theta)$ is the quality consumed by type-$\theta$ households and $N$ is the number of households. Let $q_w(\cdot)$ denote the quality-consumption schedule that maximizes welfare.

**Lemma 1** *Welfare is maximized by: (i) excluding a type-$\theta$ household if the cost of the minimum quality product, $c(\underline{q})$, exceeds the gross benefit that type*

[4] Below, we discuss how our results can be extended to discrete type spaces.

*would receive, $\theta\underline{q}$; and (ii) providing a unit of quality $q_w(\theta)$ to all types not excluded by (i), where $q_w(\theta)$ is the solution to*

$$\max_{q \geq \underline{q}} \theta q - c(q)\,.$$

*Under the first-best outcome, there is a positive measure of types excluded and a positive measure of types served.*

**Proof:** Recall that $\frac{c(q)}{q}$ is increasing. Hence, if

$$\theta\underline{q} - c(\underline{q}) < 0\,, \tag{2}$$

then

$$\theta q - c(q) < 0\,,$$

for all $q \geq \underline{q}$. Thus, for all feasible $q$, welfare is greater if these types are excluded than if they are served.

If

$$\theta\underline{q} - c(\underline{q}) \geq 0\,, \tag{3}$$

then it is welfare enhancing (at least weakly) to serve these types. Note that the minimum $\theta$ satisfying (3) is the least upper bound of $\theta$ satisfying (2). It follows that there is a marginal type, $\underline{\theta}_w$, such that $\theta < \underline{\theta}_w$ should be excluded and $\theta \geq \underline{\theta}_w$ should be served. Clearly, $\underline{\theta}_w > 0$, and the assumption that $\bar{\theta}\underline{q} > c(\underline{q})$ implies $\underline{\theta}_w < \bar{\theta}$. Hence, there is a positive measure of types who should be excluded and a positive measure of types who should be served. If a type is served, then its contribution to welfare is maximized choosing $q$ to maximize $\theta q - c(q)$ subject to $q \geq \underline{q}$. ■

Note the proof establishes the existence of a marginal type $\underline{\theta}_w$ such that this and all higher types are served and lower types are not. Because the marginal contribution of quality to welfare is increasing in type, it follows that $\theta > \theta'$ implies $q_w(\theta) \geq q_w(\theta')$, with the inequality being strict if $q_w(\theta) > \underline{q}$. Recalling our notational convention, set $q_w(\theta) = 0$ for those types who are excluded (not served).

## 2.2 The unrestricted equilibrium

We first characterize the equilibrium when a profit-maximizing monopolist is not subject to any product-line restriction. Standard analysis (*e.g.*, Mussa and Rosen, 1978; Caillaud and Hermalin, 2000) demonstrates that the monopolist's problem of finding a profit-maximizing, incentive-compatible menu of qualities and prices, $\langle q(\theta), p(\theta)\rangle$, where a type-$\theta$ household is induced to buy a unit of quality $q(\theta)$ at price $p(\theta)$, is equivalent to the following program:

$$\max_{\{q(\theta)|q(\theta)\in\mathcal{Q}\}} N\int_0^{\bar{\theta}}\Big(\theta q(\theta) - c\big(q(\theta)\big) - m(\theta)q(\theta)\Big)f(\theta)d\theta \tag{4}$$

$$\text{subject to } \underline{q} \leq q(\theta') \leq q(\theta) \quad \forall \theta, \theta' \text{ such that } \theta' < \theta\,, \tag{5}$$

where

$$m(\theta) \equiv \frac{1-F(\theta)}{f(\theta)}$$

is the inverse of the hazard rate.[5] Let $q_u(\cdot)$ denote the solution to (4).

The price schedule is given by

$$p_u(\theta) = \theta q_u(\theta) - \int_0^{\theta} q_u(t)dt\,. \tag{6}$$

As we will show shortly, there exists a lowest type, $\underline{\theta}_u > \underline{\theta}_w$, that is served (*i.e.,* $q_u(\theta) = 0$ for all $\theta < \underline{\theta}_u$. Equation (6) implies $p_u(\underline{\theta}_u) = \underline{\theta}_u q_u(\underline{\theta}_u)$; as is well known, the lowest type served enjoys no consumer surplus.

The marginal contribution to profits of an increase in $q(\theta)$ beyond $\underline{q}$ is proportional to

$$\theta - m(\theta) - c'\big(q(\theta)\big)\,. \tag{7}$$

The presence of $m(\theta)$ captures the fact that increasing the quality level offered to type-$\theta$ households increases the information rents that have to be given to all higher types to keep them from purchasing the product intended for type-$\theta$ households. For $q(\theta) = q_u(\theta)$, expression (7) equals zero if the order restriction, condition (5), doesn't bind for that type; is less than zero if the downward order restriction binds; and is greater than zero if the upward order restriction binds. As is well known (see, *e.g.,* Caillaud and Hermalin, 2000), if $m(\theta)$ is non-increasing (*i.e.,* the standard non-decreasing hazard property holds), then (5) is not binding except, possibly, at $\underline{q}$.[6] In the latter case, if $q_u(\theta) \leq \underline{q}$, then expression (7) is less than zero.

[5] We use $m$ as a mnemonic because the inverse hazard rate is also known as the Mills ratio.

[6] Examples of distributions with non-decreasing hazard rates are any distributions with affine densities (including the uniform) and power function distributions, $F(\theta) = \big(\theta/\bar{\theta}\big)^{\alpha}$, for $\alpha \geq 1$.

Because the quality allocated to the top type cannot affect the information rent received by any other type ($m(\bar{\theta}) = 0$), the monopolist, as is well known, has no incentive to distort the quality allocated the top type away from its efficient level: $q_u(\bar{\theta}) = q_w(\bar{\theta})$.

Next, consider household types at the bottom of the distribution.

**Lemma 2** *A positive measure of the population is unserved by the unrestricted monopolist. Specifically, there exists a type* $\underline{\theta}_u$, $\underline{\theta}_w \leq \underline{\theta}_u < \bar{\theta}$, *such that* $q_u(\theta) = 0$ *for all* $\theta < \underline{\theta}_u$.

**Proof:** By (6), $p_u(\theta) \leq \theta q_u(\theta)$. It follows that it cannot be profitable to serve any $\theta < \underline{\theta}_w$. Hence, $\underline{\theta}_u \geq \underline{\theta}_w$. ■

Total surplus under the unrestricted equilibrium is

$$W_u = N \int_0^{\bar{\theta}} \Big(\theta q_u(\theta) - c\big(q_u(\theta)\big)\Big) f(\theta) d\theta = N \int_{\underline{\theta}_u}^{\bar{\theta}} \Big(\theta q_u(\theta) - c\big(q_u(\theta)\big)\Big) f(\theta) d\theta \,. \quad (8)$$

## 2.3 The restricted equilibrium

Now suppose that the monopolist is restricted to offering only a single level of quality, and let $q_r$ denote the monopolist's choice.

Clearly, it would be suboptimal for the monopolist to charge a price, $p_r$, such that $p_r > \bar{\theta} q_r$ or $p_r \leq 0$. It also would never be profit maximizing to offer a price and quality such that all consumers enjoyed positive surplus. Instead, the monopolist chooses a price and quality such that there is a marginal household type just indifferent between buying and not buying. Rather than view the monopolist's problem as one of choosing an optimal price and quality, we can view it as one of choosing an optimal cutoff type and quality. That is, the monopolist's problem can be expressed as[7]

$$\max_{\{\theta \in \Theta, q \in \mathcal{Q}\}} N\big(\theta q - c(q)\big)\big(1 - F(\theta)\big) \,, \quad (9)$$

Maximization with respect to $q$ is equivalent to

$$\max_{q \in \mathcal{Q}} \theta q - c(q) \,,$$

[7]We could replace $F(\cdot)$ with $\underline{F}(\theta) = \lim_{\varepsilon \downarrow 0} F(\theta - \varepsilon)$ in expression (9) to make it generalizable to the discrete-type case.

from which we conclude that if $\underline{\theta}_r$ is the marginal type, then $q_r = q_w(\underline{\theta}_r)$.

Recall that $\underline{\theta}_w q(\underline{\theta}_w) = c(\underline{\theta}_w)$. Hence, the firm cannot earn positive profits if $\underline{\theta}_r \leq \underline{\theta}_w$. By assumption, $\bar{\theta}\underline{q} > c(\underline{q})$ and there thus exists a $\theta > \underline{\theta}_w$ such that the restricted monopolist earns positive profits from charging $\theta q_w(\theta)$.

In summary, we have established:

**Lemma 3** *Under a single-product restriction, a positive measure of the population is unserved by the monopolist. Specifically, $\underline{\theta}_w < \underline{\theta}_r < \bar{\theta}$. The marginal household type consumes the efficient quality.*

Although the marginal household type consumes the efficient quality, the restricted equilibrium outcome is inefficient relative to the first best for three reasons. First, there is only one quality even though variety is efficient. Second, there is the standard monopoly output distortion due to the price's being above marginal cost, so that too few households choose to make purchases (*i.e.,* $\underline{\theta}_r > \underline{\theta}_w$). Third, for reasons identified by Spence (1975), quality is typically too low *conditional on the households served.* Intuitively, the monopolist's profits depend on how the marginal customer values quality, but total surplus depends on how the average customer values quality. Here, the marginal customer values quality less than any other household purchasing the product. Formally, this third point is readily seen by considering welfare under a single-product restriction:

$$W_r = N \int_{\underline{\theta}_r}^{\infty} \big(\theta q - c(q)\big) dF(\theta)\,, \tag{10}$$

where $\underline{\theta}_r$ is the marginal type. Were $q$ chosen to maximize (10), the first-order condition would be equivalent to

$$\mathbb{E}\{\theta | \theta > \underline{\theta}_r\} - c'(q) \leq 0\,.$$

If this condition is an equality (*i.e.,* $\underline{q}$ is not so large that it is the efficient level), then, because $\mathbb{E}\{\theta | \theta > \underline{\theta}_r\} > \underline{\theta}_r$, it follows that the efficient quality for serving types $\theta \geq \underline{\theta}_r$ is greater than the profit-maximizing quality, $q_w(\underline{\theta}_r)$.[8]

[8] Were the type space discrete, this argument would require that the marginal household type not be the maximal type in $\Theta$.

## 2.4 The effects of a product-line restriction

We now compare the equilibrium in which the monopolist is restricted to a single product to the equilibrium in which the firm is free to offer a full product line. One consequence of imposing a single-product restriction is that low-value households types can get priced out of the market.

**Lemma 4** *Suppose that at least one of the following conditions hold:*

*(i) There is a unique solution to the restricted monopolist's problem;*

*(ii) There is a unique solution to the unrestricted monopolist's problem;*

*(iii) If there are multiple solutions to the unrestricted monopolist's problem, then the firm chooses the one that maximizes the set of households served (i.e., that minimizes $\underline{\theta}_u$);*

*(iv) It is not a solution to the restricted monopolist's problem to set quality equal to the minimum feasible quality (i.e., $q_r > \underline{q}$);*

*(v) The hazard rate associated with the distribution over household types is everywhere non-decreasing.*

*Then a single-product restriction weakly reduces the set of households that consume the service in equilibrium (i.e., $\underline{\theta}_r \geq \underline{\theta}_u$).*[9]

**Proof:** The proof of the sufficiency of condition (v) is relegated to Appendix A. Assume, here, that at least one of conditions (i)–(iv) holds.

Define quality $q'$ such that $\underline{q} \leq q' \leq \min\{q_u(\underline{\theta}_u), q_r\}$. Suppose, counterfactually, that $\underline{\theta}_r < \underline{\theta}_u$. Consider an extension of the the *un*restricted firm's equilibrium product line in which $q(\theta)$ remains the same for $\theta \geq \underline{\theta}_u$, but now $q = q'$ for $\theta \in [\underline{\theta}_r, \underline{\theta}_u)$. By construction, this extended product line satisfies the order restriction and, thus, it would have been feasible for the unrestricted firm to have offered this product line. By (6), this extension is incentive compatible if and only if the prices charged to all types $\theta \geq \underline{\theta}_u$ are reduced by $(\underline{\theta}_u - \underline{\theta}_r)q'$ per household. Because this extended line was not the line chosen by the unrestricted monopolist, the change in profits from adopting the extended product line cannot be positive:

$$\big(\underline{\theta}_r q' - c(q')\big)\big(F(\underline{\theta}_u) - F(\underline{\theta}_r)\big) - \big(1 - F(\underline{\theta}_u)\big)(\underline{\theta}_u - \underline{\theta}_r)q' \leq 0\,. \tag{11}$$

[9] This proposition also holds for a discrete type space.

Note this inequality is strict if conditions (ii) or (iii) are satisfied. By revealed preference under the restricted regime,

$$\big(\underline{\theta}_r q_r - c(q_r)\big)\big(F(\underline{\theta}_u) - F(\underline{\theta}_r)\big) - \big(1 - F(\underline{\theta}_u)\big)(\underline{\theta}_u - \underline{\theta}_r)q_r \geq 0\,. \tag{12}$$

Note this inequality is strict if condition (i) is satisfied.

We can consider the left-hand sides of these two expressions to be a function of $q$. For $q > 0$, the sign of this function is the same as the sign of

$$\left(\underline{\theta}_r - \frac{c(q)}{q}\right)\big(F(\underline{\theta}_u) - F(\underline{\theta}_r)\big) - \big(1 - F(\underline{\theta}_u)\big)(\underline{\theta}_u - \underline{\theta}_r)\,. \tag{13}$$

Because $c(q)/q$ is increasing and $\underline{\theta}_u > \underline{\theta}_r$ by supposition, it follows that (13) is strictly decreasing in $q$.

If $q_r > \underline{q}$, then $q'$ can be chosen so $q' < q_r$. Hence, (12) implies that (11) is positive and the unrestricted monopolist would strictly prefer to serve the lower household types, a contradiction.

If condition (iv) does not hold and $q_r = \underline{q}$, then, the left-hand sides of (11) and (12) are identical. Moreover, because either (i), (ii), or (iii) holds, one of the inequalities is strict, which yields a contradiction. Therefore, $\underline{\theta}_r \geq \underline{\theta}_u$. ■

Lemma 4 leaves open the possibility that, if there are multiple equilibria absent the single-product restriction *and* there are multiple equilibria with the single-product restriction, including one in which the firm offers only the lowest technically feasible quality, *and* the hazard rate is decreasing on at least one interval, then the lowest household type served in some of the restricted equilibria may be lower than the lowest type served in some—but not all—of the unrestricted equilibria. Given the highly restrictive set of conditions this case would have to satisfy, we doubt that this possibility is of empirical importance.

Lemma 4 also leaves open the possibility that the lowest type served would be unaffected by the single-product restriction. As we now show, this can happen only if a single-product monopolist would maximize its profits by offering the lowest technologically feasible quality. This seems an unlikely scenario in many markets and is certainly not the scenario envisioned by proponents of product-line regulation, who worry that households with low types will receive worse service ("the Internet dirt road") absent regulation.

**Proposition 1** *If the restricted monopolist does not offer the product with the lowest technically feasible quality level, then a single-product restriction strictly reduces the set of households served in comparison with the unrestricted equilibrium.*

**Proof:** From Lemma 4, $\underline{\theta}_r \geq \underline{\theta}_u$. Suppose, counterfactually, that $\underline{\theta}_r = \underline{\theta}_u$. Select a $q'$ such that $\underline{q} \leq q' < q_r$ and $q' \leq q_u(\underline{\theta}_u)$. The same revealed preference argument used to establish (11) and (12) in the proof of Lemma 4 allows us to conclude

$$\big(\theta q' - c(q')\big)\big(F(\underline{\theta}_u) - F(\theta)\big) - \big(1 - F(\underline{\theta}_u)\big)(\underline{\theta}_u - \theta)q' \leq 0 \tag{14}$$

for all $\theta < \underline{\theta}_u$, and

$$\big(\underline{\theta}_r q_r - c(q_r)\big)\big(F(\theta) - F(\underline{\theta}_r)\big) - \big(1 - F(\theta)\big)(\theta - \underline{\theta}_r)q_r \geq 0 \tag{15}$$

for all $\theta > \underline{\theta}_r$. Observe (14) implies

$$\theta - \frac{c(q')}{q'} \leq \big(1 - F(\underline{\theta}_u)\big)\frac{\underline{\theta}_u - \theta}{F(\underline{\theta}_u) - F(\theta)}$$

for all $\theta < \underline{\theta}_u$. Hence, in the limit as $\theta \to \underline{\theta}_u$,

$$\underline{\theta}_u - \frac{c(q')}{q'} \leq m(\underline{\theta}_u)\,. \tag{16}$$

Similarly, we can reëxpress (15) as

$$\underline{\theta}_r - \frac{c(q_r)}{q_r} \geq \big(1 - F(\theta)\big)\frac{\theta - \underline{\theta}_r}{F(\theta) - F(\underline{\theta}_r)}$$

for all $\theta > \underline{\theta}_r$. Hence, in the limit as $\theta \to \underline{\theta}_r$,

$$\underline{\theta}_r - \frac{c(q_r)}{q_r} \geq m(\underline{\theta}_r)\,. \tag{17}$$

Recalling that (a) $c(q)/q$ is a strictly increasing function of $q$; (b) $q_r > q'$; and (c) we're supposing $\underline{\theta}_u = \underline{\theta}_r$, expressions (16) and (17) imply

$$\underline{\theta}_r - m(\underline{\theta}_r) \geq \frac{c(q_r)}{q_r} > \frac{c(q')}{q'} \geq \underline{\theta}_r - m(\underline{\theta}_r)\,,$$

which is impossible. Hence, the supposition $\underline{\theta}_u = \underline{\theta}_r$ must be false *reductio ad absurdum*. Given $\underline{\theta}_u \leq \underline{\theta}_r$, we are left with $\underline{\theta}_u < \underline{\theta}_r$, as was to be shown. ■

The intuition underlying this result is the following. When a multiproduct monopolist serves additional household types at the bottom of the market, it must lower the prices charged to higher-type consumers, but it does *not* have to distort their quality levels further. When a single-product firm serves additional household types at the bottom of the market, it does so by lowering the the price and the quality level consumed by all households making positive purchases. There is, thus, an additional, costly distortion.

As an example, to show that Proposition 1 does not apply to an empty set, suppose that $c(q) = q^2/2$, $\Theta = [0,1]$, $F(\theta) = \theta$ (*i.e.,* the distribution is uniform), and $\underline{q} < 2/3$. Supposing that $q \geq \underline{q}$ will not prove binding, the first-order conditions for the restricted monopolist's profit-maximization problem are

$$q\big(1 - F(\theta)\big) - \big(\theta q - c(q)\big)f(\theta) = q(1-\theta) - (\theta q - \frac{1}{2}q^2) = 0$$

and

$$\theta - c'(q) = \theta - q = 0$$

with respect to $\theta$ and $q$ respectively. The solution is $\underline{\theta}_r = 2/3$ and $q_r = 2/3 > \underline{q}$.

Restricting the firm to offering a single product results in some households' consuming lower-quality products. One response could be to require the firm to offer only the product with a quality equal to the highest level offered prior to imposition of the regulation.[10] Recall that the highest quality product offered by the multiproduct monopolist has the quality that is efficient for the household type that most values quality, $q_w(\bar{\theta})$.

**Proposition 2** *Consider two policies, each of which requires the monopolist to offer at most one quality level. The first policy allows the firm to choose any technologically feasible quality level. The second requires the firm to offer only the highest quality variant that would be offered by an unregulated firm. In equilibrium, strictly fewer households will be served under the second policy than the first.*

[10] We are assuming here that the regulation is unanticipated, so that the firm did not strategically reduce the highest quality offered prior to the imposition of regulation.

**Proof:** First, we assemble some facts. Recall that $q_w(\bar{\theta}) > q_w(\theta)$ for all $\theta < \bar{\theta}$, including $\underline{\theta}_r$, where $\underline{\theta}_r$ is the marginal type served under the first policy. By Lemma 3, $q_r = q_w(\underline{\theta}_r)$. Hence, $q_r < q_w(\bar{\theta})$.

Let $\underline{\theta}_R$ denote the marginal type under the second policy. We can establish that $\underline{\theta}_R \geq \underline{\theta}_r$ by mimicking the proof of Lemma 4. Specifically, suppose $\underline{\theta}_R < \underline{\theta}_r$. Let $\underline{\theta}_R$ play the role played by $\underline{\theta}_r$ in the proof of Lemma 4 and let $\underline{\theta}_r$ play the role played by $\underline{\theta}_u$ in that proof. Note, because $q_r < q_w(\bar{\theta})$, the equivalent of condition (iv) holds.

To establish that $\underline{\theta}_R > \underline{\theta}_r$, suppose, counterfactually, that $\underline{\theta}_R = \underline{\theta}_r$. Then $\underline{\theta}_r$ must satisfy the first-order conditions for both of the following programs,

$$\max_{\theta} \big(\theta q_r - c(q_r)\big)\big(1 - F(\theta)\big) \quad \text{and} \quad \max_{\theta} \bigg(\theta q_w(\bar{\theta}) - c\big(q_w(\bar{\theta})\big)\bigg)\big(1 - F(\theta)\big)\,.$$

Hence,

$$q_r\big(1 - F(\underline{\theta}_r)\big) - \big(q_r - c(q_r)\big) f(\underline{\theta}_r) = 0$$

and

$$q_w(\bar{\theta})\big(1 - F(\underline{\theta}_r)\big) - \bigg(q_w(\bar{\theta}) - c\big(q_w(\bar{\theta})\big)\bigg) f(\underline{\theta}_r) = 0\,.$$

Rearranging and combining, we have

$$\frac{c(q_r)}{q_r} = \underline{\theta}_r - m(\underline{\theta}_r) = \frac{c\big(q_w(\bar{\theta})\big)}{q_w(\bar{\theta})}\,. \tag{18}$$

But, as established earlier, $c(q)/q$ is a strictly increasing function of $q$. Hence, (18) contradicts the fact that $q_w(\bar{\theta}) > q_r$. The result follows *reductio ad absurdum.* ■

A corollary of Lemma 4 and Proposition 2 is the following.

**Corollary 1** *Consider a policy that requires the monopolist to offer only the highest quality variant that would be offered by an unregulated monopolist. Compared to the unregulated outcome, this policy strictly reduces the set of households served.*

Return to the case of a single-product restriction that grants the firm the freedom to choose the quality. As Proposition 1 shows, one consequence of this restriction could be greater exclusion of low-type households in comparison with the unrestricted equilibrium. Consider those households that are excluded under the single-product equilibrium, but which would have been served under the unrestricted equilibrium. All of these households, with the possible exception of the lowest type served, enjoyed consumption benefits strictly greater than the production costs of serving them. Hence, the increased exclusion must reduce total surplus *ceteris paribus*. We refer to this as the *exclusion effect* of a single-product restriction.

There are two other effects of a single-product restriction. One is that the quality enjoyed by the the marginal type, $\underline{\theta}_r$, is efficient given the restriction, whereas it will typically be inefficient in the unrestricted equilibrium. It follows, therefore, that there is a positive measure of households that enjoy more efficient quality in the sense that

$$\theta q_w(\underline{\theta}_r) - c\big(q_w(\underline{\theta}_r)\big) > \theta q_u(\theta) - c\big(q_u(\theta)\big) .$$

We refer to this welfare benefit as the *improved-quality effect* of a single-product restriction.

On the other hand, the quality enjoyed by the top type falls from $q_w(\bar{\theta})$ to $q_w(\underline{\theta}_r)$. It follows, therefore, that there is a positive measure of types that enjoy less efficient quality in the sense that

$$\theta q_w(\underline{\theta}_r) - c\big(q_w(\underline{\theta}_r)\big) < \theta q_u(\theta) - c\big(q_u(\theta)\big) .$$

We refer to this reduction in welfare as the *reduced-quality effect* of a single-product restriction.

Because the improved-quality effect is positive, whereas the exclusion and reduced-quality effects are negative, the overall welfare impact of imposing a single-product restriction is, in general, ambiguous. The following examples illustrate the possibilities.

**Example 1:** Consider a discrete type space with three elements.[11] Let

$$\theta_1 = \frac{5}{2} \qquad \theta_2 = 2\sqrt{2} \qquad \theta_3 = 3$$
$$f_1 = \frac{7}{60} \qquad f_2 = \frac{13}{60} \qquad f_3 = \frac{40}{60}\,.$$

Suppose $c(q) = q^2/2$, which implies $q_w(\theta) = \theta$, and assume $\underline{q} < 1/100$. Using expression (57) from Appendix B and the fact that $c'(q) = q$, we have

$$\begin{aligned} q_u(\theta_1) &= \frac{5}{2} - \frac{53}{7}\left(2\sqrt{2} - \frac{5}{2}\right) \approx .0133 \\ q_u(\theta_2) &= 2\sqrt{2} - \frac{40}{13}(3 - 2\sqrt{2}) \approx 2.301 \\ q_u(\theta_1) &= q_w(3) = 3\,. \end{aligned}$$

Consequently, unrestricted welfare is

$$W_u = N \sum_{t=1}^{3} \big(\theta_t q_w(\theta_t) - c(q_w(\theta_t))\big) f_t = N\frac{249,785\sqrt{2} - 332,281}{5460} \approx 3.840N\,.$$

Now consider the imposition of single-product restriction. The monopolist's possible *per-capita* profits are

$$\text{profit} = \begin{cases} \theta_1 q_w(\theta_1) - c\big(q_w(\theta_1)\big) = q_w(\theta_1)^2/2 = 25/8\,, \text{ if } \underline{\theta}_r = \theta_1 \\ \Big(\theta_2 q_w(\theta_2) - c\big(q_w(\theta_2)\big)\Big)\frac{53}{60} = 53 q_w(\theta_2)^2/120 = 53/15\,, \text{ if } \underline{\theta}_r = \theta_2 \\ \Big(\theta_3 q_w(\theta_3) - c\big(q_w(\theta_3)\big)\Big)\frac{40}{60} = 40 q_w(\theta_3)^2/120 = 3\,, \text{ if } \underline{\theta}_r = \theta_3 \end{cases}.$$

Given that 53/15 is the largest of the possible profits, the single-product restriction induces the firm to exclude the lowest type household. Nevertheless, welfare increases:

$$W_r = N \sum_{t=2}^{3} \Big(\theta_t q_w(\theta_2) - c\big(q_w(\theta_2)\big)\Big) f_t = N\left(4\sqrt{2} - \frac{9}{5}\right) \approx 3.857N > W_u\,.$$

[11] This example and some of the others below assume a discrete type space. This is done to simplify the exposition. A more general analysis of the discrete-type case is presented in Appendix B.

□

In this example, the improved-quality effect dominates the combination of the exclusion effect and the reduced-quality effect. The next example leads to the opposite welfare ranking.

**Example 2:** Maintain all the assumptions of Example 1, except assume $f_1 = f_2 = 1/6$. Applying the same analysis as before, one finds that $q_u(\theta_1) = 15 - 10\sqrt{2} \approx .8579$, $q_u(\theta_2) = 10\sqrt{2} - 12 \approx 2.142$, and welfare is

$$W_u = \frac{17N}{6}(13\sqrt{2} - 17) \approx 3.924N\,.$$

If a single-product restriction is imposed, then the monopolist's potential profits are the same as in Example 1, except now $\underline{\theta}_r = \theta_2$ implies profits of $5/6 \times (2\sqrt{2})^2/2 = 10/3$. Hence, as in the earlier example, the monopolist would set quality at $2\sqrt{2}$ and price at 8. Welfare would, accordingly, be

$$W_r = N\big(4\sqrt{2} - 2\big) \approx 3.657N\,,$$

and the single-product restriction harms welfare. □

The next result provides a technical condition for determining the net effect of single-product restriction on total surplus. We apply that condition after stating the result.

**Proposition 3** *Suppose: the distribution of household types has a non-decreasing hazard rate; $c(q) = q^2/2$; $\theta - m(\theta) = c(\underline{q})/\underline{q}$ implies $\theta > c'(\underline{q})$; and $\frac{1}{2}\theta = \theta - m(\theta)$ implies $\theta - m(\theta) > c'(\underline{q})$. Then, if*

$$\int_{\underline{\theta}_r}^{\bar{\theta}} \big((\theta - \underline{\theta}_r)^2 - m(\theta)^2\big) f(\theta) d\theta \geq 0\,, \tag{19}$$

*a single-product restriction lowers welfare relative to the unrestricted monopoly outcome.*

Here and elsewhere, proofs not given in the text may be found in Appendix A.

The exclusion effect is operating and, thus, welfare is reduced by a single-product restriction if the improved-quality effect is not too large relative to the reduced-quality effect. Condition (19) ensures this is the case. Because the reduced quality to high types generates the reduced-efficiency effect, and the improved quality to the "middle types" (those in an interval above $\underline{\theta}_r$) generates the improved-efficiency effect, the balance between the two is a function of the difference in quality under the two regimes,

$$\big(\theta - m(\theta)\big) - \underline{\theta}_r\,,$$

and the distribution of types greater than $\underline{\theta}_r$. If the former is not too small for middle types and large enough for high types, while the distribution is appropriately balanced across the types, then improved-quality effect does not dominate the reduced-quality effect. This intuition is formalized by the following corollary.

**Corollary 2** *Suppose: the distribution of household types has a non-decreasing hazard rate; $c(q) = q^2/2$; $\theta - m(\theta) = c(\underline{q})/\underline{q}$ implies $\theta > c'(\underline{q})$; and $\frac{1}{2}\theta = \theta - m(\theta)$ implies $\theta - m(\theta) > c'(\underline{q})$. Then, if*

*(i) $m'(\theta) \geq -1$ for all $\theta \geq \underline{\theta}_r$; and*

*(ii) $\int_{\underline{\theta}_r}^{\bar{\theta}} \Big( \big(\theta - m(\theta)\big) - \underline{\theta}_r \Big) f(\theta) d\theta \geq 0\,,$*

*then a single-product restriction lowers welfare relative to the unrestricted monopoly outcome.*

**Proof:** Observe

$$\begin{aligned}\big((\theta - \underline{\theta}_r)^2 - m(\theta)^2\big) &= \big((\theta - \underline{\theta}_r) - m(\theta)\big) \times \big((\theta - \underline{\theta}_r) + m(\theta)\big)\\ &= \Big(\big(\theta - m(\theta)\big) - \underline{\theta}_r\Big) \times \big((\theta - \underline{\theta}_r) + m(\theta)\big)\,.\end{aligned}$$

Given (i) and the fact that $m(\cdot)$ is non-increasing, both terms are monotonically increasing in $\theta$. It follows from Chebyshev's inequality (see, *e.g.,* Theorem 7.1 of Pečarić et al., 1992) that condition (19) is satisfied, and we can apply Proposition 3. ■

As an example, suppose $c(q) = q^2/2$ and $\theta$ is uniformly distributed over $[0, \bar{\theta}]$.[12] For the uniform distribution, $m(\theta) = \bar{\theta} - \theta$, so condition (i) of the corollary is satisfied. Observe that

$$\begin{aligned}\int_{\underline{\theta}_r}^{\bar{\theta}} \Big( \big(\theta - m(\theta)\big) - \underline{\theta}_r \Big) f(\theta) d\theta &= \int_{\underline{\theta}_r}^{\bar{\theta}} \big(2\theta - (\bar{\theta} + \underline{\theta}_r)\big) \frac{1}{\bar{\theta}} d\theta\\ &= \left.\frac{\theta^2 - \theta(\bar{\theta} + \underline{\theta}_r)}{\bar{\theta}}\right|_{\underline{\theta}_r}^{\bar{\theta}} = 0\,.\end{aligned}$$

[12] To simplify the exposition, here and in other examples below we sometimes assume $\underline{q} = 0$. This assumption does not affect the conclusions drawn from these examples.

Hence, condition (ii) of the corollary is satisfied. It follows that welfare is reduced by the imposition of a single-product restriction.

A single-product restriction reduces the monopolist's profits.

Hence, a necessary condition for such a restriction to raise total surplus is that it raise consumer surplus. Conversely, if a single-product restriction reduces aggregate consumer surplus, then it must also reduce welfare.

It is possible, however, that welfare can fall as a consequence of a single-product restriction, while aggregate consumer surplus can increase, as the following example illustrates.

**Example 3:** Assume that $c(q) = q^2/2$ and $\theta$ is uniformly distributed on $[\gamma, \gamma{+}1]$, $\gamma \geq 0$.[13] From Corollary 2, we know that a single-product restriction reduces welfare if $\gamma < 2(\gamma+1)/3$ or $\gamma < 2$. Applying the above formulæ, it can be shown that

$$q(\theta) = \max\{0, 2\theta - \gamma - 1\} \text{ and}$$
$$q_r = \max\left\{\gamma, \frac{2}{3}(\gamma+1)\right\},$$

where $q_r$ is equilibrium quality under a single-product restriction. Further calculations reveal:

$$W_u = \begin{cases} \frac{1}{8}(1+\gamma)^3\,, \text{ if } \gamma < 1 \\ \frac{1}{2}\gamma(1+\gamma)\,, \text{ if } \gamma \geq 1 \end{cases} \quad \text{and} \quad W_r = \begin{cases} \frac{1}{9}(1+\gamma)^3\,, \text{ if } \gamma < 2 \\ \frac{1}{2}\gamma(1+\gamma)\,, \text{ if } \gamma \geq 2 \end{cases}.$$

Observe that $W_u \geq W_r$ and, consistent with Corollary 2, is strictly greater for $\gamma < 2$. In other words, in this example a single-product restriction can never increase welfare, but can decrease it.

We can also consider consumer surplus by type. From (6) it follows that

$$CS_u(\theta) = \int_0^\theta q(z)dz = \begin{cases} 0\,, \text{ if } \theta < \frac{1}{2}(1+\gamma) \\ \theta^2 - \theta + \frac{1}{4}(1-\gamma^2)\,, \text{ if } \gamma < 1 \;\&\; \theta \geq \frac{1}{2}(1+\gamma) \\ (\theta-1)(\theta-\gamma)\,, \text{ if } \gamma \geq 1 \end{cases}$$

[13] Observe that, contrary to the assumption stated at the outset of our analysis, the support of $\theta$ in this example is bounded away from 0. At the cost of complication, one can readily extend the support to include 0 by having $\theta$ uniformly distributed on the interval $[0, \gamma)$ with density $\frac{\epsilon}{\gamma}$ and uniformly distributed on the interval $[\gamma, \gamma{+}1]$ with density $1-\epsilon$. For $\epsilon$ sufficiently small, the two examples will lead to identical equilibria.

absent restriction. Similarly,

$$CS_r(\theta) = (\theta - \theta_m)q^{\mathrm{SL}} = \begin{cases} 0\,, \text{ if } \theta < \theta_m \\ \theta\left(\frac{2}{3}(1+\gamma)\right) - \left(\frac{2}{3}(1+\gamma)\right)^2\,, \text{ if } \gamma < 2 \;\&\; \theta \geq \theta_m \\ \gamma(\theta - \gamma)\,, \text{ if } \gamma \geq 2 \end{cases}$$

under a single-product restriction. Depending on the value of $\gamma$, the imposition of a single-product restriction can make all household types worse off, all better off, or simultaneously generate winners and losers. For instance, if $\gamma = 0$, then $CS_u(\theta) = CS_r(\theta) = 0$ if $\theta \leq 1/2$ and $CS_u(\theta) \geq CS_r(\theta)$ if $\theta > 1/2$ (with equality only for $\theta = 5/6$). On the other hand, if $\gamma \geq 2$, then, because $\gamma \geq \theta - 1$, we have $CS_u(\theta) \leq CS_r(\theta)$ (and equal only for $\theta = \gamma$ and $\theta = \gamma + 1$). Finally, consider $\gamma = 3/2$. Straightforward algebra reveals that $\theta \in (\frac{3}{2}, \frac{11}{6}) \cup (\frac{7}{3}, \frac{5}{2}]$ are strictly worse off from a single-product restriction, $\theta \in (\frac{11}{6}, \frac{7}{3})$ are strictly better off, with the remaining types (3/2, 11/6, and 7/3) left indifferent.

Observe that, in the last case ($\gamma = 3/2$), the lowest and highest types are the ones made worse off by a single-product restriction. The harming of the lowest types stems, in part, from the exclusion of the very lowest from the market (*i.e.,* $\theta \in [3/2, 5/3)$), but it also stems from the higher price the remaining low types must pay (*i.e.,* $\theta \in [5/3, 11/6)$). The highest types lose because the information rent they capture to induce them not to mimic moderate types is lost. The intermediate types win because they tend to get a superior quality good at a relatively low price (for instance, a type with $\theta = 25/12$ purchases the same quality, but pays 1/16 less under a single-product restriction.

We summarize the preceding analysis as follows.

**Proposition 4** *Suppose that $c(q) = q^2/2$ and household types are distributed uniformly on the interval $[\gamma, \gamma{+}1]$. A single-product restriction strictly lowers total surplus if $\gamma < 2$ and has no effect on total surplus otherwise. With respect to consumer surplus, the restriction:*

*(i) harms almost every household type if $\gamma \leq 1$;*

*(ii) results in a positive measure of consumer types doing worse, specifically,*

$$\theta \in \big(\gamma, (4+\gamma)/3\big) \cup \big((1+4\gamma)/3, \gamma+1\big)$$

*and a positive measure of consumer types doing better specifically,*

$$\big((4+\gamma)/3, (1+4\gamma)/3\big)$$

*if* $\gamma \in (1,2)$*; and*

*(iii) benefits almost every household type if* $\gamma \geq 2$.

□

With regard to the political economy of single-product restrictions, Example 3 shows it is possible to have situations in which the majority of households favor a single-product restriction, the producer opposes it, and imposition of the restriction would reduce overall welfare.

## 2.5 Alternative cost assumptions

In this subsection, we briefly discuss three extensions with respect to our modeling of costs. First, our model assumes that there are no product-specific fixed costs. In the presence of fixed costs, variety is costly as well as potentially beneficial. Katz (1980) has shown that, in the presence of such fixed costs, a profit-maximizing monopolist may offer more or fewer than the total-surplus-maximizing number of products. Thus, in some cases, restricting the number of products offered by the monopolist would exacerbate the distortion, and in other cases, it would ameliorate it.

Second, we have assumed that there are no economies of scale in production. Hence, marginal-cost pricing would allows the firm to cover its costs. The presence of economies of scale can give rise to an additional social benefit of allowing the firm to offer a product line: a multiproduct firm has a greater ability to cover its costs. Moreover, in a model that required investment in production facilities, an unrestricted monopolist would have greater investment incentives than would a restricted firm.

Third, we have assumed that unit costs rise with quality. Some participants in the network neutrality debate have argued that increased quality is essentially costless, at least up to some point. We doubt the empirical validity of this claim, but it is nonetheless of interest to examine the case in which $c(q) \equiv c$ for all $q$ not exceeding some maximum possible quantity, $\bar{q}$.

Observe that, when $c(q) \equiv c$, there is no social benefit of variety in the following sense. Under the first-best outcome, all households that make purchases consume the highest quality available, $\bar{q}$. Now, consider the profit-maximizing outcomes. Recall that the marginal contribution to the unrestricted monopolist's profits by an increase in $q(\theta)$ is proportional to $\theta - m(\theta) - c'\big(q(\theta)\big)$ (see equation (7) above). It can be seen by inspection that the monopolist's problem of picking a quality has a bang-bang solution: when $c'(\theta) = 0$, a type-$\theta$ household either consumes $\bar{q}$ (if $\theta > m(\theta)$) or is excluded from the market (if $\theta < m(\theta)$). Given that the monopolist offers only a single product absent any restriction, imposing a single-product restriction has no effect on the equilibrium outcome.

Summarizing this analysis,

**Proposition 5** *Suppose $c(q) \equiv c$ for all $q \in [\underline{q}, \bar{q}]$. A single-product restriction has no effect on the set of equilibrium outcomes.*

It should be noted that this result depends on the functional form we have assumed for consumer benefits. Specifically, let $b(q, \theta)$ denote the gross consumption benefits enjoyed by a type-$\theta$ household from consumption of a single unit of the product with quality $q$. The standard screening condition requires that $b_{q\theta}(q, \theta) > 0$. Our functional form imposes the stronger condition that $b_{q\theta}(q, \theta)$ is a positive constant. When this cross partial can vary with $q$, there are cases in which the unrestricted monopolist would offer a non-degenerate product line even though there is no marginal cost of higher quality. Although there is no social value to variety, the monopolist would offer a product line as a screening device (*i.e.,* as a form of second-degree price discrimination). Relative to the first best, this is a privately profitable action that is socially wasteful (some households consume products with inefficiently low quality levels).

A monopolist restricted to offering a single product would offer the efficient quality level, $\bar{q}$. It does not follow, however, that the single-line restriction would raise welfare. One would also have to check whether the single-product monopolist would serve more or fewer households than would the unrestricted monopolist. This remains a question for future research. Observe that—because the quality consumed under a single-product restriction is efficient, but those consumed when a product line is offered are not—a sufficient condition for a single-product restriction to raise welfare is that it not reduce the total number of households served in equilibrium.

The contrapositive of this result is that a necessary condition for offering a product line to raise total surplus is that it lead to more households' being served. This finding parallels the well-known result that a necessary condition for third-degree price discrimination to raise welfare is that total output rise under discrimination.[14] There are, however, important differences between the effects of product lines (or second-degree price discrimination) and third-degree discrimination. First, a simple measure of total output lacks economic meaning when products have different quality levels and quality is costly. Second, the distributions of the welfare effects across market segments are somewhat different. In the case of third-degree price discrimination toward households, discrimination lowers efficiency in high-value markets (prices rise), raises efficiency in low-value markets (prices fall), and has ambiguous effects in middle-value markets. In contrast, "discrimination" in the form of a product line increases efficiency at both the high and low ends of the market, and efficiency losses occur in the middle.

## 2.6 Technologically restricted quality levels

Heretofore, we have allowed the producer to choose the quality level from a continuum of possibilities. In this section, we assume that there are only two technologically feasible quality levels, high ($h$) and low ($\ell$).

We make this assumption for two reasons. One is that often quality levels (*e.g.,* bandwidth, speed, picture quality, etc.) are limited and largely determined by forces beyond the producer's control (*e.g.,* capabilities of routers and computers, the installed base of complementary products, or network effects). The second reason is that our analysis of oligopoly in Section 3 below examines a similar setting because of well-known difficulties of analyzing competition between competing multiproduct firms. Considering a similar monopoly model helps identify the role competition plays in that analysis.

A household of type $\theta$ obtains gross benefit $h\theta$ from consuming a unit of the high-quality good (service) and $\ell\theta$ from consuming a unit of the low-quality good (service). Let $c > 0$ be the cost of providing a unit of the high-quality good. For convenience, we set the cost of providing a unit of the low-quality good to 0. So that there is a welfare benefit to high quality, we assume $h\bar{\theta} - c > \bar{\theta}\ell$. Observe that there is also a welfare benefit from offering the low-quality product (*i.e.,* $h\theta - c < \ell\theta$ for $\theta$ sufficiently close to zero).

At prices $p_h$ and $p_\ell$, a type-$\theta$ household is indifferent between the two qualities if and only if

$$h\theta - p_h = \ell\theta - p_\ell \, .$$

[14]See, *e.g.,* Varian (1985) and references therein.

Solving this expression for $\theta$ and noting that the left-hand side increases faster in $\theta$ than the right-hand side, we can conclude that all types such that

$$\theta \geq \frac{p_h - p_\ell}{h - \ell}$$

prefer high to low quality. If the ratio on the right-hand side exceeds 1, then there is no demand for the high-quality product.

A household prefers a unit of low quality to no units at all if and only if

$$\ell\theta - p_\ell \geq 0\,.$$

Provided

$$\frac{p_h - p_\ell}{h - \ell} > \frac{p_\ell}{\ell}\,, \tag{20}$$

demand by type can be given as

$$\text{demand of type } \theta = \begin{cases} \text{no units}\,, \text{ if } \theta < \frac{p_\ell}{\ell} \\ \text{one unit of } \ell \text{ quality}\,, \text{ if } \frac{p_\ell}{\ell} \leq \theta < \frac{p_h - p_\ell}{h-\ell} \\ \text{one unit of } h \text{ quality}\,, \text{ if } \frac{p_h - p_\ell}{h-\ell} \leq \theta \end{cases}. \tag{21}$$

We first characterize the equilibrium absent a single-product restriction.

**Lemma 5** *Suppose only two quality levels are technologically feasible and the distribution of household types has a non-decreasing hazard rate. An unregulated monopolist will offer both qualities and set prices defined by*

$$p_\ell = \ell m\left(\frac{p_\ell}{\ell}\right) \qquad \text{and} \qquad p_h = p_\ell + c + (h - \ell)m\left(\frac{p_h - p_\ell}{h - \ell}\right). \tag{22}$$

It follows from this result that the marginal household types for consumption of the low- and high-quality products satisfy

$$\theta_\ell = m(\theta_\ell) \tag{23}$$

and

$$\theta_h = \frac{c}{h - \ell} + m(\theta_h)\,, \tag{24}$$

respectively.

Welfare absent a single-product restriction is

$$N \int_{\theta_\ell}^{\theta_h} \ell\theta f(\theta)d\theta + N \int_{\theta_h}^{\bar{\theta}} (\theta h - c) f(\theta) d\theta \,. \tag{25}$$

The derivative of this expression with respect to $\theta_h$ is

$$\big(c - \theta_h(h-\ell)\big) f(\theta_h) \,.$$

By (24), this last expression is equal to $-m(\theta_h)(h-\ell)f(\theta_h) < 0$. Hence, welfare would rise if the monopolist lowered $\theta_h$ while holding $\theta_\ell$ constant. In other words, conditional on the total population it serves, the monopolist provides the high-quality product to too few households.

Now, suppose that regulation forces the firm to offer at most one of the two possible products. A single-product monopolist chooses price, $p_r$, to maximize

$$N\left(1 - F\big(\frac{p_r}{q_r}\big)\right)\big(p_r - c(q_r)\big) \,, \tag{26}$$

where, recall, $c(h) = c$ and $c(\ell) = 0$.

**Lemma 6** *Suppose that only two quality levels are technologically feasible and the distribution of household types has a non-decreasing hazard rate. If the monopolist offers a single product of quality $q_r$ it maximizes its profits by charging a price, $p_r$, that satisfies*

$$p_r = c(q_r) + qm\left(\frac{p_r}{q_r}\right) \,. \tag{27}$$

**Proof:** The proof is similar to that of Lemma 5 and is omitted. ■

It follows from this result that

$$\underline{\theta}_r = \frac{c(q_r)}{q_r} + m(\underline{\theta}_r) \,.$$

When $q_r = \ell$, this expression is identical to (23). Thus, if a single-product restriction induces the monopolist to sell only the low-quality product, then the restriction has no effect on the set of household types served in equilibrium. However, the restriction reduces the set of households consuming the high-quality product to zero from a level that would be too low absent the restriction. We can thus conclude:

**Proposition 6** *Suppose that only two quality levels are technologically feasible and the distribution of household types has a non-decreasing hazard rate. If the monopolist chooses the low-quality product when restricted to offering a single product, then the restriction lowers total surplus.*

The next result relates the restricted monopolist's product choice to the underlying, exogenous parameters of the model.

**Corollary 3** *Suppose that only two quality levels are technologically feasible and the distribution of household types has a non-decreasing hazard rate. A single-product restriction will induce the firm to offer only the low-quality product and reduce welfare if*

$$\Bigl(1 - F\bigl(u^{-1}(0)\bigr)\Bigr)u^{-1}(0)\ell > \Bigl(1 - F\bigl(u^{-1}(\frac{c}{h})\bigr)\Bigr)\left(u^{-1}(\frac{c}{h})h - c\right) , \tag{28}$$

*where* $u(x) = x - m(x)$.

**Proof:** $u(\cdot)$ is strictly monotonic, so its inverse exists. Expression (28) follows from (26) and (27). ■

Lastly, suppose that the restricted monopolist offers only the high-quality product. We have found numerous examples (*e.g.,* household types are uniformly distributed) in which the restriction lowers total surplus, but have been unable to construct an example in which it raises total surplus. However, we also have been unable to construct a proof that such an example does not exist.

# 3 Oligopoly

We now examine a market in which there are two firms (*e.g.,* a local telephone company and a cable company). An important issue is whether the firms will choose to compete head to head or offer different products than one another to avoid direct competition.

A number of previous authors have addressed this question. For single-product firms, Hotelling (1929) found that minimum differentiation would result as each firm staked out the middle of the famous Hotelling line. Shaked and Sutton (1982), however, analyzed a multi-stage game in which single-product firms choose to differentiate themselves in the first stage (*i.e.,* choose product locations that are not near one another) in order to relax second-stage price competition. Brander and Eaton (1984), Champsaur and Rochet (1989), and others have extended this analysis to multiproduct firms. However, DeFraja (1996) finds that firms offer identical product lines in a single-stage game with competition in quantities and vertically differentiated products.

## 3.1 A Cournot Model

Because our focus in this section is on competitive effects and the possible consequences of single-product restrictions, we assume now—as we did in Section 2.6—that there are only two technologically feasible qualities, high ($h$) and low ($\ell$). We maintain all the assumptions of Section 2.6. In addition, for tractability, we assume household types are uniformly distributed on the unit interval.

The industry is made up of two Cournot competitors, $A$ and $B$. Let $x_q^i$ denote the number of units of the $q$-quality good supplied by firm $i$. We assume that the firms make their quantity choices simultaneously and that prices are determined by the Walrasian auctioneer to equate quantity supplied with demand. Let $X_q$ denote the total supply of the $q$-quality good. Let $p_q$ denote the price for the $q$-quality good.

Much of the demand analysis from Section 2.6 carries over to our model of Cournot duopoly.

**Lemma 7** *If $X_h < N$ and $X_\ell > 0$, then the market-clearing prices satisfy* (20).

**Proof:** Suppose not. If (20) doesn't hold at the equilibrium prices, then the equilibrium quantity demanded of the low-quality product is 0. Given there is positive supply of the low-quality good, this implies $p_\ell = 0$. Given that $p_h = 0$ leads to excess demand for high quality, $p_h > 0$. But $p_h > 0 = p_\ell$ satisfy (20), a contradiction. ■

We will proceed under the assumption, which we will verify holds in equilibrium, that $X_h + X_\ell \le N$; that is, there is not an excess supply of units. By Lemma 7, we can use (21) to determine the market-clearing prices. These prices satisfy

$$N \times \left(1 - \frac{p_h - p_\ell}{h - \ell}\right) = X_h \quad \text{and} \quad N \times \left(\frac{p_h - p_\ell}{h - \ell} - \frac{p_\ell}{\ell}\right) = X_\ell \,. \tag{29}$$

Solving (29) for $p_h$ and $p_\ell$ yields

$$p_h = \frac{hN - hX_h - \ell X_\ell}{N} \quad \text{and} \quad p_\ell = \frac{\ell(N - X_h - X_\ell)}{N}. \tag{30}$$

## 3.2 The unrestricted equilibrium

Consider firm $A$'s choice of how much of the two products to supply as a best response to $x_h^B$ and $x_\ell^B$. Firm $A$'s choices of $x_h^A$ and $x_\ell^A$ maximize

$$\begin{aligned} &\left(\frac{hN - h(x_h^A + x_h^B) - \ell(x_\ell^A + x_\ell^B)}{N} - c\right) x_h^A \\ &\qquad + \frac{\ell\big(N - (x_h^A + x_h^B) - (x_\ell^A + x_\ell^B)\big)}{N} x_\ell^A, \end{aligned} \tag{31}$$

where we have used (30) to substitute out $p_h$ and $p_\ell$. The first-order conditions for an interior maximum of (31) with respect to $x_h^A$ and $x_\ell^A$ are:

$$\frac{h(N - 2x_h^A - x_h^B) - \ell(2x_\ell^A + x_\ell^B)}{N} - c = 0 \tag{32}$$

and

$$\frac{\ell(N - 2x_h^A - x_h^B - 2x_\ell^A - x_\ell^B)}{N} = 0, \tag{33}$$

respectively.[15] Using (33) to simplify (32), we have

$$(h - \ell)(N - 2x_h^A - x_h^B) = Nc.$$

Solving for $x_h^A$ yields

$$x_h^A = \frac{(N - x_h^B)(h - \ell) - Nc}{2(h - \ell)}. \tag{34}$$

Substituting (34) into (33) and solving for $x_\ell^A$ yields

$$x_\ell^A = \frac{Nc - x_\ell^B(h - \ell)}{2(h - \ell)}. \tag{35}$$

[15] Direct calculations show that there are no corner equilibria.

It is readily verified that the second-order conditions are satisfied.

Given the symmetry of the firms, a strategy for solving for the Nash equilibrium is to set $x_h^A = x_h^B$ and $x_\ell^A = x_\ell^B$ and, then, solve expressions (34) and (35). Doing so yields

$$x_\ell^A = x_\ell^B = \frac{Nc}{3(h-\ell)}$$

and

$$x_h^A = x_h^B = \frac{N(h-\ell-c)}{3(h-\ell)}\,.$$

The linearity of (34) and (35) ensure this solution is unique.

Observe that, as required, there is no excess supply in equilibrium:

$$x_h^A + x_h^B + x_\ell^A + x_\ell^B = \frac{2N(h-\ell-c)}{3(h-\ell)} + \frac{2Nc}{3(h-\ell)} = \frac{2}{3}N\,.$$

Equilibrium prices are

$$p_\ell = \frac{\ell}{3} \quad \text{and} \quad p_h = \frac{h+2c}{3}\,.$$

Observe, as a consequence, that equilibrium purchases as a function of type are

$$\text{type } \theta \text{ purchases} = \begin{cases} \text{no units}\,, \text{ if } \theta < \frac{1}{3} \\ \text{one unit of } \ell \text{ quality}\,, \text{ if } \frac{1}{3} \leq \theta < \frac{1}{3} + \frac{2c}{3(h-\ell)} \\ \text{one unit of } h \text{ quality}\,, \text{ if } \frac{1}{3} + \frac{2c}{3(h-\ell)} \leq \theta \end{cases}. \tag{36}$$

Realized welfare is

$$\begin{aligned} W_C \equiv N & \left( \int_{\frac{1}{3}}^{\frac{1}{3}+\frac{2c}{3(h-\ell)}} \ell\theta d\theta + \int_{\frac{1}{3}+\frac{2c}{3(h-\ell)}}^{1} (h\theta - c) d\theta \right) \\ &= \frac{4N\big((h-c)^2 + \ell(2c-h)\big)}{9(h-\ell)}\,. \end{aligned} \tag{37}$$

## 3.3 The restricted equilibrium

Now suppose that a firm is limited to offering a single quality. There are three possible market configurations: both firms offer low quality, both offer high quality, and one offers low while the other offers high. In what follows, we limit attention to pure-strategy equilibria.

### 3.3.1 Both firms offer the same quality

If the firms offer products of the same quality, then this is the standard Cournot model with linear demand

$$D(p) = N\left(1 - \frac{p}{q}\right) .$$

As is well known, each firm, $i$, supplies

$$x_q^i = \frac{N\big(q - c(q)\big)}{3q} . \tag{38}$$

Correspondingly, each firm's profit is

$$\frac{N\big(q - c(q)\big)^2}{9q} .$$

We also need to address the possibility of a deviation to supplying the other product. Consider a deviation from a symmetric equilibrium in which quality is $q$. Let $q'$ be the alternative quality. Then the price the deviating firm receives is, from (30),

$$\begin{aligned} p_{q'} &= \frac{q'N - q'x_{q'} - \ell x_q}{N} \\ &= \frac{q'N - q'x_{q'} - \ell\frac{N\left(q-c(q)\right)}{3q}}{N} \end{aligned}$$

Familiar analysis reveals that, for a firm deviating to positive production of the other quality, the profit-maximizing deviation is

$$x_{q'} = \begin{cases} \frac{N(2h+c)}{6h}, & \text{if } q' = \ell \\ \frac{N(3h-3c-\ell)}{6h}, & \text{if } q' = h \end{cases} = \frac{N\big(2h + c - \ell + q' - 4c(q')\big)}{6h} .$$

Further calculations reveal that a deviating firm's profits would be $x_{q'}^2 \frac{q'}{N}$. A same-quality equilibrium exists, therefore, if

$$x_{q'}^2 \leq \frac{N^2\big(q - c(q)\big)^2}{9} \times \frac{1}{q'q}$$

or, equivalently, if

$$\frac{2h + c - \ell + q' - 4c(q')}{6h} \leq \frac{q - c(q)}{3} \times \frac{1}{\sqrt{h}\sqrt{\ell}}\,. \tag{39}$$

If $q = h$ and $q' = \ell$ (*i.e.,* if both firms are to offer high quality in the purported equilibrium), then (39) becomes

$$c \leq \frac{2\sqrt{h}\big(h - \sqrt{h}\sqrt{\ell}\big)}{2\sqrt{h} + \sqrt{\ell}}\,. \tag{40}$$

This condition is clearly satisfied for $c$ small enough.

If $q = \ell$ and $q' = h$ (*i.e.,* if both firms are to offer low quality in the purported equilibrium), then (39) becomes

$$c \geq h - \frac{1}{3}\big(\ell + 2\sqrt{h}\sqrt{\ell}\big)\,. \tag{41}$$

Because $\ell < h$, the right-hand side of (41) is less than $h - \ell$. Hence, there exist $c$ large enough (but not exceeding the upper bound $h - \ell$) such that (41) is satisfied.

**Lemma 8** *The conditions* (40) *and* (41) *cannot be simultaneously met; that is, the existence of symmetric equilibrium in which both firms offer the same quality rules out the existence of another symmetric equilibrium in which both firms offer the other quality.*

**Proof:** We need to establish that the right-hand side of (41) is greater than the right-hand side of (40). Note the right-hand side of (41) is greater than $h - \sqrt{h}\sqrt{\ell}$ because $h > \ell$. Given $\ell > 0$, the right-hand side of (40) is less than $h - \sqrt{h}\sqrt{\ell}$. ■

To summarize the analysis of this section.

**Proposition 7** *There exist parameter values for the Cournot model such that a pure-strategy equilibrium exists in which both firms offer the high-quality product under a single-product restriction (i.e., the values satisfy* (40)*). There exist parameter values such that a pure-strategy equilibrium exists in which both firms offer the low-quality product under a single-product restriction (i.e., the values satisfy* (41)*). The two sets of parameter values do not intersect.*

### 3.3.2 The firms offer different qualities

Without loss of generality, assume that, if an asymmetric equilibrium exists, firm $A$ offers high quality and $B$ offers low quality.

To begin, we derive the equilibrium in quantities given no deviation from the firms' "assigned" quality choices. Expression (30) still applies, so firm $A$ determines its best response by maximizing

$$\left(\frac{hN - hx_h^A - \ell x_\ell^B}{N} - c\right) x_h^A$$

with respect to $x_h^A$. This yields the best response

$$x_h^A = \frac{N(h-c) - \ell x_\ell^B}{2h}\,.$$

Similarly, firm $B$ determines its best response by maximizing

$$\frac{\ell(N - x_h^A - x_\ell^B)}{N} x_\ell^B$$

with respect to $x_\ell^B$. This yields the best response

$$x_\ell^B = \frac{N - x_h^A}{2}\,.$$

Mutual best responses are

$$x_h^A = \frac{2N(h-c) - N\ell}{4h - \ell} \quad \text{and} \quad x_\ell^B = \frac{N(h+c)}{4h-\ell}\,,$$

and the corresponding equilibrium prices are

$$p_h = \frac{(2h-\ell)(h+c)}{4h-\ell} \quad \text{and} \quad p_\ell = \frac{(h+c)\ell}{4h-\ell}\,. \tag{42}$$

The corresponding equilibrium profits are

$$\pi^A = \frac{Nh(2h - 2c - \ell)^2}{(4h - \ell)^2} \quad \text{and} \quad \pi^B = \frac{N\ell(h + c)^2}{(4h - \ell)^2}$$

Now consider deviations where a firm shifts the product of which it offers a positive quantity. First, suppose $B$ deviated by offering high quality. From standard Cournot analysis, its best response to $x_h^A$ is

$$\tilde{x}_h^B = \frac{Nh - Nc - hx_h^A}{2h} = x_\ell^B - \frac{Nc}{2h}\,.$$

Corresponding profits are

$$\begin{aligned}\tilde{\pi}^B &= \frac{Nh - (x_h^A + \tilde{x}_h^B)h - Nc}{N}\tilde{x}_h^B \\ &= \frac{h}{\ell}\pi^B - cx_\ell^B + \frac{Nc^2}{4h}\,.\end{aligned}$$

Observe that, for $c$ small enough, firm $B$ would do better to deviate than to play the purported equilibrium. Straightforward calculations reveal

$$\frac{d(\pi^B - \tilde{\pi}^B)}{dc} = \frac{\big(2h^2(2h + \ell) - c(4h^2 - 8h\ell + \ell^2)\big)N}{2h(4h - \ell)^2}\,.$$

It is readily shown that this expression is positive for $c \le h - \ell$. Hence, we can establish that equilibria exist with mixed qualities for a range of $c$ if there exists a $c^* < h - \ell$ such that deviation profits, $\tilde{\pi}^B$, do not exceed equilibrium profits, $\pi^B$ for all $c \in [c^*, h - \ell)$. Observe the condition that deviating be unprofitable can be expressed as $\sqrt{\tilde{\pi}^B} \le \sqrt{\pi^B}$. Expanding both terms and canceling a common $\sqrt{N}$ from both sides, this translates to

$$\frac{2h^2 - c(2h - \ell)}{2\sqrt{h}(4h - \ell)} \le \frac{(h + c)\sqrt{\ell}}{4h - \ell}\,.$$

Solving, we have

$$c^* = \frac{2h\big(h - \sqrt{h}\sqrt{\ell}\big)}{2\sqrt{h}\big(\sqrt{h} + \sqrt{\ell}\big) - \ell}\,.$$

Observe

$$c^* < \frac{2h(h - \ell)}{2\sqrt{h}(\sqrt{h} + \sqrt{\ell}) - \ell} < h - \ell\,,$$

where the inequalities follow because $\sqrt{h\ell} > \ell > 0$. For future reference, we establish that $3c^* > h - \ell$:

$$3c^* - (h-\ell) = \frac{6h(h-\sqrt{h}\sqrt{\ell}) - (h-\ell)\big(2\sqrt{h}(\sqrt{h}+\sqrt{\ell})-\ell\big)}{2\sqrt{h}(\sqrt{h}+\sqrt{\ell})-\ell}$$

Simplified, the numerator is

$$\big(\sqrt{h}-\sqrt{\ell}\big)^2(4h-\ell) > 0\,;$$

hence, $3c^* > h - \ell$ as was to be shown.

The other possible deviation from the mixed-quality equilibrium is that $A$ chooses to offer low quality. From standard Cournot analysis, its best response to $x_\ell^B$ is

$$\tilde{x}_\ell^A = \frac{N - x_\ell^B}{2} = \frac{x_h^A + x_\ell^B}{2}\,.$$

Corresponding profits are

$$\begin{aligned}\tilde{\pi}^A &= \frac{N\ell - (x_\ell^B + \tilde{x}_\ell^A)\ell}{N}\tilde{x}_\ell^A \\ &= \frac{\ell}{N}\left(\frac{x_h^A + x_\ell^B}{2}\right)^2.\end{aligned}$$

Note that

$$\pi^A = \frac{h}{N}\big(x_h^A\big)^2\,.$$

It follows that $A$ won't find it profitable to deviate to low quality if

$$\sqrt{\ell}\frac{x_h^A + x_\ell^B}{2} \le \sqrt{h}x_h^A\,.$$

Solving with respect to $c$, this becomes

$$c \le \frac{h(4\sqrt{h}-3\sqrt{\ell}) - \ell(2\sqrt{h}-\sqrt{\ell})}{4\sqrt{h}-\sqrt{\ell}}\,. \tag{43}$$

Straightforward, albeit tedious, algebra reveals that the right-hand side of (43) is strictly greater than $c^*$. We can, therefore, conclude the following.

**Proposition 8** *There exist parameter values for the Cournot model such that, under a single-product restriction, a pure-strategy equilibrium exists in which one firm offers high quality and the other offers low quality.*

From Proposition 7, we know parameter values exist such that there are no pure-strategy equilibria in which the two firms choose the same quality. A question, therefore, is whether a mixed-quality equilibrium exists in such cases. The answer is yes. Straightforward, albeit wicked tedious, algebra reveals that $c^*$ is less than the right-hand side of (40) and the right-hand side of (43) exceeds the right-hand side of (41). We can conclude as follows.

**Proposition 9** *For all parameter values, at least one pure-strategy equilibrium exists in the Cournot model when firms are subject to a single-product restriction.*

## 3.4 The welfare effects of a product restriction

If, in equilibrium, the firms choose to offer identical products when each is subject to a single-product restriction, then the marginal household type that purchases the product is

$$\theta_{\mathrm{sym}}(q) = 1 - \frac{x_q^A + x_q^B}{N}\,,$$

where $x_q^i$ is given by (38). The resulting welfare level is

$$W_{\mathrm{sym}}(q) = \int_{\theta_{\mathrm{sym}}(q)}^{1} \big(\theta q - c(q)\big) d\theta = \begin{cases} \frac{4N\ell}{9}\,, & \text{if } q = \ell \\ \frac{4N(h-c)^2}{9h}\,, & \text{if } q = h \end{cases}.$$

Observe that $\theta_{\mathrm{sym}}(\ell) = 1/3$. Hence, by (36), the same set of households have positive consumption levels (albeit with possibly different qualities) in the unrestricted equilibrium and the restricted equilibrium in which only the low-quality product is offered. From (36), the types who consume the high-quality product in the unrestricted equilibrium are such that $\theta h - c > \theta \ell$. Therefore:

**Lemma 9** *In the Cournot model, if the consequence of imposing a single-product restriction is that both firms offer the low-quality product in equilibrium, then this restriction strictly reduces welfare vis-à-vis the unrestricted equilibrium.*

Next, suppose that both firms offer the high-quality product in the restricted equilibrium.

**Lemma 10** *In the Cournot model, if the consequence of imposing a single-product restriction is that both firms offer the high-quality product in equilibrium, then this restriction strictly reduces welfare vis-à-vis the unrestricted equilibrium.*

**Proof:** By (37),

$$\begin{aligned} W_C - W_{\text{sym}}(h) &= \frac{4}{9}N\frac{h\big((h-c)^2 + \ell(2c-h)\big) - (h-\ell)(h-c)^2}{h(h-\ell)} \\ &= \frac{4}{9}N\frac{\ell\big((h-c)^2 + 2ch - h^2\big)}{h(h-\ell)} \\ &= \frac{4}{9}N\frac{\ell c^2}{h(h-\ell)} > 0\,. \end{aligned}$$

■

Lastly, consider the restricted equilibrium in which the two firms offer different qualities. In this equilibrium, realized demand satisfies

$$\text{type } \theta \text{ purchases} = \begin{cases} \text{no units}\,, \text{ if } \theta < \frac{h+c}{4h-\ell} \\ \text{one unit of } \ell \text{ quality}\,, \text{ if } \frac{h+c}{4h-\ell} \leq \theta < \frac{2(h+c)}{4h-\ell} \\ \text{one unit of } h \text{ quality}\,, \text{ if } \frac{2(h+c)}{4h-\ell} \leq \theta \end{cases}.$$

Recall that this restricted equilibrium exists only if $c \geq c^*$. Earlier, we showed $3c^* > h - \ell$. This implies

$$\frac{h+c}{4h-\ell} > \frac{1}{3}\,,$$

which—when coupled with (36)—implies that fewer households have positive consumption levels in the restricted equilibrium than in the unrestricted equilibrium.

**Lemma 11** *In the Cournot model, if the consequence of imposing a single-product restriction is that one firm offers the high-quality product and the other offers the low-quality product in equilibrium, then this restriction strictly reduces welfare vis-à-vis the unrestricted equilibrium.*

**Proof:** Because we have already established that fewer households are served overall, the result follows if we can also show that the number of households consuming the high-quality product does not rise. By (36) this entails showing that

$$\frac{2(h+c)}{4h-\ell} \geq \frac{1}{3} + \frac{2c}{3(h-\ell)} \,.$$

Straightforward algebra reveals that this inequality holds if

$$c \leq \frac{(h-\ell)(2h+\ell)}{2h+4\ell} \,. \tag{44}$$

Tedious algebra reveals that the right-hand side of (43) is less than the right-hand side of (44). Using the fact that the restricted equilibrium exists only if (43) holds, the result follows. ■

Observe that this proof also establishes that a single-product restriction raises prices and, thus, lowers consumer surplus when it induces the firms to choose vertically differentiated products.

Combining the previous three lemmas, we have:

**Proposition 10** *In the Cournot model, total surplus is higher in the pure strategy equilibrium without a single-product restriction than under any pure strategy equilibrium with a single-product restriction.*

There are two mechanisms through which a single-product restriction harms welfare in our duopoly model. In the unrestricted equilibrium, both firms offer both products. In the restricted equilibrium, the firms sometimes offer identical products and sometimes offer vertically differentiated products. When the firms offer identical products, the single-product restriction reduces welfare by eliminating what would have been efficient variety. When the firms offer vertically differentiated products the loss of direct competition leads to inefficient reductions in consumption levels. Consequently, both consumer and total surplus fall.

# 4 Conclusion

We have formally modeled the effects of product-line restrictions such as those sought by some proponents of network neutrality regulation. For the case of a monopoly service provider, we find that a single-product restriction results in: (a) consumers who would otherwise have consumed a low-quality variant being excluded from the market; (b) consumers "in the middle" of the market consuming a higher and more efficient quality; and (c) consumers at the top of the market consuming a lower and less efficient quality. We find that the net welfare effects can be positive or negative, although the analysis suggests to us that harm is the more likely outcome. Moreover, consumers at the bottom of the market—the ones that a single-product restriction is typically intended to aid—are almost always harmed by the restriction.

In our duopoly analysis, imposition of a single-product restriction always reduces welfare. Absent the restriction, the two firms engage in head-to-head competition across full product lines. In some circumstances, the single-product restriction induces the two firms to offer identical products. The resulting loss of variety reduces welfare. In other circumstances, a restriction on the number of products that each firm is allowed to offer induces the firms to offer non-overlapping, or vertically differentiated, products. Here, the resulting loss of competition harms both consumers and economic efficiency. Lastly, we find that, to the extent that the regulation is intended to eliminate low-quality products, it may fail. Even though any one firm can offer only a single product, various firms can collectively offer a menu of products.

## Appendix A: Proofs

**Proof of Lemma 4 (sufficiency of condition (v):** Assume, counterfactually, that $\underline{\theta}_u > \underline{\theta}_r$ and define $q'$ as in the in-text portion of the proof.[16] By the same revealed preference arguments used in that portion, we know

$$\big(\theta q' - c(q')\big)\big(F(\underline{\theta}_u) - F(\theta)\big) - \big(1 - F(\underline{\theta}_u)\big)(\underline{\theta}_u - \theta)q' \leq 0 \tag{45}$$

for all $\theta \in [\underline{\theta}_r, \underline{\theta}_u]$ (*i.e.,* the unrestricted monopolist would not wish to extend down to $\theta$). Similarly,

$$\big(\underline{\theta}_r q_r - c(q_r)\big)\big(F(\theta) - F(\underline{\theta}_r)\big) - \big(1 - F(\theta)\big)(\theta - \underline{\theta}_r)q_r \geq 0 \tag{46}$$

for all $\theta \in [\underline{\theta}_r, \underline{\theta}_u]$ (*i.e.,* the restricted monopolist would not wish to cutoff

[16]This proof is for the case in which $\Theta = [0, \bar{\theta}]$. The proof for the case in which the type space is discrete is similar. For the sake of brevity, we omit it.

sales at $\theta$). Observe (45) implies

$$\frac{\theta q' - c(q')}{q'} \leq \big(1 - F(\underline{\theta}_u)\big)\frac{\underline{\theta}_u - \theta}{F(\underline{\theta}_u) - F(\theta)}$$

for all $\theta \in [\underline{\theta}_r, \underline{\theta}_u]$. Hence, it is true in the limit as $\theta \to \underline{\theta}_u$:

$$\frac{\underline{\theta}_u q' - c(q')}{q'} \leq m(\underline{\theta}_u)\,. \tag{47}$$

Similarly, we can write (46) as

$$\frac{\underline{\theta}_r q_r - c(q_r)}{q_r} \geq \big(1 - F(\theta)\big)\frac{\theta - \underline{\theta}_r}{F(\theta) - F(\underline{\theta}_r)}$$

for all $\theta \in [\underline{\theta}_r, \underline{\theta}_u]$. Hence, it is true in the limit as $\theta \to \underline{\theta}_r$:

$$\frac{\underline{\theta}_r q_r - c(q_r)}{q_r} \geq m(\underline{\theta}_r)\,. \tag{48}$$

Recall that $-c(q)/q$ is a decreasing function of $q$ and $q' \leq q_r$. By assumption $\underline{\theta}_u > \underline{\theta}_r$, hence the left-hand side of (47) is strictly greater than the left-hand side of (48). Hence,

$$m(\underline{\theta}_u) > m(\underline{\theta}_r)\,;$$

but this means the hazard rate evaluated at $\underline{\theta}_u$ is strictly less than it is evaluated at $\underline{\theta}_r$, which contradicts the assumption that the hazard rate is non-decreasing. The result follows by contradiction. ■

**Proof of Proposition 3:** The first-order condition for the monopolist's choice of marginal household type when the firm is restricted to $q = \underline{q}$ is

$$\underline{q}\big(1 - F(\theta)\big) - \big(\theta - c(\underline{q})\big)f(\theta) = 0\,,$$

which can be rewritten as

$$\theta - m(\theta) = \frac{c(\underline{q})}{\underline{q}}\,. \tag{49}$$

The non-decreasing hazard rate assumption implies that the left-hand side is increasing in $\theta$ and, hence, (49) has a unique solution. But at that solution,

$q_w(\theta) > \underline{q}$ because, by hypothesis, $\theta - m(\theta) = c(\underline{q})/\underline{q}$ implies $\theta > c'(\underline{q})$ means that $q_r > \underline{q}$. Hence, a monopolist limited to a single quality would choose a $q_r > \underline{q}$.

Given this and the fact that $c(q) = q^2/2$, it follows that $q_w(\underline{\theta}_r) = \underline{\theta}_r$.

Because $m(\cdot)$ is non-increasing and $c'(q) = q$, it follows that

$$q_u(\theta) = \theta - m(\theta) \tag{50}$$

if $\theta - m(\theta) \geq c'(\underline{q})$. Observe

$$\underline{\theta}_r = c'(q_r) = q_r \text{ and } \underline{\theta}_r - m(\underline{\theta}_r) = \frac{c(q_r)}{q_r} = \frac{1}{2}q_r\,.$$

It follows that $q_u(\underline{\theta}_r) > \underline{q}$. By Proposition 1, $\underline{\theta}_u < \underline{\theta}_r$.

The difference in welfare is

$$W_u - W_r = N\int_{\underline{\theta}_u}^{\bar{\theta}}\Big(\theta q_u(\theta) - c\big(q_u(\theta)\big)\Big)f(\theta)d\theta - N\int_{\underline{\theta}_r}^{\bar{\theta}}\big(\theta\cdot\underline{\theta}_r - \frac{1}{2}\underline{\theta}_r^2\big)f(\theta)d\theta\,.$$

Simplifying, we have

$$\begin{aligned} W_u - W_r &> N\int_{\underline{\theta}_r}^{\bar{\theta}}\left(\frac{1}{2}\big(\theta^2 - m(\theta)^2\big) - \big(\theta\cdot\underline{\theta}_r - \frac{1}{2}\underline{\theta}_r^2\big)\right)f(\theta)d\theta \\ &\qquad\qquad \text{(because } \underline{\theta}_r > \underline{\theta}_u\text{)} \\ &= \frac{N}{2}\int_{\underline{\theta}_r}^{\bar{\theta}}\big((\theta - \underline{\theta}_r)^2 - m(\theta)^2\big)f(\theta)d\theta \geq 0 \qquad \text{(by condition (19))} \end{aligned}$$

■

**Proof of Lemma 5:** Consider the monopolist's maximization problem

$$\max_{\{p_\ell,p_h\}} N\left(1 - F\big(\frac{p_h - p_\ell}{h - \ell}\big)\right)(p_h - c) + N\left(F\big(\frac{p_h - p_\ell}{h - \ell}\big) - F\big(\frac{p_\ell}{\ell}\big)\right)p_\ell\,. \tag{51}$$

For the moment, we ignore the constraints (*i.e.,* that (20) hold and that both fractions in (20) be between 0 and $\bar{\theta}$). As we will demonstrate, they are not binding. In what follows, we are free to ignore $N$. Make the change of variables $\Delta_p = p_h - p_\ell$ and let $\Delta_q = h - \ell$. Then the program (51) is equivalent to the following program.

$$\max_{\{p_\ell,\Delta_p\}} \left(1 - F\big(\frac{\Delta_p}{\Delta_q}\big)\right)(\Delta_p + p_\ell - c) + \left(F\big(\frac{\Delta_p}{\Delta_q}\big) - F\big(\frac{p_\ell}{\ell}\big)\right) p_\ell$$
$$\equiv \max_{\{p_\ell,\Delta_p\}} \left(1 - F\big(\frac{\Delta_p}{\Delta_q}\big)\right)(\Delta_p - c) + \left(1 - F\big(\frac{p_\ell}{\ell}\big)\right) p_\ell\,. \qquad (52)$$

If the solution to the unconstrained problem satisfies the constraints, then that solution is also the solution to the constrained problem. Observe that (52) can be seen as two independent optimization problems. The first-order conditions for these can written as

$$m\left(\frac{p_\ell}{\ell}\right) - \frac{p_\ell}{\ell} = 0 \qquad (53)$$

and

$$m\left(\frac{\Delta_p}{\Delta_q}\right) - \frac{\Delta_p - c}{\Delta_q} = 0\,. \qquad (54)$$

Because $m(x)$ is non-increasing in $x$, while $x$ (or $x - c/\Delta_q$) is strictly increasing in $x$, we know (i) that if solutions exist they must be unique and (ii) they satisfy the second-order condition because

$$m(x-\varepsilon) - (x-\varepsilon) > 0 \qquad \text{and} \qquad m(x+\varepsilon) - (x+\varepsilon) < 0$$

for $\varepsilon > 0$.

Given (i) the continuity of $m(\cdot)$ and (ii) that $m(0) > 0 = m(\bar{\theta})$, (53) must have a solution such that

$$0 < \frac{p_\ell}{\ell} < \bar{\theta}\,.$$

Because $m(\cdot)$ is non-increasing and $c > 0$, it follows that, if (54) has a solution, then it satisfies

$$\frac{\Delta_p}{\Delta_q} > \frac{p_\ell}{\ell}\,;$$

that is, (20). Hence, we're done if we can show that there exists a $\Delta_p$ such that $\Delta_p/\Delta_q < \bar{\theta}$. Suppose there weren't. Then

$$\frac{c - \Delta_p}{\Delta_q} > 0 \quad \text{or, equivalently,} \quad c > \Delta_p$$

for any $\Delta_p \geq \bar{\theta}\Delta_q$. In particular, this would require that

$$c > \bar{\theta}\Delta_q = \bar{\theta}(h - \ell)\,,$$

but that violates the assumption that there is a welfare benefit to high quality. By contradiction, (54) has a solution.

We have shown that the solutions to (53) and (54) maximize (52) and satisfy the relevant constraints. Algebra yields (22). ■

## Appendix B: The Discrete Case

Assume that there is a finite number, $T$, of household types indexed so that $s < t$ implies $\theta_s < \theta_t$. Let $f_t$ denote the probability a randomly drawn household is type $\theta_t$. The distribution, correspondingly, can be denoted

$$F_t = \sum_{\tau=1}^{t} f_t\,.$$

Define

$$m_t \equiv \frac{1 - F_t}{f_t}\,.$$

Note $m_T = 0$. Define

$$R_t(q) \equiv \theta_{t+1}q - \theta_t q\,.$$

The fact that $R_T(\cdot)$ is not defined is not, as will become apparent, an issue. The function $R_t(\cdot)$ is positive, strictly increasing, and convex on $\mathbb{R}_+$. As a consequence, we know from Proposition 2 of Caillaud and Hermalin (1993) that the monopolist's profit-maximization problem with respect to the choice of qualities reduces to the following.[17]

$$\max_{\{q_1,\dots,q_T\}} N \sum_{t=1}^{T} f_t\big(\theta_t q_t - c(q_t) - m_t R_t(q_t)\big) \tag{55}$$

$$\text{subject to } \underline{q} \leq q_1 \leq \cdots \leq q_T\,. \tag{56}$$

[17] Caillaud and Hermalin's Proposition 2 is essentially a fairly straightforward extension of standard results in mechanism design for two types or for a continuum of types to an arbitrary, but finite, number of types.

Let $\{\hat{q}_1, \ldots, \hat{q}_T\}$ denote the solution. Observe this solution must satisfy, for each $t$, the following condition.

$$\theta_t - m_t(\theta_{t+1} - \theta_t) - c'(\hat{q}_t) \gtreqless 0\,, \tag{57}$$

where the expression is an equality if the relevant order restriction, condition (56), doesn't bind, is greater than zero if the upward restriction binds, and is less than if the downward restriction binds.

If we impose the assumption that $m_t$ is non-increasing in $t$ (*i.e.,* a monotone hazard rate), then it can be shown that (56) is not binding except, possibly, at $\underline{q}$. In that case, if $\hat{q}_t = 0$ or $\underline{q}$, then the left-hand side of expression (57) is less than zero.

Welfare when the monopolist is unrestricted is

$$W_u = N \sum_{t=1}^{T} \bigl(\theta_t \hat{q}_t - c(\hat{q}_t)\bigr) f_t\,. \tag{58}$$

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